INTERVAL REPLACEMENTS OF PERSISTENCE MODULES

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Abstract. We define (1) a notion of a compression system ξ for a finite poset $\mathbf{P}$, which assigns each interval subposet I to a poset morphism $\xi_I : Q_I \to \mathbf{P}$ and (2) an I-rank of a persistence module M with respect to ξ, the family of which is called the interval rank invariant. A compression system ξ makes it possible to define the interval replacement (also called the interval-decomposable approximation) not only for 2D persistence modules but also for any persistence modules over any finite poset. We will show that the forming of the interval replacement preserves the interval rank invariant that is a stronger property than the preservation of the usual rank invariant. Moreover, we will give an explicit formula of the I-rank of M with respect to ξ in terms of the structure linear maps of M under a mild existence condition of joins and meets in I in the case where ξ_I is the inclusion of I into $\mathbf{P}$, or more generally, ξ_I "essentially covers" I.

Contents

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1. Introduction

Persistent homology is one of the main tools used in topological data analysis. In this context, the data are usually given in the form of a point cloud (a finite subset of a finite-dimensional Euclidean space). Persistent homology plays an important role in examining the topological property of the data [14]. Given one-parameter filtrations arising from the data, it yields representations of a Dynkin quiver Q of type A, thus modules over the path category of Q, that are sometimes called 1-*dimensional persistence modules* [20, 6, 8]. The product quiver of d Dynkin quiver of type A with full commutativity relations for some $d \geq 1$ is called dD-grid. By considering multi-parameter filtrations, representations of dD-grid naturally arise in practical settings, which are called d-dimensional persistence modules [8, 12]. Since the linear category defined by this quiver with relations can be regarded as the incidence category (Definition 2.2 (1)) of a poset, persistence modules are understood as modules over the incidence category of a poset in general, or equivalently, functors from the poset (regarded as a category) to the category $\operatorname{mod} \Bbbk$ of finite-dimensional vector spaces over a field $\Bbbk$. Except for only a few cases, the category of d-dimensional persistence modules has infinitely many indecomposables up to isomorphisms if $d > 1$ [5]. In these cases, dealing with all indecomposable persistence modules is very difficult and is usually inefficient. Therefore, to avoid it, we restrict ourselves to a finite subset of indecomposables, and try to approximate the original persistence module by those selected ones. As in our previous papers [1, 3], we choose as this subset the set of all interval modules because they are easy to handle, have nice properties, and comprise a large portion of direct summands in the indecomposable decomposition of each persistence module in practical data analysis.

Let $\mathbf{P}$ be a finite poset, and I an interval subposet (namely, a connected and convex subposet). The set of all interval subposets is denoted by $\mathbb{I}$. As mentioned above, we sometimes regard $\mathbf{P}$ as a category (Definition 2.2). The category of finite-dimensional modules over a category $\mathscr{C}$ is denoted by $\operatorname{mod} \mathscr{C}$. Hence $\operatorname{mod} \Bbbk[\mathbf{P}]$ denotes the category of persistence modules over $\mathbf{P}$. We denote by V_I the interval module defined by I (Definition 2.7). In this paper, we assume that the Dynkin quivers of type A used to define a dD-grid are equioriented. Then it is isomorphic to the incidence category of the product poset of d totally ordered finite sets.

In [3], the notion of *interval replacement* $\delta^*(M)$ of a persistence module M over a 2D-grid was introduced, which is an element of the split Grothendieck group, and is given as a pair of interval decomposable modules. The important points are that M and $\delta^*(M)$ share the same rank invariants (and hence also dimension vectors) for all $* = \mathrm{tot}, \mathrm{ss}, \mathrm{cc}$, three kinds of compression to define it, and that the interval replacement gives a way to examine the persistence module M by using interval modules.

1.1. **Purposes.** In this paper, we generalize the notion of interval replacement in three ways. The first generalization is to broaden the setting from 2D-grids to any finite posets, the second is to generalize the three kinds of compression to a compression system ξ (Definition 3.1). Roughly speaking, a compression system ξ assigns each interval I to a poset morphism $\xi_I\colon Q_I \to \mathbf{P}$ factoring through the inclusion of I into $\mathbf{P}$ having some properties with $\min I$, $\max I$. Then ξ_I gives the restriction functor $R_I := R_I^\xi\colon \operatorname{mod} \mathbf{P} \to \operatorname{mod} I$. For example, the family tot of the inclusions $\mathrm{tot}_I\colon I \hookrightarrow \mathbf{P}$ for all intervals I turns out to be a compression system, called the total compression system.

Finally, the third is to extend the rank invariants that are regarded as the invariants with respect to rectangles to the invariants with respect to *any intervals*, called the *interval rank invariant*. This is done as follows. Let $M \in \operatorname{mod} \mathbf{P}$. Then the multiplicity of $R_I(V_I)$ in the indecomposable decomposition of $R_I(M)$ is denoted by $c_M^\xi(I) = \operatorname{rank}_I^\xi M$, and is called the compression multiplicity of M at I or I-rank of M with respect to ξ (Definitions 3.8 and 4.10). In particular, $\operatorname{rank}_I^{\mathrm{tot}} M$ is simply called the *total* I-rank of M. The family $\operatorname{rank}^\xi M := (\operatorname{rank}_I^\xi M)_{I\in\mathbb{I}}$ is called the interval rank invariant of M with respect to ξ.

The Möbius inversion δ_M^ξ of $c_M^\xi\colon \mathbb{I} \to \mathbb{R}$ is called the *signed interval multiplicity* of M at I (Definition 3.18), which defines the interval replacement $\delta^\xi(M)$ with respect to ξ (Definition 4.1). The I-rank $\operatorname{rank}_I^\xi \delta^\xi(M)$ of the *interval replacement* $\delta^\xi(M)$ of M can also be naturally defined (Definition 4.1). Then we will prove that the forming of δ^ξ preserves the I-ranks as stated in the following.

Theorem 1.1 (Therorem 4.11)**.** *Let $M \in \operatorname{mod} \mathbf{P}$, and I an interval of $\mathbf{P}$. Then*

$$\operatorname{rank}_I^\xi \delta^\xi(M) = \operatorname{rank}_I^\xi M.$$

Moreover, we will give an explicit formula of the interval rank invariant of M under some existence conditions of joins and meets in the intervals. More precisely, we have the following two theorems. The first one is for the total compression, and the second is for a compression system with essential cover property defined below.

Theorem 1.2 (Theorem 5.20)**.** *Let $M \in \operatorname{mod} \mathbf{P}$, and I an interval of $\mathbf{P}$ with $\min I := \{a_1, \dots, a_n\}$, $\max I := \{b_1, \dots, b_m\}$ (elements are pairwise distinct) for some $m, n \geq 1$. Assume that for any pair $a, a' \in \min I$ (resp. $b, b' \in \max I$) there exists the join (resp. meet) of them in I. Obviously, for each $a \in \min I$,* *there exists some* $b \in \max I$ *such that* $a \leq b$. *Hence we may assume that* $a_1 \leq b_1$ *without loss of generality. Then we have*

$$\operatorname{rank}_I^{\mathrm{tot}} M = \operatorname{rank} \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \begin{bmatrix} M_{b_1,a_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & \mathbf{N} \end{bmatrix} - \operatorname{rank} \mathbf{M} - \operatorname{rank} \mathbf{N}, \tag{1.1}$$

where $\mathbf{M}, \mathbf{N}$ *are the matrices defined in Theorems 5.7 and 5.16, whose nonzero entries are given by structure linear maps* $M_{b,a} \colon M(a) \to M(b)$ *of* M *corresponding to the unique morphism from* a *to* b *in* $\mathbf{P}$ *for all* $a, b \in \mathbf{P}$. *If* $m = 1$ *(resp.* $n = 1$*), then* $\mathbf{N}$ *(resp.* $\mathbf{M}$*) is an empty matrix, and hence the formula has one of the special forms given in Theorems 5.7, 5.16, and Proposition 4.7.*

Let M and I be as above, and ξ a compression system. Then we say that ξ_I *essentially covers* I if there exists a formula of $\operatorname{rank}_I^{\mathrm{tot}} M$ in terms of linear maps $M(p)$ ($p \in S$) for some subset S of morphisms in I such that for each $p \in S$ there exists a morphism $q \in Q_I$ with $p = \xi_I(q)$. Then we will prove the following.

Theorem 1.3 (Theorem 5.25)**.** *Let* M*,* I *and* ξ *be as above. If* ξ_I *essentially covers* I*, then we have*

$$\operatorname{rank}_I^{\xi} M = \operatorname{rank}_I^{\mathrm{tot}} M.$$

Hence Theorem 1.2 gives the formula also for $\operatorname{rank}_I^{\xi} M$.

1.2. **Related works.** In [17], Kim and Mémoli introduced the generalized rank invariant for persistence modules over posets. In fact, the generalized rank invariant coincides with our proposed interval rank invariant using a specified compression system, namely the total compression system (see Example 3.3 and Lemma 5.29). However, we provide a more general theory of defining the interval rank invariant and interval replacement of persistence modules with any compression system ξ involving not only the maximum compression system (i.e., the total compression system) but also some other compression systems (for instance, the minimum and simplest source-sink compression system, see Example 3.4), allowing one to use different compression systems simultaneously or independently in practice. For example, one can combine both the source-sink and total compression systems to retrieve the original algebraic information if necessary. Also, it is sufficient to utilize the simplest source-sink compression in some situations.

In [9], Botnan, Oppermann, and Oudot introduced a general framework mainly focusing on decomposing any persistence modules using the signed barcodes in the (generalized) rank level. In detail, given any collection $\mathscr{I}$ of intervals of a poset and arbitrary map $r : \mathscr{I} \to \mathbb{Z}$, there uniquely exist two disjoint multi-sets R and S of elements of $\mathscr{I}$ such that r equals to the generalized rank invariant of interval-decomposable module $\bigoplus_{I \in R} V_I$ subtracts the generalized rank invariant of interval-decomposable module $\bigoplus_{I \in S} V_I$ (see [9, Corollary 2.5]). From this result, one can obtain a specified part of our Theorem 1.1, that is, the persistence module and its signed barcodes decomposition share the same generalized rank invariant once we let $\mathscr{I} = \mathbb{I}$ and take r to be our interval rank invariant $\mathrm{rank}^{\mathrm{tot}} M$ (viewed as the map $\mathrm{rank}^{\mathrm{tot}} M : \mathbb{I} \to \mathbb{Z}, I \mapsto \mathrm{rank}_I^{\mathrm{tot}} M$). However, our Theorem 1.1 shows that the interval replacement preserves the interval rank invariant, not only using the total compression system (i.e., generalized rank invariant) but also using other different compression systems. Another remarkable note is that they do not only focus on the locally finite collection but also on the larger collection $\mathscr{I}$ of intervals of arbitrary poset. Compared with their results, we shed light on the concept of the compression system and propose a new rank invariant of persistence modules based on the compression system. In our framework, the proposed interval rank invariant of persistence module M can be naturally regarded as the multiplicity of the interval modules in the decomposition of the "compressed" module of M. From this viewpoint, we could compute theoretically and give explicit formulas for this new interval rank invariant by utilizing the powerful Auslander–Reiten theory.

Concerning the computation aspect. The generalized rank invariant is reasonably simple because [11, Theorem 3.12] reduces its computation to the zigzag path (boundary cap in their terminology) that concatenates the lower and upper zigzags of each interval. This way of computing has the benefit of utilizing many mature algorithms to compute the rank invariant in the 1D persistence context. Nevertheless, our work provides explicit formulas for directly computing the interval rank invariant with the total compression system and other compression systems having the essential cover property as explained just before Theorem 1.3 (see Definition 5.21). Moreover, since the zigzag compression system zz (Example 5.23) corresponding to their zigzag path above has this essential cover property, Theorem 1.3 above gives an alternative proof of [11, Theorem 3.12] because as mentioned above, $\mathrm{rank}_I^{\mathrm{tot}}(M)$ coincides with their generalized rank invariant of M. The latter statement follows by [10, Lemma 3.1], but the description of the proof was imprecise, and in the process of making it accurate we found a small gap in the proof. Therefore, we give a precise proof of it by filling the gap. Combining the explicit formulas of the source-sink compression system given in [3, Proposition 6.4] for the $(n \times 2)$-grid, we have the capability to completely compute these two different families more straightforwardly and easily when the ambient poset is induced by the $(n \times 2)$-grid (see Example 4.9). In addition, these explicit formulas give an intuition for choosing which types of compression systems would induce the same interval rank invariant.

