# CHARACTERIZATIONS OF STANDARD DERIVED EQUIVALENCES OF DIAGRAMS OF DG CATEGORIES AND THEIR GLUINGS

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Abstract. A diagram consisting of differential graded (dg for short) categories and dg functors is formulated in this paper as a colax functor $X$ from a small category $I$ to the 2-category $\Bbbk$-**dgCat** of small dg categories, dg functors and dg natural transformations for a fixed commutative ring $\Bbbk$. If $I$ is a group regarded as a category with only one object $*$, then $X$ is nothing but a colax action of the group $I$ on the dg category $X(*)$. In this sense, this $X$ can be regarded as a generalization of a dg category with a colax action of a group. We define a notion of standard derived equivalence between such colax functors by generalizing the corresponding notion between dg categories with a group action. Our first main result gives some characterizations of this notion, one of which is given in terms of generalized versions of a tilting object and a quasi-equivalence. On the other hand, for such a colax functor $X$, the dg categories $X(i)$ with $i$ objects of $I$ can be glued together to have a single dg category $\int X$, called the Grothendieck construction of $X$. Our second main result insists that for such colax functors $X$ and $X'$, the Grothendieck construction $\int X'$ is derived equivalent to $\int X$ if there exists a standard derived equivalence from $X'$ to $X$. These results generalize the main results of [7] and [8] to the dg case, respectively. These are new even for dg categories with group actions. In particular, the second result gives a new tool to show the derived equivalence between the orbit categories of dg categories with group actions, which will be illustrated in some examples.

Contents



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## 1. Introduction

Throughout this paper we fix a commutative ring $\Bbbk$, and all linear categories and all linear functors are considered to be over $\Bbbk$. For any category $\mathscr{C}$, we denote by $\mathscr{C}_0$ and $\mathscr{C}_1$ the collections of all objects of $\mathscr{C}$ and of all morphisms in $\mathscr{C}$, respectively. Let $\mathscr{A}$ be a linear category. Then we have canonical embeddings $\mathscr{A} \hookrightarrow \operatorname{Mod}\mathscr{A} \hookrightarrow \mathscr{D}(\operatorname{Mod}\mathscr{A})$, where $\operatorname{Mod}\mathscr{A}$ denotes the category of (right) $\mathscr{A}$-modules, and $\mathscr{D}(\operatorname{Mod}\mathscr{A})$ stands for the derived module category of $\mathscr{A}$ that turns out to be a triangulated category. Two linear categories $\mathscr{A}$ and $\mathscr{A}'$ are said to be *derived equivalent* if $\mathscr{D}(\operatorname{Mod}\mathscr{A})$ and $\mathscr{D}(\operatorname{Mod}\mathscr{A}')$ are equivalent as triangulated categories. If $\mathscr{A}$ and $\mathscr{A}'$ are *Morita equivalent*, i.e., if $\operatorname{Mod}\mathscr{A}$ and $\operatorname{Mod}\mathscr{A}'$ are equivalent as linear categories, then $\mathscr{A}$ and $\mathscr{A}'$ are derived equivalent, but the converse is not true in most cases. Thus, the derived equivalence classification is usually rougher than the Morita equivalence classification. Broué's abelian defect conjecture in [15] made this notion of derived equivalences more important. In this connection, Rickard classified Brauer tree algebras up to derived equivalences in [45], and one of the authors gave the derived equivalence classification for representation-finite selfinjective algebras in [3]. An essential tool for the classifications above was given by Rickard's Morita type theorem for derived categories of rings in [44], which was generalized later by Keller in [28] to differential graded (dg for short) categories with an alternative proof. Both theorems give very useful criteria to check for rings or dg categories to be derived equivalent in terms of tilting complexes or tilting subcategories, which will be also used in this paper as a fundamental tool.

Recall that a dg category is a graded linear category whose morphism spaces are endowed with differentials satisfying suitable compatibility with the grading, and note that a dg category with a single object is nothing but a dg algebra.

Dg categories are used to enhance triangulated categories by Bondal–Kapranov in [14], which was motivated by the study of exceptional collections of coherent sheaves on projective varieties. Also, they are efficiently used in [30] by Keller to compute derived invariants such as K-theory, Hochschild (co-)homology and cyclic homology associated with a ring or a variety.

Now, we come back to derived equivalences of linear categories. If $\mathscr{A}$ and $\mathscr{A}'$ are derived equivalent linear categories, then they share invariants under derived equivalences, such as the center, the Grothendieck group, and those listed above. If we have the classification of a class $\mathscr{S}$ of linear categories under derived equivalences, then the computation of an invariant under derived equivalences in question for a complete set of representatives gives the invariant for all linear categories in the class $\mathscr{S}$. To obtain such a classification we need a tool that produces a lot of derived equivalent pairs $\mathscr{A}$ and $\mathscr{A}'$. In [8], for this purpose we have considered a diagram of linear categories over a small category $I$, which is formulated as a colax functor $X$ from $I$ to the 2-category $\Bbbk$-**Cat** of small linear categories, linear functors and natural transformations. Then for each morphism $a\colon i \to j$ in $I$, we have a linear functor $X(a)\colon X(i) \to X(j)$ between linear categories. We take the Grothendieck construction $\int X$ of $X$ as a gluing of these $X(i)$'s along $X(a)$'s. For two such colax functors $X$ and $X'$, suppose that a derived equivalence from $X'(i)$ to $X(i)$ is given for each $i \in I_0$. Then we have given a way to glue together these derived equivalences from $X'(i)$ to $X(i)$ to have a derived equivalence between the gluings $\int X'$ and $\int X$ if the derived equivalences are "compatible" with $X'(a)$ and $X(a)$ $(a \in I_1)$. The latter condition was shown to follow if $X'$ and $X$ are derived equivalent in a natural sense. This also shows us how to produce a glued linear category $\int X'$ that is derived equivalent to $\int X$, by using $X$ and derived equivalences from $X'(i)$ to $X(i)$ with $i \in I_0$. The class of colax functors from $I$ to $\Bbbk$-**Cat** is naturally extended to a 2-category $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{Cat})$, and hence it is possible to define a notion of equivalences between its objects. A colax functor $X$ is said to be $\Bbbk$-*projective* (resp. $\Bbbk$-*flat*) if the $\Bbbk$-modules $X(i)(x, y)$ are projective (resp. flat) for all $i \in I_0$ and for all objects $x, y$ of $X(i)$. After defining a tilting colax functor for $X$, the derived equivalence of colax functors are characterized in the main result in [7] as follows.

**Theorem 1.1.** *Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{Cat})_0$. If $X$ and $X'$ are derived equivalent, then $X'$ is equivalent in the 2-category $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{Cat})$ to a tilting colax functor $\mathscr{T}$ for $X$. If $X'$ is $\Bbbk$-projective, then the converse holds.*

The main result in [8] gives a sufficient condition for two colax functors to have derived equivalent Grothendieck constructions as follows :

**Theorem 1.2.** *Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{Cat})_0$. Assume that $X$ is $\Bbbk$-flat and that $X'$ is equivalent to a tilting colax functor $\mathscr{T}$ for $X$ in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{Cat})$. Then $\int X$ and $\int X'$ are derived equivalent.*

As a special case when $I$ is a group $G$ (regarded as a category with a single object $*$), $\int X = \mathscr{A}/G$ is the orbit category (also called the skew group category and denoted by $\mathscr{A} * G$) of the linear category $\mathscr{A} := X(*)$ with the $G$-action $X$, and hence it tells us when a derived equivalence between linear categories $\mathscr{A}$ and $\mathscr{A}'$ with $G$-actions have derived equivalent orbit categories $\mathscr{A}/G$ and $\mathscr{A}'/G$.

To consider derived equivalences of linear categories it is natural to deal with it in the setting of dg categories as is seen in [28]. Therefore it is natural to investigate the same problem for dg categories. In this paper, we will do it by considering the 2-category $\Bbbk$-**dgCat** of small dg $\Bbbk$-categories instead of $\Bbbk$-**Cat**. For two colax functors $X$ and $X'$ from a small category $I$ to $\Bbbk$-**dgCat**, instead of a derived equivalence from $X'$ to $X$, we consider a "standard derived equivalence" (a special form of a derived equivalence), and (i) characterize it by using the notions of quasi-equivalence bimodules over colax functors and tilting colax functors; and (ii) in that case we show that their gluings $\int X$ and $\int X'$ are derived equivalent. More precisely, the first result is stated as follows.

**Theorem 1.3** (Theorem 9.17 in the text)**.** *Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})_0$. Then among the statements below, we have implications (1) ⇒ (2) ⇒ (3) ⇒ (4). If $X$ is $\Bbbk$-flat, then we also have the implication (4) ⇒ (1).*

(1) *There exists an $X'$-$X$-bimodule $Z$ such that $\text{-}\overset{\mathbf{L}}{\otimes}_{X'} Z \colon \mathscr{D}(X') \to \mathscr{D}(X)$ is an equivalence in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$.*
(2) *There exists a 1-morphism $(\mathbf{F}, \boldsymbol{\psi}) \colon \mathscr{C}_{\mathrm{dg}}(X') \to \mathscr{C}_{\mathrm{dg}}(X)$ in the 2-category $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCAT})$ such that $\mathbf{L}(\mathbf{F}, \boldsymbol{\psi}) \colon \mathscr{D}(X') \to \mathscr{D}(X)$ is an equivalence in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$.*
(3) *There exists a tilting colax functor $\mathscr{T}$ for $X$, and there exists a quasi-equivalence $X'$-$\mathscr{T}$-bimodule $E$.*
(4) *There exists a tilting colax functor $\mathscr{T}$ for $X$, and there exist 1-morphisms*
$$(\mathbf{G}, \boldsymbol{\psi}') \colon \mathscr{C}_{\mathrm{dg}}(X') \to \mathscr{C}_{\mathrm{dg}}(\mathscr{T}) \text{ and } (\mathbf{F}, \boldsymbol{\psi}) \colon \mathscr{C}_{\mathrm{dg}}(\mathscr{T}) \to \mathscr{C}_{\mathrm{dg}}(X)$$
*in the 2-category $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCAT})$ such that*
$$\mathbf{L}(\mathbf{G}, \boldsymbol{\psi}') \colon \mathscr{D}(X') \to \mathscr{D}(\mathscr{T}) \text{ and } \mathbf{L}(\mathbf{F}, \boldsymbol{\psi}) \colon \mathscr{D}(\mathscr{T}) \to \mathscr{D}(X)$$
*are equivalences in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$.*

The equivalence of the form $\text{-}\overset{\mathbf{L}}{\otimes}_{X'} Z \colon \mathscr{D}(X') \to \mathscr{D}(X)$ in (1) above is called a *standard derived equivalence* from $X'$ to $X$, and we say that $X'$ *is standardly derived equivalent to* $X$ if one of the conditions in the theorem above holds. We denote this fact by $X' \overset{\mathrm{sd}}{\rightsquigarrow} X$ or $X \overset{\mathrm{sd}}{\leftsquigarrow} X'$. The second result is stated as follows.

**Theorem 1.4** (Theorem 10.6 in the text)**.** *Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})_0$, and assume that $X$ is $\Bbbk$-flat. If $X'$ is standardly derived equivalent to $X$, or equivalently, if there exists a quasi-equivalence $X'$-$\mathscr{T}$-bimodule for some tilting colax functor $\mathscr{T}$ for $X$, then $\int X'$ is derived equivalent to $\int X$.*

We remark that for any $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$, we do not know whether the relation $X' \overset{\mathrm{sd}}{\rightsquigarrow} X$ is symmetric or not at present. Therefore, when we say that "$X$ and $X'$ are standardly derived equivalent", this means that there exists a zigzag chain form $X$ to $X'$ of the form $X = X_0 \overset{\text{sd}}{\rightsquigarrow} X_1 \overset{\text{sd}}{\leftsquigarrow} X_2 \overset{\text{sd}}{\rightsquigarrow} \cdots \overset{\text{sd}}{\leftsquigarrow} X_{2n} = X'$ for some $n \geq 1$ and $X_1, X_2, \dots, X_{2n} \in \operatorname{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. In contrast, for dg categories $\mathscr{A}$ and $\mathscr{A}'$ the relation that "$\mathscr{A}'$ is derived equivalent to $\mathscr{A}$" is symmetric, which allows us to express it by saying that "$\mathscr{A}$ and $\mathscr{A}'$ are derived equivalent". We also remark the following two special cases.

The first one is the case where $I$ is a category with only one object $*$ and one morphism $\mathbb{1}_*$. In this case, $X$ is identified with a dg category $X(*)$, and hence Theorem 1.3 generalizes Keller's theorem [28, Theorem 8.1]. The second one is the case when $I = G$ is a group. In this case, $\int X = \mathscr{A}/G$ is the orbit dg category of a dg category $\mathscr{A} := X(*)$ with a $G$-action, and hence Theorem 1.4 gives us a sufficient condition for a derived equivalence between dg categories $\mathscr{A}$ and $\mathscr{A}'$ with $G$-actions to have derived equivalent dg orbit categories $\mathscr{A}/G$ and $\mathscr{A}'/G$. We will apply this to the complete Ginzburg dg algebras of quivers with potentials having a $G$-action. Recall that a quiver with potentials was introduced by Derksen, Weyman and Zelevinsky in [17] to study the theory of cluster algebras. From a quiver with potentials $(Q, W)$, the Jacobian algebra $J(Q, W)$ and the completed Ginzburg dg algebra $\widehat{\Gamma}(Q, W)$ are defined, which are related as $H^0(\widehat{\Gamma}(Q, W)) = J(Q, W)$. Therefore, $\widehat{\Gamma}(Q, W)$ is regarded as an extension of Jacobian algebra to a dg algebra.

The orbit category (the skew group algebra) $J(Q, W)/G$ was computed up to Morita equivalence as the form $J(Q_G, W_G)$ for some quiver with potentials $(Q_G, W_G)$ by Paquette–Schiffler in [42] in the case that $G$ is a finite subgroup of the automorphism group of $J(Q_G, W_G)$ acting freely on vertices. On the other hand, the orbit dg category (the skew group dg algebra) $\widehat{\Gamma}(Q, W)/G$ was computed up to Morita equivalence as the form $\widehat{\Gamma}(Q_G, W_G)$ for some quiver with potentials $(Q_G, W_G)$ by Le Meur in [34] in the case that $G$ is a finite group (see also Amiot–Plamondon [1] for the case that $G = \mathbb{Z}/2\mathbb{Z}$, Giovannini and Pasquali [22] for the cyclic case, and Giovannini, Pasquali and Plamondon [23] for the finite abelian case). We remark that for both $J(Q, W)$ and $\widehat{\Gamma}(Q, W)$, the quiver $Q_G$ can be computed by using a result by Demonet in [16] on the computation of the skew group algebra of the path algebra of a quiver with an action of a finite group, and in the arbitrary group case, $Q_G$ can be computed from a non-admissible presentation given in [9] by making it as an admissible presentation.

By Keller–Yang [31], if $(Q', W')$ is obtained as a mutation of $(Q, W)$, then the dg algebras $\widehat{\Gamma}(Q, W)$ and $\widehat{\Gamma}(Q', W')$ are derived equivalent. Using our main theorem above, we can show that this derived equivalence sometimes induces a derived equivalence between $\widehat{\Gamma}(Q_G, W_G)$ and $\widehat{\Gamma}(Q'_G, W'_G)$, where even if $(Q_G, W_G)$ and $(Q'_G, W'_G)$ do not need to be obtained by a mutation from each other. For this phenomenon, an example will be given at the end of the paper.

The paper is organized as follows. In Section 2, we shall fix notations and prepare some basic facts for our proofs. In Section 3, we collect basic facts about enriched categories that will be needed later. In section 4, we will introduce the notion of $I$-coverings that is a generalization of that of $G$-coverings for a group $G$ introduced in [5], which was obtained by generalizing the notion of Galois coverings introduced by Gabriel in [19]. This will be used in the proof of our second main theorem. In Section 5, we define a 2-functor $\int\colon \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat}) \to \Bbbk\text{-}\mathbf{dgCat}$ whose correspondence on objects is a dg version of (the opposite version of) the original Grothendieck construction. In Section 6, we will show that the Grothendieck construction is a strict left adjoint to the diagonal 2-functor, and that $I$-coverings are essentially given by the unit of the adjunction. In Section 7, we will give the definition of derived colax functors together with necessary pseudofunctors. In Section 8, we define necessary terminologies such as 2-quasi-isomorphisms for 2-morphisms, quasi-equivalences for 1-morphisms, and the derived 1-morphism $\mathbf{L}(F, \psi)\colon \mathscr{D}(X') \to \mathscr{D}(X)$ of a 1-morphism $(F, \psi)\colon X' \to X$ between colax functors, and show the fact that the derived 1-morphism of a quasi-equivalence 1-morphism between colax functors $X$, $X'$ turns out to be an equivalence between derived dg module colax functors of $X$, $X'$. Also, we give definitions of tilting colax functors and of derived equivalences. In Section 9, we define standard derived equivalences of colax functors from $I$ to $\Bbbk$-$\mathbf{dgCat}$ and quasi-equivalence bimodules, and prove our first main theorem, and in Section 10, we prove our second main theorem. Two examples are given in Section 11 that illustrate our second main theorem in the group action case. In Appendix, we review the quasi-equivalences and derived equivalences for dg categories for convenience of the reader.

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## 2. Preliminaries

In this section, after set theoretic remarks, we recall the definition of the 2-category of colax functors from a small category $I$ to a 2-category from [7] (see also Tamaki [47]).

First of all, we make set theoretic remarks (see [33] or [12, Appendix A] for details). In this paper, we adopt ZFC (Zermelo-Fraenkel set-theory (ZF) with the axiom of choice (C)) as axioms of set theory, and we do not assume the existence of urelements. In addition, we assume *the axiom of universe* stating that any set is an element of a (Grothendieck) universe. The class of all sets is denoted by $\mathsf{SET}$. The power set of a set $A$ is denoted by $\mathscr{P}A$, and the set of all non-negative integers by $\mathbb{N}$. Recall the following (see e.g., [48]):

**Fact.** *The class* $\mathsf{Univ}$ *of all universes are well-ordered.*

We fix a universe $\mathfrak{U}$ with $\mathbb{N} \in \mathfrak{U}$ once for all. A set $S$ is called a $\mathfrak{U}$-*small* set (resp. a $\mathfrak{U}$-*class*) if $S \in \mathfrak{U}$ (resp. $S \subseteq \mathfrak{U}$). Following Levy [33], we use the hierarchy of $\mathsf{SET}$: $\mathbf{Class}_0^0 \subset \mathbf{Class}_0^1 \subset \mathbf{Class}_0^2 \subset \cdots \subset \mathsf{SET}$.

**Definition 2.1.** Let $A$ be a set, and assume that $\mathfrak{U} \subseteq A$. By axiom of universe and Fact above, there exists the smallest universe $\mathfrak{U}'$ such that $A \in \mathfrak{U}'$. We define $\Psi A$ to be the smallest set $X \in \mathfrak{U}'$ satisfying the condition that

$$X \supseteq A \cup (X \times X) \cup (\bigcup_{I \in A} X^I).$$

Therefore in particular, since $X = \mathfrak{U}$ satisfies this condition for $A = \mathfrak{U}$, we have $\Psi\mathfrak{U} = \mathfrak{U}$.

**Remark 2.2.** The existence of $\Psi A$ is proved in [12, Proposition A.2.2], and it satisfies

$$\Psi A = A \cup (\Psi A \times \Psi A) \cup \left( \bigcup_{I \in A} (\Psi A)^I \right).$$

**Definition 2.3.** Let $k \in \mathbb{N}$.

(1) An element of $(\mathscr{P}\Psi)^k\mathfrak{U}$ is called a $k$-*class*, and an element of $((\mathscr{P}\Psi)^k\mathfrak{U}) \setminus ((\mathscr{P}\Psi)^{k-1}\mathfrak{U})$ is called a *proper* $k$-*class*.
(2) The category of the $k$-classes (and the maps between them) is denoted by $\mathbf{Class}^k$. Therefore we have $\mathbf{Class}_0^k = (\mathscr{P}\Psi)^k\mathfrak{U}$.

**Remark 2.4.** The following are immediate from the definition above:

(1) A 0-class is nothing but a $\mathfrak{U}$-small set.
(2) A 1-class is nothing but a $\mathfrak{U}$-class (indeed, $\Psi\mathfrak{U} = \mathfrak{U}$ shows that $(\mathscr{P}\Psi)\mathfrak{U} = \mathscr{P}\mathfrak{U}$).
(3) A $k$-class is nothing but a subset of $\Psi\mathbf{Class}_0^{k-1}$ for all $k \geq 1$.

In the following, we call $\mathfrak{U}$-small sets and $\mathfrak{U}$-classes simply small sets and 1-classes, respectively.

In this paper, all categories $\mathscr{C}$ are assumed to be "small" categories in the sense that $\mathscr{C}_0, \mathscr{C}(x, y) \in \mathsf{SET}$ for all $x, y \in \mathscr{C}_0$. We now define $\mathfrak{U}$-small categories, which are simply called small categories below.

**Definition 2.5.** Let $\mathscr{C}$ be a category, and $k \in \mathbb{N}$.

(1) $\mathscr{C}$ is called a *small category* if $\mathscr{C}_0$ and all local morphism sets $\mathscr{C}(x, y)$ $(x, y \in \mathscr{C}_0)$ are small.
(2) $\mathscr{C}$ is called a *light category* if $\mathscr{C}_0$ is a 1-class, and all local morphism sets are small sets.
(3) $\mathscr{C}$ is called a *moderate category* if $\mathscr{C}_0$ and all local morphism sets are 1-classes.
(4) More generally, $\mathscr{C}$ is called a $k$-*moderate category* if $\mathscr{C}_0$ and all local morphism sets are $k$-classes. $\mathscr{C}$ is called a *properly* $k$-moderate category if it is $k$-moderate but not $(k-1)$-moderate.

**Remark 2.6.** A 0-moderate category is nothing but a small category. A 1-moderate category is just a moderate category. A small category is a light category, and a light category is a 1-moderate category.

We next summarize necessary facts on 2-categories that will be used later.

**Definition 2.7.** A 2-category $\mathbf{C}$ is a sequence of the following data:

- A set $\mathbf{C}_0$ of objects,
- A family of categories $(\mathbf{C}(x,y))_{x,y\in\mathbf{C}_0}$,
- A family of functors $\circ := (\circ_{x,y,z} : \mathbf{C}(y,z) \circ \mathbf{C}(x,y) \to \mathbf{C}(x,z))_{x,y,z\in\mathbf{C}_0}$,
- A family of functors $(\mu_x : \mathbb{1}_x \to \mathbf{C}(x,x))_{x\in\mathbf{C}_0}$.

These data are required to satisfy the following axioms:

- (Associativity) The following diagram is commutative for all $x, y, z \in \mathbf{C}_0$

$$\begin{tikzcd} \mathbf{C}(z,w) \circ \mathbf{C}(y,z) \circ \mathbf{C}(x,y) \arrow[r, "\circ\times\mathbf{1}"] \arrow[d, "\mathbf{1}\times\circ"'] & \mathbf{C}(y,w) \circ \mathbf{C}(x,y) \arrow[d, "\circ"] \\ \mathbf{C}(z,w) \circ \mathbf{C}(x,z) \arrow[r, "\circ"] & \mathbf{C}(x,w). \end{tikzcd}$$

- (Unitality) The following diagram is commutative for all $x, y \in \mathbf{C}_0$

$$\begin{tikzcd} \mathbb{1} \times \mathbf{C}(x,y) \arrow[dr, "\mathrm{prj}_1"] \arrow[dd, "\mu_y\times\mathbf{C}(x,y)"'] & & \mathbf{C}(x,y) \times \mathbb{1} \arrow[dl, "\mathrm{prj}_2"'] \arrow[dd, "\mathbf{C}(x,y)\times\mu_x"] \\ & \mathbf{C}(x,y) & \\ \mathbf{C}(y,y) \times \mathbf{C}(x,y) \arrow[ur, "\circ"] & & \mathbf{C}(x,y) \times \mathbf{C}(x,x). \arrow[ul, "\circ"'] \end{tikzcd}$$

**Remark 2.8.** Elements of $\mathbf{C}_0$ are called objects of $\mathbf{C}$, objects (resp. morphisms, compositions) of $\mathbf{C}(x,y)$ are called are called 1-morphisms (resp. 2-morphisms, vertical compositions) of $\mathbf{C}$ with $x, y \in \mathbf{C}_0$. We sometimes abbreviate $x \in \mathbf{C}$ for $x \in \mathbf{C}_0$ if there seems to be no risk of confusion, and do the same even when $\mathbf{C}$ is a usual category. To distinguish the vertical composition from the horizontal composition, we use the notation $\bullet$ for the former, and $\circ$ for the latter. Sometimes $\circ$ is omitted.

**Definition 2.9.** Let $k \in \mathbb{N}$.

(1) The 2-category of all small categories is denoted by $\mathbf{Cat}$.
(2) The 2-category of all light categories is denoted by $\mathbf{CAT}$.
(3) The 2-category of all moderate categories is denoted by $\underline{\mathbf{CAT}}$.
(4) The 2-category of all $k$-moderate categories is denoted by $\underline{\mathbf{CAT}}^k$.

When $\mathscr{C}$ is a 2-category, if $x, y \in \mathscr{C}_0$ and $f, g \in \mathscr{C}(x,y)_0$, then $\mathscr{C}(x,y)_0$ (resp. $\mathscr{C}(x,y)(f,g)$) is called a *local* 1-*morphism set* (resp. *local* 2-*morphism set*) of $\mathscr{C}$.

**Definition 2.10.** Let $\mathscr{C}$ be a 2-category.

(1) $\mathscr{C}$ is said to be *small* if $\mathscr{C}_0$ and all of its local $r$-morphism sets are small for each $r = 1, 2$.
(2) $\mathscr{C}$ is said to be *light* if $\mathscr{C}_0$ is a class, and all of its local $r$-morphism sets are small for each $r = 1, 2$.
(3) $\mathscr{C}$ is said to be $k$-*moderate* if $\mathscr{C}_0$ and all of its local $r$-morphism sets are $k$-classes for each $r = 1, 2$, namely if $\mathscr{C}_0$ is a $k$-class, and categories $\mathscr{C}(x, y)$ are $k$-moderate for all $x, y \in \mathscr{C}_0$.

We cite the following from [33] (see [12, Proposition A.4.2] for the proof).

**Proposition 2.11.** *The following hold.*

(1) *The* 2*-category* **Cat** *is light;*
(2) *The* 2*-category* **CAT** *is* 2*-moderate; and*
(3) *The* 2*-category* $\underline{\mathbf{CAT}}^k$ *is* $(k+1)$*-moderate for all* $1 \le k \in \mathbb{N}$.

**Definition 2.12.** Let $I$ be a small category and $\mathbf{C}$ a 2-category. A *colax functor* (or an *oplax* functor) from $I$ to $\mathbf{C}$ is a triple $(X, X_i, X_{b,a})$ of data:

- a quiver morphism $X\colon I \to \mathbf{C}$, where $I$ and $\mathbf{C}$ are regarded as quivers by forgetting additional data such as 2-morphisms or compositions;
- a family $(X_i)_{i \in I_0}$ of 2-morphisms $X_i\colon X(\mathbb{1}_i) \Rightarrow \mathbb{1}_{X(i)}$ in $\mathbf{C}$ indexed by $i \in I_0$; and
- a family $(X_{b,a})_{(b,a)}$ of 2-morphisms $X_{b,a}\colon X(ba) \Rightarrow X(b)X(a)$ in $\mathbf{C}$ indexed by $(b, a) \in \operatorname{com}(I) := \{(b, a) \in I_1 \times I_1 \mid ba \text{ is defined}\}$

satisfying the axioms:

(a) For each $a\colon i \to j$ in $I$ the following are commutative:

$$\begin{tikzcd} X(a\mathbb{1}_i) \arrow[r, Rightarrow, "X_{a,\mathbb{1}_i}"] \arrow[dr, equal] & X(a)X(\mathbb{1}_i) \arrow[d, Rightarrow, "X(a)X_i"] \\ & X(a)\mathbb{1}_{X(i)} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X(\mathbb{1}_j a) \arrow[r, Rightarrow, "X_{\mathbb{1}_j,a}"] \arrow[dr, equal] & X(\mathbb{1}_j)X(a) \arrow[d, Rightarrow, "X_jX(a)"] \\ & \mathbb{1}_{X(j)}X(a) \end{tikzcd} \; ; \text{ and}$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in $I$ the following is commutative:

$$\begin{tikzcd} X(cba) \arrow[r, Rightarrow, "X_{c,ba}"] \arrow[d, Rightarrow, "X_{cb,a}"'] & X(c)X(ba) \arrow[d, Rightarrow, "X(c)\theta_{b,a}"] \\ X(cb)X(a) \arrow[r, Rightarrow, "X_{c,b}X(a)"'] & X(c)X(b)X(a). \end{tikzcd}$$

**Definition 2.13.** Let $\mathbf{C}$ be a 2-category and $X = (X, X_i, X_{b,a})$, $X' = (X', X'_i, X'_{b,a})$ be colax functors from $I$ to $\mathbf{C}$. A 1-*morphism* (called a *left transformation*) from $X$ to $X'$ is a pair $(F, \psi)$ of data

- a family $F := (F(i))_{i \in I_0}$ of 1-morphisms $F(i)\colon X(i) \to X'(i)$ in $\mathbf{C}$ ; and

- a family $\psi := (\psi(a))_{a\in I_1}$ of 2-morphisms $\psi(a)\colon X'(a)F(i) \Rightarrow F(j)X(a)$

$$\begin{tikzcd} X(i) \arrow[r, "F(i)"] \arrow[d, "X(a)"'] & X'(i) \arrow[d, "X'(a)"] \arrow[dl, Rightarrow, "\psi(a)"'] \\ X(j) \arrow[r, "F(j)"'] & X'(j) \end{tikzcd}$$

  in $\mathbf{C}$ indexed by $a\colon i \to j$ in $I_1$

satisfying the axioms

(a) For each $i \in I_0$ the following is commutative:

$$\begin{tikzcd} X'(\mathbb{1}_i)F(i) \arrow[r, Rightarrow, "\psi(\mathbb{1}_i)"] \arrow[d, Rightarrow, "X'_iF(i)"'] & F(i)X(\mathbb{1}_i) \arrow[d, Rightarrow, "F(i)X_i"] \\ \mathbb{1}_{X'(i)}F(i) \arrow[r, equal] & F(i)\mathbb{1}_{X(i)} \end{tikzcd} \quad ; \text{ and}$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k$ in $I$ the following is commutative:

$$\begin{tikzcd} X'(ba)F(i) \arrow[r, Rightarrow, "X'_{b,a}F(i)"] \arrow[d, Rightarrow, "\psi(ba)"'] & X'(b)X'(a)F(i) \arrow[r, Rightarrow, "X'(b)\psi(a)"] & X'(b)F(j)X(a) \arrow[d, Rightarrow, "\psi(b)X(a)"] \\ F(k)X(ba) \arrow[rr, Rightarrow, "F(k)\,X_{b,a}"'] & & F(k)X(b)X(a). \end{tikzcd}$$

A 1-morphism $(F, \psi)\colon X \to X'$ is said to be *$I$-equivariant* if $\psi(a)$ is a 2-isomorphism in $\mathbf{C}$ for all $a \in I_1$.

**Definition 2.14.** Let $\mathbf{C}$ be a 2-category, $X = (X, X_i, X_{b,a})$, $X' = (X', X'_i, X'_{b,a})$ be colax functors from $I$ to $\mathbf{C}$, and $(F, \psi)$, $(F', \psi')$ 1-morphisms from $X$ to $X'$. A 2-*morphism* from $(F, \psi)$ to $(F', \psi')$ is a family $\zeta = (\zeta(i))_{i\in I_0}$ of 2-morphisms $\zeta(i)\colon F(i) \Rightarrow F'(i)$ in $\mathbf{C}$ indexed by $i \in I_0$ such that the following is commutative for all $a\colon i \to j$ in $I$:

$$\begin{tikzcd} X'(a)F(i) \arrow[r, Rightarrow, "X'(a)\zeta(i)"] \arrow[d, Rightarrow, "\psi(a)"'] & X'(a)F'(i) \arrow[d, Rightarrow, "\psi'(a)"] \\ F(j)X(a) \arrow[r, Rightarrow, "\zeta(j)X(a)"] & F'(j)X(a). \end{tikzcd}$$

**Definition 2.15.** Let $\mathbf{C}$ be a 2-category, $X = (X, X_i, X_{b,a})$, $X' = (X', X'_i, X'_{b,a})$ and $X'' = (X'', X''_i, X''_{b,a})$ colax functors from $I$ to $\mathbf{C}$, and let $(F, \psi)\colon X \to X'$, $(F', \psi')\colon X' \to X''$ be 1-morphisms. Then the composite $(F', \psi')(F, \psi)$ of $(F, \psi)$ and $(F', \psi')$ is a 1-morphism from $X$ to $X''$ defined by

$$(F', \psi')(F, \psi) := (F'F, \psi' \circ \psi),$$

where $F'F := (F'(i)F(i))_{i\in I_0}$ and for each $a\colon i \to j$ in $I$, $(\psi' \circ \psi)(a) := F'(j)\psi(a) \bullet \psi'(a)F(i)$ is the pasting of the diagram

$$\begin{tikzcd}
X(i) \arrow[r, "F(i)"] \arrow[d, "X(a)"'] & X'(i) \arrow[r, "F'(i)"] \arrow[d, "X'(a)"] \arrow[dl, Rightarrow, "\psi(a)"'] & X''(i) \arrow[d, "X''(a)"] \arrow[dl, Rightarrow, "\psi'(a)"'] \\
X(j) \arrow[r, "F(j)"'] & X'(j) \arrow[r, "F'(j)"'] & X''(j).
\end{tikzcd}$$

The following is straightforward to verify.

**Proposition 2.16.** *Let* $\mathbf{C}$ *be a* 2*-category. Then colax functors* $I \to \mathbf{C}$, 1*-morphisms between them, and* 2*-morphisms between* 1*-morphisms* (*defined above*) *define a* 2*-category, which we denote by* $\mathrm{Colax}(I, \mathbf{C})$.

**Notation 2.17.** Let $\mathbf{C}$ be a 2-category. Then we denote by $\mathbf{C}^{\mathrm{op}}$ (resp. $\mathbf{C}^{\mathrm{co}}$) the 2-category obtained from $\mathbf{C}$ by reversing the 1-morphisms (resp. the 2-morphisms), and we set $\mathbf{C}^{\mathrm{coop}} := (\mathbf{C}^{\mathrm{co}})^{\mathrm{op}} = (\mathbf{C}^{\mathrm{op}})^{\mathrm{co}}$.

## 3. Enriched categories

In this section we collect basic facts about enriched categories which will be needed later. Throughout this section, we fix a symmetric monoidal category $\mathbb{V}$ and work in $\mathbb{V}$. Before starting our discussion we recall the definition of symmetric monoidal categories.

**Definition 3.1.** (1) A *monoidal category* is a sequence of the data

- a category $\mathbb{V}$,
- an object 1 of $\mathbb{V}$,
- a functor $\otimes\colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}$,
- a family of natural isomorphisms $a_{A,B,C}\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ indexed by the triples $A, B, C$ of objects of $\mathbb{V}$, called the *associator*,
- a family of natural isomorphisms $\ell_A\colon 1 \otimes A \to A$ indexed by the objects $A$ of $\mathbb{V}$,
- a family of natural isomorphisms $r_A\colon A \otimes 1 \to A$ indexed by the objects $A$ of $\mathbb{V}$

that satisfies the following axioms:

(a) For any $A, B, C, D \in \mathbb{V}_0$, the following is commutative:

$$\begin{tikzcd}
 & A \otimes (B \otimes (C \otimes D)) \arrow[dl, "a"'] \arrow[dr, "1\otimes a"] & \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "a"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[d, "a"] \\
((A \otimes B) \otimes C) \otimes D & & (A \otimes (B \otimes C)) \otimes D \arrow[ll, "a\otimes 1"]
\end{tikzcd} ;$$

(b) For any $A, B \in \mathbb{V}_0$, the following is commutative:

$$\begin{tikzcd} A \otimes (B \otimes 1) \arrow[r, "1\otimes r"] \arrow[d, "a"'] & A \otimes B \\ (A \otimes B) \otimes 1 \arrow[ur, "r"'] & \end{tikzcd} \text{; and}$$

(c) $\ell_1 = r_1 \colon 1 \otimes 1 \to 1$.

According to [32], it is known that both of the following diagrams automatically turn out to be commutative for all objects $A, B$ in a monoidal category $\mathbb{V}$:

$$\begin{tikzcd} 1 \otimes (A \otimes B) \arrow[r, "\ell"] \arrow[d, "a"'] & A \otimes B \\ (1 \otimes A) \otimes B \arrow[ur, "\ell\otimes 1"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \otimes (1 \otimes B) \arrow[r, "1\otimes \ell"] \arrow[d, "a"'] & A \otimes B \\ (A \otimes 1) \otimes B \arrow[ur, "r\otimes 1"'] & \end{tikzcd} .$$

(2) A *switching operation* on $\mathbb{V}$ is a family $t = (t_{A,B} \colon A \otimes B \to B \otimes A)_{(A,B)\in \mathbb{V}_0 \times \mathbb{V}_0}$ such that the following is commutative:

$$\begin{tikzcd} A \otimes B \arrow[r, "t_{A,B}"] \arrow[d, "f\otimes g"'] & B \otimes A \arrow[d, "g\otimes f"] \\ C \otimes D \arrow[r, "t_{C,D}"'] & D \otimes C \end{tikzcd}$$

for all morphisms $f \colon A \to C$ and $g \colon B \to D$ in $\mathbb{V}$.

(3) A monoidal category $\mathbb{V}$ with a switching operation $t$ is called a *symmetric monoidal category* if the following hold:

(a) $t_{A,B} \circ t_{B,A} = 1$ for all $A, B \in \mathbb{V}_0$; and

(b) For any $A, B, C \in \mathbb{V}_0$, the following is commutative:

$$\begin{tikzcd} & A \otimes (B \otimes C) \arrow[dl, "t_{A,B\otimes C}"'] \arrow[dr, "a_{A,B,C}"] & \\ (B \otimes C) \otimes A & & (A \otimes B) \otimes C \arrow[d, "t_{A,B}\otimes 1"] \\ B \otimes (C \otimes A) \arrow[u, "a_{B,C,A}"] & & (B \otimes A) \otimes C \arrow[dl, "a^{-1}_{B,A,C}"] \\ & B \otimes (A \otimes C) \arrow[ul, "1\otimes t_{A,C}"] & \end{tikzcd} .$$

**Example 3.2.** The following give examples of symmetric monoidal categories:

(1) $\mathbb{V} := \mathbf{Cat}$, the category of small caetegories. Here, 1 is given by a category with only one object and one morphism, $\otimes$ is given by the direct product of small categories and $a, \ell, r, t$ are given as the canonical isomorphisms.

(2) $\mathbb{V} := \operatorname{Mod}\Bbbk$, the category of $\Bbbk$-modules. In this case, $1$ is given by $\Bbbk$, $\otimes$ is given by the tensor product over $\Bbbk$, and $a, \ell, r, t$ are also given as the canonical isomorphisms.

(3) $\mathbb{V} := \mathscr{C}(\Bbbk)$, the category of the (unbounded) chain complexes (here we use cocomplexes) of $\Bbbk$-modules and the chain morphisms, i.e., the degree-preserving morphisms commuting with differentials. In this case, $1$ is given by the complex $\Bbbk$ concentrated in degree $0$, for $A, B \in \mathbb{V}_0$, $A \otimes B$ is given as the tensor chain complex over $\Bbbk$, and also $a, \ell, r, t$ are given as the canonical isomorphisms. Note that for each $A \in \mathbb{V}_0$, the "underlying set" $\mathscr{C}(\Bbbk)(\Bbbk, A)$ is the set of 0-cocycles $Z^0(A)$ of $A$.

**Definition 3.3.** A category $\mathscr{A}$ *enriched over* $\mathbb{V}$, or simply a $\mathbb{V}$-*category* consists of the following data:

- a class of objects $\mathscr{A}_0$;
- for two objects $x, y$ in $\mathscr{A}$, an object $\mathscr{A}(x, y)$ in $\mathbb{V}$;
- for three objects $x, y, z$ in $\mathscr{A}$, a morphism
$$\circ : \mathscr{A}(y, z) \otimes \mathscr{A}(x, y) \to \mathscr{A}(x, z)$$
in $\mathbb{V}$; and
- for an object $x$ in $\mathscr{A}$, a morphism in $\mathbb{V}$
$$1_x : 1 \to \mathscr{A}(x, x)$$

satisfying the following conditions:

(1) For any objects $x, y, z, w$, the following diagram is commutative:

$$\begin{tikzcd}
(\mathscr{A}(z,w) \otimes \mathscr{A}(y,z)) \otimes \mathscr{A}(x,y) \arrow[rr, "a"] \arrow[d, "\circ\times 1"'] & & \mathscr{A}(z,w) \otimes (\mathscr{A}(y,z) \otimes \mathscr{A}(x,y)) \arrow[d, "1\times\circ"] \\
\mathscr{A}(y,w) \otimes \mathscr{A}(x,y) \arrow[dr, "\circ"'] & & \mathscr{A}(z,w) \otimes \mathscr{A}(x,z) \arrow[dl, "\circ"'] \\
& \mathscr{A}(x,w) &
\end{tikzcd}$$
; and

(2) For any objects $x, y$, the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{A}(y,y) \otimes \mathscr{A}(x,y)) \arrow[r, "\circ"] & \mathscr{A}(x,y) & \mathscr{A}(x,y) \otimes \mathscr{A}(x,x) \arrow[l, "\circ"'] \\
1 \otimes \mathscr{A}(x,y) \arrow[u] \arrow[ur] & & \mathscr{A}(x,y) \otimes 1 \arrow[u] \arrow[ul]
\end{tikzcd}$$
.

**Definition 3.4.** Given $\mathbb{V}$-categories $\mathscr{A}, \mathscr{B}$, a $\mathbb{V}$-functor or an enriched functor $F : \mathscr{A} \to \mathscr{B}$ consists of the following data:

- for each $x \in \mathscr{A}_0$, an object $F(x)$ of $\mathscr{B}$;
- for any $x, y \in \mathscr{A}_0$, a morphism in $\mathbb{V}$,
$$F_{x,y} : \mathscr{A}(x, y) \to \mathscr{B}(F(x), F(y))$$

that satisfies the following axioms:

(1) For any $x, y, z \in \mathscr{A}_0$, the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{A}(y,z) \otimes \mathscr{A}(x,y) \arrow[r, "\circ"] \arrow[d, "F_{y,z} \times F_{x,y}"'] & \mathscr{A}(x,z) \arrow[d, "F_{x,z}"] \\
\mathscr{B}(F(y),F(z)) \otimes \mathscr{B}(F(x),F(y)) \arrow[r, "\circ"] & \mathscr{B}(F(x),F(z))
\end{tikzcd} \quad ; \text{and}$$

(2) For each $x \in \mathscr{A}_0$, the following diagram is commutative:

$$\begin{tikzcd}
1 \arrow[r, "1_x"] \arrow[dr, "1_{F(x)}"'] & \mathscr{A}(x,x) \arrow[d, "F_{x,x}"] \\
& \mathscr{A}(F(x),F(x))
\end{tikzcd} .$$

**Definition 3.5.** Let $F, G\colon \mathscr{A} \to \mathscr{B}$ be $\mathbb{V}$-functors between $\mathbb{V}$-categories. A $\mathbb{V}$-*natural transformation* $\alpha$ from $F$ to $G$, denoted by $\alpha\colon F \Rightarrow G$, is a family $\alpha = (\alpha(x))_{x \in \mathscr{A}_0}$ of morphisms $\alpha(x)\colon 1 \to \mathscr{B}(F(x), G(x))$ in $\mathbb{V}$ making the following diagram commutative for all $x, y \in \mathscr{A}_0$:

$$\begin{tikzcd}[column sep=small]
& \mathscr{A}(x,y) \arrow[dl, "r^{-1}"'] \arrow[dr, "\ell^{-1}"] & \\
\mathscr{A}(x,y) \otimes 1 \arrow[d, "G \otimes \alpha(x)"'] & & 1 \otimes \mathscr{A}(x,y) \arrow[d, "\alpha(y) \otimes F"] \\
\mathscr{B}(G(x),G(y)) \otimes \mathscr{B}(F(x),G(x)) \arrow[dr, "\circ"'] & & \mathscr{B}(F(y),G(y)) \otimes \mathscr{B}(F(x),F(y)) \arrow[dl, "\circ"] \\
& \mathscr{B}(F(x),G(y)) &
\end{tikzcd} . \tag{3.1}$$

The composition of $\mathbb{V}$-natural transformations is defined in an obvious way.

**Definition 3.6.** The 2-category of *small* $\mathbb{V}$-categories, $\mathbb{V}$-functors, and $\mathbb{V}$-natural transformations is denoted by $\mathbb{V}$-**Cat**.

**Example 3.7.** The following are examples of $\mathbb{V}$-categories.

(1) In the case where $\mathbb{V} = \mathbf{Cat}$, $\mathbb{V}$-categories are nothing but (strict) 2-categories. $\mathbb{V}$-functors are called 2-functors.
(2) In the case where $\mathbb{V} = \operatorname{Mod} \Bbbk$, $\mathbb{V}$-categories are nothing but $\Bbbk$-linear categories. In this case, $\mathbb{V}$-**Cat** is denoted by $\Bbbk$-**Cat**.
(3) In the case where $\mathbb{V} = \mathscr{C}(\Bbbk)$, $\mathbb{V}$-categories are called dg (differential graded) categories over $\Bbbk$. In this case, $\mathbb{V}$-**Cat** is denoted by $\Bbbk$-**dgCat**. In most cases we only deal with small dg categories, therefore we sometimes omit the word "small" if there seems to be no confusion.

**Definition 3.8.** A dg category $\mathscr{C}_{\mathrm{dg}}(\Bbbk)$ is defined as follows. Objects are the chain complexes of $\Bbbk$-modules, thus the same as $\mathscr{C}(\Bbbk)$. Let $X, Y$ be objects of

$\mathscr{C}_{\mathrm{dg}}(\Bbbk)$. Then $\mathscr{C}_{\mathrm{dg}}(\Bbbk)(X,Y)$ is a complex of $\Bbbk$-modules defined by

$$\mathscr{C}_{\mathrm{dg}}(\Bbbk)(X,Y) := \bigoplus_{n\in\mathbb{Z}} \mathscr{C}_{\mathrm{dg}}(\Bbbk)^n(X,Y), \text{where}$$
$$\mathscr{C}_{\mathrm{dg}}(\Bbbk)^n(X,Y) := \prod_{p\in\mathbb{Z}} (\operatorname{Mod}\Bbbk)(X^p, Y^{p+n}),$$

and with a differential $d = (d^n \colon \mathscr{C}_{\mathrm{dg}}(\Bbbk)^n(X,Y) \to \mathscr{C}_{\mathrm{dg}}(\Bbbk)^{n+1}(X,Y))_{n\in\mathbb{Z}}$ defined by

$$d^n(f) := (d_Y^{p+n} f^p - (-1)^n f^{p+1} d_X^p)_{p\in\mathbb{Z}}$$

for all $f = (f^p)_{p\in\mathbb{Z}} \in \mathscr{C}_{\mathrm{dg}}(\Bbbk)^n(X,Y)$.

**Definition 3.9.** Let $\mathscr{C}$ be a dg category, $x, y \in \mathscr{C}_0$, and take $f = (f^i)_{i\in\mathbb{Z}} \in \mathscr{C}(x,y) = \bigoplus_{i\in\mathbb{Z}} \mathscr{C}^i(x,y)$. If $f^i = 0$ for all $i \neq 0$ and $d_{\mathscr{C}}(f) = 0$, then we call $f$ a 0-*cocycle morphism*. We identify each 0-cocycle element $g \in Z^0(\mathscr{C}(x,y))$ of $\mathscr{C}(x,y)$ with the 0-cocycle morphism $f \in \mathscr{C}(x,y)$ defined by $f^0 := g$ and $f^i = 0$ for all $i \neq 0$.

**Remark 3.10.** We here remind the explicit form of compositions in a dg category. Let $\mathscr{C}$ be a dg category, $x, y, z \in \mathscr{C}$, and $f = (f^i)_{i\in\mathbb{Z}} \in \mathscr{C}(x,y) = \bigoplus_{i\in\mathbb{Z}} \mathscr{C}^i(x,y)$, $g = (g^j)_{j\in\mathbb{Z}} \in \mathscr{C}(y,z) = \bigoplus_{j\in\mathbb{Z}} \mathscr{C}^j(y,z)$. Then we have the formula

$$g \circ f := \left( \sum_{i\in\mathbb{Z}} g^{n-i} \circ f^i \right)_{n\in\mathbb{Z}}. \tag{3.2}$$

On the other hand, in the opposite category $\mathscr{C}^{\mathrm{op}}$ of $\mathscr{C}$ having the composition $*$, we have $f \in \mathscr{C}^{\mathrm{op}}(y,x), g \in \mathscr{C}^{\mathrm{op}}(z,y)$, and

$$f * g = \left( \sum_{i\in\mathbb{Z}} (-1)^{(n-i)i} g^{n-i} \circ f^i \right)_{n\in\mathbb{Z}}. \tag{3.3}$$

Note that the representable functor $\mathscr{C}(\text{-}, z) = \mathscr{C}^{\mathrm{op}}(z, \text{-})$ is a functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{C}_{\mathrm{dg}}(\Bbbk)$, and hence $\mathscr{C}(f,z) \colon \mathscr{C}(y,z) \to \mathscr{C}(x,z)$ is defined as $\mathscr{C}^{\mathrm{op}}(z,f) \colon \mathscr{C}^{\mathrm{op}}(z,y) \to \mathscr{C}^{\mathrm{op}}(z,x)$ by

$$\mathscr{C}(f,z)(g) := \mathscr{C}^{\mathrm{op}}(z,f)(g) := f * g = \left( \sum_{i\in\mathbb{Z}} (-1)^{(n-i)i} g^{n-i} \circ f^i \right)_{n\in\mathbb{Z}}.$$

**Remark 3.11.** Consider the case that $\mathbb{V} = \mathscr{C}(\Bbbk)$, and let $F, G \colon \mathscr{A} \to \mathscr{B}$ be dg functors between dg categories. Then a $\mathbb{V}$-natural transformation is called a *dg natural transformation*. By definition, a dg natural transformation $\alpha \colon F \Rightarrow G$ is a family $\alpha = (\alpha(x))_{x\in\mathscr{A}_0}$ of morphisms $\alpha(x) \colon \Bbbk \to \mathscr{B}(F(x), G(x))$ in $\mathscr{C}(\Bbbk)$ making the diagram (3.1) commutative. We set $\alpha_x := \alpha(x)(1_\Bbbk)$, where $1_\Bbbk$ is the identitiy of $\Bbbk$, and make the identification $\alpha = (\alpha_x)_{x\in\mathscr{A}_0}$. As in Exmaple 3.2 (3), $\alpha_x \in Z^0(\mathscr{B}(F(x), G(x)))$ for all $x \in \mathscr{A}_0$, and the commutativity of (3.1) is equivalent to saying that the following is commutative in $\mathscr{B}$ for all morphisms $f\colon x \to y$ in $\mathscr{A}$:

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha_x"'] & F(y) \arrow[d, "\alpha_y"] \\ G(x) \arrow[r, "G(f)"'] & G(y) \end{tikzcd} .$$

Here we have to remark that both $\alpha_x$ and $\alpha_y$ are 0-cocycles in $\mathscr{B}(F(x), G(x))$ and in $\mathscr{B}(F(y), G(y))$, respectively. Thus, we can set $F(f) = (F(f)^n)_{n\in\mathbb{Z}}$, $G(f) = (G(f)^n)_{n\in\mathbb{Z}}$, $\alpha_x = (\alpha_x)^0$, and $\alpha_y = (\alpha_y)^0$, and the commutativity of the diagram above is equivalent to the equality

$$\alpha_y F(f)^n = G(f)^n \alpha_x$$

for all $n \in \mathbb{Z}$. In particular, this is used in the case where $\mathscr{B} = \mathscr{C}_{\mathrm{dg}}(\Bbbk)$, the dg category of dg $\Bbbk$-modules, later. In this case the 0-cocycles are the chain morphisms.

