The symmetry of finite group schemes, Watanabe type theorem, and the a-invariant of the ring of invariants

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Dedicated to Professor Mel Hochster and Professor Craig Huneke

Abstract

Let k be a field, and G be a k-group scheme of finite type. Let G_{ad} be the k-scheme G with the adjoint action of G. We call $\lambda_{G,G} = H^0(\operatorname{Spec} k, e^*(\omega_{G_{\mathrm{ad}}}))$ the Knop character of G, where $e : \operatorname{Spec} k \to G_{\mathrm{ad}}$ is the unit element, and $\omega_{G_{\mathrm{ad}}}$ is the G-canonical module. We prove that $\lambda_{G,G}$ is trivial in the following cases: (1) G is finite, and $k[G]^*$ is a symmetric algebra; (2) G is finite and étale; (3) G is finite and constant; (4) G is smooth and connected reductive; (5) G is abelian; (6) G is finite, and the identity component G° of G is linearly reductive; (7) G is finite and linearly reductive. Let V be a small G-module of dimension $n < \infty$. We assume that $\lambda_{G,G}$ is trivial. Let $H = \mathbb{G}_m$ be the one-dimensional torus, and let V be of degree one as an H-module so that $S = \operatorname{Sym} V^*$ is a $\tilde{G}$-algebra generated by degree one elements, where $\tilde{G} = G \times H$. We set $A = S^G$. Then we have (i) $\omega_A \cong \omega_S^G$ as (H, A)-modules; (ii) $a(A) \le -n$ in general, where $a(A)$ denotes the a-invariant. Moreover, the following are equivalent: (1) The action $G \to GL(V)$ factors through $SL(V)$; (2) $\omega_S \cong S(-n)$ as $(\tilde{G}, S)$-modules; (3) $\omega_S \cong S$ as (G, S)-modules; (4) $\omega_A \cong A(-n)$ as (H, A)-modules; (5) A is quasi-Gorenstein; (6) A is quasi-Gorenstein and $a(A) = -n$; (7) $a(A) = -n$. This partly generalizes recent results of Liedtke–Yasuda `arXiv:2304.14711v2` and Goel–Jeffries–Singh `arXiv:2306.14279v1`.

1. Introduction

Let k be a field, V a finite-dimensional k-vector space, and G a finite subgroup of $GL(V)$. Let $S = \operatorname{Sym} V^* = k[V]$, and $A = S^G$. Hochster and Eagon proved that in non-modular case (that is, the case that the order $|G|$ of G is not divisible by the characteristic of k), A is Cohen–Macaulay. K.-i. Watanabe proved that in non-modular case, $G \subset SL(V)$ if and only if G does not have a pseudo-reflection and A is Gorenstein [Wat1, Wat2]. Since

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then, his result has been generalized by several authors. Fleischmann and Woodcock [FW] and Braun [Bra] proved that if $G \subset GL(V)$ is a finite subgroup without pseudo-reflection, then A is quasi-Gorenstein (or equivalently, $\omega_A \cong A$) if and only if $G \subset SL(V)$.

It has been known that the condition that the finite group G does not have a pseudo-reflection sometimes can be generalized to more general G. The condition is replaced by the condition that $\pi : V = \operatorname{Spec} S \to \operatorname{Spec} A = V/\!/G$ is a principal G-bundle off codimension two or more, and called an almost principal bundle or quasi-torsor [Has4, C-R], and we call this condition 'V is small.' Namely, we say that V is small if there exist some open subset W of $\operatorname{Spec} A$ and G-stable open subset U of $\pi^{-1}(W)$ such that $\operatorname{codim}(V \setminus U, V) \geq 2$, $\operatorname{codim}(V/\!/G \setminus W, V/\!/G) \geq 2$, and $\pi : U \to W$ is a principal G-bundle (or a G-torsor) in the sense that π is faithfully flat, and $\Phi : G \times U \to U \times_W U$ given by $\Phi(g, u) = (gu, u)$ is an isomorphism. Note that if G is a finite constant group, then V is small if and only if $G \subset GL(V)$, and G does not have a pseudo-reflection.

Knop [Kno] pointed out that the equivalence $\omega_A \cong A \iff G \subset SL(V)$ is not true any more even if G is a (disconnected) reductive group over an algebraically closed field of characteristic zero, and the action is small. Letting λ_{ad} be the top exterior power of $\operatorname{Lie}(G)^*$, the dual of the adjoint representation, the triviality of $\det_V \otimes \lambda_{\mathrm{ad}}^*$ was important [Kno, Satz 2]. Note that λ_{ad} is trivial if G is finite, and we can recover Watanabe's original result.