In [16], Hiraoka, Nakashima, Obayashi, and Xu also established the general theory for approximating any persistence modules over a finite fully commutative acyclic quiver $G = (Q, R)$ by interval decomposable modules, which shares the same spirit with ours. They defined the so-called interval approximation (which, essentially, coincides with our interval replacement $\delta_M^{\xi}(\cdot)$). For the sake of fast computation, they consider defining interval approximation on the restriction of the collection $\mathbb{I}$ of all intervals, called the partial interval approximation (which shares the similar idea of considering those intervals having "good" shapes in [9]). For instance, they define the partial interval approximation restricted to the collection of k-essential intervals and estimate the computational complexity of (partial) interval approximation. Their remarkable distinction is treating the collection of interval approximations as a rank invariant of persistence modules (see [16, Definition 3.36, Example 3.37]). On the contrary, the collection of compression multiplicities is treated as a rank invariant in our work. Moreover, our Theorem 1.1 extends their [16, Theorem 3.29], in the sense that forming the interval replacements preserves I-ranks not only for all segments I but also for all intervals I. Another main contribution in [16] is providing a new way of visualizing interval approximation in the commutative ladder setting, called the connected persistence diagram (see [16, Definition 4.7]).

Due to the flexibility of selecting compression systems, one advantage is that for representation-finite commutative ladders, the paper [16] designed a way of efficiently computing the complete invariant $d_M(\cdot) : \mathscr{L} \to \mathbb{Z}$ by choosing finitely many compression systems (explicitly, the cardinality of $\mathscr{L}$) and solving a linear equations system. Another advantage of the compression system is developing the so-called interval resolution of given persistence modules based on the well-selected compression system [2]. Thanks to this technique, the relationship between interval replacement and interval resolution can be well-established in the context of commutative ladders.

1.3. **Our contributions.** (1) We introduce the compression system and the interval rank invariant with respect to the compression system. These allow us to extend the concept of interval replacement defined on the commutative grid in [3] to the finite poset (Theorem 1.1). We follow the convention in [3] to view the interval replacement of the persistence module as an element in the split Grothendieck group.

(2) Based on Proposition 4.7, we see an interesting phenomenon (Corollary 4.8) and exhibit such an example (Example 4.9).

(3) We provide explicit formulas in Theorem 1.2 to directly compute the interval rank invariant, utilizing the Auslander–Reiten theory. To this end, we first gave a formula to compute the dimension of $\mathrm{Hom}(X, Y)$ in terms of a projective presentation of X, and then for each interval I, we computed the almost split sequence starting from V_I over the incidence category $\Bbbk[I]$, and also gave the projective presentations of all the terms in the sequence to compute the Hom dimensions of them. These computations can also be used for later research. In addition, the explicit formulas for the interval rank invariant give an intuition for choosing which types of compression systems would induce the same interval rank invariant.

(4) We give a sufficient condition for the I-rank of a persistence module M with respect to a compression system ξ to coincide with the total I-rank (Theorem 1.3). As stated above, this gives an alternative proof of [11, Theorem 3.12].

1.4. **Organization.** The paper is organized as follows. Section 2 is devoted to collecting necessary terminologies and fundamental properties for the later use, in particular, incidence categories and incidence algebras defined by a finite poset, and the Möbius inversion.

In Sect. 3, we introduce the notion of compression systems ξ, the compression multiplicity, and the signed interval multiplicities with respect to ξ.

The latter makes it possible to define the interval replacement and the interval rank invariant of a persistence module in Sect. 4, where we prove the preservation of interval rank invariant under forming the interval replacement (Theorem 1.1).

In Sect. 5, we give an explicit formula for the total interval rank invariant (Theorem 1.2) by computing the almost split sequences starting from V_I for any interval $I \in \mathbb{I}$ and projective presentations of all the terms in the sequence. In addition, we give a sufficient condition for a compression system to have the same interval rank invariant as tot (Theorem 1.3) to give it also an explicit formula.

Finally, in Sect. 6, we give some examples showing the incompleteness of the interval rank invariant.

2. Preliminaries

Throughout this paper, $\Bbbk$ is a field, $\mathbf{P} = (\mathbf{P}, \leq)$ is a finite poset. The category of finite-dimensional $\Bbbk$-vector spaces is denoted by $\operatorname{mod} \Bbbk$. For a quiver Q, a path p from x to y in Q is expressed by $p\colon x \rightsquigarrow y$, and the ideal of the path category $\Bbbk[Q]$ generated by all the commutativity relations is denoted by com_Q.

2.1. Incidence categories.

Definition 2.1. A $\Bbbk$-linear category $\mathscr{C}$ is said to be *finite* if it has only finitely many objects and for each pair (x, y) of objects, the Hom-space $\mathscr{C}(x, y)$ is finite-dimensional.

Covariant functors from $\mathscr{C}$ to $\operatorname{mod} \Bbbk$ are called *left $\mathscr{C}$-modules*. They together with natural transformations between them as morphisms form a $\Bbbk$-linear category, which is denoted by $\operatorname{mod} \mathscr{C}$.

Definition 2.2. The poset $\mathbf{P}$ is regarded as a category as follows. The set $\mathbf{P}_0$ of objects is defined by $\mathbf{P}_0 := \mathbf{P}$. For each pair $(x, y) \in \mathbf{P} \times \mathbf{P}$, the set $\mathbf{P}(x, y)$ of morphisms from x to y is defined by $\mathbf{P}(x, y) := \{p_{y,x}\}$ if $x \leq y$, and $\mathbf{P}(x, y) := \emptyset$ otherwise, where we set $p_{y,x} := (y, x)$. The composition is defined by $p_{z,y} p_{y,x} = p_{z,x}$ for all $x, y, z \in \mathbf{P}$ with $x \leq y \leq z$. The identity $\mathbb{1}_x$ at an object $x \in \mathbf{P}$ is given by $\mathbb{1}_x = p_{x,x}$.

(1) The *incidence category* $\Bbbk[\mathbf{P}]$ of $\mathbf{P}$ is defined as the $\Bbbk$-linearization of the category $\mathbf{P}$. Namely, it is a $\Bbbk$-linear category defined as follows. The set of objects $\Bbbk[\mathbf{P}]_0$ is equal to $\mathbf{P}$, for each pair $(x, y) \in \mathbf{P} \times \mathbf{P}$, the set of morphisms $\Bbbk[\mathbf{P}](x, y)$ is the vector space with basis $\mathbf{P}(x, y)$; thus it is a one-dimensional vector space $\Bbbk p_{y,x}$ if $x \leq y$, or zero otherwise. The

composition is defined as the $\Bbbk$-bilinear extension of that of $\mathbf{P}$. Note that $\Bbbk[\mathbf{P}]$ is a finite $\Bbbk$-linear category.

(2) Covariant ($\Bbbk$-linear) functors $\Bbbk[\mathbf{P}] \to \operatorname{mod} \Bbbk$ are called *persistence modules*.

In the sequel, we set $[\leq]_{\mathbf{P}} := \{(x, y) \in \mathbf{P} \times \mathbf{P} \mid x \leq y\}$, and $A := \Bbbk[\mathbf{P}]$ (therefore, $A_0 = \mathbf{P}$), and so the category of finite-dimensional persistence modules is denoted by $\operatorname{mod} A$.

Definition 2.3. Let I be a nonempty full subposet of $\mathbf{P}$.

(1) For any $(x, y) \in [\leq]_{\mathbf{P}}$, we set $[x, y] := \{z \in \mathbf{P} \mid x \leq z \leq y\}$, and call it the *segment* from x to y in $\mathbf{P}$. The set of all segments in $\mathbf{P}$ is denoted by $\operatorname{Seg}(\mathbf{P})$.

(2) The *Hasse quiver* $H(\mathbf{P})$ of $\mathbf{P}$ is a quiver defined as follows. The set $H(\mathbf{P})_0$ of vertices is equal to $\mathbf{P}$, the set $H(\mathbf{P})_1$ of arrows is given by the set $\{a_{y,x} \mid (x, y) \in [\leq]_{\mathbf{P}}, [x, y] = \{x, y\}\}$, and the source and the target of $a_{y,x}$ are x and y, respectively, where we set $a_{y,x} := p_{y,x}$. In the sequel, we set $Q := H(\mathbf{P})$. Thus we have $Q_0 = \mathbf{P}$, and we can regard $\Bbbk[\mathbf{P}] = \Bbbk[Q]/\mathrm{com}_Q$ by identifying the coset of each path $p \colon x \rightsquigarrow y$ with the morphism $p_{y,x}$.

(3) A *source* (resp. *sink*) of I is nothing but a minimal (resp. maximal) element in I, which is characterized as an element $x \in I$ such that in the Hasse quiver $H(I)$, there is no arrow with target (resp. source) x. The set of all sources (sinks) in I is denoted by $\operatorname{sc}(I)$ (resp. $\operatorname{sk}(I)$).

(4) I is said to be *connected* if the full subquiver of $H(\mathbf{P})$ whose vertex set is equal to I is connected.

(5) I is said to be *convex* if for any $x, y \in I$ with $x \leq y$, we have $[x, y] \subseteq I$.

(6) The *convex hull* $\operatorname{conv}(I)$ of I is defined as the smallest (with respect to the inclusion) convex subset of $\mathbf{P}$ that contains I. Equivalently, $\operatorname{conv}(I)$ is the union of all segments between elements of I.

(7) I is called an *interval* if I is connected and convex.

(8) The set of all intervals of $\mathbf{P}$ is denoted by $\mathbb{I}(\mathbf{P})$, or simply by $\mathbb{I}$. We regard $\mathbb{I}$ as a poset $\mathbb{I} = (\mathbb{I}, \leq)$ with the inclusion relation: $I \leq J \Leftrightarrow I \subseteq J$ for all $I, J \in \mathbb{I}$. Since $\mathbf{P}$ is finite, $\mathbb{I}$ is also finite.

(9) Let $I \in \mathbb{I}$. The *cover* of I is defined as

$$\operatorname{Cov}(I) := \{L \in \mathbb{I} \mid I < L \text{ and } [I, L] = \{I, L\}\}.$$

(10) Let U be a subset of $\mathbb{I}$. The least upper bound of U is called the *join* of U, and is denoted by $\bigvee U$. As the least element, it is unique if it exists.

Remark 2.4. Note that for a subposet I of $\mathbf{P}$ to be connected is not equivalent to saying that the Hasse quiver $H(I)$ of I is connected. Namely, the former implies the latter, but the converse does not hold in general. For example, $\mathbf{P} = \{1 < 2 < 3\}$ and $I := \{1, 3\}$. However, if I is convex, then the two notions coincide.

Note also that any segment $[x, y]$ in $\mathrm{Seg}(\mathbf{P})$ is an interval with source x and sink y. Hence $\mathrm{Seg}(\mathbf{P}) \subseteq \mathbb{I}(\mathbf{P})$ (see the statements just after Lemma 3.10 for more precise relation).

Remark 2.5. Since the factor category $A = \Bbbk[Q]/\mathrm{com}_Q$, the category $\mathrm{mod}\, A$ of persistence modules is isomorphic to the category $\mathrm{rep}_{\Bbbk}(Q, \rho)$ of $\Bbbk$-representations of the bound quiver (Q, ρ). We usually identify these categories. Thus persistence modules are given as representations of the bound quiver (Q, ρ).

Note that in general, the join of a poset might not exist. In our setting, we have the following:

Proposition 2.6. *Let U be a subset of $\mathbb{I}$. If U has a lower bound, then the join of U exists.*

Proof. Let I be a lower bound of U. Let us write $U := \{I_1, ..., I_n\}$ with $n \geq 1$. Then the subset of $\mathbf{P}$ defined by $\bigcup_{k=1}^n I_k$ is connected since $I \leq I_k$ for all $k = 1, ..., n$. It follows that $\mathrm{conv}\,(\bigcup_{k=1}^n I_k)$ is connected, convex, and containing U, and hence it is an upper bound of U. Now, let W be an upper bound of U. Since $\bigcup_{k=1}^n I_k \subseteq W$ and W is convex, we have $\mathrm{conv}\,(\bigcup_{k=1}^n I_k) \subseteq W$. Thus $\mathrm{conv}\,(\bigcup_{k=1}^n I_k) = \bigvee U$. □

Definition 2.7 (Interval modules)**.** Let I be an interval of $\mathbf{P}$.

