Quantized superalgebras and generalized quantum groups

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QUANTIZED SUPERALGEBRAS AND GENERALIZED QUANTUM GROUPS

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（量子スーパー代数と一般化量子群）

理学研究科

数物系専攻

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Contents

1. Introduction

In this thesis, we study generalized quantum groups $\mathcal{U}_X(\epsilon)$ $(X = A, B, C, D)$. $\mathcal{U}_A$ and $\mathcal{U}_B$, introduced in [3] without Serre relations, are certain extension of the well known quantum group of type A and type B respectively. The similarity between these algebras and the quantized enveloping algebra in [5] was pointed out. In Section 2, we first review the quantized superalgebra $U[\epsilon]$ from [5] and the algebra $\mathcal{U}_A(\epsilon)$ from [3]. We then establish an isomorphism between them, which enable us to transport Serre relations of the former to the latter. We also study the isomorphism that switching adjacent components of ϵ. In Section 3, we examine the Hopf algebra structure of $\mathcal{U}_A(\epsilon)$. In Section 4, we study $\mathcal{U}_B(\epsilon)$ in similar manner to Section 2. Based on the studies of $\mathcal{U}_A(\epsilon)$ and $\mathcal{U}_B(\epsilon)$, we introduce new algebras $\mathcal{U}_C(\epsilon)$, $\mathcal{U}_D(\epsilon)$, and we show there is an isomorphism between them under certain conditions. In Section 6, we consider representations of $\mathcal{U}_A$, $\mathcal{U}_B$, $\mathcal{U}_C$ and $\mathcal{U}_D$. Throughout the paper we use the notations : $[i,j] = \{k \in \mathbb{Z} \mid i \le k \le j\}$, the q-integer $[m] = [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$, $[X,Y]_t = XY - t\,YX$, where $[X,Y]_1$ is simply denoted by $[X,Y]$. We assume that $n \ge 4$.

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2. $U[\epsilon]$ and $\mathcal{U}_A(\epsilon)$

For a sequence $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ of 0 or 1, we introduce two algebras $U[\epsilon]$ and $\mathcal{U}_A(\epsilon)$ and show they are isomorphic as a $\mathbb{C}(q)$-algebra to each other by adding elements having simple commutation relations with generators. $U[\epsilon]$ was defined in [5] and $\mathcal{U}_A(\epsilon)$ was defined in [3] without Serre relations.

2.1. Quantized superalgebra $U[\epsilon]$.

For $i, j \in [1, n-1]$, Set

$$C_{ij} = C_{ji} = \delta_{ij}((-1)^{\epsilon_i} + (-1)^{\epsilon_{i+1}}) - \delta_{i,j+1}(-1)^{\epsilon_i} - \delta_{i+1,j}(-1)^{\epsilon_{i+1}}. \tag{2.1}$$

We also set $\epsilon_{i,j} = \epsilon_i - \epsilon_j$. Let $U[\epsilon]$ be a $\mathbb{C}(q)$-algebra with generators $K_i^{\pm 1}, E_i, F_i$ $(i \in [1, n-1])$ and relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1,\ K_i K_j = K_j K_i, \tag{2.2}$$

$$K_i E_j K_i^{-1} = q^{C_{ij}} E_j,\ K_i F_j K_i^{-1} = q^{-C_{ij}} F_j, \tag{2.3}$$

$$[E_i, F_j]_{(-1)^{\epsilon_{i,i+1}\epsilon_{j,j+1}}} = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \tag{2.4}$$

$$E_i^2 = F_i^2 = 0 \quad \text{if } \epsilon_i \ne \epsilon_{i+1}, \tag{2.5}$$

$$[E_i, E_j]_{(-1)^{\epsilon_{i,i+1}\epsilon_{j,j+1}}} = [F_i, F_j]_{(-1)^{\epsilon_{i,i+1}\epsilon_{j,j+1}}} = 0 \quad \text{if } |i-j| \ge 2, \tag{2.6}$$

$$[E_i, [E_i, E_j]_q]_{q^{-1}} = [F_i, [F_i, F_j]_q]_{q^{-1}} = 0 \quad \text{if } \epsilon_i = \epsilon_{i+1}, |i-j| = 1, \tag{2.7}$$

$$\begin{aligned}&[E_i, [[E_{i-1}, E_i]_{(-1)^{\epsilon_{i-1,i}}q}, E_{i+1}]_{(-1)^{\epsilon_{i-1,i+1}\epsilon_{i+1,i+2}}q^{-1}}]_{(-1)^{\epsilon_{i-1,i+2}}} \\ &= (E \to F) = 0 \quad \text{if } \epsilon_i \ne \epsilon_{i+1}.\end{aligned} \tag{2.8}$$

Remark 2.1. The notations $p(\alpha_i)$, d_i in [5, P328,P331] are replaced with $\epsilon_{i,i+1}$, $(-1)^{\epsilon_i}$.

2.2. **Hopf algebra $\mathcal{U}_A$.** For $i,j \in [1,n-1]$, Set

$$q_i = \begin{cases} q & (\epsilon_i = 0) \\ -q^{-1} & (\epsilon_i = 1), \end{cases} \qquad D_{ij} = D_{ji} = \begin{cases} q_i q_{i+1} & (j = i) \\ q^{-1}_{\max(i,j)} & (j = i \pm 1) \\ 1 & (\text{otherwise}). \end{cases} \tag{2.9}$$

Let $\mathcal{U}_A(\epsilon)$ be a $\mathbb{C}(q)$-algebra generated by $e_i, f_i, k_i^{\pm 1}$ $(i \in [1,n-1])$ obeying the relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1 \ , \ k_i k_j = k_j k_i, \tag{2.10}$$

$$k_i e_j k_i^{-1} = D_{ij} e_j \ , \ k_i f_j k_i^{-1} = D_{ij}^{-1} f_j, \tag{2.11}$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \tag{2.12}$$

$$e_i^2 = f_i^2 = 0 \quad \text{if } \epsilon_i \neq \epsilon_{i+1}, \tag{2.13}$$

$$[e_i, e_j] = [f_i, f_j] = 0 \quad \text{if } |i-j| \geq 2, \tag{2.14}$$

$$e_i^2 e_j - (-1)^{\epsilon_i} [2] e_i e_j e_i + e_j e_i^2 = (e \to f) = 0 \quad \text{if } \epsilon_i = \epsilon_{i+1}, |i-j| = 1, \tag{2.15}$$

$$\begin{aligned} & e_i e_{i-1} e_i e_{i+1} + (-1)^{\epsilon_i} [2] e_i e_{i-1} e_{i+1} e_i - e_i e_{i+1} e_i e_{i-1} \\ & \quad - e_{i-1} e_i e_{i+1} e_i + e_{i+1} e_i e_{i-1} e_i = (e \to f) = 0 \quad \text{if } \epsilon_i \neq \epsilon_{i+1}. \end{aligned} \tag{2.16}$$

Remark 2.2. The algebra $\mathcal{U}_A(\epsilon)$ was introduced in [3] without relations (2.13)-(2.16).

Example 2.3. For $U[1,1,0,0,1]$ and $\mathcal{U}_A(1,1,0,0,1)$ one has

$$(q^{C_{i,j}})_{i,j=1}^4 = \begin{pmatrix} q^{-2} & q & 1 & 1 \\ q & 1 & q^{-1} & 1 \\ 1 & q^{-1} & q^2 & q^{-1} \\ 1 & 1 & q^{-1} & 1 \end{pmatrix}, \quad (D_{i,j})_{i,j=1}^4 = \begin{pmatrix} q^{-2} & -q & 1 & 1 \\ -q & -1 & q^{-1} & 1 \\ 1 & q^{-1} & q^2 & q^{-1} \\ 1 & 1 & q^{-1} & -1 \end{pmatrix}.$$

We see from this example that the difference of $q^{C_{i,j}}$ and $D_{i,j}$ are just by signs.

2.3. **Isomorphism as a $\mathbb{C}(q)$-algebra.** We add to $U[\epsilon]$ invertible elements θ_i, θ'_i $(i \in [1,n-1])$ such that

$$\theta_i^4 = {\theta'_i}^4 = 1, \qquad (\theta_i \theta'_i)^2 = (-1)^{\epsilon_i + \epsilon_{i+1}},$$

and that they commute with K_j and have the following commutation relations with E_j and F_j.

$$\theta_i E_j = \alpha_{ij} E_j \theta_i, \qquad \theta'_i E_j = \alpha'_{ij} E_j \theta'_i, \tag{2.17}$$

$$\theta_i F_j = \alpha_{ij}^{-1} F_j \theta_i, \qquad \theta'_i F_j = {\alpha'}_{ij}^{-1} F_j \theta'_i. \tag{2.18}$$

Here

$$\alpha_{ij} = (-1)^{\delta_{i+1,j}\epsilon_{i+1} + \epsilon_i \epsilon_{j+1}} \sqrt{-1}^{\,\mathrm{sgn}\,(i-j)(\epsilon_i \epsilon_j + \epsilon_{i+1}\epsilon_{j+1})}, \tag{2.19}$$

$$\alpha'_{ij} = (-1)^{\delta_{i-1,j}\epsilon_i + \epsilon_i \epsilon_{j+1}} \sqrt{-1}^{\,\mathrm{sgn}\,(j-i)(\epsilon_i \epsilon_j + \epsilon_{i+1}\epsilon_{j+1})}, \tag{2.20}$$

and $\mathrm{sgn}\,(i) = 1\ (i \geq 0),\ = -1\ (i < 0)$.