We do not use the following notion explicitely, but we introduce it here to make clear the relationship between our setting and other general setting.

**Definition 3.12.** Let $\mathscr{A}$ and $\mathscr{B}$ be dg categories. Then the *functor dg category* $\mathsf{Hom}(\mathscr{A}, \mathscr{B})$ is defined as follows. Objects are the dg functors $\mathscr{A} \to \mathscr{B}$. Let $F, G\colon \mathscr{A} \to \mathscr{B}$ be two dg functors, and $n \in \mathbb{Z}$. A *derived transformation* $\alpha^n\colon F \Rightarrow G$ *of degree* $n$ from $F$ to $G$ is a family $\alpha^n = (\alpha^n_x)_{x\in\mathscr{A}_0}$ of morphisms $\alpha^n_x \in \mathscr{B}(F(x), G(x))^n$ such that for any morphism $f \in \mathscr{A}(x, y)^m$, $x, y \in \mathscr{A}_0$, we have

$$\alpha^n_y F(f) = (-1)^{mn} G(f)\alpha^n_x.$$

Then we denote by $\mathsf{Hom}(\mathscr{A}, \mathscr{B})^n(F, G)$ the set of all derived transformations of degree $n$ from $F$ to $G$, and set

$$\mathsf{Hom}(\mathscr{A}, \mathscr{B})(F, G) := \bigoplus_{n\in\mathbb{Z}} \mathsf{Hom}(\mathscr{A}, \mathscr{B})^n(F, G),$$

elements of which are called *derived transformation* from $F$ to $G$. The differential $d$ is given by $d(\alpha^n_x) := d_{\mathscr{B}}(\alpha^n_x)$ for all $\alpha \in \mathsf{Hom}(\mathscr{A}, \mathscr{B})(F, G)$, $n \in \mathbb{Z}$, and $x \in \mathscr{A}_0$.

**Remark 3.13.** In Definition 3.12, the category $\mathsf{Hom}(\mathscr{A}, \mathscr{B})$ is *small* if both $\mathscr{A}$ and $\mathscr{B}$ are small; is *light* if $\mathscr{A}$ is small and $\mathscr{B}$ is light; and is $k$*-moderate* if $\mathscr{A}$ is $(k-1)$-moderate and $\mathscr{B}$ is $k$-moderate for all $k \geq 1$.

**Definition 3.14.** We denote by $\Bbbk$-**DGCat** the 2-category whose objects are the small dg categories, whose 1-morphisms are the dg-functors between these objects, and whose 2-morphisms are the derived transformations bvetween these dg functors. The vertical composition of 2-morphisms is defined in an obvious way by a formula similar to (3.2), and the horizontal composition of 2-morphisms is defined by

$$\beta^n \circ \alpha^m := (\beta^n \circ F) \bullet (E' \circ \alpha^m) = (-1)^{mn}(F' \circ \alpha^m) \bullet (\beta^n \circ E)$$

for all 2-morphisms $\alpha = (\alpha^n)_{n\in\mathbb{Z}}$ and $\beta = (\beta^n)_{n\in\mathbb{Z}}$ in a diagram

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "E"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \arrow[r, bend left, "E'"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathscr{C} \end{tikzcd}$$

in $\Bbbk$-**DGCat**, where $(E' \circ \alpha^m)_x := E'(\alpha^m_x)$ and $(\beta^n \circ E)_y := \beta^n_{E(y)}$ for all $x \in \mathscr{A}_0, y \in \mathscr{B}_0$. Note the difference with $\Bbbk$-**dgCat**. Objects and 1-morphisms are the same, but 2-morphisms are different: they are dg natural transformations in $\Bbbk$-**dgCat**, and derived transformations in $\Bbbk$-**DGCat**. Therefore, for left part of the diagram above, we have

$$\Bbbk\text{-}\mathbf{dgCat}(\mathscr{A},\mathscr{B})(E,F) = Z^0(\Bbbk\text{-}\mathbf{DGCat}(\mathscr{A},\mathscr{B})(E,F)). \tag{3.4}$$

We note that both $\Bbbk$-**dgCat** and $\Bbbk$-**DGCat** are light 2-categories.

**Definition 3.15.** By replacing small dg categories with light dg categories in the definitions of $\Bbbk$-**dgCat** and $\Bbbk$-**DGCat**, we define the 2-categories $\Bbbk$-**dgCAT** and $\Bbbk$-**DGCAT**, respectively. We remark that these are 2-moderate 2-categories.

## 4. $I$-coverings

In this section we introduce the notion of $I$-coverings that is a generalization of that of $G$-coverings for a group $G$ introduced in [5], which was obtained by generalizing the notion of Galois coverings introduced by Gabriel in [19]. This will be used in the proof of our main theorem.

In the following, we will consider $I$-coverings in $\Bbbk$-**dgCat**, i.e., in the case that $\mathbb{V} = \mathscr{C}(\Bbbk)$. The precise form in this case is described as follows.

**Definition 4.1.** We define a 2-functor $\Delta\colon \Bbbk\text{-}\mathbf{dgCat} \to \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$ as follows, which is called the *diagonal* 2-functor:

- Let $\mathscr{C} \in \Bbbk$-**dgCat**. Then $\Delta(\mathscr{C})$ is defined to be a functor sending each morphism $a\colon i \to j$ in $I$ to $\mathbb{1}_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$.
- Let $E\colon \mathscr{C} \to \mathscr{C}'$ be a 1-morphism in $\Bbbk$-**dgCat**. Then $\Delta(E)\colon \Delta(\mathscr{C}) \to \Delta(\mathscr{C}')$ is a 1-morphism $(F, \psi)$ in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$ defined by $F(i) := E$ and $\psi(a) := \mathbb{1}_E$ for all $i \in I_0$ and all $a \in I_1$:

$$\begin{tikzcd} \mathscr{C} \arrow[r, "E"] \arrow[d, "\mathbf{1}_{\mathscr{C}}"'] \arrow[dr, phantom, "\mathbf{1}_E"] & \mathscr{C}' \arrow[d, "\mathbf{1}_{\mathscr{C}'}"] \arrow[dl, Rightarrow] \\ \mathscr{C} \arrow[r, "E"] & \mathscr{C}'. \end{tikzcd}$$

- Let $E, E'\colon \mathscr{C} \to \mathscr{C}'$ be 1-morphisms (that is, dg functors) in $\Bbbk$-**dgCat**, and $\alpha\colon E \Rightarrow E'$ a 2-morphism in $\Bbbk$-**dgCat**. Then $\Delta(\alpha)\colon \Delta(E) \Rightarrow \Delta(E')$ is a 2-morphism in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$ defined by $\Delta(\alpha) := (\alpha)_{i\in I_0}$.

**Remark 4.2.** Let $\mathbf{C}$ be a 2-category, $X = (X, X_i, X_{b,a}) \in \mathrm{Colax}(I, \mathbf{C})_0$, and $C \in \mathbf{C}_0$. Further let

- $F$ be a family of 1-morphisms $F(i)\colon X(i) \to C$ in $\mathbf{C}$ indexed by $i \in I_0$; and

- $\psi$ be a family of 2-morphisms $\psi(a)\colon F(i) \Rightarrow F(j)X(a)$ indexed by $a\colon i \to j$ in $I$:

$$
\begin{tikzcd}
X(i) \arrow[r, "F(i)"] \arrow[d, "X(a)"'] & C \arrow[dl, Rightarrow, "\psi(a)"'] \arrow[d, equal] \\
X(j) \arrow[r, "F(j)"'] & C
\end{tikzcd}
$$

Then $(F, \psi)$ is in $\mathrm{Colax}(I, \mathbf{C})(X, \Delta(C))$ if and only if the following hold.

(a) For each $i \in I_0$ the following is commutative:

$$
\begin{tikzcd}
F(i) \arrow[r, Rightarrow, "\psi(\mathbb{1}_i)"] \arrow[dr, equal] & F(i)X(\mathbb{1}_i) \arrow[d, Rightarrow, "F(i)X_i"] \\
 & F(i)\mathbb{1}_{X(i)}
\end{tikzcd}
; \text{ and}
$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k$ in $I$ the following is commutative:

$$
\begin{tikzcd}
F(i) \arrow[r, Rightarrow, "\psi(a)"] \arrow[d, Rightarrow, "\psi(ba)"'] & F(j)X(a) \arrow[d, Rightarrow, "\psi(b)X(a)"] \\
F(k)X(ba) \arrow[r, Rightarrow, "F(k)\,X_{b,a}"'] & F(k)X(b)X(a).
\end{tikzcd}
$$

**Definition 4.3.** Let $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}$ and $(F, \psi)\colon X \to \Delta(\mathscr{C})$ be in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then

(1) $(F, \psi)$ is called an *$I$-precovering* (of $\mathscr{C}$) if for any $i, j \in I_0$, $x \in X(i), y \in X(j)$, the morphism

$$(F, \psi)^{(1)}_{x,y}\colon \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) \to \mathscr{C}(F(i)x, F(j)y)$$

of $\Bbbk$-complexes defined by the following is an isomorphism:

$$
\begin{aligned}
\bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) &\xrightarrow{\bigoplus_{a \in I(i,j)} F(j)} \bigoplus_{a \in I(i,j)} \mathscr{C}(F(j)X(a)x, F(j)y) \\
&\xrightarrow{\bigoplus_{a \in I(i,j)} \mathscr{C}(\psi(a)_x, F(j)y)} \bigoplus_{a \in I(i,j)} \mathscr{C}(F(i)x, F(j)y) \\
&\xrightarrow{\text{summation}} \mathscr{C}(F(i)x, F(j)y),
\end{aligned}
$$

the precise form of which is given as follows:

$$
\begin{aligned}
(F,\psi)^{(1)}_{x,y}(((f_a^n)_{n\in\mathbb{Z}})_{a\in I(i,j)}) &= \sum_{a\in I(i,j)} \psi(a)_x * F(j)(f_a) \\
&= \left(\sum_{a\in I(i,j)}\sum_{r\in\mathbb{Z}}(-1)^{(n-r)r}F(j)(f_a)^{n-r}\circ\psi(a)_x^r\right)_{n\in\mathbb{Z}} \\
&= \left(\sum_{a\in I(i,j)} F(j)(f_a)^n\circ\psi(a)_x\right)_{n\in\mathbb{Z}},
\end{aligned}
\tag{4.5}
$$

where the second term is computed by using (3.3), and the last term uses the fact that $\psi(a)_x$ is concentrated in degree 0 (Remark 3.11).

(2) $(F,\psi)$ is called an *$I$-covering* if it is an $I$-precovering and is *dense*, i.e., for each $c\in\mathscr{C}_0$ there exists an $i\in I_0$ and $x\in X(i)_0$ such that $F(i)(x)$ is isomorphic to $c$ in $\mathscr{C}$.

## 5. Grothendieck constructions

In this section we define a 2-functor $\int\colon \mathrm{Colax}(I,\mathbb{V}\text{-}\mathbf{Cat})\to\mathbb{V}\text{-}\mathbf{Cat}$ whose correspondence on objects is a $\mathbb{V}$-enriched version of (the opposite version of) the original Grothendieck construction (cf. [47]). In particular, we deal with the case of $\Bbbk$-**dgCat** later.

**Definition 5.1.** We define a 2-functor $\int\colon \mathrm{Colax}(I,\mathbb{V}\text{-}\mathbf{Cat})\to\mathbb{V}\text{-}\mathbf{Cat}$, which is called the *Grothendieck construction*.

**On objects.** Let $X=(X(i),X_i,X_{b,a})\in\mathrm{Colax}(I,\mathbb{V}\text{-}\mathbf{Cat})_0$. Then $\int X\in\mathbb{V}\text{-}\mathbf{Cat}_0$ is defined as follows.

- $(\int X)_0 := \bigcup_{i\in I_0}\{i\}\times X(i)_0 = \{{}_ix := (i,x)\mid i\in I_0, x\in X(i)_0\}$.
- For each ${}_ix, {}_jy\in(\int X)_0$, we set

$$(\textstyle\int X)({}_ix,{}_jy) := \bigoplus_{a\in I(i,j)} X(j)(X(a)x,y).$$

- For any ${}_ix,{}_jy,{}_kz\in(\int X)_0$ and each $f=(f_a)_{a\in I(i,j)}\in(\int X)({}_ix,{}_jy)$, $g=(g_b)_{b\in I(j,k)}\in(\int X)({}_jy,{}_kz)$, we set

$$g\circ f := \left(\sum_{\substack{a\in I(i,j)\\ b\in I(j,k)\\ c=ba}} g_b\circ X(b)f_a\circ X_{b,a}x\right)_{c\in I(i,k)},$$

which is the composite of the following:

$$
\begin{tikzcd}
(\int X)({}_jy, {}_kz) \times (\int X)({}_ix, {}_jy) \arrow[r, dashed] \arrow[d, equal] & (\int X)({}_ix, {}_kz) \arrow[d, equal] \\
\bigoplus_{b\in I(j,k)} X(k)(X(b)y, z) \times \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y) \arrow[d, equal] & \bigoplus_{c\in I(i,k)} X(k)(X(c)x, z) \\
\bigoplus_{b,a}\{X(k)(X(b)y, z) \times X(j)(X(a)x, y)\} \arrow[d, "\bigoplus_{b,a}(\mathbf{1}\times X(b))"'] & \bigoplus_{b,a} X(k)(X(ba)x, z) \arrow[u, "\text{summation}"'] \\
\bigoplus_{b,a}\{X(k)(X(b)y, z) \times X(j)(X(b)X(a)x, X(b)y)\} \arrow[r] & \bigoplus_{b,a} X(k)(X(b)X(a)x, z), \arrow[u, "\bigoplus_{b,a} X(k)(X_{b,a}x,\, z)"']
\end{tikzcd}
\tag{5.6}
$$

where elements are mapped as follows:

$$
\begin{tikzcd}
((g_b)_b, (f_a)_a) \arrow[r, mapsto, dashed] \arrow[d, mapsto] & \left(\sum_{c=ba} g_b \circ X(b)f_a \circ X_{b,a}x\right)_c \\
(g_b, f_a)_{b,a} \arrow[d, mapsto] & (g_b \circ X(b)f_a \circ X_{b,a}x)_{b,a} \arrow[u, mapsto] \\
(g_b, X(b)f_a)_{b,a} \arrow[r, mapsto] & (g_b \circ X(b)f_a)_{b,a}. \arrow[u, mapsto]
\end{tikzcd}
$$

Note here that the composition with $X_{b,a}x$ is "contravariant", which is used in (5.8).

- For each ${}_ix \in (\int X)_0$ the identity $\mathbb{1}_{ix}$ is given by

$$
\mathbb{1}_{ix} = (\delta_{a,\mathbf{1}_i} X_i\, x)_{a\in I(i,i)} \in \bigoplus_{a\in I(i,i)} X(i)(X(a)x, x),
$$

where $\delta$ is the Kronecker delta[1].

**On 1-morphisms.** Let $X = (X, X_i, X_{b,a})$ and $X' = (X', X'_i, X'_{b,a})$ be objects of $\mathrm{Colax}(I, \mathbb{V}\text{-}\mathbf{Cat})$, and let $(F, \psi)\colon X \to X'$ be a 1-morphism in $\mathrm{Colax}(I, \mathbb{V}\text{-}\mathbf{Cat})$. Then a 1-morphism

$$
\textstyle\int(F, \psi)\colon \int X \to \int X'
$$

in $\mathbb{V}$-**Cat** is defined as follows.

- For each ${}_ix \in (\int X)_0$, $\int(F, \psi)({}_ix) := {}_i(F(i)x)$.
- Let ${}_ix, {}_jy \in (\int X)_0$. Then we define

$$
\textstyle\int(F, \psi)\colon (\int X)({}_ix, {}_jy) \to (\int X')({}_i(F(i)x), {}_j(F(j)y))
$$

[1]This is used to mean that the $a$-th component is $\eta_i\, x$ if $a = \mathbb{1}_i$, or 0 otherwise.

as the composite

$$\bigoplus_{a\in I(i,j)} X(j)(X(a)x,y) \xrightarrow{\bigoplus_{a\in I(i,j)} F(j)} \bigoplus_{a\in I(i,j)} X'(j)(F(j)X(a)x, F(j)y)$$
$$\xrightarrow{\bigoplus_{a\in I(i,j)} X'(j)(\psi(a)_x, F(j)y)} \bigoplus_{a\in I(i,j)} X'(j)(X'(a)F(i)x, F(j)y). \tag{5.7}$$

Namely, for each $f = (f_a)_{a\in I(i,j)} \in (\int X)({}_ix, {}_jy)$, we set

$$\textstyle\int(F,\psi)(f) := (F(j)f_a \circ \psi(a)x)_{a\in I(i,j)}.$$

**On 2-morphisms.** Let $X = (X, X_i, X_{b,a})$ and $X' = (X', X'_i, X'_{b,a})$ be objects of $\mathrm{Colax}(I, \mathbb{V}\text{-}\mathbf{Cat})$, $(F,\psi)\colon X \to X'$ and $(F',\psi')\colon X' \to X''$ 1-morphisms in $\mathrm{Colax}(I, \mathbb{V}\text{-}\mathbf{Cat})$, and let $\zeta\colon (F,\psi) \Rightarrow (F',\psi')$ be a 2-morphism in $\mathrm{Colax}(I, \mathbb{V}\text{-}\mathbf{Cat})$. Then a 2-morphism

$$\textstyle\int\zeta\colon \int(F,\psi) \Rightarrow \int(F',\psi')$$

in $\mathbb{V}$-$\mathbf{Cat}$ is defined by

$$(\textstyle\int\zeta)_i x := \begin{cases} \zeta(i)_x \circ X'_i(F(i)x) & \text{if } a = \mathbb{1}_i \\ 0 & \text{if } a \neq \mathbb{1}_i \end{cases}$$

in $\int X'$ for each ${}_ix \in (\int X)_0$.

In particular, in the case that $\mathbb{V} = \mathscr{C}(\Bbbk)$, i.e., that $\mathbb{V}\text{-}\mathbf{Cat} = \Bbbk\text{-}\mathbf{dgCat}$, the precise form of the Grothendieck construction

$$\textstyle\int\colon \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat}) \to \Bbbk\text{-}\mathbf{dgCat}$$

is described as follows.

**On objects.** Let $X = (X, X_i, X_{b,a}) \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})_0$. Then $\int X \in \Bbbk\text{-}\mathbf{dgCat}_0$ is defined as follows.

- $(\int X)_0 := \bigcup_{i\in I_0}\{i\} \times X(i)_0 = \{{}_ix := (i,x) \mid i \in I_0, x \in X(i)_0\}$.
- For each ${}_ix, {}_jy \in (\int X)_0$, we set

$$(\textstyle\int X)({}_ix, {}_jy) := \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y) = \bigoplus_{a\in I(i,j)} \bigoplus_{n\in\mathbb{Z}} X(j)^n(X(a)x, y),$$

where note that $X(j)(X(a)x,y)$ is a dg $\Bbbk$-module.

- For any ${}_ix, {}_jy, {}_kz \in (\int X)_0$ and each $f = (f_a^p)_{a\in I(i,j), p\in\mathbb{Z}} \in (\int X)({}_ix, {}_jy)$, $g = (g_b^q)_{b\in I(j,k), q\in\mathbb{Z}} \in (\int X)({}_jy, {}_kz)$, it turns out that

$$g \circ f = \left( \sum_{\substack{a\in I(i,j)\\ b\in I(j,k)\\ c=ba}} X_{b,a}x * (g_b \circ (X(b)f_a)) \right)_{c\in I(i,k), n\in\mathbb{Z}}$$
$$= \left( \sum_{\substack{a\in I(i,j)\\ b\in I(j,k)\\ c=ba}} \sum_{p,r\in\mathbb{Z}} (-1)^{(n-r)r} g_b^{n-r-p} \circ (X(b)f_a)^p \circ (X_{b,a}x)^r \right)_{c\in I(i,k), n\in\mathbb{Z}} \tag{5.8}$$

because of the contravariant part in (5.6).
- For each ${}_ix \in (\int X)_0$ the identity $\mathbb{1}_{{}_ix}$ is given by

$$\mathbb{1}_{{}_ix} = (\delta_{a,\mathbf{1}_i} X_i\, x)_{a\in I(i,i)} \in \bigoplus_{a\in I(i,i)} X(i)(X(a)x, x) = \bigoplus_{a\in I(i,j)} \bigoplus_{p\in\mathbb{Z}} X(i)^p(X(a)x, x).$$

**On 1-morphisms.** Let $X = (X, X_i, X_{b,a})$ and $X' = (X', X'_i, X'_{b,a})$ be objects of $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$, and let $(F, \psi)\colon X \to X'$ be a 1-morphism in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then a 1-morphism

$$\textstyle\int(F,\psi)\colon\ \int X \to \int(X')$$

in $\Bbbk$-**dgCat** is defined as follows.

- For each ${}_ix \in (\int X)_0$, $\int(F,\psi)({}_ix) := {}_i(F(i)x)$.
- Let ${}_ix, {}_jy \in (\int X)_0$. Then we define

$$\textstyle\int(F,\psi)\colon (\int X)({}_ix, {}_jy) \to (\int X')({}_i(F(i)x), {}_j(F(j)y))$$

as in (5.7). Namely, for each $f = ((f_a^n)_{n\in\mathbb{Z}})_{a\in I(i,j)} \in (\int X)({}_ix, {}_jy) = \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y)$, we have

$$\begin{aligned}((f_a^n)_{n\in\mathbb{Z}})_{a\in I(i,j)} &\mapsto ((F(j)(f_a^n))_{n\in\mathbb{Z}})_{a\in I(i,j)}\\ &\mapsto \psi(a)_x * ((F(j))(f_a^n))_{n\in\mathbb{Z}})_{a\in I(i,j)}\\ &= ((F(j)(f_a)^n \circ \psi(a)_x)_{n\in\mathbb{Z}})_{a\in I(i,j)} \quad (\text{cf. } (3.3))\end{aligned}$$

Thus we have

$$\textstyle\int(F,\psi)(f) = ((F(j)(f_a)^n \circ \psi(a)_x)_{n\in\mathbb{Z}})_{a\in I(i,j)}. \tag{5.9}$$

**On 2-morphisms.** Let $X = (X, X_i, X_{b,a})$ and $X' = (X', X'_i, X'_{b,a})$ be objects of $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$, $(F,\psi)\colon X \to X'$ and $(F',\psi')\colon X \to X'$ 1-morphisms in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$, and let $\zeta\colon (F,\psi) \Rightarrow (F',\psi')$ a 2-morphism in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then a 2-morphism

$$\textstyle(\int\zeta)\colon\ \int(F,\psi) \Rightarrow \int(F',\psi')$$

in $\Bbbk$-**dgCat** is defined by

$$(\textstyle\int \zeta)({}_i x) = \begin{cases} \zeta(i)_x \circ X'_i(F(i)x) = (\sum_{r\in\mathbb{Z}} \zeta(i)_x^{n-r} \circ X'_i(F(i)x)^r)_{n\in\mathbb{Z}} & \text{if } a = \mathbb{1}_i \\ 0 & \text{if } a \neq \mathbb{1}_i \end{cases}$$

in $\int X'$ for each ${}_i x \in (\int X)_0$.

**Example 5.2.** Let $A$ be a dg $\Bbbk$-algebra with the differential $d_A$ regarded as a dg $\Bbbk$-category with a single object. Then $A \in \Bbbk\text{-}\mathbf{dgCat}_0$. Consider the functor $X := \Delta(A)\colon I \to \Bbbk\text{-}\mathbf{dgCat}$. Then it is straightforward to verify the following.

(1) If $I$ is a free category defined by the quiver $1 \to 2$, then $\int X$ is isomorphic to the triangular dg algebra $\begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$.

(2) If $I$ is a free category $\mathbb{P}Q$ defined by a quiver $Q$, then $\int X$ is isomorphic to the dg path-category $AQ$ of $Q$ over $A$ defined as follows:
   - $(AQ)_0 := Q_0$.
   - For any $i, j \in Q_0$,

$$AQ(i,j) := \bigoplus_{\mu\in\mathbb{P}Q(i,j)} A\mu = \left\{ \sum_{\mu\in\mathbb{P}Q(i,j)} a_\mu \mu \;\middle|\; (a_\mu)_{\mu\in\mathbb{P}Q(i,j)} \in \bigoplus_{\mu\in\mathbb{P}Q(i,j)} A \right\}.$$

   - For any $i, j, k \in Q_0$, the composition $AQ(j,k) \times AQ(i,j) \to AQ(i,k)$ is given by

$$\sum_{\nu\in\mathbb{P}Q(j,k)} b_\nu \nu \times \sum_{\mu\in\mathbb{P}Q(i,j)} a_\mu \mu \mapsto \sum_{\substack{\mu\in\mathbb{P}Q(i,j),\\ \nu\in\mathbb{P}Q(j,k)}} b_\nu a_\mu \nu\mu = \sum_{\lambda\in\mathbb{P}Q(i,k)} \left( \sum_{\lambda=\nu\mu} b_\nu a_\mu \right) \lambda.$$

   - For any $i, j \in Q_0$ and any $n \in \mathbb{Z}$, we set $(AQ)^n(i,j) = \bigoplus_{\mu\in\mathbb{P}Q(i,j)} A^n\mu$.
   - For any $i, j \in Q_0$ and any $n \in \mathbb{Z}$, the differential $d : (AQ)^n(i,j) \to (AQ)^{n+1}(i,j)$ is given by

$$d\left( \sum_{\mu\in\mathbb{P}Q(i,j)} a_\mu \mu \right) = \sum_{\mu\in\mathbb{P}Q(i,j)} d_A(a_\mu)\mu,$$

   which automatically satisfies the graded Leibniz rule.

Indeed, we can define an isomorphism $\phi\colon AQ \to \int X$ as follows: We regard $A$ as a category with a single objects $*$. Then for each $i \in Q_0$, we have $X(i)_0 = \{*\}$ and $X(i)_1 = A$. Then $(\int X)_0 = \bigsqcup_{i\in Q_0} X(i)_0 = \bigcup_{i\in Q_0}\{{}_i*\} = \{{}_i* \mid i \in Q_0\}$. Therefore, we define a bijection $\phi_0\colon (AQ)_0 \to (\int X)_0$ by $i \mapsto {}_i*$. For any $i, j \in Q_0$, since we have $(AQ)(i,j) = \bigoplus_{\mu\in\mathbb{P}Q(i,j)} A\mu$, and

$$(\textstyle\int X)({}_i*, {}_j*) := \bigoplus_{\mu\in I(i,j)} X(j)(X(\mu)*, *) = \bigoplus_{\mu\in I(i,j)} X(j)_1 = \bigoplus_{\mu\in\mathbb{P}Q(i,j)} A,$$

we define a bijection $\phi_1\colon (AQ)(i,j) \to (\int X)({}_i*, {}_j*)$ by $\sum_{\mu\in\mathbb{P}Q} a_\mu\mu \mapsto (a_\mu)_{\mu\in\mathbb{P}Q}$. Then $\phi := (\phi_0, \phi_1)\colon AQ \to \int X$ turns out to be an isomorphism.

(3) If $I$ is a poset $S$, then $\int X$ is isomorphic to the incidence dg category $AS$ of $S$ over $A$ defined as follows:
- $(AS)_0 := S$ as a set.
- For any $i,j \in S$, $(AS)(i,j) := \begin{cases} A & \text{if } i \le j, \\ 0 & \text{otherwise.} \end{cases}$
- For any $i,j,k \in S$, the composition $AS(j,k) \times AS(i,j) \to AS(i,k)$ is given by the multiplication of $A$ for the case that $i \le j \le k$, and as zero otherwise.
- For any $i,j \in Q_0$ and any $n \in \mathbb{Z}$, we set $(AS)^n(i,j) := \begin{cases} A^n & \text{if } i \le j, \\ 0 & \text{otherwise.} \end{cases}$
- For any $i,j \in Q_0$ and any $n \in \mathbb{Z}$, the differential $d : (AS)^n(i,j) \to (AS)^{n+1}(i,j)$ is given by $d_A\colon A^n \to A^{n+1}$ if $i \le j$, and as zero otherwise, which automatically satisfies the graded Leibniz rule.

Indeed, we can define an isomorphism $\phi\colon AS \to \int X$ as follows: We regard $A$ as a category with a single objects $*$. Then for each $i \in S$, we have $X(i)_0 = \{*\}$ and $X(i)_1 = A$. Then $(\int X)_0 = \bigsqcup_{i\in I_0} X(i)_0 = \bigcup_{i\in I_0}\{{}_i*\} = \{{}_i* \mid i \in S\}$. Therefore, we define a bijection $\phi_0\colon (AS)_0 \to \int(X)_0$ by $i \mapsto {}_i*$. For any $i,j \in S$, since we have $(AS)(i,j) = \begin{cases} A & \text{if } i \le j, \\ 0 & \text{otherwise} \end{cases}$, and

$$(\textstyle\int X)({}_i*, {}_j*) := \bigoplus_{\mu\in S(i,j)} X(j)(X(\mu)*, *) = \bigoplus_{\mu\in S(i,j)} X(j)_1 = \bigoplus_{\mu\in S(i,j)} A = A, \text{if } i \le j,$$

we define a bijection $\phi_1\colon (AS)(i,j) \to \int(X)({}_i*, {}_j*)$ by $\sum_{\mu\in S} a_\mu\mu \mapsto (a_\mu)_{\mu\in S}$. Then $\phi := (\phi_0, \phi_1)\colon AQ \to \int X$ turns out to be an isomorphism.

(4) If $I$ is a monoid $G$, then $\int X$ is isomorphic to the monoid dg algebra[2] $AG$ of $G$ over $A$ defined as follows:
- $AG := \bigoplus_{g\in G} Ag$.
- The multiplication $AG \times AG \to AG$ is defined by

$$\left(\sum_{g\in G} a_g g\right)\cdot\left(\sum_{h\in G} b_h h\right) := \sum_{g,h\in G}(a_g b_h)gh = \sum_{f\in G}\left(\sum_{gh=f} a_g b_h\right) f.$$

- For each $n \in \mathbb{Z}$, $(AG)^n := \bigoplus_{g\in G} A^n g$.
- The differential $d : (AG)^n \to (AG)^{n+1}$ is given by $d\left(\sum_{g\in G} a_g g\right) := \sum_{g\in G} d_A(a_g)g$, which automatically satisfies the graded Leibniz rule.

In (3) above, $AS$ is defined to be the factor category of the dg path-category $AQ$ modulo the ideal generated by the full commutativity relations in $Q$, where

[2]Since $AG$ has the identity $1_A 1_G$, this is regarded as a category with a single object.

$Q$ is the Hasse diagram of $S$ regarded as a quiver by drawing an arrow $x \to y$ if $x \leq y$ in $Q$. If $S$ is a finite poset, then $AS$ is identified with the usual incidence dg algebra.

See [9] for further examples of the Grothendieck constructions of functors, further examples of the Grothendieck constructions of a functor $X\colon I \to \Bbbk\textbf{-dgCat}$ will be done in the forthcoming paper.

**Definition 5.3.** Let $X \in \mathrm{Colax}(I, \mathbb{V}\textbf{-Cat})$. We define a left transformation $(P_X, \phi_X) := (P, \phi)\colon X \to \Delta(\int X)$ (called the *canonical morphism*) as follows.

- For each $i \in I_0$, the functor $P(i)\colon X(i) \to \int X$ is defined by
$$\begin{cases} P(i)x := {}_ix \\ P(i)f := (\delta_{a,\mathbf{1}_i} f \circ (X_i\, x))_{a \in I(i,i)}\colon {}_ix \to {}_iy \text{ in } \int(X) \end{cases}$$
for all $f\colon x \to y$ in $X(i)$.
- For each $a\colon i \to j$ in $I$, the natural transformation $\phi(a)\colon P(i) \Rightarrow P(j)X(a)$
$$\begin{tikzcd} X(i) \arrow[r, "P(i)"] \arrow[d, "X(a)"'] & \int(X) \arrow[d, equal] \arrow[dl, phantom, "\overset{\phi(a)}{\swarrow}"] \\ X(j) \arrow[r, "P(j)"'] & \int(X) \end{tikzcd}$$
is defined by $\phi(a)x := (\delta_{b,a} \mathbb{1}_{X(a)x})_{b \in I(i,j)}$ for all $x \in X(i)_0$.

Now let $X \in \mathrm{Colax}(I, \Bbbk\textbf{-dgCat})$. The left transformation $(P_X, \phi_X) := (P, \phi)\colon X \to \Delta(\int X)$ is as follows.

- For each $i \in I_0$, the dg functor $P(i)\colon X(i) \to \int(X)$ is defined by $P(i)x := {}_ix$ for all $x \in X(i)_0$, and by setting $P(i)f\colon {}_ix \to {}_iy$ as
$$\begin{aligned} P(i)f &:= (\delta_{a,\mathbf{1}_i}(X_i\, x) * f)_{a \in I(i,i)} \\ &= \left(\left(\delta_{a,\mathbf{1}_i} \sum_{r \in \mathbb{Z}} (-1)^{(n-r)r} f^{n-r} \circ (X_i\, x)^r\right)_{n \in \mathbb{Z}}\right)_{a \in I(i,i)} \end{aligned} \tag{5.10}$$
for all $f\colon x \to y$ in $X(i)$. Note here that the map $\mathscr{C}(X_i(x), y)\colon \mathscr{C}(x,y) \to \mathscr{C}(X(\mathbb{1}_i)x, y)$, $f \mapsto f \circ X_i x$ is given by the contravariant functor $\mathscr{C}(\text{-}, y)$ at $X_i x$.
- For each $a\colon i \to j$ in $I$, the dg natural transformation $\phi(a)\colon P(i) \Rightarrow P(j)X(a)$
$$\begin{tikzcd} X(i) \arrow[r, "P(i)"] \arrow[d, "X(a)"'] & \int X \arrow[d, equal] \arrow[dl, phantom, "\overset{\phi(a)}{\swarrow}"] \\ X(j) \arrow[r, "P(j)"'] & \int X \end{tikzcd}$$
is defined by $\phi(a)x := (\delta_{b,a} \mathbb{1}_{X(a)x})_{b \in I(i,j)}$ for all $x \in X(i)_0$.

**Lemma 5.4.** *The* $(P, \phi)$ *defined above is a* 1*-morphism in* $\mathrm{Colax}(I, \Bbbk\textbf{-dgCat})$.

*Proof.* This is straightforward by using Remark 4.2. □

Consider the cases that $X$ in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{DGCat})$ and that $X$ in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. In both cases, we have the canonical $I$-covering $(P, \phi)\colon X \to \Delta(\int X)$ as shown below.

**Proposition 5.5.** *Let $X \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{DGCat})_0$. Then the canonical morphism $(P, \phi)\colon X \to \Delta(\int X)$ is an $I$-covering. More precisely, the morphism*

$$(P, \phi)^{(1)}_{x,y}\colon \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) \to (\textstyle\int X)(P(i)x, P(j)y)$$

*is the identity for all $i, j \in I_0$ and all $x \in X(i)_0$, $y \in X(j)_0$.*

*Proof.* By the definitions of $\int(X)_0$ and of $P$ it is obvious that $(P, \phi)$ is dense. Let $i, j \in I_0$ and $x \in X(i)$, $y \in X(j)$. We only have to show that

$$(P, \phi)^{(1)}_{x,y}\colon \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) \to (\textstyle\int X)(P(i)x, P(j)y)$$

is the identity. Let $f = (f_a)_{a \in I(i,j)} \in \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y)$. Then by noting the form of $f_a\colon X(a)x \to y$ in $X(j)$, we have the following equalities for each $n \in \mathbb{Z}$ by (4.5), (5.10) and (5.8):

$$
\begin{aligned}
&(P, \phi)^{(1)}_{x,y}(f)^n = \sum_{a \in I(i,j)} P(j)(f_a)^n \circ \phi(a)_x \\
&= \sum_{a \in I(i,j)} \left( \delta_{b,\mathbf{1}_j} \sum_{s \in \mathbb{Z}} (-1)^{(n-s)s} f_a^{n-s} \circ X_j(X(a)x)^s \right)_{b \in I(j,j)} \circ \phi(a)_x \\
&= \sum_{a \in I(i,j)} \left( \delta_{b,\mathbf{1}_j} \sum_{s \in \mathbb{Z}} (-1)^{(n-s)s} f_a^{n-s} \circ X_j(X(a)x)^s \right)_{b \in I(j,j)} \circ (\delta_{c,a} \mathbb{1}_{X(a)x})_{c \in I(i,j)} \\
&= \sum_{a \in I(i,j)} \left( \sum_{\substack{b \in I(j,j) \\ c \in I(i,j) \\ d = bc}} \delta_{b,\mathbf{1}_j} \sum_{r,s \in \mathbb{Z}} (-1)^{(n-r)r} (-1)^{(n-r-s)s} f_a^{n-r-s} \circ X_j(X(a)x)^s \circ X(b)(\delta_{c,a} \mathbb{1}_{X(a)x})^0 \circ (X_{b,c}x)^r \right)_{d \in I(i,j)} \\
&= \sum_{a \in I(i,j)} \left( \delta_{d,a} \sum_{r,s \in \mathbb{Z}} (-1)^{(n-r)r} (-1)^{(n-r-s)s} f_a^{n-r-s} \circ X_j(X(a)x)^s \circ X(\mathbb{1}_j)(\mathbb{1}_{X(a)x})^0 \circ (X_{\mathbf{1}_j,a}x)^r \right)_{d \in I(i,j)} \\
&= \sum_{a \in I(i,j)} \left( \delta_{d,a} \sum_{r,s \in \mathbb{Z}} (-1)^{(n-r)r} (-1)^{(n-r-s)s} f_a^{n-r-s} \circ X_j(X(a)x)^s \circ (X_{\mathbf{1}_j,a}x)^r \right)_{d \in I(i,j)} \\
&= \sum_{a \in I(i,j)} \left( \delta_{d,a} \sum_{\substack{r,s,t \in \mathbb{Z} \\ n = r+s+t}} (-1)^{rs+rt+st} f_a^t \circ X_j(X(a)x)^s \circ (X_{\mathbf{1}_j,a}x)^r \right)_{d \in I(i,j)} \qquad (m := r+s)
\end{aligned}
$$

$$= \sum_{a\in I(i,j)} \left( \delta_{d,a} \sum_{\substack{r,m,t\in\mathbb{Z}\\ n=m+t}} (-1)^{r(m-r)+mt} f_a^t \circ X_j(X(a)x)^{(m-r)} \circ (X_{\mathbf{1}_j,a}x)^r \right)_{d\in I(i,j)}$$

$$= \sum_{a\in I(i,j)} \left( \delta_{d,a} \sum_{\substack{m,t\in\mathbb{Z}\\ n=m+t}} (-1)^{mt} f_a^t \circ \sum_{r\in\mathbb{Z}} (-1)^{(m-r)r} X_j(X(a)x)^{(m-r)} \circ (X_{\mathbf{1}_j,a}x)^r \right)_{d\in I(i,j)}$$

$$= \sum_{a\in I(i,j)} \big(\delta_{d,a}((X_{\mathbf{1}_j,a}x * X_j(X(a)x)) * f_a)^n\big)_{d\in I(i,j)}$$

$$\stackrel{*}{=} \sum_{a\in I(i,j)} \big(\delta_{d,a}(\mathbb{1}_{X(a)x} * f_a)^n\big)_{d\in I(i,j)} = f^n.$$

In the above the equality $\stackrel{*}{=}$ holds. Indeed, let $(\text{-})^{\mathrm{op}} \colon X(j) \to X(j)^{\mathrm{op}}$ be the canonical contaravariant functor defined by $u^{\mathrm{op}} := u$ for all $u \in X(j)_0 \cup X(j)_1$, and $(h \circ g)^{\mathrm{op}} = g * h$ for all morphisms $g\colon u \to v, h\colon v \to w$ in $X(j)$. If we have an equality $h \circ g = \mathbb{1}_u$ in $X(j)$, then we have $g * h = (h \circ g)^{\mathrm{op}} = \mathbb{1}_u^{\mathrm{op}} = \mathbb{1}_u$. By applying this fact to the case that $g = X_{\mathbf{1}_j,a}x, h = X_j(X(a)x), u = X(a)x$, we have $X_{\mathbf{1}_j,a}x * X_j(X(a)x) = \mathbb{1}_{X(a)x}$. $\square$

**Proposition 5.6.** *Let $X \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})_0$. Then the canonical morphism $(P, \phi)\colon X \to \Delta(\int X)$ is an $I$-covering. More precisely, the morphism*

$$(P,\phi)^{(1)}_{x,y}\colon \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y) \to (\textstyle\int X)(P(i)x, P(j)y)$$

*is the identity for all $i, j \in I_0$ and all $x \in X(i)_0$, $y \in X(j)_0$.*

*Proof.* The proof is almost the same. The difference is that $X_j$ and $X_{b,c}$ are dg natural transformations, and thus their degrees are 0. This makes the long computation above simpler as follows.

$$
\begin{aligned}
(P,\phi)^{(1)}_{x,y}(f)^n &= \sum_{a\in I(i,j)} P(j)(f_a)^n \circ \phi(a)_x \\
&= \sum_{a\in I(i,j)} \left(\delta_{b,\mathbf{1}_j}\sum_{s\in\mathbb{Z}}(-1)^{(n-s)s} f_a^{n-s}\circ X_j(X(a)x)^s\right)_{b\in I(j,j)} \circ \phi(a)_x \\
&= \sum_{a\in I(i,j)} \left(\delta_{b,\mathbf{1}_j} f_a^n \circ X_j(X(a)x)\right)_{b\in I(j,j)} \circ (\delta_{c,a}\mathbb{1}_{X(a)x})_{c\in I(i,j)} \\
&= \sum_{a\in I(i,j)} \left(\sum_{\substack{b\in I(j,j)\\ c\in I(i,j)\\ d=bc}}\delta_{b,\mathbf{1}_j}\sum_{r\in\mathbb{Z}}(-1)^{(n-r)r} f_a^{n-r}\circ X_j(X(a)x)\circ X(b)(\delta_{c,a}\mathbb{1}_{X(a)x})^0\circ (X_{b,c}x)^r\right)_{d\in I(i,j)} \\
&= \sum_{a\in I(i,j)} \left(\delta_{d,a} f_a^n \circ X_j(X(a)x)\circ X(\mathbb{1}_j)(\mathbb{1}_{X(a)x})^0\circ (X_{\mathbf{1}_j,a}x)^0\right)_{d\in I(i,j)} \\
&= \sum_{a\in I(i,j)} \left(\delta_{d,a} f_a^n \circ X_j(X(a)x)\circ (X_{\mathbf{1}_j,a}x)\right)_{d\in I(i,j)} \\
&= \sum_{a\in I(i,j)} \left(\delta_{d,a} f_a^n \circ (X_j(X(a)x)\circ X_{\mathbf{1}_j,a}x)\right)_{d\in I(i,j)} \\
&\overset{*}{=} \sum_{a\in I(i,j)} \left(\delta_{d,a} f_a^n \circ \mathbb{1}_{X(a)x}\right)_{d\in I(i,j)} = f^n.
\end{aligned}
$$

The equality $\overset{*}{=}$ holds since $X_j(X(a)x)\circ X_{\mathbf{1}_j,a}x = \mathbb{1}_{X(a)x}$.

□

**Lemma 5.7.** *Let $X \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})_0$ and $H\colon \int X \to \mathscr{C}$ be in $\Bbbk$-$\mathbf{dgCat}$ and consider the composite 1-morphism $(F,\psi)\colon X \xrightarrow{(P,\phi)} \Delta(\int X) \xrightarrow{\Delta(H)} \Delta(\mathscr{C})$. Then $(F,\psi)$ is an $I$-covering if and only if $H$ is an equivalence.*

*Proof.* Obviously $(F,\psi)$ is dense if and only if so is $H$. Further for each $i,j\in I_0$, $x\in X(i)$ and $y \in X(j)$, $(F,\psi)^{(1)}_{x,y}$ is an isomorphism if and only if so is $H_{{}_ix,{}_jy}$ because we have a commutative diagram

$$
\begin{tikzcd}
\bigoplus_{a\in I(i,j)} X(j)(X(a)x,y) \arrow[r, "(F,\psi)^{(1)}_{x,y}"] \arrow[d, equal, "(P,\phi)^{(1)}_{x,y}"'] & \mathscr{C}(F(i)x, F(j)y) \\
\int(X)({}_ix,{}_jy) \arrow[ur, "H_{{}_ix,{}_jy}"'] &
\end{tikzcd}
$$

by Proposition 5.6. □

## 6. Adjoints

In this section we will show that the Grothendieck construction is a strict left adjoint to the diagonal 2-functor, and that $I$-coverings are essentially given by the unit of the adjunction.

**Definition 6.1.** Let $\mathscr{C} \in \mathbb{V}\text{-}\mathbf{Cat}$. We define a functor $Q_{\mathscr{C}} \colon \int \Delta(\mathscr{C}) \to \mathscr{C}$ by

- $Q_{\mathscr{C}}({}_ix) := x$ for all ${}_ix \in (\int \Delta(\mathscr{C}))_0$; and
- $Q_{\mathscr{C}}((f_a)_{a\in I(i,j)}) := \sum_{a\in I(i,j)} f_a$ for all $(f_a)_{a\in I(i,j)} \in (\int \Delta(\mathscr{C}))({}_ix, {}_jy)$ and for all ${}_ix, {}_jy \in (\int \Delta(\mathscr{C}))_0$.

It is easy to verify that $Q_{\mathscr{C}}$ is a $\mathbb{V}$-functor.

**Theorem 6.2.** *The* 2*-functor* $\int \colon \operatorname{Colax}(I, \mathbb{V}\text{-}\mathbf{Cat}) \to \mathbb{V}\text{-}\mathbf{Cat}$ *is a strict left* 2*-adjoint to the* 2*-functor* $\Delta \colon \mathbb{V}\text{-}\mathbf{Cat} \to \operatorname{Colax}(I, \mathbb{V}\text{-}\mathbf{Cat})$. *The unit is given by the family of canonical morphisms* $(P_X, \phi_X) \colon X \to \Delta(\int X)$ *indexed by* $X \in \operatorname{Colax}(I, \mathbb{V}\text{-}\mathbf{Cat})$, *and the counit is given by the family of* $Q_{\mathscr{C}} \colon \int \Delta(\mathscr{C}) \to \mathscr{C}$ *defined as above indexed by* $\mathscr{C} \in \mathbb{V}\text{-}\mathbf{Cat}$.

*In particular,* $(P_X, \phi_X)$ *has a strict universality in the comma category* $(X\downarrow \Delta)$, *i.e., for each* $(F, \psi) \colon X \to \Delta(\mathscr{C})$ *in* $\operatorname{Colax}(I, \mathbb{V}\text{-}\mathbf{Cat})$ *with* $\mathscr{C} \in \mathbb{V}\text{-}\mathbf{Cat}$, *there exists a unique* $H \colon \int X) \to \mathscr{C}$ *in* $\mathbb{V}\text{-}\mathbf{Cat}$ *such that the following is a commutative diagram in* $\operatorname{Colax}(I, \mathbb{V}\text{-}\mathbf{Cat})$*:*

$$\begin{tikzcd} X \arrow[r, "(F,\psi)"] \arrow[d, "(P_X,\phi_X)"'] & \Delta(\mathscr{C}). \\ \Delta(\int X) \arrow[ur, dashed, "\Delta(H)"'] & \end{tikzcd}$$

*Proof.* For simplicity set $\eta := ((P_X, \phi_X))_{X\in \operatorname{Colax}(I,\mathbb{V}\text{-}\mathbf{Cat})_0}$ and $\varepsilon := (Q_{\mathscr{C}})_{\mathscr{C}\in\mathbb{V}\text{-}\mathbf{Cat}_0}$.

**Claim 1.** $\Delta\varepsilon \cdot \eta\Delta = \mathbb{1}_\Delta$.

Indeed, let $\mathscr{C} \in \mathbb{V}\text{-}\mathbf{Cat}$. It is enough to show that $\Delta(Q_{\mathscr{C}}) \cdot (P_{\Delta(\mathscr{C})}, \phi_{\Delta(\mathscr{C})}) = \mathbb{1}_{\Delta(\mathscr{C})}$. Now

$$\begin{aligned} \text{LHS} &= \left((Q_{\mathscr{C}} P_{\Delta(\mathscr{C})}(i))_{i\in I_0}, (Q_{\mathscr{C}} \phi_{\Delta(\mathscr{C})}(a))_{a\in I_1}\right), \text{ and} \\ \text{RHS} &= \left((\mathbb{1}_{\mathscr{C}})_{i\in I_0}, (\mathbb{1}_{\mathbf{1}_{\mathscr{C}}})_{a\in I_1}\right). \end{aligned}$$

*First entry*: Let $i \in I$. Then $Q_{\mathscr{C}} P_{\Delta(\mathscr{C})}(i) = \mathbb{1}_{\mathscr{C}}$ because for each $x, y \in \mathscr{C}_0$ and each $f \in \mathscr{C}(x, y)$ we have $(Q_{\mathscr{C}} P_{\Delta(\mathscr{C})}(i))(x) = Q_{\mathscr{C}}({}_ix) = x$; and $(Q_{\mathscr{C}} P_{\Delta(\mathscr{C})}(i))(f) = (\delta_{a,\mathbf{1}_i} f \cdot ((\eta_{\Delta(\mathscr{C})})_i\, x))_{a\in I_1} = \sum_{a\in I(i,i)} \delta_{a,\mathbf{1}_i} f = f$.