(1) A persistence module V_I is defined as a representation of Q as follows: For each $x \in Q_0$, $V_I(x) = \Bbbk$ if $x \in I$, and $V_I(x) = 0$ otherwise; For each $a_{y,x} \in Q_1$, $V_I(a_{y,x}) = \mathbb{1}_{\Bbbk}$ if $\{x, y\} \subseteq I$, and $V_I(a_{y,x}) = 0$ otherwise. It is obvious that V_I satisfies the full commutativity relations, and hence V_I is, in fact, a persistence module. It is also easy to see that V_I is indecomposable.

(2) A persistence module isomorphic to V_I for some $I \in \mathbb{I}$ is called an *interval module.*

(3) A persistence module is said to be *interval decomposable* if it is isomorphic to a finite direct sum of interval modules. Thus 0 is trivially interval decomposable.

We will use the notation $d_M(L)$ to denote the multiplicity of an indecomposable direct summand L of a module M in its indecomposable decomposition as explained in the following well-known theorem.

Theorem 2.8 (Krull–Schmidt)**.** *Let $\mathscr{C}$ be a finite $\Bbbk$-linear category, and fix a complete set $\mathscr{L} = \mathscr{L}_{\mathscr{C}}$ of representatives of isoclasses of indecomposable objects in $\mathrm{mod}\,\mathscr{C}$. Then every finite-dimensional left $\mathscr{C}$-module M is isomorphic to the direct sum $\bigoplus_{L \in \mathscr{L}} L^{d_M(L)}$ for some unique function $d_M \colon \mathscr{L} \to \mathbb{Z}_{\geq 0}$. Therefore another finite-dimensional left $\mathscr{C}$-module N is isomorphic to M if and only if $d_M = d_N$. In this sense, the function d_M is a complete invariant of M under isomorphisms.*

In one-parameter persistent homology, this function d_M corresponds to the persistence diagram of M, which is a graph plotting each $d_M(L)$ as a colored point on $\mathscr{L}$.

Let us recall some basic facts about Möbius functions. For more details we refer the reader to [19]. Following Definition 2.3 (1), we denote by $\mathrm{Seg}(\mathbb{I})$ the set of all segments of $\mathbb{I}$. Here, we want to emphasize that we are considering segments of the poset $(\mathbb{I}, \leq)$ and not $(\mathbf{P}, \leq)$.

2.2. **Incidence algebra and Möbius inversion.**

Definition 2.9 (Incidence algebra of $\mathbb{I}$)**.** We define the incidence algebra $\mathbb{R}\mathbb{I}$ of $\mathbb{I}$ over the field $\mathbb{R}$ of real numbers by using the incidence category $\mathbb{R}[\mathbb{I}]$ as the matrix algebra

$$\mathbb{R}\mathbb{I} := \bigoplus_{J,I\in\mathbb{I}} \mathbb{R}[\mathbb{I}](I,J), \text{ where } \mathbb{R}[\mathbb{I}](I,J) = \begin{cases} \mathbb{R}p_{J,I} & (I \leq J), \\ 0 & (\text{otherwise}) \end{cases}$$

with the usual matrix multiplication:

$$(m_{J,I})_{J,I\in\mathbb{I}}(m'_{J,I})_{J,I\in\mathbb{I}} := (\sum_{K\in\mathbb{I}} m_{J,K}m'_{K,I})_{J,I\in\mathbb{I}}$$

for all $(m_{J,I})_{J,I\in\mathbb{I}}, (m'_{J,I})_{J,I\in\mathbb{I}} \in \mathbb{R}\mathbb{I}$. Note that $\mathbb{R}\mathbb{I}$ is also obtained as

$$\mathbb{R}\mathbb{I} = \bigoplus_{I\leq J} \mathbb{R}p_{J,I}$$

with the multiplication defined by $p_{L,K}p_{J,I} = \delta_{K,J}p_{L,I}$ for all $I, J, K, L \in \mathbb{I}$ with $I \leq J$ and $K \leq L$, where $\delta_{K,J}$ is Kronecker's delta symbol.

Remark 2.10. For the definition above, we have the following remarks.

(1) (As a matrix algebra with blocks $\mathbb{R}$ or 0) To express each element of $\mathbb{R}\mathbb{I}$ as a matrix, we fix a total order on $\mathbb{I}$ extending the original partial order. By regarding the isomorphism $\mathbb{R} \to \mathbb{R}p_{J,I}$ sending 1 to $p_{J,I}$ as the identity map, we can regard $\mathbb{R}\mathbb{I}$ as a matrix algebra over $\mathbb{R}$ with the set of (J,I)-entries $\mathbb{R}$ if $I \leq J$, and 0 otherwise.

(2) (As a set of functions from $\mathrm{Seg}(\mathbb{I})$ to $\mathbb{R}$) Note that we have a bijection $\mathrm{Seg}(\mathbb{I}) \to \{p_{J,I} \mid I, J \in \mathbb{I}, I \leq J\}$ sending $[I,J]$ to $p_{J,I}$, and that each element m of $\mathbb{R}\mathbb{I}$ can be regarded as a function $\{p_{J,I} \mid I, J \in \mathbb{I}, I \leq J\} \to \mathbb{R}$ sending each $p_{J,I}$ to the (J,I)-entry $m_{J,I}$ of m. By combining these we can also regard $\mathbb{R}\mathbb{I}$ as the set $\mathbb{R}^{\mathrm{Seg}(\mathbb{I})}$ of functions $\mathrm{Seg}(\mathbb{I}) \to \mathbb{R}$, namely, by identifying an element $m \in \mathbb{R}\mathbb{I}$ with the function sending each segment $[I,J]$ to the (J,I)-entry $m_{J,I}$ of m.

(3) (Right action on $\mathbb{R}^{\mathbb{I}}$) Consider the opposite poset $\mathbb{I}^{\mathrm{op}}$ of $\mathbb{I}$, and the incidence algebra $\mathbb{R}(\mathbb{I}^{\mathrm{op}})$, which can be regarded as the opposite algebra of $\mathbb{R}\mathbb{I}$: $\mathbb{R}(\mathbb{I}^{\mathrm{op}}) = (\mathbb{R}\mathbb{I})^{\mathrm{op}}$. Since $\mathbb{I}^{\mathrm{op}}$ itself is an interval of $\mathbb{I}^{\mathrm{op}}$, we can define the interval module $V := V_{\mathbb{I}^{\mathrm{op}}}$, which is isomorphic to the set $\mathbb{R}^{\mathbb{I}}$ of functions as a vector space over $\mathbb{R}$. The isomorphism is given by sending $(v_I \in V(I) = \mathbb{R})_{I\in\mathbb{I}}$ to the map $I \mapsto v_I$ $(I \in \mathbb{I})$. By identifying $\mathbb{R}^{\mathbb{I}}$ with V under this isomorphism, $\mathbb{R}^{\mathbb{I}}$ has a left $\mathbb{R}(\mathbb{I}^{\mathrm{op}})$-module structure, that

is, a right $\mathbb{R}\mathbb{I}$-module structure, the explicit definition of which is given as follows: Let $f \in \mathbb{R}^{\mathbb{I}}$ and $(I, J) \in \mathbb{I}^2$ with $I \leq J$ in $\mathbb{I}$. Then

$$(f \cdot p_{J,I})(K) = \delta_{I,K} f(J) \tag{2.2}$$

for all $K \in \mathbb{I}$.

Definition 2.11 (Zeta and Möbius functions)**.** We set

$$\zeta := \sum_{I \leq J} p_{J,I} \in \bigoplus_{I \leq J} \mathbb{R} p_{J,I} = \mathbb{R}\mathbb{I} \cong \mathbb{R}^{\mathrm{Seg}(\mathbb{I})}$$

(see Remark 2.10 (2)), and call it the *zeta function*. Then note that ζ is expressed as a lower triangular matrix with all diagonal entries 1 in $\mathbb{R}\mathbb{I}$ as a matrix algebra (see Remark 2.10 (1)). Thus it is invertible in $\mathbb{R}\mathbb{I}$, the inverse is given by the adjoint matrix of ζ, which is denoted by μ, and called the *Möbius function*.

Note that for any $f \in \mathbb{R}^{\mathbb{I}}$, we have

$$(f \cdot \zeta)(K) = \sum_{I \leq J} \delta_{I,K} f(J) = \sum_{K \leq J} f(J) \tag{2.3}$$

for all $K \in \mathbb{I}$ by (2.2).

Theorem 2.12 (Möbius Inversion Formula)**.** *For any $f, g \in \mathbb{R}^{\mathbb{I}}$ and $I \in \mathbb{I}$, the following are equivalent:*

(1) $f(I) = \sum_{I \leq J \in \mathbb{I}} g(J)$*; and*
(2) $g(I) = \sum_{I \leq J \in \mathbb{I}} f(J)\mu([I, J])$.

Proof. Since $\mu = \sum_{I \leq J} \mu([I, J]) p_{J,I}$, we have

$$(f \cdot \mu)(K) = \sum_{I \leq J} \delta_{I,K} f(J)\mu([I, J]) = \sum_{K \leq J} f(J)\mu([K, J]).$$

By this together with (2.3), the equivalence follows from the fact that $f = g\zeta$ if and only if $f\mu = g$. □

3. Compressions and multiplicities

For a functor $F : \mathscr{C} \to \mathscr{D}$ between categories, we denote by $U(F): U(\mathscr{C}) \to U(\mathscr{D})$ the underlying quiver morphism between the underlying quivers of $\mathscr{C}$, $\mathscr{D}$.

3.1. **Compression systems.**

Definition 3.1. A *compression system* for A is a family $\xi := (\xi_I)_{I \in \mathbb{I}}$ of quiver morphisms $\xi_I : Q_I \to U(A)$ from an acyclic connected finite quiver Q_I without multiple arrows satisfying the following three conditions for each $I \in \mathbb{I}$:

(1) ξ_I factors through the inclusion morphism $U(\Bbbk[I]) \hookrightarrow U(A)$ of quivers;
(2) The image $\xi_I((Q_I)_0)$ of vertices contains all the sources and sinks of I; and
(3) If $I = [x, y] \in \mathrm{Seg}(\mathbf{P})$, then there exists a path q in Q_I such that $\overline{\xi_I}(q) = p_{y,x}$, where $\overline{\xi_I} : \Bbbk[Q_I] \to A$ is the linear functor that is a unique extension of ξ_I.

Let $I \in \mathbb{I}$. Then we set $B_I := \Bbbk[Q_I]/\operatorname{Ker}\overline{\xi_I}$ (note that $\mathrm{com}_{Q_I} \subseteq \operatorname{Ker}\overline{\xi_I}$). For each morphism p in $\Bbbk[Q_I]$, we often write the morphism $p + \operatorname{Ker}\overline{\xi_I}$ in B_I simply by p if there seems to be no confusion. Then $\overline{\xi_I}$ induces a functor $F_I^\xi := \widetilde{\xi_I} \colon B_I \to A$. The restriction functor $R_I^\xi \colon \operatorname{mod} A \to \operatorname{mod} B_I$ is defined by sending M to $M \circ F_I^\xi$ for all $M \in \operatorname{mod} A$. The functors F_I^ξ and R_I^ξ are simply denoted by F_I and R_I, respectively, if there seems to be no confusion. Then $R_I(V_I)$ is isomorphic to the interval B_I-module $V_{(Q_I)_0}$ of the poset $((Q_I)_0, \leq)$ defined by Q_I, where $x \leq y$ if and only if there exists a path from x to y in Q_I, and hence $R_I(V_I)$ is indecomposable.

Remark 3.2. We have another simplified version of the definition above that was used in the abstract. If we restrict ourselves to the case where ξ_I factors through the inclusion $U(I) \hookrightarrow U(A)$ instead of (1) above, then a simplified version can be obtained. Note here that Q_I turns out to be a finite poset as described in Definition 3.1, and that $\operatorname{Ker}\overline{\xi_I}$ is generated by full commutativity relations in Q_I. Hence in this case $B_I = \Bbbk[Q_I]/\mathrm{com}_{Q_I}$ can be seen as just the incidence category of the poset Q_I. Therefore, the simplified version is obtained as follows:

A compression system for A is a family $\xi = (\xi_I)_{I\in\mathbb{I}}$ of poset morphisms $\xi_I \colon Q_I \to \mathbf{P}$ from a connected finite poset Q_I satisfying the following conditions for each $I \in \mathbb{I}$.