We define $\lambda_{G,G} = H^0(\operatorname{Spec} k, e^*(\omega_{G_{\mathrm{ad}}}))$, and call it the Knop character of G, where $e : \operatorname{Spec} k \to G_{\mathrm{ad}}$ is the unit element, and $\omega_{G_{\mathrm{ad}}}$ is the G-equivariant canonical module of G_{ad}. If, moreover, G is a normal closed subgroup scheme of another affine k-group scheme $\tilde{G}$ of finite type, then $\lambda_{G,G}$ is a character of $\tilde{G}$, and we denote it by $\lambda_{\tilde{G},G}$. Note that $\lambda_{G,G} \cong \lambda_{\mathrm{ad}}^*$, if G is k-smooth. By [Has4, (11.22)], it is easy to see that if V is small, then $\omega_A \cong A(a)$ if and only if $\omega_S = S \otimes \det_{V^*} \cong S(a) \otimes_k \lambda_{\tilde{G},G}$ if and only if $\det_V \cong \lambda_{G,G}$ as G-modules, and $a = -n$. If, moreover, $\lambda_{G,G} \cong k$, then A is quasi-Gorenstein if and only if $G \subset SL(V)$, and if these conditions are satisfied, then $a(A) = -n$. So it is natural to ask, when $\lambda_{G,G}$ is trivial. In [Has4, (11.21)], it is pointed out that if G is finite and linearly reductive, étale, or connected reductive, then $\lambda_{G,G}$ is trivial, but $\lambda_{G,G}$ is nontrivial if k is a field of characteristic not two and $G = O(2)$.

In this paper, we discuss when $\lambda_{G,G}$ is trivial, assuming that G is finite (but not étale). It is well-known that the group algebra kG is symmetric [SY, Example IV.2.6]. A finite dimensional k-Hopf algebra is Frobenius in general [SY, Theorem VI.3.6]. In general, a finite dimensional k-Hopf algebra H is not symmetric even if H is cocommutative, or equivalently, $H = k[G]^*$ for some finite k-group scheme G, see [LS, p. 85]. We prove that $\lambda_{G,G}$ is trivial if and only if the notions of the left integral and the right integral agree in $H = k[G]^*$. The latter condition is called the unimodular property of H. As the square s_H^2 of the antipode s_H of H is the identity, H is unimodular if and only if H is a symmetric algebra, see [Hum, Rad].

As an application of the G-triviality of $\lambda_{G,G}$, we prove the following.

Theorem 3.6. *Let k be a field, G be an affine k-group scheme of finite type, and V be a small G-module of dimension $n < \infty$. We assume that $\lambda_{G,G}$ is trivial. Let $H = \mathbb{G}_m$ be the one-dimensional torus, and let V be of degree one as an H-module so* *that* $S = \operatorname{Sym} V^*$ *is a* $\tilde{G}$*-algebra generated by degree one elements, where* $\tilde{G} = G \times H$*. We set* $A = S^G$*. Then we have*

(i) $\omega_A \cong \omega_S^G$ *as* (H, A)*-modules;*

(ii) $a(A) \leq -n$ *in general, where* $a(A)$ *denotes the a-invariant.*

Moreover, the following are equivalent:

(1) *The action* $G \to GL(V)$ *factors through* $SL(V)$*;*

(2) $\omega_S \cong S(-n)$ *as* $(\tilde{G}, S)$*-modules;*

(3) $\omega_S \cong S$ *as* (G, S)*-modules;*

(4) $\omega_A \cong A(-n)$ *as* (H, A)*-modules;*

(5) A *is quasi-Gorenstein;*

(6) A *is quasi-Gorenstein and* $a(A) = -n$*;*

(7) $a(A) = -n$*.*

For the case that G is finite and constant, the theorem was proved (in a stronger form) by Goel, Jeffries, and Singh [GJS]. Note that they do not require that the action of G on V is small. They proved that $a(A) \leq a(S) = -n$ in general. They also proved that the equality $a(A) = -n$ holds if and only if the image of $G \to GL(V)$ is a subgroup of $SL(V)$ without pseudo-reflections for the case that G is finite and constant [GJS, Proposition 4.1, Theorem 4.4]. It is interesting to ask if these are true for any finite group scheme G. Note also that the equivalence (1)⇔(5) for the case that G is finite linearly reductive (but not necessarily constant) was proved recently by Liedtke and Yasuda [LY]. It also follows from [Has4, (7.61),(11.22)**2**].

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2. Preliminaries

(2.1) Let k be a field, $f : \tilde{G} \to H$ be a homomorphism between affine k-group schemes of finite type with $G = \operatorname{Ker} f$. Let $\mathcal{F}(\tilde{G})$ be the category of $\tilde{G}$-schemes separated of finite type over k. For $(h_Z : Z \to \operatorname{Spec} k) \in \mathcal{F}(\tilde{G})$, *the* $\tilde{G}$-dualizing complex of Z (or better, of h_Z) is $h_Z^!(k)$ by definition, and we denote it by $\mathbb{I}_Z = \mathbb{I}_Z(G)$, where $(-)^!$ denotes the twisted inverse [Has2]. The $\tilde{G}$-canonical module ω_Z is the lowest nonzero cohomology group of $\mathbb{I}_Z$. It is a coherent $\tilde{G}$-module. If $Z = \operatorname{Spec} B$ is affine, $H^0(Z, \omega_Z)$ is denoted by ω_B, and is called the $\tilde{G}$-equivariant canonical module of B. When we forget the $\tilde{G}$-structure, $\mathbb{I}_Z$ is the dualizing complex of the scheme Z without the $\tilde{G}$-action [Has2, (31.17)].