*Second entry*: Let $a \colon i \to j$ in $I$. Then $Q_{\mathscr{C}} \phi_{\Delta(\mathscr{C})}(a) = \mathbb{1}_{\mathbf{1}_{\mathscr{C}}}$ because for each $x \in \mathscr{C}_0$ we have $Q_{\mathscr{C}}\left(\phi_{\Delta(\mathscr{C})}(a)x\right) = Q_{\mathscr{C}}\left((\delta_{b,a} \mathbb{1}_{\Delta(\mathscr{C})(a)x})_{b\in I(i,j)}\right) = \sum_{b\in I(i,j)} \delta_{b,a} \mathbb{1}_x = \mathbb{1}_x = \mathbb{1}_{\mathbf{1}_{\mathscr{C}} x}$. This shows that LHS = RHS.

**Claim 2.** $\varepsilon \int \cdot \int \eta = \mathbb{1}_{\int}$.

Indeed, let $X \in \operatorname{Colax}(I, \mathbb{V}\text{-}\mathbf{Cat})$. It is enough to show that $Q_{\int X} \cdot \int (P_X, \phi_X) = \mathbb{1}_{\int X}$.

*On objects*: Let ${}_ix \in (\int X)_0$. Then $Q_{\int X}\big(\int(P_X,\phi_X)(x)\big) = Q_{\int X}({}_i(P_X(i)x)) = {}_ix$.

*On morphisms*: Let $f = (f_a)_{a\in I(i,j)} \colon {}_ix \to {}_jy$ be in $\int(X)$. Then we have

$$\begin{aligned} Q_{\int X}\int(P_X,\phi_X)(f) &= Q_{\int X}((P_X(j)(f_a)\circ \phi_X(a)x)_{a\in I(i,j)}) \\ &= \sum_{a\in I(i,j)} P_X(j)(f_a)\circ\phi_X(a)x = (P_X,\phi_X)^{(1)}_{x,y}(f) = f. \end{aligned}$$

Thus the claim holds. The two claims above prove the assertion. □

**Corollary 6.3.** *Let $(F,\psi)\colon X \to \Delta(\mathscr{C})$ be in* $\mathrm{Colax}(I,\mathbb{V}\text{-}\mathbf{Cat})$*. Then the following are equivalent.*

1. *$(F,\psi)$ is an $I$-covering;*
2. *There exists an equivalence $H\colon \int X \to \mathscr{C}$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "{(F,\psi)}"] \arrow[d, "{(P_X,\phi_X)}"'] & \Delta(\mathscr{C}) \\ \Delta(\int(X)) \arrow[ur, "\Delta(H)"'] & \end{tikzcd}$$

*is strictly commutative.*

*Proof.* This immediately follows by Theorem 6.2 and Lemma 5.7. More precisely,

$$\begin{aligned} (F,\psi)^{(1)}_{x,y}(((f_a^n)_{n\in\mathbb{Z}})_{a\in I(i,j)}) &= \sum_{a\in I(i,j)} \psi(a)_x * F(j)(f_a) \\ &= \sum_{a\in I(i,j)} H\phi(a)_x * HP(j)(f_a) \\ &= H(\sum_{a\in I(i,j)} \phi(a)_x * P(j)(f_a)) \\ &= H(P,\phi)^{(1)}_{x,y}(f). \end{aligned} \tag{6.11}$$

□

In particular, in the case that $\mathbb{V} = \mathscr{C}(\Bbbk)$, i.e., $\mathbb{V}$-**Cat** $= \Bbbk$-**dgCat**, we have the following.

The 2-functor $\int\colon \mathrm{Colax}(I,\Bbbk\text{-}\mathbf{dgCat}) \to \Bbbk\text{-}\mathbf{dgCat}$ is a strict left 2-adjoint to the 2-functor $\Delta\colon \Bbbk\text{-}\mathbf{dgCat} \to \mathrm{Colax}(I,\Bbbk\text{-}\mathbf{dgCat})$. The unit is given by the family of canonical morphisms $(P_X,\phi_X)\colon X \to \Delta(\int X)$ indexed by $X \in \mathrm{Colax}(I,\Bbbk\text{-}\mathbf{dgCat})$, and the counit is given by the family of $Q_{\mathscr{C}}\colon \int\Delta(\mathscr{C}) \to \mathscr{C}$ defined as above indexed by $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}$.

In particular, $(P_X,\phi_X)$ has a strict universality in the comma category $(X{\downarrow}\Delta)$, i.e., for each $(F,\psi)\colon X \to \Delta(\mathscr{C})$ in $\mathrm{Colax}(I,\Bbbk\text{-}\mathbf{dgCat})$ with $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}$, there exists a unique $H\colon \int(X) \to \mathscr{C}$ in $\Bbbk$-**dgCat** such that the following is a commutative diagram in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$:

$$\begin{tikzcd}
X \arrow[r, "{(F,\psi)}"] \arrow[d, "{(P_X,\phi_X)}"'] & \Delta(\mathscr{C}). \\
\Delta(\int X) \arrow[ur, dashed, "\Delta(H)"'] &
\end{tikzcd}$$

## 7. The derived colax functors

Let $X\colon I \to \Bbbk$-**dgCat** be a colax functor. In this section we formulate the definition of the "derived category $\mathscr{D}(X)$" of $X$ as a colax functor from $I$ to a 2-category of triangulated categories by modifying the definition given in the previous paper [7]. We first recall the definition of colax functors between 2-categories.

**Definition 7.1.** Let $\mathbf{B}$ and $\mathbf{C}$ be 2-categories.

(1) A *colax functor* from $\mathbf{B}$ to $\mathbf{C}$ is a triple $(X, \eta, \theta)$ of data:

- a triple $X = (X_0, X_1, X_2)$ of maps $X_i\colon \mathbf{B}_i \to \mathbf{C}_i$ ($\mathbf{B}_i$ denotes the collection of $i$-morphisms of $\mathbf{B}$ for each $i = 0, 1, 2$) preserving domains and codomains of all 1-morphisms and 2-morphisms (i.e. $X_1(\mathbf{B}_1(i,j)) \subseteq \mathbf{C}_1(X_0 i, X_0 j)$ for all $i, j \in \mathbf{B}_0$ and $X_2(\mathbf{B}_2(a,b)) \subseteq \mathbf{C}_2(X_1 a, X_1 b)$ for all $a, b \in \mathbf{B}_1$ (we omit the subscripts of $X$ below));
- a family $\eta := (\eta_i)_{i\in\mathbf{B}_0}$ of 2-morphisms $\eta_i\colon X(\mathbb{1}_i) \Rightarrow \mathbb{1}_{X(i)}$ in $\mathbf{C}$ indexed by $i \in \mathbf{B}_0$; and
- a family $\theta := (\theta_{b,a})_{(b,a)}$ of 2-morphisms $\theta_{b,a}\colon X(ba) \Rightarrow X(b)X(a)$ in $\mathbf{C}$ indexed by $(b,a) \in \mathrm{com}(\mathbf{B}) := \{(b,a) \in \mathbf{B}_1 \times \mathbf{B}_1 \mid ba \text{ is defined}\}$

satisfying the axioms:

(i) $(X_1, X_2)\colon \mathbf{B}(i,j) \to \mathbf{C}(X_0 i, X_0 j)$ is a functor for all $i, j \in \mathbf{B}_0$;

(ii) For each $a\colon i \to j$ in $\mathbf{B}_1$ the following are commutative:

$$\begin{tikzcd}
X(a\mathbb{1}_i) \arrow[r, Rightarrow, "\theta_{a,\mathbb{1}_i}"] \arrow[dr, equal] & X(a)X(\mathbb{1}_i) \arrow[d, Rightarrow, "X(a)\eta_i"] \\
& X(a)\mathbb{1}_{X(i)}
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
X(\mathbb{1}_j a) \arrow[r, Rightarrow, "\theta_{\mathbb{1}_j,a}"] \arrow[dr, equal] & X(\mathbb{1}_j)X(a) \arrow[d, Rightarrow, "\eta_j X(a)"] \\
& \mathbb{1}_{X(j)}X(a)
\end{tikzcd}\quad ;$$

(iii) For each $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in $\mathbf{B}_1$ the following is commutative:

$$\begin{tikzcd}
X(cba) \arrow[r, Rightarrow, "\theta_{c,ba}"] \arrow[d, Rightarrow, "\theta_{cb,a}"'] & X(c)X(ba) \arrow[d, Rightarrow, "X(c)\theta_{b,a}"] \\
X(cb)X(a) \arrow[r, Rightarrow, "\theta_{c,b}X(a)"'] & X(c)X(b)X(a)
\end{tikzcd}\quad ; \text{ and}$$

(iv) For each $a, a' \colon i \to j$ and $b, b' \colon j \to k$ in $\mathbf{B}_1$ and each $\alpha \colon a \to a'$, $\beta \colon b \to b'$ in $\mathbf{B}_2$ the following is commutative:

$$\begin{tikzcd} X(ba) \arrow[r, Rightarrow, "\theta_{b,a}"] \arrow[d, Rightarrow, "X(\beta*\alpha)"'] & X(b)X(a) \arrow[d, Rightarrow, "X(\beta)*X(\alpha)"] \\ X(b'a') \arrow[r, Rightarrow, "\theta_{b',a'}"] & X(b')X(a'). \end{tikzcd}$$

(2) A *lax functor* from $\mathbf{B}$ to $\mathbf{C}$ is a colax functor from $\mathbf{B}$ to $\mathbf{C}^{\mathrm{co}}$ (see Notation 2.17).

(3) A *pseudofunctor* from $\mathbf{B}$ to $\mathbf{C}$ is a colax functor with all $\eta_i$ and $\theta_{b,a}$ 2-isomorphisms.

(4) We define a 2-category $\mathrm{Colax}(\mathbf{B}, \mathbf{C})$ having all the colax functors $\mathbf{B} \to \mathbf{C}$ as the objects as follows.

**1-morphisms.** Let $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ be colax functors from $\mathbf{B}$ to $\mathbf{C}$. A 1-*morphism* (called a *left transformation*) from $X$ to $X'$ is a pair $(F, \psi)$ of data

- a family $F := (F(i))_{i \in \mathbf{B}_0}$ of 1-morphisms $F(i) \colon X(i) \to X'(i)$ in $\mathbf{C}$ ; and
- a family $\psi := (\psi(a))_{a \in \mathbf{B}_1}$ of 2-morphisms $\psi(a) \colon X'(a)F(i) \Rightarrow F(j)X(a)$

$$\begin{tikzcd} X(i) \arrow[r, "F(i)"] \arrow[d, "X(a)"'] & X'(i) \arrow[d, "X'(a)"] \arrow[dl, phantom, "\swarrow\psi(a)"] \\ X(j) \arrow[r, "F(j)"'] & X'(j) \end{tikzcd}$$

in $\mathbf{C}$ indexed by $a \colon i \to j$ in $\mathbf{B}_1$ that satisfies the following three conditions:

(0) for each $\alpha \colon a \Rightarrow b$ in $\mathbf{B}(i, j)$ the following is commutative:

$$\begin{tikzcd} X'(a)F(i) \arrow[r, Rightarrow, "X'(\alpha)F(i)"] \arrow[d, Rightarrow, "\psi(a)"'] & X'(b)F(i) \arrow[d, Rightarrow, "\psi(b)"] \\ F(j)X(a) \arrow[r, Rightarrow, "F(j)X(\alpha)"'] & F(j)X(b), \end{tikzcd} \tag{7.12}$$

thus $\psi$ gives a family of natural transformations of functors:

$$\begin{tikzcd} \mathbf{B}(i,j) \arrow[r, "X'"] \arrow[d, "X"'] & \mathbf{C}(X'(i), X'(j)) \arrow[d, "\mathbf{C}(F(i), X'(j))"] \arrow[dl, phantom, "\swarrow\psi_{ij}"] \\ \mathbf{C}(X(i), X(j)) \arrow[r, "\mathbf{C}(X(i), F(j))"'] & \mathbf{C}(X(i), X'(j)) \end{tikzcd} \quad (i, j \in \mathbf{B}_0),$$

(a) For each $i \in \mathbf{B}_0$ the following is commutative:

$$\begin{tikzcd} X'(\mathbb{1}_i)F(i) \arrow[r, Rightarrow, "\psi(\mathbf{1}_i)"] \arrow[d, Rightarrow, "\eta'_i F(i)"'] & F(i)X(\mathbb{1}_i) \arrow[d, Rightarrow, "F(i)\eta_i"] \\ \mathbb{1}_{X'(i)}F(i) \arrow[r, equal] & F(i)\mathbb{1}_{X(i)} \end{tikzcd} \quad ; \text{ and}$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k$ in $\mathbf{B}_1$ the following is commutative:

$$\begin{tikzcd} X'(ba)F(i) \arrow[r, Rightarrow, "\theta'_{b,a}F(i)"] \arrow[d, Rightarrow, "\psi(ba)"'] & X'(b)X'(a)F(i) \arrow[r, Rightarrow, "X'(b)\psi(a)"] & X'(b)F(j)X(a) \arrow[d, Rightarrow, "\psi(b)X(a)"] \\ F(k)X(ba) \arrow[rr, Rightarrow, "F(k)\,\theta_{b,a}"'] & & F(k)X(b)X(a). \end{tikzcd}$$

**2-morphisms.** Let $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ be colax functors from $\mathbf{B}$ to $\mathbf{C}$, and $(F, \psi)$, $(F', \psi')$ 1-morphisms from $X$ to $X'$. A 2-*morphism* from $(F, \psi)$ to $(F', \psi')$ is a family $\zeta = (\zeta(i))_{i \in \mathbf{B}_0}$ of 2-morphisms $\zeta(i) \colon F(i) \Rightarrow F'(i)$ in $\mathbf{C}$ indexed by $i \in \mathbf{B}_0$ such that the following is commutative for all $a \colon i \to j$ in $\mathbf{B}_1$:

$$\begin{tikzcd} X'(a)F(i) \arrow[r, Rightarrow, "X'(a)\zeta(i)"] \arrow[d, Rightarrow, "\psi(a)"'] & X'(a)F'(i) \arrow[d, Rightarrow, "\psi'(a)"] \\ F(j)X(a) \arrow[r, Rightarrow, "\zeta(j)X(a)"] & F'(j)X(a). \end{tikzcd}$$

**Composition of 1-morphisms.** Let $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ and $X'' = (X'', \eta'', \theta'')$ be colax functors from $\mathbf{B}$ to $\mathbf{C}$, and let $(F, \psi) \colon X \to X'$, $(F', \psi') \colon X' \to X''$ be 1-morphisms. Then the composite $(F', \psi')(F, \psi)$ of $(F, \psi)$ and $(F', \psi')$ is a 1-morphism from $X$ to $X''$ defined by

$$(F', \psi')(F, \psi) := (F'F, \psi' \circ \psi),$$

where $F'F := ((F'(i)F(i))_{i \in \mathbf{B}_0}$ and for each $a \colon i \to j$ in $\mathbf{B}$, $(\psi' \circ \psi)(a) := F'(j)\psi(a) \circ \psi'(a)F(i)$ is the pasting of the diagram

$$\begin{tikzcd} X(i) \arrow[r, "F(i)"] \arrow[d, "X(a)"'] & X'(i) \arrow[r, "F'(i)"] \arrow[d, "X'(a)"] \arrow[dl, phantom, "\swarrow \psi(a)"] & X''(i) \arrow[d, "X''(a)"] \arrow[dl, phantom, "\swarrow \psi'(a)"] \\ X(j) \arrow[r, "F(j)"'] & X'(j) \arrow[r, "F'(j)"'] & X''(j). \end{tikzcd}$$

**Remark 7.2.** We make the following remarks.

(1) Note that a (strict) 2-functor from $\mathbf{B}$ to $\mathbf{C}$ is a pseudofunctor with all $\eta_i$ and $\theta_{b,a}$ identities.

(2) By regarding the category $I$ as a 2-category with all 2-morphisms identities, the definition (1) of colax functors above coincides with Definition 2.12.

(3) When $\mathbf{B} = I$, the definition (4) of $\mathrm{Colax}(\mathbf{B}, \mathbf{C})$ above coincides with that of $\mathrm{Colax}(I, \mathbf{C})$ given before.
(4) It is well-known that the composite of pseudofunctors turns out to be a pseudofunctor.

**Notation 7.3.** We introduce the following 2-categories.

(1) $\Bbbk$-**AB** denotes the 2-category of light abelian $\Bbbk$-categories, $\Bbbk$-functors between them, and natural transformations between these $\Bbbk$-functors.
(2) $\Bbbk$-**FRB** denotes the 2-category of light Frobenius $\Bbbk$-categories, $\Bbbk$-functors between them, and natural transformations between these $\Bbbk$-functors.
(3) $\Bbbk$-**TRI** denotes the 2-category of light triangulated $\Bbbk$-categories, triangle $\Bbbk$-functors between them, and natural transformations between these triangle $\Bbbk$-functors.
(4) $\Bbbk$-$\underline{\mathbf{TRI}}^2$ denotes the 2-category of 2-moderate triangulated $\Bbbk$-categories, triangle $\Bbbk$-functors between them, and natural transformations between these triangle $\Bbbk$-functors.

**Definition 7.4.** Let $\mathscr{A} \in \Bbbk\text{-}\mathbf{dgCat}_0 = \Bbbk\text{-}\mathbf{DGCat}_0$. A dg functor $\mathscr{A}^{\mathrm{op}} \to \mathscr{C}_{\mathrm{dg}}(\Bbbk)$ is called a *right dg $\mathscr{A}$-module.* We set

$$\mathscr{C}(\mathscr{A}) := \Bbbk\text{-}\mathbf{dgCat}(\mathscr{A}^{\mathrm{op}}, \mathscr{C}_{\mathrm{dg}}(\Bbbk)), \text{and}$$
$$\mathscr{C}_{\mathrm{dg}}(\mathscr{A}) := \Bbbk\text{-}\mathbf{DGCat}(\mathscr{A}^{\mathrm{op}}, \mathscr{C}_{\mathrm{dg}}(\Bbbk)).$$

$\mathscr{C}(\mathscr{A})$ is called the *category of* (*right*) *dg $\mathscr{A}$-modules*, and $\mathscr{C}_{\mathrm{dg}}(\mathscr{A})$ is called the *dg category of* (*right*) *dg $\mathscr{A}$-modules.* Thus in particular, we have

$$\mathscr{C}(\mathscr{A})_0 = \mathscr{C}_{\mathrm{dg}}(\mathscr{A})_0,$$

which consists of the right dg $\mathscr{A}$-modules. Note that $\mathscr{C}(\mathscr{A})$ is in $\Bbbk$-**AB**, or more presicely, in $\Bbbk$-**FRB**, whereas $\mathscr{C}_{\mathrm{dg}}(\mathscr{A})$ is in $\Bbbk$-**dgCAT** and in $\Bbbk$-**DGCAT**. By (3.4), they have the following relation for all objects $X, Y$:

$$\mathscr{C}(\mathscr{A})(X, Y) = Z^0(\mathscr{C}_{\mathrm{dg}}(\mathscr{A})(X, Y)).$$

Thus a morphism $X \to Y$ in $\mathscr{C}(\mathscr{A})$ is given as a dg natural transformation, but in $\mathscr{C}_{\mathrm{dg}}(\mathscr{A})$ it is given as a derived transformation.

**Definition 7.5.** Let $\mathscr{A}$ and $\mathscr{B}$ be small dg categories.

(1) A dg functor $\mathscr{B} \to \mathscr{C}_{\mathrm{dg}}(\Bbbk)$ is called a *left dg $\mathscr{B}$-module.* A $\mathscr{B}$-$\mathscr{A}$-bimodule is a dg functor $M\colon \mathscr{A}^{\mathrm{op}} \otimes_\Bbbk \mathscr{B} \to \mathscr{C}_{\mathrm{dg}}(\Bbbk)$.
(2) For each $B \in \mathscr{B}_0$ and $A \in \mathscr{A}_0$, we set ${}_BM := M(\text{-}, B)\colon \mathscr{A}^{\mathrm{op}} \to \mathscr{C}_{\mathrm{dg}}(\Bbbk)$ and $M_A := M(A, \text{-})\colon \mathscr{B} \to \mathscr{C}_{\mathrm{dg}}(\Bbbk)$, and ${}_BM_A := M(A, B)$. Note that ${}_BM$ is a right dg $\mathscr{A}$-module, $M_A$ is a left dg $\mathscr{B}$-module, and ${}_BM_A$ is a dg $\Bbbk$-module.
(3) If $f\colon A' \to A$ is a morphism in $\mathscr{A}$, and $g\colon B' \to B$ is a morphism in $\mathscr{B}$, then we set ${}_gM := M(\text{-}, g)$ and $M_f := M(f, \text{-})$. Note that $M_f\colon M_A \to M_{A'}$ is a derived transformation between dg functors, and that ${}_gM\colon {}_{B'}M \to {}_BM$ is a derived transformation between dg functors.

(4) To emphasise that $M$ is a $\mathscr{B}$-$\mathscr{A}$-bimodule, we sometimes use the notation ${}_{\mathscr{B}}M_{\mathscr{A}}$.

(5) The dg category $\mathscr{A}$ defines an $\mathscr{A}$-$\mathscr{A}$-bimodule $\mathscr{A}(\text{-}, ?)\colon \mathscr{A}^{\mathrm{op}} \otimes_{\Bbbk} \mathscr{A} \to \mathscr{C}_{\mathrm{dg}}(\Bbbk)$ by $(x, y) \mapsto \mathscr{A}(x, y)$. We denote this bimodule by ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$, and we use the same convention that ${}_x\mathscr{A} := \mathscr{A}(\text{-}, x)$, $\mathscr{A}_y := \mathscr{A}(y, \text{-})$, and ${}_x\mathscr{A}_y := \mathscr{A}(y, x)$ for all $x, y \in \mathscr{A}_0$; and for any morpisms $f\colon x' \to x$, $g\colon y' \to y$ in $\mathscr{A}$, we write ${}_g\mathscr{A} : {}_{y'}\mathscr{A} \to {}_y\mathscr{A}$, and $\mathscr{A}_f\colon \mathscr{A}_x \to \mathscr{A}_{x'}$. Sometimes we abbreviate them as $g^\wedge\colon y'^\wedge \to y^\wedge$ and ${}^\wedge f\colon {}^\wedge x \to {}^\wedge x'$, respectively.

**Remark 7.6.** In Definition 7.5 (3), note that 0-cocycle morphisms are preserved by the correspondences $f \mapsto M_f$ and $g \mapsto {}_gM$, namely, if $f \in Z^0(\mathscr{A}(A', A))$ (resp. $g \in Z^0(\mathscr{B}(B', B))$), then $M_f \in Z^0(\mathscr{C}_{\mathrm{dg}}(\mathscr{B}^{\mathrm{op}})(M_A, M_{A'})) = \mathscr{C}(\mathscr{B}^{\mathrm{op}})(M_A, M_{A'})$ (resp. ${}_gM \in Z^0(\mathscr{C}_{\mathrm{dg}}(\mathscr{A})\ ({}_{B'}M, {}_BM)) = \mathscr{C}(\mathscr{A})({}_{B'}M, {}_BM)$).

Indeed, since $M_A$ is a left dg $\mathscr{B}$-module, the dg functor $M\colon \mathscr{A}^{\mathrm{op}} \otimes_{\Bbbk} \mathscr{B} \to \mathscr{C}_{\mathrm{dg}}(\Bbbk)$ induces a dg functor $M : \mathscr{A}^{\mathrm{op}} \to \mathscr{C}_{\mathrm{dg}}(\mathscr{B}^{\mathrm{op}})$. Therefore, for any $A, A' \in \mathscr{A}_0$, by noting that $\mathscr{A}^{\mathrm{op}}(A, A') = \mathscr{A}(A', A)$, it induces a chain map

$$M_{A,A'} : \mathscr{A}(A', A) \to \mathscr{C}_{\mathrm{dg}}(\mathscr{B}^{\mathrm{op}})(M_A, M_{A'}).$$

Hence if $f \in Z^0(\mathscr{A}(A', A))$, then $M_f \in Z^0(\mathscr{C}_{\mathrm{dg}}(\mathscr{B}^{\mathrm{op}})(M_A, M_{A'}))$. Thus in this case, $M_f\colon M_A \to M_{A'}$ is a dg natural transformation between dg functors. Similar argument works for the remaining case.

**Notation 7.7.** Let $\mathscr{A}, \mathscr{B}, \mathscr{A}', \mathscr{B}'$ be small dg categories, $E\colon \mathscr{A}' \to \mathscr{A}$, $F\colon \mathscr{B}' \to \mathscr{B}$ dg functors, and $M$ an $\mathscr{A}$-$\mathscr{B}$-bimodule.

(1) We denote by ${}_EM$, $M_F$ and ${}_EM_F$ the $\mathscr{A}'$-$\mathscr{B}$-bimodule, $\mathscr{A}$-$\mathscr{B}'$-bimodule, and $\mathscr{A}'$-$\mathscr{B}'$-bimodule defined as follows, respectively:

$$\begin{aligned}
{}_EM &:= M(\text{-}, E(?)) = M \circ (\mathbb{1}_{\mathscr{B}^{\mathrm{op}}} \otimes_k E)\colon \mathscr{B}^{\mathrm{op}} \otimes_{\Bbbk} \mathscr{A}' \xrightarrow{\mathbf{1}_{\mathscr{B}^{\mathrm{op}}} \otimes_k E} \mathscr{B}^{\mathrm{op}} \otimes_{\Bbbk} \mathscr{A} \xrightarrow{M} \mathscr{C}_{\mathrm{dg}}(\Bbbk),\\
M_F &:= M(F(\text{-}), ?) = M \circ (F \otimes_k \mathbb{1}_{\mathscr{A}})\colon \mathscr{B}'^{\mathrm{op}} \otimes_{\Bbbk} \mathscr{A} \xrightarrow{F \otimes_k \mathbf{1}_{\mathscr{A}}} \mathscr{B}^{\mathrm{op}} \otimes_{\Bbbk} \mathscr{A} \xrightarrow{M} \mathscr{C}_{\mathrm{dg}}(\Bbbk), \text{and}\\
{}_EM_F &:= M(F(\text{-}), E(?)) = M \circ (F \otimes_k E)\colon \mathscr{B}'^{\mathrm{op}} \otimes_{\Bbbk} \mathscr{A}' \xrightarrow{F \otimes_k E} \mathscr{B}^{\mathrm{op}} \otimes_{\Bbbk} \mathscr{A} \xrightarrow{M} \mathscr{C}_{\mathrm{dg}}(\Bbbk).
\end{aligned}$$

(2) Moreover, if $E'\colon \mathscr{A}' \to \mathscr{A}$ and $F'\colon \mathscr{B}' \to \mathscr{B}$ are dg functors, and $\alpha\colon E \Rightarrow E'$, $\beta\colon F \Rightarrow F'$ are derived transformations, then $M$ defines morphisms of bimodules as follows:

$$\begin{aligned}
{}_\alpha M &:= M(\text{-}, \alpha_{(?)}) = M \circ (\mathbb{1}_{\mathscr{B}^{\mathrm{op}}} \otimes_k \alpha)\colon {}_EM \to {}_{E'}M\\
M_\beta &:= M(\beta_{(\text{-})}, ?) = M \circ (\beta \otimes_k \mathbb{1}_{\mathscr{A}})\colon M_{F'} \to M_F, \text{and}\\
{}_\alpha M_\beta &:= M(\beta_{(\text{-})}, \alpha_{(?)}) = M \circ (\beta \otimes_{\Bbbk} \alpha)\colon {}_EM_{F'} \to {}_{E'}M_F.
\end{aligned}$$

(3) We often abbreviate the morphism of $\mathscr{A}'$-$\mathscr{A}$-bimodules (induced from the bimodule ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$)

$${}_\alpha\mathscr{A} : {}_E\mathscr{A} \to {}_{E'}\mathscr{A} \quad \text{as} \quad \overline{\alpha}\colon \overline{E} \to \overline{E'},$$

and the morphism of $\mathscr{B}$-$\mathscr{B}'$-bimodules (induced from the bimodule ${}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}}$)

$$\mathscr{B}_\beta\colon \mathscr{B}_{F'} \to \mathscr{B}_F \quad \text{as} \quad \overline{\beta}^* : \overline{F'}^* \to \overline{F}^*.$$

**Definition 7.8.** Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ and $\mathscr{D}$ be small dg categories, and ${}_{\mathscr{D}}W_{\mathscr{C}}, {}_{\mathscr{C}}V_{\mathscr{B}}$, ${}_{\mathscr{B}}U_{\mathscr{A}}$ bimodules. Then the canonical isomorphism

$$\boldsymbol{a} = \boldsymbol{a}_{W,V,U} \colon W \otimes_{\mathscr{C}} (V \otimes_{\mathscr{B}} U) \to (W \otimes_{\mathscr{C}} V) \otimes_{\mathscr{B}} U$$

that represent the associativity of the tensor products is called the *associator* of tensor products.

**Definition 7.9.** Since both $\Bbbk$-**dgCat** and $\Bbbk$-**DGCat** are 2-categories, the correspondences $\mathscr{A} \mapsto \mathscr{C}_{\mathrm{dg}}(\mathscr{A})$ and $\mathscr{A} \mapsto \mathscr{C}(\mathscr{A})$ are extended to representable 2-functors

$$\mathscr{C}'_{\mathrm{dg}} := \Bbbk\text{-}\mathbf{DGCat}((\text{-})^{\mathrm{op}}, \mathscr{C}_{\mathrm{dg}}(\Bbbk)) \colon \Bbbk\text{-}\mathbf{DGCat} \to \Bbbk\text{-}\mathbf{DGCAT}^{\mathrm{coop}} \text{ and}$$
$$\mathscr{C}' := \Bbbk\text{-}\mathbf{dgCat}((\text{-})^{\mathrm{op}}, \mathscr{C}_{\mathrm{dg}}(\Bbbk)) \colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{FRB}^{\mathrm{coop}},$$

respectively. By modifying these, we define pseudofunctors

$$\mathscr{C}_{\mathrm{dg}} \colon \Bbbk\text{-}\mathbf{DGCat} \to \Bbbk\text{-}\mathbf{DGCAT} \text{ and}$$
$$\mathscr{C} \colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{FRB}$$

as follows. For any diagram

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "E"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \end{tikzcd}$$

in $\Bbbk$-**DGCat** (resp. in $\Bbbk$-**dgCat**), we define $\mathscr{C}_{\mathrm{dg}}(E) := \text{-} \otimes_{\mathscr{A}} {}_E\mathscr{B}$ and $\mathscr{C}_{\mathrm{dg}}(\alpha) := \text{-} \otimes_{\mathscr{A}} {}_\alpha\mathscr{B}$ (resp. $\mathscr{C}(E) := \text{-} \otimes_{\mathscr{A}} {}_E\mathscr{B}$ and $\mathscr{C}(\alpha) := \text{-} \otimes_{\mathscr{A}} {}_\alpha\mathscr{B}$) (see Notation 7.7). Note that $\mathscr{C}(\alpha) := \text{-} \otimes_{\mathscr{A}} {}_\alpha\mathscr{B}$ is defined by regarding $\alpha$ in $\Bbbk$-**dgCat** as a 2-morphism in $\Bbbk$-**DGCat** concentrated in degree 0.

We define the structures of pseudofunctors for $\mathscr{C}_{\mathrm{dg}}$ and $\mathscr{C}$ as follows.

- For each $\mathscr{A} \in \Bbbk\text{-}\mathbf{DGCat}_0 = \Bbbk\text{-}\mathbf{dgCat}_0$, we define $\eta_{\mathscr{A}} \colon \mathscr{C}_{\mathrm{dg}}(\mathbb{1}_{\mathscr{A}}) \Rightarrow \mathbb{1}_{\mathscr{C}_{\mathrm{dg}}(\mathscr{A})}$ (resp. $\eta_{\mathscr{A}} \colon \mathscr{C}(\mathbb{1}_{\mathscr{A}}) \Rightarrow \mathbb{1}_{\mathscr{C}(\mathscr{A})}$) by setting
$$\eta_{\mathscr{A}} M \colon M \otimes_{\mathscr{A}} \mathscr{A}(?, \text{-}) \to M$$
to be the canonical isomorphisms for all $M \in \mathscr{C}_{\mathrm{dg}}(\mathscr{A})_0 = \mathscr{C}(\mathscr{A})_0$.
- For each pair of dg functors $\mathscr{A} \xrightarrow{F} \mathscr{A}' \xrightarrow{G} \mathscr{A}''$ in $\Bbbk\text{-}\mathbf{DGCat}_1 = \Bbbk\text{-}\mathbf{dgCat}_1$, we define
$$\theta_{G,F} \colon \mathscr{C}_{\mathrm{dg}}(GF) \Rightarrow \mathscr{C}_{\mathrm{dg}}(G) \circ \mathscr{C}_{\mathrm{dg}}(F)$$
$$(\text{resp. } \theta_{G,F} \colon \mathscr{C}(GF) \Rightarrow \mathscr{C}(G) \circ \mathscr{C}(F))$$
as the canonical isomorphism
$$\text{-} \otimes_{\mathscr{A}} {}_{GF}\mathscr{A}'' \Rightarrow (\text{-} \otimes_{\mathscr{A}} {}_F\mathscr{A}') \otimes_{\mathscr{A}'} {}_G\mathscr{A}''$$

given as the composite of the canonical isomorphisms (see Definition 7.8)

$$\begin{aligned} \text{-} \otimes_{\mathscr{A}} {}_{GF}\mathscr{A}'' &\cong \text{-} \otimes_{\mathscr{A}} (\mathscr{A}'(?, F(\text{-})) \otimes_{\mathscr{A}'} \mathscr{A}''(?, G(\text{-}))) \\ &\xrightarrow{\boldsymbol{a}^{-1}_{(\text{-}),\mathscr{A}'(?,F(\text{-})),\mathscr{A}''(?,G(\text{-}))}} (\text{-} \otimes_{\mathscr{A}} \mathscr{A}'(?, F(\text{-}))) \otimes_{\mathscr{A}'} \mathscr{A}''(?, G(\text{-})). \end{aligned}$$

It is straightforward to check that this defines pseudofunctors $\mathscr{C}_{\mathrm{dg}}$ and $\mathscr{C}$.

**Remark 7.10.** In the definition above, note that $\mathscr{C}(E)$ is a left adjoint to $\mathscr{C}'(E)$ and that $\mathscr{C}'(E)$ has also a right adjoint. Therefore, $\mathscr{C}'(E)$ is an exact functor.

**Remark 7.11.** Using the notation in Definition 7.5 (5), the Yoneda embedding $Y_{\mathscr{A}} \colon \mathscr{A} \to \mathscr{C}_{\mathrm{dg}}(\mathscr{A})$ can be defined as the functor sending a morphism $f \colon x \to y$ in $\mathscr{A}$ to the morphism $f^{\wedge} \colon x^{\wedge} \to y^{\wedge}$ in $\mathscr{C}_{\mathrm{dg}}(\mathscr{A})$.

**Definition 7.12.** As is easily seen, the stable category construction $\mathscr{F} \mapsto \underline{\mathscr{F}}$ can be extended to a 2-functor $\mathrm{st} \colon \Bbbk\text{-}\mathbf{FRB} \to \Bbbk\text{-}\mathbf{TRI}$ in an obvious way. Then we set

$$\mathscr{H} := \mathrm{st} \circ \mathscr{C} \colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{TRI},$$

which turns out to be a pseudofunctor as a composite of a pseudofunctor and a 2-functor. For each $\mathscr{A} \in \Bbbk\text{-}\mathbf{dgCat}$, $\mathscr{H}(\mathscr{A})$ is called the *homotopy category* of $\mathscr{A}$.

For each $\mathscr{A} \in \Bbbk\text{-}\mathbf{DGCat}_0$, we set

$$\mathscr{D}(\mathscr{A}) := \mathscr{H}(\mathscr{A})[\mathrm{qis}^{-1}]$$

to be the quotient category of the homotopy cagtegory of $\mathscr{A}$ with respect to quasi-isomorphisms, and call it the *derived category* of $\mathscr{A}$.

**Remark 7.13.** We note that for each $\mathscr{A} \in \Bbbk\text{-}\mathbf{DGCat}$, the homotopy category $\mathscr{H}(\mathscr{A})$ is a light triangulated category, but the derived category $\mathscr{D}(\mathscr{A})$ is a properly 2-moderate triangulated category although there exist isomorphisms (7.14) below.

**Definition 7.14.** Let $\mathscr{A} \in \Bbbk\text{-}\mathbf{DGCat}_0$.

(1) We denote by $\mathscr{H}_{\mathrm{p}}(\mathscr{A})$ the full subcategory of the homotopy category $\mathscr{H}(\mathscr{A})$ of $\mathscr{A}$ consisting of the *homotopically projective* objects $M$, i.e., objects $M$ such that $\mathscr{H}(\mathscr{A})(M, A) = 0$ for all acyclic objects $A$.

(2) Let $\mathscr{H}_{\mathrm{p}}(\mathscr{A}) \xrightarrow{\sigma_{\mathscr{A}}} \mathscr{H}(\mathscr{A}) \xrightarrow{Q_{\mathscr{A}}} \mathscr{D}(\mathscr{A})$ be the inclusion functor and the quotient functor, respectively, and set $\mathbf{j}_{\mathscr{A}} := Q_{\mathscr{A}} \circ \sigma_{\mathscr{A}}$. Then there exists a functor $\mathbf{p}_{\mathscr{A}} \colon \mathscr{D}(\mathscr{A}) \to \mathscr{H}_{\mathrm{p}}(\mathscr{A})$ giving a left adjoint $\sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}}$ to $Q_{\mathscr{A}}$:

$$\begin{tikzcd}
& \mathscr{H}_{\mathrm{p}}(\mathscr{A}) \arrow[dl, "\sigma_{\mathscr{A}}"'] \arrow[d, phantom, "\perp"] & \\
\mathscr{H}(\mathscr{A}) \arrow[rr, bend right, "Q_{\mathscr{A}}"'] & {} & \mathscr{D}(\mathscr{A}) \arrow[ul, "\mathbf{p}_{\mathscr{A}}"']
\end{tikzcd}$$

such that

$$\mathbf{p}_{\mathscr{A}} \mathbf{j}_{\mathscr{A}} = \mathbb{1}_{\mathscr{H}_{\mathrm{p}}(\mathscr{A})} \tag{7.13}$$

is satisfied[3] and the counit $\varepsilon_{\mathscr{A}} \colon (\sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}}) \circ Q_{\mathscr{A}} \Rightarrow \mathbb{1}_{\mathscr{H}(\mathscr{A})}$ consists of quasi-isomorphisms $\varepsilon_{\mathscr{A},M} \colon \mathbf{p}_{\mathscr{A}} M \to M$ for all $M \in \mathscr{H}(\mathscr{A})_0$. In particular,

[3] This can be done by taking $\mathbf{p}_{\mathscr{A}}(P) := P$ for all $P \in \mathscr{H}_{\mathrm{p}}(\mathscr{A})$.

both $\mathbf{p}_{\mathscr{A}}$ and $\mathbf{j}_{\mathscr{A}}$ are equivalences and quasi-inverses to each other, and by the adjoint above we have a canonical isomorphism

$$\mathscr{D}(\mathscr{A})(L, M) \cong \mathscr{H}(\mathscr{A})(\mathbf{p}_{\mathscr{A}}(L), M) \tag{7.14}$$

for all $L, M \in \mathscr{D}(\mathscr{A})_0 = \mathscr{H}(\mathscr{A})_0$. Here, note that the right hand sides are always small, but the left hand sides are not.

**Definition 7.15.** We define two 2-categories $\mathscr{C}(\Bbbk\text{-}\mathbf{dgCat})$ and $\mathscr{H}(\Bbbk\text{-}\mathbf{dgCat})$ as follows:

- $\mathscr{C}(\Bbbk\text{-}\mathbf{dgCat})_0 := \{\mathscr{C}(\mathscr{A}) \mid \mathscr{A} \in \Bbbk\text{-}\mathbf{dgCat}_0\}$.
- For any objects $\mathscr{C}(\mathscr{A}), \mathscr{C}(\mathscr{B})$ of $\mathscr{C}(\Bbbk\text{-}\mathbf{dgCat})$, 1-morphisms from $\mathscr{C}(\mathscr{A})$ to $\mathscr{C}(\mathscr{B})$ are the $\Bbbk$-functors $F\colon \mathscr{C}(\mathscr{A}) \to \mathscr{C}(\mathscr{B})$ satisfying the condition
$$F(\mathscr{H}_{\mathrm{p}}(\mathscr{A})_0) \subseteq \mathscr{H}_{\mathrm{p}}(\mathscr{B})_0 \tag{7.15}$$
When this is the case, we say that $F$ *preserves homotomically projectives*.
- $\mathscr{H}(\Bbbk\text{-}\mathbf{dgCat})_0 := \{\mathscr{H}(\mathscr{A}) \mid \mathscr{A} \in \Bbbk\text{-}\mathbf{dgCat}_0\}$.
- For any objects $\mathscr{H}(\mathscr{A}), \mathscr{H}(\mathscr{B})$ of $\mathscr{H}(\Bbbk\text{-}\mathbf{dgCat})$), 1-morphisms from $\mathscr{H}(\mathscr{A})$ to $\mathscr{H}(\mathscr{B})$ are the $\Bbbk$-functors of the form $\underline{F}$ ($:= \mathrm{st}(F)$) for some $\Bbbk$-functors $F\colon \mathscr{C}(\mathscr{A}) \to \mathscr{C}(\mathscr{B})$ satisfying the condition (7.15).
- In both 2-categories, the 2-morphisms are the natural transformations between those 1-morphisms.

**Remark 7.16.** (1) By definition, the 2-functor $\mathrm{st}\colon \Bbbk\text{-}\mathbf{FRB} \to \Bbbk\text{-}\mathbf{TRI}$ restricts to a 2-functor

$$\mathrm{st}\colon \mathscr{C}(\Bbbk\text{-}\mathbf{dgCat}) \to \mathscr{H}(\Bbbk\text{-}\mathbf{dgCat}).$$

(2) In the definition above, unlike objects, note that we defined the 1-morphisms in $\mathscr{C}(\Bbbk\text{-}\mathbf{dgCat})$ not as the "image" of 1-morphisms in $\Bbbk\text{-}\mathbf{dgCat}$ under $\mathscr{C}$.

Nevertheless, pseudofunctors

$$\mathscr{C} : \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{FRB} \text{ and } \mathscr{H} : \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{TRI}$$

restrict to pseudofunctors

$$\mathscr{C} : \Bbbk\text{-}\mathbf{dgCat} \to \mathscr{C}(\Bbbk\text{-}\mathbf{dgCat}) \text{ and } \mathscr{H} : \Bbbk\text{-}\mathbf{dgCat} \to \mathscr{H}(\Bbbk\text{-}\mathbf{dgCat}).$$

Indeed, let $F\colon \mathscr{A} \to \mathscr{B}$ be a 1-morphism in $\Bbbk\text{-}\mathbf{dgCat}$. It is enough to show that $\mathscr{C}(F)$ is a 1-morphism in $\mathscr{C}(\Bbbk\text{-}\mathbf{dgCat})$, i.e. that $\mathscr{C}(F)(\mathscr{H}_{\mathrm{p}}(\mathscr{A})_0) \subseteq \mathscr{H}_{\mathrm{p}}(\mathscr{B})_0$. Let $P \in \mathscr{H}_{\mathrm{p}}(\mathscr{A})_0$. Then since $\mathscr{C}(F)$ is a left adjoint to $\mathscr{C}'(F)$ , we have

$$\mathscr{H}(\mathscr{B})(\mathscr{C}(F)(P), A) \cong \mathscr{H}(\mathscr{A})(P, \mathscr{C}'(F)(A))$$

for all acyclic objects $A$ of $\mathscr{C}(\mathscr{B})$. The right hand side is zero because $\mathscr{C}'(F)$ is exact by Remark 7.10, and hence $\mathscr{C}'(F)(A)$ is acyclic in $\mathscr{C}(\mathscr{A})$. This shows that $\mathscr{C}(F)(P) \in \mathscr{H}_{\mathrm{p}}(\mathscr{B})_0$. Noting that $\mathscr{C}(F) = \text{-}\otimes_{\mathscr{A}} {}_F\mathscr{B}$ and that ${}_A({}_F\mathscr{B}) = \mathscr{B}(\text{-}, F(A))$ is a projective $\mathscr{B}$-module for all $A \in \mathscr{A}_0$, this argument is generalized in Lemma 7.18 below.

Since we would like to make the domain of a pseudofunctor $\mathbf{L}$ wider, we adapted this definition of 1-morphisms.

**Definition 7.17.** Let $\mathscr{A}$ and $\mathscr{B}$ be small dg categories. A $\mathscr{B}$-$\mathscr{A}$-bimodule $U$ is said to be *right homotopically projective* if ${}_BU$ is a homotopically projective right dg $\mathscr{A}$-module for all $B \in \mathscr{B}_0$.

**Lemma 7.18.** *Let $U$ be a $\mathscr{B}$-$\mathscr{A}$-bimodule for dg categories $\mathscr{A}$ and $\mathscr{B}$. Then the following are equivalent:*

(1) *$U$ is right homotopically projective.*
(2) *The dg functor $\text{-} \otimes_{\mathscr{B}} U \colon \mathscr{C}_{\mathrm{dg}}(\mathscr{B}) \to \mathscr{C}_{\mathrm{dg}}(\mathscr{A})$ preserves homotopically projective objects.*

*Proof.* (1) $\Rightarrow$ (2). For any $P \in \mathscr{H}_{\mathrm{p}}(\mathscr{B})_0$ and any acyclic object $A \in \mathscr{H}(\mathscr{A})_0$, taking $H^0$ to the isomorphism $\mathscr{C}_{\mathrm{dg}}(\mathscr{A})(P \otimes_{\mathscr{B}} U, A) \cong \mathscr{C}_{\mathrm{dg}}(\mathscr{B})(P, \mathscr{C}_{\mathrm{dg}}(\mathscr{A})(U, A))$, we have

$$\mathscr{H}(\mathscr{A})(P \otimes_{\mathscr{B}} U, A) \cong \mathscr{H}(\mathscr{B})(P, \mathscr{C}_{\mathrm{dg}}(\mathscr{A})(U, A)).$$

The right hand side is 0 because for each $B \in \mathscr{B}_0$, $(\mathscr{C}_{\mathrm{dg}}(\mathscr{A})(U, A))(B)$ becomes acyclic, which is shown by

$$H^i((\mathscr{C}_{\mathrm{dg}}(\mathscr{A})(U, A))(B)) = H^i(\mathscr{C}_{\mathrm{dg}}(\mathscr{A})({}_BU, A)) = \mathscr{H}(\mathscr{A})({}_BU, A[i]) = 0$$

for all $i \in \mathbb{Z}$. Hence $P \otimes_{\mathscr{B}} U \in \mathscr{H}_{\mathrm{p}}(\mathscr{A})$.

(2) $\Rightarrow$ (1). Let $B \in \mathscr{B}_0$. Then by (2), ${}_BU \cong {}_B\mathscr{B} \otimes_{\mathscr{B}} U$ is homotopically projective because so is ${}_B\mathscr{B}$. $\square$

**Definition 7.19.** We furthre define pseudofunctors $\mathbf{L}$ and $\mathscr{D}$ in the diagram

$$\begin{tikzcd}
 & \mathscr{C}(\Bbbk\text{-}\mathbf{dgCat}) \arrow[r, hook] \arrow[d, "\mathrm{st}"] & \Bbbk\text{-}\mathbf{FRB} \arrow[d, "\mathrm{st}"] \\
\Bbbk\text{-}\mathbf{dgCat} \arrow[ur, "\mathscr{C}"] \arrow[r, "\mathscr{H}"] \arrow[dr, "\mathscr{D}"'] & \mathscr{H}(\Bbbk\text{-}\mathbf{dgCat}) \arrow[r, hook] \arrow[d, "\mathbf{L}"] & \Bbbk\text{-}\mathbf{TRI} \\
 & \Bbbk\text{-}\underline{\mathbf{TRI}}^2 &
\end{tikzcd}.$$

To define $\mathbf{L}$, consider a diagram

$$\begin{tikzcd}
\mathscr{H}(\mathscr{A}) \arrow[r, bend left, "E"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{H}(\mathscr{B})
\end{tikzcd}$$

in $\mathscr{H}(\Bbbk\text{-}\mathbf{dgCat})$. We set $\mathbf{L}(\mathscr{H}(\mathscr{A})) := \mathscr{D}(\mathscr{A})$. Since $E(\mathscr{H}_{\mathrm{p}}(\mathscr{A})_0) \subseteq \mathscr{H}_{\mathrm{p}}(\mathscr{B})_0$ by definition of $\mathscr{H}(\Bbbk\text{-}\mathbf{dgCat})$, $E$ restricts to a functor $E| \colon \mathscr{H}_{\mathrm{p}}(\mathscr{A}) \to \mathscr{H}_{\mathrm{p}}(\mathscr{B})$, and we can define $\mathbf{L}(E)$ as the composite $\mathbf{L}(E) := \mathbf{j}_{\mathscr{B}} \circ E| \circ \mathbf{p}_{\mathscr{A}}$ as in the diagram

$$\begin{tikzcd}
\mathscr{H}_{\mathrm{p}}(\mathscr{A}) \arrow[r, "E|"] & \mathscr{H}_{\mathrm{p}}(\mathscr{B}) \arrow[d, "\mathbf{j}_{\mathscr{B}}"] \\
\mathscr{D}(\mathscr{A}) \arrow[u, "\mathbf{p}_{\mathscr{A}}"] \arrow[r, dashed, "\mathbf{L}(E)"'] & \mathscr{D}(\mathscr{B})
\end{tikzcd}.$$

Moreover, using the restriction $\alpha| \colon E| \Rightarrow F|$ of $\alpha$, we define $\mathbf{L}(\alpha)$ by setting $\mathbf{L}(\alpha) := \mathbf{j}_{\mathscr{B}} \circ \alpha| \circ \mathbf{p}_{\mathscr{A}}$.

Then for any functors $\mathscr{H}(\mathscr{A}) \xrightarrow{E} \mathscr{H}(\mathscr{B}) \xrightarrow{E'} \mathscr{H}(\mathscr{C})$ in $\mathscr{H}(\Bbbk\text{-}\mathbf{dgCat})$, we have

$$\mathbf{L}(E') \circ \mathbf{L}(E) = \mathbf{L}(E' \circ E). \tag{7.16}$$

Indeed, since $\mathbf{p}_{\mathscr{B}} \circ \mathbf{j}_{\mathscr{B}} = \mathbb{1}_{\mathscr{H}_{\mathrm{p}}(\mathscr{B})}$ (see (7.13)), we have

$$\begin{aligned}\mathbf{L}(E') \circ \mathbf{L}(E) &= j_{\mathscr{C}} \circ E'| \circ \mathbf{p}_{\mathscr{B}} \circ \mathbf{j}_{\mathscr{B}} \circ E| \circ \mathbf{p}_{\mathscr{A}} \\ &= j_{\mathscr{C}} \circ E'| \circ E| \circ \mathbf{p}_{\mathscr{A}} = \mathbf{L}(E' \circ E).\end{aligned}$$

Also, for each $\mathscr{H}(\mathscr{A}) \in \mathscr{H}(\Bbbk\text{-}\mathbf{dgCat})$, we have a natural isomorphism

$$\eta_{\mathscr{A}} \colon \mathbb{1}_{\mathscr{D}(\mathscr{A})} \Longrightarrow Q_{\mathscr{A}} \circ \sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}} = Q_{\mathscr{A}} \circ \mathbb{1}_{\mathscr{H}(\mathscr{A})} \circ \sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}} = \mathbf{L}(\mathbb{1}_{\mathscr{H}(\mathscr{A})})$$

given by the unit of the adjoint $\sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}} \dashv Q_{\mathscr{A}}$. These natural isomorphisms define a structure of a pseudofunctor for $\mathbf{L}$, and it is easy to check that $\mathbf{L}$ is in fact a pseudofunctor. Finally, we define $\mathscr{D}$ as the composite

$$\mathscr{D} := \mathbf{L} \circ \mathscr{H} \colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\underline{\mathbf{TRI}}^2,$$

which is a pseudofunctor as the composite of pseudofunctors.