(1) ξ_I factors through the poset inclusion $I \hookrightarrow \mathbf{P}$;
(2) The image $\xi_I(Q_I)$ contains $\min I \cup \max I$.
(3) If $I = [x, y] \in \operatorname{Seg}(\mathbf{P})$, then there exists a pair $(x', y') \in [\leq]_{Q_I}$ such that $(\xi_I(x'), \xi_I(y')) = (x, y)$.

In this simplified version, $B_I := \Bbbk[I]$ is the incidence category of I, and the functor $F_I^\xi \colon \Bbbk[I] \to \Bbbk[\mathbf{P}]$ is given as the linearization of ξ_I.

We have kept the present version of the definition for the future use.

Example 3.3 (tot)**.** Let $I \in \mathbb{I}$, and $Q_I := H(I)$, the Hasse quiver of I. Then we may identify $\Bbbk[I] = \Bbbk[Q_I]/\mathrm{com}_{Q_I}$. Now define a quiver morphism $\mathrm{tot}_I \colon H(I) \to U(A)$ as the composite

$$H(I) \hookrightarrow U(I) \hookrightarrow U(\Bbbk[I]) \hookrightarrow U(A).$$

Then $F_I = \widetilde{\mathrm{tot}_I}$ is the inclusion $\Bbbk[I] \hookrightarrow A$, and $B_I = \Bbbk[I]$ in this case. (In the simplified version, tot_I is just the inclusion $Q_I := I \hookrightarrow \mathbf{P}$.) This defines a compression system $\mathrm{tot} := (\mathrm{tot}_I)_{I\in\mathbb{I}}$ for A, which is called the *total* compression system for A.

Example 3.4 (ss)**.** Let $I \in \mathbb{I}$, and $Q_I^{\mathrm{ss}} := H(\mathrm{sk}(I) \cup \mathrm{sc}(I))$, the Hasse quiver of the full subposet $\mathrm{sk}(I) \cup \mathrm{sc}(I)$ of I. Define a quiver morphism $\mathrm{ss}_I \colon Q_I^{\mathrm{ss}} \to U(A)$ as the composite

$$Q_I^{\mathrm{ss}} \hookrightarrow U(I) \hookrightarrow U(\Bbbk[I]) \hookrightarrow U(A),$$

where each arrow $x \to y$ in Q_I^{ss} is sent to the unique morphism $p_{y,x}$ from x to y in I. Then $F_I = \widetilde{\mathrm{ss}_I}$ is the inclusion $\Bbbk[Q_I^{\mathrm{ss}}] \hookrightarrow A$, and $B_I = \Bbbk[Q_I^{\mathrm{ss}}]$ in this case. (In the simplified version, ss_I is given as the inclusion $Q_I^{\mathrm{ss}} := \mathrm{sk}(I) \cup \mathrm{sc}(I) \hookrightarrow \mathbf{P}$.) This defines a compression system $\mathrm{ss} := (\mathrm{ss}_I)_{I\in\mathbb{I}}$ for A, which is called the *source-sink* compression system for A.

Example 3.5. Let $\mathbf{P} := G_{5,2}$ as in Example 4.9, and I be the interval with $\mathrm{sc}(I) := \{(1,2),(2,1)\}$ and $\mathrm{sk}(I) := \{(3,2)\}$. Take $Q_I := H(\{(1,2),(2,2),(3,2),(2,1)\})$, the Hasse quiver of the full subposet $\{(1,2),(2,2),(3,2),(2,1)\}$ of I, and take $\xi_I \colon Q_I \to U(A)$ to be the inclusion, then this ξ_I can be taken as a component of a compression system for A, which satisfies $\mathrm{ss}_I \neq \xi_I \neq \mathrm{tot}_I$.

Remark 3.6. The compression system ξ in all the examples above satisfies the following condition for all $I \in \mathbb{I}$ that is stronger than the condition (1) in Definition 3.1:

(1′) ξ_I factors through the inclusion morphism $U(I) \hookrightarrow U(A)$.

In such cases, we have $B_I = \Bbbk[Q_I]/\mathrm{com}_{A_I}$. The condition (1) allows for B_I to have relations other than commutativity relations such as zero relations. In this paper, however, we only consider compression systems satisfying (1′) instead of (1).

In the sequel, we let $\xi := (\xi_I \colon Q_I \to U(A))_{I\in\mathbb{I}}$ be a compression system for A. For later use, we fix the following notations.

Notation 3.7. Let $I \in \mathbb{I}$. Set $F = F_I^{\mathrm{tot}} \colon \Bbbk[I] \to A$ to be the inclusion functor defined by the inclusion map $I \to \mathbf{P}$ of posets, and $R = R_I^{\mathrm{tot}} \colon \operatorname{mod} A \to \operatorname{mod} \Bbbk[I]$ the restriction functor defined by F. Then the quiver morphism $\xi_I \colon Q_I \to U(A)$ factors through $U(F)$ by the condition (1) in Definition 3.1, and the functor $F_I \colon B_I \to A$ factors through F as in the following diagrams:

$$\begin{tikzcd} Q_I \arrow[rr, "\xi_I"] \arrow[dr, "\xi'"'] & & U(A) \\ & U(\Bbbk[I]) \arrow[ur, "U(F)"'] & \end{tikzcd}, \qquad \begin{tikzcd} B_I \arrow[rr, "F_I"] \arrow[dr, "F'"'] & & A \\ & \Bbbk[I] \arrow[ur, "F"'] & \end{tikzcd}.$$

Hence we have the corresponding factorization of R_I by R as in the diagram

$$\begin{tikzcd} \operatorname{mod} B_I & & \operatorname{mod} A \arrow[ll, "R_I"'] \arrow[dl, "R"] \\ & \operatorname{mod} \Bbbk[I] \arrow[ul, "R'"] & \end{tikzcd}.$$

3.2. **Compression multiplicities.** Let $\mathscr{L}_I := \mathscr{L}_{B_I}$ be a complete set of representatives of isoclasses of indecomposable left B_I-modules (see the notations in Theorem 2.8). Since $R_I(V_I)$ is indecomposable, we may assume that $R_I(V_I) \in \mathscr{L}_I$.

Definition 3.8. Let $M \in \operatorname{mod} A$, and $I \in \mathbb{I}$. Then the number

$$c_M^{\xi}(I) := d_{R_I(M)}(R_I(V_I))$$

is called the *compression multiplicity* of I in M with respect to ξ.

Following [7], we introduce the subsequent definition.

Definition 3.9. A subset K of $\mathbf{P}$ is called an *antichain* in $\mathbf{P}$ if any distinct elements of K are incomparable. We denote by $\mathrm{Ac}(\mathbf{P})$ the set of all antichains in $\mathbf{P}$. For any $K, L \in \mathrm{Ac}(\mathbf{P})$, we define $K \le L$ if for all $x \in K$, there exists $z_x \in L$ such that $x \le z_x$, and for all $z \in L$, there exists $x_z \in K$ such that $x_z \le z$. In this case, we define $[K, L] := \{y \in \mathbf{P} \mid x \le y \le z \text{ for some } x \in K \text{ and for some } z \in L\}$.

Lemma 3.10. $\{[K, L] \mid K, L \in \mathrm{Ac}(\mathbf{P}),\ K \le L\}$ *forms the set of all convex subsets in* $\mathbf{P}$.

Proof. Let $K, L \in \mathrm{Ac}(\mathbf{P})$ such that $K \le L$. First, let us show that $[K, L] = \mathrm{conv}(K \cup L)$. Let $y \in \mathrm{conv}(K \cup L)$. By definition, there exist $x_0, z_0 \in K \cup L$ such that $x_0 \le y \le z_0$. Now assume both $x_0 \in K$ and $z_0 \in K$. In this case, since $K \in \mathrm{Ac}(\mathbf{P})$ and x_0 and z_0 are comparable, then necessarily $x_0 = z_0$ and so $y = x_0 = z_0 \in K \subseteq [K, L]$. Similarly, if both $x_0 \in L$ and $z_0 \in L$, we have $y \in L \subseteq [K, L]$. So either $x_0 \in K, z_0 \in L$ or $x_0 \in L, z_0 \in K$. If $x_0 \in K, z_0 \in L$, then by definition we have $y \in [K, L]$. Now assume we have $x_0 \in L, z_0 \in K$. Since $K \le L$, there exists $k \in K$ such that $k \le x_0$. So we have $k \le x_0 \le y \le z_0$ with both $k, z_0 \in K$. Therefore $k = z_0$ because $K \in \mathrm{Ac}(\mathbf{P})$, and so $y = k = z_0 \in K \subseteq [K, L]$. This proves that $[K, L] \supseteq \mathrm{conv}(K \cup L)$, and so $[K, L] = \mathrm{conv}(K \cup L)$. In particular, $[K, L]$ is a convex set. Now let S be a convex subset in $\mathbf{P}$. Because $\mathbf{P}$ is finite, we can define $K := \min S$, $L := \max S$, and we have $K \le L$. Then it is clear that $S = \mathrm{conv}(K \cup L)$, and so $S = [K, L]$. $\square$

Hence we have $\mathbb{I} = \{[K, L] \mid K, L \in \mathrm{Ac}(\mathbf{P}),\ K \le L,\ [K, L] \text{ is connected}\}$. In particular, we have $I = [\min I, \max I]$ for all $I \in \mathbb{I}$ and $\mathrm{Seg}(\mathbf{P}) = \{I \in \mathbb{I}(P) \mid |\min I| = 1 = |\max I|\}$. The following is immediate from Lemma 3.10.

Corollary 3.11. *Let* $I \in \mathbb{I}$. *Then* $I = \mathrm{conv}(\mathrm{sc}(I) \cup \mathrm{sk}(I))$. *In particular, if* $\mathrm{sc}(I) \cup \mathrm{sk}(I) \subseteq J \in \mathbb{I}$, *then* $I \le J$. $\square$

Proposition 3.12. *Let* $I, J \in \mathbb{I}$. *Then*

$$c^{\xi}_{V_J}(I) = \begin{cases} 1 & \text{if } I \le J, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $I \le J$, then $c^{\xi}_{V_J}(I) = d_{R_I(V_J)}(R_I(V_I)) = d_{R_I(V_I)}(R_I(V_I)) = 1$. Otherwise, Corollary 3.11 ensures the existence of a vertex $x \in \mathrm{sc}(I) \cup \mathrm{sk}(I)$ that is not in J. By assumption on ξ_I, there exists $x' \in (Q_I)_0$ such that $\xi_I(x') = x$. By definition, x' satisfies $R_I(V_I)(x') = \Bbbk$ and $R_I(V_J)(x') = 0$. Hence in particular, $R_I(V_I)$ is not a direct summand of $R_I(V_J)$, so $c^{\xi}_{V_J}(I) = 0$.

$\square$

Proposition 3.13. *Let* $M, N \in \mathrm{mod}\, A$, *and* $I \in \mathbb{I}$. *Then*

$$c^{\xi}_{M \oplus N}(I) = c^{\xi}_M(I) + c^{\xi}_N(I).$$

Proof. This is a direct consequence of the additivity of R_I and the uniqueness of $d_{M\oplus N}$ in Theorem 2.8. □

When $M \in \operatorname{mod} A$ is interval decomposable, it is possible to express the compression multiplicities of interval modules by multiplicities of interval modules and vice-versa.

Proposition 3.14. *Let $M \in \operatorname{mod} A$ and $I \in \mathbb{I}$. If M is interval decomposable, then*

$$c_M^\xi(I) = \sum_{I \leq J \in \mathbb{I}} d_M(V_J).$$

This can be rewritten as

$$c_M^\xi = d_M \zeta,$$

where $d_M(I) := d_M(V_I)$.