(2.2) A morphism $\varphi : X \to Y$ of $\tilde{G}$-schemes of finite type over k is called a $\tilde{G}$-enriched principal G-bundle if G acts trivially on Y, φ is faithfully flat, and the morphism $\Phi : G \times X \to X \times_Y X$ given by $\Phi(g, x) = (gx, x)$ is an isomorphism. As G is affine, flat, and Gorenstein over $\operatorname{Spec} k$, φ is affine, flat, and Gorenstein.

(2.3) Let X be a scheme and U its open subset. We say that U is n-large if $\operatorname{codim}(X \setminus U, X) \geq n+1$, where we regard that the codimension of the empty set in X is $\infty \geq n+1$.

Definition 2.4 (cf. [Has4, (10.2)]). A diagram of $\tilde{G}$-schemes of finite type

$$X \xleftarrow{i} U \xrightarrow{\rho} V \xrightarrow{j} Y \tag{1}$$

is called a $\tilde{G}$-enriched n-almost rational principal G-bundle if (1) G acts trivially on Y; (2) j is an open immersion, and $j(V)$ is n-large in Y; (3) i is an open immersion, and $i(U)$ is n-large in X; (4) $\rho : U \to V$ is a principal G-bundle. That is, ρ is faithfully flat, and $\Phi : G \times U \to U \times_V U$ given by $\Phi(n, u) = (nu, u)$ is an isomorphism.

(2.5) In what follows, 1-large and 1-almost will simply be called large and almost, respectively. A $\tilde{G}$-morphism $\varphi : X \to Y$ is said to be a $\tilde{G}$-enriched n-almost principal G-bundle with respect to U and V, if U is a $\tilde{G}$-stable open subset of X, V is an H-stable open subset of Y, and the diagram (1) is a $\tilde{G}$-enriched n-almost rational principal G-bundle, where $\rho : U \to V$ is the restriction of φ. We say that a $\tilde{G}$-morphism $\varphi : X \to Y$ is a $\tilde{G}$-enriched n-almost principal G-bundle if it is so with respect to U and V for some U and V.

Lemma 2.6. *Let $\varphi : X \to Y$ be a $\tilde{G}$-enriched almost principal G-bundle between $\tilde{G}$-schemes of finite over k. Assume that X is normal, and that $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism. Let $\operatorname{Ref}(\tilde{G}, X)$ be the category of coherent $(\tilde{G}, \mathcal{O}_X)$-modules which are reflexive as $\mathcal{O}_X$-modules, and Let $\operatorname{Ref}(H, Y)$ be the category of coherent $(H, \mathcal{O}_Y)$-modules which are reflexive as $\mathcal{O}_Y$-modules. Then we have: The functor $\mathcal{G} : \operatorname{Ref}(\tilde{G}, X) \to \operatorname{Ref}(H, Y)$ given by $\mathcal{G}(\mathcal{M}) = (\varphi_* \mathcal{M})^G$ is an equivalence, and $\mathcal{F} : \operatorname{Ref}(H, Y) \to \operatorname{Ref}(\tilde{G}, X)$ given by $\mathcal{F}(\mathcal{N}) = (\varphi^* \mathcal{N})^{**}$ is its quasi-inverse, where $(-)^{**}$ is the double dual.*

Proof. Follows immediately from [Has4, (10.13),(11.3)]. A short and self-contained proof for the case that everything is affine can be found in [HK, (2.4)]. □

(2.7) Let X be a $\tilde{G}$-scheme, and L be a $\tilde{G}$-module. Let $h_X : X \to \operatorname{Spec} k$ be the structure map. Then for a quasi-coherent $(\tilde{G}, \mathcal{O}_X)$-module $\mathcal{M}$, we denote $\mathcal{M} \otimes_{\mathcal{O}_X} h_X^* L$ by $\mathcal{M} \otimes_k L$. Note that G is a normal closed subgroup scheme of $\tilde{G}$. So $\tilde{G}$ acts on G by the adjoint action. We denote this scheme by G_{ad}. Let $e : \operatorname{Spec} k \to G_{\mathrm{ad}}$ be the unit element. It is a $\tilde{G}$-stable closed immersion. We denote the $\tilde{G}$-module $H^0(\operatorname{Spec} k, e^*(\omega_{G_{\mathrm{ad}}}))$ by $\lambda_{\tilde{G},G}$, and we call it the *Knop character* of G (enriched by $\tilde{G}$). If G is k-smooth, then $\lambda_{G,G} \cong \det_{\mathfrak{g}}^*$ by [Has2, (28.11)], where $\mathfrak{g} = \operatorname{Lie} G$ is the adjoint representation of G, and det denotes the top exterior power. Its dual $\det_{\mathfrak{g}} = \lambda_{G,G}^*$ is denoted by λ_{ad} in [Kno], and played an important role in studying Gorenstein property of invariant subrings [Kno, Satz 2].