Let $\tilde{U}[\epsilon]$ be the algebra enlarged from $U[\epsilon]$ by adding $\theta_i^{\pm 1}, {\theta'_i}^{\pm 1}$ $(i \in [1,n-1])$.

Proposition 2.4. *The $\mathbb{C}(q)$-linear map $\iota : \mathcal{U}_A(\epsilon) \to \tilde{U}[\epsilon]$ defined by*

$$\iota(e_i) = E_i\theta_i, \quad \iota(f_i) = F_i\theta'_i, \quad \iota(k_i^{\pm 1}) = \alpha_{ii}^{-1} K_i^{\pm 1}\theta_i\theta'_i$$

gives an $\mathbb{C}(q)$-algebra homomorphism.

Proof. We show that ι preserves (2.12).

$$\begin{aligned}\iota([e_i, f_j]) =& E_i\theta_i F_j\theta'_j - F_j\theta'_j E_i\theta_i \\ =& \alpha_{ij}^{-1}(E_iF_j - \alpha_{ij}\alpha'_{ji}F_jE_i)\theta_i\theta'_j \\ =& \alpha_{ij}^{-1}([E_i, F_j]_{(-1)^{(\epsilon_i+\epsilon_{i+1})(\epsilon_j+\epsilon_{j+1})}})\theta_i\theta'_j \\ =& \delta_{ij}\alpha_{ii}^{-1}\left(\frac{K_i - K_i^{-1}}{q - q^{-1}}\right)\theta_i\theta'_i \\ =& \iota\left(\delta_{ij}\frac{k_i - k_i^{-1}}{q - q^{-1}}\right).\end{aligned}$$

The other cases are similar. □

We also enlarge $\mathcal{U}_A(\epsilon)$ by adding θ_i, θ'_i that commute with k_j and have the same commutation relations with e_j, f_j as in (2.17),(2.18). Let $\tilde{\mathcal{U}}_A(\epsilon)$ be the resulting algebra. Then the previous proposition immediately implies

Theorem 2.5. *$\tilde{U}[\epsilon]$ is isomorphic to $\tilde{\mathcal{U}}_A(\epsilon)$ as a $\mathbb{C}(q)$-algebra.*

Remark 2.6. The $\mathbb{C}(q)$-isomorphism $\iota^{\pm 1}$ is defined by

$$\iota^{\pm 1}(\theta_i) = \theta_i, \quad \iota^{\pm 1}(\theta'_i) = \theta'_i, \quad \iota^{\pm 1}(\theta_i^{-1}) = \theta_i^{-1}, \quad \iota^{\pm 1}(\theta_i'^{-1}) = \theta_i'^{-1},$$
$$\iota^{-1}(e_i) = E_i\theta_i^{-1}, \quad \iota^{-1}(f_i) = F_i\theta_i'^{-1}, \quad \iota(k_i^{\pm 1}) = \alpha_{ii}K_i^{\pm 1}\theta_i^{-1}\theta_i'^{-1}.$$

2.4. **Isomorphism switching adjacent components of ϵ.** For $\epsilon = (\epsilon_1, \ldots, \epsilon_i, \epsilon_{i+1}, \ldots, \epsilon_n)$ let $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_i, \epsilon'_{i+1}, \ldots, \epsilon'_n)$ denote the new sequence $(\epsilon_1, \ldots, \epsilon_{i+1}, \epsilon_i, \ldots, \epsilon_n)$ obtained by switching the i-th and $(i+1)$-th components.

Theorem 2.7. *For $i \in [1, n-1]$ define a map τ_i by*

$$\tau_i(k_i) = k_i^{-1}, \qquad \tau_i(e_i) = -f_ik_i, \qquad \tau_i(f_i) = -k_i^{-1}e_i,$$
$$\tau_i(k_j) = k_ik_j, \qquad \tau_i(e_j) = [e_i, e_j]_{D_{ij}}, \qquad \tau_i(f_j) = [f_j, f_i]_{D_{ij}^{-1}} \quad (|i-j| = 1),$$
$$\tau_i(X_l) = X_l \quad \text{for } X = e, f, k \quad (|i-l| \geq 2).$$

Then τ_i gives an isomorphism from $\mathcal{U}_A(\epsilon)$ to $\mathcal{U}_A(\epsilon')$. Moreover, τ_i^{-1} is given by

$$\tau_i^{-1}(k_i) = k_i^{-1}, \qquad \tau_i^{-1}(e_i) = -k_i^{-1}f_i, \qquad \tau_i^{-1}(f_i) = -e_ik_i,$$
$$\tau_i^{-1}(k_j) = k_ik_j, \qquad \tau_i^{-1}(e_j) = [e_j, e_i]_{D_{ij}}, \qquad \tau_i^{-1}(f_j) = [f_i, f_j]_{D_{ij}^{-1}} \quad (|i-j| = 1),$$
$$\tau_i^{-1}(X_l) = X_l \quad \text{for } X = e, f, k \quad (|i-l| \geq 2).$$

Proof of this theorem is given in Appendix B.

3. Hopf algebra structure

$\mathcal{U}_A(\epsilon)$ is a Hopf algebra with coproduct Δ, counit ε and antipode $\mathcal{S}$ given by

(3.1)

$$\Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \Delta(e_i) = 1 \otimes e_i + e_i \otimes k_i, \quad \Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i,$$
$$\varepsilon(k_i) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0, \quad \mathcal{S}(k_i^{\pm 1}) = k_i^{\mp 1}, \quad \mathcal{S}(e_i) = -e_ik_i^{-1}, \quad \mathcal{S}(f_i) = -k_if_i.$$

The homogeneous cases are identified with the usual quantized enveloping algebras as

$$\mathcal{U}_A(0,\dots,0) = U_q(A_{n-1}), \quad \mathcal{U}_A(1,\dots,1) = U_{-q^{-1}}(A_{n-1}). \tag{3.2}$$

The most difficult part for checking it is a Hopf algebra lies in the proof that Δ is an algebra homomorphism. It is given in Appendix A.

Remark 3.1. $U[\epsilon]$, extended by an element σ, also becomes a Hopf algebra with coproduct Δ'. σ is an involutive element ($\sigma^2 = 1$) and commutes with the generators of $U[\epsilon]$ in the following manner.

$$\sigma E_i = (-1)^{\epsilon_{i,i+1}} E_i \sigma, \quad \sigma F_i = (-1)^{\epsilon_{i,i+1}} F_i \sigma, \quad \sigma K_i^{\pm 1} = K_i^{\pm 1}\sigma.$$

With this σ the coproduct Δ' is given by

$$\Delta'(\sigma) = \sigma \otimes \sigma, \qquad \Delta'(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1},$$
$$\Delta'(E_i) = \sigma^{\epsilon_{i,i+1}} \otimes E_i + E_i \otimes K_i, \qquad \Delta'(F_i) = F_i \otimes 1 + K_i^{-1}\sigma^{\epsilon_{i,i+1}} \otimes F_i.$$

Although both $\mathcal{U}_A(\epsilon)$ and $U[\epsilon]$ are Hopf algebras, ι does not induce a Hopf algebra homomorphism.

Set $e_i^{(m)} = \dfrac{e_i^m}{[m]!}, f_i^{(m)} = \dfrac{f_i^m}{[m]!}$ and define

$$R_i = \sum_{m=0}^{\infty} q^{m(m-1)/2} \prod_{a=1}^{m} (q^a - q^{-a}) e_i^{(m)} \otimes f_i^{(m)},$$
$$R_i' = \sum_{m=0}^{\infty} (-1)^m q^{-m(m-1)/2} \prod_{a=1}^{m} (q^a - q^{-a}) e_i^{(m)} \otimes f_i^{(m)}.$$

Let V, W be $\mathcal{U}_A$-modules. If there exist $m_0 \geq 1$ such that $e_i^{(m_0)} V = 0$ or $f_i^{(m_0)} W = 0$, then $R_i' = R_i^{-1}$ on $V \otimes W$.