Proof. By assumption, M can be decomposed as a direct sum of interval modules: $M = \bigoplus_{J\in\mathbb{I}} V_J^{d_M(V_J)}$. Now, Proposition 3.13 yields

$$c_M^\xi(I) = \sum_{J\in\mathbb{I}} d_M(V_J)\ c_{V_J}^\xi(I).$$

Proposition 3.12 leads to the desired formula. □

Corollary 3.15. *Let $M \in \operatorname{mod} A$. If M is interval decomposable, then*

$$d_M = c_M^\xi \mu.$$

Proof. This follows directly from Theorem 2.12. □

By adopting the argument used in [3, Theorem 4.23], it is possible to write μ explicitly.

Theorem 3.16. *Let us define $\mu' \in \mathbb{R}\mathbb{I}$ by*

$$\mu'([I, J]) := \sum_{S\in E} (-1)^{|S|},$$

for $I, J \in \mathbb{I}$ with $I \leq J$, and where E is the set of all sets S such that $S \subseteq$ $\operatorname{Cov}(I)$ and $\bigvee S = J$. Note that if S is nonempty, then $\bigvee S$ is well defined by Proposition 2.6. We artificially define $\bigvee \emptyset := I$ to simplify notations. Then

$$\mu = \mu'.$$

Proof. Let us prove that $\zeta\mu' = 1_{\mathbb{R}\mathbb{I}}$. Let $I, J \in \mathbb{I}$ with $I \leq J$. We have

$$\begin{aligned}
(\zeta\mu')([I,J]) &= \sum_{I\le L\le J} \mu'([I,L]) \\
&= \sum_{I\le L\le J} \sum_{S\in E} (-1)^{|S|} \\
&= \sum_{\substack{S\subseteq \mathrm{Cov}(I) \\ \bigvee S\le J}} (-1)^{|S|} \\
&= 1 - \sum_{\substack{\emptyset\neq S\subseteq \mathrm{Cov}(I) \\ \bigvee S\le J}} (-1)^{|S|-1}
\end{aligned}$$

Now, write ${\uparrow_{\mathbb{I}}} I := \{L \in \mathbb{I} \mid I \le L\}$ for $I \in \mathbb{I}$, and define the function f as follows:

$$\begin{aligned}
f\colon 2^{\uparrow_{\mathbb{I}} I} &\to \mathbb{R} \\
Z &\mapsto \sum_{L\in Z} d_{V_J}(V_L),
\end{aligned}$$

where $2^{\uparrow_{\mathbb{I}} I}$ is the power set of ${\uparrow_{\mathbb{I}}} I$. Note that by definition

$$\bigcap_{L\in S} {\uparrow_{\mathbb{I}}} L = {\uparrow_{\mathbb{I}}} \bigvee S.$$

Therefore, we have

$$\begin{aligned}
f\left(\bigcap_{L\in S} {\uparrow_{\mathbb{I}}} L\right) &= f\left({\uparrow_{\mathbb{I}}} \bigvee S\right) \\
&= \sum_{\bigvee S\le L} d_{V_J}(V_L) \\
&= c^{\xi}_{V_J}\left(\bigvee S\right) \\
&= \begin{cases} 1 & \text{if } \bigvee S \le J, \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

where the last two equalities come from Propositions 3.14 and 3.12, respectively. Thus we can write:

$$(\zeta\mu')([I,J]) = 1 - \sum_{\emptyset\neq S\subseteq \mathrm{Cov}(I)} (-1)^{|S|-1} f\left(\bigcap_{L\in S} {\uparrow_{\mathbb{I}}} L\right).$$

It is easily seen that $(\uparrow_{\mathbb{I}} I, 2^{\Uparrow_{\mathbb{I}} I}, f)$ is a finite measure space. So, by the inclusion-exclusion principle

$$\begin{aligned}
(\zeta\mu')([I,J]) &= 1 - f\left(\bigcup_{L\in \mathrm{Cov}(I)} \uparrow_{\mathbb{I}} L\right)\\
&= 1 - \sum_{I<L} d_{V_J}(V_L)\\
&= 1 - \left(\sum_{I\leq L} d_{V_J}(V_L) - d_{V_J}(V_I)\right)\\
&= 1 - (c^{\xi}_{V_J}(V_I) - d_{V_J}(V_I))\\
&= d_{V_J}(V_I)
\end{aligned}$$

where the last two equalities also come from Propositions 3.14 and 3.12 respectively. Finally, since $d_{V_J}(V_I) = 1$ if and only if $I = J$, we have $(\zeta\mu')([I,J]) = 1_{\mathbb{R}\mathbb{I}}([I,J])$, so $\zeta\mu' = 1_{\mathbb{R}\mathbb{I}}$ and we deduce that $\mu = \mu'$. □

3.3. **Signed interval multiplicities.** It is now possible to rewrite Corollary 3.15 in the following way:

Corollary 3.17. *Let $M \in \operatorname{mod} A$ and $I \in \mathbb{I}$. If M is interval decomposable, then*

$$d_M(V_I) = \sum_{S\subseteq \mathrm{Cov}(I)} (-1)^{|S|}\ c^{\xi}_M\left(\bigvee S\right).$$

Definition 3.18. Let $M \in \operatorname{mod} A$ and $I \in \mathbb{I}$. We define the *signed interval multiplicity* δ^{ξ}_M of M with respect to ξ as the function $\delta^{\xi}_M\colon \mathbb{I} \to \mathbb{Z}$ by setting

$$\delta^{\xi}_M(I) := \sum_{S\subseteq \mathrm{Cov}(I)} (-1)^{|S|}\ c^{\xi}_M\left(\bigvee S\right)$$

for all $I \in \mathbb{I}$. By Theorem 3.16, this can be rewritten as

$$\delta^{\xi}_M := c^{\xi}_M\mu.$$

Remark 3.19. Note that in Definition 3.18, M is not necessarily interval decomposable anymore. If M is interval decomposable, it is clear that $\delta^{\xi}_M = d_M(V_{(\cdot)})$ as functions on $\mathbb{I}$ by Corollary 3.17.

Proposition 3.20. *Let $M \in \operatorname{mod} A$. For all $I \in \mathbb{I}$, we have*

$$c^{\xi}_M(I) = \sum_{I\leq J\in\mathbb{I}} \delta^{\xi}_M(J),$$

that is to say

$$c^{\xi}_M = \delta^{\xi}_M\zeta.$$

Proof. This is a direct consequence of Theorem 2.12. □

Remark 3.21. Proposition 3.20 gives an alternative definition of the signed interval multiplicity δ_M^ξ without using Möbius Inversion Formula. Indeed, it is possible to define δ_M^ξ by induction in the following way: first, define $\delta_M^\xi(I) := c_M^\xi(I)$ for every maximal interval I. Then define inductively $\delta_M^\xi(I) := c_M^\xi(I) - \sum_{I<J} \delta_M^\xi(J)$.

4. Interval replacement and interval rank invariant

Noting that for each $I \in \mathbb{I}$, $\delta_M^\xi(I)$ can be defined even for modules M that are not necessarily interval decomposable, we introduce the following.

4.1. Interval replacement.

Definition 4.1. Let $M \in \operatorname{mod} A$. We set

$$\delta^\xi(M)_+ := \bigoplus_{\substack{I\in\mathbb{I} \\ \delta_M^\xi(I)>0}} V_I{}^{\delta_M^\xi(I)}, \quad \delta^\xi(M)_- := \bigoplus_{\substack{I\in\mathbb{I} \\ \delta_M^\xi(I)<0}} V_I{}^{(-\delta_M^\xi(I))}, \text{ and}$$

$$\delta^\xi(M) := \left[\!\left[\delta^\xi(M)_+\right]\!\right] - \left[\!\left[\delta^\xi(M)_-\right]\!\right],$$

where $[\![X]\!]$ is the element of the split Grothendieck group $K^\oplus(A)$ of A corresponding to a module X. We call $\delta^\xi(M)$ the *interval replacement* of M, $\delta^\xi(M)_+, \delta^\xi(M)_-$ the *positive part* and the *negative part* of $\delta^\xi(M)$, respectively. Note that $\delta^\xi(M)$ is not a module, just an element of the split Grothendieck group, while both $\delta^\xi(M)_+$ and $\delta^\xi(M)_-$ are interval decomposable modules, and that $\delta^\xi(M)$ can be presented by the pair of these interval decomposable modules.

Definition 4.2. Let $M \in \operatorname{mod} A$ and $[x, y] \in \operatorname{Seg}(\mathbf{P})$. Then we have a unique morphism $p_{y,x} : x \to y$ in $\mathbf{P}$ (see Definition 2.2), and M yields a linear map

$$M_{y,x} := M(p_{y,x}) : M(x) \to M(y).$$

Using this we set $\operatorname{rank}_{[x,y]} M := \operatorname{rank} M_{y,x}$. This is call the $[x, y]$-*rank* of M. Then the family $\operatorname{rank}_{\operatorname{Seg}(\mathbf{P})} M := (\operatorname{rank}_{[x,y]} M)_{[x,y]\in\operatorname{Seg}(\mathbf{P})}$ is just the so-called *rank invariant* of M.

Definition 4.3. For each $[x, y] \in \operatorname{Seg}(\mathbf{P})$, we define the $[x, y]$-*rank* of $\delta^\xi(M)$ to be

$$\operatorname{rank}_{[x,y]} \delta^\xi(M) := \operatorname{rank}_{[x,y]} \delta^\xi(M)_+ - \operatorname{rank}_{[x,y]} \delta^\xi(M)_-$$

and the *dimension vector* of $\delta^\xi(M)$ to be

$$\underline{\dim}\, \delta^\xi(M) := \underline{\dim}\, \delta^\xi(M)_+ - \underline{\dim}\, \delta^\xi(M)_-$$

Then by Definition 4.1, we have

$$\operatorname{rank}_{[x,y]} \delta^\xi(M) = \sum_{I\in\mathbb{I}} \delta_M^\xi(I) \cdot \operatorname{rank}_{[x,y]} V_I, \text{and}$$

$$\underline{\dim}\, \delta^\xi(M) = \sum_{I\in\mathbb{I}} \delta_M^\xi(I) \cdot \underline{\dim}(V_I).$$

Notation 4.4. Let $I \in \mathbb{I}$, $M \in \operatorname{mod} \Bbbk[I]$, and $x, y \in I$.

(1) We set $P_x := \Bbbk[I](x,\text{-})$ (resp. $P'_x := \Bbbk[I^{\mathrm{op}}](x,\text{-})$) to be the projective indecomposable $\Bbbk[I]$-module (resp. $\Bbbk[I^{\mathrm{op}}]$-module) corresponding to the vertex x, and $I_x := D(\Bbbk[I](\text{-},x))$ (resp. $I'_x := D(\Bbbk[I^{\mathrm{op}}](\text{-},x))$) to be the injective indecomposable $\Bbbk[I]$-module (resp. $\Bbbk[I^{\mathrm{op}}]$-module) corresponding to the vertex x, where D denotes the usual $\Bbbk$-duality $\mathrm{Hom}_{\Bbbk}(\text{-},\Bbbk)$.

(2) By the Yoneda lemma, we have an isomorphism

$$M(x) \to \mathrm{Hom}_{\Bbbk[I]}(P_x, M), \quad m \mapsto \rho_m \ (m \in M(x)),$$

where $\rho_m \colon P_x \to M$ is defined by $\rho_m(p) := M(p)(m)\ (= p \cdot m)$, the right multiplication by m.

(3) Since $p_{y,x} \in \Bbbk[I](x,y) = P_x(y)$, we can set $P_{y,x} := \rho_{p_{y,x}} \colon P_y \to P_x$. Similarly, we set $p^{\mathrm{op}}_{x,y} := p_{y,x} \in \mathbf{P}^{\mathrm{op}}(y,x) = \mathbf{P}(x,y)$ for all $(x,y) \in [\leq]_{\mathbf{P}}$. It induces a morphism $P'_{x,y} := \rho_{p^{\mathrm{op}}_{x,y}} \colon P'_x \to P'_y$ in $\mathrm{mod}\,\Bbbk[I^{\mathrm{op}}]$.

Notation 4.5. For each positive integer n, we set $[n] := \{1, 2, \dots, n\}$.

For each $I \in \mathbb{I}$ and any $C, M \in \mathrm{mod}\,\Bbbk[I]$, the following lemma makes it possible to compute the dimension of $\mathrm{Hom}_{\Bbbk[I]}(C, M)$ by using a projective presentation of C and the module structure of M. Later we apply this to the case where M is given in the form $M = R(L)$ for some $L \in \mathrm{mod}\,A$. Note in that case that $M(p) = L(p)$ for all paths p inside I and all $p \in I_0$.