Lemma 2.8. *Let $\varphi : X = \operatorname{Spec} T \to Y = \operatorname{Spec} B$ be a $\tilde{G}$-enriched almost principal G-bundle which is also a morphism in $\mathcal{F}(\tilde{G})$ with X and Y affine normal. Then $\omega_T = (T \otimes_B \omega_B)^{**} \otimes_k \lambda_{\tilde{G},G}$, where $(-)^{**} = \operatorname{Hom}_T(\operatorname{Hom}_T(-, T), T)$ is the double dual. We also have that $\omega_B = (\omega_T \otimes_k \lambda^*_{\tilde{G},G})^G$. In particular, if moreover, $\lambda_{\tilde{G},G} \cong k$, then $\omega_T \cong (T \otimes_B \omega_B)^{**}$ and $\omega_B \cong \omega_T^G$.*

Proof. This is a special case of [Has4, (11.22)]. □

(2.9) Let Λ be a finite-dimensional k-algebra. We say that Λ is *Frobenius* if ${}_\Lambda\Lambda \cong D(\Lambda_\Lambda)$ as left Λ-modules, where $D = \operatorname{Hom}_k(-, k)$. This is equivalent to say that there is a nondegenerate bilinear form $\beta : \Lambda \times \Lambda \to k$ such that $\beta(ac, b) = \beta(a, cb)$. If, moreover, we can take such a β to be symmetric, we say that Λ is *symmetric*. This is equivalent to say that the bimodule ${}_\Lambda\Lambda_\Lambda$ is isomorphic to $D({}_\Lambda\Lambda_\Lambda)$.

(2.10) Let Γ be a finite dimensional k-Hopf algebra. We define

$$\textstyle\int_\Gamma^l := \{x \in \Gamma^* \mid \forall y \in \Gamma^* \ yx = \epsilon(y)x\},$$

where $\epsilon : \Gamma^* \to k$ is given by $\epsilon(y) = y(1_\Gamma)$. An element of $\int_\Gamma^l$ is called a *left integral* on Γ (or *in* Γ^*, according to the terminology in [Mon]).

(2.11) Note that

$$\textstyle\int_\Gamma^l = (\Gamma^*)^\Gamma = \{\psi \in \Gamma^* \mid \omega_{\gamma^*}(\psi) = \psi \otimes 1\} = \operatorname{Hom}_\Gamma(\Gamma, k).$$

Indeed, if $\gamma_1, \ldots, \gamma_n$ is a k-basis of Γ and $\gamma_1^*, \ldots, \gamma_n^*$ is its dual basis, then the comodule structure ω_{Γ^*} of Γ^* is given by $\omega_{\Gamma^*}(\alpha) = \sum_{i=1}^n \gamma_i^* \alpha \otimes \gamma_i$ for $\alpha \in \Gamma^*$. In other words, $\omega(\alpha) = \sum_{(\alpha)} \alpha_{(0)} \otimes \alpha_{(1)}$ is given by $\sum_{(\alpha)} \langle \beta, \alpha_{(1)} \rangle \alpha_{(0)} = \beta\alpha$ for $\beta \in \Gamma^*$. So $\omega(\psi) = \psi \otimes 1$ is equivalent to say that $\rho\psi = \epsilon(\rho)\psi$ for $\rho \in \Gamma^*$, as desired.

(2.12) We also define $\int_\Gamma^r = \{x \in \Gamma^* \mid \forall y \in \Gamma^* \ xy = \varepsilon(y)x\}$, and an element of $\int_\Gamma^r$ is called a *right integral* on Γ. It is known that $\dim_k \int_\Gamma^l = \dim_k \int_\Gamma^r = 1$ [Swe1, Corollary 5.1.6]. If $\int_\Gamma^l = \int_\Gamma^r$, then we say that Γ^* is unimodular. Radford proved [Rad] that Γ^* is a symmetric algebra if and only if Γ^* is unimodular and $\mathcal{S}^2$ is an inner automorphism of Γ^*, where $\mathcal{S}$ is the antipode of Γ. Suzuki [Suz] constructed an example of a finite dimensional unimodular k-Hopf algebra which is not symmetric.

Lemma 2.13. *Let G be a finite k-group scheme, H an affine k-group scheme of finite type, and $\tilde{G} = G \times H$ is the direct product. Let $\Gamma = k[G]$ be the coordinate ring of G. Then the following are equivalent.*

(1) Γ^* *is symmetric.*

(2) Γ^* *is unimodular.*

(3) $\lambda_{\tilde{G},G} \cong k$ *as $\tilde{G}$-modules.*

(4) $\lambda_{G,G} \cong k$ *as G-modules.*

Proof. As Γ^* is cocommutative, $s^2 = \mathrm{id}_{\Gamma^*}$, where s is the antipode of Γ^*. By [Hum, Theorem 1,2] (see also [Rad]), (1)⇔(2) holds.