Since $\mathbf{L}$ is a pseudofunctor satisfying (7.16), we have the following. This will be used in the proof of Theorem 9.17.

**Lemma 7.20.** *The pseudofunctor $\mathbf{L} \colon \mathscr{H}(\Bbbk\text{-}\mathbf{dgCat}) \to \Bbbk\text{-}\underline{\mathbf{TRI}}^2$ strictly preserves the vertical and the horizontal compositions of 2-morphisms. More precisely, let $E, F, G \colon \mathscr{H}(\mathscr{A}) \to \mathscr{H}(\mathscr{A})$, $E', F' \colon \mathscr{H}(\mathscr{B}) \to \mathscr{H}(\mathscr{C})$, and $\alpha \colon E \Rightarrow F$, $\beta \colon F \Rightarrow G, \gamma \colon E' \Rightarrow F'$ be in $\mathscr{H}(\Bbbk\text{-}\mathbf{dgCat})$. Then we have*

$$\begin{aligned}&\mathbf{L}(\beta \bullet \alpha) = \mathbf{L}(\beta) \bullet \mathbf{L}(\alpha), \text{ and} \\ &\mathbf{L}(\gamma \circ \alpha) = \mathbf{L}(\gamma) \circ \mathbf{L}(\alpha).\end{aligned}$$

*Proof.* Straightforward by (7.13). □

**Definition 7.21.** (1) Define a 2-subcategory $\mathscr{C}_{\mathrm{dg}}(\Bbbk\text{-}\mathbf{dgCat})$ of $\Bbbk\text{-}\mathbf{dgCAT}$ as follows: Objects are the dg categories of the form $\mathscr{C}_{\mathrm{dg}}(\mathscr{A})$ for some $\mathscr{A} \in \Bbbk\text{-}\mathbf{dgCat}_0$, 1-morphisms are the dg functors $F \colon \mathscr{C}_{\mathrm{dg}}(\mathscr{A}) \to \mathscr{C}_{\mathrm{dg}}(\mathscr{B})$ preserving homotopically projective objects with $\mathscr{A}, \mathscr{B} \in \Bbbk\text{-}\mathbf{dgCat}_0$, and 2-morphisms are the dg natural transformations between these 1-morphisms.

(2) By noting the fact that for any $\mathscr{A} \in \Bbbk\text{-}\mathbf{dgCat}_0$, we have

$$\begin{aligned}\mathscr{C}(\mathscr{A})(X, Y) &= Z^0(\mathscr{C}_{\mathrm{dg}}(\mathscr{A})(X, Y)) \\ \mathscr{H}(\mathscr{A})(X, Y) &= H^0(\mathscr{C}_{\mathrm{dg}}(\mathscr{A})(X, Y))\end{aligned}$$

for all objects $X, Y \in \mathscr{C}_{\mathrm{dg}}(\mathscr{A})_0 = \mathscr{C}(\mathscr{A})_0 = \mathscr{H}(\mathscr{A})_0$, we define 2-functors $Z^0$ and $H^0$ in the the following diagram so that it turns out to be strictly commutative:

$$\begin{tikzcd} & \mathscr{C}_{\mathrm{dg}}(\Bbbk\text{-}\mathbf{dgCat}) \arrow[d, "Z^0"] \arrow[dd, bend left, "H^0"] \\ \Bbbk\text{-}\mathbf{dgCat} \arrow[ur, "\mathscr{C}_{\mathrm{dg}}"] \arrow[r, "\mathscr{C}"] \arrow[dr, "\mathscr{H}"'] & \mathscr{C}(\Bbbk\text{-}\mathbf{dgCat}) \arrow[d, "\mathrm{st}"] \\ & \mathscr{H}(\Bbbk\text{-}\mathbf{dgCat}) \end{tikzcd} .$$

For each $\mathscr{C}_{\mathrm{dg}}(\mathscr{A}) \in \mathscr{C}_{\mathrm{dg}}(\Bbbk\text{-}\mathbf{dgCat})_0$, $Z^0(\mathscr{C}_{\mathrm{dg}}(\mathscr{A})) := \mathscr{C}(\mathscr{A})$ and $H^0(\mathscr{C}_{\mathrm{dg}}(\mathscr{A})) := \mathscr{H}(\mathscr{A})$. Next, let $E, F\colon \mathscr{C}_{\mathrm{dg}}(\mathscr{A}) \to \mathscr{C}_{\mathrm{dg}}(\mathscr{B})$ be dg functors in $\mathscr{C}_{\mathrm{dg}}(\Bbbk\text{-}\mathbf{dgCat})$ and $\alpha\colon E \Rightarrow F$ a dg natural transformation. We define $Z^0(E), Z^0(\alpha)$ and $H^0(E), H^0(\alpha)$ as follows. We set

$$(Z^0(E))(M) := E(M), \text{and } (H^0(E))(M) := E(M)$$

for all $M \in \mathscr{C}_{\mathrm{dg}}(\mathscr{A})_0 = \mathscr{C}(\mathscr{A})_0 = \mathscr{H}(\mathscr{A})_0$. For each $M, N \in \mathscr{C}_{\mathrm{dg}}(\mathscr{A})_0$, the dg functor $E$ induces a chain map

$$E(M,N)\colon \mathscr{C}_{\mathrm{dg}}(\mathscr{A})(M,N) \to \mathscr{C}_{\mathrm{dg}}(\mathscr{B})(E(M), E(N)).$$

Using this we set

$$(Z^0(E))(M,N) := Z^0(E(M,N))\colon \mathscr{C}(\mathscr{A})(M,N) \to \mathscr{C}(\mathscr{B})(E(M), E(N)), \text{and}$$
$$(H^0(E))(M,N) := H^0(E(M,N))\colon \mathscr{H}(\mathscr{A})(M,N) \to \mathscr{H}(\mathscr{B})(E(M), E(N)).$$

Finally, by noting that $\alpha_M \in \mathscr{C}(\mathscr{B})(E(M), F(M))$, we set

$$(Z^0(\alpha))_M := \alpha_M \in \mathscr{C}(E(M), F(M)), \text{and}$$
$$(H^0(\alpha))_M := H^0(\alpha_M) \in \mathscr{H}(E(M), F(M))$$

Note here that if $\alpha$ above were just a general derived transformation, then neither $Z^0(\alpha)$ nor $H^0(\alpha)$ is defined. This is a reason why we consider colax functors $X\colon I \to \Bbbk\text{-}\mathbf{dgCat}$ rather than $X\colon I \to \Bbbk\text{-}\mathbf{DGCat}$ below.

**Remark 7.22.** Consider a dg functor $F\colon \mathscr{C}_{\mathrm{dg}}(\mathscr{A}) \to \mathscr{C}_{\mathrm{dg}}(\mathscr{B})$ with $\mathscr{A}, \mathscr{B} \in \Bbbk\text{-}\mathbf{dgCat}_0$. Here, we do not assume the condition $F(\mathscr{H}_{\mathrm{p}}(\mathscr{A})_0) \subseteq \mathscr{H}_{\mathrm{p}}(\mathscr{B})_0$. Even in this case, $F$ induces a triangle functor $H^0(F) : \mathscr{H}(\mathscr{A}) \to \mathscr{H}(\mathscr{B})$, and it is possible to define

$$\mathbf{L}(H^0(F))\colon \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$$

by $\mathbf{L}(H^0(F)) := Q_{\mathscr{B}} \circ H^0(F) \circ \sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}}$ as in the left part of the diagram

$$\begin{tikzcd} \mathscr{H}(\mathscr{A}) \arrow[r, "H^0(F)"] & \mathscr{H}(\mathscr{B}) \arrow[r, "H^0(F')"] \arrow[dd, bend left, "Q_{\mathscr{B}}"] & \mathscr{H}(\mathscr{C}) \arrow[dd, bend left, "Q_{\mathscr{C}}"] \\ \mathscr{H}_{\mathrm{p}}(\mathscr{A}) \arrow[u, "\sigma_{\mathscr{A}}"] & \mathscr{H}_{\mathrm{p}}(\mathscr{B}) \arrow[u, "\sigma_{\mathscr{B}}"] & \mathscr{H}_{\mathrm{p}}(\mathscr{C}) \arrow[u, "\sigma_{\mathscr{C}}"] \\ \mathscr{D}(\mathscr{A}) \arrow[u, "\mathbf{p}_{\mathscr{A}}"] & \mathscr{D}(\mathscr{B}) \arrow[u, "\mathbf{p}_{\mathscr{B}}"] & \mathscr{D}(\mathscr{C}) \arrow[u, "\mathbf{p}_{\mathscr{C}}"] \end{tikzcd} ,$$

although $H^0(F)$ may not in the domain of the pseudofunctor $\mathbf{L}$.

Of course, if $F$ preserves homotopically projective objects, then these two definitions of $\mathbf{L}(H^0(F))$ coincide because we have the restriction $H^0F|\colon \mathscr{H}_{\mathrm{p}}(\mathscr{A}) \to \mathscr{H}_{\mathrm{p}}(\mathscr{B})$ satisfying the commutativity $H^0(F) \circ \sigma_{\mathscr{A}} = \sigma_{\mathscr{B}} \circ H^0(F)|$, and $\mathbf{j}_{\mathscr{B}} = Q_{\mathscr{B}} \circ \sigma_{\mathscr{B}}$. In the following, we simply set $\mathbf{L}(F) := \mathbf{L}(H^0(F))$ (e.g., see Theorem 9.1).

Let $\mathscr{C}_{\mathrm{dg}}(\mathscr{A}) \xrightarrow{F} \mathscr{C}_{\mathrm{dg}}(\mathscr{B}) \xrightarrow{F'} \mathscr{C}_{\mathrm{dg}}(\mathscr{C})$ be dg functors with $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ small dg categories. Then we can definie a natural transformation $\mathbf{L}_{F',F}\colon \mathbf{L}(F')\circ\mathbf{L}(F) \to \mathbf{L}(F' \circ F)$ by

$$\begin{aligned}\mathbf{L}_{F',F} = (Q_{\mathscr{C}} \circ H^0F') \circ \varepsilon_{\mathscr{B}} \circ (H^0F \circ \sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}}) :& \\ \overbrace{Q_{\mathscr{C}} \circ H^0F' \circ \sigma_{\mathscr{B}} \circ \mathbf{p}_{\mathscr{B}}}^{\mathbf{L}(F')} \circ \overbrace{Q_{\mathscr{B}} \circ H^0F \circ \sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}}}^{\mathbf{L}(F)}& \\ \Longrightarrow \overbrace{Q_{\mathscr{C}} \circ (H^0F' \circ \mathbb{1}_{\mathscr{H}(\mathscr{B})} \circ H^0F) \circ \sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}}}^{\mathbf{L}(F'\circ F)},&\end{aligned} \tag{7.17}$$

where $\varepsilon_{\mathscr{B}}\colon (\sigma_{\mathscr{B}} \circ \mathbf{p}_{\mathscr{B}}) \circ Q_{\mathscr{B}} \Rightarrow \mathbb{1}_{\mathscr{H}(\mathscr{B})}$ is the the counit. Then in general, the equality (7.16) does not need to hold. There are following two special cases.

Case 1. In the case that $F$ preserves homotopically projective objects. In this case, the equality (7.16) holds, Indeed,

$$\begin{aligned}\mathbf{L}(F') \circ \mathbf{L}(F) &= Q_{\mathscr{C}} \circ H^0F' \circ \sigma_{\mathscr{B}} \circ \mathbf{p}_{\mathscr{B}} \circ Q_{\mathscr{B}} \circ H^0F \circ \sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}} \\ &= Q_{\mathscr{C}} \circ H^0F' \circ \sigma_{\mathscr{B}} \circ \mathbf{p}_{\mathscr{B}} \circ Q_{\mathscr{B}} \circ \sigma_{\mathscr{B}} \circ (H^0F|) \circ \mathbf{p}_{\mathscr{A}} \\ &= Q_{\mathscr{C}} \circ H^0F' \circ \sigma_{\mathscr{B}} \circ \mathbf{p}_{\mathscr{B}} \circ \mathbf{j}_{\mathscr{B}} \circ (H^0F|) \circ \mathbf{p}_{\mathscr{A}} \\ &= Q_{\mathscr{C}} \circ H^0F' \circ \sigma_{\mathscr{B}} \circ (H^0F|) \circ \mathbf{p}_{\mathscr{A}} \\ &= Q_{\mathscr{C}} \circ H^0F' \circ H^0F \circ \sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}} = \mathbf{L}(F' \circ F).\end{aligned}$$

Case 2. In the case that $F'$ preserves acyclic objects (hence preserves quasi-isomorphisms). In this case, $\mathbf{L}_{F',F}\colon \mathbf{L}(F')\circ\mathbf{L}(F) \xrightarrow{\sim} \mathbf{L}(F'\circ F)$ is an isomorphism because for each object $M \in \mathscr{D}(\mathscr{A})$, we have

$$(\mathbf{L}_{F',F})_M = Q_{\mathscr{C}}(H^0F'(\varepsilon_{\mathscr{B},H^0F(\sigma_{\mathscr{A}}(\mathbf{p}_{\mathscr{A}}(M)))})),$$

which turns out to be an isomorphism.

To make the remark above more precisely, we introduce the following three 2-categories $\tilde{\mathscr{C}_{\mathrm{dg}}}(\Bbbk\text{-}\mathbf{dgCat})$, $\tilde{\mathscr{C}}(\Bbbk\text{-}\mathbf{dgCat})$ and $\tilde{\mathscr{H}}(\Bbbk\text{-}\mathbf{dgCat})$ that have more 1-morphisms than $\mathscr{C}_{\mathrm{dg}}(\Bbbk\text{-}\mathbf{dgCat})$, $\mathscr{C}(\Bbbk\text{-}\mathbf{dgCat})$ and $\mathscr{H}(\Bbbk\text{-}\mathbf{dgCat})$ have, resplectively.

**Definition 7.23.** We define 2-categories $\tilde{\mathscr{C}_{\mathrm{dg}}}(\Bbbk\text{-}\mathbf{dgCat})$, $\tilde{\mathscr{C}}(\Bbbk\text{-}\mathbf{dgCat})$ and $\tilde{\mathscr{H}}(\Bbbk\text{-}\mathbf{dgCat})$ as follows:

- $\tilde{\mathscr{C}_{\mathrm{dg}}}(\Bbbk\text{-}\mathbf{dgCat})_0 := \{\mathscr{C}_{\mathrm{dg}}(\mathscr{A}) \mid \mathscr{A} \in \Bbbk\text{-}\mathbf{dgCat}_0\}$.
- For any objects $\mathscr{C}_{\mathrm{dg}}(\mathscr{A}), \mathscr{C}_{\mathrm{dg}}(\mathscr{B})$ of $\tilde{\mathscr{C}_{\mathrm{dg}}}(\Bbbk\text{-}\mathbf{dgCat})$, 1-morphisms from $\mathscr{C}_{\mathrm{dg}}(\mathscr{A})$ to $\mathscr{C}_{\mathrm{dg}}(\mathscr{B})$ are the dg functors $F\colon \mathscr{C}_{\mathrm{dg}}(\mathscr{A}) \to \mathscr{C}_{\mathrm{dg}}(\mathscr{B})$ with $\mathscr{A}, \mathscr{B} \in \Bbbk\text{-}\mathbf{dgCat}_0$ (here we do not assume that they satisfy (7.15)).

- $\widetilde{\mathscr{C}}(\Bbbk\text{-}\mathbf{dgCat})_0 := \{\mathscr{C}(\mathscr{A}) \mid \mathscr{A} \in \Bbbk\text{-}\mathbf{dgCat}_0\}$.
- For any objects $\mathscr{C}(\mathscr{A}), \mathscr{C}(\mathscr{B})$ of $\widetilde{\mathscr{C}}(\Bbbk\text{-}\mathbf{dgCat})$, 1-morphisms from $\mathscr{C}(\mathscr{A})$ to $\mathscr{C}(\mathscr{B})$ are the $\Bbbk$-functors $F\colon \mathscr{C}(\mathscr{A}) \to \mathscr{C}(\mathscr{B})$.
- $\widetilde{\mathscr{H}}(\Bbbk\text{-}\mathbf{dgCat})_0 := \{\mathscr{H}(\mathscr{A}) \mid \mathscr{A} \in \Bbbk\text{-}\mathbf{dgCat}_0\}$.
- For any objects $\mathscr{H}(\mathscr{A}), \mathscr{H}(\mathscr{B})$ of $\widetilde{\mathscr{H}}(\Bbbk\text{-}\mathbf{dgCat})$), 1-morphisms from $\mathscr{H}(\mathscr{A})$ to $\mathscr{H}(\mathscr{B})$ are the $\Bbbk$-functors of the form $\underline{F}\,(:= \mathrm{st}(F))$ for some $\Bbbk$-functors $F\colon \mathscr{C}(\mathscr{A}) \to \mathscr{C}(\mathscr{B})$.
- In each 2-category, the 2-morphisms are the natural transformations between those 1-morphisms.

The following makes up Remark 7.22.

**Proposition 7.24.** *The 2-functor $H^0\colon \mathscr{C}_{\mathrm{dg}}(\Bbbk\text{-}\mathbf{dgCat}) \to \mathscr{H}(\Bbbk\text{-}\mathbf{dgCat})$ is extended to a 2-functor $H^0\colon \widetilde{\mathscr{C}_{\mathrm{dg}}}(\Bbbk\text{-}\mathbf{dgCat}) \to \widetilde{\mathscr{H}}(\Bbbk\text{-}\mathbf{dgCat})$, and the pseudofunctor $\mathbf{L}\colon \mathscr{H}(\Bbbk\text{-}\mathbf{dgCat}) \to \Bbbk\text{-}\underline{\mathbf{TRI}}^2$ is extended to a lax functor $\mathbf{L}\colon \widetilde{\mathscr{H}}(\Bbbk\text{-}\mathbf{dgCat}) \to \Bbbk\text{-}\underline{\mathbf{TRI}}^2$.*

*Proof.* The first assertion is straightforward. For the second assertion, we summarize the structures of the lax functor $\mathbf{L}$. For dg functors $\mathscr{C}_{\mathrm{dg}}(\mathscr{A}) \xrightarrow{F} \mathscr{C}_{\mathrm{dg}}(\mathscr{B}) \xrightarrow{F'} \mathscr{C}_{\mathrm{dg}}(\mathscr{C})$ with $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ small dg categories. The structure morphism $\mathbf{L}_{F',F}\colon \mathbf{L}(F') \circ \mathbf{L}(F) \to \mathbf{L}(F' \circ F)$ is given in (7.17). Also, for each $\mathscr{H}(\mathscr{A}) \in \widetilde{\mathscr{H}}(\Bbbk\text{-}\mathbf{dgCat})$, we have a natural isomorphism

$$\eta_{\mathscr{A}}\colon \mathbb{1}_{\mathscr{D}(\mathscr{A})} \Longrightarrow Q_{\mathscr{A}} \circ \sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}} = Q_{\mathscr{A}} \circ H^0(\mathbb{1}_{\mathscr{C}_{\mathrm{dg}}(A)}) \circ \sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}} = \mathbf{L}(\mathbb{1}_{\mathscr{C}_{\mathrm{dg}}(\mathscr{A})})$$

given by the unit of the adjoint $\sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}} \dashv Q_{\mathscr{A}}$, which defines the remaining structure of the lax functor $\mathbf{L}$. It is easy to check that $\mathbf{L}$ is in fact a lax functor which satisfies the axioms:

(i) For each $a\colon i \to j$ in $I$ the following are commutative:

$$\begin{tikzcd}
\mathbf{L}F\mathbb{1}_{\mathscr{D}(\mathscr{A})} \arrow[r, Rightarrow, "\mathbf{L}F\circ\eta_{\mathscr{A}}"] \arrow[dr, equal] & \mathbf{L}F\mathbf{L}(\mathbb{1}) \arrow[d, Rightarrow, "\mathbf{L}_{F,1}"] \\
& \mathbf{L}(F\mathbb{1})
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{1}_{\mathscr{D}(\mathscr{A})}\mathbf{L}F \arrow[r, Rightarrow, "\eta_{\mathscr{A}}\circ\mathbf{L}F"] \arrow[dr, equal] & \mathbf{L}(\mathbb{1})\mathbf{L}F \arrow[d, Rightarrow, "\mathbf{L}_{1,F}"] \\
& \mathbf{L}(\mathbb{1}F)
\end{tikzcd}
\quad ; \text{ and}$$

$$\begin{aligned}
\mathbf{L}_{F,1} = (Q_{\mathscr{C}} \circ H^0F) \circ \varepsilon_{\mathscr{B}} \circ (H^0\mathbb{1} \circ \sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}}) :\ 
&\overbrace{Q_{\mathscr{C}} \circ H^0F \circ \sigma_{\mathscr{B}} \circ \mathbf{p}_{\mathscr{B}}}^{\mathbf{L}(F)} \circ \overbrace{Q_{\mathscr{B}} \circ H^0\mathbb{1} \circ \sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}}}^{\mathbf{L}(\mathbf{1})} \\
&\Longrightarrow \overbrace{Q_{\mathscr{C}} \circ (H^0F \circ \mathbb{1}_{\mathscr{H}(\mathscr{B})} \circ H^0\mathbb{1}) \circ \sigma_{\mathscr{A}} \circ \mathbf{p}_{\mathscr{A}}}^{\mathbf{L}(F\mathbf{1})},
\end{aligned}$$

(ii) For dg functors $\mathscr{C}_{\mathrm{dg}}(\mathscr{A}) \xrightarrow{F} \mathscr{C}_{\mathrm{dg}}(\mathscr{B}) \xrightarrow{F'} \mathscr{C}_{\mathrm{dg}}(\mathscr{C}) \xrightarrow{F''} \mathscr{C}_{\mathrm{dg}}(\mathscr{E})$ with $\mathscr{A}, \mathscr{B}, \mathscr{C}$ and $\mathscr{E}$ small dg categories. The following is commutative:

$$\begin{tikzcd}
\mathbf{L}(F'') \circ \mathbf{L}(F') \circ \mathbf{L}(F) \arrow[r, Rightarrow, "\mathbf{L}(F'')\circ \mathbf{L}_{F',F}"] \arrow[d, Rightarrow, "\mathbf{L}_{F'',F'}\circ \mathbf{L}F"'] & \mathbf{L}F'' \circ \mathbf{L}(F' \circ F) \arrow[d, Rightarrow, "\mathbf{L}_{F'',(F',F)}"] \\
\mathbf{L}(F'' \circ F') \circ \mathbf{L}F \arrow[r, Rightarrow, "\mathbf{L}_{(F'',F'),F}"'] & \mathbf{L}(F'' \circ F' \circ F).
\end{tikzcd}$$

□

**Definition 7.25.** (1) The pseudofunctor $\mathscr{H} \colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{TRI}$ restricts to a pseudofunctor $\mathscr{H}_{\mathrm{p}} \colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{TRI}$ sending $\mathscr{A}$ to $\mathscr{H}_{\mathrm{p}}(\mathscr{A})$ for all $\mathscr{A} \in \Bbbk\text{-}\mathbf{dgCat}_0$.

(2) For each $\mathscr{A} \in \Bbbk\text{-}\mathbf{dgCat}$, we define $\mathrm{per}(\mathscr{A})$ to be the smallest full triangulated subcategory of $\mathscr{D}(\mathscr{A})$ closed under direct summands (and isomorphisms), and containing the representable functors $\mathscr{A}(\text{-}, M)$ for all $M \in \mathscr{A}_0$. Hence note that the class of objects of $\mathrm{per}(\mathscr{A})$ coincides with the class of compact objects in $\mathscr{D}(\mathscr{A})$ by a theorem explained in [28, Sect. 5]. $\mathrm{per}(\mathscr{A})$ is called the *perfect derived category* of $\mathscr{A}$. Then the pseudofunctor $\mathscr{D} \colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\underline{\mathbf{TRI}}^2$ restricts to a pseudofunctor $\mathrm{per} \colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\underline{\mathbf{TRI}}^2$ sending $\mathscr{A}$ to $\mathrm{per}(\mathscr{A})$ for all $\mathscr{A} \in \Bbbk\text{-}\mathbf{dgCat}_0$.

We cite the following theorem from [8], which is a useful tool to define new colax functors from an old one by composing with pseudofunctors.

**Theorem 7.26.** *Let $\mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ be 2-categories and $V \colon \mathbf{C} \to \mathbf{D}$ a pseudofunctor. Then the obvious correspondence*

$$\mathrm{Colax}(\mathbf{B}, V) \colon \mathrm{Colax}(\mathbf{B}, \mathbf{C}) \to \mathrm{Colax}(\mathbf{B}, \mathbf{D})$$

*turns out to be a pseudofunctor.*

**Corollary 7.27.** *Let $X \colon I \to \Bbbk\text{-}\mathbf{dgCat}$ be a colax functor. Then the following are colax functors again*

*The dg colax functor of $X$ :* $\mathscr{C}_{\mathrm{dg}}(X) := \mathscr{C}_{\mathrm{dg}} \circ X \colon I \to \Bbbk\text{-}\mathbf{DGCAT}$,

*The complex colax functor of $X$ :* $\mathscr{C}(X) := \mathscr{C} \circ X \colon I \to \Bbbk\text{-}\mathbf{FRB}$,

*The homotopy colax functor of $X$ :* $\mathscr{H}(X) := \mathscr{H} \circ X \colon I \to \Bbbk\text{-}\mathbf{TRI}$,

*The homotopically projecitve colax functor of $X$ :* $\mathscr{H}_{\mathrm{p}}(X) := \mathscr{H}_{\mathrm{p}} \circ X \colon I \to \Bbbk\text{-}\mathbf{TRI}$,

*The derived colax functor of $X$ :* $\mathscr{D}(X) := \mathscr{D} \circ X \colon I \to \Bbbk\text{-}\underline{\mathbf{TRI}}^2$, *and*

*The perfect derived colax functor of $X$ :* $\mathrm{per}(X) := \mathrm{per} \circ X \colon I \to \Bbbk\text{-}\underline{\mathbf{TRI}}^2$.

**Remark 7.28.** Let $X = (X, X_i, X_{b,a}) \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$.

(1) An explicit description of the complex colax functor

$$\mathscr{C}(X) := \mathscr{C} \circ X = (\mathscr{C}(X), \mathscr{C}(X)_i, \mathscr{C}(X)_{b,a}) \colon I \to \Bbbk\text{-}\mathbf{FRB}$$

of $X$ is given as follows.

- for each $i \in I_0$, $\mathscr{C}(X)(i) = \mathscr{C}(X(i))$; and
- for each $a\colon i \to j$ in $I$, the functor $\mathscr{C}(X)(a)\colon \mathscr{C}(X)(i) \to \mathscr{C}(X)(j)$ is given by $\mathscr{C}(X)(a) = \text{-}\otimes_{X(i)}\overline{X(a)}$, where $\overline{X(a)}$ is the $X(i)$-$X(j)$-bimodule $\overline{X(a)} := {}_{X(a)}X(j)$ (Notation 7.7 (3)).

(2) An explicit description of the derived colax functor $\mathscr{D}(X)\colon I \to \Bbbk\text{-}\underline{\mathbf{TRI}}^2$ of $X$ is as follows.

- for each $i \in I_0$, $\mathscr{D}(X)(i) = \mathscr{D}(X(i))$; and
- For each $a\colon i \to j$ in $I$, $\mathscr{D}(X)(a)\colon \mathscr{D}(X)(i) \to \mathscr{D}(X)(j)$ is given by

$$\text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)} := \mathbf{L}(\text{-} \otimes_{X(i)} \overline{X(a)})\colon \mathscr{D}(X(i)) \to \mathscr{D}(X(j)).$$

Note that by the remark in Definition 7.25 (2), $\mathrm{per}(X)$ is a colax subfunctor of $\mathscr{D}(X)$.

**Remark 7.29.** Let $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}_0$. Then it is obvious by definitions that

$$\Delta(\mathscr{H}_{\mathrm{p}}(\mathscr{C})) = \mathscr{H}_{\mathrm{p}}(\Delta(\mathscr{C})) \quad \text{and} \quad \Delta(\mathrm{per}(\mathscr{C})) = \mathrm{per}(\Delta(\mathscr{C})).$$

**Proposition 7.30.** *The pseudofunctor* per *preserves $I$-precoverings, that is, if $(F,\psi)\colon X \to \Delta(\mathscr{C})$ is an $I$-precovering in* $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$ *with* $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}_0$, *then so is*

$$\mathrm{per}(F,\psi)\colon\ \mathrm{per}(X) \to \Delta(\mathrm{per}(\mathscr{C}))$$

*in* $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$.

*Proof.* Let $i, j \in I_0$ and $M \in (\mathrm{per}\, X(i))_0, N \in (\mathrm{per}\, X(j))_0$. It suffices to show that $\mathrm{per}(F,\psi)$ induces an isomorphism

$$\mathrm{per}(F,\psi)^{(1)}_{M,N}\colon \coprod_{a\in I(i,j)} \mathrm{per}\, X(j)(M\overset{\mathbf{L}}{\otimes}_{X(i)}\overline{X(a)}, N) \to \mathrm{per}\,\mathscr{C}(M\overset{\mathbf{L}}{\otimes}_{X(i)}\overline{F(i)}, N\overset{\mathbf{L}}{\otimes}_{X(j)}\overline{F(j)}).$$

By assumption, $(F,\psi)$ induces an isomorphism $(F,\psi)^{(1)}_{x,y}\colon \coprod_{a\in I(i,j)} X(j)(X(a)x, y) \to \mathscr{C}(F(i)x, F(j)y)$ for all $x \in X(i)_0, y \in X(j)_0$. Namely,

$$(F,\psi)^{(1)}\colon \coprod_{a\in I(i,j)} X(j)_{X(a)} \to {}_{F(j)}\mathscr{C}_{F(i)}$$

is a morphism of $X(j)$-$X(i)$-bimodules. We first show the following.

**Claim.** *There exists an isomorphism*

$$\mathbf{R}\,\mathrm{Hom}_{\mathscr{C}}(\overline{F(i)}, N \overset{\mathbf{L}}{\otimes}_{X(j)} \overline{F(j)}) \to \coprod_{a\in I(i,j)} \mathbf{R}\,\mathrm{Hom}_{X(j)}(\overline{X(a)}, N)).$$

Indeed, this is given by the composite of the following isomorphisms:

$$
\begin{aligned}
\mathbf{R}\operatorname{Hom}_{\mathscr{C}}(\overline{F(i)}, N \overset{\mathbf{L}}{\otimes}_{X(j)} \overline{F(j)}) &= \mathbf{R}\operatorname{Hom}_{\mathscr{C}}({}_{F(i)}\mathscr{C}, N \overset{\mathbf{L}}{\otimes}_{X(j)} {}_{F(j)}\mathscr{C}) \\
&\overset{\text{(a)}}{\to} N \overset{\mathbf{L}}{\otimes}_{X(j)} {}_{F(j)}\mathscr{C}_{F(i)} \\
&\overset{\text{(b)}}{\to} N \overset{\mathbf{L}}{\otimes}_{X(j)} \coprod_{a\in I(i,j)} X(j)_{X(a)} \\
&\overset{\text{(c)}}{\to} \coprod_{a\in I(i,j)} N \overset{\mathbf{L}}{\otimes}_{X(j)} X(j)_{X(a)} \\
&\overset{\text{(d)}}{\to} \coprod_{a\in I(i,j)} N_{X(a)} \\
&\overset{\text{(e)}}{\to} \coprod_{a\in I(i,j)} \mathbf{R}\operatorname{Hom}_{X(j)}({}_{X(a)}X(j), N) \\
&= \coprod_{a\in I(i,j)} \mathbf{R}\operatorname{Hom}_{X(j)}(\overline{X(a)}, N),
\end{aligned}
$$

where (a) is obtained by the Yoneda lemma, (b) is an isomorphism, induced from $((F,\psi)^{(1)})^{-1}$, (c) is the natural isomorphism induced by the cocontinuity of the tensor product, (d) comes from the property of the tensor product, and (e) is given by the Yoneda lemma. Now, it is not hard to verify the commutativity of the following diagram:

$$
\begin{tikzcd}
\coprod\limits_{a\in I(i,j)} \operatorname{per} X(j)(M \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}, N) \arrow[r, "\operatorname{per}(F,\psi)^{(1)}_{M,N}"] \arrow[d, "\text{(a)}"', "\simeq"] & \operatorname{per}\mathscr{C}(M \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{F(i)}, N \overset{\mathbf{L}}{\otimes}_{X(j)} \overline{F(j)}) \arrow[d, "\text{(b)}"', "\simeq"] \\
\coprod\limits_{a\in I(i,j)} \operatorname{per} X(i)(M, \mathbf{R}\operatorname{Hom}_{X(j)}(\overline{X(a)}, N)) \arrow[d, "\text{(c)}"', "\simeq"] & \operatorname{per} X(i)(M, \mathbf{R}\operatorname{Hom}_{\mathscr{C}}(\overline{F(i)}, N \overset{\mathbf{L}}{\otimes}_{X(j)} \overline{F(j)})) \arrow[dl, "\simeq", "\text{(d)}"'] \\
\operatorname{per} X(i)(M, \coprod\limits_{a\in I(i,j)} \mathbf{R}\operatorname{Hom}_{X(j)}(\overline{X(a)}, N)), &
\end{tikzcd}
$$

where the isomorphisms (a) and (b) are given by adjoints, and (c) is the natural morphism, which is an isomorphism because $M$ is compact, and (d) is an isomorphism given by the claim above. Hence $\operatorname{per}(F,\psi)^{(1)}_{M,N}$ is an isomorphism. □

**Proposition 7.31.** *The homotopically projecitve colax functor $\mathscr{H}_{\mathrm{p}}$ induce a colax funtor*

$$\mathscr{H}_{\mathrm{p}}(P,\psi)\colon\ \mathscr{H}_{\mathrm{p}}(X) \to \Delta(\mathscr{H}_{\mathrm{p}}(\textstyle\int X))$$

*in* $\operatorname{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$.

*Proof.* For each $i \in I_0$, if $U$ is in $\mathscr{H}_{\mathrm{p}}(X(i))$, then

$$\mathscr{H}_{\mathrm{p}}(P(i))(U) \cong U \otimes_{X(i)} \overline{P(i)} = U \otimes_{X(i)} (\textstyle\int X)(\text{-}, P(i)(?))$$

Therefore, if $Y$ is an acyclic $\int X$-module,

$$\begin{aligned}\mathscr{H}(\textstyle\int X)(\mathscr{H}_{\mathrm{p}}(P(i))(U), Y) &\cong \mathscr{H}(\textstyle\int X)(U \otimes_{X(i)} (\textstyle\int X)(\text{-}, P(i)(?)), Y)\\ &\cong \mathscr{H}(X(i))(U, \mathscr{H}(\textstyle\int X)((\textstyle\int X)(\text{-}, P(i)(?)), Y)) = 0.\end{aligned}$$

Then $\mathscr{H}_{\mathrm{p}}(P(i))(U)$ is in $\mathscr{H}_{\mathrm{p}}(\int X)$. For each $a\colon i \to j$ in $I_1$, the dg natural transformation $\phi(a)\colon P(i) \Rightarrow P(j)X(a)$

$$\begin{tikzcd} X(i) \arrow[r, "P(i)"] \arrow[d, "X(a)"'] & \int X \arrow[d, equal] \arrow[dl, Rightarrow, "\phi(a)"'] \\ X(j) \arrow[r, "P(j)"'] & \int X \end{tikzcd}$$

is defined by $\phi(a)x := (\delta_{b,a}\mathbb{1}_{X(a)x})_{b\in I(i,j)}$ for all $x \in X(i)_0$, then we have the following diagram

$$\begin{tikzcd} \mathscr{H}_{\mathrm{p}}(X(i)) \arrow[r, "\mathscr{H}_{\mathrm{p}}(P(i))"] \arrow[d, "\mathscr{H}_{\mathrm{p}}(X(a))"'] & \mathscr{H}_{\mathrm{p}}(\int X) \arrow[d, equal] \arrow[dl, Rightarrow, "\mathscr{H}_{\mathrm{p}}(\phi(a))"'] \\ \mathscr{H}_{\mathrm{p}}(X(j)) \arrow[r, "\mathscr{H}_{\mathrm{p}}(P(j))"'] & \mathscr{H}_{\mathrm{p}}(\int X). \end{tikzcd}$$

□

**Definition 7.32** (Quasi-equivalences [30]). Let $\mathscr{A}, \mathscr{B}$ be small dg categories and $E\colon \mathscr{A} \to \mathscr{B}$ a dg functor. Then $E$ is called a *quasi-equivalence* if

1. The restriction $E_{X,Y}\colon \mathscr{A}(X,Y) \to \mathscr{B}(E(X), E(Y))$ of $E$ to $\mathscr{A}(X,Y)$ is a quasi-isomorphism for all $X, Y \in \mathscr{A}_0$; and
2. The induced functor $H^0(E) : H^0(\mathscr{A}) \to H^0(\mathscr{B})$ is an equivalence.

**Remark 7.33.** By definition, it is clear that the relation defined by quasi-equivalence is reflexive and transitive, but it is known to be not symmetric.

**Definition 7.34.** For a triangulated category $\mathscr{U}$ and a class of objects $\mathscr{V}$ in $\mathscr{U}$, we denote by $\mathrm{thick}_{\mathscr{U}}\,\mathscr{V}$ (resp. $\mathrm{Loc}_{\mathscr{U}}\,\mathscr{V}$) the smallest full triangulated subcategory of $\mathscr{U}$ closed under direct summands (resp. infinite direct sums) that contains $\mathscr{V}$.

Let $\mathscr{A}$ be a small dg category, and $\mathscr{T}$ a full dg subcategory of $\mathscr{C}_{\mathrm{dg}}(\mathscr{A})$. Then $\mathscr{T}$ is called a *tilting dg subcategory* for $\mathscr{A}$, if

1. $\mathscr{T}_0 \subseteq \mathrm{per}(\mathscr{A})_0 \bigcap \mathscr{H}_{\mathrm{p}}(\mathscr{A})_0$, (hence every $T \in \mathrm{per}(\mathscr{A})_0$ is a compact object in $\mathscr{D}(\mathscr{A})$[4]); and
2. $\mathrm{thick}_{\mathscr{D}(\mathscr{A})}(\mathscr{T}_0) = \mathrm{per}(\mathscr{A})$ (equivalently $\mathrm{Loc}_{\mathscr{D}(\mathscr{A})}(\mathscr{T}) = \mathscr{D}(\mathscr{A})$).

[4]Note that an object $T$ of $\mathscr{D}(\mathscr{A})$ is in $\mathrm{per}(\mathscr{A})$ if and only if $T$ is a compact object in $\mathscr{D}(\mathscr{A})$ by Neeman's theorem [28, Theorem 5.3].

Thus, in Keller's words in [28], $\mathscr{T}$ is tilting if and only if $\mathscr{T}_0$ forms a set of small (= compact) generators for $\mathscr{D}(\mathscr{A})$.

## 8. Derived equivalences of colax functors

In this section, we define necessary terminologies such as 2-quasi-isomorphisms for 2-morphisms, quasi-equivalences for 1-morphisms, and the derived 1-morphism $\mathbf{L}(\dot{F}, \dot{\psi})\colon \mathscr{D}(X) \to \mathscr{D}(X')$ of a 1-morphism $(F, \psi)\colon X \to X'$ between colax functors, and show the fact that the derived 1-morphism of a quasi-equivalence between colax functors $X$, $X'$ turns out to be an equivalence between derived dg module colax functors of $X$, $X'$. Finally, we give definitions of tilting subfunctors and of derived equivalences.

**Definition 8.1.** Let $\mathbf{C}$ be a 2-category and $(F, \psi)\colon X \to X'$ a 1-morphism in the 2-category $\mathrm{Colax}(I, \mathbf{C})$. Then $(F, \psi)$ is called *$I$-equivariant* if for each $a \in I_1$, $\psi(a)$ is a 2-isomorphism in $\mathbf{C}$.

We cite the following without a proof.

**Lemma 8.2** ([7]). *Let $\mathbf{C}$ be a 2-category and $(F, \psi)\colon X \to X'$ a 1-morphism in the 2-category $\mathrm{Colax}(I, \mathbf{C})$. Then $(F, \psi)$ is an equivalence in $\mathrm{Colax}(I, \mathbf{C})$ if and only if*

(1) *For each $i \in I_0$, $F(i)$ is an equivalence in $\mathbf{C}$; and*
(2) *For each $a \in I_1$, $\psi(a)$ is a 2-isomorphism in $\mathbf{C}$ (namely, $(F, \psi)$ is $I$-equivariant).*

To define the notion of 2-quasi-isomorphisms in $\Bbbk$-$\mathbf{dgCat}$ , we need the following statement.

**Lemma 8.3.** *Consider a 2-morphism $\alpha$ in the 2-category $\Bbbk$-$\mathbf{dgCat}$ as in*

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "E"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \end{tikzcd} .$$

*We adapt Notation 7.7 (3), e.g., $\overline{E} := {}_E\mathscr{B}$, $\overline{\alpha} := {}_\alpha\mathscr{B}$ and $\overline{E}^* := \mathscr{B}_E, \overline{\alpha}^* := \mathscr{B}_\alpha$. Note that since $\alpha$ is a dg natural transformation, both $\overline{\alpha}$ and $\overline{\alpha}^*$ are 0-cocyle morphisms by Remark 7.6, and hence it is possible to define $\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{\alpha}$, $H^0{}_{\alpha_x}\mathscr{B}$ and so on. Then the following are equivalent.*

(1) $\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}} \overline{\alpha}$ *in the diagram*

$$\begin{tikzcd}[column sep=huge] \mathscr{D}(\mathscr{A}) \arrow[r, bend left, "\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{E}"] \arrow[r, bend right, "\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{F}"'] \arrow[r, phantom, "\Downarrow\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{\alpha}"] & \mathscr{D}(\mathscr{B}) \end{tikzcd}$$

*is a 2-isomorphism in $\Bbbk$-$\underline{\mathbf{TRI}}^2$.*

(2) *$H^0{}_{\alpha_x}\mathscr{B}\colon {}_{E(x)}\mathscr{B} \to {}_{F(x)}\mathscr{B}$ is a quasi-isomorphism in $\mathscr{H}(\mathscr{B})$ for all $x \in \mathscr{A}_0$.*

(3) *$\overline{\alpha}^* \overset{\mathbf{L}}{\otimes}_{\mathscr{A}}$ - in the diagram*

$$\begin{tikzcd}[column sep=huge] \mathscr{D}(\mathscr{A}^{\mathrm{op}}) \arrow[r, bend left, "\overline{F}^* \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} -"] \arrow[r, bend right, "\overline{E}^* \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} -"'] \arrow[r, phantom, "\Downarrow \overline{\alpha}^* \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} -"] & \mathscr{D}(\mathscr{B}^{\mathrm{op}}) \end{tikzcd}$$

*is a 2-isomorphism in $\Bbbk$-$\underline{\mathbf{TRI}}^2$.*

(4) *$H^0\mathscr{B}_{\alpha_x}\colon \mathscr{B}_{E(x)} \to \mathscr{B}_{F(x)}$ is a quasi-isomorphism in $\mathscr{H}(\mathscr{B}^{\mathrm{op}})$.*

*Proof.* (1) $\Rightarrow$ (2). Let $x \in \mathscr{A}_0$. Note that we have ${}_x\mathscr{A} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} \overline{\alpha} \cong Q_{\mathscr{B}} H^0{}_{\alpha_x}\mathscr{B}$, which is an isomorphism in $\mathscr{D}(\mathscr{B})$ if and only if $H^0{}_{\alpha_x}\mathscr{B}$ is a quasi-isomorphism in $\mathscr{H}(\mathscr{B})$. Hence (2) follows from (1) by applying (1) to the representable functor ${}_x\mathscr{A}$.

(2) $\Rightarrow$ (1). Let $\mathscr{U}$ be the full subcategory of $\mathscr{D}(\mathscr{A})$ consisting of objects $M$ satisfying the condition that $M \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} \overline{\alpha}\colon M \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} \overline{E} \to M \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} \overline{F}$ is an isomorphism. Then by (2) we have ${}_x\mathscr{A} \in \mathscr{U}$ for all $x \in \mathscr{A}_0$. Here, it is easy to show that $\mathscr{U}$ is a triangulated subcategory of $\mathscr{D}(\mathscr{A})$ and that $\mathscr{U}$ is closed under isomorphisms and direct sums with small index sets. Therefore we have $\mathscr{U} = \mathscr{D}(\mathscr{A})$, which means that (1) holds.

(2) $\Rightarrow$ (4). Assume that $H^0{}_{\alpha_x}\mathscr{B}\colon {}_{E(x)}\mathscr{B} \to {}_{F(x)}\mathscr{B}$ is a quasi-isomorphism in $\mathscr{H}(\mathscr{B})$. Then $Q_{\mathscr{B}} H^0{}_{\alpha_x}\mathscr{B}$ is an isomorphism in $\mathscr{D}(\mathscr{B})$. We set $\mathrm{Hom}_{\mathscr{B}}(\cdot, \text{-}) := \mathscr{C}_{\mathrm{dg}}(\mathscr{B})(\cdot, \text{-})$. Then the functor

$$\mathbf{R}\mathscr{C}_{\mathrm{dg}_{\mathscr{B}}}(\cdot, {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}}) := Q_{\mathscr{B}} H^0 \mathscr{C}_{\mathrm{dg}}(\mathscr{B})(\mathbf{p}_{\mathscr{B}}(\cdot), {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}})\colon \mathscr{D}(\mathscr{B}) \to \mathscr{D}(\mathscr{B}^{\mathrm{op}})$$

sends the isomorphism $Q_{\mathscr{B}} H^0{}_{\alpha_x}\mathscr{B}$ to an isomorphism

$$\begin{aligned} \mathbf{R}\mathscr{C}_{\mathrm{dg}_{\mathscr{B}}}({}_{\alpha_x}\mathscr{B}, {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}})\colon \mathbf{R}\mathscr{C}_{\mathrm{dg}_{\mathscr{B}}}&({}_{F(x)}\mathscr{B}, {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}}) \\ &\to \mathbf{R}\mathscr{C}_{\mathrm{dg}_{\mathscr{B}}}({}_{E(x)}\mathscr{B}, {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}}), \end{aligned}$$

in $\mathscr{D}(\mathscr{B}^{\mathrm{op}})$, which is given by

$$\begin{aligned} Q_{\mathscr{B}^{\mathrm{op}}} H^0 \operatorname{Hom}_{\mathscr{B}}({}_{\alpha_x}\mathscr{B}, {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}})\colon\ \operatorname{Hom}_{\mathscr{B}}&({}_{F(x)}\mathscr{B}, {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}}) \\ &\to \operatorname{Hom}_{\mathscr{B}}({}_{E(x)}\mathscr{B}, {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}}), \end{aligned}$$

in $\mathscr{D}(\mathscr{B}^{\mathrm{op}})$ (see Remark 7.21 for $H^0$). By the Yoneda lemma, it is isomorphic to

$$Q_{\mathscr{B}^{\mathrm{op}}} H^0 \mathscr{B}_{\alpha_x}\colon \mathscr{B}_{F(x)} \to \mathscr{B}_{E(x)}$$

and is an isomorphism in $\mathscr{D}(\mathscr{B}^{\mathrm{op}})$. As a consequence, $H^0\mathscr{B}_{\alpha_x}$ is a quasi-isomorphism in $\mathscr{H}(\mathscr{B}^{\mathrm{op}})$.

(4) $\Rightarrow$ (2). This is proved in the same way as in the converse direction.

(3) $\Leftrightarrow$ (4). The same proof for the equivalence (1) $\Leftrightarrow$ (2) works also for this case. $\square$

**Definition 8.4.** Let $E, F\colon \mathscr{A} \to \mathscr{B}$ be 1-morphisms and $\alpha\colon E \Rightarrow F$ a 2-morphism in the 2-category $\Bbbk$-**dgCat**. Then $\alpha$ is called a 2-*quasi-isomorphism* in $\Bbbk$-**dgCat** if one of the statements (1), ..., (4) in Lemma 8.3 holds.

**Remark 8.5.** We can use the condition (2) above to check whether $\alpha$ is a 2-quasi-equivalence. Once it is checked, we can use the property (1).

**Definition 8.6.** Let $(F, \psi)\colon X \to X'$ be a 1-morphism in Colax$(I, \Bbbk\text{-}\mathbf{dgCat})$. Then $(F, \psi)$ is called a *quasi-equivalence* 1-*morphism* if

(1) For each $i \in I_0$, $F(i) : X(i) \to X'(i)$ is a quasi-equivalence; and
(2) For each $a \in I_1$, $\psi(a)$ is a 2-quasi-isomorphism (see Definition 8.4).

See the diagram below to understand the situation:

$$\begin{tikzcd}
X(i) \arrow[r, "F(i)", "\text{q-eq}"'] \arrow[d, "X(a)"'] & X'(i) \arrow[d, "X'(a)"] \arrow[dl, Rightarrow, "\psi(a)"', "\text{2-qis}"] \\
X(j) \arrow[r, "\text{q-eq}", "F(j)"'] & X'(j).
\end{tikzcd}$$

**Remark 8.7.** In the above, consider the condition

($2'$) For each $a \in I_1$, $\psi(a)$ is a 2-isomorphiusm.

Then obviously ($2'$) implies (2). Therefore, a 1-morphism $(F, \psi)$ satisfying (1) and ($2'$) can be called an $I$-*equivariant quasi-equivalence* 1-*morphism*.