Lemma 4.6. *Let $I \in \mathbb{I}$ and $C, M \in \mathrm{mod}\,\Bbbk[I]$. Assume that C has a projective presentation*

$$\bigoplus_{j=1}^{n} P_{y_j} \xrightarrow{\mu} \bigoplus_{i=1}^{m} P_{x_i} \xrightarrow{\varepsilon} C \to 0$$

for some $x_1, x_2 \dots, x_m, y_1, y_2, \dots, y_n \in I$, and $\mu := [a_{ji} P_{y_j,x_i}]_{(i,j)\in[m]\times[n]}$ with $a_{ij} \in \Bbbk$, $((i,j) \in [m] \times [n])$. Then we have

$$\dim \mathrm{Hom}_{\Bbbk[I]}(C, M) = \sum_{i=1}^{m} \dim M(x_i) - \mathrm{rank}\,{}^t\!\left([a_{ji} M_{y_j,x_i}]_{(i,j)\in[m]\times[n]}\right).$$

Proof. Set $Y := \bigoplus_{j=1}^{n} P_{y_j}, X := \bigoplus_{i=1}^{m} P_{x_i}$ for short. Then we have an exact sequnce $Y \xrightarrow{\mu} X \xrightarrow{\varepsilon} C \to 0$, which yields an exact sequence

$$0 \to \mathrm{Hom}_{\Bbbk[I]}(C, M) \to \mathrm{Hom}_{\Bbbk[I]}(X, M) \xrightarrow{\mathrm{Hom}_{\Bbbk[I]}(\mu, M)} \mathrm{Hom}_{\Bbbk[I]}(Y, M).$$

Hence $\operatorname{Hom}_{\Bbbk[I]}(C, M) \cong \operatorname{Ker}\operatorname{Hom}_{\Bbbk[I]}(\mu, M)$. Now we have

$$\begin{aligned}
&\operatorname{Ker}\operatorname{Hom}_{\Bbbk[I]}(\mu, M) = \{f \in \operatorname{Hom}_{\Bbbk[I]}(X, M) \mid f\mu = 0\}\\
&\cong \left\{ (f_1, \dots, f_m) \in \bigoplus_{i=1}^{m} \operatorname{Hom}_{\Bbbk[I]}(P_{x_i}, M) \,\middle|\, (f_1, \dots, f_m)[a_{ji}P_{y_j,x_i}]_{(i,j)} = 0 \right\}\\
&= \left\{ (f_1, \dots, f_m) \in \bigoplus_{i=1}^{m} \operatorname{Hom}_{\Bbbk[I]}(P_{x_i}, M) \,\middle|\, \left(\sum_{i=1}^{m} a_{ji} f_i P_{y_j,x_i} \right)_{j\in[n]} = 0 \right\}\\
&\cong \left\{ \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \bigoplus_{i=1}^{m} M(x_i) \,\middle|\, \left(\sum_{i=1}^{m} a_{ji} M_{y_j,x_i}(b_i) \right)_{j\in[n]} = 0 \right\}\\
&= \left\{ \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \bigoplus_{i=1}^{m} M(x_i) \,\middle|\, {}^t\left([a_{ji} M_{y_j,x_i}]_{(i,j)} \right) \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = 0 \right\}\\
&= \operatorname{Ker}\left({}^t\left([a_{ji} M_{y_j,x_i}]_{(i,j)} \right) : \bigoplus_{i=1}^{m} M(x_i) \to \bigoplus_{j=1}^{n} M(y_j) \right).
\end{aligned}$$

Hence $\dim \operatorname{Hom}_{\Bbbk[I]}(C, M) = \sum_{i=1}^{m} \dim M(x_i) - \operatorname{rank} {}^t\left([a_{ji} M_{y_j,x_i}]_{(i,j)} \right)$. □

Proposition 4.7. *Let $M \in \operatorname{mod} A$ and $[x, y] \in \operatorname{Seg}(\mathbf{P})$. Then we have*

$$c_M^{\xi}([x, y]) = \operatorname{rank}_{[x,y]} M.$$

In particular, $c_M^{\xi}([x, y])$ does not depend on ξ.

Proof. For simplicity, we put $I := [x, y]$. We use Notation 3.7. Then we have

$$c_M^{\xi}(I) = d_{R_I(M)}(R_I(V_I)) = d_{R'(R(M))}(R'(R(V_I))) = d_{R'(R(M))}(R'(V_I))$$

because $R(V_I) = V_I$. Note that as a $\Bbbk[I]$-module, we have

$$V_I \cong P_x \cong I_y.$$

We first compute $d_{R(M)}(V_I)$. By applying the formula given in [4] to $V_I = I_y$, we have

$$d_{R(M)}(V_I) = \dim \operatorname{Hom}_{\Bbbk[I]}(I_y, R(M)) - \dim \operatorname{Hom}_{\Bbbk[I]}(I_y/\operatorname{soc} I_y, R(M)). \tag{4.4}$$

Here, the first term is given by

$$\dim \operatorname{Hom}_{\Bbbk[I]}(I_y, R(M)) = \dim \operatorname{Hom}_{\Bbbk[I]}(P_x, R(M)) = \dim R(M)(x) = \dim M(x).$$

For the second term, consider the canonical short exact sequence

$$0 \to \operatorname{soc} I_y \xrightarrow{\mu} I_y \xrightarrow{\varepsilon} I_y/\operatorname{soc} I_y \to 0$$

in $\operatorname{mod} \Bbbk[I]$. Since $I_y \cong P_x$ and $\operatorname{soc} I_y = \Bbbk p_{y,x} \cong P_y$, we see that this turns out to be a projective presentation of $I_y/\operatorname{soc} I_y$, where μ is given by $P_{y,x}$:

$$0 \to P_y \xrightarrow{P_{y,x}} P_x \xrightarrow{\varepsilon} I_y/\operatorname{soc} I_y \to 0.$$

Hence by Lemma 4.6, we see that the second term of (4.4) is given by

$$\dim \operatorname{Hom}_A(I_y/\operatorname{soc} I_y, R(M)) = \dim M(x) - \operatorname{rank} M_{y,x} = \dim M(x) - \operatorname{rank}_I M.$$

Therefore, we have

$$\begin{aligned} d_{R(M)}(V_I) &= \dim M(x) - (\dim M(x) - \operatorname{rank}_I M) \\ &= \operatorname{rank}_I M. \end{aligned}$$

This means that $R(M)$ has a decomposition of the form $R(M) \cong V_I^{(\operatorname{rank}_I M)} \oplus N$ for some $N \in \operatorname{mod} \Bbbk[I]$. Then $R'(R(M)) \cong R'(V_I)^{(\operatorname{rank}_I M)} \oplus R'(N)$, which shows that $c_M^{\xi}(I) = d_{R'(R(M))}(R'(V_I)) \geq \operatorname{rank}_I M$.

Next, we show the converse inequality. Set $c := c_M^{\xi}(I) = d_{R_I(M)}(R_I(V_I))$. Then we have an isomorphism

$$R_I(M) \cong R_I(V_I)^c \oplus N' \text{ in } \operatorname{mod} B_I. \tag{4.5}$$

By assumption on ξ_I, we have $x, y \in \xi_I((Q_I)_0)$, and there exists a path q in Q_I with $F_I(q) = p_{y,x}$. Then from (4.5), we have $R_I(M)(q) \cong R_I(V_I)(q)^c \oplus N'(q)$ and hence

$$\begin{aligned} \operatorname{rank}_I M &= \operatorname{rank} M(p_{y,x}) = \operatorname{rank} M(F_I(q)) \\ &= \operatorname{rank} R_I(M)(q) \geq c \cdot \operatorname{rank} R_I(V_I)(q) = c \end{aligned}$$

because $\operatorname{rank} R_I(V_I)(q) = \operatorname{rank} V_I(p_{y,x}) = 1$. Thus, $\operatorname{rank}_I M \geq c_M^{\xi}(I)$. $\square$

The following is an immediate consequence of the proposition above.

Corollary 4.8. *Let $M \in \operatorname{mod} A$ and $I = [x, y] \in \operatorname{Seg}(\mathbf{P})$. If $R(M) \in \operatorname{mod} \Bbbk[I]$ is indecomposable and not isomorphic to V_I, then $M(p_{y,x}) = 0$.*

Proof. Here we take the total compression system tot in Example 3.3. Then by assumption, we have $c_M^{\mathrm{tot}}(I) = 0$. Hence by the proposition above, we obtain $\operatorname{rank} M(p_{y,x}) = \operatorname{rank}_I M = 0$, which shows the assertion. $\square$

Example 4.9. For each positive integer n, we denote by $\mathbb{A}_n$ the set $\{1, 2, \ldots, n\}$ with the usual linear order $i < i+1$ $(i = 1, 2, \ldots, n-1)$, and for each poset P_1, P_2, we regard the direct product $P_1 \times P_2$ as the poset with the partial order defined by $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$ for all $(x, y), (x', y') \in P_1 \times P_2$. We set $G_{m,n} := \mathbb{A}_m \times \mathbb{A}_n$, and call it a *2D-grid*. For example, $\mathbf{P} := G_{5,2}$ has the following Hasse quiver:

$$\begin{tikzcd} (1,2) \arrow[r] & (2,2) \arrow[r] & (3,2) \arrow[r] & (4,2) \arrow[r] & (5,2) \\ (1,1) \arrow[r] \arrow[u] & (2,1) \arrow[r] \arrow[u] & (3,1) \arrow[r] \arrow[u] & (4,1) \arrow[r] \arrow[u] & (5,1) \arrow[u] \end{tikzcd}.$$

For $I := \mathbf{P}$, it is known that there exists an indecomposable $\Bbbk[I]$-module M with dimension vector $\left[\begin{smallmatrix} 2 & 3 & 3 & 2 & 1 \\ 1 & 2 & 3 & 3 & 2 \end{smallmatrix}\right]$. Then since $M \not\cong V_I$, we have to have $M(p) = 0$ for any path p from $(1, 1)$ to $(5, 2)$ by the corollary above.

Indeed, it is not hard to check that the following module M is indecomposable with this dimension vector (thus this dimension vector is realized as this M):

$$\begin{tikzcd}
\Bbbk^2 \arrow[r, "\left[\begin{smallmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]"] & \Bbbk^3 \arrow[r, "\mathbf{1}"] & \Bbbk^3 \arrow[r, "\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]"] & \Bbbk^2 \arrow[r, "{[1\ 0]}"] & \Bbbk \\
\Bbbk \arrow[u, "\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]"] \arrow[r, "\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]"'] & \Bbbk^2 \arrow[u, "\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{smallmatrix}\right]"'] \arrow[r, "\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{smallmatrix}\right]"'] & \Bbbk^3 \arrow[u, "\mathbf{1}"'] \arrow[r, "\mathbf{1}"'] & \Bbbk^3 \arrow[u, "\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]"'] \arrow[r, "\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]"'] & \Bbbk^2 \arrow[u, "{[1\ 0]}"']
\end{tikzcd}.$$

It is certain that M satisfies the condition $M(p) = 0$ stated above. We note that this M is obtained as the Auslander–Reiten translation τM_λ of the indecomposable module M_λ with $\lambda = 1$ in Example 6.2.

4.2. **Interval rank invariant.** Proposition 4.7 suggests us to define the following.

Definition 4.10. Let $M \in \operatorname{mod} A$ and $I \in \mathbb{I}$. Then we set

$$\operatorname{rank}_I^\xi M := c_M^\xi(I),$$

and call it the I-rank of M with respect to ξ, and the family $\operatorname{rank}_{\mathbb{I}}^\xi M := (\operatorname{rank}_I^\xi M)_{I \in \mathbb{I}}$ is called the *interval rank invariant* of M with respect to ξ. Note that since for each $J \in \mathbb{I}$, $\operatorname{rank}_I^\xi V_J$ does not depend on ξ by Proposition 3.12, we see that for every interval decomposable module N, $\operatorname{rank}_I^\xi N$ does not depend on ξ, and hence we may write it $\operatorname{rank}_I N$.