We prove (2)⇒(3). Note that Γ^* is a $(G, k[G])$-module. In other words, Γ^* is a Γ-Hopf module. Let $\zeta : \Gamma^* \to \Gamma^* \otimes_k \Gamma$ be the map given by $\zeta(\gamma) = \sum_{(\gamma)} \gamma_{(0)}(\mathcal{S}\gamma_{(1)}) \otimes \gamma_{(2)}$, where $\mathcal{S}$ is the antipode of Γ. By the proof of [Swe1, Theorem 4.1.1], ζ is injective and $\operatorname{Im}\zeta = \int_\Gamma^l \otimes k[G]$. Now let us consider the same map $\zeta : k[G_{\mathrm{ad}}]^* \to k[G_{\mathrm{ad}}]^* \otimes k[G_{\mathrm{ad}}]$. It is easy to see that this is a $(G, k[G_{\mathrm{ad}}])$-homomorphism. Note also that $\int_\Gamma^l$ is a G-submodule of both $k[G_{\mathrm{ad}}]$ and $k[G_r]$, where G_r is the right regular action. As $\int_\Gamma^l = \int_\Gamma^r$, we have that $\int_\Gamma^l$ as the submodule of $k[G_{\mathrm{ad}}]$ is also G-trivial (isomorphic to k). Hence ζ induces an isomorphism $k[G_{\mathrm{ad}}]^* \cong k[G_{\mathrm{ad}}]$ of $(G, k[G_{\mathrm{ad}}])$-modules. As H acts trivially on G_{ad}, the isomorphism is that of $(\tilde{G}, k[G_{\mathrm{ad}}])$-modules. Pulling back this isomorphism by the unit element $e : \operatorname{Spec} k \to G_{\mathrm{ad}}$, we get $\lambda_{\tilde{G},G} \cong k$, as we have $k[G_{\mathrm{ad}}]^* \cong \omega_{k[G_{\mathrm{ad}}]}$ by the duality of finite morphisms, see [Has2, (27.8)].

(3)⇒(4) is trivial.

(4)⇒(2). The argument above shows that $k[G_{\mathrm{ad}}]^* \cong \lambda' \otimes k[G_{\mathrm{ad}}]$, where λ' is $\int_\Gamma^l$ as a G-submodule of $k[G_{\mathrm{ad}}]^*$. As $\int_\Gamma^l$ is trivial as a G-submodule of $k[G_l]$, where G_l is G with the left regular action, we have that λ' agrees with $\int_\Gamma^l$ as a G-submodule of $k[G_r]$. The assumption (4) means $k \cong \lambda \cong \lambda'$. So $\int_\Gamma^l \subset \int_\Gamma^r$. As we know that both $\int_\Gamma^l$ and $\int_\Gamma^r$ are one-dimensional, we have that Γ^* is unimodular. □

3. Main theorem

(3.1) Let S be a k-algebra of finite type on which G acts. Let $A = S^G$ be the ring of invariants. If the canonical map $\operatorname{Spec} S \to \operatorname{Spec} A$ is an almost principal G-bundle, then we say that the G-action on S is small. If V is a G-module and $S = k[V] = \operatorname{Sym} V^*$ is small, then we say that the representation V of G is small. If G is a finite (constant) group, then V is small if and only if the action is faithful, and $G \subset GL(V)$ does not have a pseudo-reflection. Letting each element of V^* of degree one, $S = \operatorname{Sym} V^*$ is a graded G-algebra. So letting $H = \mathbb{G}_m$ and $\tilde{G} = G \times H$, we have that S is a $\tilde{G}$-algebra.

Lemma 3.2 (cf. [Has4, Remark 11.21])**.** *In the following cases, we have that the Knop character $\lambda_{\tilde{G},G}$ is trivial as $\tilde{G}$-modules.*

(1) *G is finite, and $k[G]^*$ is a symmetric algebra;*

(2) *G is finite and étale;*

(3) *G is finite and constant;*

(4) *G is smooth and connected reductive;*

(5) *G is abelian;*

(6) *G is finite, and the identity component G° of G is linearly reductive;*

(7) *G is finite and linearly reductive.*

Proof. By Lemma 2.13, (1) is already proved, and it suffices to show that $\lambda_{G,G} \cong k$ for (2)–(7).

For the case that (2), (3), or (4) is assumed, G is k-smooth, and hence $\lambda_{G,G} = \det_{\mathfrak{g}^*} = \bigwedge^{\text{top}} \mathfrak{g}^*$. As the 0^{th} exterior power is always trivial, (4) has been proved. The assertion (3) is a special case of (2) (also, direct proofs are well-known, see for example, [SY, Example IV.2.6]).

We prove (4). We may assume that k is algebraically closed. Let T be a maximal torus of G. As $\lambda_{G,G} = \det_{\mathfrak{g}}^*$ is one-dimensional, it suffices to show that $\det_{\mathfrak{g}}$ is trivial as a T-module. We have $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, where Φ is the set of roots (that is, nonzero weights of $\mathfrak{g}$). It is known that $\dim \mathfrak{g}_\alpha = 1$ for each $\alpha \in \Phi$, and $\Phi = -\Phi$. Thus $\det_{\mathfrak{g}}$ has weight $0 + \sum_{\alpha\in\Phi} \alpha = 0$.

We prove (5). If G is abelian, then the action of G on G_{ad} is trivial. So $\omega_{G_{\text{ad}}}$ and $\lambda_{G,G}$ are G-trivial.