Theorem 3.2. *For any $x \in \mathcal{U}_A(\epsilon)$ we have*

$$(\tau_i^{-1} \otimes \tau_i^{-1})(\Delta(\tau_i(x))) = R_i \Delta(x) R_i'.$$

Proof. The case of $\epsilon_i = \epsilon_{i+1} = 0$ is proved in [4, Prop 37.3.2.], and the case of $\epsilon_i = \epsilon_{i+1} = 1$ is similar by replacing q with $-q^{-1}$. So, we check this when $\epsilon_i \neq \epsilon_{i+1}$. In this case, thanks to (2.13), R_i and R_i' become finite sum. Therefore, this can be checked directly. For instance,

$$\begin{aligned}(\tau_i^{-1} \otimes \tau_i^{-1})(\Delta(\tau_i(e_i))) =&(\tau_i^{-1} \otimes \tau_i^{-1})(\Delta(-f_i k_i))\\ =&(\tau_i^{-1} \otimes \tau_i^{-1})(\tau_i(e_i) \otimes k_i + 1 \otimes \tau_i(e_i))\\ =&e_i \otimes k_i^{-1} + 1 \otimes e_i.\end{aligned}$$

On the other hand,

$$\begin{aligned}R_i \Delta(e_i) R_i' =&(1 \otimes 1 + (q - q^{-1}) e_i \otimes f_i)(1 \otimes e_i + e_i \otimes k_i)(1 \otimes 1 - (q - q^{-1}) e_i \otimes f_i)\\ =&1 \otimes e_i + e_i \otimes k_i + (q - q^{-1}) e_i \otimes f_i e_i - (q - q^{-1}) e_i \otimes e_i f_i\\ =&1 \otimes e_i + e_i \otimes k_i - (q - q^{-1}) e_i \otimes \left(\frac{k_i - k_i^{-1}}{q - q^{-1}}\right)\\ =&e_i \otimes k_i^{-1} + 1 \otimes e_i.\end{aligned}$$

The other cases are checked similarly. □

4. $\mathcal{U}_B(\epsilon)$

In this section we define an algebra $\mathcal{U}_B(\epsilon)$ and state a similar result to Theorem 2.5. First, D_{ij} $(i,j \in [1,n])$ for this case is the same as (2.9) except when $i=j=n$, in which case

$$D_{nn} = \begin{cases} q & (\epsilon_n = 0) \\ -q^{-1} & (\epsilon_n = 1). \end{cases}$$

We define $\mathcal{U}_B(\epsilon)$ as a $\mathbb{C}(q^{\frac{1}{2}})$-algebra generated by $e_i, f_i, k_i^{\pm 1} (i \in [1,n])$ obeying the following relations.

$$(4.1)\quad k_i k_i^{-1} = k_i^{-1} k_i = 1\ ,\ k_i k_j = k_j k_i,$$

$$(4.2)\quad k_i e_j k_i^{-1} = D_{ij} e_j\ ,\ k_i f_j k_i^{-1} = D_{ij}^{-1} f_j,$$

$$(4.3)\quad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} \quad \text{if } (i,j) \neq (n,n),$$

$$(4.4)\quad e_i^2 = f_i^2 = 0 \quad \text{if } \epsilon_i \neq \epsilon_{i+1}, i \neq n,$$

$$(4.5)\quad [e_i, e_j] = [f_i, f_j] = 0 \quad \text{if } |i-j| \geq 2,$$

$$(4.6)\quad e_i^2 e_j - (-1)^{\epsilon_i}[2] e_i e_j e_i + e_j e_i^2 = (e \to f) = 0 \quad \text{if } \epsilon_i = \epsilon_{i+1}, |i-j| = 1, i \neq n,$$

$$(4.7)\quad e_i e_{i-1} e_i e_{i+1} + (-1)^{\epsilon_i}[2] e_i e_{i-1} e_{i+1} e_i - e_i e_{i+1} e_i e_{i-1} - e_{i-1} e_i e_{i+1} e_i + e_{i+1} e_i e_{i-1} e_i = (e \to f) = 0 \quad \text{if } \epsilon_i \neq \epsilon_{i+1},$$

$$(4.8)\quad [e_n, f_n] = \frac{k_n - k_n^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$

$$(4.9)\quad e_{n-1} e_n^3 - e_n^3 e_{n-1} + (1 + (-1)^{\epsilon_n}[2])(e_n^2 e_{n-1} e_n - e_n e_{n-1} e_n^2) = (e \to f) = 0.$$

As seen from above, the first 7 relations are the same as those in section 2.2 for $\mathcal{U}_A(\epsilon)$ except a possible restriction on i. $\mathcal{U}_B$ is also a Hopf algebra with (3.1).

Remark 4.1. The algebra $\mathcal{U}_B(\epsilon)$ was introduced in [3] without relations (4.4)-(4.9). We also made the replacements $q \to -1/q, \epsilon_i \to 1 - \epsilon_i$.

Theorem 4.2. *Theorem 2.7 holds as it is with $\mathcal{U}_A(\epsilon)$ and $\mathcal{U}_A(\epsilon')$ replaced by $\mathcal{U}_B(\epsilon)$ and $\mathcal{U}_B(\epsilon')$.*

5. $\mathcal{U}_C(\epsilon)$ AND $\mathcal{U}_D(\epsilon)$

In this section, we introduce $\mathcal{U}_C(\epsilon)$ and $\mathcal{U}_D(\epsilon)$. For $\epsilon_{n-1} \neq \epsilon_n$, τ_{n-1} becomes an isomorphism between $\mathcal{U}_C(\epsilon)$ and $\mathcal{U}_D(\epsilon')$.

5.1. $\mathcal{U}_C(\epsilon)$.

We define the constants $D_{i,j} (= D_{j,i})$ as follows. First, D_{ij} $(i,j \in [1,n])$ for this case is the same as (2.9) except

$$D_{n,n} = \begin{cases} q^4 & (\epsilon_n = 0) \\ q^{-4} & (\epsilon_n = 1), \end{cases} \qquad D_{n,n-1} = D_{n-1,n} = \begin{cases} q^{-2} & (\epsilon_n = 0) \\ q^2 & (\epsilon_n = 1). \end{cases}$$

Let $\mathcal{U}_C(\epsilon)$ be the $\mathbb{C}(q)$-algebra generated by $e_i, f_i, k_i^{\pm 1}(i \in [1,n])$ obeying the relations below.

(5.1) $k_i k_i^{-1} = k_i^{-1} k_i = 1 \ , \ k_i k_j = k_j k_i,$

(5.2) $k_i e_j k_i^{-1} = D_{ij} e_j \ , \ k_i f_j k_i^{-1} = D_{ij}^{-1} f_j,$

(5.3) $[e_i, f_j] = \delta_{ij} \dfrac{k_i - k_i^{-1}}{q - q^{-1}} \quad \text{if } (i,j) \neq (n,n),$

(5.4) $e_i^2 = f_i^2 = 0 \quad \text{if } \epsilon_i \neq \epsilon_{i+1}, i \neq n,$

(5.5) $[e_i, e_j] = [f_i, f_j] = 0 \quad \text{if } |i-j| \geq 2,$

(5.6) $e_i^2 e_j - (-1)^{\epsilon_i}[2] e_i e_j e_i + e_j e_i^2 = (e \to f) = 0 \text{ if } \epsilon_i = \epsilon_{i+1}, |i-j| = 1, i,j \neq n,$

(5.7) $e_i e_{i-1} e_i e_{i+1} + (-1)^{\epsilon_i}[2] e_i e_{i-1} e_{i+1} e_i - e_i e_{i+1} e_i e_{i-1}$
$\qquad - e_{i-1} e_i e_{i+1} e_i + e_{i+1} e_i e_{i-1} e_i = (e \to f) = 0 \quad \text{if } \epsilon_i \neq \epsilon_{i+1}, 1 \leq i \leq n-2,$

(5.8) $[e_n, f_n] = \dfrac{k_n - k_n^{-1}}{q^2 - q^{-2}},$

(5.9) $[e_{n-1}, [e_{n-1}, [e_{n-1}, e_n]_{q^2}]_{q^{-2}}] = (e \to f) = 0 \quad \text{if } \epsilon_{n-1} = \epsilon_n,$

(5.10)
$$e_n^2 e_{n-1} - (q^2 + q^{-2}) e_n e_{n-1} e_n + e_{n-1} e_n^2 = (e \to f) = 0,$$

(5.11)
$$[[[e_{n-2}, e_{n-1}]_D, [[e_{n-2}, e_{n-1}]_D, e_n]_{D^{-2}}], e_{n-1}] = (e \to f) = 0$$
$$\text{if } \epsilon_{n-2} \neq \epsilon_{n-1} \neq \epsilon_n, D := D_{n-2,n-1},$$

(5.12)
$$[[[[[[e_{n-3}, e_{n-2}]_D, e_{n-1}]_D, e_n]_{D^{-2}}, e_{n-1}]_{-D^{-1}}, e_{n-2}]_D, e_{n-1}] = (e \to f) = 0$$
$$\text{if } \epsilon_{n-2} = \epsilon_{n-1} \neq \epsilon_n, D := D_{n-2,n-1}.$$

Example 5.1. For $\mathcal{U}_C(1,1,0,0,1)$ one has

$$(D_{i,j})_{i,j=1}^4 = \begin{pmatrix} q^{-2} & -q & 1 & 1 & 1 \\ -q & -1 & q^{-1} & 1 & 1 \\ 1 & q^{-1} & q^2 & q^{-1} & 1 \\ 1 & 1 & q^{-1} & -1 & q^2 \\ 1 & 1 & 1 & q^2 & q^{-4} \end{pmatrix}.$$

$\mathcal{U}_C(\epsilon)$ becomes a Hopf algebra with (3.1).

5.2. $\mathcal{U}_D(\epsilon)$**.** We define the constants $D_{i,j}(= D_{j,i})$ as follows. First, D_{ij} $(i,j \in [1,n])$ for this case is the same as (2.9) except

$$D_{n-2,n} = \begin{cases} q^{-1} & (\epsilon_{n-1} = 0) \\ -q & (\epsilon_{n-1} = 1) \end{cases}, \quad D_{n,n} = \begin{cases} q^2 & (\epsilon_{n-1} = \epsilon_n = 0) \\ -1 & (\epsilon_{n-1} \neq \epsilon_n) \\ q^{-2} & (\epsilon_{n-1} = \epsilon_n = 1) \end{cases},$$

$$D_{n,n-1} = \begin{cases} -q^2 & (\epsilon_{n-1} = 0, \epsilon_n = 1) \\ 1 & (\epsilon_{n-1} = \epsilon_n) \\ -q^{-2} & (\epsilon_{n-1} = 1, \epsilon_n = 0) \end{cases}.$$

Let $\mathcal{U}_D$ be the $\mathbb{C}(q)$-algebra generated by $e_i, f_i, k_i^{\pm 1} (i \in [1,n])$ obeying the relations below.