**Definition 8.8.** Let $X, X' \in \text{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$, and $(\mathbf{F}, \boldsymbol{\psi})\colon \mathscr{C}_{\text{dg}}(X) \to \mathscr{C}_{\text{dg}}(X')$ be in Colax$(I, \Bbbk\text{-}\mathbf{dgCAT})$. Then we define a 1-morphism

$$\mathbf{L}(\mathbf{F}, \boldsymbol{\psi}) := (\mathbf{LF}, \mathbf{L}\boldsymbol{\psi})\colon \mathscr{D}(X') \to \mathscr{D}(X)$$

in $\Bbbk$-$\underline{\mathbf{TRI}}^2$ as follows. For each $a\colon i \to j$ in $I$, $(\mathbf{LF})(i) := \mathbf{L}(\mathbf{F}(i))$, and $(\mathbf{L}\boldsymbol{\psi})(a)$ is defined by the following commutative diagram:

$$\begin{tikzcd}
(\text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X'(a)}) \circ \mathbf{LF}(i) \arrow[r, Rightarrow, "(\mathbf{L}\boldsymbol{\psi})(a)"] \arrow[d, Rightarrow] & \mathbf{LF}(j) \circ (\text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}) \arrow[d, equal] \\
\mathbf{L}((\text{-} \otimes_{X(i)} \overline{X'(a)} \circ \mathbf{F}(i)) \arrow[r, Rightarrow, "\mathbf{L}(\boldsymbol{\psi}(a))"] & \mathbf{L}(\mathbf{F}(j) \circ (\text{-} \otimes_{X(i)} \overline{X(a)})),
\end{tikzcd}$$

where the vertical arrow on the right is the identity (see Case 1 in Remark 7.22), and that on the left is $\mathbf{L}_{(\text{-} \otimes_{X(i)} \overline{X(a)}), \mathbf{F}(i)}$, the structure morphism of the lax functor $\mathbf{L}$ (see Remark 7.22), which turns out to be the identity if $\mathbf{F}(i)$ preserves homotopically projective objects.

**Definition 8.9.** Let $(F, \psi)\colon X \to X'$ be a 1-morhism in Colax$(I, \Bbbk\text{-}\mathbf{dgCat})$. This yields a 1-morphism

$$(\dot{F}, \dot{\psi}) := ((\dot{F}(i))_{i \in I_0}, (\dot{\psi}(a))_{a \in I_1})\colon \mathscr{C}_{\text{dg}}(X) \to \mathscr{C}_{\text{dg}}(X'),$$

in Colax$(I, \Bbbk\text{-}\mathbf{dgCAT})$, which defines a 1-morphism

$$\mathbf{L}(\dot{F}, \dot{\psi}) = \mathscr{D}((F, \psi))\colon \mathscr{D}(X) \to \mathscr{D}(X')$$

in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$. The explicit forms of $(\dot{F}, \dot{\psi})$ and $\mathbf{L}(\dot{F}, \dot{\psi}) := (\mathbf{L}\dot{F}, \mathbf{L}\dot{\psi})$ are given as follows. For each $i \in I_0$, using the dg $X(i)$-$X'(i)$-bimodule $\overline{F(i)} := {}_{F(i)}X'(i)$, we define a dg functor

$$\dot{F}(i) := \text{-} \otimes_{X'(i)} \overline{F(i)} \colon \mathscr{C}_{\mathrm{dg}}(X(i)) \to \mathscr{C}_{\mathrm{dg}}(X'(i)).$$

Note here that the bimodule $\overline{F(i)}$ is right homotopically projective. Hence by Lemma 7.18, $\dot{F}(i)$ preserves homotopically projective objects. This defines a triangle functor

$$(\mathbf{L}\dot{F})(i) := \mathbf{L}(\dot{F}(i)) = \text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{F(i)} \colon \mathscr{D}(X(i)) \to \mathscr{D}(X'(i)).$$

Next let $a \colon i \to j$ be a morphism in $I$. Then $\psi(a) \colon X'(a)F(i) \Rightarrow F(j)X(a)$ induces a morphism of $X'(j)$-$X(i)$-bimodules

$$\overline{\psi(a)} \colon \overline{X'(a)F(i)} \to \overline{F(j)X(a)},$$

where we adapt Notation 7.7 (3), e.g., $\overline{X'(a)F(i)} := {}_{X'(a)F(i)}X'(j)$, which induces the diagram

$$\begin{tikzcd}
(\text{-} \otimes_{X(i)} \overline{F(i)}) \otimes_{X'(i)} \overline{X'(a)} \arrow[r, Rightarrow, dashed, "\dot{\psi}(a)"] \arrow[d, Rightarrow, "\sim"'] & (\text{-} \otimes_{X(i)} \overline{X(a)}) \otimes_{X(j)} \overline{F(j)} \arrow[d, Rightarrow, "\sim"] \\
\text{-} \otimes_{X(i)} \overline{X'(a)F(i)} \arrow[r, Rightarrow, "\text{-} \otimes_{X(i)} \overline{\psi(a)}"'] & \text{-} \otimes_{X(i)} \overline{F(j)X(a)}
\end{tikzcd} \tag{8.18}$$

of 2-morphisms in $\mathscr{C}_{\mathrm{dg}}(\Bbbk\text{-}\mathbf{dgCat})$, where the vertical morphisms are natural isomorphisms. Then $\dot{\psi}(a)$ is defined as the unique 2-morphism making this diagram commutative, which is usually identified with $\text{-} \otimes_{X(i)} \overline{\psi(a)}$. This gives us the diagram

$$\begin{tikzcd}
\mathscr{C}_{\mathrm{dg}}(X(i)) \arrow[r, "\dot{F}(i)"] \arrow[d, "\text{-} \otimes_{X(i)} \overline{X(a)} = \mathscr{C}_{\mathrm{dg}}(X(a))"'] & \mathscr{C}_{\mathrm{dg}}(X'(i)) \arrow[d, "\text{-} \otimes_{X'(i)} \overline{X'(a)} = \mathscr{C}_{\mathrm{dg}}(X'(a))"] \arrow[dl, Rightarrow, "\dot{\psi}(a)"'] \\
\mathscr{C}_{\mathrm{dg}}(X(j)) \arrow[r, "\dot{F}(j)"'] & \mathscr{C}_{\mathrm{dg}}(X'(j)).
\end{tikzcd}$$

and the 1-morphism $(\dot{F}, \dot{\psi}) \colon \mathscr{C}_{\mathrm{dg}}(X) \to \mathscr{C}_{\mathrm{dg}}(X')$. By Lemma 7.20, the pseudo-functor $\mathbf{L} \circ H^0$ sends the diagram (8.18) to the commutative diagram

$$\begin{tikzcd}
(\text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{F(i)}) \overset{\mathbf{L}}{\otimes}_{X'(i)} \overline{X'(a)} \arrow[r, Rightarrow, "\mathbf{L}(\dot{\psi}(a))"] \arrow[d, Rightarrow, "\sim"'] & (\text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}) \overset{\mathbf{L}}{\otimes}_{X(j)} \overline{F(j)} \arrow[d, Rightarrow, "\sim"] \\
\text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X'(a)F(i)} \arrow[r, Rightarrow, "\text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{\psi(a)}"'] & \text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{F(j)X(a)}
\end{tikzcd} \tag{8.19}$$

in $\Bbbk$-$\underline{\mathbf{TRI}}^2$. Using this we set

$$(\mathbf{L}\dot{\psi})(a) := \mathbf{L}(\dot{\psi}(a))\colon (\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} \overline{X'(a)}) \circ \mathbf{L}\dot{F}(i) \Rightarrow \mathbf{L}\dot{F}(j) \circ (\text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}),$$

which gives us the diagram

$$\begin{tikzcd}
\mathscr{D}(X(i)) \arrow[r, "(\mathbf{L}\dot{F})(i)"] \arrow[d, "\text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)} = \mathscr{D}(X(a))"'] & \mathscr{D}(X'(i)) \arrow[d, "\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} \overline{X'(a)} = \mathscr{D}(X'(a))"] \arrow[dl, Rightarrow, "(\mathbf{L}\dot{\psi})(a)"'] \\
\mathscr{D}(X(j)) \arrow[r, "(\mathbf{L}\dot{F})(j)"'] & \mathscr{D}(X'(j)).
\end{tikzcd}$$

and the 1-morphism $(\mathbf{L}\dot{F}, \mathbf{L}\dot{\psi})\colon \mathscr{D}(X) \to \mathscr{D}(X')$.

The following says that a quasi-equivalence between colax functors induces a derived equivlence between them, which will be important for our main result.

**Proposition 8.10.** *Let $(F, \psi)\colon X \to X'$ be a quasi-equivalence 1-morphism in* Colax$(I, \Bbbk$-$\mathbf{dgCat})$. *Then $\mathbf{L}(\dot{F}, \dot{\psi})\colon \mathscr{D}(X) \to \mathscr{D}(X')$ is an equivalence in* Colax$(I, \Bbbk$-$\underline{\mathbf{TRI}}^2)$.

*Proof.* Let $i \in I_0$. Then since $F(i)\colon X(i) \to X'(i)$ is a quasi-equivalence, we have

$$(\mathbf{L}\dot{F})(i) := \text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{F(i)}\colon \mathscr{D}(X(i)) \to \mathscr{D}(X'(i))$$

is an equivalence of triangulated categories by Theorem A.1.

Let $a\colon i \to j$ be a morphism in $I$. Then since

$$\psi(a)\colon X'(a)F(i) \Rightarrow F(j)X(a)$$

is a 2-quasi-isomorphism, $\text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{\psi(a)}$ is a 2-isomorphism by definition. Hence by the commutative diagram (8.19),

$$(\mathbf{L}\dot{\psi})(a)\colon (\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} \overline{X'(a)}) \circ \mathbf{L}\dot{F}(i) \Rightarrow \mathbf{L}\dot{F}(j) \circ (\text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}).$$

is a 2-isomorphism. Therefore, by Lemma 8.2, $\mathbf{L}(\dot{F}, \dot{\psi})$ is an equivalence in Colax$(I, \Bbbk$-$\underline{\mathbf{TRI}}^2)$. □

A dg $\Bbbk$-category $\mathscr{A}$ is called $\Bbbk$-*projective* (resp. $\Bbbk$-*flat*) if $\mathscr{A}(x, y)$ are dg projective (resp. flat) $\Bbbk$-modules for all $x, y \in \mathscr{A}_0$.

**Definition 8.11.** Let $X\colon I \to \Bbbk$-$\mathbf{dgCat}$ be a colax functor.

(1) $X$ is called $\Bbbk$-*projective* (resp. $\Bbbk$-*flat*) if $X(i)$ are $\Bbbk$-projective (resp. $\Bbbk$-flat) for all $i \in I_0$.
(2) Let $Y, Y'\colon I \to \Bbbk$-$\mathbf{dgCAT}$ be colax functors. Then $Y'$ is called a *colax subfunctor* of $Y$ if there exists a *$I$-equivariant inclusion* 1-morphism $Y' \to Y$, namely, a 1-morphism $(\sigma, \rho)\colon Y' \to Y$ such that $\sigma(i)\colon Y'(i) \to Y(i)$ is the inclusion for each $i \in I_0$, and $\rho(a)\colon Y(a)\sigma(i) \Rightarrow \sigma(j)Y'(a)$ is an 2-isomorphism (i.e., a dg natural isomorphism) for each morphism $a\colon i \to j$ in $I$.

(3) A colax subfunctor $\mathscr{T}$ of $\mathscr{C}_{\mathrm{dg}}(X)$ is called a *tilting colax functor* for $X$ if for each $i \in I_0$, $\mathscr{T}(i) \subseteq \mathscr{C}_{\mathrm{dg}}(X(i))$ is a tilting dg subcategory for $X(i)$ (see Definition 7.34). See the diagram below for $(\sigma, \rho)$:

$$\begin{tikzcd}
\mathscr{T}(i) \arrow[r, hook, "\sigma(i)"] \arrow[d, "\mathscr{T}(a)"'] & \mathscr{C}_{\mathrm{dg}}(X(i)) \arrow[d, "\mathscr{C}_{\mathrm{dg}}(X(a))"] \arrow[dl, Rightarrow, "\sim"', "\rho(a)"] \\
\mathscr{T}(j) \arrow[r, hook, "\sigma(j)"'] & \mathscr{C}_{\mathrm{dg}}(X(j)).
\end{tikzcd}$$

**Definition 8.12.** Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then $X$ and $X'$ are said to be *derived equivalent* (we denoted this fact by $X \overset{\mathrm{der}}{\sim} X$) if $\mathscr{D}(X)$ and $\mathscr{D}(X')$ are equivalent in the 2-category $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$. Note by Lemma 8.2 that this is the case if and only if there exists a 1-morphism $(\mathbf{F}, \boldsymbol{\psi}) : \mathscr{D}(X) \to \mathscr{D}(X')$ in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$ such that

(1) For each $i \in I_0$, $\mathbf{F}(i) : \mathscr{D}(X(i)) \to \mathscr{D}(X'(i))$ is a triangle equivalence in $\Bbbk\text{-}\underline{\mathbf{TRI}}^2$; and
(2) For each $a \in I_1$, $\boldsymbol{\psi}(a)$ is a 2-isomorphism in $\Bbbk\text{-}\underline{\mathbf{TRI}}^2$ (i.e., $(\mathbf{F}, \boldsymbol{\psi})$ is $I$-equivariant).

In the next section, we will characterize a derived equivalence between colax functors in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$ given by a left derived functor between dg module categories or given by the left derived tensor functor of a bimodule.

## 9. Characterizations of standard derived equivalences of colax functors

In this section, we define standard derived equivalences between colax functors from $I$ to $\Bbbk$-**dgCat**, and give its characterizations as our first main result in this paper.

We first cite the following from [28, Theorem 8.1] without a proof.

**Theorem 9.1.** *Let $\mathscr{A}$ and $\mathscr{C}$ be small dg categories. Consider the following conditions.*

(1) *There is a dg functor $H : \mathscr{C}_{\mathrm{dg}}(\mathscr{C}) \to \mathscr{C}_{\mathrm{dg}}(\mathscr{A})$ such that $\mathbf{L}H : \mathscr{D}(\mathscr{C}) \to \mathscr{D}(\mathscr{A})$ is an equivalence (see Remark 7.22).*
(2) *$\mathscr{C}$ is quasi-equivalent to a tilting dg subcategory for $\mathscr{A}$.*
(3) *There exists a dg category $\mathscr{B}$ and dg functors*

$$\mathscr{C}_{\mathrm{dg}}(\mathscr{C}) \xrightarrow{G} \mathscr{C}_{\mathrm{dg}}(\mathscr{B}) \xrightarrow{F} \mathscr{C}_{\mathrm{dg}}(\mathscr{A})$$

*such that $\mathbf{L}G$ and $\mathbf{L}F$ are equivalences.*

*Then*

(a) (1) *implies* (2).
(b) (2) *implies* (3).

Next, we cite the statement [28, Theorem 8.2] in the $\Bbbk$-flat case.

**Theorem 9.2** (Keller). *Let $\mathscr{A}$ and $\mathscr{B}$ be small dg $\Bbbk$-categories and assume that $\mathscr{A}$ is $\Bbbk$-flat. Then the following are equivalent.*

(1) *There exists a $\mathscr{B}$-$\mathscr{A}$-bimodue $Y$ such that $\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{B}} Y : \mathscr{D}(\mathscr{B}) \to \mathscr{D}(\mathscr{A})$ is a triangle equivalence.*

(2) *There is a dg functor $H : \mathscr{C}_{\rm dg}(\mathscr{C}) \to \mathscr{C}_{\rm dg}(\mathscr{A})$ such that $\mathbf{L}H : \mathscr{D}(\mathscr{C}) \to \mathscr{D}(\mathscr{A})$ is an equivalence (see Remark 7.22).*

(3) *$\mathscr{B}$ is quasi-equivalent to a tilting dg subcategory for $\mathscr{A}$.*

The following lemma given by Keller characterizes dg bimdoules which induced an equivalences of derived categories.

**Lemma 9.3** ([30, Lemma 3.10]). *Let $\mathscr{A}$ and $\mathscr{B}$ be dg categories and $E$ an $\mathscr{A}$-$\mathscr{B}$ bimodule. Then $\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}} E \colon \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$ is an equivalence of triangulated categories, if and only if*

(1) *the dg $\mathscr{B}$-module $E(\text{-}, A)$ is perfect for all $A \in \mathscr{A}$,*

(2) *the morphism*

$$\mathscr{A}(A, A') \to \mathbf{R}\mathscr{C}_{\rm dg}(\mathscr{B})(E(\text{-}, A), E(\text{-}, A'))$$

*is a quasi-isomorphism for all $A, A' \in \mathscr{A}$ and*

(3) *the dg $\mathscr{B}$-module $E(\text{-}, A)$, $A \in \mathscr{A}$, form a tilting dg subcategory for $\mathscr{B}$ (Definition 7.34).*

**Definition 9.4.** Let $\mathscr{A}$ and $\mathscr{B}$ be dg categories and $E$ an $\mathscr{A}$-$\mathscr{B}$ bimodule. We denote by $\underline{\mathscr{A}}$ the full subcategory of $\mathscr{D}(\mathscr{A})$ with

$$\underline{\mathscr{A}}_0 = \{D \in \mathscr{D}(\mathscr{A}) \mid D \cong C^\wedge \text{ for some } C \in \mathscr{A}_0\}.$$

Then $E$ is called a *quasi-equivalence bimodule* if (a) $\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}} E \colon \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$ is an equivalence of triangulated categories, and (b) it gives rise to an equivalence $\underline{\mathscr{A}} \to \underline{\mathscr{B}}$, that is, $\underline{\mathscr{A}} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} E \subseteq \underline{\mathscr{B}}$.

**Definition 9.5.** The derived equivalence of the form $\mathbf{L}H \colon \mathscr{D}(\mathscr{C}) \to \mathscr{D}(\mathscr{A})$ or $\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{B}} Y \colon \mathscr{D}(\mathscr{C}) \to \mathscr{D}(\mathscr{A})$ above is called a *standard derived equivalence* from $\mathscr{B}$ to $\mathscr{A}$, and if the statements of Theorem 9.2 hold, then we say that $\mathscr{B}$ is *standardly derived equivalent* to $\mathscr{A}$. Here it seems that $\mathscr{A}$ and $\mathscr{B}$ are not symmetric, but in that case, the $\mathscr{A}$-$\mathscr{B}$-bimodule $Y^T := \mathscr{C}_{\rm dg}(\mathscr{A})(Y, \mathscr{A})$ induces a triangle equivalence $\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}} Y^T : \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$. Thus this relation is symmetric for $\mathscr{A}$ and $\mathscr{B}$.

**Definition 9.6.** Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$.

(1) An $X'$-$X$-*bimodule* is a pair $Z = ((Z(i))_{i \in I_0}, (Z(a))_{a \in I_1})$, where $Z(i)$ is an $X'(i)$-$X(i)$-bimodule for all $i \in I_0$, and $Z(a) \colon Z(i) \otimes_{X(i)} \overline{X(a)} \to \overline{X'(a)} \otimes_{X'(j)} Z(j)$ is a 0-cocycle morphism of $X'(i)$-$X(j)$-bimodules for all morphisms $a \colon i \to j$ in $I$, such that

$$\text{-} \otimes_{X'} Z := ((\text{-} \otimes_{X'(i)} Z(i))_{i \in I_0}, (\text{-} \otimes_{X'(\mathrm{dom}(a))} Z(a))_{a \in I_1}) \colon \mathscr{C}_{\rm dg}(X') \to \mathscr{C}_{\rm dg}(X)$$

is a 1-morphism in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$, where $\mathrm{dom}(a) = i$ is the domain of $a \in I_1$, and $\text{-} \otimes_{X'(i)} Z(a)$ is given up to associators (see Definition 7.8), i.e., it is identified with the composite with associators as in the diagram

$$\begin{tikzcd}
(\text{-} \otimes_{X'(i))} Z(i)) \otimes_{X(i)} \overline{X(a)} \arrow[r, dashed] \arrow[d, "\boldsymbol{a}^{-1}_{(\text{-}), Z(i), \overline{X(a)}}"] & (\text{-} \otimes_{X'(i)} \overline{X'(a)}) \otimes_{X'((j)} Z(j) \\
\text{-} \otimes_{X'(i)} (Z(i) \otimes_{X(i)} \overline{X(a)}) \arrow[r, "\text{-} \otimes_{X'(i)} Z(a)"] & \text{-} \otimes_{X'(i)} (\overline{X'(a)} \otimes_{X'((j))} Z(j)) \arrow[u, "\boldsymbol{a}_{(\text{-}), \overline{X'(a)}, Z(j)}"']
\end{tikzcd}.$$

(2) If ${}_B Z(i)$ is a homotopically projective dg $X(i)$-module for all $B \in X'(i)_0$ and $i \in I_0$, then $Z$ is said to be *right homotopically projective*.

(3) A 1-morphism $(F, \psi)\colon X' \to X$ in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$ is said to *preserve homotopically projective objects* if so does $F(i)$ for all $i \in I_0$.

Recall that there is another definition of quasi-equivalences between dg categories by using "quasi-functor" given by Keller. We will give a similar definition in our setting.

**Definition 9.7.** Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$, and $Z$ be a $X'$-$X$-bimodule.

(1) We denote by $\underline{X}$ the colax subfunctor of $\mathscr{D}(X)$ such that $\underline{X}(i)$ is the full subcategory of $\mathscr{D}(X(i))$ with $\underline{X}(i) := \underline{X(i)}$. The structure of $\underline{X}$ is as follows: $X = ((X(i))_{i \in I_0}, (X(a))_{a \in I_1})$, where for each $a\colon i \to j$ in $I$, we have a strict commutative diagram

$$\begin{tikzcd}
\mathscr{D}(X(i)) \arrow[r, "\mathscr{D}(X(a)) = \text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}"] & \mathscr{D}(X(j)) \\
\underline{X}(i) \arrow[r, "\mathscr{D}(X(a)) = \text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}"'] \arrow[u, hook, "\sigma(i)"] & \underline{X}(j) \arrow[u, hook, "\sigma(j)"']
\end{tikzcd},$$

Namely, it is required that for any $C \in X(i)_0$, there exists some $D \in X(j)_0$ such that $C^\wedge \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)} \cong D^\wedge$.

(2) We denote by $\mathbb{Z}\underline{X}$ the colax subfunctor of $\mathscr{D}(X)$ such that $\mathbb{Z}\underline{X}(i)$ is the full subcategory of $\mathscr{D}(X(i))$ with

$$\mathbb{Z}\underline{X}(i)_0 = \{D \in \mathscr{D}(X(i)) \mid D \cong C^\wedge[n] \text{ for some } C \in X(i)_0, n \in \mathbb{Z}\}.$$

(3) The $X'$-$X$-bimodule $Z$ is called a *quasi-functor* if $\text{-} \overset{\mathbf{L}}{\otimes}_{X'} Z\colon \mathscr{D}(X') \to \mathscr{D}(X)$ gives rise to a 1-morphism $\underline{X'} \to \underline{X}$ in the senses that for each $i \in I_0$, we have a strictly commutative diagram

$$\begin{tikzcd}
\mathscr{D}(X'(i)) \arrow[r, "\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} Z(i)"] & \mathscr{D}(X(i)) \\
\underline{X'}(i) \arrow[r, "\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} Z(i)"'] \arrow[u, hook, "\sigma'(i)"] & \underline{X}(i) \arrow[u, hook, "\sigma(i)"']
\end{tikzcd}.$$

Namely, it is required that for any $x \in X'(i)_0$, there exists some $y \in X(i)_0$ such that $Z(\text{-}, x) \cong x^\wedge \overset{\mathbf{L}}{\otimes}_{X'(i)} Z \cong y^\wedge$.

(4) The $X'$-$X$-bimodule $Z$ is called a *quasi-equivalence bimodule* if (a)

$$\text{-}\overset{\mathbf{L}}{\otimes}_{X'} Z \colon \mathscr{D}(X') \to \mathscr{D}(X)$$

is an equivalence in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$, and (b) it gives rise to an equivalence $\underline{X'} \to \underline{X}$. Here in the diagram

$$\begin{tikzcd}
& \mathscr{D}(X'(i)) \arrow[rr, "\text{-}\overset{\mathbf{L}}{\otimes}_{X'(i)} Z(i)"] \arrow[dl, "\mathscr{D}X'(a)"'] & & \mathscr{D}(X(i)) \arrow[dl, "\mathscr{D}X(a)"] \arrow[dlll, Rightarrow, "\text{-}\overset{\mathbf{L}}{\otimes}_{X'(i)} Z(a)"] \\
\mathscr{D}(X'(j)) \arrow[rr] & & \mathscr{D}(X(j)) & \\
& \underline{X'}(i) \arrow[rr] \arrow[dl] \arrow[uu, hook, "\sigma'(i)"'] & & \underline{X}(i) \arrow[dl] \arrow[uu, hook] \arrow[dlll, Rightarrow, "\text{-}\overset{\mathbf{L}}{\otimes}_{X'(i)} Z(a)"] \\
\underline{X'}(j) \arrow[rr] \arrow[uu, hook] & & \underline{X}(j) \arrow[uu, hook, "\sigma(j)"'] &
\end{tikzcd}$$

the condition (b) is equivalent to saying that $\text{-}\overset{\mathbf{L}}{\otimes}_{X'(i)} Z(i) \colon \underline{X'}(i) \to \underline{X}(i)$ is an equivalence for all $i \in I_0$, and that $\text{-}\overset{\mathbf{L}}{\otimes}_{X'(i)} Z(a)$ in the bottom square is a 2-isomorphism for all $a \colon i \to j$ in $I_1$. Note that it also required that $(\text{-}\overset{\mathbf{L}}{\otimes}_{X'(i)} Z(a)) \circ \sigma'(i) = \sigma(j) \circ (\text{-}\overset{\mathbf{L}}{\otimes}_{X'(i)} Z(a))$ for all morphisms $a \colon i \to j$ in $I$ (note that both hand sides are horizontal composites), but this is automatically satisfied [5].

**Remark 9.8.** Note that in Definition 8.6 another notion of *quasi-equivalence* is defined for a 1-morphsim, which we have to distinguish. Any quasi-equivalence 1-morphism $X \to X'$ induces a quasi-equivalence $X$-$X'$-bimdoule by Proposition 8.10.

Indeed, in this proposition, the $X$-$X'$-bimodule $E := (\overline{F(i)}, \overline{\psi(a)})_{i \in I_0, a \in I_1}$ is equal to $\mathbf{L}(\dot{F}, \dot{\psi})$ and is a quasi-equivalence bimodule.

**Lemma 9.9.** *The following are equivalent*

(1) $\text{-}\overset{\mathbf{L}}{\otimes}_{X'} Z \colon \mathscr{D}(X') \to \mathscr{D}(X)$ *is an equivalence in* $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$ *giving rise to an equivalence* $\underline{X'} \to \underline{X}$.

(2) $\text{-}\overset{\mathbf{L}}{\otimes}_{X'} Z$ *gives rise to equivalences* $\mathbb{Z}\underline{X'} \to \mathbb{Z}\underline{X}$ *and* $\underline{X'} \to \underline{X}$.

*In this case, there exists a quasi-equivalence* $X' \to X$.

*Proof.* (1) $\Rightarrow$ (2). This is trivial by the definition of quasi-equivalence.

[5] The left/right faces, and the front/back faces are strictly commutative by (3) and (1), respectively

(2) $\Rightarrow$ (1). Assume that the statement (2) holds. To show that $\text{-}\overset{\mathbf{L}}{\otimes}_{X'} Z\colon \mathscr{D}(X') \to \mathscr{D}(X)$ is an equivalence in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$, it remains that $\text{-}\overset{\mathbf{L}}{\otimes}_{X'(i)} Z(a))$ is a 2-natural isomorphism. It follows that $C^{\wedge}\overset{\mathbf{L}}{\otimes}_{X'(i)} Z(a)$ is invertible for all $C \in X'(i)$ by the assumption.

$\square$

**Lemma 9.10.** *Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$, and $Z$ a $X'$-$X$-bimodule. Then for any $a\colon i \to j$ in $I$, $Z$ naturally induces a 2-morphism*

$$\widehat{Z(a)}\colon (\text{-} \otimes_{X'(i)} \overline{X'(a)}) \circ \mathscr{C}_{\mathrm{dg}}(X(i))(Z(i), \text{-}) \Rightarrow \mathscr{C}_{\mathrm{dg}}(X(j))(Z(j), \text{-}) \circ (\text{-} \otimes_{X(i)} \overline{X(a)}).$$

*Proof.* We first define a 2-morphism

$$\begin{aligned} &\mathscr{C}_{\mathrm{dg}}(X(i))(Z(i), \text{-}) \circ \mathscr{C}_{\mathrm{dg}}(X(j))(\overline{X(a)}, \text{-}) \\ &\qquad\qquad \Rightarrow \mathscr{C}_{\mathrm{dg}}(X'(j))(\overline{X'(a)}, \text{-}) \circ \mathscr{C}_{\mathrm{dg}}(X(j))(Z(j), \text{-}). \end{aligned}$$

This is defined as a composite of the following 2-morphisms

$$\begin{aligned} &\mathscr{C}_{\mathrm{dg}}(X(i))(Z(i), \text{-}) \circ \mathscr{C}_{\mathrm{dg}}(X(j))(\overline{X(a)}, \text{-}) \\ =\ &\mathscr{C}_{\mathrm{dg}}(X(i))(Z(i), \mathscr{C}_{\mathrm{dg}}(X(j))(\overline{X(a)}, \text{-})) \\ \overset{\cong}{\Rightarrow}\ &\mathscr{C}_{\mathrm{dg}}(X(j))(Z(i) \otimes_{X(i)} \overline{X(a)}, \text{-}) \\ \overset{(*)}{\Rightarrow}\ &\mathscr{C}_{\mathrm{dg}}(X(j))(\overline{X'(a)} \otimes_{X'(j)} Z(j), \text{-}) \\ \overset{\cong}{\Rightarrow}\ &\mathscr{C}_{\mathrm{dg}}(X'(j))(\overline{X'(a)}, \mathscr{C}_{\mathrm{dg}}(X(j))(Z(j), \text{-})) \\ =\ &\mathscr{C}_{\mathrm{dg}}(X'(j))(\overline{X'(a)}, \text{-}) \circ \mathscr{C}_{\mathrm{dg}}(X(j))(Z(j), \text{-}), \end{aligned}$$

where $(*)$ stands for $\mathscr{C}_{\mathrm{dg}}(X(j))(M(a), \text{-})$, and the isomorphisms are given by adjunctions.

Then we can define $\widehat{Z(a)}$ as the composite of the following (the second and the third ones are obtained by applying the functors $(\text{-} \otimes_{X'(i)} \overline{X'(a)})$ from the left, and $(\text{-} \otimes_{X(i)} \overline{X(a)})$ from the right to the above.)

$$\begin{aligned} &(\text{-} \otimes_{X'(i)} \overline{X'(a)}) \circ \mathscr{C}_{\mathrm{dg}}(X(i))(Z(i), \text{-}) \circ \mathbb{1} \\ \overset{\text{(a)}}{\Rightarrow} &(\text{-} \otimes_{X'(i)} \overline{X'(a)}) \circ \mathscr{C}_{\mathrm{dg}}(X(i))(Z(i), \text{-}) \circ \mathscr{C}_{\mathrm{dg}}(X(j))(\overline{X(a)}, \text{-}) \circ (\text{-} \otimes_{X(i)} \overline{X(a)}) \\ \overset{\text{(b)}}{\Rightarrow} &(\text{-} \otimes_{X'(i)} \overline{X'(a)}) \circ \mathscr{C}_{\mathrm{dg}}(X'(j))(\overline{X'(a)}, \text{-}) \circ \mathscr{C}_{\mathrm{dg}}(X(j))(Z(j), \text{-}) \circ (\text{-} \otimes_{X(i)} \overline{X(a)}) \\ \overset{\text{(c)}}{\Rightarrow} &\mathbb{1} \circ \mathscr{C}_{\mathrm{dg}}(X(j))(Z(j), \text{-}) \circ (\text{-} \otimes_{X(i)} \overline{X(a)}), \end{aligned}$$

where (a) is given by the unit, (b) is given by the 2-morphism defined above, and (c) is given by the counit. $\square$

**Definition 9.11.** Let $X \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$, and let $\mathscr{U}$ be a colax subfunctor of $\mathscr{D}(X)$ such that $\mathscr{U}(i)$ is a full subcategory of $\mathscr{D}(X(i))$ for all $i \in I_0$.

(1) We denote by $\mathbb{Z}\mathscr{U}$ the colax subfunctor of $\mathscr{D}(X)$ such that $\mathbb{Z}\mathscr{U}(i)$ is the full subcategory of $\mathscr{D}(X(i))$ with $\mathbb{Z}\mathscr{U}(i)_0 = \{U[n] \mid U \in \mathscr{U}(i)_0, n \in \mathbb{Z}\}$.

(2) A *lift* of $\mathscr{U}$ is a a pair $(X', Z)$ of a colax functor $X'$ and an $X'$-$X$-bimodule $Z$ such that $\text{-} \overset{\mathbf{L}}{\otimes}_{X'} Z \colon \mathscr{D}(X') \to \mathscr{D}(X)$ gives rise to equivalences $\mathbb{Z}\underline{X'} \to \mathbb{Z}\mathscr{U}$ and $\underline{X'} \to \mathscr{U}$.

(3) A *standard lift* of $\mathscr{U}$ is a lift $(\mathscr{B}, M)$ of $\mathscr{U}$ constructed as follows: Take $\mathscr{B}$ as the colax subfunctor of $\mathscr{C}_{\text{dg}}(X)$ such that for each $i \in I$, $\mathscr{B}(i)$ is the full subcategory of $\mathscr{C}_{\text{dg}}(X(i))$ with $\mathscr{B}(i)_0 := \{\mathbf{p}_{X(i)}U \mid U \in \mathscr{U}(i)_0\}, \mathscr{B}(a) := \mathscr{C}_{\text{dg}}(X(a))|_{\mathscr{B}(i)}$ for all $a \in I_1$ and $M$ as a $\mathscr{B}$-$X$-bimodule definied by ${}_BM(i)_A := B(A) \cong \mathscr{C}_{\text{dg}}(X(i))(A^\wedge, B)$ for all $B \in \mathscr{B}_0, A \in X(i)_0$, and we define a 0-cocycle morphism

$$M(a) \colon M(i) \otimes_{X(i)} \overline{X(a)} \to \overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} M(j)$$

of $\mathscr{B}(i)$-$X(j)$-bimodules by

$$\begin{aligned} M(a)_{C^\wedge} \colon C^\wedge \otimes_{\mathscr{T}(i)} M(i) \otimes_{X(i)} \overline{X(a)} &\cong M(i)(\text{-}, C) \otimes_{X(i)} \overline{X(a)} \\ &\cong C \otimes_{X(i)} \overline{X(a)} \cong \mathscr{T}(a)(C) \cong C^\wedge \otimes \overline{\mathscr{T}(a)} \otimes M(j) \end{aligned}$$

for all $a \in I_1$ and $C \in \mathscr{B}(i)$.

**Remark 9.12.** The definition of a lift implies that in particular that $\text{-} \overset{\mathbf{L}}{\otimes}_{X'} Z$ induces an equivalence from $\mathscr{H}_{\text{p}}{}^b(X')$ onto the colax subfunctor of $\mathscr{C}_{\text{dg}}(X)$ generated by $\mathscr{U}$, where $\mathscr{H}_{\text{p}}{}^b(X')$ is a colax subfuncor of $\mathscr{H}_{\text{p}}(X')$ such that $\mathscr{H}_{\text{p}}{}^b(X'(i))$ is the smallest strictly full subcategory of $\mathscr{H}_{\text{p}}\,X'(i)$ containing of the $A^\wedge, A \in X'(i)$ for $i \in I$. If $Z(i)_B$ is homotopically projective for each $B \in X'(i)$, a quasi-inverse is induced by $\mathbf{R}\mathscr{C}_{\text{dg}}(X(i))(Z(i), -)$. We now define a 2-isomorphism

$$\widetilde{Z(a)} \colon (\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} \overline{X'(a)}) \circ \mathbf{R}\mathscr{C}_{\text{dg}}(X(i))(Z(i), \text{-}) \Rightarrow \mathbf{R}\mathscr{C}_{\text{dg}}(X(j))(Z(j), \text{-}) \circ (\text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{U}(i)} \mathscr{U}(a))$$

as the converse of the composite of the following 2-isomorphisms

$$\begin{aligned} &\mathbf{R}\mathscr{C}_{\text{dg}}(X(j)(Z(j), \text{-}) \circ \mathscr{U}(a) \\ &\overset{\text{(a)}}{\Rightarrow} \mathbf{R}\mathscr{C}_{\text{dg}}(X(j)(Z(j), \text{-}) \circ \mathscr{U}(a) \circ (\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} Z(i)) \circ \mathbf{R}\mathscr{C}_{\text{dg}}(X(i))(Z(i), \text{-}) \\ &\overset{\text{(b)}}{\Rightarrow} \mathbf{R}\mathscr{C}_{\text{dg}}(X(j)(Z(j), \text{-}) \circ (\text{-} \overset{\mathbf{L}}{\otimes}_{X'(j)} Z(j)) \circ (\text{-} \overset{\mathbf{L}}{\otimes}_{X'(a)} \overline{X(a)}) \circ \mathbf{R}\mathscr{C}_{\text{dg}}(X(i))(Z(i), \text{-}) \\ &\overset{\text{(c)}}{\Rightarrow} (\text{-} \overset{\mathbf{L}}{\otimes}_{X'(a)} \overline{X(a)}) \circ \mathbf{R}\mathscr{C}_{\text{dg}}(X(i))(Z(i), \text{-}), \end{aligned}$$

where (a) and (c) are induced by $\mathbb{1} \Rightarrow (\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} Z(i)) \circ \mathbf{R}\mathscr{C}_{\text{dg}}(X(i))(Z(i), \text{-})$ and $(\text{-} \overset{\mathbf{L}}{\otimes}_{X'(a)} \overline{X(a)} \circ \mathbf{R}\mathscr{C}_{\text{dg}}(X(i))(Z(i), \text{-}) \Rightarrow \mathbb{1}$, respectively, and (b) is given by

$$\mathbf{R}\mathscr{C}_{\text{dg}}(X(j)(Z(j), \text{-}) \circ Z(a) \circ \mathbf{R}\mathscr{C}_{\text{dg}}(X(i))(Z(i), \text{-}).$$

Then we have the following diagram

$$\begin{tikzcd}
\mathscr{H}_{\mathrm{p}}{}^b(X'(i)) \arrow[r, "- \overset{\mathbf{L}}{\otimes}_{X'(i)} Z(i)"] \arrow[d, "\mathscr{H}_{\mathrm{p}}{}^b X'(a)"'] & \mathscr{U}(i) \arrow[r, "\mathbf{R}\mathscr{C}_{\mathrm{dg}}(X(i))(Z(i),-)"] \arrow[d, "\mathscr{U}(a)"] \arrow[dl, phantom, "\swarrow Z(a)"] & \mathscr{H}_{\mathrm{p}}{}^b(X'(i)) \arrow[d, "\mathscr{H}_{\mathrm{p}}{}^b X'(a)"] \arrow[dl, phantom, "\swarrow \widetilde{Z(a)}"] \\
\mathscr{H}_{\mathrm{p}}{}^b(X'(j)) \arrow[r, "- \overset{\mathbf{L}}{\otimes}_{X'(j)} Z(j)"'] & \mathscr{U}(j) \arrow[r, "\mathbf{R}\mathscr{C}_{\mathrm{dg}}(X(j))(Z(j),-)"'] & \mathscr{H}_{\mathrm{p}}{}^b(X'(j))
\end{tikzcd}.$$

We use the following notion to prove Proposition 9.14 below.

**Definition 9.13.** Let $\mathscr{C}$ be a small dg category, and $M$ a (right) dg $\mathscr{C}$-module. Then the *category* $\mathbf{c}(M)$ *of elements* of $M$ is a category defined as follows.

For each $i \in \mathscr{C}_0$, recall that $M(i)$ is a $\mathbb{Z}$-graded vector space $M(i) = \bigoplus_{p\in\mathbb{Z}} M(i)^p$ with a differential $d_{M(i)}$ of degree 1. By forgetting both the $\mathbb{Z}$-graded vector space structure and the differential, we regard $M(i)$ just as a set, and we set

$$\mathbf{c}(M)_0 := \bigsqcup_{j\in\mathscr{C}_0} M(j) = \{j_x := (j,x) \mid j \in \mathscr{C}_0, x \in M(j)\}, \text{and}$$

$$\mathbf{c}(M)_1 := \bigsqcup_{a\in\mathscr{C}_1} M(\mathrm{cod}(a)) = \{a_x := (a,x) \mid a \in \mathscr{C}_1, x \in M(\mathrm{cod}(a))\}.$$

For each $a_x \in \mathbf{c}(M)_1$ with $a\colon i \to j$ in $\mathscr{C}$, we set its source to be $s(a_x) := i_{M(a)(x)}$ and its target to be $t(a_x) := j_x$, namely we regard $a_x$ as an arrow

$$a_x\colon i_{M(a)(x)} \to j_x.$$

In this way we define a quiver $(\mathbf{c}(M)_0, \mathbf{c}(M)_1, s, t)$. Let $i_z \xrightarrow{a_y} j_y \xrightarrow{b_x} k_x$ be a path of length 2 in this quiver. Then by definition, $i \xrightarrow{a} j \xrightarrow{b} k$ is in $\mathscr{C}$ and $z = M(a)(y), y = M(b)(x)$, thus we have $z = M(a)M(b)(x) = M(ba)(x)$. Therefore, we have an arrow $(ba)_x\colon i_z \to k_x$ in this quiver. The composition is then defined by setting

$$b_x \circ a_y := (ba)_x.$$

The identity of $j_x \in \mathbf{c}(M)_0$ is given by $\mathbb{1}_{j_x} = (\mathbb{1}_j)_x$.

We define a functor $P_M\colon \mathbf{c}(M) \to \mathscr{C}$ as the first projection, namely by sending $a_x\colon i_{M(a)(x)} \to j_x$ to $a\colon i \to j$.

**Proposition 9.14.** *Let $\mathscr{C}$ be a small dg category. Then every dg $\mathscr{C}$-module is given as a colimit of representable dg $\mathscr{C}$-modules.*

*Proof.* Let $Y_{\bm{.}}\colon \mathscr{C} \to \mathscr{C}_{\mathrm{dg}}(\mathscr{C})$ be the dg Yoneda embedding, which is defined by sending $a\colon i \to j$ in $\mathscr{C}$ to $\mathscr{C}(\text{-}, a)\colon \mathscr{C}(\text{-}, i) \to \mathscr{C}(\text{-}, j)$. Let $M$ be a dg $\mathscr{C}$-module. We show that $M$ is isomorphic to the colimit of $Y_{\bm{.}} \circ P_M$:

$$M \cong \varinjlim(Y_{\bm{.}} \circ P_M).$$

For simplicity, we set $F := Y_{\bm{.}} \circ P_M$. We first construct a cocone $(M, (\alpha_{j_x})_{j_x\in\mathbf{c}(M)_0})$. For each $j_x \in \mathbf{c}(M)_0$ with $j \in \mathscr{C}_0, x \in M(j)$, we define $\alpha_{j_x}\colon F(j_x) \to M$ as follows. By definition, we have

$$F(j_x) = (Y_{\bm{.}} \circ P_M)(j_x) = Y_{\bm{.}}(j) = \mathscr{C}(\text{-}, j). \tag{9.20}$$

By the dg Yoneda lemma, there exists an isomorphism $\phi_{j,M}\colon \mathscr{C}_{\rm dg}(\mathscr{C})(\mathscr{C}(\text{-},j),M) \to M(j)$. Then we define $\alpha_{j_x}$ by

$$\alpha_{j_x} := \phi_{j,M}^{-1}(x) = M(\text{-})(x),$$

which sends any $a\colon i \to j$ in $\mathscr{C}$ to $\alpha_{j_x}(a) := M(a)(x)$, in particular, $\mathbb{1}_j$ to $x$. We show that $(M, (\alpha_{j_x})_{j_x \in \mathbf{c}(M)_0})$ is a cocone of $F$. Let $\alpha_x\colon i_y \to j_x$ be a morphism in $\mathbf{c}(M)$ with $a\colon i \to j$ in $\mathscr{C}$ and $x \in M(j), y = M(a)(x)$. Then by noting (9.20) and that $F(a_x) = \mathscr{C}(\text{-}, a)$, it is enough to show the commutativity of the diagram

$$\begin{tikzcd}
\mathscr{C}(\text{-},i) \arrow[d, "\mathscr{C}(\text{-},a)"'] \arrow[dr, "\alpha_{i_y}"] & \\
& M \\
\mathscr{C}(\text{-},j) \arrow[ur, "\alpha_{j_x}"'] &
\end{tikzcd} .$$

By the dg Yoneda lemma, it is enough to check the commutativity for the $\mathbb{1}_i$, which is shown by $\alpha_{i_y}(\mathbb{1}_i) = y = M(a)(x) = \alpha_{j_x}(a) = \alpha_{j_x}(\mathscr{C}(\text{-},a)(\mathbb{1}_i))$.

Finally, we show that this cocone is a colimit. Let $(N, (\beta_{i_x}\colon F(i_x) \to N)_{i_x \in \mathbf{c}(M)_0})$ be any cocone of $F$. It is enough to show the unique existence of a $\gamma \in \mathscr{C}_{\rm dg}(\mathscr{C})(M,N)$ that makes the diagram

$$\begin{tikzcd}
\mathscr{C}(\text{-},i) \arrow[rr, "\alpha_{i_x}"] \arrow[dr, "\beta_{i_x}"'] & & M \arrow[dl, "\gamma"] \\
& N &
\end{tikzcd} .$$

commutative for all $i_x \in \mathbf{c}(M)_0$. By the commutativity for $\mathbb{1}_i$, we must have $\gamma_i(x) = \beta_{i_x}(\mathbb{1}_i)$ for all $i \in \mathscr{C}_0, x \in M(i)$. Conversely we can define a morphism $\gamma\colon M \to N$ by setting $\gamma_i(x) := \beta_{i_x}(\mathbb{1}_i)$ for all $i \in \mathscr{C}_0, x \in M(i)$, which makes the diagram above commutative. Thus the unique existence of such a $\gamma$ is verified. □

**Lemma 9.15.** *If* $(\mathbf{F}, \boldsymbol{\psi})\colon \mathscr{C}_{\rm dg}(X') \to \mathscr{C}_{\rm dg}(X)$ *is a* 1*-morphism in the* 2*-category* $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCAT})$*, then we can define an* $X'$*-*$X$*-bimodule* $N$ *by setting* $N(i)(D, C) = \mathbf{F}(i)(C^\wedge)(D)$ *for* $C \in X'(i)$ *and* $B \in X(i)$*.*

*Proof.* Let $N(i)$ be the bimodule $N(i)(D,C) = F(i)(C^\wedge)(D)$ for $C \in X'(i)$ and $B \in X(i)$. For any $C \in X'(i)$, we define a 2-morphism

$$N(a)_{C^\wedge}\colon C^\wedge \otimes N(i) \otimes_{X(i)} \overline{X(a)} \to C^\wedge \otimes \overline{X'(a)} \otimes_{X'(j)} N(j)$$

as the composite of the following 2-morphisms:

$$\begin{aligned} C^\wedge \otimes N(i) \otimes_{X(i)} \overline{X(a)} &\cong N(i)(\text{-}, C) \otimes_{X(i)} \overline{X(a)} \\ &= F(i)(C^\wedge) \otimes_{X(i)} \overline{X(a)} \\ &\overset{\psi(a)_{C^\wedge}}{\Longrightarrow} F(j)(C^\wedge \otimes \overline{X'(a)}) \\ &\cong F(j)((X'(a)C)^\wedge) \\ &= N(j)(\text{-}, X'(a)C) \\ &\cong X'(j)(\text{-}, X'(a)C) \otimes N(j) \\ &\cong C^\wedge \otimes \overline{X'(a)} \otimes_{X'(j)} N(j) \end{aligned}$$

By Proposition 9.14, every dg $X'(i)$-module $T$ is a colimit of representable functors, we chose an isomorphism $\xi_T \colon \varinjlim T_\lambda \to T$ with $(T_\lambda)_{\lambda\in\Lambda}$ a family of representable functors once for all, we can define $N(a)_T$ by the following commutative diagram:

$$\begin{tikzcd} \varinjlim T_\lambda \otimes_{X'(i)} N(i) \otimes_{X(i)} \overline{X(a)} \arrow[r, "\varinjlim N(a)_{T_\lambda}"] \arrow[d, "\cong"', "(\varinjlim \xi_T)\otimes_{X'(i)} N(i) \otimes_{X(i)} \overline{X(a)}"] & \varinjlim T_\lambda \otimes_{X'(i)} \overline{X'(a)} \arrow[d, "\cong"', "(\varinjlim \xi_T)\otimes_{X'(i)} \overline{X'(a)}"] \\ T \otimes_{X'(i)} N(i) \otimes_{X(i)} \overline{X(a)} \arrow[r, "N(a)_T"] & T \otimes_{X'(i)} \overline{X'(a)} \otimes_{X'(j)} N(j), \end{tikzcd}$$

and we obtain a 2-morphism

$$N(a) := (N(a)_T)_{T\in\mathscr{C}_{\mathrm{dg}}(X'(i))} \colon \text{-} \otimes_{X'(i)} N(i) \otimes_{X(i)} \overline{X(a)} \Rightarrow \text{-} \otimes_{X'(i)} \overline{X'(a)} \otimes_{X'(j)} N(j).$$

□

Now let $(\mathscr{B}, M)$ be any lift of $\mathscr{U}$ such that ${}_B M(i)$ is homotopically projective for each $B \in \mathscr{B}(i)$. Let $X \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$ and $(\mathbf{F}, \boldsymbol{\psi}) \colon \mathscr{C}_{\mathrm{dg}}(X') \to \mathscr{C}_{\mathrm{dg}}(X)$ be a 1-morphism in the 2-category $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCAT})$ such that $\mathbf{L}(\mathbf{F}, \boldsymbol{\psi}) \colon \mathscr{D}(X') \to \mathscr{D}(X)$ induce a 1-morphism $\underline{X'} \to \mathscr{U}$. Then we have the following diagram

$$\begin{tikzcd} & & & & \mathscr{D}(X'(i)) \arrow[dl, "\mathbf{L}F(i)"'] \arrow[dd, "\mathscr{D}X'(a)"] & & \underline{X'}(i) \arrow[ll] \arrow[dl] \arrow[dd] \\ & \mathscr{D}(\mathscr{B}(i)) \arrow[rr, "\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{B}(i)} M(i)"] \arrow[dd] & & \mathscr{D}(X(i)) \arrow[dd, "\mathscr{D}X(a)"] & & \mathscr{U}(i) \arrow[ll] \arrow[dd] & \\ \underline{\mathscr{B}}(i) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathscr{U}(i) \arrow[ur] \arrow[dd] & & \mathscr{D}(X'(j)) \arrow[dl, "\mathbf{L}F(j)"] & & \underline{X'}(j) \arrow[ll] \arrow[dl] \\ & \mathscr{D}(\mathscr{B}(j)) \arrow[rr] & & \mathscr{D}(X(j)) & & \mathscr{U}(j) \arrow[ll] & \\ \underline{\mathscr{B}}(j) \arrow[ur] \arrow[rr] & & \mathscr{U}(j) \arrow[ur] & & & & \end{tikzcd} \tag{9.21}$$

Keeping the notations above, let $Y$ be an $X'$-$\mathscr{B}$-bimodule given by

$$Y(i)(B,C) = \mathscr{C}_{\rm dg}(X(i))({}_BM(i), F(i)(C^\wedge))$$

for $i \in I, B \in \mathscr{B}(i), C \in X'(i)$.