We prove (6). We may assume that k is algebraically closed of characteristic $p > 0$. By [Swe2, (3.11)], $G^\circ = \operatorname{Spec} kM$ for some abelian p-group M. Note that $\pi^0(G) = G_{\text{red}}$ is a closed subgroup scheme, and is a constant finite group. Note that $G = G^\circ \rtimes G_{\text{red}}$ is a semidirect product. As G_{red} acts on G° by the adjoint action, it acts on the character group $\chi(G^\circ) = M$. As the action is that of groups, it fixes the unit element 1_M of M. As a G°-module, $k[G^\circ_r] = k[G^\circ_l]$ is decomposed into the sum of one-dimensional G°-modules as $\bigoplus_{m\in M} k \cdot m$. Note that $k \cdot m$ is isomorphic to k if and only if $m = 1_M$, and that $\int^r_{k[G^\circ]} = \int^l_{k[G^\circ]}$ is generated by the projection $\pi : k[G] \to k$ given by $\pi(1_M) = 1$ and $\pi(m) = 0$ for $m \in M \setminus \{1_M\}$. As $gm \neq 1_M$ if $m \neq 1_M$ and $g1_M = 1_M$ for any $g \in G_{\text{red}}$, we have that $g\pi = \pi$ for any $g \in G_{\text{red}}$, where $(g\pi)(m) = \pi(g^{-1}(m))$. This shows that $\lambda_{G,G^\circ} \cong k$. As G° is a G-stable (closed and) open neighborhood of the unit element e in G_{ad}, we have that $\lambda_{G,G} = \lambda_{G,G^\circ} \cong k$, as desired.

We prove (7). By [Has3, Lemma 2.2], G° is linearly reductive. By (6), the assertion is clear now. □

Example 3.3 (cf. [Kno, p. 51])**.** $\lambda_{G,G}$ is not G-trivial in general, even if k is an algebraically closed field of characteristic zero, and G is k-smooth. Let $k = \mathbb{C}$, and consider

$$G = O_2 = \{A \in GL_2(\mathbb{C}) \mid {}^tAA = E_2\},$$

where E_2 is the identity matrix. Then the Lie algebra $\mathfrak{g}$ of G is

$$\{B \in \mathfrak{gl}_2(\mathbb{C}) = \operatorname{Mat}_2(\mathbb{C}) \mid {}^tB + B = O\},$$

on which G acts by the action $(A, B) \mapsto AB^tA$. It is easy to see that the action is nontrivial, and hence $\lambda_{G,G} = \mathfrak{g}^*$ is also nontrivial.

Example 3.4. $\lambda_{G,G}$ is not G-trivial in general, even if G is finite. Consider the restricted Lie algebra (see [Jac, (V.7)] for definition) L over a field k of characteristic $p > 0$ with the basis e, f with the relations $[f, e] = e$, $f^p = f$, and $e^p = 0$. Take the restricted universal enveloping algebra V of L. Letting each element of $x \in L$ primitive (i.e., $\Delta(x) = x \otimes 1 + 1 \otimes x$), V is a p^2-dimensional cocommutative Hopf algebra which is not unimodular, see [LS, p. 85]. Letting $G = \operatorname{Spec} V^*$, we have that $\lambda_{G,G}$ is not trivial by Lemma 2.13.

Lemma 3.5. *Let k be a field, G and H be affine k-group schemes of finite type, and $\tilde{G} = G \times H$. Let S be a $\tilde{G}$-algebra, and assume that the action of G on S is small. We assume that S is normal, and $\lambda_{G,G}$ is trivial. Let L be an (H, A)-module which is projective as an A-module. Then the following are equivalent:*

(1) $\omega_S \cong S \otimes_A L$ *as $(\tilde{G}, S)$-modules;*

(2) $\omega_A \cong L$ *as (H, A)-modules,*

where the action of G on L is trivial.

Proof. (1)⇒(2). By Lemma 2.8, $\omega_A \cong \omega_S^G \cong (S \otimes_A L)^G \cong A \otimes_A L \cong L$, since L is G-trivial and A-flat.

(2)⇒(1). We have $\omega_S^G \cong \omega_A \cong L$. Applying the functor $(S \otimes_A -)^{**}$, which is the quasi-inverse of $(-)^G : \operatorname{Ref}(\tilde{G}, S) \to \operatorname{Ref}(H, A)$, we get isomorphisms

$$\omega_S \cong (S \otimes_A \omega_S^G)^{**} \cong (S \otimes_A L)^{**} \cong S \otimes_A L$$

of $(\tilde{G}, S)$-modules. □

Theorem 3.6. *Let k be a field, G be an affine k-group scheme of finite type, and V be a small G-module of dimension $n < \infty$. We assume that $\lambda_{G,G}$ is trivial. Let $H = \mathbb{G}_m$ be the one-dimensional torus, and let V be of degree one as an H-module so that $S = \operatorname{Sym} V^*$ is a $\tilde{G}$-algebra generated by degree one elements, where $\tilde{G} = G \times H$. We set $A = S^G$. Then we have*