(5.13) $k_i k_i^{-1} = k_i^{-1} k_i = 1\ ,\ k_i k_j = k_j k_i,$

(5.14) $k_i e_j k_i^{-1} = D_{ij} e_j\ ,\ k_i f_j k_i^{-1} = D_{ij}^{-1} f_j,$

(5.15) $[e_i, f_j] = \delta_{ij} \dfrac{k_i - k_i^{-1}}{q - q^{-1}},$

(5.16) $e_i^2 = f_i^2 = 0 \quad \text{if } \epsilon_i \neq \epsilon_{i+1}, i \neq n,,$

(5.17) $e_n^2 = f_n^2 = 0 \quad \text{if } \epsilon_{n-1} \neq \epsilon_n,$

(5.18) $[e_i, e_j] = [f_i, f_j] = 0 \quad \text{if } D_{i,j} = 1,$

(5.19) $e_i^2 e_j - (-1)^{\epsilon_i}[2] e_i e_j e_i + e_j e_i^2 = (e \to f) = 0 \text{ if } \epsilon_i = \epsilon_{i+1}, |i-j| = 1, j \neq n,$

(5.20) $e_{n-2}^2 e_n - (-1)^{\epsilon_{n-1}}[2] e_{n-2} e_n e_{n-2} + e_n e_{n-2}^2 = (e \to f) = 0 \text{ if } \epsilon_{n-2} = \epsilon_{n-1},$

(5.21) $e_n^2 e_{n-2} - (-1)^{\epsilon_{n-1}}[2] e_n e_{n-2} e_n + e_{n-2} e_n^2 = (e \to f) = 0 \text{ if } \epsilon_{n-1} = \epsilon_n,$

(5.22) $$e_i e_{i-1} e_i e_{i+1} + (-1)^{\epsilon_i}[2] e_i e_{i-1} e_{i+1} e_i - e_i e_{i+1} e_i e_{i-1} - e_{i-1} e_i e_{i+1} e_i + e_{i+1} e_i e_{i-1} e_i = (e \to f) = 0 \quad \text{if } \epsilon_i \neq \epsilon_{i+1}, 1 \leq i \leq n-2,$$

(5.23) $$e_{n-2} e_{n-3} e_{n-2} e_n + (-1)^{\epsilon_{n-2}}[2] e_{n-2} e_{n-3} e_n e_{n-2} - e_{n-2} e_n e_{n-2} e_{n-3} - e_{n-3} e_{n-2} e_n e_{n-2} + e_n e_{n-2} e_{n-3} e_{n-2} = (e \to f) = 0 \quad \text{if } \epsilon_{n-2} \neq \epsilon_{n-1},$$

(5.24) $$e_{n-2} e_n e_{n-1} - e_{n-2} e_{n-1} e_n + (-1)^{\epsilon_{n-1}}[2](e_{n-1} e_{n-2} e_n - e_n e_{n-2} e_{n-1}) + e_n e_{n-1} e_{n-2} - e_{n-1} e_n e_{n-2} = (e \to f) = 0 \quad \text{if } \epsilon_{n-1} \neq \epsilon_n.$$

Example 5.2. For $\mathcal{U}_D(1,0,1,0)$ and $\mathcal{U}_D(0,1,0,0)$ one has

$$(D_{i,j})_{i,j=1}^4 = \begin{pmatrix} -1 & q^{-1} & 1 & 1 \\ q^{-1} & -1 & -q & -q \\ 1 & -q & -1 & -q^{-2} \\ 1 & -q & -q^{-2} & -1 \end{pmatrix},\ (D_{i,j})_{i,j=1}^4 = \begin{pmatrix} -1 & -q & 1 & 1 \\ -q & -1 & q^{-1} & q^{-1} \\ 1 & q^{-1} & q^2 & 1 \\ 1 & q^{-1} & 1 & q^2 \end{pmatrix}.$$

$\mathcal{U}_D(\epsilon)$ becomes a Hopf algebra with (3.1). The homogeneous cases are identified with the usual quantized enveloping algebras as

(5.25) $$\mathcal{U}_X(0,\dots,0) = U_q(X_n), \quad \mathcal{U}_X(1,\dots,1) = U_{-q^{-1}}(X_n) \quad (X = B, C, D).$$

5.3. **Isomorphism between $\mathcal{U}_C(\epsilon)$ and $\mathcal{U}_D(\epsilon)$ as a $\mathbb{C}(q)$-algebra.** In the case of $\mathcal{U}_C(\epsilon)$, for $i \in [1, n-2]$ Theorem 2.7 holds as it is with $\mathcal{U}_A(\epsilon)$ and $\mathcal{U}_A(\epsilon')$ replaced by $\mathcal{U}_C(\epsilon)$ and $\mathcal{U}_C(\epsilon')$. If $\epsilon_{n-1} = \epsilon_n$, a $\mathbb{C}(q)$-algebra automorphism τ_{n-1} is defined by

(5.26) $$\begin{aligned}
\tau_{n-1}(e_n) &= \frac{(-1)^{\epsilon_{n-1}}}{[2]}(e_{n-1}^2 e_n - (1+D) e_{n-1} e_n e_{n-1} + D e_n e_{n-1}^2), \\
\tau_{n-1}(f_n) &= \frac{(-1)^{\epsilon_{n-1}}}{[2]}(D^{-1} f_{n-1}^2 f_n - (1+D^{-1}) f_{n-1} f_n f_{n-1} + f_n f_{n-1}^2), \\
\tau_{n-1}^{-1}(e_n) &= \frac{(-1)^{\epsilon_{n-1}}}{[2]}(D e_{n-1}^2 e_n - (1+D) e_{n-1} e_n e_{n-1} + e_n e_{n-1}^2), \\
\tau_{n-1}^{-1}(f_n) &= \frac{(-1)^{\epsilon_{n-1}}}{[2]}(f_{n-1}^2 f_n - (1+D^{-1}) f_{n-1} f_n f_{n-1} + D^{-1} f_n f_{n-1}^2), \\
\tau_{n-1}(k_n) &= k_{n-1}^2 k_n, \quad \tau_{n-1}^{-1}(k_n) = k_{n-1}^2 k_n \qquad (D := D_{n-1,n} \in \mathcal{U}_C).
\end{aligned}$$

For other elements, τ_{n-1} is defined in the same manner as Theorem 2.7. If $\epsilon_{n-1} \neq \epsilon_n$, $\mathbb{C}(q)$-algebra isomorphism $\tau_{n-1} : \mathcal{U}_C(\epsilon_1, \dots, \epsilon_{n-1}, \epsilon_n) \to \mathcal{U}_D(\epsilon_1, \dots, \epsilon_n, \epsilon_{n-1})$ is defined by

$$\begin{aligned}(5.27)\quad &\tau_{n-1}(e_n) = \frac{1}{[2]}[e_{n-1}, e_n]_{D_{n-1,n}}, \quad \tau_{n-1}(f_n) = \frac{(-1)^{\epsilon_{n-1}}}{[2]}[f_n, f_{n-1}]_{D^{-1}_{n,n-1}},\\ &\tau^{-1}_{n-1}(e_n) = (-1)^{\epsilon_n}[e_n, e_{n-1}]_{D_{n-1,n}}, \quad \tau^{-1}_{n-1}(f_n) = [f_{n-1}, f_n]_{D^{-1}_{n-1,n}},\\ &\tau_{n-1}(k_n) = k_{n-1}k_n, \quad \tau^{-1}_{n-1}(k_n) = k_{n-1}k_n.\end{aligned}$$

For other elements, τ_{n-1} is defined in the same manner as Theorem 2.7.
In the case of $\mathcal{U}_D(\epsilon)$, for $i \in [1, n-2]$ Theorem 2.7 holds as it is with $\mathcal{U}_A(\epsilon)$ and $\mathcal{U}_A(\epsilon')$ replaced by $\mathcal{U}_D(\epsilon)$ and $\mathcal{U}_D(\epsilon')$ except next cases:

(5.28)

$$\begin{aligned}&\tau_{n-2}(e_n) = [e_{n-2}, e_n]_{D_{n,n-2}}, \quad \tau_{n-2}(f_n) = [f_n, f_{n-2}]_{D^{-1}_{n,n-2}}, \quad \tau_{n-2}(k_n) = k_nk_{n-2},\\ &\tau^{-1}_{n-2}(e_n) = [e_n, e_{n-2}]_{D_{n,n-2}}, \quad \tau^{-1}_{n-2}(f_n) = [f_{n-2}, f_n]_{D^{-1}_{n,n-2}}, \quad \tau^{-1}_{n-2}(k_n) = k_nk_{n-2}.\end{aligned}$$

If $\epsilon_{n-1} = \epsilon_n$, a $\mathbb{C}(q)$-algebra automorphism τ_{n-1} is defined by

$$\begin{aligned}(5.29)\quad &\tau_{n-1}(e_n) = e_n, \quad \tau_{n-1}(f_n) = f_n, \quad \tau_{n-1}(k_n) = k_n,\\ &\tau^{-1}_{n-1}(e_n) = e_n, \quad \tau^{-1}_{n-1}(f_n) = f_n, \quad \tau^{-1}_{n-1}(k_n) = k_n.\end{aligned}$$

For other elements, τ_{n-1} is defined in the same manner as Theorem 2.7.