By Lemma 9.10, we have

$$\overline{Y(a)}\colon \text{-}(\otimes_{\mathscr{B}(i)}\overline{\mathscr{B}(a)}) \circ \mathscr{C}_{\rm dg}(\mathscr{B}(i))(M(i),\text{-}) \Rightarrow \mathscr{C}_{\rm dg}(\mathscr{B}(j))(M(j),\text{-}) \circ (\text{-} \otimes_{X(i)}\overline{X(a)}).$$

Then we have the following

$$\begin{aligned}&(\text{-} \otimes_{\mathscr{B}(i)}\overline{\mathscr{B}(a)}) \circ \mathscr{C}_{\rm dg}(\mathscr{B}(i))(M(i),\text{-}) \circ \mathbf{F}(i)\\ &\overset{\text{(a)}}{\Rightarrow} \mathscr{C}_{\rm dg}(\mathscr{B}(j))(M(j),\text{-}) \circ (\text{-} \otimes_{X(i)}\overline{X(a)}) \circ \mathbf{F}(i)\\ &\overset{\text{(b)}}{\Rightarrow} \mathscr{C}_{\rm dg}(\mathscr{B}(j))(M(j),\text{-}) \circ \mathbf{F}(j) \circ (\text{-} \otimes_{X'(i)}\overline{X'(a)}),\end{aligned}$$

where (a) is given by $\overline{Y(a)} \circ \mathbf{F}(i)$, (b) is given by $\mathscr{C}_{\rm dg}(\mathscr{B}(j))(M(j),\text{-}) \circ \boldsymbol{\psi}(a)$, $\boldsymbol{\psi}(a)\colon (\text{-} \otimes_{X(i)}\overline{X(a)}) \circ \mathbf{F}(i) \Rightarrow \mathbf{F}(j) \circ (\text{-} \otimes_{X'(i)}\overline{X'(a)})$.

Now for $a \in I_1$, and any $x' \in X'(i), b \in \mathscr{B}(i)$, we have

$$\begin{aligned}(x'^\wedge) \otimes_{X'(i)} Y(i) \otimes_{\mathscr{B}(i)} \overline{\mathscr{B}(a)} &\cong Y(i)(\text{-}, x') \otimes_{\mathscr{B}(i)} \overline{\mathscr{B}(a)}\\ &= \mathscr{C}_{\rm dg}(X(i))(M(i), F(i)(x'^\wedge)) \otimes_{\mathscr{B}(i)} \overline{\mathscr{B}(a)}\\ &\overset{\text{(c)}}{\Rightarrow} \mathscr{C}_{\rm dg}(\mathscr{B}(j))(M(j),\text{-}) \circ \mathbf{F}(j) \circ ((x'^\wedge) \otimes_{X'(i)} \overline{X'(a)})\\ &= \mathscr{C}_{\rm dg}(\mathscr{B}(j))(M(j), \mathbf{F}(j)(X'(a)x')^\wedge))\\ &\cong (x'^\wedge) \otimes_{X'(i)} \overline{X'(a)} \otimes_{X'(j)} Y(j)\end{aligned}$$

where (c) is given the above morphism. By Lemma 9.15, we have a bimodule morphism

$$Y(a)\colon \text{-} \otimes_{X'(i)}Y(i) \otimes_{\mathscr{B}(i)} \overline{\mathscr{B}(a)} \to \text{-} \otimes_{X'(i)}\overline{X'(a)} \otimes_{X'(j)} Y(j).$$

**Lemma 9.16.** (1) $\text{-} \overset{\mathbf{L}}{\otimes}_{X'}Y$ *induces a* 1*-morphism* $\underline{X'} \to \underline{\mathscr{B}}$*, hence* $Y$ *is a quasi-functor. It is a quasi-equivalence if* $\mathbf{L}\mathbf{F}$ *induces an equivalence* $\mathbb{Z}\underline{X'} \to \mathbb{Z}\mathscr{U}$*.*

(2) *There is a canonical* 2*-morphism*

$$\text{-} \overset{\mathbf{L}}{\otimes}_{X'}Y \overset{\mathbf{L}}{\otimes}_{\mathscr{B}} M \to \mathbf{L}F,$$

*which is invertible for* $Q \in \mathscr{H}_{\rm p}{}^b(X'(i))$*, that is,*

$$Q \overset{\mathbf{L}}{\otimes}_{X'(i)} Y(i) \overset{\mathbf{L}}{\otimes}_{\mathscr{B}(i)} M(i) \xrightarrow{\sim} \mathbf{L}F(i)(Q)$$

*where* $\mathscr{H}_{\rm p}{}^b(X'(i))$ *is the smallest strictly full subcategory of* $\mathscr{H}_{\rm p}(X'(i))$ *containing of the* $A^\wedge, A \in X'(i)$*. It is invertible for arbitrary* $Q \in \mathscr{D}(X(i))$ *if and only if* $\mathbf{L}F(i)$ *commutes with direct sums for each* $i \in I_0$*.*

(3) *If* $(X', Z)$ *is a lift of* $\mathscr{U}$ *and* $F = \text{-} \otimes Z$*, then* $Y$ *is a quasi-equivalence bimodule* $X' \to \mathscr{B}$ *and we have* $\text{-} \overset{\mathbf{L}}{\otimes}_{X'}Y \overset{\mathbf{L}}{\otimes}_{\mathscr{B}} M \cong \mathbf{L}F$*.*

*Proof.* Let $Y$ be an $X'$-$\mathscr{B}$-bimodule given by

$$Y(i)(B,C) = \mathscr{C}_{\mathrm{dg}}(X(i))({}_BM(i), F(i)(C^{\wedge}))$$

for $i \in I, B \in \mathscr{B}(i), C \in X'(i)$, and for each $a : i \to j$, there is a bimodule morphism

$$Y(a)\colon \text{-} \otimes_{X'(i)} Y(i) \otimes_{\mathscr{B}(i)} \overline{\mathscr{B}(a)} \to \text{-} \otimes_{X'(i)} \overline{X'(a)} \otimes_{X'(j)} Y(j).$$

(1) $\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} Y(i)$ induces a functor $\underline{X'}(i) \to \underline{\mathscr{B}}(i)$, hence $Y(i)$ is a quasi-functor. It is a quasi-equivalence if $\mathbf{L}(\mathbf{F}(i))$ induces an equivalence $\mathbb{Z}\underline{X'}(i) \to \mathbb{Z}\mathscr{U}(i)$. For any $x' \in X'(i)$, we have

$$\begin{aligned}
x'^{\wedge} \overset{\mathbf{L}}{\otimes}_{X(i)} Y(i) \overset{\mathbf{L}}{\otimes}_{\mathscr{B}(i)} \overline{\mathscr{B}(a)} &\cong \\
R\mathscr{C}_{\mathrm{dg}}(X(i))(M(i), \mathbf{F}(i)(x'^{\wedge})) \overset{\mathbf{L}}{\otimes}_{\mathscr{B}(i)} \overline{\mathscr{B}(a)}) &\cong \\
R\mathscr{C}_{\mathrm{dg}}(X(j))(M(j), \mathbf{LF}(i)(x'^{\wedge}) \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}) &\cong \\
R\mathscr{C}_{\mathrm{dg}}(X(j))(M(j), \mathbf{LF}(j)((x'^{\wedge}) \overset{\mathbf{L}}{\otimes}_{X'(i)} \overline{X'(a)})) &\cong \\
Y(j)(\text{-}, X'(a)x') &\cong \\
(x'^{\wedge}) \overset{\mathbf{L}}{\otimes}_{X'(i)} \overline{X'(a)} \overset{\mathbf{L}}{\otimes}_{X'(j)} Y(j)
\end{aligned}$$

Then $\text{-} \overset{\mathbf{L}}{\otimes}_{X'} Y$ induces a 1-morphism $\underline{X'} \to \underline{\mathscr{B}}$, hence $Y$ is a quasi-functor, that is, we have the following diagram

$$\begin{tikzcd}
 & \mathscr{D}(X'(i)) \arrow[rr, "\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} Y(i)"] \arrow[dl, "\mathscr{D}(X'(a))"'] & & \mathscr{D}(\mathscr{B}(i)) \arrow[dl, "\mathscr{D}(\mathscr{B}(a))"] \arrow[dlll, Rightarrow, "\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} Y(a)"'] \\
\mathscr{D}(X'(j)) \arrow[rr, "\overset{\mathbf{L}}{\otimes}_{X'(j)} Y(j)"'] & & \mathscr{D}(\mathscr{B}(j)) & \\
 & \underline{X'}(i) \arrow[uu] \arrow[rr] \arrow[dl] & & \underline{\mathscr{B}}(i) \arrow[uu] \arrow[dl] \arrow[dlll, Rightarrow, "\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} Y(a)"'] \\
\underline{X'}(j) \arrow[uu] \arrow[rr] & & \underline{\mathscr{B}}(j) \arrow[uu] &
\end{tikzcd}$$

Note that $\text{-} \overset{\mathbf{L}}{\otimes}_{X'} M$ induces an equivalence $\mathbb{Z}\underline{X'} \to \mathbb{Z}\mathscr{B}$ if and only if $\mathbf{LF}$ induces an equivalence $\mathbb{Z}\underline{X'} \to \mathbb{Z}\mathscr{U}$. The second assertion now follows from Lemma 9.9.

(2) There is a canonical morphism $Q \overset{\mathbf{L}}{\otimes}_{X'(i)} Y(i) \overset{\mathbf{L}}{\otimes}_{\mathscr{B}(i)} M(i) \to \mathbf{L}F(i)(Q)$ which is invertible for $Q \in \mathscr{H}_{\mathrm{p}}{}^{b}(X'(i))$. It is invertible for arbitrary $Q \in \mathscr{D}(X(i))$ if and only if $\mathbf{L}F(i)$ commutes with direct sums.

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\mathscr{D}(X'(i)) \arrow[r, "- \overset{\mathbf{L}}{\otimes}_{X'(i)} Y(i)"] \arrow[d, "\mathscr{D}(X'(a))"'] \arrow[rr, bend left, "\mathbf{LF}(i)"] & \mathscr{D}(\mathscr{B}(i)) \arrow[r, "- \overset{\mathbf{L}}{\otimes}_{\mathscr{B}(i)} M(i)"] \arrow[d, "\mathscr{D}(\mathscr{B}(a))"] \arrow[dl, Rightarrow, "(\mathbf{L}Y)(a)"'] & \mathscr{D}(X(i)) \arrow[d, "\mathscr{D}(X(a))"] \arrow[dl, Rightarrow, "(\mathbf{L}M)(a)"] \\
\mathscr{D}(X'(j)) \arrow[r, "- \overset{\mathbf{L}}{\otimes}_{X'(j)} Y(j)"'] \arrow[rr, bend right, "\mathbf{LF}(j)"'] & \mathscr{D}(\mathscr{B}(j)) \arrow[r, "- \overset{\mathbf{L}}{\otimes}_{\mathscr{B}(j)} M(j)"'] & \mathscr{D}(X(j))
\end{tikzcd}
\cdot
$$

For each $a \in I$, there is a commutative diagram

$$
\begin{tikzcd}
(- \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X'(a)}) \circ (- \overset{\mathbf{L}}{\otimes} M(i)) \circ (- \overset{\mathbf{L}}{\otimes} Y(i)) \arrow[r, Rightarrow, "(\mathbf{L}M \circ \mathbf{L}Y)(a)"] \arrow[d, Rightarrow] & (- \overset{\mathbf{L}}{\otimes} M(j)) \circ (- \overset{\mathbf{L}}{\otimes} Y(j)) \circ (- \overset{\mathbf{L}}{\otimes} \overline{X(a)}) \arrow[d, Rightarrow] \\
(- \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X'(a)}) \circ \mathbf{LF}(i) \arrow[r, Rightarrow, "(\mathbf{L}\boldsymbol{\psi})(a)"] & \mathbf{L}(\mathbf{F}(j) \circ (- \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}))
\end{tikzcd}
.
$$

(3) If $(X'(i), Z(i))$ is a lift of $\mathscr{U}(i)$ and $F(i) = - \otimes Z(i)$, then $Y(i)$ is a quasi-equivalence $X'(i) \to \mathscr{B}(i)$ and $- \overset{\mathbf{L}}{\otimes}_{X'(i)} Y(i) \overset{\mathbf{L}}{\otimes}_{\mathscr{B}(i)} M(i) \cong \mathbf{L}F(i)$ by [28, Lemma 7.3]. Therefore the assertions of (3) are from (1) and (2).

$\square$

Recall that the Morita theory for dg categories was given by Keller. We will give a similar theorem in our setting.

**Theorem 9.17.** *Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then we have implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). If $X$ is $\Bbbk$-flat, then we also have the implication (4) $\Rightarrow$ (1).*

(1) *There exists an $X'$-$X$-bimodule $Z$ such that $- \overset{\mathbf{L}}{\otimes}_{X'} Z \colon \mathscr{D}(X') \to \mathscr{D}(X)$ is an equivalence in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$.*

(2) *There exists a 1-morphism $(\mathbf{F}, \boldsymbol{\psi}) \colon \mathscr{C}_{\mathrm{dg}}(X') \to \mathscr{C}_{\mathrm{dg}}(X)$ in the 2-category $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCAT})$ such that $\mathbf{L}(\mathbf{F}, \boldsymbol{\psi}) \colon \mathscr{D}(X') \to \mathscr{D}(X)$ is an equivalence in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$.*

(3) *There exists a tilting colax functor $\mathscr{T}$ for $X$, and there exists a quasi-equivalence $X'$-$\mathscr{T}$-bimodule $E$.*

(4) *There exists a tilting colax functor $\mathscr{T}$ for $X$, and there exist 1-morphisms*
$$(\mathbf{G}, \boldsymbol{\psi}') \colon \mathscr{C}_{\mathrm{dg}}(X') \to \mathscr{C}_{\mathrm{dg}}(\mathscr{T}) \text{ and } (\mathbf{F}, \boldsymbol{\psi}) \colon \mathscr{C}_{\mathrm{dg}}(\mathscr{T}) \to \mathscr{C}_{\mathrm{dg}}(X)$$
*in the 2-category $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCAT})$ such that*
$$\mathbf{L}(\mathbf{G}, \boldsymbol{\psi}') \colon \mathscr{D}(X') \to \mathscr{D}(\mathscr{T}) \text{ and } \mathbf{L}(\mathbf{F}, \boldsymbol{\psi}) \colon \mathscr{D}(\mathscr{T}) \to \mathscr{D}(X)$$
*are equivalences in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$.*

*Proof.* (1) $\Rightarrow$ (2). We can take $(\mathbf{F}, \boldsymbol{\psi}) := \text{-} \otimes_{X'} Z$.

(2) $\Rightarrow$ (3). Assume that the statement (2) holds. Then for each $a\colon i \to j$ in $I$, we have a diagram

$$\begin{tikzcd} \mathscr{C}_{\mathrm{dg}}(X'(i)) \arrow[r, "\mathbf{F}(i)"] \arrow[d, "\mathscr{C}_{\mathrm{dg}}(X'(a))"'] & \mathscr{C}_{\mathrm{dg}}(X(i)) \arrow[d, "\mathscr{C}_{\mathrm{dg}}(X(a))"] \arrow[dl, Rightarrow, "\boldsymbol{\psi}(a)"'] \\ \mathscr{C}_{\mathrm{dg}}(X'(j)) \arrow[r, "\mathbf{F}(j)"'] & \mathscr{C}_{\mathrm{dg}}(X(j)) \end{tikzcd}, \tag{9.22}$$

where note that $\boldsymbol{\psi}(a)$ is a dg natural transformation by assumption. For each $i \in I_0$, we set $\mathscr{U}(i)$ to be the full dg subcategory of $\mathscr{C}_{\mathrm{dg}}(X(i))$ with

$$\mathscr{U}(i)_0 = \{\mathbf{LF}(i)(C^\wedge) \mid C \in X'(i)_0\}.$$

Let $(\mathscr{T}, M)$ be the standard lift of $\mathscr{U}$ (see Definition 9.11(3)):

$$\begin{aligned} \mathscr{T}(i)_0 &:= \{\mathbf{p}_{X(i)}U \mid U \in \mathscr{U}(i)_0\}, \\ M(i)(A, \mathbf{p}_{X(i)}U) &:= \mathbf{p}_{X(i)}U(A) \end{aligned}$$

for all $i \in I_0$. By Lemma 9.16, we define a quasi-equivalence $X'$-$\mathscr{T}$-bimodule $E$ by setting

$$E(i)(B, C) = \mathscr{C}_{\mathrm{dg}}(X(i))(M(i), \mathbf{F}(i)(C^\wedge)(B)).$$

for all $i \in I_0$, $B \in \mathscr{T}(i), C \in X'(i)$, and a bimodule morphism

$$E(a)\colon \text{-} \otimes_{X'(i)} E(i) \otimes_{\mathscr{B}(i)} \overline{\mathscr{B}(a)} \to \text{-} \otimes_{X'(i)} \overline{X'(a)} \otimes_{X'(j)} E(j).$$

It follows from [28, Lemma 7.3] that $E(i)$ is a quasi-equivalence $X'(i) \to \mathscr{T}(i)$, and it follows from Lemma 9.16 that we have the following diagram such that $\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} E(a)$ is an 2-isomorphism

$$\begin{tikzcd} & \mathscr{D}(X'(i)) \arrow[rr, "\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} E(i)"] \arrow[dl, "\mathscr{D}(X'(a))"'] & & \mathscr{D}(\mathscr{T}(i)) \arrow[dl, "\mathscr{D}(\mathscr{T}(a))"] \arrow[lll, Rightarrow, "\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} E(a)"'] \\ \mathscr{D}(X'(j)) \arrow[rr, "\overset{\mathbf{L}}{\otimes}_{X'(j)} Y(j)"'] & & \mathscr{D}(\mathscr{T}(j)) & \\ & \underline{X'}(i) \arrow[uu] \arrow[rr] \arrow[dl] & & \underline{\mathscr{T}}(i) \arrow[uu] \arrow[dl] \arrow[lll, Rightarrow, "\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} E(a)"'] \\ \underline{X'}(j) \arrow[uu] \arrow[rr] & & \underline{\mathscr{T}}(j). \arrow[uu] & \end{tikzcd}$$

(3) $\Rightarrow$ (4). Assume that the statement (3) holds. Then there exists a quasi-equivalence $X'$-$\mathscr{T}$-bimodule $E$ for $X$, which gives us the necessary 1-morphism $(\mathbf{G}, \boldsymbol{\psi}') := \text{-} \otimes_{X'} E\colon \mathscr{C}_{\mathrm{dg}}(X') \to \mathscr{C}_{\mathrm{dg}}(\mathscr{T})$ in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCAT})$ and an equivalence $\mathbf{L}(\mathbf{G}, \boldsymbol{\psi}') = \text{-} \overset{\mathbf{L}}{\otimes}_{X'} E\colon \mathscr{D}(X') \to \mathscr{D}(\mathscr{T})$ in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$.

We define a $\mathscr{T}$-$X$-bimodule $M$ as follows. For each $i \in I_0$, we set

$${}_B M(i)_A := B(A) \cong \mathscr{C}_{\mathrm{dg}}(X(i))(A^\wedge, B)$$

for all $B \in \mathscr{T}(i)_0, A \in X(i)_0$. For each $a\colon i \to j$ in $I$, we define a 0-cocycle morphism

$$M(a)\colon M(i) \otimes_{X(i)} \overline{X(a)} \to \overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} M(j)$$

of $\mathscr{T}(i)$-$X(j)$-bimodules by

$$\begin{aligned} M(a)_{C^\wedge}\colon C^\wedge \otimes_{\mathscr{T}(i)} M(i) \otimes_{X(i)} \overline{X(a)} &\cong M(i)(\text{-}, C) \otimes_{X(i)} \overline{X(a)} \\ &\cong C \otimes_{X(i)} \overline{X(a)} \cong \mathscr{T}(a)(C) \cong C^\wedge \otimes \overline{\mathscr{T}(a)} \otimes M(j) \end{aligned}$$

for all $a \in I_1$ and $C \in \mathscr{T}(i)$.

Let $(\sigma, \rho)\colon \mathscr{T} \to \mathscr{C}_{\mathrm{dg}}(X)$ be an $I$-equivariant inclusion, and $i \in I_0$. We define a dg functor

$$F(i)\colon \mathscr{C}_{\mathrm{dg}}(\mathscr{T}(i)) \to \mathscr{C}_{\mathrm{dg}}(X(i))$$

by setting $F(i) := \text{-} \otimes_{\mathscr{T}(i)} M(i)$. Then since for each $B \in \mathscr{T}(i)_0$, the right dg $X(i)$-module ${}_BM(i) = B$ $(\in \mathscr{T}(i)_0 \subseteq \mathscr{H}_{\mathrm{p}}(X(i))_0)$ is homotopically projective, $F(i)$ preserves homotopically projectives by Lemma 7.18. Now for any $B, C \in \mathscr{T}(i)_0$, the bimodule $M(i)$ defines a morphism ${}_?M(i)\colon \mathscr{T}(i)(B,C) \to \mathscr{C}_{\mathrm{dg}}(X(i))({}_BM(i), {}_CM(i))$ in $\mathscr{C}_{\mathrm{dg}}(\Bbbk)$ by sending each $f\colon B \to C$ to ${}_fU(i)\colon {}_BM(i) \to {}_CM(i)$. Here since we have ${}_BM(i) = B$, ${}_fM(i) = f$ by definition, ${}_?M(i)$ is the identity of $\mathscr{T}(i)(B,C)$. Hence it induces an isomorphism in homology. Moreover, $\{{}_BM(i) \mid B \in \mathscr{T}(i)_0\} = \mathscr{T}(i)_0$ and $\mathscr{T}(i)$ is a tilting dg category for $X(i)$. Hence by [28, Lemma 6.1(a)], $\mathbf{L}F(i) = \text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{T}(i)} M(i)\colon \mathscr{D}(\mathscr{T}(i)) \to \mathscr{D}(X(i))$ is a triangle equivalence. Next for each $a\colon i \to j$ in $I$, we construct a 2-morphism $\boldsymbol{\psi}(a)$ in the diagram

$$\begin{tikzcd} \mathscr{C}_{\mathrm{dg}}(\mathscr{T}(i)) \arrow[r, "\mathbf{F}(i)"] \arrow[d, "\mathscr{C}_{\mathrm{dg}}(\mathscr{T}(a))"'] & \mathscr{C}_{\mathrm{dg}}(X(i)) \arrow[d, "\mathscr{C}_{\mathrm{dg}}(X(a))"] \arrow[dl, Rightarrow, "\boldsymbol{\psi}(a)"'] \\ \mathscr{C}_{\mathrm{dg}}(\mathscr{T}(j)) \arrow[r, "\mathbf{F}(j)"'] & \mathscr{C}_{\mathrm{dg}}(X(j)) \end{tikzcd} .$$

where

$$\begin{aligned} \boldsymbol{\psi}(a) := \text{-} \otimes_{\mathscr{T}(i)} M(a) :& \\ \text{-} \otimes_{\mathscr{T}(i)} M(i) \otimes_{X(i)} \overline{X(a)} \Rightarrow \text{-} \otimes_{\mathscr{T}(i)} \overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} M(j).& \end{aligned}$$

Then we have the following diagram such that $\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} E(i), \mathbf{L}F(i)$ are triangle equivalences and that $\mathbf{L}E(a), \mathbf{L}\boldsymbol{\psi}(a)$ are 2-isomorphisms

$$\begin{tikzcd} \mathscr{D}(X'(i)) \arrow[r, "\text{-} \overset{\mathbf{L}}{\otimes}_{X'(i)} E(i)"] \arrow[d, "\mathscr{D}(X'(a))"'] & \mathscr{D}(\mathscr{T}(i)) \arrow[r, "\mathbf{L}F(i)"] \arrow[d, "\mathscr{D}(\mathscr{T}(a))"] \arrow[dl, Rightarrow, "\mathbf{L}E(a)"'] & \mathscr{D}(X(i)) \arrow[d, "\mathscr{D}(X(a))"] \arrow[dl, Rightarrow, "\mathbf{L}\boldsymbol{\psi}(a)"'] \\ \mathscr{D}(X'(j)) \arrow[r, "\text{-} \overset{\mathbf{L}}{\otimes}_{X'(j)} E(j)"'] & \mathscr{D}(\mathscr{T}(j)) \arrow[r, "\mathbf{L}F(i)"'] & \mathscr{D}(X(j)) \end{tikzcd} .$$

(4) $\Rightarrow$ (1). Assume that the statement (4) holds.

There is a diagram

$$
\begin{tikzcd}
\mathscr{C}_{\mathrm{dg}}(X'(i)) \arrow[r, "G(i)"] \arrow[d, "\mathscr{C}_{\mathrm{dg}}(X'(a))"'] & \mathscr{C}_{\mathrm{dg}}(\mathscr{T}(i)) \arrow[r, "F(i)"] \arrow[d, "\mathscr{C}_{\mathrm{dg}}(\mathscr{T}(a))"] \arrow[dl, Rightarrow, "\phi(a)"'] & \mathscr{C}_{\mathrm{dg}}(X(i)) \arrow[d, "\mathscr{C}_{\mathrm{dg}}(X(a))"] \arrow[dl, Rightarrow, "\psi(a)"] \\
\mathscr{C}_{\mathrm{dg}}(X'(j)) \arrow[r, "G(j)"'] & \mathscr{C}_{\mathrm{dg}}(\mathscr{T}(j)) \arrow[r, "F(j)"'] & \mathscr{C}_{\mathrm{dg}}(X(j))
\end{tikzcd},
$$

such that we have the following diagram

$$
\begin{tikzcd}
\mathscr{D}(X'(i)) \arrow[r, "\mathbf{L}G(i)"] \arrow[d, "\mathscr{D}(X'(a))"'] & \mathscr{D}(\mathscr{T}(i)) \arrow[r, "\mathbf{L}F(i)"] \arrow[d, "\mathscr{D}(\mathscr{T}(a))"] \arrow[dl, Rightarrow, "(\mathbf{L}\phi)(a)"'] & \mathscr{D}(X(i)) \arrow[d, "\mathscr{D}(X(a))"] \arrow[dl, Rightarrow, "(\mathbf{L}\psi)(a)"] \\
\mathscr{D}(X'(j)) \arrow[r, "\mathbf{L}G(j)"'] & \mathscr{D}(\mathscr{T}(j)) \arrow[r, "\mathbf{L}F(j)"'] & \mathscr{D}(X(j))
\end{tikzcd}.
$$

with $\mathbf{L}G(i)$ and $\mathbf{L}F(i)$ are triangle equivalences and

$$
\begin{aligned}
(\mathbf{L}\phi)(a) \colon \mathscr{D}(\mathscr{T}(a)) \circ \mathbf{L}G(i) &\cong \mathbf{L}G(j) \circ \mathscr{D}(X'(a)) \\
(\mathbf{L}\psi)(a) \colon \mathscr{D}(X(a)) \circ \mathbf{L}F(i) &\cong \mathbf{L}F(j) \circ \mathscr{D}(\mathscr{T}(a))
\end{aligned}
$$

are 2-isomorphisms.

Let $M(i)$ be the bimodule $M(i)(B, A) = G(i)(A^\wedge)(B)$ for $A \in X'(i)$ and $B \in \mathscr{T}(i)$, and let $N(i)$ be the bimodule $N(i)(D, C) = F(i)(C^\wedge)(D)$ for $C \in \mathscr{T}(i)$ and $B \in X(i)$. By Lemma 9.15, we have 2-morphisms

$$
N(a) := (N(a)_T)_{T \in \mathscr{C}_{\mathrm{dg}}(\mathscr{T}(i))} \colon \text{-} \otimes_{\mathscr{T}(i)} N(i) \otimes_{X(i)} \overline{X(a)} \Rightarrow \text{-} \otimes_{\mathscr{T}(i)} \overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} N(j)
$$

and

$$
M(a) := (M(a)_{T'})_{T' \in \mathscr{C}_{\mathrm{dg}}(X'(i))} \colon \text{-} \otimes_{X'(i)} M(i) \otimes_{\mathscr{T}(i)} \overline{\mathscr{T}(a)} \Rightarrow \overline{X'(a)} \otimes_{X(j)} M(j).
$$

Then by [28, Lemma 6.3(b)], $\mathbf{L}F(i) \circ \mathbf{L}G(i) = (\text{-} \overset{\mathbf{L}}{\otimes} N(i)) \circ (\text{-} \overset{\mathbf{L}}{\otimes} M(i)) \cong \text{-} \overset{\mathbf{L}}{\otimes} (M(i) \otimes \mathbf{p}N(i))$, where we set $\mathbf{p}N(i) := \mathbf{p}_{\mathscr{T}(i)\text{-}X(i)} N(i)$ for short, which is a triangle equivalence as the composite of triangle equivalences.

We define an $X'$-$X$-bimodule $Z$ up to associators. It is given as follows:

$$
\begin{aligned}
Z(i) :&= M(i) \otimes_{\mathscr{T}(i)} \mathbf{p}N(i) \text{ for all } i \in I_0, \\
Z(a) :&= \boldsymbol{a}^{-1}_{M(i), \mathbf{p}N(i), \overline{X(a)}} \circ (M(i) \otimes_{\mathscr{T}(i)} \mathbf{p}N(a)) \circ \boldsymbol{a}_{M(i), \overline{\mathscr{T}(a)}, \mathbf{p}N(j)} \\
&\quad \circ (M(a) \otimes_{\mathscr{T}(j)} \mathbf{p}N(j)) \circ \boldsymbol{a}^{-1}_{\overline{X'(a)}, M(j), \mathbf{p}N(j)} \\
&\qquad\qquad\qquad\qquad \text{for all } a \colon i \to j \text{ in } I,
\end{aligned}
$$

where the $X'(i)$-$X(i)$-bimodule morphism $Z(a)$ is defined by the commutative diagram

$$
\begin{tikzcd}
Z(i) \otimes_{X(i)} \overline{X(a)} \arrow[r, dashed, "Z(a)"] \arrow[d, equal] & \overline{X'(a)} \otimes_{X'(j)} Z(j) \\
(M(i) \otimes_{\mathscr{T}(i)} \mathbf{p}N(i)) \otimes_{X(i)} \overline{X(a)} \arrow[d, "\boldsymbol{a}^{-1}_{M(i),\mathbf{p}N(i),\overline{X(a)}}"] & \overline{X'(a)} \otimes_{X'(j)} (M(j) \otimes_{\mathscr{T}(j)} \mathbf{p}N(j)) \arrow[u, equal] \\
M(i) \otimes_{\mathscr{T}(i)} (\mathbf{p}N(i) \otimes_{X(i)} \overline{X(a)}) \arrow[d, dashed, "M(i)\otimes_{\mathscr{T}(i)}\mathbf{p}N(a)"] & (\overline{X'(a)} \otimes_{X'(j)} M(j) \otimes_{\mathscr{T}(j)} \mathbf{p}N(j) \arrow[u, "\boldsymbol{a}^{-1}_{\overline{X'(a)},M(j),\mathbf{p}N(j)}"'] \\
M(i) \otimes_{\mathscr{T}(i)} (\overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} \mathbf{p}N(j)) \arrow[r, "\boldsymbol{a}_{M(i),\overline{\mathscr{T}(a)},\mathbf{p}N(j)}"'] & (M(i) \otimes_{\mathscr{T}(i)} \overline{\mathscr{T}(a)}) \otimes_{\mathscr{T}(j)} \mathbf{p}N(j) \arrow[u, "M(a)\otimes_{\mathscr{T}(j)}\mathbf{p}N(j)"']
\end{tikzcd}
,
\tag{9.23}
$$

where the morphism $\mathbf{p}N(a)$ is defined as follows: we consider the triangle

$$
\mathbf{p}N(i) \xrightarrow{\varepsilon_{N(i)}} N(i) \xrightarrow{\delta_{N(i)}} \boldsymbol{a}N(i) \to
$$

of $\mathscr{T}(i)$-$X(i)$-bimodules. By tensoring with $\overline{X(a)}$ from the right, and with $\overline{\mathscr{T}(a)}$ from the left, this induces two triangles

$$
\begin{tikzcd}
\mathbf{p}N(i) \otimes_{X(i)} \overline{X(a)} \arrow[r, "\varepsilon"] \arrow[d, dashed, "\mathbf{p}N(a)"'] & N(i) \otimes_{X(i)} \overline{X(a)} \arrow[r, "\delta"] \arrow[d, "N(a)"] & \boldsymbol{a}N(i) \otimes_{X(i)} \overline{X(a)} \arrow[r] \arrow[d, dashed, "\boldsymbol{a}N(a)"] & {} \\
\overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} \mathbf{p}N(j) \arrow[r, "\varepsilon"'] & \overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} N(j) \arrow[r, "\delta"'] & \overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} \boldsymbol{a}N(j) \arrow[r] & {}
\end{tikzcd}
\tag{9.24}
$$

of $\mathscr{T}(i)$-$X(j)$-bimodules. We now show that

$$
\mathscr{C}_{\mathrm{dg}}(\mathscr{T}(i)^{\mathrm{op}} \otimes_{\Bbbk} X(j))(\mathbf{p}N(i) \otimes_{X(i)} \overline{X(a)}, \overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} \boldsymbol{a}N(j)) = 0.
$$

For any $t \in \mathscr{T}(i)_0$, we have isomorphisms

$$
t^\wedge \otimes \overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} \boldsymbol{a}N(j) \cong \mathscr{T}(j)(\text{-}, \mathscr{T}(a)t) \otimes_{\mathscr{T}(j)} \boldsymbol{a}N(j) \cong \boldsymbol{a}N(j)(\text{-}, \mathscr{T}(a)t),
$$

where the last term is an acyclic right dg $X(j)$-module. Hence we have

$$
\begin{aligned}
&\mathscr{C}_{\mathrm{dg}}(X(j))(t^\wedge \otimes \mathbf{p}N(i) \otimes_{X(i)} \overline{X(a)}, t^\wedge \otimes \overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} \boldsymbol{a}N(j)) \\
\cong\,&\mathscr{C}_{\mathrm{dg}}(X(j))(t^\wedge \otimes \mathbf{p}N(i) \otimes_{X(i)} \overline{X(a)}, \boldsymbol{a}N(j)(\text{-}, \mathscr{T}(a)t)) \\
\cong\,&\mathscr{C}_{\mathrm{dg}}(X(i)(t^\wedge \otimes \mathbf{p}N(i), \mathscr{C}_{\mathrm{dg}}(X(j))(\overline{X(a)}, \boldsymbol{a}N(j)(\text{-}, \mathscr{T}(a)t)).
\end{aligned}
$$

Here we have $\mathscr{C}_{\mathrm{dg}}(X(j))(\overline{X(a)}, \boldsymbol{a}N(j)(\text{-}, \mathscr{T}(a)t)) = 0$ because $\overline{X(a)}$ is a homotopically projective right dg $X(j)$-module. As a consequence, we have

$$
\mathscr{C}_{\mathrm{dg}}(X(j))(t^\wedge \otimes \mathbf{p}N(i) \otimes_{X(i)} \overline{X(a)}, t^\wedge \otimes \overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} \boldsymbol{a}N(j)) = 0. \tag{9.25}
$$

Take any $\alpha \in \mathscr{C}_{\mathrm{dg}}(\mathscr{T}(i)^{\mathrm{op}} \otimes_{\Bbbk} X(j))(\mathbf{p}N(i) \otimes_{X(i)} \overline{X(a)}, \overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} \boldsymbol{a}N(j))$. Then the equality (9.25) means that for any $t \in \mathscr{T}(i)_0$, we have

$$
0 = \alpha(?, t)\colon \mathscr{C}_{\mathrm{dg}}(X(j))((\mathbf{p}N(i) \otimes_{X(i)} \overline{X(a)})(?, t), (\overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} \boldsymbol{a}N(j))(?, t)).
$$

Thus $\alpha = 0$, and hence we have

$$\mathscr{C}_{\mathrm{dg}}(\mathscr{T}(j)^{\mathrm{op}} \otimes_{\Bbbk} X(j))(\mathbf{p}N(i) \otimes_{X(i)} \overline{X(a)}, \overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} \boldsymbol{a}N(j)) = 0.$$

Then by the diagaram (9.24), we see that there is a unique morphism $\mathbf{p}N(a) \colon \mathbf{p}N(i) \otimes_{X(i)} \overline{X(a)} \to \overline{\mathscr{T}(a)} \otimes_{\mathscr{T}(j)} \mathbf{p}N(j)$ such that the diagram (9.24) commutes. In this way $\mathbf{p}N(a)$ is defined.

Then by the diagram (9.23), the $X'$-$X$-bimodule $Z$ is defined, and we have a 1-morphism $\text{-} \otimes_{X'} Z \colon \mathscr{C}_{\mathrm{dg}}(X') \to \mathscr{C}_{\mathrm{dg}}(X)$ with the following diagram

$$\begin{tikzcd}
\mathscr{C}_{\mathrm{dg}}(X'(i)) \arrow[r, "\text{-}\otimes Z(i)"] \arrow[d, "\mathscr{C}_{\mathrm{dg}}(X'(a))"'] & \mathscr{C}_{\mathrm{dg}}(X(i)) \arrow[d, "\mathscr{C}_{\mathrm{dg}}(X(a))"] \arrow[dl, Rightarrow, "\text{-}\otimes Z(a)"] \\
\mathscr{C}_{\mathrm{dg}}(X'(j)) \arrow[r, "\text{-}\otimes Z(j)"'] & \mathscr{C}_{\mathrm{dg}}(X(j))
\end{tikzcd}.$$

By Definition 2.15, the composite $(\mathbf{L}F, \mathbf{L}\psi) \circ (\mathbf{L}G, \mathbf{L}\phi)$ of $(\mathbf{L}F, \mathbf{L}\psi)$ and $(\mathbf{L}G, \mathbf{L}\phi)$ is a 1-morphism from $\mathscr{D}(X')$ to $\mathscr{D}(X)$ defined by

$$(\mathbf{L}F, \mathbf{L}\psi) \circ (\mathbf{L}G, \mathbf{L}\phi) := (\mathbf{L}F \circ \mathbf{L}G, \mathbf{L}\psi \circ \mathbf{L}\phi),$$

where $\mathbf{L}F \circ \mathbf{L}G := (\mathbf{L}F(i) \circ \mathbf{L}G(i))_{i \in I_0}$ and for each $a \colon i \to j$ in $I$, $(\mathbf{L}\psi \circ \mathbf{L}\phi)(a) := (\mathbf{L}F(j) \circ \mathbf{L}\phi(a)) \bullet (\mathbf{L}\psi(a) \circ \mathbf{L}G(i))$ is the pasting of the diagram

$$\begin{tikzcd}
\mathscr{D}(X'(i)) \arrow[r, "\mathbf{L}G(i)"] \arrow[d, "\mathscr{D}(X'(a))"'] & \mathscr{D}(\mathscr{T}(i)) \arrow[r, "\mathbf{L}F(i)"] \arrow[d, "\mathscr{D}(\mathscr{T}(a))"] \arrow[dl, Rightarrow, "(\mathbf{L}\phi)(a)"'] & \mathscr{D}(X(i)) \arrow[d, "\mathscr{D}(X(a))"] \arrow[dl, Rightarrow, "(\mathbf{L}\psi)(a)"] \\
\mathscr{D}(X'(j)) \arrow[r, "\mathbf{L}G(j)"'] & \mathscr{D}(\mathscr{T}(j)) \arrow[r, "\mathbf{L}F(j)"'] & \mathscr{D}(X(j))
\end{tikzcd}. \tag{9.26}$$

Therefore,

$$\begin{aligned}
\mathbf{L}F(j)(\mathbf{L}\phi)(a) \colon \mathbf{L}F(j) \circ \mathscr{D}(\mathscr{T}(a)) \circ \mathbf{L}G(i) &\overset{\cong}{\Rightarrow} \mathbf{L}F(j) \circ \mathbf{L}G(j) \circ \mathscr{D}(X'(a)) \\
(\mathbf{L}\psi)(a)\mathbf{L}G(i) \colon \mathscr{D}(X(a)) \circ \mathbf{L}F(i) \circ \mathbf{L}G(i) &\overset{\cong}{\Rightarrow} \mathbf{L}F(j) \circ \mathscr{D}(\mathscr{T}(a)) \circ \mathbf{L}G(i)
\end{aligned}.$$

Consequently,

$$\begin{aligned}
&(\mathbf{L}\psi \circ \mathbf{L}\phi)(a) := (\mathbf{L}F(j) \circ \mathbf{L}\phi(a)) \bullet (\mathbf{L}\psi(a) \circ \mathbf{L}G(i)) \colon \\
&\mathscr{D}(X(a)) \circ \mathbf{L}F(i) \circ \mathbf{L}G(i) \overset{\cong}{\Rightarrow} \mathbf{L}F(j) \circ \mathbf{L}G(j) \circ \mathscr{D}(X'(a))
\end{aligned}$$

We show that $\text{-} \overset{\mathbf{L}}{\otimes} Z(a)$ is equivalent to $(\mathbf{L}\psi \circ \mathbf{L}\phi)(a)$.

From the definition of $Z(a)$, we have the following diagram:

$$
\begin{tikzcd}
\text{-}\overset{\mathbf{L}}{\otimes} Z(i) \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)} \arrow[r, dashed, "\text{-}\overset{\mathbf{L}}{\otimes} Z(a)"] \arrow[d, equal] & \text{-}\overset{\mathbf{L}}{\otimes} \overline{X'(a)} \overset{\mathbf{L}}{\otimes}_{X'(j)} Z(j) \arrow[d, equal] \\
\text{-}\overset{\mathbf{L}}{\otimes}(M(i) \otimes_{\mathscr{T}(i)} \mathbf{p}N(i)) \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)} \arrow[d, "\cong"] & \text{-}\overset{\mathbf{L}}{\otimes} \overline{X'(a)} \overset{\mathbf{L}}{\otimes}_{X'(j)} (M(j) \otimes_{\mathscr{T}(j)} \mathbf{p}N(j)) \\
\text{-}\overset{\mathbf{L}}{\otimes}(M(i) \overset{\mathbf{L}}{\otimes}_{\mathscr{T}(i)} N(i)) \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)} \arrow[d, "\boldsymbol{a}^{-1}_{M(i),N(i),\overline{X(a)}}"] & \text{-}\overset{\mathbf{L}}{\otimes} \overline{X'(a)} \overset{\mathbf{L}}{\otimes}_{X'(j)} (M(j) \overset{\mathbf{L}}{\otimes}_{\mathscr{T}(j)} N(j)) \arrow[u, "\cong"] \\
\text{-}\overset{\mathbf{L}}{\otimes} M(i) \overset{\mathbf{L}}{\otimes}_{\mathscr{T}(i)} (N(i) \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}) \arrow[d, dashed, "M(i)\overset{\mathbf{L}}{\otimes}_{\mathscr{T}(i)} N(a)"] & (\overline{X'(a)} \overset{\mathbf{L}}{\otimes}_{X'(j)} M(j)) \overset{\mathbf{L}}{\otimes}_{\mathscr{T}(j)} N(j) \arrow[u, "\boldsymbol{a}^{-1}_{\overline{X'(a)},M(j),N(j)}"] \\
\text{-}\overset{\mathbf{L}}{\otimes} M(i) \overset{\mathbf{L}}{\otimes}_{\mathscr{T}(i)} (\overline{\mathscr{T}(a)} \overset{\mathbf{L}}{\otimes}_{\mathscr{T}(j)} N(j)) \arrow[r, "\boldsymbol{a}_{M(i),\overline{\mathscr{T}(a)},N(j)}"'] & (\text{-}\overset{\mathbf{L}}{\otimes} M(i) \overset{\mathbf{L}}{\otimes}_{\mathscr{T}(i)} \overline{\mathscr{T}(a)}) \overset{\mathbf{L}}{\otimes}_{\mathscr{T}(j)} N(j), \arrow[u, "M(a)\overset{\mathbf{L}}{\otimes}_{\mathscr{T}(j)} N(j)"]
\end{tikzcd}
$$

where $\text{-}\overset{\mathbf{L}}{\otimes} N(a)$ and $\text{-}\overset{\mathbf{L}}{\otimes} M(a)$ are 2-isomorphisms. Note that for each $i \in I_0$, we have a 2-isomorphism

$$\xi_i \colon \mathbf{L}F(i) \circ \mathbf{L}G(i) = (\text{-}\overset{\mathbf{L}}{\otimes} N(i)) \circ (\text{-}\overset{\mathbf{L}}{\otimes} M(i)) \overset{\cong}{\Rightarrow} \text{-}\overset{\mathbf{L}}{\otimes}(M(i) \otimes \mathbf{p}N(i)) = \text{-}\overset{\mathbf{L}}{\otimes} Z(i),$$

we get the following commutative diagram

$$
\begin{tikzcd}
(\text{-}\overset{\mathbf{L}}{\otimes} X(a)) \circ \mathbf{L}F(i) \circ \mathbf{L}G(i) \arrow[r, "(\mathbf{L}\psi \circ \mathbf{L}\phi)(a)"] \arrow[d, "\xi_i \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}"', "\cong"] & \mathbf{L}F(j) \circ \mathbf{L}G(j) \circ (\text{-}\overset{\mathbf{L}}{\otimes} \overline{X'(a)}) \arrow[d, "\text{-}\overset{\mathbf{L}}{\otimes}_{X'(i)} \overline{X'a)} \overset{\mathbf{L}}{\otimes}_{X'(j)} \xi_i"', "\cong"] \\
\text{-}\overset{\mathbf{L}}{\otimes} Z(i) \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)} \arrow[r, "\text{-}\overset{\mathbf{L}}{\otimes} Z(a)"] & \text{-}\overset{\mathbf{L}}{\otimes}_{X'((i)} \overline{X'(a)} \overset{\mathbf{L}}{\otimes}_{X'(j)} Z(j)
\end{tikzcd}
\quad . \tag{9.27}
$$

Therefore, we have

$$
\begin{aligned}
&(\mathbf{L}F, \mathbf{L}\psi) \circ (\mathbf{L}G, \mathbf{L}\phi) := (\mathbf{L}F \circ \mathbf{L}G, \mathbf{L}\psi \circ \mathbf{L}\phi) \\
=&((\mathbf{L}F \circ \mathbf{L}G)(i), (\mathbf{L}\psi \circ \mathbf{L}\phi)(a))_{i \in I_0, a \in I_1} \\
=&(\mathbf{L}G(i) \circ \mathbf{L}F(i), (\mathbf{L}F(j) \circ \mathbf{L}\phi(a)) \bullet (\mathbf{L}\psi(a) \circ \mathbf{L}G(i)))_{i \in I_0, a \in I_1} \quad \text{by (9.26)} \\
\cong&(\text{-}\overset{\mathbf{L}}{\otimes}_{X'(i)} Z(i), \text{-}\overset{\mathbf{L}}{\otimes} Z(a))_{i \in I_0, a \in I_1} \quad \text{by (9.27)} \\
=&\,\text{-}\overset{\mathbf{L}}{\otimes}_{X'} Z.
\end{aligned}
$$

Then $\text{-}\overset{\mathbf{L}}{\otimes}_{X'} Z \colon \mathscr{D}(X') \to \mathscr{D}(X)$ is an equivalence in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$. □

**Definition 9.18.** Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. The equivalence of the form in the statement (1) in Theorem 9.17 above (namely, of the form $\text{-}\overset{\mathbf{L}}{\otimes}_{X'} Z \colon \mathscr{D}(X') \to \mathscr{D}(X)$ in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$ for an $X'$-$X$-bimodule $Z$) is called a *standard derived equivalence* from $X'$ to $X$, and if it exists, then we say that $X'$ *is standardly* *derived equivalent to* $X$. We denote this fact by $X' \overset{\mathrm{sd}}{\rightsquigarrow} X$. At present we do not know whether this relation is symmetric or not. See Problem 9.20.

**Remark 9.19.** For dg categories $\mathscr{A}, \mathscr{B}$, the relation that $\mathscr{B}$ is standardly derived equivalent to $\mathscr{A}$ (see Definition 9.5), is known to be symmetric (take $Z^\top$ instead of $Z$, see [28, 6.2]).

**Problem 9.20.** *Let* $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$*. Under which condition,* $X' \overset{\mathrm{sd}}{\rightsquigarrow} X$ *implies* $X \overset{\mathrm{sd}}{\rightsquigarrow} X'$*?*

## 10. Derived equivalences of Grothendieck constructions

In this section, we give our second main result stating that for $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$, if $X'$ is standardly derived equivalent to $X$, then their Grothendieck constructions are derived equivalent.

10.1. **Quasi-equivalence 1-morphisms.** In this subsection, we use the contents in Appendix A.

**Proposition 10.1.** *Let* $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$*. Assume that* $(F, \psi)\colon X \to X'$ *in* $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$ *is a quasi-equivalence 1-morphism. Then* $\int(F,\psi)\colon \int X \to \int X'$ *is a quasi-equivalence.*

*Proof.* Recall that a 1-morphism

$$\textstyle\int(F,\psi)\colon \int X \to \int X'$$

in $\Bbbk$-$\mathbf{dgCat}$ is defined by

- for each ${}_ix \in \int(X)_0$, $\int(F,\psi)({}_ix) := {}_i(F(i)x)$, and
- for each ${}_ix, {}_jy \in (\int X)_0$ and each $f = (f_a)_{a\in I(i,j)} \in (\int X)({}_ix, {}_jy)$, $\int(F,\psi)(f) := (F(j)f_a \circ \psi(a)x)_{a\in I(i,j)}$, where each entry is the composite of
$$X'(a)F(i)x \xrightarrow{\psi(a)x} F(j)X(a)x \xrightarrow{F(j)f_a} F(j)y.$$

Then we have the following

$$\begin{array}{ccc} (\int X)({}_ix, {}_jy) & \xrightarrow{\int(F,\psi)} & (\int X')(\int(F,\psi)({}_ix), \int(F,\psi)({}_jy)) \\ \| & & \| \\ \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y) & \xrightarrow[\int(F,\psi)]{} & \bigoplus_{a\in I(i,j)} X'(j)(X'(a)F(i)x, F(j)y) \end{array} \tag{10.28}$$

Assume that $(F, \psi) : X \to X'$ is a quasi-equivalence, that is

(1) For each $i \in I_0$, $F(i) : X(i) \to X'(i)$ is a quasi-equivalence; and
(2) For each $a \in I_1$, $\psi(a)$ is a 2-quasi-isomorphism.