(i) $\omega_A \cong \omega_S^G$ *as (H, A)-modules;*

(ii) $a(A) \leq -n$ *in general, where $a(A)$ denotes the a-invariant.*

Moreover, the following are equivalent:

(1) *The action $G \to GL(V)$ factors through $SL(V)$;*

(2) $\omega_S \cong S(-n)$ *as $(\tilde{G}, S)$-modules;*

(3) $\omega_S \cong S$ *as (G, S)-modules;*

(4) $\omega_A \cong A(-n)$ *as (H, A)-modules;*

(5) *A is quasi-Gorenstein;*

(6) *A is quasi-Gorenstein and* $a(A) = -n$*;*

(7) $a(A) = -n$*.*

Proof. The assertion (i) is clear by Corollary 2.8. We prove (ii). We have an (H, A)-linear isomorphism $\omega_A \to \omega_S^G$ by assumption, and $\omega_S^G \subset \omega_S = S \otimes_k \det_V$. So $a(A) \leq a(S) = -n$ in general. The equality holds only if $\det_V$ is trivial. Namely, we have (7)⇒(1).

(1) is equivalent to say that $\det_V \cong k(-n)$. Combining this with the fact $\omega_S \cong S \otimes_k \det_V$, we get (1)⇒(2).

(2)⇒(3) is trivial.

(3)⇒(1). $S \otimes_k \det_V \cong \omega_S \cong S$ as (G, S)-modules. So

$$\det_V \cong S/S_+ \otimes_S (S \otimes_k \det_V) \cong S/S_+ \otimes_S \omega_S \cong S/S_+ \otimes_S S \cong S/S_+ \cong k$$

as G-modules. This shows (1).

(2)⇒(4). $\omega_A \cong \omega_S^G \cong S(-n)^G \cong A(-n)$ as (H, A)-modules.

(4)⇒(6)⇒(5) is trivial.

(5)⇒(3). By assumption, ω_A is projective. As A is positively graded and ω_A is a graded finitely generated module of rank one, we have that $\omega_A \cong A(a)$ for some $a \in \mathbb{Z}$. By Lemma 3.5, we have that $\omega_S \cong S \otimes_A A(a) \cong S(a)$ as $(\tilde{G}, S)$-modules. Forgetting the grading, we have that $\omega_S \cong S$ as (G, S)-modules, as desired.

(6)⇒(7) is trivial. □

Remark 3.7. Goel–Jeffries–Singh [GJS] proved better theorems than Theorem 3.6 for the case that G is finite and constant. They proved the inequality $a(A) \leq -n$ without assuming that the action is small. They also prove there that $a(A) = -n$ implies that the action is small (and hence $G \subset SL(V)$), see [GJS, Proposition 4.1, Theorem 4.4]. The author does not know if these are true for a general finite group scheme G.

The equivalence (1)⇔(5) for the case that G is finite and constant was first proved by Fleischmann and Woodcock [FW] and Braun [Bra]. The author proved that $\omega_S^G \cong \omega_A$ if G is finite linearly reductive, without assuming that the action is small [Has2, (32.4)]. The equivalence (1)⇔(5) for the case that G is finite linearly reductive was proved by Liedtke–Yasuda [LY, Proposition 4.7] (A is strongly F-regular this case, and hence quasi-Gorenstein is equivalent to Gorenstein there).

Example 3.8. We give an example of higher-dimensional G. Let m, n and t be positive integers such that $2 \leq t \leq m \leq n$. Let $W_1 = k^n$, $W_2 = k^m$, $E = k^{t-1}$, and $G = GL(E)$. We consider that G acts on E as a vector representation, while the actions of G on W_1 and W_2 are trivial. We set $V = \operatorname{Hom}_k(E, W_2) \oplus \operatorname{Hom}_k(W_1, E)$, $S = \operatorname{Sym} V^*$, and $A = S^G$. We define $X = V = \operatorname{Hom}(E, W_2) \times \operatorname{Hom}(W_1, E) = \operatorname{Spec} S = E^n \times (E^*)^m$, and $Y = \operatorname{Spec} A = X/\!/G$. The quotient map $\pi : X \to Y$ is identified with the map $\Pi : X \to Y_t$ given by $(\varphi, \psi) \mapsto \varphi \circ \psi$, where $Y_t = \{\rho \in \operatorname{Hom}(W_1, W_2) \mid \operatorname{rank} \rho < t\}$ is the determinantal variety, see [DP]. Note that Π is a $GL(W_1) \times G \times GL(W_2)$-enriched almost principal G-bundle, see [Has1]. So by the theorem, we have that $a(A) \leq a(S) = -(m + n)(t-1)$, and the equality holds if and only if A is Gorenstein. Note also that the usual grading of $A = k[\mathrm{Hom}(W_1, W_2)^*]/I_t$, where I_t is the determinantal ideal, is the one such that each element of $\mathrm{Hom}(W_1, W_2)^*$ is of degree one. However, the grading used here is the one which is inherited from the grading of S, and each element of $\mathrm{Hom}(W_1, W_2)^*$ is of degree two. For the case that k is of characteristic zero, Lascoux's resolution [Las] tells us that $a(A) = 2(-mn + n(m - t + 1)) = -2n(t - 1) \le -(m + n)(t - 1) = a(S)$, doubling the degree to adopt our grading inherited from S. Being a graded ASL over a distributive lattice, A is Cohen–Macaulay, and the Hilbert series of A is independent of k, see [BH]. So $a(A)$ is also independent of k, and we always have $a(A) = -2n(t - 1)$. So $a(A) = a(S)$ if and only if $m = n$. This shows that A is Gorenstein if and only if $m = n$, and this is the well-known theorem by Svanes [Sva].