6. Reprsentations of $\mathcal{U}_A(\epsilon)$, $\mathcal{U}_B(\epsilon)$, $\mathcal{U}_C(\epsilon)$ and $\mathcal{U}_D(\epsilon)$

Set

$$\begin{aligned}&\mathcal{W} := \mathbb{C}(q)\left\langle |\mathbf{m}\rangle := |m_1, \dots, m_n\rangle \,\middle|\, \forall i \in [1, n], m_i \in \mathbb{Z}, 0 \le m_i \le \frac{1}{\epsilon_i}\right\rangle,\\ &\mathcal{W}_l := \mathbb{C}(q)\left\langle |\mathbf{m}\rangle := |m_1, \dots, m_n\rangle \,\middle|\, |\mathbf{m}\rangle \in \mathcal{W}, \sum_{i=1}^{n} m_i = l\right\rangle,\\ &|\mathbf{e}_i\rangle := |0, \dots, 0, \overset{i}{1}, 0, \dots, 0\rangle \in \mathcal{W} \quad \text{(See also [3, Section 2])}.\end{aligned}$$

Proposition 6.1. *The map $\pi : \mathcal{U}_A(\epsilon) \to \mathrm{End}(\mathcal{W}_l)$ defined by*

$$\begin{aligned}e_i|\mathbf{m}\rangle &= [m_i]|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle,\\ f_i|\mathbf{m}\rangle &= [m_{i+1}]|\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle,\\ k_i|\mathbf{m}\rangle &= q_i^{-m_i}q_{i+1}^{m_{i+1}}|\mathbf{m}\rangle\end{aligned}$$

for $i \in [1, n-1]$ is an irreducible representation, where $0 \le l \le n$ if $\epsilon_1 \cdots \epsilon_n = 1$ and $l \in \mathbb{Z}_{\ge 0}$ otherwise.

Proposition 6.2. *The map* $\pi : \mathcal{U}_B(\epsilon) \to \mathrm{End}(\mathcal{W})$ *defined by*

$$\begin{aligned} e_i|\mathbf{m}\rangle &= [m_i]|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle \quad (1 \le i < n), \\ f_i|\mathbf{m}\rangle &= [m_{i+1}]|\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle \quad (1 \le i < n), \\ k_i|\mathbf{m}\rangle &= q_i^{-m_i} q_{i+1}^{m_{i+1}}|\mathbf{m}\rangle \quad (1 \le i < n), \\ e_n|\mathbf{m}\rangle &= \sqrt{-1}\kappa[m_n]|\mathbf{m} - \mathbf{e}_n\rangle, \\ f_n|\mathbf{m}\rangle &= |\mathbf{m} + \mathbf{e}_n\rangle, \\ k_n|\mathbf{m}\rangle &= \sqrt{-1}q^{-\frac{1}{2}} q_n^{-m_n}|\mathbf{m}\rangle \end{aligned}$$

is an irreducible representation, where

$$\kappa = \frac{q+1}{q-1}. \tag{6.1}$$

Proposition 6.3. *The map* $\pi : \mathcal{U}_C(\epsilon) \to \mathrm{End}(\mathcal{W})$ *defined by*

$$\begin{aligned} e_i|\mathbf{m}\rangle &= [m_i]|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle \quad (1 \le i < n), \\ f_i|\mathbf{m}\rangle &= [m_{i+1}]|\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle \quad (1 \le i < n), \\ k_i|\mathbf{m}\rangle &= q_i^{-m_i} q_{i+1}^{m_{i+1}}|\mathbf{m}\rangle \quad (1 \le i < n), \\ e_n|\mathbf{m}\rangle &= \frac{[m_n][m_n-1]}{[2]^2}|\mathbf{m} - \mathbf{e}_n\rangle, \\ f_n|\mathbf{m}\rangle &= |\mathbf{m} + \mathbf{e}_n\rangle, \\ k_n|\mathbf{m}\rangle &= -q^{-2m_n-1}|\mathbf{m}\rangle \end{aligned}$$

is an irreducible representation, if $\epsilon_n = 0$.

Proposition 6.4. *The map* $\pi : \mathcal{U}_D(\epsilon) \to \mathrm{End}(\mathcal{W})$ *defined by*

$$\begin{aligned} e_i|\mathbf{m}\rangle &= [m_i]|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle \quad (1 \le i < n), \\ f_i|\mathbf{m}\rangle &= [m_{i+1}]|\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle \quad (1 \le i < n), \\ k_i|\mathbf{m}\rangle &= q_i^{-m_i} q_{i+1}^{m_{i+1}}|\mathbf{m}\rangle \quad (1 \le i < n), \\ e_n|\mathbf{m}\rangle &= [m_{n-1}]|\mathbf{m} - \mathbf{e}_{n-1} - \mathbf{e}_n\rangle, \\ f_n|\mathbf{m}\rangle &= |\mathbf{m} + \mathbf{e}_{n-1} + \mathbf{e}_n\rangle, \\ k_n|\mathbf{m}\rangle &= (-q^{-1})^{1-m_n}(q_{n-1})^{-m_{n-1}}|\mathbf{m}\rangle \end{aligned}$$

is an irreducible representation, if $\epsilon_n = 1$.

Proposition 6.1 and 6.2 are stated in [3], and Proposition 6.3 is stated in [1], where $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_n = 0$. We define $\pi(k_i^{-1})$ to be $\pi(k_i)^{-1}$, and the image $\pi(g)$ is denoted by g for simplicity. In the above propositions, vectors $|\mathbf{m}'\rangle = |m_1', \ldots, m_n'\rangle$ are understood as zero unless $0 \le m_i' \le \frac{1}{\epsilon_i}$.

Proposition 6.1, 6.2, 6.3, and 6.4 can be directly checked. For example, we show that π preserves (5.11) when $\epsilon_n = 0$.

$$\begin{aligned} &[[[e_{n-2}, e_{n-1}]_D, [[e_{n-2}, e_{n-1}]_D, e_n]_{D^{-2}}], e_{n-1}] \\ =&[3]e_{n-1}e_{n-2}e_{n-1}e_{n-2}e_n e_{n-1} - [3]e_{n-1}e_{n-2}e_n e_{n-1}e_{n-2}e_{n-1} \\ &+ (-1)^{\epsilon_{n-1}}[2]e_{n-2}e_{n-1}e_n e_{n-1}e_{n-2}e_{n-1} - (-1)^{\epsilon_{n-1}}[2]e_{n-1}e_{n-2}e_{n-1}e_n e_{n-1}e_{n-2} \\ &+ e_{n-2}e_{n-1}e_{n-2}e_{n-1}e_n e_{n-1} + e_n e_{n-1}e_{n-2}e_{n-1}e_{n-2}e_{n-1} \\ &- e_{n-1}e_{n-2}e_{n-1}e_{n-2}e_{n-1}e_n - e_{n-1}e_n e_{n-1}e_{n-2}e_{n-1}e_{n-2}. \end{aligned}$$

Here,

$$
\begin{aligned}
&[3]e_{n-1}e_{n-2}e_{n-1}e_{n-2}e_ne_{n-1}|\mathbf{m}\rangle\\
=&[3][m_{n-1}]e_{n-1}e_{n-2}e_{n-1}e_{n-2}e_n|\mathbf{m}-\mathbf{e}_{n-1}+\mathbf{e}_n\rangle\\
=&\frac{[3]}{[2]^2}[m_n+1][m_n][m_{n-1}]e_{n-1}e_{n-2}e_{n-1}e_{n-2}|\mathbf{m}-\mathbf{e}_{n-1}-\mathbf{e}_n\rangle\\
=&\frac{[3]}{[2]^2}[m_n+1][m_n][m_{n-1}][m_{n-2}]e_{n-1}e_{n-2}e_{n-1}|\mathbf{m}-\mathbf{e}_{n-2}+\mathbf{e}_n\rangle\\
=&\frac{[3]}{[2]^2}[m_n+1][m_n][m_{n-1}]^2[m_{n-2}]e_{n-1}e_{n-2}|\mathbf{m}-\mathbf{e}_{n-2}+\mathbf{e}_{n-1}\rangle\\
=&\frac{[3]}{[2]^2}[m_n+1][m_n][m_{n-1}]^2[m_{n-2}][m_{n-2}-1]e_{n-1}|\mathbf{m}-2\mathbf{e}_{n-2}\rangle\\
=&\frac{[3]}{[2]^2}[m_n+1][m_n][m_{n-1}]^3[m_{n-2}][m_{n-2}-1]|\mathbf{m}-2\mathbf{e}_{n-2}-\mathbf{e}_{n-1}+\mathbf{e}_n\rangle.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&[3]e_{n-1}e_{n-2}e_ne_{n-1}e_{n-2}e_{n-1}|\mathbf{m}\rangle\\
=&\frac{[3]}{[2]^2}[m_n+2][m_n+1][m_{n-1}]^3[m_{n-2}][m_{n-2}-1]|\mathbf{m}-2\mathbf{e}_{n-2}-\mathbf{e}_{n-1}+\mathbf{e}_n\rangle,\\
&(-1)^{\epsilon_{n-1}}[2]e_{n-2}e_{n-1}e_ne_{n-1}e_{n-2}e_{n-1}|\mathbf{m}\rangle=0,\\
&(-1)^{\epsilon_{n-1}}[2]e_{n-1}e_{n-2}e_{n-1}e_ne_{n-1}e_{n-2}|\mathbf{m}\rangle=0,\quad e_{n-2}e_{n-1}e_{n-2}e_{n-1}e_ne_{n-1}|\mathbf{m}\rangle=0,\\
&e_ne_{n-1}e_{n-2}e_{n-1}e_{n-2}e_{n-1}|\mathbf{m}\rangle\\
=&\frac{1}{[2]^2}[m_n+3][m_n+2][m_{n-1}]^3[m_{n-2}][m_{n-2}-1]|\mathbf{m}-2\mathbf{e}_{n-2}-\mathbf{e}_{n-1}+\mathbf{e}_n\rangle,\\
&e_{n-1}e_{n-2}e_{n-1}e_{n-2}e_{n-1}e_n|\mathbf{m}\rangle\\
=&\frac{1}{[2]^2}[m_n][m_n-1][m_{n-1}]^3[m_{n-2}][m_{n-2}-1]|\mathbf{m}-2\mathbf{e}_{n-2}-\mathbf{e}_{n-1}+\mathbf{e}_n\rangle,\\
&e_{n-1}e_ne_{n-1}e_{n-2}e_{n-1}e_{n-2}|\mathbf{m}\rangle=0.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\frac{[m_{n-1}]^3[m_{n-2}][m_{n-2}-1]}{[2]^2}&([3][m_n+1][m_n]-[3][m_n+2][m_n+1]\\
&+[m_n+3][m_n+2]-[m_n][m_n-1])=0.
\end{aligned}
$$