**Claim 1.** *Let* ${}_ix, {}_jy \in (\int X)_0$*. Then the restriction*

$$\textstyle\int(F,\psi)_{{}_ix,{}_jy}\colon (\int X)({}_ix, {}_jy) \to (\int X')(\int(F,\psi)({}_ix), \int(F,\psi)({}_jy))$$

*of* $\int(F,\psi)$ *to* $(\int X)({}_ix, {}_jy)$ *is a quasi-isomorphism.*

Indeed, we have to show that for each $k \in \mathbb{Z}$, the following is an isomorphism:

$$H^k(\textstyle\int X)({}_ix, {}_jy) \xrightarrow{H^k(\int(F,\psi)_{{}_ix,{}_jy})} H^k(\textstyle\int X')(\int(F,\psi)({}_ix), \int(F,\psi)({}_jy)).$$

By (10.28), it is decomposed as follows:

$$\begin{tikzcd}[column sep=small]
H^k((\int X)({}_ix,{}_jy)) \arrow[rr, "H^k(\int(F,\psi)_{{}_ix,{}_jy})"] \arrow[d, equal] & & H^k((\int X')(\int(F,\psi)({}_ix), \int(F,\psi)({}_jy))) \arrow[d, equal] \\
\bigoplus_{a\in I(i,j)} H^k(X(j)(X(a)x,y)) \arrow[dr, "\bigoplus_{a\in I(i,j)} H^k(F(j)_{X(a)x,y})"'] & & \bigoplus_{a\in I(i,j)} H^k(X'(j)(X'(a)F(i)x,F(j)y)) \\
& \bigoplus_{a\in I(i,j)} H^k(X'(j)(F(j)X(a)x,F(j)y) \arrow[ur, "\bigoplus_{a\in I(i,j)} H^k(X'(j)(\psi(a)_x,F(j)y))"'] &
\end{tikzcd} \tag{10.29}$$

By assumption, $H^k(F(j)_{X(a)x,y})$ is an isomorphism for all $a \in I(i,j)$, and hence so is $\bigoplus_{a\in I(i,j)} H^k(F(j))$. Therefore, it remains to show that

$$\bigoplus_{a\in I(i,j)} H^k(X'(j)(\psi(a)_x, F(j)y))$$

is an isomorphism. Let $a\colon i \to j$ be a morphism in $I$. Then since

$$\psi(a)\colon X'(a)F(i) \Rightarrow F(j)X(a)$$

is a 2-quasi-isomorphism, by definition, we have a 2-isomorphism

$$\text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{\psi(a)}\colon \text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X'(a)F(i)} \Rightarrow \text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{F(j)X(a)}.$$

By specializing at $x^\wedge \in \mathscr{D}(X(i))_0$, this yields an isomorphism

$$x^\wedge \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{\psi(a)}\colon x^\wedge \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X'(a)F(i)} \to x^\wedge \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{F(j)X(a)},$$

i.e., an isomorphism

$$(\psi(a)_x)^\wedge\colon (X'(a)F(i)(x))^\wedge \xrightarrow{\sim} (F(j)X(a)(x))^\wedge$$

in $\mathscr{D}(X'(j))$. Since we have a commutative diagram

$$\begin{tikzcd}
\mathscr{D}(X'(j))((F(j)X(a)(x))^\wedge, (F(j)(y))^\wedge[k]) \arrow[dr, "\mathscr{D}(X'(j))((\psi(a)_x)^\wedge, (F(j)(y))^\wedge[k])"] \arrow[dd, "\cong"'] & \\
& \mathscr{D}(X'(j))((X'(a)F(i)(x))^\wedge, (F(j)(y))^\wedge[k]) \arrow[dd, "\cong"] \\
H^k X'(j)(F(j)X(a)(x), F(j)(y)) \arrow[dr, "H^k(X'(j)(\psi(a)_x, F(j)y))"] & \\
& H^k(X'(j)(X'(a)F(i)(x), F(j)y)
\end{tikzcd}$$

with the vertical canonical isomorphisms, we see that

$$\begin{aligned} H^k(X'(j)(\psi(a)_x, F(j)y))\colon H^k(X'(j)(F(j)X(a)(x), F(j)(y)) \\ \to H^k(X'(j)(X'(a)F(i)(x), F(j)y) \end{aligned}$$

is an isomorphism, and hence so is $\bigoplus_{a\in I(i,j)} H^k(X'(j)(\psi(a)_x, F(j)y))$, as desired. Therefore, we conclude that $H^k(\int(F,\psi)_{{}_ix,{}_jy})$ is an isomorphim by the commutative diagram (10.29). Hence it follows that $\int(F,\psi)_{{}_ix,{}_jy}$ is a quasi-isomorphism for all ${}_ix$ and ${}_jy$.

Next we show the following:

**Claim 2.** $H^0(\int X) \xrightarrow{H^0((\int(F,\psi))} H^0(\int X')$ *is an equivalence.*

By Claim 1 for $k = 0$, we have that

$$\bigoplus_{a\in I(i,j)} H^0(X(j)(X(a)x, y)) \xrightarrow{H^0(\int(F,\psi)_{{}_ix,{}_jy})} \bigoplus_{a\in I(i,j)} H^0((X'(j)(X'(a)F(i)x, F(j)y))$$

is bijective for all ${}_ix$ and ${}_jy$. Thus,

$$H^0(\textstyle\int(F,\psi))\colon H^0(\int X) \to H^0(\int X')$$

is fully faithful. It only remains to show that it is dense. By the definition of Grothendieck construction, we have

$$H^0(\textstyle\int X')_0 = H^0(\bigsqcup_{i\in I_0} X'(i)_0) = \bigsqcup_{i\in I_0} H^0(X'(i))_0 = \bigsqcup_{i\in I_0} X'(i)_0.$$

For any ${}_ix' \in \bigsqcup_{i\in I_0} X'(i)_0$ with $i \in I_0$ and $x' \in X'(i)_0$, note that

$$H^0(X(i)) \xrightarrow{H^0(F(i))} H^0(X'(i))$$

is dense by (1) above. Thus there exists $x \in X(i)_0$ such that $y := F(i)(x) = H^0(F(i)(x)) \cong x'$ in $H^0(X'(i))$. Thus there exists $f : x' \xrightarrow{\sim} y$ in $H^0(X'(i))$. Since

$$H^0(\textstyle\int(F,\psi))({}_ix) = \int(F,\psi)({}_ix) = {}_iF(i)(x) = {}_iy,$$

it suffices to show that ${}_iy \cong {}_ix'$ in $H^0(\int(X'))$. Noting that

$$\begin{aligned}
H^0(\textstyle\int X')({}_ix', {}_iy) &= H^0(\textstyle\int X')({}_ix', {}_iy)) = H^0(\bigoplus_{a\in I(i,i)}(X'(i)(X'(a)x', y)) \\
&= \bigoplus_{a\in I(i,i)} H^0((X'(i)(X'(a)x', y)), \text{ and} \\
H^0(\textstyle\int X')({}_iy, {}_ix') &= H^0(\textstyle\int X')({}_iy, {}_ix')) = H^0(\bigoplus_{a\in I(i,i)}(X'(i)(X'(a)y, x')) \\
&= \bigoplus_{a\in I(i,i)} H^0((X'(i)(X'(a)y, x')),
\end{aligned}$$

we can take elements

$$\begin{aligned}
(\delta_{b,\mathbf{1}_i} f^{-1} \circ X'_i(y))_{b\in I(i,i)} &\in \bigoplus_{a\in I(i,i)} H^0((X'(i)(X'(a)y, x')), \text{and} \\
(\delta_{a,\mathbf{1}_i} f \circ X'_i(x))_{a\in I(i,i)} &\in \bigoplus_{a\in I(i,i)} H^0((X'(i)(X'(a)x', y)),
\end{aligned}$$

where entries are of the following forms

$$X'(\mathbb{1}_i)y \xrightarrow{X'_i(y)} y \xrightarrow{f^{-1}} x', \quad X'(\mathbb{1}_i)x' \xrightarrow{X'_i(x')} x' \xrightarrow{f} y,$$

respectively. A direct calculation shows that

$$\begin{aligned}
(\delta_{b,\mathbf{1}_i} f^{-1} \circ X'_i(y))_{b\in I(i,i)} \circ (\delta_{a,\mathbf{1}_i} f \circ X'_i(x'))_{a\in I(i,i)} &= \mathbb{1}_{{}_ix'}, \\
(\delta_{a,\mathbf{1}_i} f \circ X'_i(x'))_{a\in I(i,i)} \circ (\delta_{b,\mathbf{1}_i} f^{-1} \circ X'_i(y))_{b\in I(i,i)} &= \mathbb{1}_{{}_iy}
\end{aligned}$$

Then we have ${}_iy \cong {}_ix'$ in $H^0(\int X')$. Therefore $H^0(\int(F,\psi))$ is dense. $\square$

The following is the main result in this subsection.

**Theorem 10.2.** *Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Assume that $X$ is $\Bbbk$-flat. If there exists a tilting colax functor $\mathscr{T}$ for $X$ and there exists a quasi-equivalence 1-morphism from $X'$ to $\mathscr{T}$, then $\int X'$ is derived equivalent to $\int X$.*

*Proof.* Note that $\int(X)$ is also $\Bbbk$-flat by definition of $\int(X)$. Let $\mathscr{T}$ be a tilting colax subfunctor of $\mathscr{C}_{\mathrm{dg}}(X)$ with an $I$-equivariant inclusion $(\sigma,\rho)\colon \mathscr{T} \hookrightarrow \mathscr{C}_{\mathrm{dg}}(X)$. Put $(P,\phi) := (P_X, \phi_X)$ for short. Let $\mathscr{T}'$ be the full dg subcategory of $\mathscr{C}_{\mathrm{dg}}(\int X)$ consisting of the objects $\mathrm{per}(P(i))(U)$ ($\in \mathrm{per}(\int X)$) with $i \in I_0$ and $U \in \mathscr{T}(i)_0$, which is called the *gluing* of $\mathscr{T}(i)$'s.

We now show that $\mathscr{T}'$ is a tilting dg subcategory for $\int(X)$. By Proposition 7.31, the objects in $\mathscr{T}'(i)$ are homotopically projective for all $i \in I_0$. For each $i \in I_0$ and $x \in X(i)$, we have

$$\begin{aligned}
\mathrm{per}(P(i))(X(i)(\text{-}, x)) &\cong X(i)(\text{-}, x)) \otimes_{X(i)} \overline{P(i)} \\
&= X(i)(\text{-}, x)) \otimes_{X(i)} (\textstyle\int X)(\text{-}, P(i)(?)) \\
&\cong (\textstyle\int X)(\text{-}, P(i)(x)) = (\textstyle\int X)(\text{-}, {}_ix).
\end{aligned}$$

Thus

$$\begin{aligned}(\textstyle\int X)(\text{-}, {}_i x) &\cong \operatorname{per}(P(i))(X(i)(\text{-}, x))\\ &\in \operatorname{per}(P(i))(\operatorname{thick}\mathscr{T}(i))\\ &\subseteq \operatorname{thick}\{\operatorname{per}(P(i))(U) \mid U \in \mathscr{T}(i)\}\\ &\subseteq \operatorname{thick}\mathscr{T}'.\end{aligned}$$

Therefore, $\operatorname{thick}\mathscr{T}' = \operatorname{per}(\int X)$, and hence $\mathscr{T}'$ is a tilting dg subcategory for $\int X$, as desired. In particular, we see that $\int X$ is derived equivalent to $\mathscr{T}'$. Let $(F, \psi)$ be the restriction of $\operatorname{per}((P, \phi))$ to $\mathscr{T}$. Then by construction $(F, \psi)\colon \mathscr{T} \to \Delta(\mathscr{T}')$ is a dense functor, and it is an $I$-precovering because so is

$$\operatorname{per}((P, \phi))\colon \operatorname{per}(X) \to \Delta(\operatorname{per}(\textstyle\int X))$$

by Proposition 7.30. Thus $(F, \psi)$ is an $I$-covering, which shows that $\mathscr{T}' \simeq \int \mathscr{T}$ by Corollary 6.3. Thus we have

$$\textstyle\int X \overset{\text{der}}{\sim} \int \mathscr{T}. \tag{10.30}$$

Since there exists a quasi-equivalence from $X'$ to $\mathscr{T}$ in $\operatorname{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$, we have a quasi-equivalence from $\int X'$ to $\int \mathscr{T}$ ($\simeq \mathscr{T}'$) in $\Bbbk\text{-}\mathbf{dgCat}$ by Proposition 10.1, and hence $\int X'$ and $\int \mathscr{T}$ are derived equivalent by Theorem A.1. As a consequence, $\int X'$ is derived equivalent to $\int X$. □

Let $X, X' \in \operatorname{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Since the relation that $X'$ is quasi-equivalent to $X$ is not symmetric with respect to $X'$ and $X$, we consider a zigzag chain of quasi-equivalences between them defined as follows.

**Definition 10.3.** Let $X, X' \in \operatorname{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then a *zigzag chain of quasi-equivalence 1-morphisms* between $X$ and $X'$ is a chain of 1-morphisms of the form

$$X =: X_0 \xleftarrow{(F_1,\psi_1)} X_1 \xrightarrow{(F_2,\psi_2)} \cdots \xleftarrow{(F_{n-1},\psi_{n-1})} X_{n-1} \xrightarrow{(F_n,\psi_n)} X_n := X'$$

in $\operatorname{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$ with $n$ even $\geq 2$, where $(F_i, \psi_i)$ are quasi-equivalence 1-morphisms for all $i = 1, \dots, n$. Note that a quasi-equivalence 1-morphism $X \xrightarrow{(F_2,\psi_2)} X'$ is also regarded as a zigzag chain of quasi-equivalence 1-morphism by setting $n = 2$ and $(F_1, \psi_1)$ to be the identity 1-morphism.

The following is immediate from Proposition 10.5, Theorems A.1 and 10.2.

**Corollary 10.4.** *Let $X, X' \in \operatorname{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Assume that $X$ is $\Bbbk$-flat and that there exists a tilting colax functor $\mathscr{T}$ for $X$ such that there exists a zigzag chain of quasi-equivalence 1-morphisms between $X'$ and $\mathscr{T}$ in $\operatorname{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then $\int X'$ and $\int X$ are derived equivalent.*

*Proof.* By assumption, we have a zigzag chain of quasi-equivalences between $X'$ and $\mathscr{T}$ of the form

$$\mathscr{T} =: X_0 \xleftarrow{(F_1,\psi_1)} X_1 \xrightarrow{(F_2,\psi_2)} \cdots \xleftarrow{(F_{n-1},\psi_{n-1})} X_{n-1} \xrightarrow{(F_n,\psi_n)} X_n := X'$$

in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$ with $n$ even $\geq 2$, where $(F_i, \psi_i)$ are quasi-equivalences for all $i = 1, \dots, n$. Then by Proposition 10.1, we have a zigzag chain

$$\int \mathscr{T} =: \int X_0 \leftarrow \int X_1 \to \cdots \leftarrow \int X_{n-1} \to \int X_n := \int X'$$

of quasi-equivalences of dg categories, and hence a sequence of derived equivalences

$$\int \mathscr{T} =: \int X_0 \overset{\text{der}}{\sim} \int X_1 \overset{\text{der}}{\sim} \cdots \overset{\text{der}}{\sim} \int X_{n-1} \overset{\text{der}}{\sim} \int X_n := \int X'$$

by Theorem A.1. Since to be derived equivalent is an equivalence relation, we have $\int X' \overset{\text{der}}{\sim} \int \mathscr{T}$. It follows from (the proof of) Theorem 10.2 (see (10.32)) that $\int X \overset{\text{der}}{\sim} \int \mathscr{T}$. Therefore, $\int X'$ and $\int X$ are derived equivalent. $\square$

### 10.2. **Quasi-equivalence bimodules.**

**Proposition 10.5.** *Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Assume that there exists a quasi-equivalence $X'$-$X$-bimodule $E$. Then we have a quasi-equivalence $\int X'$-$\int X$-bimodule $\int E$.*

*Proof.* We construct a $\int X'$-$\int X$-bimodule $\int E$, namely a dg functor

$$\int E \colon (\textstyle\int X)^{\mathrm{op}} \otimes_{\Bbbk} (\int X') \to \mathscr{C}_{\mathrm{dg}}(\Bbbk).$$

as follows.

**On objects.** Let $({}_ix, {}_{i'}x') \in (\int X)^{\mathrm{op}}_0 \times (\int X')_0$, where $i, i' \in I_0$, $x \in X(i)_0$, $x' \in X'(i')_0$. Then we set

$$(\textstyle\int E)({}_ix, {}_{i'}x') := \bigoplus_{a \in I(i,i')} E(i')(X(a)x, x') \in \mathscr{C}_{\mathrm{dg}}(\Bbbk)_0.$$

**On morphisms.** Let $f \in \int(X)^{\mathrm{op}}({}_ix, {}_jy) = \int(X)({}_jy, {}_ix) = \bigoplus_{c \in I(j,i)} X(i)(X(c)y, x)$ and $g \in \int(X')({}_{i'}x', {}_{j'}y') = \bigoplus_{d \in I(i',j')} X(j')(X'(d)x', y')$. Then by noting that $f_c \in X(j)(X(c)y, x)$ and $g_d \in X(j')(X'(d)x', y')$ for all $c \in I(j,i), d \in I(i',j')$, we define $(\int E)(f \otimes g)$ as follows (see the diagram below).

$$\begin{tikzcd}
(\int E)({}_ix, {}_{i'}x') \arrow[r, "(\int E)(f \otimes g)"] \arrow[d, equal] & (\int E)({}_jy, {}_{j'}y') \arrow[d, equal] \\
\bigoplus_{a \in I(i,i')} E(i')(X(a)x, x') \arrow[r, dashed] & \bigoplus_{b \in I(j,j')} E(j')(X(b)y, y')
\end{tikzcd}$$

Let $(v_a)_{a \in I(i,i')} \in \bigoplus_{a \in I(i,i')} E(i')(X(a)x, x')$, where for each $a \in I(i,i')$, we have $v_a \in E(i')(X(a)x, x')$. Here in $I$, we have the following setting:

$$\begin{tikzcd}
i \arrow[d, "a"'] & j \arrow[l, "c"'] \arrow[d, "b"] \\
i' \arrow[r, "d"'] & j'
\end{tikzcd}$$

We then set

$$
\begin{aligned}
&(\textstyle\int E)(f \otimes g)((v_a)_{a \in I(i,i')}) \\
&:= \left( \sum_{b=dac} ((X(d) \circ X_{ac}) \bullet X_{d,ac})_y \cdot g_d \cdot E(d)(v_a) \cdot X(d)X(a)(f_c) \right)_{b \in I(j,j')},
\end{aligned}
$$

where the sum is taken over all triples $(c, a, d) \in I(j, i) \times I(i, i') \times I(i', j')$ such that $b = dac$. The right half of the terms in the summation can be visualized in the following diagram, where the dashed arrows stand for "elements" of $E$:

$$
\begin{tikzcd}
X(a)X(c)y \arrow[r, "X(a)(f_c)"] \arrow[d, "X(d)"] \arrow[dr, phantom, "\downarrow X(d)"] & X(a)x \arrow[r, dashed, "v_a"] \arrow[d, "X(d)"] \arrow[dr, phantom, "\downarrow E(d)"] & x' \arrow[d, "X'(d)"] & \\
X(d)X(a)X(c)y \arrow[r, "X(d)X(a)(f_c)"'] & X(d)X(a)x \arrow[r, dashed, "E(d)(v_a)"'] & X'(d)x' \arrow[r, "g(d)"'] & y'
\end{tikzcd},
$$

and the left half in the following:

$$
X(b) = X(dac) \xrightarrow{(X_{d,ac})_y} X(d)X(ac) \xrightarrow{X(d)\circ(X_{ac})_y} X(d)X(a)X(c).
$$

Note that $E(d)\colon E(i') \to E(j')$ is an $X'(i')$-$X(i')$-bimodule morphism, where the $X'(i')$-$X(i')$-bimodule structure of $E(j')$ is defined from its $X'(j')$-$X(j')$-bimodule structure by using the functors $X(d)\colon X(i') \to X(j')$ and $X'(d)\colon X'(i') \to X'(j')$, which explains the vertical correspondence in the diagram above.

We next show that $\int E$ is a quasi-equivalence bimodule. We have to show the conditions (a) and (b) in Definition 9.4. We show the condition (a) by using Lemma 9.3. Consider the canonical morphism $(P, \phi)\colon X \to \Delta(\int X)$. Then for each $i \in I_0$ and $x \in X(i)$, we have

$$
\begin{aligned}
\operatorname{per}(P(i))(X(i)(\text{-}, x)) &\cong X(i)(\text{-}, x) \otimes_{X(i)} \overline{P(i)} \\
&= X(i)(\text{-}, x) \otimes_{X(i)} (\textstyle\int X)(\text{-}, P(i)(?)) \\
&\cong (\textstyle\int X)(\text{-}, P(i)(x)) = (\textstyle\int X)(\text{-}, {}_i x).
\end{aligned}
$$

For each $i \in I_0$ and $x' \in X'(i)$, there exists some $x \in X(i)_0$ such that $E(i)(\text{-}, x') \cong X(i)(\text{-}, x)$.

$$
\begin{aligned}
\operatorname{per}(P(i))(E(i)(\text{-}, x')) &\cong E(i)(\text{-}, x') \otimes_{X(i)} \overline{P(i)} \\
&= E(i)(\text{-}, x') \otimes_{X(i)} (\textstyle\int X)(\text{-}, P(i)(?)) \\
&\cong X(i)(\text{-}, x) \otimes_{X(i)} (\textstyle\int X)(\text{-}, P(i)(?)) \\
&\cong (\textstyle\int X)(\text{-}, P(i)(x)) = (\textstyle\int X)(\text{-}, {}_i x).
\end{aligned}
$$

$$
\begin{aligned}
(\textstyle\int E)({}_?\text{-}, {}_i x') &:= \bigoplus_{a \in I(?,i)} E(i)(X(a)(\text{-}), x') \\
&\cong \bigoplus_{a \in I(?,i)} X(i)(X(a)(\text{-}), x) = (\textstyle\int X)({}_?\text{-}, {}_i x).
\end{aligned}
\tag{10.31}
$$

Thus

$$\begin{aligned}(\textstyle\int X)(\text{-}, {}_i x) &\cong \operatorname{per}(P(i))(X(i)(\text{-}, x)) \\ &\in \operatorname{per}(P(i))(\operatorname{thick} E(i)(\text{-}, x')) \\ &\subseteq \operatorname{thick}\{\operatorname{per}(P(i))E(i)(\text{-}, x')\} \\ &\subseteq \operatorname{thick}(\textstyle\int E)(\text{-}, {}_i x').\end{aligned}$$

Therefore, $\operatorname{thick}(\int E) = \operatorname{per}(\int X)$. Namely, the conditions (1) and (3) in Lemma 9.3 hold. It remains to show the condition (2) of this lemma, that is, to show that

$$(\textstyle\int X')({}_i x', {}_j y') \to \mathbf{R}\mathscr{C}_{\mathrm{dg}\int X}((\textstyle\int E)(\text{-}, {}_i x'), (\int E)(\text{-}, {}_j y'))$$

is a quasi-isomorphism.

$$\begin{aligned}&\mathbf{R}\mathscr{C}_{\mathrm{dg}\int X}((\textstyle\int E)(\text{-}, {}_i x'), (\int E)(\text{-}, {}_j y')) \cong \mathscr{C}_{\mathrm{dg}}(\int X)((\int X)(\text{-}, {}_i x), (\int X)(\text{-}, {}_j y)) \\ &\cong (\textstyle\int X)({}_i x, {}_j y) = \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) \\ &\cong \bigoplus_{a \in I(i,j)} \mathscr{C}_{\mathrm{dg}}(X(j))(X(j)(\text{-}, X(a)x), X(j)(\text{-}, y))) \\ &\cong \bigoplus_{a \in I(i,j)} \mathscr{C}_{\mathrm{dg}}(X(j))(X(i)(\text{-}, x) \otimes_{X(i)} \overline{X(a)}, X(j)(\text{-}, y))) \\ &\cong \bigoplus_{a \in I(i,j)} \mathscr{C}_{\mathrm{dg}}(X(j))((E(i)(\text{-}, x') \otimes_{X(i)} \overline{X(a)}, E(j)(\text{-}, y'))) \\ &\cong \bigoplus_{a \in I(i,j)} \mathscr{C}_{\mathrm{dg}}(X(j))(X'(i)(\text{-}, x') \otimes_{X'(i)} \overline{X'(a)} \otimes_{X'(j)} E(j), E(j)(\text{-}, y'))) \\ &\cong \bigoplus_{a \in I(i,j)} \mathscr{C}_{\mathrm{dg}}(X(j))(X'(j)(\text{-}, X'(a)x') \otimes_{X'(j)} E(j), E(j)(\text{-}, y'))) \\ &\cong \bigoplus_{a \in I(i,j)} \mathscr{C}_{\mathrm{dg}}(X(j))((E(j)(\text{-}, X'(a)x'), E(j)(\text{-}, y'))).\end{aligned}$$

See the diagram for the last three lines:

$$\begin{tikzcd}
& \mathscr{D}(X'(i)) \arrow[rr, "\text{-}\overset{\mathbf{L}}{\otimes}_{X'(i)} E(i)"] \arrow[dl, "\mathscr{D}X'(a)"'] & & \mathscr{D}(X(i)) \arrow[dl, "\mathscr{D}X(a)"] \arrow[dll, Rightarrow, "\text{-}\overset{\mathbf{L}}{\otimes}_{X'(i)} E(a)"'] \\
\mathscr{D}(X'(j)) \arrow[rr] & & \mathscr{D}(X(j)) & \\
& \underline{X'(i)} \arrow[uu, "\sigma'(i)"] \arrow[rr] \arrow[dl] & & \underline{X}(i) \arrow[uu] \arrow[dl] \arrow[dll, Rightarrow, "\text{-}\overset{\mathbf{L}}{\otimes}_{X'(i)} E(a)"'] \\
\underline{X'}(j) \arrow[uu] \arrow[rr] & & \underline{X}(j) \arrow[uu, "\sigma(j)"'] &
\end{tikzcd}.$$

Since

$$X'(j)(X'(a)x', y') \xrightarrow{\text{qis}} \mathscr{C}_{\mathrm{dg}}(X(j))((E(j)(\text{-}, X'(a)x'), E(j)(\text{-}, y'))$$

is a quasi-isomorphism induced by the quasi-equivalence bimodule $E$, it follows that

$$\begin{aligned}(\textstyle\int X')({}_ix', {}_jy') &= \bigoplus_{a\in I(i,j)} X'(j)(X'(a)x', y') \\ &\xrightarrow{\text{qis}} \bigoplus_{a\in I(i,j)} \mathscr{C}_{\text{dg}}(X(j))((E(j)(\text{-}, X'(a)x'), E(j)(\text{-}, y'))) \\ &\cong \mathbf{R}\mathscr{C}_{\text{dg}\int X}((\textstyle\int E)(\text{-}, {}_ix'), (\textstyle\int E)(\text{-}, {}_jy'))\end{aligned}$$

is a quasi-isomorphism. By Lemma 9.3, $\int E$ induces a derived equivalence between $\int X'$ and $\int X$. Note that the condition (b) is trivially satisfied by (10.31). Hence $\int E$ is a quasi-equivalence bimodule. □

The following is our second main result in this paper.

**Theorem 10.6.** *Let $X, X' \in \text{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Assume that $X$ is $\Bbbk$-flat. If there exist a tilting colax functor $\mathscr{T}$ for $X$ and a quasi-equivalence $X'$-$\mathscr{T}$-bimodule (namely, if $X'$ is standardly derived equivalent to $X$ by Theorem 9.17), then $\int X'$ is derived equivalent to $\int X$.*

*Proof.* Note that $\int X$ is also $\Bbbk$-flat by definition of $\int X$. Let $\mathscr{T}$ be a tilting colax subfunctor of $\mathscr{C}_{\text{dg}}(X)$ with an $I$-equivariant inclusion $(\sigma, \rho)\colon \mathscr{T} \hookrightarrow \mathscr{C}_{\text{dg}}(X)$. Put $(P, \phi) := (P_X, \phi_X)$ for short. Let $\mathscr{T}'$ be the full dg subcategory of $\mathscr{C}_{\text{dg}}(\int X)$ consisting of the objects $\text{per}(P(i))(U)$ ($\in \text{per}(\int X)$) with $i \in I_0$ and $U \in \mathscr{T}(i)_0$, which is called the *gluing* of $\mathscr{T}(i)$'s.

We now show that $\mathscr{T}'$ is a tilting dg subcategory for $\int X$. By Proposition 7.31, the objects in $\mathscr{T}'(i)$ are homotopically projective for all $i \in I_0$. For each $i \in I_0$ and $x \in X(i)$, we have

$$\begin{aligned}\text{per}(P(i))(X(i)(\text{-}, x)) &\cong X(i)(\text{-}, x) \otimes_{X(i)} \overline{P(i)} \\ &= X(i)(\text{-}, x) \otimes_{X(i)} (\textstyle\int X)(\text{-}, P(i)(?)) \\ &\cong (\textstyle\int X)(\text{-}, P(i)(x)) = (\textstyle\int X)(\text{-}, {}_ix).\end{aligned}$$

Thus

$$\begin{aligned}(\textstyle\int X)(\text{-}, {}_ix) &\cong \text{per}(P(i))(X(i)(\text{-}, x)) \\ &\in \text{per}(P(i))(\text{thick}\,\mathscr{T}(i)) \\ &\subseteq \text{thick}\{\text{per}(P(i))(U) \mid U \in \mathscr{T}(i)\} \\ &\subseteq \text{thick}\,\mathscr{T}'.\end{aligned}$$

Therefore, $\text{thick}\,\mathscr{T}' = \text{per}(\int X)$, and hence $\mathscr{T}'$ is a tilting dg subcategory for $\int X$, as desired. In particular, we see that $\int X$ is derived equivalent to $\mathscr{T}'$. Let $(F, \psi)$ be the restriction of $\text{per}((P, \phi))$ to $\mathscr{T}$. Then by construction $(F, \psi)\colon \mathscr{T} \to \Delta(\mathscr{T}')$ is a dense functor, and it is an $I$-precovering because so is

$$\text{per}((P, \phi))\colon \text{per}(X) \to \Delta(\text{per}(\textstyle\int X))$$

by Proposition 7.30. Thus $(F, \psi)$ is an $I$-covering, which shows that $\mathscr{T}' \simeq \int \mathscr{T}$ by Corollary 6.3. Thus we have

$$\int X \overset{\text{der}}{\sim} \int \mathscr{T}. \tag{10.32}$$

Since there exists a quasi-equivalence $X'$-$\mathscr{T}$-bimodule $E$, we have a quasi-equivalence $\int X'$-$\int \mathscr{T}$-bimodule $\int E$ by Proposition 10.5, and hence $\int X'$ and $\int \mathscr{T}$ are derived equivalent (see Definition 9.7). As a consequence, $\int X'$ is derived equivalent to $\int X$. □

10.3. **Group actions.** In the special case that $I = G$ is a group, which has a unique object $*$, Theorems 10.2 and 10.6 have the forms below.

**Definition 10.7.** Let $\mathscr{A}$ and $\mathscr{B}$ be dg categories with $G$-actions.

(1) A tilting dg subcategory $\mathscr{T}$ for $\mathscr{A}$ is called *$G$-equivariant* if there exists a $G$-equivariant inclusion $(\sigma, \rho) \colon \mathscr{T} \to \mathscr{C}_{\text{dg}}(\mathscr{A})$.
(2) An $\mathscr{A}$-$\mathscr{B}$-bimodule $Z$ is called a *quasi-equivalence* if (a) the functor

$$\text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} Z \colon \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$$

is an equivalence in $\mathrm{Colax}(G, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$, and (b) it gives rise to a equivalence $\underline{\mathscr{A}} \to \underline{\mathscr{B}}$.

**Corollary 10.8.** *Let $\mathscr{A}$ and $\mathscr{B}$ be dg categories with $G$-actions. Assume that $\mathscr{A}$ is $\Bbbk$-flat. If there exist a $G$-equivariant tilting dg subcategory $\mathscr{T}$ for $\mathscr{A}$, and quasi-equivalence 1-morphism from $\mathscr{B} \to \mathscr{T}$, then the orbit categories $\mathscr{A}/G$ and $\mathscr{B}/G$ are derived equivalent.*

This corollary will be applied in Example 11.8.

**Corollary 10.9.** *Let $\mathscr{A}$ and $\mathscr{B}$ be dg categories with $G$-actions. Assume that $\mathscr{A}$ is $\Bbbk$-flat. If there exist a $G$-equivariant tilting dg subcategory $\mathscr{T}$ for $\mathscr{A}$, and a quasi-equivalence $\mathscr{B}$-$\mathscr{T}$-bimodule, then the orbit categories $\mathscr{A}/G$ and $\mathscr{B}/G$ are derived equivalent.*

**Remark 10.10.** Remark 9.8 also proves Corollary 10.8 by Corollary 10.9.

10.4. **Diagonal functors.** The following is easy to verify.

**Lemma 10.11.** *Let $\mathscr{C}, \mathscr{C}'$ be in $\Bbbk$-$\mathbf{dgCat}$. If $\mathscr{C}$ and $\mathscr{C}'$ are* standardly *derived equivalent, then so are $\Delta(\mathscr{C})$ and $\Delta(\mathscr{C}')$.*

*Proof.* Since $\mathscr{C}'$ is standardly derived equivalent to $\mathscr{C}$, there exists a dg functor $H : \mathscr{C}_{\text{dg}}(\mathscr{C}') \to \mathscr{C}_{\text{dg}}(\mathscr{C})$ such that $\mathbf{L}H : \mathscr{D}(\mathscr{C}') \to \mathscr{D}(\mathscr{C})$ is an equivalence. Then $\Delta(\mathbf{L}H) : \Delta(\mathscr{D}(\mathscr{C}')) \to \Delta(\mathscr{D}(\mathscr{C}))$ is an equivalence, and it yields an equivalence $\mathbf{L}\Delta(H) : \mathscr{D}(\Delta(\mathscr{C}')) \to \mathscr{D}(\Delta(\mathscr{C}))$, where $\Delta(H) : \mathscr{C}_{\text{dg}}(\Delta(\mathscr{C}')) \to \mathscr{C}_{\text{dg}}(\Delta(\mathscr{C}))$ is a dg functor. □

Theorem 10.2 together with the lemma above and Example 5.2 gives us a unified proof of the following fact.

**Theorem 10.12.** *Assume that $\Bbbk$ is a field and that dg $\Bbbk$-algebras $A$ and $A'$ are derived equivalent. Then the following pairs are derived equivalent as well:*

(1) *dg path categories $AQ$ and $A'Q$ for any quiver $Q$;*
(2) *incidence dg categories $AS$ and $A'S$ for any poset $S$; and*
(3) *monoid dg algebras $AG$ and $A'G$ for any monoid $G$.*

*Proof.* Assume that $A$ is derived equivalent to $A'$. Then $A'$ is standardly derived equivalent to $A$. By Lemma 10.11, $\Delta(A')$ is standardly derived equivalent to $\Delta(A)$. Hence by Theorem 10.2, $\Delta(A')$ and $\Delta(A)$ are derived equivalent. $\square$

**Remark 10.13.** We remark that for a colax functor $X\colon I \to \Bbbk\text{-}\mathbf{dgCat}$, the following does not hold in general:

(*) $\mathscr{D}(\int X)$ is equivalent to $\int \mathscr{D}X$ as triangulated categories.

If this would be true, then it immediately follows that if $\mathscr{D}(X)$ and $\mathscr{D}(X')$ are equivalent, then $\mathscr{D}(\int(X))$ and $\mathscr{D}(\int(X'))$ are equivalent. Thus, our main theorem becomes trivial.

However, in many cases, (*) even does not have sense. Since $\int X \in \Bbbk\text{-}\mathbf{dgCat}_0$, the left hand side $\mathscr{D}(\int X)$ is a triangulated category. On the other hand, since $\mathscr{D}X \in \mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$, we do not know how to define the right hand side $\int \mathscr{D}X$ as a triangulated category. For example, let $I$ be a free category defined by the quiver $1 \to 2$, $A$ a dg algebra regared as a dg category with only one object, and $\Delta(A)\colon I \to \Bbbk\text{-}\mathbf{dgCat}$ the diagonal functor of $A$. Then by Example 5.2, $\int \Delta(A)$ is isomorphic to the lower triagular matrix dg algebra of $A$, or equivalently, the morphism category $\mathrm{Mor}(A)$ of the category $A$. If (*) is true, then the following holds:

(#) $\mathscr{D}(\mathrm{Mor}(A))$ is equivalent to $\mathrm{Mor}(\mathscr{D}(A))$ as categories.

Indeed, since $\mathscr{D}(\Delta(A)) \cong \Delta(\mathscr{D}(A))$ (i.e., the derived tensor product by the $A$-$A$-bimodule $A$ is isomorphic to the identity of $\mathscr{D}(A)$), we heve $\mathscr{D}(\mathrm{Mor}(A)) \cong \mathscr{D}(\int(\Delta(A)))$ is equivalent (by (*)) to $\int \mathscr{D}(\Delta(A)) \cong \int \Delta(\mathscr{D}(A)) \cong \mathrm{Mor}\,\mathscr{D}(A)$.

We do not know how to define a triangulated category structure on $\mathrm{Mor}(\mathscr{D}(A))$. Even in the case that it was possible, as is easily seen, (#) does not hold even for $A = \Bbbk$, the base field concentrated in degree 0.

## 11. Examples

In this section, we give two examples that illustrate our main theorem.

**Remark 11.1.** Let $G$ be a group, which we regard as a groupoid with only one object $*$. Let $(Q, W)$ be a quiver with potentials. Regard the complete Ginzburg dg algebra $\widehat{\Gamma}(Q, W)$ as a dg category with only one object, and a $G$-action on it as a functor $X_{Q,W}\colon G \to \Bbbk\text{-}\mathbf{dgCat}$ with $X_{Q,W}(*) = \widehat{\Gamma}(Q, W)$. Then $\int(X_{Q,W})$ is nothing but the orbit category $\widehat{\Gamma}(Q, W)/G$, which is also equivalent to the skew group dg algebra $\widehat{\Gamma}(Q, W) * G$, and is calculated as $\widehat{\Gamma}(Q_G, W_G)$ up to Morita equivaleces in the case that $G$ is a finite group in [34] (see also [23] for the finite abelian case). Therefore in this case note that $\int(X_{Q,W})$ is calculated as $\widehat{\Gamma}(Q_G, W_G)$ up to Morita equivalences.

11.1. **Mutations, the complete Ginzburg dg algebras and derived equivalences.** In the example below we will use the constructions of mutations and the Ginzburg dg algebras, and a "tilting" bimodule given by Keller–Yang. To make it easy to understand these examples, we recall these constructions and fix our notations.

11.1.1. *Mutations.* Let $Q$ be a quiver. A path in $Q$ is said to be *cyclic* if its source and target coincide. A potential on $Q$ is an element of the closure $\mathrm{Pot}(\Bbbk Q)$ of the subspace of $\Bbbk Q$ generated by all non-trivial cyclic paths in $Q$. We say that two potentials are *cyclically equivalent* if their difference is in the closure of the subspace generated by the differences $a_1 \cdots a_s - a_2 \cdots a_s a_1$ for all cycles $a_1 \cdots a_s$ in $Q$.

The complete path algebra $\widehat{\Bbbk Q}$ is the completion of the path algebra $\Bbbk Q$ with respect to the ideal generated by the arrows of $Q$. Let $\mathfrak{m}$ be the ideal of $\widehat{\Bbbk Q}$ generated by the arrows of $Q$. A *quiver with potential* is a pair $(Q, W)$ of a quiver $Q$ and a potential $W$ of $Q$ such that $W$ is in $\mathfrak{m}^2$ and no two cyclically equivalent cyclic paths appear in the decomposition of $W$.

A quiver with potential is called *trivial* if its potential is a linear combination of cyclic paths of length 2 and its Jacobian algebra is the product of copies of the base field $\Bbbk$. A quiver with potential is called *reduced* if $\partial_a W$ is contained in $\mathfrak{m}^2$ for all arrows $a$ of $Q$.

Let $(Q', W')$ and $(Q'', W'')$ be two quivers with potentials such that $Q'$ and $Q''$ have the same set of vertices. Their direct sum, denoted by $(Q', W') \oplus (Q'', W'')$, is the new quiver with potential $(Q, W)$, where $Q$ is the quiver whose vertex set is the same as the vertex set of $Q'$ (and $Q''$) and whose arrow set is the disjoint union of the arrow set of $Q'$ and the arrow set of $Q''$, and $W = W' + W''$.

Two quivers with potentials $(Q, W)$ and $(Q', W')$ are *right-equivalent* if $Q$ and $Q'$ have the same set of vertices and there exists an algebra isomorphism $\phi : \Bbbk Q \to \Bbbk Q'$ whose restriction on vertices is the identity map and $\phi(W)$ and $W'$ are cyclically equivalent. Such an isomorphism $\phi$ is called a right-equivalence.

For any quiver with potential $(Q, W)$, there exist a trivial quiver with potential $(Q_{\mathrm{tri}}, W_{\mathrm{tri}})$ and a reduced quiver with potential $(Q_{\mathrm{red}}, W_{\mathrm{red}})$ such that $(Q, W)$ is right-equivalent to the direct sum $(Q_{\mathrm{tri}}, W_{\mathrm{tri}}) \oplus (Q_{\mathrm{red}}, W_{\mathrm{red}})$. Furthermore, the right-equivalence class of each of $(Q_{\mathrm{tri}}, W_{\mathrm{tri}})$ and $(Q_{\mathrm{red}}, W_{\mathrm{red}})$ is uniquely determined by the right equivalence class of $(Q, W)$. We call $(Q_{\mathrm{tri}}, W_{\mathrm{tri}})$ and $(Q_{\mathrm{red}}, W_{\mathrm{red}})$ the *trivial part* and the *reduced part* of $(Q, W)$, respectively.

**Definition 11.2.** Let $(Q, W)$ be a quiver with potential, and $i$ a vertex of $Q$. Assume the following conditions:

(1) the quiver $Q$ has no loops;
(2) the quiver $Q$ does not have 2-cycles at $i$;
(3) no cyclic path occurring in the expansion of $W$ starts and ends at $i$.

Note that under the condition (1), any potential is cyclically equivalent to a potential satisfying (3). We define a new quiver with potential $\widetilde{\mu}_i(Q, W) =$

$(Q', W')$ as follows. The new quiver $Q'$ is obtained from $Q$ by the following procedure:

**Step 1:** For each arrow $\beta$ with target $i$ and each arrow $\alpha$ with source $i$, add a new arrow $[\alpha\beta]$ from the source of $\beta$ to the target of $\alpha$ .

**Step 2:** Replace each arrow $\alpha$ with source or target $i$ with an arrow $\alpha^*$ in the opposite direction.

The new potential $W'$ is the sum of two potentials $W'_1$ and $W'_2$, where the potential $W'_1$ is obtained from $W$ by replacing each composition $\alpha\beta$ by $[\alpha\beta]$, where $\beta$ is an arrow with target $i$, and the potential $W'_2$ is given by

$$W'_2 = \sum_{\alpha,\beta \in Q_1} [\alpha\beta]\beta^*\alpha^*,$$

where the sum ranges over all pairs of arrows $\alpha$ and $\beta$ such that $\beta$ ends at $i$ and $\alpha$ starts at $i$. It is easy to see that $\widetilde{\mu_i}(Q, W)$ satisfies (1), (2) and (3). We define $\mu_i(Q, W)$ as the reduced part of $\widetilde{\mu_i}(Q, W)$, and call $\mu_i$ the *mutation* at the vertex $i$.

11.1.2. *The complete Ginzburg dg algebras.*

**Definition 11.3.** Let $(Q, W)$ be a quiver with potential. The *complete Ginzburg dg algebra* $\widehat{\Gamma}(Q, W)$ is constructed as follows [21]: Let $\widetilde{Q}$ be the graded quiver with the same vertices as $Q$ and whose arrows are

- the arrows of $Q$ (they all have degree 0),
- an arrow $\overline{\alpha} : j \to i$ of degree $-1$ for each arrow $\alpha : i \to j$ of $Q$,
- a loop $t_i : i \to i$ of degree $-2$ for each vertex $i$ of $Q$.

The underlying graded algebra of $\widehat{\Gamma}(Q, W)$ is the completion of the graded path algebra $k\widetilde{Q}$ in the category of graded vector spaces with respect to the ideal generated by the arrows of $\widetilde{Q}$. Thus, the $n$-th component of $\widehat{\Gamma}(Q, W)$ consists of elements of the form $\sum_p \lambda_p p$ with $\lambda_p \in \Bbbk$, where $p$ runs over all paths of degree $n$. The differential of $\widehat{\Gamma}(Q, W)$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule

$$d(uv) = d(u)v + (-1)^p ud(v),$$

for all homogeneous $u$ of degree $p$ and all $v$, and takes the following values on the arrows of $\widetilde{Q}$:

- $da = 0$ for each arrow $a$ of $Q$,
- $d(\overline{a}) = \partial_a W$ for each arrow $\overline{a}$ of $Q$,
- $d(t_i) = e_i(\sum_a [a, \overline{a}])e_i$ for each vertex $i$ of $Q$, where $e_i$ is the trivial path at $i$ and the sum is taken over the set of arrows of $Q$.

**Remark 11.4.** We regard the complete Ginzburg dg algebra $\widehat{\Gamma}(Q, W)$ as a dg category as follows.

- The objects are the vertices of $\widetilde{Q}$ (namely the vertices of $Q$).
- $\widehat{\Gamma}(Q, W)(i, j) := e_j\widehat{\Gamma}(Q, W)e_i$ for all objects $i, j$.

- The composition is given by the multiplication of $\widehat{\Gamma}(Q,W)$.
- The grading and the differential are naturally defined from those of the dg algebra structure.

The following lemma is an easy consequence of the definition (cf. [31, Lemma 2.8]).

**Lemma 11.5.** *Let $(Q,W)$ be a quiver with potential. Then the Jacobian algebra $\mathrm{Jac}(Q,W)$ is the 0-th cohomology of the complete Ginzburg dg algebra $\widehat{\Gamma}(Q,W)$, i.e.*

$$\mathrm{Jac}(Q,W) = H^0(\widehat{\Gamma}(Q,W)).$$

11.1.3. *Derived equivalences.* Let $(Q,W)$ be a quiver with potential and $i$ a fixed vertex of $Q$. We assume (1), (2) and (3) as above. Write $\widetilde{\mu_i}(Q,W) = (Q',W')$. Let $\Gamma = \widehat{\Gamma}(Q,W)$ and $\Gamma' = \widehat{\Gamma}(Q',W')$ be the complete Ginzburg dg algebras associated to $(Q,W)$ and $(Q',W')$, respectively. We set $P_j = e_j\Gamma$ and $P'_j = e_j\Gamma'$ for all vertices $j$ of $Q$.

We cite the following from [31, Theorem 3.2] without a proof.

**Theorem 11.6.** *There is a triangle equivalence*

$$F : \mathscr{D}(\Gamma') \to \mathscr{D}(\Gamma)$$

*which sends the $P'_j$ to $P_j$ for $j \neq i$, and sends $P'_i$ to the cone $T_i$ over the morphism*

$$P_i \to \bigoplus_{\alpha\in Q_1, s(\alpha)=i} P_{t(\alpha)}$$

$$a \mapsto \sum_{\alpha\in Q_1, s(\alpha)=i} e_{t(\alpha)}\alpha a,$$

*The functor $F$ restricts to triangle equivalences from $\mathrm{per}(\Gamma')$ to $\mathrm{per}(\Gamma)$ and from $\mathscr{D}_{fd}(\Gamma')$ to $\mathscr{D}_{fd}(\Gamma)$.*

The proof is based on a construction of a $\Gamma'$-$\Gamma$-bimodule $T$, and $F$ is defined by $F := \text{-}\overset{\mathbf{L}}{\otimes}_\Gamma T \colon \mathscr{D}(\Gamma') \to \mathscr{D}(\Gamma)$. We recall the construction of $T$ by Keller-Yang below. As a right $\Gamma$-module, let $T$ be the direct sum of $T_i$ and $P_j$ for all $j \in Q_0$ with $j \neq i$. A left $\Gamma'$-module structure on $T$ will be defined in the next proposition. To this end we define a map $f \colon \{e_j \mid j \in Q_0\} \cup (\widetilde{Q'})_1 \to \mathrm{End}_\Gamma(T)$ as follows. First, we set $f(e_j) := f_j : T_j \to T_j$ to be the identity map for all $j \in Q_0$.

We denote by $\lambda_a$ the left multiplication $x \mapsto ax$ by $a$ below when this makes sense, and by $e_{\Sigma i}$ the unique idempotent in $\Gamma$ such that $e_{\Sigma i}\Gamma = \Sigma P_i = P_i[1]$, the shift of $P_i$, for all $i \in Q_0$.

Let $\alpha \in Q_1$ with $s(\alpha) = i$. Then define $f_{\alpha^*} : T_{t(\alpha)} \to T_i$ of degree 0 as the cannonical embedding $T_{t(\alpha)} = P_{t(\alpha)} \hookrightarrow T_i$, that is,

$$f_{\alpha^*} := \lambda_{e_{t(\alpha)}} : T_{t(\alpha)} \to T_i, \quad a \mapsto e_{t(\alpha)}a.$$

Define also the morphsim $f_{\overline{\alpha^*}} : T_i \to T_{t(\alpha)}$ of degree $-1$ by

$$f_{\overline{\alpha^*}}((e_{\Sigma i})a_i + \sum_{\rho\in Q_1, s(\rho)=i} e_{t(\rho)}a_\rho) = -\alpha t_i a_i - \sum_{\rho\in Q_1, s(\rho)=i} \alpha\overline{\rho}a_\rho$$

Let $\beta \in Q_1$ with $t(\beta) = i$. Then define the morphism $f_{\beta^*} : T_i \to T_{s(\beta)}$ of degree 0 by

$$f_{\beta^*}((e_{\Sigma i})a_i + \sum_{\rho\in Q_1, s(\rho)=i} e_{t(\rho)}a_\rho) = -\overline{\beta}a_i - \sum_{\rho\in Q_1, s(\rho)=i} (\partial_{\rho\beta}W)a_\rho.$$

Define also the morphsim $f_{\overline{\beta^*}} : T_{s(\beta)} \to T_i$ of degree $-1$ as the composite of the morphism $\lambda_{e_{\Sigma i}\beta}\colon T_{s(\beta)} \to \Sigma P_i$ and the cannonical embedding $\Sigma P_i \hookrightarrow T_i$, that is,

$$f_{\overline{\beta^*}} := \lambda_{e_{\Sigma i}\beta} : T_{s(\beta)} \to T_i, \quad a \mapsto e_{\Sigma i}\beta a.$$

Let $\alpha, \beta \in Q_1$ with $s(\alpha) = i, t(\beta) = i$. Then define

$$f_{[\alpha\beta]} := \lambda_{\alpha\beta} : T_{s(\beta)} \to T_{t(\alpha)}, \quad a \mapsto \alpha\beta a.$$

and

$$f_{\overline{[\alpha\beta]}} := 0 : T_{t(\alpha)} \to T_{s(\beta)}.$$

Let $\gamma \in Q_1$ be an arrow not incident to $i$. Then define

$$\begin{aligned} f_\gamma &:= \lambda_\gamma : T_{s(\gamma)} \to T_{t(\gamma)}, \quad a \mapsto \gamma a, \\ f_{\overline{\gamma}} &:= \lambda_{\overline{\gamma}} : T_{t(\gamma)} \to T_{s(\gamma)}, \quad a \mapsto \overline{\gamma}a. \end{aligned}$$

Let $j \in Q_0$ with $j \neq i$. Then define

$$f_{t'_j} := \lambda_{t_j} : T_j \to T_j, \quad a \mapsto t_j a.$$

It is a morphism of degree $-2$. Finally, define $f_{t'_i}$ as the linear morphism of degree $-2$ from $T_i$ to itself given by

$$f_{t'_i}((e_{\Sigma i})a_i + \sum_{\rho\in Q_1, s(\rho)=i} e_\rho a_\rho) = -e_{\Sigma i}(t_i a_i + \sum_{\rho\in Q_1, s(\rho)=i} \overline{\rho}a_\rho).$$

By [31, Proposition 3.5] we have the following.