Example 3.9. Let k be an algebraically closed field of characteristic $p > 0$, and ℓ be a prime number which does not divide $p(p - 1)$. In particular, ℓ is odd. Let

$$G = \left\{ \begin{bmatrix} t & \alpha \\ 0 & 1 \end{bmatrix} \;\middle|\; t \in \boldsymbol{\mu}_\ell,\ \alpha \in \boldsymbol{\alpha}_p \right\},$$

where $\boldsymbol{\alpha}_p = \mathrm{Spec}\, k[a]/(a^p)$ is the first Frobenius kernel of the additive group $\mathbb{G}_a = \mathrm{Spec}\, k[a] = \mathbb{A}^1$, and $\boldsymbol{\mu}_\ell = \mathrm{Spec}\, k[T]/(T^\ell - 1) \subset GL_1$. Note that G acts on the vector representation $W = k^2$ in a natural way. Let $V = W \oplus W^*$. It is easy to see that V is small and $G \subset SL(V)$. It is also easy to see that $\lambda_{G,G} = \mathrm{soc}\, k[(\boldsymbol{\alpha}_p)_{\mathrm{ad}}]^* = (\mathrm{soc}\, k[a]/(a^p))^* = (ka^{p-1})^*$. With the adjoint action, we have $\sigma_t \cdot a = t^{-2}a$, where $\sigma = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$. By assumption, $\lambda_{G,G}$ is nontrivial. By [Has4, Corollary 11.22], $A = S^G$ is not quasi-Gorenstein, where $A = k[V] = \mathrm{Sym}\, V^*$.

(3.10) Let G be a finite k-group scheme, H a k-group scheme of finite type, and set $\tilde{G} = G \times H$. Let S be a $\tilde{G}$-algebra, and $A = S^G$. In [C-R], the trace map $\mathrm{Tr}_{S/A} : S \to A$ is defined. Let $\delta_G : k[G] \to k$ be a non-zero left integral (that is, $\delta_G \in \int^l_{k[G]} \setminus \{0\}$). This is equivalent to say that $\delta_G \in \mathrm{Hom}_G(k[G_l], k) \setminus \{0\}$. For any G-algebra S, let $\mathrm{Tr}_{S/A} : S \to S'$ be the composite

$$S \xrightarrow{\omega_S} S' \otimes_k k[G] \xrightarrow{1_{S'} \otimes \delta_G} S' \otimes_k k = S',$$

where S' is the A-module S with the trivial G-action. By [C-R, Definition-Proposition 3.6], the image of $\mathrm{Tr}_{S/A}$ is contained in $A = S^G$, and hence the map $\mathrm{Tr}_{S/A} : S \to A$ is induced. It is easy to see that $\mathrm{Tr}_{S/A}$ is A-linear.

(3.11) Assume that the Hopf algebra $k[G]^*$ is unimodular. Then δ_G is also a right integral. That is, $\delta_G : k[G_r] \to k$ is G-linear (note that $k[G_r]$ is a $k[G]$-comodule algebra letting the coproduct $\Delta : k[G_r] \to k[G_r] \otimes k[G_r]$ the coaction). As $\omega_S : S \to S' \otimes_k k[G_r]$ is also G-linear, we have that $\mathrm{Tr}_{S/A} : S \to A$ is (G, A)-linear. Moreover, δ_G is H-linear, since H acts trivially on G_r. Letting the action of H on S' be the same as that on S, we have that $\omega_S : S \to S' \otimes_k k[G_r]$ is also H-linear. Thus $\mathrm{Tr}_{S/A}$ is $(\tilde{G}, A)$-linear.

Theorem 3.12. *Let S be a k-algebra of finite type. Let G be a finite k-group scheme, H be a k-group scheme of finite type, and $\tilde{G} = G \times H$. Assume that $\tilde{G}$ acts on S. If $k[G]^*$ is symmetric and either*

(1) *The map $\operatorname{Spec} S \to \operatorname{Spec} A$ is a $\tilde{G}$-enriched principal G-bundle; or*

(2) *The action of G on S is small, and S satisfies the (S_2)-condition,*

then $\zeta : S \to \operatorname{Hom}_A(S, A)$ $(s \mapsto (t \mapsto \operatorname{Tr}_{S/A}(st)))$ is an isomorphism of $(\tilde{G}, S)$-modules.

Proof. This is [C-R, Corollary 3.13] except that we need to prove that the map ζ is $\tilde{G}$-linear. This is done in the discussion above. □

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