Therefore, $[[[e_{n-2},e_{n-1}]_D,[[e_{n-2},e_{n-1}]_D,e_n]_{D^{-2}}],e_{n-1}]|\mathbf{m}\rangle=0$.

Remark 6.5. In view of (3.2) and (5.25), the representations of Proposition 6.1, 6.2, 6.3, and 6.4 reduce to the known ones in the homogeneous case $\epsilon_1 = \cdots = \epsilon_n$:

$\mathcal{W}_l \simeq l$-fold symmetric tensor rep. of $U_q(A_{n-1})$ for $\epsilon_1 = \cdots = \epsilon_n = 0$,
$\mathcal{W}_l \simeq l$-fold anti-symmetric tensor rep. of $U_{-q^{-1}}(A_{n-1})$ for $\epsilon_1 = \cdots = \epsilon_n = 1$,
$\mathcal{W} \simeq q$-oscillator rep. of $U_q(B_n)$ [1] for $\epsilon_1 = \cdots = \epsilon_n = 0$ in Prop 6.2,
$\mathcal{W} \simeq$ spin rep. of $U_{-q^{-1}}(B_n)$ [2] for $\epsilon_1 = \cdots = \epsilon_n = 1$ in Prop 6.2,
$\mathcal{W} \simeq q$-oscillator rep. of $U_q(C_n)$ [1] for $\epsilon_1 = \cdots = \epsilon_n = 0$ in Prop 6.3,
$\mathcal{W} \simeq$ spin rep. of $U_{-q^{-1}}(D_n)$ [2] for $\epsilon_1 = \cdots = \epsilon_n = 1$ in Prop 6.4.

Appendix A. Proof of Δ

We prove that Δ is an algebra homomorphism. Suppose that $|i-j| = 1, |i-l| \geq 2$.

Proof. We show that Δ ppreserves relation (2.13). Assume $\epsilon_i \neq \epsilon_{i+1}$.

$$\begin{aligned}\Delta(e_i^2) =&(1 \otimes e_i + e_i \otimes k_i)^2\\ =&1 \otimes e_i^2 + e_i \otimes e_i k_i + e_i \otimes k_i e_i + e_i^2 \otimes k_i^2\\ =&e_i \otimes e_i k_i - e_i \otimes e_i k_i\\ =&0.\end{aligned}$$

We show that Δ preserves (2.14). Assume $\epsilon_i \neq \epsilon_{i+1}$.

$$\begin{aligned}\Delta([e_i, e_l]) =&(1 \otimes e_i + e_i \otimes k_i)(1 \otimes e_l + e_l \otimes k_l) - (1 \otimes e_l + e_l \otimes k_l)(1 \otimes e_i + e_i \otimes k_i)\\ =&1 \otimes e_i e_l + e_l \otimes e_i k_l + e_i \otimes k_i e_l + e_i e_l \otimes k_i k_l\\ &- (1 \otimes e_l e_i + e_i \otimes e_l k_i + e_l \otimes k_l e_i + e_l e_i \otimes k_l k_i)\\ =&0.\end{aligned}$$

We show that Δ preserves (2.15). Assume $\epsilon_i = \epsilon_{i+1}$.

$$\begin{aligned}&\Delta(e_i^2 e_j - (-1)^{\epsilon_i}[2] e_i e_j e_i + e_j e_i^2)\\ =&(1 \otimes e_i + e_i \otimes k_i)^2 (1 \otimes e_j + e_j \otimes k_j)\\ &- (-1)^{\epsilon_i}[2](1 \otimes e_i + e_i \otimes k_i)(1 \otimes e_j + e_j \otimes k_j)(1 \otimes e_i + e_i \otimes k_i)\\ &+ (1 \otimes e_i + e_i \otimes k_i)(1 \otimes e_j + e_j \otimes k_j)^2\\ =&1 \otimes (e_i^2 e_j - (-1)^{\epsilon_i}[2] e_i e_j e_i + e_j e_i^2)\\ &+ e_j \otimes (e_i^2 k_j - (-1)^{\epsilon_i}[2] e_i k_j e_i + k_j e_i^2)\\ &+ e_i \otimes (k_i e_i e_j - (-1)^{\epsilon_i}[2] e_i e_j k_i + e_i k_i e_j)\\ &+ e_i \otimes (e_j k_i e_i - (-1)^{\epsilon_i}[2] k_i e_j e_i + e_j e_i k_i)\\ &+ e_i e_j \otimes (k_i e_i k_j - (-1)^{\epsilon_i}[2] k_i k_j e_i + e_i k_i k_j)\\ &+ e_j e_i \otimes (k_j k_i e_i - (-1)^{\epsilon_i}[2] e_i k_i k_j + k_j e_i k_i)\\ &+ e_i^2 \otimes (k_i^2 e_j - (-1)^{\epsilon_i}[2] k_i e_j k_i + e_j k_i^2)\\ &+ (e_i^2 e_j - (-1)^{\epsilon_i}[2] e_i e_j e_i + e_j e_i^2) \otimes k_i^2 k_j.\end{aligned}$$

Now, we pay an attention to $D_{i,j} + D_{i,j}^{-1} = (-1)^{\epsilon_i}[2]$. The second term vanishes as,

$$e_j \otimes (e_i^2 k_j - (-1)^{\epsilon_i}[2]e_i k_j e_i + k_j e_i^2) = D_{i,j}(D_{i,j} + D_{i,j}^{-1} - (-1)^{\epsilon_i}[2])(e_j \otimes e_i^2 k_j) = 0.$$

The other terms vanish similarly.
We show that Δ preserves (2.16). Assume $\epsilon_i \neq \epsilon_{i+1}$.