**Proposition 11.7.** *The map $f\colon \{e_j \mid j \in Q_0\} \cup (\widetilde{Q'})_1 \to \mathrm{End}_\Gamma(T)$ defined above extends to a homomorphism of dg algebras from $\Gamma'$ to $\mathrm{End}_\Gamma(T)$. In this way, $T$ becomes a left dg $\Gamma'$-module, and also a dg $\Gamma'$-$\Gamma$-bimodule.*

11.2. **Examples.**

**Example 11.8.** Let $(Q, W)$ be the quiver with potential given as follows:

$$\begin{tikzcd}
& c_1 \arrow[dr, "\beta_1"] & & & b_2 \arrow[dr, "\alpha_2"] & \\
b_1 \arrow[ur, "\alpha_1"] & & a_1 \arrow[ll, "\gamma_1"] \arrow[r, "\delta_1"] & a_2 \arrow[ur, "\gamma_2"] \arrow[d, "\delta_2"] & & c_2 \arrow[ll, "\beta_2"] \\
c_4 \arrow[rr, "\beta_4"] & & a_4 \arrow[u, "\delta_4"] \arrow[dl, "\gamma_4"] & a_3 \arrow[l, "\delta_3"] \arrow[rr, "\gamma_3"] & & b_3 \arrow[dl, "\alpha_3"] \\
& b_4 \arrow[ul, "\alpha_4"] & & & c_3 \arrow[ul, "\beta_3"] &
\end{tikzcd}$$

$W = \delta_4\delta_3\delta_2\delta_1 + \sum_{i=1}^{3} \gamma_i\beta_i\alpha_i$. If we do mutations at $c_1$ and $c_3$ for $(Q, W)$, we get the following quiver with potential $(Q', W')$

$$\begin{tikzcd}
& c_1 \arrow[dl, "\alpha_1^*"'] \arrow[dr, "\beta_1^*"] & & & b_2 \arrow[dr, "\alpha_2"] & \\
b_1 \arrow[rr, bend right, "{[\beta_1\alpha_1]}"'] & & a_1 \arrow[ll, bend right, "\gamma_1"'] \arrow[r, "\delta_1"] & a_2 \arrow[ur, "\gamma_2"] \arrow[d, "\delta_2"] & & c_2 \arrow[ll, "\beta_2"] \\
c_4 \arrow[rr, "\beta_4"] & & a_4 \arrow[u, "\delta_4"] \arrow[dl, "\gamma_4"] & a_3 \arrow[l, "\delta_3"] \arrow[rr, bend right, "\gamma_3"'] \arrow[dr, "\beta_3^*"'] & & b_3 \arrow[ll, bend right, "{[\beta_3\alpha_3]}"'] \\
& b_4 \arrow[ul, "\alpha_4"] & & & c_3 \arrow[ur, "\alpha_3^*"'] &
\end{tikzcd}$$

$W' = \delta_4\delta_3\delta_2\delta_1 + \gamma_1[\beta_1\alpha_1] + \gamma_3[\beta_3\alpha_3] + \gamma_2\beta_2\alpha_2 + \gamma_4\beta_4\alpha_4 + [\beta_1\alpha_1]\alpha_1^*\beta_1^* + [\beta_3\alpha_3]\alpha_3^*\beta_3^*$.

The reduced part $(Q'_{\rm red}, W'_{\rm red})$ of $(Q', W')$ is given as follows:

$$\begin{tikzcd}
& c_1 \arrow[dl, "\alpha_1^*"'] \arrow[dr, "\beta_1^*"] & & & b_2 \arrow[dr, "\alpha_2"] & \\
b_1 & & a_1 \arrow[r, "\delta_1"] & a_2 \arrow[ur, "\gamma_2"] \arrow[d, "\delta_2"] & & c_2 \arrow[ll, "\beta_2"] \\
c_4 \arrow[rr, "\beta_4"] & & a_4 \arrow[u, "\delta_4"] \arrow[dl, "\gamma_4"] & a_3 \arrow[l, "\delta_3"] \arrow[dr, "\beta_3^*"'] & & b_3 \\
& b_4 \arrow[ul, "\alpha_4"] & & & c_3 \arrow[ur, "\alpha_3^*"'] &
\end{tikzcd}$$

$W'_{\rm red} = \delta_4\delta_3\delta_2\delta_1 + \gamma_2\beta_2\alpha_2 + \gamma_4\beta_4\alpha_4$.

Consider the cyclic group $G$ of order 2 with generator $g$, and define a $G$-action on $(Q, W)$ as a unique quiver automorphism induced by the permutation of indexes $i = 1, 2, 3, 4$:

$$i \mapsto i - 2 \pmod 4. \tag{11.33}$$

Then the quiver with potential $(Q_G, W_G)$ is given as follows:

$$
\begin{tikzcd}
 & c_1 \arrow[dr, "\beta"] & & & b_2 \arrow[dr, "\alpha'"] & \\
b_1 \arrow[ur, "\alpha"] & & a_1 \arrow[ll, "\gamma"] \arrow[r, bend right, "\delta"'] & a_2 \arrow[l, bend right, "\delta'"'] \arrow[ur, "\gamma'"] & & c_2 \arrow[ll, "\beta'"]
\end{tikzcd}
$$

$W_G = (\delta'\delta)^2 + 2\gamma\beta\alpha + 2\gamma'\beta'\alpha'$. Define also a $G$-action on $(Q'_{\mathrm{red}}, W'_{\mathrm{red}})$ by the same permutation of indexes as (11.33). Then the quiver with potential $((Q'_{\mathrm{red}})_G, (W'_{\mathrm{red}})_G)$ is given as follows:

$$
\begin{tikzcd}
 & c_1 \arrow[dl, "\alpha^*"'] & & & b_2 \arrow[dr, "\alpha'"] & \\
b_1 & & a_1 \arrow[ul, "\beta^*"'] \arrow[r, bend right, "\delta"'] & a_2 \arrow[l, bend right, "\delta'"'] \arrow[ur, "\gamma'"] & & c_2 \arrow[ll, "\beta'"]
\end{tikzcd}
$$

$(W'_{\mathrm{red}})_G = (\delta'\delta)^2 + 2\gamma'\beta'\alpha'$.

If we do mutations at $c_1$ and $c_3$ for $(Q, W)$, then we do mutation at $c_1$ for $(Q_G, W_G)$. Then the reduced part of $\mu_{c_1}(Q_G, W_G)$ coincides with $((Q'_{\mathrm{red}})_G, (W'_{\mathrm{red}})_G)$. Indeed, the quiver with potential $\mu_{c_1}(Q_G, W_G)$ is the following

$$
\begin{tikzcd}
 & c_1 \arrow[dl, "\alpha^*"'] & & & b_2 \arrow[dr, "\alpha'"] & \\
b_1 \arrow[rr, bend right, "{[\beta\alpha]}"'] & & a_1 \arrow[ul, "\beta^*"'] \arrow[ll, bend right, "\gamma"'] \arrow[r, bend right, "\delta"'] & a_2 \arrow[l, bend right, "\delta'"'] \arrow[ur, "\gamma'"] & & c_2 \arrow[ll, "\beta'"]
\end{tikzcd}
$$

$\mu_{c_1}(W_G) = (\delta'\delta)^2 + 2\gamma[\beta\alpha] + 2\gamma'\beta'\alpha' + 2[\beta\alpha]\alpha^*\beta^*$. The potential is not reductive, so we have the following quiver with potential

$$
\begin{tikzcd}
 & c_1 \arrow[dl, "\alpha^*"'] & & & b_2 \arrow[dr, "\alpha'"] & \\
b_1 & & a_1 \arrow[ul, "\beta^*"'] \arrow[r, bend right, "\delta"'] & a_2 \arrow[l, bend right, "\delta'"'] \arrow[ur, "\gamma'"] & & c_2 \arrow[ll, "\beta'"]
\end{tikzcd}
$$

$\mu_{c_1}(W_G) = (\delta'\delta)^2 + 2\gamma'\beta'\alpha'$. Hence by Keller-Yang's result [31, Theorem 3.2 (b)] the Ginzburg dg algebras of $(Q_G, W_G)$ and $((Q'_{\mathrm{red}})_G, (W'_{\mathrm{red}})_G)$ are derived equivalent. On the other hand, by Remark 11.1 we know that $\int X_{Q,W}$ is Morita equivalent to $\widehat{\Gamma}(Q_G, W_G)$, and $\int(X_{Q',W'})$ is Morita equivalent to $\widehat{\Gamma}(Q'_G, W'_G)$, and which is isomorphic to $\widehat{\Gamma}((Q'_{\mathrm{red}})_G, (W'_{\mathrm{red}})_G)$ by Keller–Yang [31, Lemma 2.9] because $(Q', W')$ and $(Q'_{\mathrm{red}}, W'_{\mathrm{red}})$ are right-equivalent. As a consequence, $\int X_{Q,W}$ and $\int X_{Q',W'}$ are derived equivalent. The same conclusion can be obtained from our result Corollary 10.8 as in the next example.

**Example 11.9.** Let $(Q,W)$ be the quiver with potential given as follows:

$$\begin{tikzcd}
 & & 1 \arrow[dl, "a_1"'] & & \\
 & 2 \arrow[dl, "a_2"'] & & 6 \arrow[ul, "a_6"'] & \\
3 \arrow[rr, "a_3"'] & & 4 \arrow[rr, "a_4"'] & & 5, \arrow[ul, "a_5"']
\end{tikzcd}$$

$$W = a_5a_4a_3a_2a_1a_6.$$

Let $I = \{1,3,5\}$. Mizuno [40] defined successive mutation $\mu_I(Q,W) = \mu_5 \circ \mu_3 \circ \mu_1(Q,W) = (Q',W')$ given by the quiver with potential as follows:

$$\begin{tikzcd}
 & & 1 \arrow[dr, "a_6^*"] & & \\
 & 2 \arrow[ur, "a_1^*"] \arrow[dr, "{[a_3a_2]}"'] & & 6 \arrow[ll, "{[a_1a_6]}"] \arrow[dr, "a_5^*"] & \\
3 \arrow[ur, "a_2^*"] & & 4 \arrow[ll, "a_3^*"] \arrow[ur, "{[a_5a_4]}"] & & 5, \arrow[ll, "a_4^*"]
\end{tikzcd}$$

$$W' = [a_1a_6]a_6^*a_1^* + [a_3a_2]a_2^*a_3^* + [a_5a_4]a_4^*a_5^* + [a_5a_4][a_3a_2][a_1a_6].$$

By [40, Theorem 1.1], the Jacobian algebras $\mathrm{Jac}(Q,W)$ and $\mathrm{Jac}(Q',W')$ are derived equivalent.

(1) Consider the cyclic group $G$ of order 3 with generator $g$, and define the action of $g$ on $(Q,W)$ by $i \mapsto i-2$ and $a_i \mapsto a_{i-2}$ (modulo 6). Therefore, we have

$$Ga_1 = \{a_1, a_5, a_3\}, Ga_2 = \{a_2, a_6, a_4\}.$$

In this case $(Q_G, W_G)$ is the quiver with potential given as follows:

$$\begin{tikzcd}
1 \arrow[r, bend right, "\alpha"'] & 2 \arrow[l, bend right, "\beta"']
\end{tikzcd}$$

$$W_G = (\beta\alpha)^3.$$

(2) Next we define the action of $g$ on $(Q',W')$ by

$$i \mapsto i-2,\ a_i^* \mapsto a_{i-2}^*,\ \text{and } [a_ia_{i+5}] \mapsto [a_{i-2}a_{i+3}] \pmod 6$$

for all $i = 1, \dots, 6$.

Therefore, we have

$$Ga_1^* = \{a_1^*, a_5^*, a_3^*\}, Ga_2^* = \{a_2^*, a_6^*, a_4^*\}.$$

In this case $(Q'_G, W'_G)$ is the quiver with potential given as follows:

$$\begin{tikzcd} G1 \arrow[r, bend right, "Ga_2^*"'] & G2 \arrow[l, bend right, "Ga_1^*"'] \arrow[loop right, "G[a_6a_1]"] \end{tikzcd}$$

$$W'_G = 3G[a_6a_1]G(a_1^*)G(a_6^*) + G([a_6a_1])^3.$$

Here the Jacobian algebras $\mathrm{Jac}(Q_G, W_G)$ and $\mathrm{Jac}(Q'_G, W'_G)$ are representation-finite, selfinjective algebras, and by the main theorem in [3], they are derived equivalent because their derived equivalence types are the same. By Keller-Yang's result [31], the complete Ginzburg dg algebras $\widehat{\Gamma}(Q,W)$ and $\widehat{\Gamma}(Q',W')$ are derived equivalent as dg algebras. By using Corollary 10.8, we will show that $\widehat{\Gamma}(Q,W)/G$ and $\widehat{\Gamma}(Q',W')/G$ are derived equivalent as dg algebras. Therefore the complete Ginzburg dg algebras $\widehat{\Gamma}(Q_G,W_G)$ and $\widehat{\Gamma}(Q'_G,W'_G)$ are derived equivalent as dg algebras by Remark 11.1. We set $\Gamma^{(1)} := \widehat{\Gamma}(\mu_1(Q,W)), \Gamma^{(2)} := \widehat{\Gamma}(\mu_3 \circ \mu_1(Q,W)), \Gamma' := \widehat{\Gamma}(\mu_5 \circ \mu_3 \circ \mu_1(Q,W)) = \widehat{\Gamma}(Q',W')$. Then Keller–Yang's theorem (Theorem 11.6) gives us the following derived equivalences $F_3, F_2, F_1$ defined as $(\text{-}) \overset{\mathbf{L}}{\otimes}_{\Gamma'} T^{(3)}$, $(\text{-}) \overset{\mathbf{L}}{\otimes}_{\Gamma^{(2)}} T^{(2)}$, $(\text{-}) \overset{\mathbf{L}}{\otimes}_{\Gamma^{(1)}} T^{(1)}$ using the dg bimodules $T^{(3)}, T^{(2)}, T^{(1)}$ constructed as in Proposition 11.7, respectively. These functors send objects as follows:

$$\begin{array}{ccccccc}
\mathscr{D}(\Gamma') & \xrightarrow{F_3} & \mathscr{D}(\Gamma^{(2)}) & \xrightarrow{F_2} & \mathscr{D}(\Gamma^{(1)}) & \xrightarrow{F_1} & \mathscr{D}(\Gamma) \\
P'_5 & \mapsto & (P_5^{(2)} \to P_6^{(2)}) & \mapsto & (P_5^{(1)} \to P_6^{(1)}) & \mapsto & (P_5 \to P_6) =: T(5) \\
P'_3 & \mapsto & P_3^{(2)} & \mapsto & (P_3^{(1)} \to P_4^{(1)}) & \mapsto & (P_3 \to P_4) =: T(3) \\
P'_1 & \mapsto & P_1^{(2)} & \mapsto & P_1^{(1)} & \mapsto & (P_1 \to P_2) =: T(1) \\
P'_i & \mapsto & P_i^{(2)} & \mapsto & P_i^{(1)} & \mapsto & P_i =: T(i), (i = 2,4,6)
\end{array}$$

where $P'_i = e_i\Gamma', P_i^{(2)} = e_i\Gamma^{(2)}, P_i^{(1)} = e_i\Gamma^{(1)}$ for all $i \in Q_0$. Then $F := F_1 \circ F_2 \circ F_3 = \text{-} \overset{\mathbf{L}}{\otimes}_{\Gamma'} T^{(3)} \overset{\mathbf{L}}{\otimes}_{\Gamma^{(2)}} T^{(2)} \overset{\mathbf{L}}{\otimes}_{\Gamma^{(1)}} T^{(1)}$ is an equivalence from $\mathscr{D}(\Gamma')$ to $\mathscr{D}(\Gamma)$. Here $T^{(3)} \overset{\mathbf{L}}{\otimes}_{\Gamma^{(2)}} T^{(2)} \overset{\mathbf{L}}{\otimes}_{\Gamma^{(1)}} T^{(1)}$ is a dg $\Gamma'$-$\Gamma$-bimodule and is isomorphic to the direct sum $T$ of the indecomposable objects $T(i), (i = 1, \dots, 6)$ as a dg right $\Gamma$-module, by which we identify these and regard $T$ as a dg $\Gamma'$-$\Gamma$-bimodule. Let $\mathscr{T}$ be the full subcategory of $\mathscr{C}_{\mathrm{dg}}(\Gamma)$ consisting of $T(1), T(2), \cdots, T(6)$. We show that $\mathscr{T}$ is a desired tilting subcategory for $\Gamma$.

Now since $g$ acts on $P_i$ by ${}^gP_i = P_{i-2}, (i = 1, \dots, 6)$ by the $G$-action in (1) above, we have ${}^gT(i) = T(i-2), (i = 1, \dots, 6)$. On the other hand by the $G$-action in (2), $g$ acts on $P'_i$ by ${}^gP'_i = P'_{i-2}, (i = 1, \dots, 6)$.

We construct a 1-morphism $(F', \phi)\colon \Gamma' \to \mathscr{T}$ that is a $G$-quasi-equivalence. To this end we have to construct a quasi-equivalence $F'\colon \Gamma' \to \mathscr{T}$ and a 2-quasi-isomorphism $\phi(a)\colon \mathscr{T}(a) \circ F' \Rightarrow F' \circ a$ in $\Bbbk$-**dgCat** for each $a \in G$ (see

Definition 8.6):

$$\begin{tikzcd}
\Gamma' \arrow[r, "F'"] \arrow[d, "a"'] & \mathscr{T} \arrow[r, hook] \arrow[d, "\mathscr{T}(a)={}^a(\text{-})"] \arrow[dl, Rightarrow, "\phi(a)"] & \mathscr{C}_{\mathrm{dg}}(\Gamma) \arrow[d, "{}^a(\text{-})"] \\
\Gamma' \arrow[r, "F'"'] & \mathscr{T} \arrow[r, hook] & \mathscr{C}_{\mathrm{dg}}(\Gamma)
\end{tikzcd}$$

(It is trivial that the right square is strictly commutative). We now define $F'$ as follows: First recall the Yoneda embedding $Y\colon \Gamma' \to \mathscr{C}_{\mathrm{dg}}(\Gamma')$ is defined by $Y(i) := \Gamma'(\text{-}, i) = e_i\Gamma'$ for all $i \in \Gamma'_0$, and $Y(\mu) := \Gamma'(\text{-}, \mu)$ for all $\mu \in \Gamma'_1$. Let $\alpha_M\colon \Gamma' \overset{\mathbf{L}}{\otimes}_{\Gamma'} M \to M$ be the usual natural isomorphisim for all $\Gamma'$-$\Gamma$-bimodule $M$. This yields the isomorphism $e_i\alpha_M\colon e_i\Gamma' \overset{\mathbf{L}}{\otimes}_{\Gamma'} M \to e_iM$ for each $i \in \Gamma'_0$ that is natural in $i$ and in $M$. Note that the naturality in $i$ means that for each $f\colon i \to j$ in $\Gamma'$, we have a commutative diagram

$$\begin{tikzcd}
e_i\Gamma' \overset{\mathbf{L}}{\otimes}_{\Gamma'} M \arrow[r, "e_i\alpha_M"] \arrow[d, "\Gamma'(\text{-},f)\overset{\mathbf{L}}{\otimes}_{\Gamma'} M"'] & e_iM \arrow[d, "M(\text{-},f)"] \\
e_j\Gamma' \overset{\mathbf{L}}{\otimes}_{\Gamma'} M \arrow[r, "e_j\alpha_M"'] & e_jM.
\end{tikzcd}$$

We then define $F' := F \circ Y\colon \Gamma' \to \mathrm{per}(\Gamma) \subseteq \mathscr{D}(\Gamma)$, thus $F'(i) = e_i\Gamma' \overset{\mathbf{L}}{\otimes}_{\Gamma'} T \xrightarrow{e_i\alpha_T} T(i)$ for all $i \in Q_0$, and $F'(\mu) = \Gamma'(\text{-}, \mu) \overset{\mathbf{L}}{\otimes}_{\Gamma'} T \cong \lambda_\mu\colon T(i) \to T(j)$ for all $\mu \in \Gamma'_1(i,j)$ with $i, j \in \Gamma'_0$. Thus we have a commutative diagram

$$\begin{tikzcd}
F'(i) \arrow[r, "e_i\alpha_T"] \arrow[d, "F'(\mu)"'] & T(i) \arrow[d, "\lambda_\mu"] \\
F'(j) \arrow[r, "e_j\alpha_T"'] & T(j).
\end{tikzcd}$$

Next we define a 2-quasi-isomorphism $\phi(a)\colon {}^aF' \Rightarrow F'a$ for each $a \in G$. Let $i \in \Gamma'_0$, and $a \in G$. Then the isomorphism $e_i\alpha_T\colon F'(i) \to T(i)$ yields isomomrphisms ${}^a(F'(i)) \xrightarrow{{}^a(e_i\alpha_T)} {}^aT(i) = T(ai)$, and $F'(ai) \xrightarrow{e_{ai}\alpha_T} T(ai)$. Thus we have an isomorphism

$$\phi_i(a) := (e_{ai}\alpha_T)^{-1} \circ {}^a(e_i\alpha_T)\colon {}^a(F'(i)) \to F'(ai).$$

We then define $\phi(a) := (\phi_i(a))_{i\in\Gamma'_0}\colon {}^aF' \Rightarrow F'a$ for all $a \in G$ and $\phi := (\phi(a))_{a\in G}$.

**Claim 1.** *The pair* $(F', \phi)$ *is a* 1*-morphism* $\Gamma' \to \mathscr{T}$ (see Definition 2.13)*.*

Indeed, because $F'(i)$ is clearly a dg-functor, it suffices to show that $\phi(a)$ is a 2-morphism in $\Bbbk$-**dgCat** for each $a \in G$. Namely, we have to show the commutativity of the diagram

$$\begin{tikzcd} {}^aF'(u) \arrow[r, "\phi_u(a)"] \arrow[d, "{}^aF'(\mu)"'] & F'(au) \arrow[d, "F'(a\mu)"] \\ {}^aF'(v) \arrow[r, "\phi_v(a)"'] & F'(av) \end{tikzcd}$$

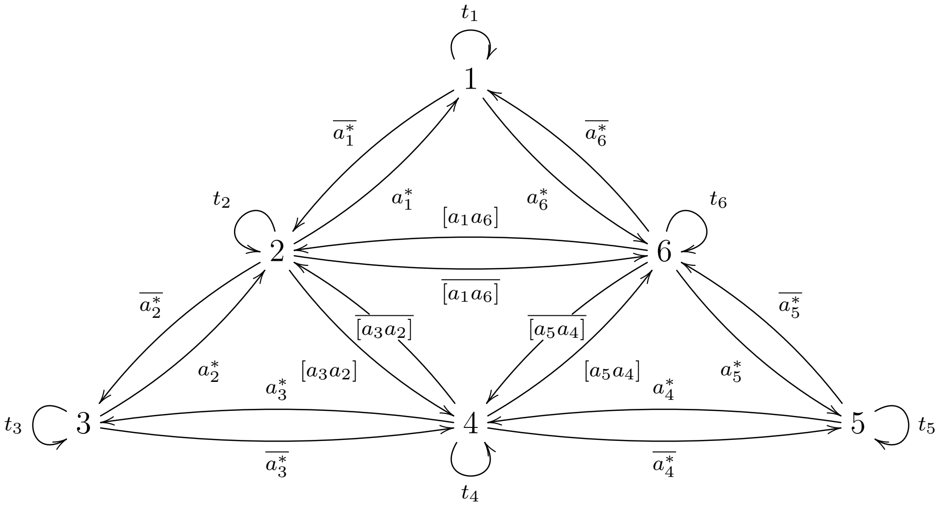
for all $\mu\colon u \to v$ in $\Gamma'_1$ and $a \in G$. It suffices to show the commutativity of this only for $a = g$ and for all $\mu \in \widetilde{Q'}_1$. Therefore finally we have only to show the commutativity of the diagram

$$\begin{tikzcd} {}^gF'(u) \arrow[r, "{}^g(e_u\alpha_T)"] \arrow[d, "{}^gF'(\mu)"'] & {}^gT(u) \arrow[r, equal] \arrow[d, "{}^gT(\mu)"] & T(gu) \arrow[d, "T(g\mu)"] & F'(gu) \arrow[l, "e_{a_u}\alpha_T"'] \arrow[d, "F'(g\mu)"] \\ {}^gF'(v) \arrow[r, "e_{a_v}\alpha_T"'] & {}^gT(v) \arrow[r, equal] & T(gv) & F'(gv) \arrow[l, "e_{a_v}\alpha_T"] \end{tikzcd} \tag{11.34}$$

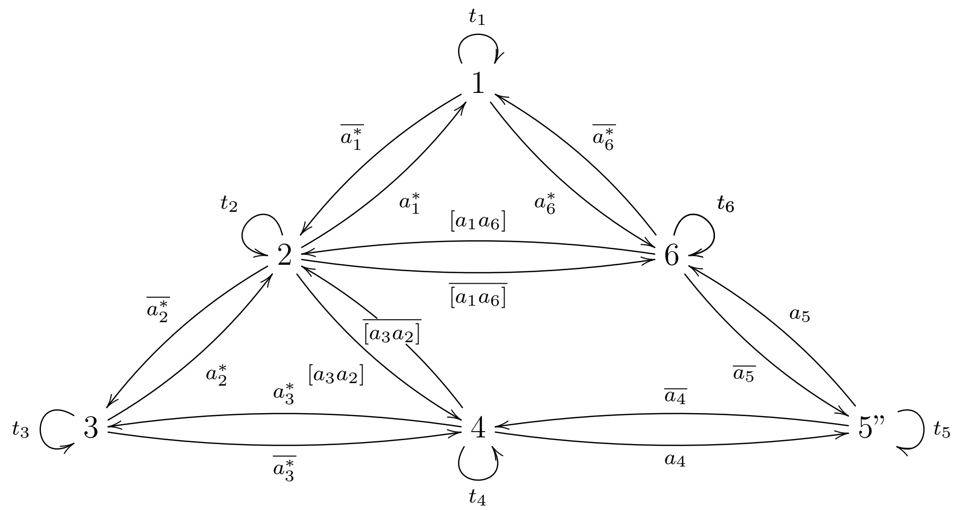
for all $\mu \in \widetilde{Q'}_1$. We check this only for three cases below. The remaining cases are checked similarly, and is left to the reader.

Now the quivers of $\Gamma', \Gamma^{(2)}, \Gamma^{(1)}, \Gamma$ are given as follows:

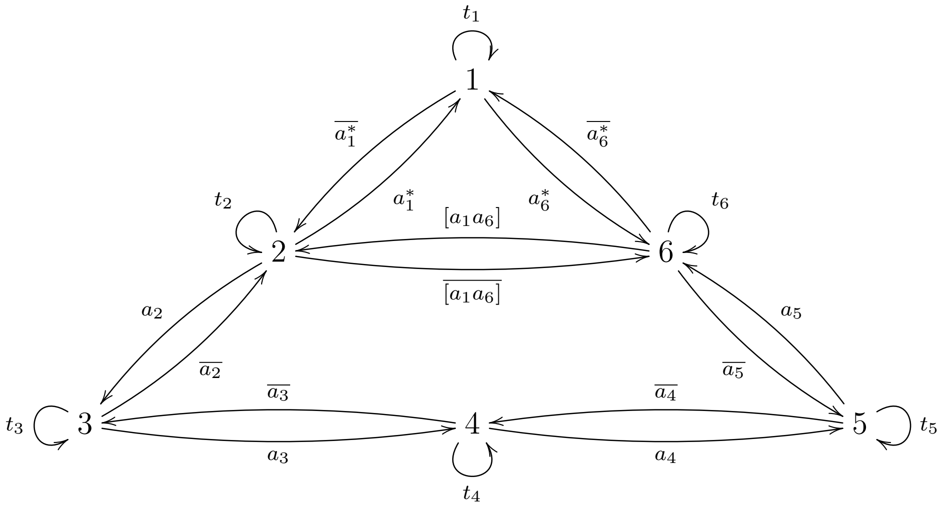
$\Gamma' =$



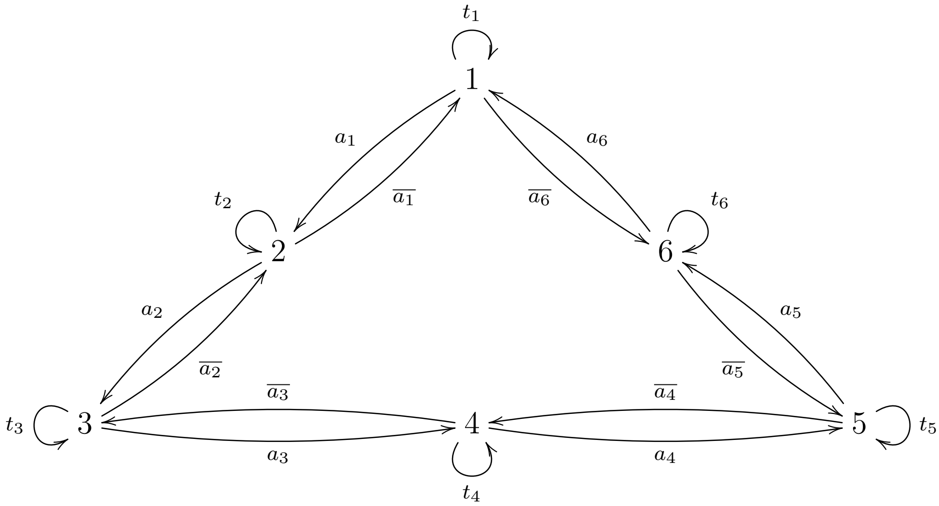
$\Gamma^{(2)} =$




$\Gamma^{(1)} =$




$\Gamma =$




**Case 1.** $\mu = a_i^* \in \widetilde{Q'}$ for some $i = 1, \dots, 6$, say $i = 1$. Then up to Yoneda embeddings (for the first three correspondences) we have $a_1^* \overset{F_3}{\mapsto} a_1^* \overset{F_2}{\mapsto} a_1^* \overset{F_1}{\mapsto}$

$f_{a_1^*} \overset{{}^g(\text{-})}{\mapsto} f_{a_5^*}$. Since we have commutative diagrams

$$\begin{tikzcd} F'(2) \arrow[r, "e_2\alpha_T"] \arrow[d, "F'(a_1^*)"'] & T(2) \arrow[d, "f_{a_1^*}"] \\ F'(1) \arrow[r, "e_1\alpha_T"'] & T(1) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T(6) \arrow[d, "f_{a_5^*}"] & F'(6) \arrow[l, "e_6\alpha_T"'] \arrow[d, "F'(a_5^*)"] \\ T(5) & F'(5) \arrow[l, "e_5\alpha_T"] \end{tikzcd},$$

we have a commutative diagram:

$$\begin{tikzcd} {}^gF'(2) \arrow[r, "{}^g(e_2\alpha_T)"] \arrow[d, "{}^gF'(a_1^*)"'] & {}^gT(2) \arrow[r, equal] \arrow[d, "{}^gf_{a_1^*}"] & T(6) \arrow[d, "f_{ga_1^*}"] & F'(g2) \arrow[l, "e_{g2}\alpha_T"'] \arrow[d, "F'(ga_1^*)"] \\ {}^gF'(1) \arrow[r, "{}^g(e_1\alpha_T)"'] & {}^gT(1) \arrow[r, equal] & T(5) & F'(g1), \arrow[l, "e_{g1}\alpha_T"] \end{tikzcd}$$

and hence (11.34) is verified in this case.

**Case 2.** $\mu = \overline{a_i^*} \in \widetilde{Q'}$ for some $i = 1, \dots, 6$, say $i = 1$. Then up to Yoneda embeddings (for the first three correspondences) we have $\overline{a_1^*} \overset{F_3}{\mapsto} \overline{a_1^*} \overset{F_2}{\mapsto} \overline{a_1^*} \overset{F_1}{\mapsto} f_{\overline{a_1^*}} \overset{{}^g(\text{-})}{\mapsto} f_{\overline{a_5^*}}$. Since we have commutative diagrams

$$\begin{tikzcd} F'(1) \arrow[r, "e_1\alpha_T"] \arrow[d, "F'(\overline{a_1^*})"'] & T(1) \arrow[d, "f_{\overline{a_1^*}}"] \\ F'(2) \arrow[r, "e_2\alpha_T"'] & T(2) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T(5) \arrow[d, "f_{\overline{a_5^*}}"] & F'(5) \arrow[l, "e_5\alpha_T"'] \arrow[d, "F'(\overline{a_5^*})"] \\ T(6) & F'(6) \arrow[l, "e_6\alpha_T"] \end{tikzcd},$$

we have a commutative diagram:

$$\begin{tikzcd} {}^gF'(1) \arrow[r, "{}^g(e_1\alpha_T)"] \arrow[d, "{}^gF'(\overline{a_1^*})"'] & {}^gT(1) \arrow[r, equal] \arrow[d, "{}^gf_{\overline{a_1^*}}"] & T(5) \arrow[d, "f_{g\overline{a_1^*}}"] & F'(g1) \arrow[l, "e_{g1}\alpha_T"'] \arrow[d, "F'(g\overline{a_1^*})"] \\ {}^gF'(2) \arrow[r, "{}^g(e_1\alpha_T)"'] & {}^gT(2) \arrow[r, equal] & T(6) & F'(g2), \arrow[l, "e_{g2}\alpha_T"] \end{tikzcd}$$

and hence (11.34) is verified in this case.

**Case 3.** $\mu = t_i \in \widetilde{Q'}$ for some $i = 1, \dots, 6$, say $i = 1$. Then up to Yoneda embeddings (for the first three correspondences) we have $t_1 \overset{F_3}{\mapsto} t_1 \overset{F_2}{\mapsto} t_1 \overset{F_1}{\mapsto} f_{t'_1} \overset{{}^g(\text{-})}{\mapsto} f_{t'_5}$. Therefore we have ${}^gF'(t_1) = f_{t'_5} = F'(t_5) = F'(gt_1)$. Hence ${}^g(F'(t_1)) = F'(gt_1)$.

Since we have commutative diagrams

$$\begin{tikzcd} F'(1) \arrow[r, "e_1\alpha_T"] \arrow[d, "F'(t_1)"'] & T(1) \arrow[d, "f_{t'_1}"] \\ F'(1) \arrow[r, "e_1\alpha_T"'] & T(1) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T(5) \arrow[d, "f_{t'_5}"] & F'(5) \arrow[l, "e_5\alpha_T"'] \arrow[d, "F'(t_5)"] \\ T(5) & F'(5) \arrow[l, "e_5\alpha_T"] \end{tikzcd},$$

we have a commutative diagram:

$$
\begin{tikzcd}
{}^{g}F'(1) \arrow[r, "{}^{g}(e_1\alpha_T)"] \arrow[d, "{}^{g}F'(t_1)"'] & {}^{g}T(1) \arrow[r, equal] \arrow[d, "{}^{g}f_{t'_1}"] & T(5) \arrow[d, "f_{gt'_1}"] & F'(g1) \arrow[l, "e_{g1}\alpha_T"'] \arrow[d, "F'(gt_1)"] \\
{}^{g}F'(1) \arrow[r, "e_{a_1}\alpha_T"'] & {}^{g}T(1) \arrow[r, equal] & T(5) & F'(g1), \arrow[l, "e_{g1}\alpha_T"]
\end{tikzcd}
$$

and hence (11.34) is verified in this case. We check the conditions (a) and (b) in Definition 2.13.

**Verifications of (a):** This is equivalent to the equation that $\phi(1) = \mathbb{1}_{F'}$, which follows from the construction of $\phi$ and the fact that both $\Gamma'$ and $\mathscr{T}$ have strict $G$-actions.

**Verification of (b):** This condition is equivalent to saying that the following diagram is commutative:

$$
\begin{tikzcd}
{}^{b}({}^{a}(F'(i))) \arrow[r, "{}^{b}(\phi_i(a))"] \arrow[dr, "\phi_i(ba)"'] & {}^{b}((F'(ai))) \arrow[d, "\phi_{(ai)}(b)"] \\
 & F'(bai)
\end{tikzcd}
\tag{11.35}
$$

for all $a, b \in G$ and $i \in \Gamma'_0$. By definition of $\phi_i(a)$, the following diagram is commutative:

$$
\begin{tikzcd}
{}^{a}(F'(i)) \arrow[r, "{}^{a}(e_i\alpha_T)"] \arrow[d, "\phi_i(a)"'] & {}^{a}T(i) \arrow[d, equal] \\
F'(ai) \arrow[r, "e_{ai}\alpha_T"'] & T(ai).
\end{tikzcd}
$$

This yields the following commutative diagram:

$$
\begin{tikzcd}
{}^{b}({}^{a}(F'(i))) \arrow[r, "{}^{b}({}^{a}(e_i\alpha_T))"'] \arrow[d, "{}^{b}(\phi_i(a))"'] \arrow[rr, equal, bend left] & {}^{b}({}^{a}(T(i))) \arrow[d, equal] & {}^{ba}(F'(i)) \arrow[l, "{}^{ba}(e_i\alpha_T)"] \arrow[dd, "\phi_i(ba)"] \\
{}^{b}(F'(ai)) \arrow[r, "{}^{b}(e_{ai}\alpha_T)"'] \arrow[d, "\phi_{ai}(b)"'] & {}^{b}(T(ai)) \arrow[d, equal] & \\
F'(bai) \arrow[r, "e_{b(ai)}\alpha_T"] \arrow[rr, equal, bend right] & T(b(ai)) & F'(bai), \arrow[l, "e_{bai}\alpha_T"']
\end{tikzcd}
$$

which shows the commutativity of the diagram (11.35).

It remains to show that $(F', \phi)$ is a quasi-equivalence. Namely we have to show the following claims:

**Claim 2.** *$F'$ is an isomorphism, and hence a quasi-equivalence.*

Indeed, we regard $\Gamma'$ as a dg category following Remark 11.4. For each $i \in Q_0$, we have $F'(i) = T(i)$. Hence $F'$ is bijective on objects. Moreover, for each $i, j \in Q_0$, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{D}(\Gamma')(\Gamma'(\text{-}, i), \Gamma'(\text{-}, j)) \arrow[r, "F"] & \mathscr{D}(\Gamma)(F(i), F(j)) \\
\Gamma'(i, j) \arrow[u, "Y"] \arrow[r, "F'"'] & \mathscr{T}(F'(i), F'(j)), \arrow[u, equal]
\end{tikzcd}$$

where $Y$ and $F$ above are bijective. Hence $F'$ above is bijective.

**Claim 3.** *$\phi(a)$ is a 2-quasi-isomorphism for all $a \in G$, i.e., $\mathscr{T}(\text{-}, \phi_i(a))\colon \mathscr{T}(\text{-}, {}^aF'(i)) \to \mathscr{T}(\text{-}, F'(ai))$ is a quasi-isomorphism in $\mathscr{C}(\mathscr{T})$ for all $a \in G$ and $i \in \Gamma'_0$.*

Indeed, by construction $\phi_i(a)\colon {}^aF'(i) \to F'(ai)$ is an isomorphism in $\mathscr{T}$. Therefore $\mathscr{T}(\text{-}, \phi_i(a))$ is an isomorpism in $\mathscr{C}(\mathscr{T})$, and thus it is a quasi-isomorphism.

As a consequence, $\widehat{\Gamma}(Q_G, W_G)$ and $\widehat{\Gamma}(Q'_G, W'_G)$ are derived equivalent. Note that the quivers with potentials $(Q_G, W_G)$ and $(Q'_G, W'_G)$ are not mutated from each other in this case. Therefore we cannot apply [31, Theorem 3.2] by Keller-Yang to have this derived equivalence.

To give an example of the case that the category $I$ is not a group, we need to give how to compute the Grothendieck construction of a functor $X\colon I \to \Bbbk$-**dgCat** at least. This will be done in the forthcoming paper, which will include such an example.

## Appendix A. Quasi-equivalence morphisms and derived equivalences

The following statement is stated in [30] without a proof in a remark after [30, Lemma 3.10]. For completeness, we give a proof of it in this appendix.

**Theorem A.1.** *Let $E : \mathscr{A} \to \mathscr{B}$ be a quasi-equivalence between dg categories $\mathscr{A}$ and $\mathscr{B}$. Then $\text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} \overline{E}\colon \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$ is an equivalence of triangulated categories, where $\overline{E}$ is the $\mathscr{A}$-$\mathscr{B}$-bimodule $\overline{E} := {}_E\mathscr{B}$. In particular, $\mathscr{A}$ and $\mathscr{B}$ are derived equivalent.*

For the proof we prepare the following three lemmas.

**Lemma A.2.** *Let $\mathscr{D}$ and $\mathscr{D}'$ be triangulated categories, and $F : \mathscr{D} \to \mathscr{D}'$ and $G : \mathscr{D}' \to \mathscr{D}$ triangle functors. Assume that the following conditions are satisfied*

(1) *$F$ is fully faithful,*
(2) *$G$ is a right adjoint to $F$, and*

(3) *$G(X)=0$ implies $X=0$ for all objects $X$ of $\mathscr{D}'$.*
*Then $F$ is an equivalence.*

*Proof.* We denote the unit and the counit of the adjoint by $\eta : \mathbb{1}_{\mathscr{D}} \Rightarrow G \circ F$ and by $\varepsilon : F \circ G \Rightarrow \mathbb{1}_{\mathscr{D}'}$, respectively. Let $D \in \mathscr{D}'$, and take a distinguished triangle

$$FG(D) \xrightarrow{\varepsilon_D} D \to Y \to FG(D)[1]$$

in $\mathscr{D}'$. Apply the functor $G$ to get

$$GFG(D) \xrightarrow{G(\varepsilon_D)} G(D) \to G(Y) \to G(D)[1].$$

Since $F$ is fully faithful, $\eta : \mathbb{1} \Rightarrow G \circ F$ is an isomorphism. In particular, $\eta_{G(D)}$ is an isomorphism. Then the equality $G(\varepsilon_D)\eta_{G(D)} = \mathbb{1}_{G(D)}$ yields a commutative diagram with triangle rows:

$$\begin{tikzcd}
GFG(D) \arrow[r, "G(\varepsilon_D)"] \arrow[d, "\eta_{G(D)}^{-1}"'] & G(D) \arrow[r] \arrow[d, equal] & G(Y) \arrow[r] \arrow[d] & G(D)[1] \arrow[d] \\
G(D) \arrow[r, "\mathbf{1}_{G(D)}"'] & G(D) \arrow[r] & G(Y) \arrow[r] & G(D)[1]
\end{tikzcd}.$$

Thus $G(Y)=0$. Therefore, $Y=0$ and $FG(D) \cong D$. Hence $F$ is an equivalence. □

**Lemma A.3.** *Let $\mathscr{A}$ and $\mathscr{B}$ be dg categories, and $N$ a dg $\mathscr{A}$-$\mathscr{B}$-bimodule. Assume that*

(1) *the dg module ${}_AN$ is compact in $\mathscr{D}(\mathscr{B})$ for all $A \in \mathscr{A}$,*
(2) *The canonical morphism $\alpha_{Y,Z,k}\colon H^k(\mathscr{A}(Y,Z)) \to \operatorname{Hom}_{\mathscr{D}(\mathscr{B})}({}_YN, {}_ZN[k])$ is an isomorphism for all $Y, Z \in \mathscr{A}$ and for all $k \in \mathbb{Z}$.*

*Then $\text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N$ is fully faithful.*

*Proof.* We know that $(\text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N, \mathbf{R}\mathscr{C}_{\mathrm{dg}_{\mathscr{B}}}(N, \text{-}))$ is an adjoint pair, say with the usual unit $\eta$. Therefore to show that $\text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N$ is fully faithful, it suffices to show the following.

**Claim.** *For each $M \in \mathscr{D}(\mathscr{A})$, $\eta_M : M \to \mathbf{R}\mathscr{C}_{\mathrm{dg}_{\mathscr{B}}}(N, M \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N)$ is an isomorphism in $\mathscr{D}(\mathscr{A})$.*

To show this, let $\mathscr{C}$ be the full subcategory of $\mathscr{D}(\mathscr{A})$ formed by those objects $M$ such that $\eta_M$ is an isomorphism. To show the claim we have only to show that $\mathscr{C} = \mathscr{D}(\mathscr{A})$. As is easily seen $\mathscr{C}$ is a triangulated subcategory of $\mathscr{D}(\mathscr{A})$. Therefore it suffices to show the following two facts:

(i) ${}_A\mathscr{A} \in \mathscr{C}$ for all $A \in \mathscr{A}$; and
(ii) $\mathscr{C}$ is closed under small coproducts.

(i) Let $A \in \mathscr{A}$. We show that ${}_A\mathscr{A} \in \mathscr{C}$, namely that

$$\eta_{{}_A\mathscr{A}} : {}_A\mathscr{A} \to \mathbf{R}\mathscr{C}_{\mathrm{dg}_{\mathscr{B}}}(N, {}_A\mathscr{A} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N) \cong \mathbf{R}\mathscr{C}_{\mathrm{dg}_{\mathscr{B}}}(N, {}_AN)$$

is an isomorphism in $\mathscr{D}(\mathscr{A})$. It suffices to show that

$$\eta_{A\mathscr{A}} : {}_A\mathscr{A} \to \mathbf{R}\mathscr{C}_{\mathrm{dg}_\mathscr{B}}(N, {}_AN)$$

is a quasi-isomorphism. For each $A' \in \mathscr{A}$ and $k \in \mathbb{Z}$ we have the following commutative diagram:

$$\begin{tikzcd} H^k(\mathscr{A}(A',A)) \arrow[rr, "H^k(\eta_{\mathscr{A}(A',A)})"] \arrow[dr, "\alpha_{A',A,k}"'] & & H^k(\mathbf{R}\mathscr{C}_{\mathrm{dg}_\mathscr{B}}({}_{A'}N, {}_AN)) \arrow[dl, "\beta_{A',A,k}"] \\ & \mathrm{Hom}_{\mathscr{D}(\mathscr{B})}({}_{A'}N, {}_AN[k]) & \end{tikzcd},$$

where $\beta_{A',A,k}$ is the canonical isomorphism. Since $\alpha_{A',A,k}$ is an isomorphism by the assumption (2), $H^k(\eta_{\mathscr{A}(A',A)})$ turns out to be an isomorphism, which shows (i).

(ii) Let $I$ be a small set and let $M_i \in \mathscr{C}$ for all $i \in I$. We have the following commutative diagram with canonical morphisms in $\mathscr{D}(\mathscr{A})$:

$$\begin{tikzcd} \bigoplus_{i\in I} M_i \arrow[r, "\eta_{\bigoplus_{i\in I} M_i}"] \arrow[dd, equal] & \mathbf{R}\mathscr{C}_{\mathrm{dg}_\mathscr{B}}(N, (\bigoplus_{i\in I} M_i) \overset{\mathbf{L}}{\otimes}_\mathscr{A} N) \\ & \mathbf{R}\mathscr{C}_{\mathrm{dg}_\mathscr{B}}(N, \bigoplus_{i\in I}(M_i \overset{\mathbf{L}}{\otimes}_\mathscr{A} N) \arrow[u, "\wr"', "\text{(a)}"] \\ \bigoplus_{i\in I} M_i \arrow[r, "\sim", "\bigoplus_{i\in I}\eta_{M_i}"'] & \bigoplus_{i\in I} \mathbf{R}\mathscr{C}_{\mathrm{dg}_\mathscr{B}}(N, M_i \overset{\mathbf{L}}{\otimes}_\mathscr{A} N) \arrow[u, "\wr"', "\text{(b)}"] \end{tikzcd},$$

where (a) is an isomorphism because - $\overset{\mathbf{L}}{\otimes}_\mathscr{A} N$ is a left adjoint and preserves small coproducts, and (b) is an isomorphism by the assumption (1). Thus

$$\eta_{\bigoplus_{i\in I} M_i} : \bigoplus_{i\in I} M_i \to \mathbf{R}\mathscr{C}_{\mathrm{dg}_\mathscr{B}}(N, \bigoplus_{i\in I} M_i \overset{\mathbf{L}}{\otimes}_\mathscr{A} N)$$

is an isomorphism, and hence we have $\bigoplus_{i\in I} M_i \in \mathscr{C}$. As a consequence, $\mathscr{C}$ is closed under small coproducts. $\square$

**Lemma A.4.** *Let $\mathscr{A}$ and $\mathscr{B}$ be dg categories and $E\colon \mathscr{A} \to \mathscr{B}$ a quasi-equivalence. Then for each right $\mathscr{B}$-module $M$ the following holds:*

$$\mathbf{R}\mathscr{C}_{\mathrm{dg}_\mathscr{B}}({}_E\mathscr{B}, M) = 0 \text{ implies } M = 0.$$

*Proof.* Let $M$ be a $\mathscr{B}$-module, and assume that $\mathbf{R}\mathscr{C}_{\mathrm{dg}_\mathscr{B}}({}_E\mathscr{B}, M) = 0$. Take any $B \in \mathscr{B}$. It is enough to show that $M(B) = 0$. Now, since $H^0(E) : H^0(\mathscr{A}) \to H^0(\mathscr{B})$ is an equivalence (the condition (2) in Definition 7.32), there exists an object $A \in \mathscr{A}$, such that $E(A) = H^0(E)(A) \cong B$ in $H^0(\mathscr{B})$. Then by the functor $H^0(\mathscr{B}) \to \mathscr{D}(\mathscr{B}), X \mapsto {}_X\mathscr{B}$ we have ${}_{E(A)}\mathscr{B} \cong {}_B\mathscr{B}$ in $\mathscr{D}(\mathscr{B})$. Hence by the dg Yoneda lemma we have

$$M(B) \cong \mathbf{R}\mathscr{C}_{\mathrm{dg}_\mathscr{B}}({}_B\mathscr{B}, M) \cong \mathbf{R}\mathscr{C}_{\mathrm{dg}_\mathscr{B}}({}_{E(A)}\mathscr{B}, M) = 0,$$

as required. $\square$

**Proof of Theorem A.1.** Define a dg $\mathscr{A}$-$\mathscr{B}$-bimodule $N$ by $N := {}_E\mathscr{B}$. Then $N$ satisfies the condition (1) in Lemma A.3, and by the assumption (in particular, by the condition (1) in Definition 7.32) $N$ also satisfies the condition (2) in Lemma A.3. Therefore $F := \text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N \colon \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$ is fully faithful by Lemma A.3. Moreover $G := \mathbf{R}\mathscr{C}_{\mathrm{dg}_{\mathscr{B}}}(N, \text{-})$ is a right adjoint to $F$ and satisfies the condition (3) in Lemma A.2 by the assumption and Lemma A.4. Hence $F$ is an equivalence between $\mathscr{D}(\mathscr{A})$ and $\mathscr{D}(\mathscr{B})$ by Lemma A.2. $\square$

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