For each $I \in \mathbb{I}$, we define the *I-rank* of $\delta^\xi(M)$ with respect to ξ to be

$$\begin{aligned} \operatorname{rank}_I^\xi \delta^\xi(M) &:= \operatorname{rank}_I^\xi \delta^\xi(M)_+ - \operatorname{rank}_I^\xi \delta^\xi(M)_- \\ &\quad (= \operatorname{rank}_I \delta^\xi(M)_+ - \operatorname{rank}_I \delta(M)_-^\xi). \end{aligned}$$

Note that $\operatorname{rank}_I^\xi \delta^\xi(M)$ may depend on ξ because $\delta^\xi(M)_\pm$ depend on ξ. Thus by Definition 4.1, we have

$$\operatorname{rank}_I^\xi \delta^\xi(M) = \sum_{J \in \mathbb{I}} \delta_M^\xi(J) \cdot \operatorname{rank}_I V_J.$$

Using the notations given above, we obtain the following.

Theorem 4.11. *Let $M \in \operatorname{mod} A$, and $I \in \mathbb{I}$. Then*

$$\operatorname{rank}_I^\xi \delta^\xi(M) = \operatorname{rank}_I^\xi M.$$

In particular, for any $[x, y] \in \operatorname{Seg}(\mathbf{P})$, we have

$$\begin{gathered} \operatorname{rank}_{[x,y]}^\xi \delta^\xi(M) = \operatorname{rank}_{[x,y]} M, \\ \underline{\dim}\, \delta^\xi(M) = \underline{\dim}\, M. \end{gathered}$$

Thus, δ^ξ preserves the interval rank invariants of all persistence modules M. In this sense, we called $\delta^\xi(M)$ an *interval replacement* of M.

Proof. It is enough to show the first equality because the second (resp. third) one follows by considering the case that $I = [x, y]$ (resp. the cases that $[x, x]$ for all $x \in \mathbf{P}$). By Definition 4.10, to show the first one, it suffices to show the following:

$$\sum_{J \in \mathbb{I}} \delta_M^{\xi}(J) \cdot \operatorname{rank}_I V_J = \operatorname{rank}_I^{\xi} M.$$

Let $I, J \in \mathbb{I}$. Then by Proposition 3.12, we have

$$\operatorname{rank}_I V_J = c_{V_J}^{\xi}(I) = \begin{cases} 1 & \text{if } I \leq J, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\sum_{J \in \mathbb{I}} \delta_M^{\xi}(J) \cdot \operatorname{rank}_I V_J = \sum_{I \leq J \in \mathbb{I}} \delta_M^{\xi}(J) = c_M^{\xi}(I) = \operatorname{rank}_I^{\xi} M,$$

where the second equality follows by Proposition 3.20, and the last one by definition. □

5. The formula of total I-rank

Definition 5.1. Let $I \in \mathbb{I}$. I is said to be of (n, m)-type if $|\operatorname{sc}(I)| = n$ and $|\operatorname{sk}(I)| = m$ $(n, m \in \mathbb{N}_+)$. If this is the case, we set $\operatorname{sc}(I) = \{a_1, \dots, a_n\}$ and $\operatorname{sk}(I) = \{b_1, \dots, b_m\}$ for $I \in \mathbb{I}$ in the sequel.

Definition 5.2. Let $I \in \mathbb{I}$. I is said to *satisfy the existence condition of pairwise joins* in $\operatorname{sc}(I)$ (resp. *meets* in $\operatorname{sk}(I)$) if $a_i \vee a_j$ (resp. $b_i \wedge b_j$) exists in $\mathbf{P}$ for every $i \neq j$. To shorten notation, we set $a_{ij} := a_i \vee a_j$ (resp. $b_{ij} := b_i \wedge b_j$) whenever the join (resp. meet) exists subsequently.

Notation 5.3. Let I be an interval in $\mathbb{I}$ of (n, m)-type satisfying the existence condition of pairwise joins in $\operatorname{sc}(I)$ and pairwise meets in $\operatorname{sk}(I)$.

(1) We set

$$\operatorname{sc}_1(I) := \{a_{i_1 i_2} \mid i_1, i_2 \in [n] \textit{ with } i_1 < i_2\},$$
$$\operatorname{sk}_1(I) := \{b_{i_1 i_2} \mid i_1, i_2 \in [m] \textit{ with } i_1 < i_2\}.$$

Furthermore, we equip $\operatorname{sc}_1(I)$ with another total order $\preceq_{\text{lex}}$, defined by $a_{i_1 i_2} \preceq_{\text{lex}} a_{j_1 j_2}$ if and only if their index words satisfy the relation $i_1 i_2 \leq_{\text{lex}} j_1 j_2$. Here $\leq_{\text{lex}}$ denotes the lexicographic order from left to right. Similarly, we give the total order to $\operatorname{sk}_1(I)$.

(2) Let us denote $\downarrow_I x := \{y \in I \mid y \leq x\}$ for $x \in I$.

(3) For each $X \subseteq \mathbf{P}$, we set $P_X := \bigoplus_{x \in X} P_x$ and $P'_X := \bigoplus_{x \in X} P'_x$.

Proposition 5.4. *Let $M \in \operatorname{mod} A$, and I be an interval in $\mathbb{I}$ of (n, m)-type with $n \geq 2$, $\operatorname{sc}(I) = \{a_1, \dots, a_n\}$. Assume I satisfies the existence condition of pairwise joins of $\operatorname{sc}(I)$. Then we have the following projective presentation (may not be minimal) of V_I in $\operatorname{mod} \Bbbk[I]$:*

$$P_{\operatorname{sc}_1(I)} \xrightarrow{\varepsilon_1} P_{\operatorname{sc}(I)} \xrightarrow{\varepsilon_0} V_I \to 0, \tag{5.6}$$

where $\varepsilon_0, \varepsilon_1$ are given by

$$\varepsilon_0 := (\rho_{1_{a_1}}, \rho_{1_{a_2}}, \dots, \rho_{1_{a_n}}),$$

where we set $1_u := 1 \in \Bbbk = V_I(u)$ for all $u \in I$, and

$$\varepsilon_1 := \begin{array}{c} \\ \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \vdots \\ \mathit{n\text{-}1} \\ \mathit{n} \end{array} \begin{array}{c} \begin{array}{cccccccc} \mathit{12} & \cdots & \mathit{1n} & \mathit{23} & \cdots & \mathit{2n} & \cdots & \mathit{n\text{-}1,n} \end{array} \\ \begin{bmatrix} P_{a_{12},a_1} & \cdots & P_{a_{1n},a_1} & 0 & \cdots & 0 & \cdots & 0 \\ -P_{a_{12},a_2} & \cdots & 0 & P_{a_{23},a_2} & \cdots & P_{a_{2n},a_2} & \cdots & 0 \\ 0 & \cdots & 0 & -P_{a_{23},a_3} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & P_{a_{n-1,n},a_{n-1}} \\ 0 & \cdots & -P_{a_{1n},a_n} & 0 & \cdots & -P_{a_{2n},a_n} & \cdots & -P_{a_{n-1,n},a_n} \end{bmatrix} \end{array}.$$

Proof. We verify the exactness of the sequence (5.6) in the following steps:

(a) showing that ε_0 is surjective; (b) showing that $\varepsilon_0\varepsilon_1 = 0$; (c) showing that $\dim \operatorname{Im} \varepsilon_1 \geq \dim \operatorname{Ker} \varepsilon_0$.

(a) This holds by $V_I = A1_{a_1} + A1_{a_2} + \cdots + A1_{a_n} = \operatorname{Im} \varepsilon_0$.

(b) It suffices to show the following composition

$$P_{a_{i_1 i_2}} \xrightarrow{\varepsilon_1'} \bigoplus_{i \in [n]} P_{a_i} \xrightarrow{\varepsilon_0} V_I \tag{5.7}$$

is zero for every $i_1, i_2 \in [n]$ with $i_1 < i_2$, where

$$\varepsilon_1' := \begin{array}{c} \\ 1 \\ \vdots \\ i_1 \\ \vdots \\ i_2 \\ \vdots \\ n \end{array} \begin{array}{c} i_1 i_2 \\ \begin{bmatrix} 0 \\ \vdots \\ P_{a_{i_1 i_2}, a_{i_1}} \\ \vdots \\ -P_{a_{i_1 i_2}, a_{i_2}} \\ \vdots \\ 0 \end{bmatrix} \end{array}.$$

Let $a_{i_1 i_2} \leq t \in I$. Then

$$\begin{aligned} (\varepsilon_0 \varepsilon_1')(p_{t,a_{i_1 i_2}}) &= (\rho_{1_{a_{i_1}}} P_{a_{i_1 i_2}, a_{i_1}})(p_{t,a_{i_1 i_2}}) - (\rho_{1_{a_{i_2}}} P_{a_{i_1 i_2}, a_{i_2}})(p_{t,a_{i_1 i_2}}) \\ &= V_I(p_{t,a_{i_1}})(1_{a_{i_1}}) - V_I(p_{t,a_{i_2}})(1_{a_{i_2}}) = 1_t - 1_t = 0. \end{aligned}$$

(c) It suffices to show $\dim \operatorname{Im} (\varepsilon_1)_x \geq \dim \operatorname{Ker} (\varepsilon_0)_x$ for each $x \in I$. Fix $x \in I$, and set $\operatorname{sc}(I) \cap {\downarrow_I} x = \{a_{i_1}, \dots, a_{i_\ell}\}$ ($\ell \in \mathbb{Z}_{\geq 1}$). By Notation 5.3, $P_{\operatorname{sc}(I)}(x) = \bigoplus_{j \in [\ell]} \Bbbk p_{x,a_{i_j}}$, and $P_{\operatorname{sc}_1(I)}(x) = \bigoplus_{a \in \operatorname{sc}_1(I)} \Bbbk p_{x,a}$.

If $\ell = 1$, then $P_{\operatorname{sc}_1(I)}(x) = 0$, and hence (5.6) becomes

$$0 \to \Bbbk p_{x,a_{i_1}} \xrightarrow{\alpha} \Bbbk \to 0,$$

where α is an isomorphism defined by $\alpha(p_{x,a_{i_1}}) := 1_{\Bbbk}$. The claim follows since $\operatorname{Im}(\varepsilon_1)_x = 0 = \operatorname{Ker}(\varepsilon_0)_x$.

If $\ell \geq 2$, then $\mathrm{sc}_1(I) \cap {\downarrow_I} x = \{a_{i_j i_k} \mid j,k \in [\ell] \text{ with } i_j < i_k\}$. Hence

$$P_{\mathrm{sc}_1(I)}(x) = \bigoplus_{\substack{j,k\in[\ell] \\ i_j<i_k}} \Bbbk p_{x,a_{i_j i_k}}.$$

Then (5.6) becomes the first row of the commutative diagram

$$\begin{tikzcd}
\bigoplus_{\substack{j,k\in[\ell] \\ i_j<i_k}} \Bbbk p_{x,a_{i_j i_k}} \arrow[r, "(\varepsilon_1)_x"] & \bigoplus_{j\in[\ell]} \Bbbk p_{x,a_{i_j}} \arrow[r, "(\varepsilon_0)_x"] & \Bbbk \arrow[r] & 0 \\
\Bbbk^{\binom{\ell}{2}} \arrow[u, "\alpha"] \arrow[r, "\varepsilon_1(x)"] & \Bbbk^{\ell} \arrow[u, "\beta"] \arrow[r, "\varepsilon_0(x)"] & \Bbbk \arrow[u, equal] &
\end{tikzcd},$$

where $\alpha = \bigoplus_a \alpha_a$ (resp. $\beta = \bigoplus_{j\in[\ell]} \beta_j$) is the isomorphism defined by $\alpha_a(1_{\Bbbk}) := p_{x,a}$ (resp. $\beta_j(1_{\Bbbk}) := p_{x,a_{i_j}}$) for all $a \in \{a_{i_j,i_k} \mid j,k \in [\ell], i_j < i_k\}$ (resp. $j \in [\ell]$), the matrix $\varepsilon_1(x)$ is given by

$$\begin{array}{c} \\ i_1 \\ i_2 \\ i_3 \\ \vdots \\ i_{\ell-1} \\ i_\ell \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} i_1i_2 & i_1i_3 & \cdots & i_1i_\ell & i_2i_3 & \cdots & i_2i_\ell & \cdots & i_{\ell-1}i_\ell \end{array} \\
\begin{bmatrix}
1 & 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 \\
-1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 \\
0 & -1 & \cdots & 0 & -1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 \\
0 & 0 & \cdots & -1 & 0 & \cdots & -1 & \cdots & -1
\end{bmatrix}
\end{array} \tag{5.8}$$

and the matrix $\varepsilon_0(x)$ is given by $(\underbrace{1, 1, \cdots, 1}_{\ell})$. The commutativity of the left square follows from the fact that $P_{a,b}(p_{x,a}) = p_{x,b}$ for all $(b,a) \in [\leq]_I$. The remaining commutativity is trivial. It is clear that $\operatorname{rank} \varepsilon_0(x) = 1$. For the matrix $\varepsilon_1(x)$, note that the last $\ell - 1$ rows are linearly independent, and that the sum of all rows is a zero row vector. This shows that $\operatorname{rank} \varepsilon_1(x) = \ell - 1$. Thus $\dim \operatorname{Im} \varepsilon_1(x) = \ell - 1 = \dim \operatorname{Ker} \varepsilon_0(x)$, and hence $\dim \operatorname{Im}(\varepsilon_1)_x = \dim \operatorname{Ker}(\varepsilon_0)_x$. $\square$

For each $I \in \mathbb{I}$, we have $DV_I \cong V_{I^{\mathrm{op}}}$ in $\operatorname{mod} \Bbbk[I^{\mathrm{op}}]$. Hence Lemma 5.4 shows the following.