$$\begin{aligned}
&\Delta(e_i e_{i-1} e_i e_{i+1} + (-1)^{\epsilon_i}[2]e_i e_{i-1} e_{i+1} e_i - e_i e_{i+1} e_i e_{i-1} - e_{i-1} e_i e_{i+1} e_i + e_{i+1} e_i e_{i-1} e_i) \\
=&1 \otimes (e_i e_{i-1} e_i e_{i+1} + e_i + (-1)^{\epsilon_i}[2]e_i e_{i-1} e_{i+1} e_i - e_i e_{i+1} e_i e_{i-1} \\
&\qquad\qquad\qquad\qquad\qquad - e_{i-1} e_i e_{i+1} e_i + e_{i+1} e_i e_{i-1} e_i) \\
&+ (1 + (-1)^{\epsilon_i}[2]D_{i,i+1} + D_{i,i+1}^2 D_{i-1,i+1}) e_{i+1} \otimes e_i e_{i-1} e_i k_{i+1} \\
&+ (D_{i,i+1} + (-1)^{\epsilon_i}[2] - D_{i,i-1}) e_i \otimes e_i e_{i-1} e_{i+1} k_i \\
&+ (D_{i-1,i} D_{i,i} D_{i,i+1} - 1) e_i \otimes e_{i-1} e_i e_{i+1} k_i \\
&+ ((-1)^{\epsilon_i}[2]D_{i,i} D_{i,i+1} D_{i,i-1} - D_{i,i} D_{i,i+1} + D_{i,i} D_{i,i+1}) e_i \otimes e_{i-1} e_{i+1} e_i k_i \\
&+ (-D_{i,i} D_{i,i+1} D_{i,i-1} + 1) e_i \otimes e_{i+1} e_i e_{i-1} k_i \\
&+ ((-1)^{\epsilon_i}[2]D_{i-1,i+1} D_{i-1,i} - D_{i-1,i}^2 D_{i-1,i+1} - 1) e_{i-1} \otimes e_i e_{i+1} e_i k_{i-1} \\
&+ (1 - D_{i,i} D_{i,i-1} D_{i,i+1}) e_i e_{i+1} \otimes e_i e_{i-1} k_i k_{i+1} \\
&+ (D_{i,i} D_{i,i-1} + (-1)^{\epsilon_i}[2]D_{i,i} D_{i,i+1} D_{i,i-1} - D_{i,i} D_{i,i+1}) e_i e_{i+1} \otimes e_{i-1} e_i k_i k_{i+1} \\
&+ (D_{i,i+1} - D_{i,i-1} + (-1)^{\epsilon_i}[2]) e_{i-1} e_i \otimes e_i e_{i+1} k_{i-1} k_i \\
&+ (-D_{i,i} D_{i,i+1} D_{i,i-1} + 1) e_{i-1} e_i \otimes e_{i+1} e_i k_{i-1} k_i \\
&+ (1 - D_{i,i} D_{i,i-1} D_{i,i+1}) e_i e_{i-1} \otimes e_i e_{i+1} k_{i-1} k_i \\
&+ ((-1)^{\epsilon_i}[2]D_{i,i} D_{i,i-1} D_{i,i+1} - D_{i,i} D_{i,i+1} + D_{i,i} D_{i,i-1}) e_i e_{i-1} \otimes e_{i+1} e_i k_{i-1} k_i \\
&+ ((-1)^{\epsilon_i}[2] - D_{i,i-1} + D_{i,i+1}) e_{i+1} e_i \otimes e_i e_{i-1} k_{i+1} k_i \\
&+ (-1 + D_{i,i} D_{i,i-1} D_{i,i+1}) e_{i+1} e_i \otimes e_{i-1} e_i k_{i+1} k_i \\
&+ (1 + D_{i,i+1}^2 + (-1)^{\epsilon_i}[2]D_{i,i+1}) e_i e_{i-1} e_i \otimes e_{i+1} k_i k_{i-1} k_i \\
&+ (D_{i,i} D_{i,i-1} + (-1)^{\epsilon_i}[2]D_{i,i} D_{i,i-1} D_{i,i+1} - D_{i,i} D_{i,i+1}) e_i e_{i-1} e_{i+1} \otimes e_i k_i k_{i-1} k_{i+1} \\
&+ (1 - D_{i,i} D_{i,i-1} D_{i,i+1}) e_{i-1} e_i e_{i+1} \otimes e_i k_i k_{i-1} k_{i+1} \\
&+ (1 - D_{i,i} D_{i,i-1} D_{i,i+1}) e_{i+1} e_i e_{i-1} \otimes e_i k_i k_{i-1} k_{i+1} \\
&+ ((-1)^{\epsilon_i}[2]D_{i,i-1} - D_{i,i-1}^2 - 1) e_i e_{i+1} e_i \otimes e_{i-1} k_i k_{i+1} k_i \\
&+ ((-1)^{\epsilon_i}[2] - D_{i,i-1} + D_{i,i+1}) e_{i-1} e_{i+1} e_i \otimes e_i k_{i-1} k_{i+1} k_i \\
&+ (e_i e_{i-1} e_i e_{i+1} + (-1)^{\epsilon_i}[2]e_i e_{i-1} e_{i+1} e_i - e_i e_{i+1} e_i e_{i-1} \\
&\qquad\qquad\qquad - e_{i-1} e_i e_{i+1} e_i + e_{i+1} e_i e_{i-1} e_i) \otimes k_i^2 k_{i-1} k_{i+1}.
\end{aligned}$$

Since $D_{i,i} D_{i,i-1} D_{i,i+1} = 1$, $(-1)^{\epsilon_i}[2] = D_{i,i-1} - D_{i,i+1}$, all coefficients become 0. □

Appendix B. Proof of τ

Theorem 2.5 is proved by direct caluculations. We show some cases. First, we show that τ_i preserves the relation (2.11).

By D'_{ij} we denote the constants D_{ij} for $\mathcal{U}_A(\epsilon')$.

$$\begin{aligned}\tau_i(k_ie_{i+1}k_i^{-1}) =&k_i^{-1}[e_i,e_{i+1}]_{D'_{i,i+1}}k_i\\ =&D'^{-1}_{i,i}D'^{-1}_{i,i+1}[e_i,e_{i+1}]_{D'_{i,i+1}}k_i^{-1}k_i\\ =&D'_{i,i-1}[e_i,e_{i+1}]_{D'_{i,i+1}}\\ =&\tau_i(D_{i,i+1}e_{i+1}).\end{aligned}$$

Other cases are proved similarly.
Next, we show that τ_i preserves the relation (2.12).

$$\begin{aligned}\tau_i([e_i,f_i]) =&f_ik_ik_i^{-1}e_i - k_i^{-1}e_if_ik_i\\ =&-[e_i,f_i]\\ =&\frac{k_i^{-1}-k_i}{q-q^{-1}}\\ =&\tau_i(\frac{k_i-k_i^{-1}}{q-q^{-1}}).\end{aligned}$$

Next, we show that τ_i preserves the relation (2.13). Assume $\epsilon_{i+1}\neq\epsilon_{i+2}$.

$$\begin{aligned}\tau_i(e_{i+1}^2) =&[e_i,e_{i+1}]^2_{D_{i,i+1}}\\ =&e_ie_{i+1}e_ie_{i+1} - D_{i,i+1}e_{i+1}e_i^2e_{i+1} - D_{i,i+1}e_ie_{i+1}^2e_i + D_{i,i+1}^2e_{i+1}e_ie_{i+1}e_i. \quad (*)\end{aligned}$$

Case1. $\epsilon_i=\epsilon_{i+1}\neq\epsilon_{i+2}$. Note that $(-1)^{\epsilon_i}[2]=D_{i,i+1}+D_{i,i+1}^{-1}$. By using relation (2.13),(2.14), and (2.15),

$$\begin{aligned}(*) =&e_ie_{i+1}e_ie_{i+1} - D_{i,i+1}e_{i+1}e_i^2e_{i+1} + D_{i,i+1}^2e_{i+1}e_ie_{i+1}e_i\\ =&\frac{e_i^2e_{i+1}^2+e_{i+1}e_i^2e_{i+1}}{D_{i,i+1}+D_{i,i+1}^{-1}} - D_{i,i+1}e_{i+1}e_i^2e_{i+1} + D_{i,i+1}^2\frac{e_{i+1}^2e_i^2+e_{i+1}e_i^2e_{i+1}}{D_{i,i+1}+D_{i,i+1}^{-1}}\\ =&\frac{1+D_{i,i+1}^2-D_{i,i+1}(D_{i,i+1}+D_{i,i+1}^{-1})}{D_{i,i+1}+D_{i,i+1}^{-1}}e_{i+1}e_i^2e_{i+1}\\ =&0.\end{aligned}$$

Case2. $\epsilon_i\neq\epsilon_{i+1}\neq\epsilon_{i+2}$. This is proved similarly.
Next, we show that τ_i preserves the relation (2.14). Suppose that $|i-j|=1, |i-l|\geq 2, |i-m|\geq 2, |j-m|\geq 2$. By using the relation (2.11),(2.12),(2.14),

$$\tau_i([e_i,e_l]) = \tau_i(e_ie_l-e_le_i) = f_ik_ie_l - e_lf_ik_i = 0,$$
$$\tau_i([e_j,e_m]) = [e_i,e_j]_{D_{ij}}e_m - e_m[e_i,e_j]_{D_{ij}} = e_m[e_i,e_j]_{D_{ij}} - e_m[e_i,e_j]_{D_{ij}} = 0.$$

$$\begin{aligned}&\tau_i([e_{i-1},e_{i+1}])\\ =&[e_i,e_{i-1}]_{D_{i,i-1}}[e_i,e_{i+1}]_{D_{i,i+1}} - [e_i,e_{i+1}]_{D_{i,i+1}}[e_i,e_{i-1}]_{D_{i,i-1}}\\ =&e_ie_{i-1}e_ie_{i+1} - D_{i,i-1}e_{i-1}e_i^2e_{i+1} - D_{i,i+1}e_ie_{i-1}e_{i+1}e_i + D_{i,i-1}D_{i,i+1}e_{i-1}e_ie_{i+1}e_i\\ &-e_ie_{i+1}e_ie_{i-1} + D_{i,i+1}e_{i+1}e_i^2e_{i-1} + D_{i,i-1}e_ie_{i+1}e_{i-1}e_i - D_{i,i-1}D_{i,i+1}e_{i+1}e_ie_{i-1}e_i. \quad (*)\end{aligned}$$

Case1. $\epsilon_i=\epsilon_{i+1}$. Since $D_{i,i-1}=D_{i,i+1}$, we put $D:=D_{i,i-1}=D_{i,i+1}$. By using (2.15),

$$\begin{aligned}D_{i,i-1}e_{i-1}e_i^2e_{i+1} =&D(-e_{i-1}e_{i+1}e_i^2 + (D+D^{-1})e_{i-1}e_ie_{i+1}e_i),\\ D_{i,i+1}e_{i+1}e_i^2e_{i-1} =&D(-e_{i+1}e_{i-1}e_i^2 + (D+D^{-1})e_{i+1}e_ie_{i-1}e_i).\end{aligned}$$

Hence,

$$\begin{aligned}(*) =& e_ie_{i-1}e_ie_{i+1} - e_{i-1}e_ie_{i+1}e_i - e_ie_{i+1}e_ie_{i-1} - e_{i+1}e_ie_{i-1}e_i\\ =&\frac{1}{D+D^{-1}}(e_{i-1}e_i^2e_{i+1} + e_i^2e_{i-1}e_{i+1} - e_{i-1}e_i^2e_{i+1} - e_{i-1}e_{i+1}e_i^2\\ &- e_{i+1}e_i^2e_{i-1} - e_i^2e_{i+1}e_{i-1} + e_{i+1}e_{i-1}e_i^2 + e_{i+1}e_i^2e_{i-1})\\ =&0.\end{aligned}$$

Case2. $\epsilon_i \neq \epsilon_{i+1}$. Note that $D_{i,i-1}D_{i,i+1} = -1, D_{i,i-1} - D_{i,i+1} = (-1)^{\epsilon_i}[2]$. By using relation (2.14),(2.16),

$$\begin{aligned}(*) =& e_ie_{i-1}e_ie_{i+1} + (-1)^{\epsilon_i}[2]e_ie_{i-1}e_{i+1}e_i - e_ie_{i+1}e_ie_{i-1} - e_{i-1}e_ie_{i+1}e_i + e_{i+1}e_ie_{i-1}e_i\\ =&0.\end{aligned}$$

Next, we show that τ_i preserves the relation (2.15). Suppose that $\epsilon_i = \epsilon_{i+1}$.

$$\begin{aligned}&\tau_i(e_i^2e_{i-1} - (-1)^{\epsilon_i}[2]e_ie_{i-1}e_i + e_{i-1}e_i^2)\\ =&f_ik_if_ik_i[e_i,e_{i-1}]_{D_{i,i-1}} - (-1)^{\epsilon_i}[2]f_ik_i[e_i,e_{i-1}]_{D_{i,i-1}}f_ik_if_ik_i[e_i,e_{i-1}]_{D_{i,i-1}}f_ik_if_ik_i.\end{aligned}$$

We focus on the first term.

$$\begin{aligned}&f_ik_i[e_i,e_{i-1}]_{D_{i,i-1}}\\ =&D_{i,i}D_{i,i-1}(f_ie_ie_{i-1} - D_{i,i-1}f_ie_{i-1}e_i)k_i\\ =&D_{i,i}D_{i,i-1}((e_if_i - \frac{k_i-k_i^{-1}}{q-q^{-1}})e_{i-1} - D_{i,i-1}e_{i-1}(e_if_i - \frac{k_i-k_i^{-1}}{q-q^{-1}}))k_i\\ =&D_{i,i}D_{i,i-1}[e_i,e_{i-1}]_{D_{i,i-1}}f_ik_i + D_{i,i}D_{i,i-1}(D_{i,i-1}e_{i-1}\frac{k_i-k_i^{-1}}{q-q^{-1}} - \frac{k_i-k_i^{-1}}{q-q^{-1}}e_{i-1})k_i\\ =&D_{i,i+1}^{-1}[e_i,e_{i-1}]_{D_{i,i-1}}f_ik_i + D_{i,i+1}^{-1}e_{i-1}.\end{aligned}$$

Hence,

$$\begin{aligned}&f_ik_if_ik_i[e_i,e_{i-1}]_{D_{i,i-1}}\\ &=f_ik_i(D_{i,i+1}^{-1}[e_i,e_{i-1}]_{D_{i,i-1}}f_ik_i + D_{i,i+1}^{-1}e_{i-1})\\ &=D_{i,i+1}^{-2}([e_i,e_{i-1}]_{D_{i,i-1}}f_ik_if_ik_i) + (1+D_{i,i+1}^{-2})e_{i-1}f_ik_i.\end{aligned}$$

$$\begin{aligned}&f_ik_i[e_i,e_{i-1}]_{D_{i,i-1}}f_ik_if_ik_i\\ &=(D_{i,i+1}^{-1}[e_i,e_{i-1}]_{D_{i,i-1}}f_ik_i + D_{i,i+1}^{-1}e_{i-1})f_ik_i\\ &=D_{i,i+1}^{-1}[e_i,e_{i-1}]_{D_{i,i-1}}f_ik_if_ik_i + D_{i,i+1}^{-1}e_{i-1}f_ik_i.\end{aligned}$$

Since $(-1)^{\epsilon_i}[2] = D_{i,i+1} + D_{i,i+1}^{-1}$, when $\epsilon_i = \epsilon_{i+1}$,

$$\begin{aligned}&\tau_i(e_i^2e_{i-1} - (-1)^{\epsilon_i}[2]e_ie_{i-1}e_i + e_{i-1}e_i^2)\\ =&(D_{i,i+1}^{-2} - D_{i,i+1}^{-1}(D_{i,i+1} + D_{i,i+1}^{-1}) - 1)[e_i,e_{i-1}]_{D_{i,i-1}}f_ik_if_ik_i\\ &+ (1+D_{i,i+1}^{-2})e_{i-1}f_ik_i - D_{i,i+1}^{-1}e_{i-1}f_ik_i\\ =&0.\end{aligned}$$

Next, we show that τ_i preserves the relation (2.16). We assume that $\epsilon_i \neq \epsilon_{i+1}$. Here,

$$k_i^{-1}e_i[f_j,f_i]_{D_{ij}^{-1}} = -D_{ij}[f_j,f_i]_{D_{ij}^{-1}}k_i^{-1}e_i - f_j.$$

So,

$$\begin{aligned}\tau_i(f_if_jf_i) =&(-k_i^{-1}e_i)[f_j,f_i]_{D_{ij}^{-1}}(-k_i^{-1}e_i)\\ =&(-D_{ij}[f_j,f_i]_{D_{ij}^{-1}}k_i^{-1}e_i-f_j)k_i^{-1}e_i\\ =&-f_jk_i^{-1}e_i\\ =&f_j\tau_i(f_i).\end{aligned}$$

Hence,

$$\begin{aligned}\tau_i(f_if_{i-1}f_if_{i+1}) =&f_{i-1}\tau_i(f_if_{i+1})\\ =&-D_{i,i+1}f_{i-1}\tau_i(f_{i+1}f_i)-f_{i-1}f_{i+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\tau_i(f_if_{i-1}f_if_{i+1}+(-1)^{\epsilon_i}[2]f_if_{i-1}f_{i+1}f_i-f_if_{i+1}f_if_{i-1}-f_{i-1}f_if_{i+1}f_i+f_{i+1}f_if_{i-1}f_i)\\ =&-D_{i,i+1}f_{i-1}\tau_i(f_{i+1}f_i)-f_{i-1}f_{i+1}\\ &+(-1)^{\epsilon_{i+1}}(-D_{i,i-1}\tau_i(f_{i-1})f_{i+1}\tau_i(f_i)-f_{i-1}\tau_i(f_{i+1}f_i))\\ &-(-D_{i,i-1}\tau_i(f_{i-1}f_i)-f_{i-1})-\tau_i(f_{i-1})f_{i+1}\tau_i(f_i)+\tau_i(f_{i+1})f_{i-1}\tau_i(f_i)\\ =&D_{i,i-1}f_{i+1}\tau_i(f_{i-1})+D_{i,i-1}^2\tau_i(f_{i-1})f_{i+1}-D_{i,i-1}f_{i-1}\tau_i(f_{i+1})+\tau_i(f_{i+1})f_{i-1}\\ =&0.\end{aligned}$$

Finally, We check the well-definedness of τ_i^{-1}.

$$\begin{aligned}&\tau_i^{-1}\circ\tau_i(f_j)\\ =&\tau_i^{-1}(f_jf_i-D_{ij}'^{-1}f_if_j)\\ =&(f_if_j-D_{ij}^{-1}f_jf_i)(-e_ik_i)-D_{ij}'^{-1}(-e_ik_i)(f_if_j-D_{ij}^{-1}f_jf_i).\end{aligned}$$

Since $D_{ij}'^{-1}D_{ii}^{-1}D_{ij}^{-1}=1$,

$$\begin{aligned}&D_{ij}'^{-1}e_ik_i(f_if_j-D_{ij}^{-1}f_jf_i)\\ =&D_{ij}'^{-1}D_{ii}^{-1}D_{ij}^{-1}e_i(f_if_j-D_{ij}^{-1}f_jf_i)k_i\\ =&\left((f_if_j-D_{ij}^{-1}f_jf_i)e_i+\frac{k_i-k_i^{-1}}{q-q^{-1}}f_j-D_{ij}^{-1}f_j\frac{k_i-k_i^{-1}}{q-q^{-1}}\right)k_i\\ =&\left((f_if_j-D_{ij}^{-1}f_jf_i)e_i+\frac{D_{ij}^{-1}-D_{ij}}{q-q^{-1}}f_jk_i^{-1}\right)k_i\\ =&(f_if_j-D_{ij}^{-1}f_jf_i)e_ik_i+f_j.\end{aligned}$$

Hence,

$$\tau_i^{-1}\circ\tau_i(f_j)=(f_if_j-D_{ij}^{-1}f_jf_i)(-e_ik_i)+(f_if_j-D_{ij}^{-1}f_jf_i)e_ik_i+f_j=f_j.$$

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