Proposition 5.5. *Let $M \in \operatorname{mod} A$, and I be an interval in $\mathbb{I}$ of (n,m)-type with $m \geq 2$, $\mathrm{sk}(I) = \{b_1, \ldots, b_m\}$. Assume that I satisfies the existence condition of pairwise meets in $\mathrm{sk}(I)$. Then we have the following projective presentation (may not be minimal) of DV_I in $\operatorname{mod} \Bbbk[I^{\mathrm{op}}]$:*

$$P'_{\mathrm{sk}_1(I)} \xrightarrow{\varepsilon'_1} P'_{\mathrm{sk}(I)} \xrightarrow{\varepsilon'_0} DV_I \to 0, \tag{5.9}$$

where $\varepsilon_0', \varepsilon_1'$ are given by

$$\varepsilon_0' := (\rho_{1_{b_1}}, \rho_{1_{b_2}}, \cdots, \rho_{1_{b_m}}),$$

where $1_u := 1 \in \Bbbk = V_I(u)$ for all $u \in I$, and

$$\varepsilon_1' := \begin{array}{cc} & \begin{array}{cccccccc} \mathit{12} & \cdots & \mathit{1m} & \mathit{23} & \cdots & \mathit{2m} & \cdots & \mathit{m\text{-}1,m} \end{array} \\ \begin{array}{c} \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \vdots \\ \mathit{m\text{-}1} \\ m \end{array} & \begin{bmatrix} P'_{b_{12},b_1} & \cdots & P'_{b_{1m},b_1} & 0 & \cdots & 0 & \cdots & 0 \\ -P'_{b_{12},b_2} & \cdots & 0 & P'_{b_{23},b_2} & \cdots & P'_{b_{2m},b_2} & \cdots & 0 \\ 0 & \cdots & 0 & -P'_{b_{23},b_3} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & P'_{b_{m-1,m},b_{m-1}} \\ 0 & \cdots & -P'_{b_m,b_{1m}} & 0 & \cdots & -P'_{b_{2m},b_m} & \cdots & -P'_{b_{m-1,m},b_m} \end{bmatrix} \end{array}.$$

We note here that ε_0' is a projective cover of DV_I because it induces an isomorphism $\operatorname{top} P'_{\mathrm{sk}(I)} \cong \operatorname{top} DV_I$*, but $\varepsilon_1' \colon P'_{\mathrm{sk}_1(I)} \to \operatorname{Im} \varepsilon_1'$ is not always a projective cover. Then we can set*

$$P'_{\mathrm{sk}_1(I)} = P_1 \oplus P_2 \tag{5.10}$$

with $\varepsilon_{11}' \colon P_1 \to \operatorname{Im} \varepsilon_1'$ a projective cover, where $\varepsilon_1' = (\varepsilon_{11}', \varepsilon_{12}')$ is a matrix expression of ε_1' with respect to this decomposition of $P'_{\mathrm{sk}_1(I)}$. Then DV_I has a minimal projective resolution

$$P_1 \xrightarrow{\varepsilon_{11}'} P'_{\mathrm{sk}(I)} \xrightarrow{\varepsilon_0'} DV_I \to 0. \tag{5.11}$$

Hence by applying $(\text{-})^t := \operatorname{Hom}_{\Bbbk[I^{\mathrm{op}}]}(\text{-}, \Bbbk[I^{\mathrm{op}}])$ *to ε_{11}' in (5.11), we have a minimal projective presentation*

$$P_{\mathrm{sk}(I)} \xrightarrow{{\varepsilon_{11}'}^t} P_1^t \xrightarrow{\operatorname{coker} {\varepsilon_{11}'}^t} \tau^{-1} V_I \to 0, \tag{5.12}$$

of $\tau^{-1} V_I = \operatorname{Tr} DV_I$ in $\operatorname{mod} \Bbbk[I]$ *and a projective presentation*

$$P_{\mathrm{sk}(I)} \xrightarrow{{\varepsilon_1'}^t = \begin{bmatrix} {\varepsilon_{11}'}^t \\ {\varepsilon_{12}'}^t \end{bmatrix}} P_1^t \oplus P_2^t = P_{\mathrm{sk}_1(I)} \xrightarrow{\operatorname{coker} {\varepsilon_{11}'}^t \oplus \mathbf{1}_{P_2^t}} \tau^{-1} V_I \oplus P_2^t \to 0, \tag{5.13}$$

of $\tau^{-1} V_I \oplus P_2^t$, where

$${\varepsilon_1'}^t = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & \cdots & \text{m-1} & \text{m} \end{array} \\ \begin{array}{c} 12 \\ 13 \\ \vdots \\ \text{1m} \\ 23 \\ \vdots \\ \text{2m} \\ \vdots \\ \text{m-1,m} \end{array} & \begin{bmatrix} P_{b_1,b_{12}} & -P_{b_2,b_{12}} & 0 & \cdots & 0 & 0 \\ P_{b_1,b_{13}} & 0 & -P_{b_3,b_{13}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ P_{b_1,b_{1m}} & 0 & 0 & \cdots & 0 & -P_{b_m,b_{1m}} \\ 0 & P_{b_{23},b_2} & -P_{b_{23},b_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & P_{b_2,b_{2m}} & 0 & \cdots & 0 & -P_{b_m,b_{2m}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & P_{b_{m-1},b_{m-1,m}} & -P_{b_m,b_{m-1,m}} \end{bmatrix} \end{array}$$

□

5.1. $(n,1)$**-type.** We first consider the case where I is an interval of $(n,1)$-type. In this case, we simply write $\mathrm{sk}(I) = \{b\}$. The following is immediate from Lemma 3.10.

Lemma 5.6. *Let I be an interval in $\mathbb{I}$ of $(n,1)$-type. Then b is the maximum element and $a_1, \dots, a_n$ are minimal elements in I. Moreover, for each $x \in I$, there exists $a_i \in \mathrm{sc}(I)$ such that $a_i \le x \le b$.* □

Theorem 5.7. *Let $M \in \operatorname{mod} A$, and I be an interval in $\mathbb{I}$ of $(n,1)$-type with $n \ge 2$, $\mathrm{sc}(I) = \{a_1, \dots, a_n\}$ and $\mathrm{sk}(I) = \{b\}$. Assume I satisfies the existence condition of pairwise joins of $\mathrm{sc}(I)$. Then using Notation 3.7, we have*

$$d_{R(M)}(V_I) = \operatorname{rank} \begin{bmatrix} \mathbf{M} \\ \beta \end{bmatrix} - \operatorname{rank} \mathbf{M}, \tag{5.14}$$

where

$$\mathbf{M} = \begin{array}{c} \\ 12 \\ 13 \\ \vdots \\ 1n \\ 23 \\ \vdots \\ 2n \\ \vdots \\ n\text{-}1,n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \cdots & n\text{-}1 & n \end{array} \\ \begin{bmatrix} M_{a_{12},a_1} & -M_{a_{12},a_2} & 0 & \cdots & 0 & 0 \\ M_{a_{13},a_1} & 0 & -M_{a_{13},a_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ M_{a_{1n},a_1} & 0 & 0 & \cdots & 0 & -M_{a_{1n},a_n} \\ 0 & M_{a_{23},a_2} & -M_{a_{23},a_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & M_{a_{2n},a_2} & 0 & \cdots & 0 & -M_{a_{2n},a_n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & M_{a_{n-1,n},a_{n-1}} & -M_{a_{n-1,n},a_n} \end{bmatrix} \end{array}$$

and

$$\beta = \begin{bmatrix} M_{b,a_1} & \underbrace{0\ 0\ \cdots\ 0\ 0}_{n-1} \end{bmatrix}.$$

Here and subsequently, the matrices are written following the lexicographic order (see Notation 5.3 (1)) of index words of rows (resp. columns).

Proof. (1) Since I has a unique sink b, V_I is isomorphic to an injective indecomposable $\Bbbk[I]$-module $I_b := D(\mathrm{Hom}_{\Bbbk[I]}(\text{-}, b))$. Then again by applying the formula in [4] to V_I, we have

$$d_{R(M)}(V_I) = \dim \mathrm{Hom}_{\Bbbk[I]}(V_I, R(M)) - \dim \mathrm{Hom}_{\Bbbk[I]}(V_I/\operatorname{soc} V_I, R(M)). \tag{5.15}$$

A projective presentation of V_I in $\operatorname{mod} \Bbbk[I]$ is given by (5.6) in Proposition 5.4. Hence by Lemma 4.6, we have

$$\dim \mathrm{Hom}_{\Bbbk[I]}(V_I, R(M)) = \sum_{i \in [n]} \dim M(a_i) - \operatorname{rank} \mathbf{M}. \tag{5.16}$$

Next, we start with the following diagram:

$$\begin{tikzcd}
P_{\mathrm{sc}_1(I)} \arrow[rr, "\varepsilon_1"] \arrow[dr, two heads, "\tau_1"'] & & P_{\mathrm{sc}(I)} \arrow[r, "\varepsilon_0"] & V_I \arrow[r] & 0 \\
& \operatorname{Ker}\varepsilon_0 \arrow[ur, hook, "\iota_0"'] & & &
\end{tikzcd}$$

Note that $\operatorname{soc} V_I \cong P_b$ is projective and $\varepsilon_0 \colon P_{\mathrm{sc}(I)} \to V_I$ is epic, which yields a morphism $\lambda : P_b \to P_{\mathrm{sc}(I)}$, defined by

$$\lambda := \begin{bmatrix} P_{b,a_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

such that $\mathrm{Im}(\varepsilon_0 \circ \lambda) = \operatorname{soc} V_I$. We have the following projective presentation (may not be minimal) of $V_I/\operatorname{soc} V_I$ in $\operatorname{mod} \Bbbk[I]$:

$$P_{\mathrm{sc}_1(I)} \oplus P_b \xrightarrow{\varepsilon_1'} P_{\mathrm{sc}(I)} \xrightarrow{\varepsilon_0'} V_I/\operatorname{soc} V_I \to 0, \tag{5.17}$$

where ε_1', ε_0' is given by

$$\varepsilon_1' := \begin{bmatrix} \varepsilon_1 & \lambda \end{bmatrix}, \ \varepsilon_0' := \pi \circ \varepsilon_0.$$

In the above, $\pi \colon V_I \to V_I/\operatorname{soc} V_I$ is the canonical projection. Here, we verify the exactness of the sequence (5.17) by using the following diagram:

$$\begin{tikzcd}
P_{\mathrm{sc}_1(I)} \oplus P_b \arrow[rr, "\varepsilon_1'"] \arrow[dr, two heads, "\tau_1'"'] & & P_{\mathrm{sc}(I)} \arrow[r, "\varepsilon_0'"] & V_I/\operatorname{soc} V_I \arrow[r] & 0 \\
& \operatorname{Ker}\varepsilon_0 \oplus P_b \arrow[ur, hook, "\iota_0'"'] & & &
\end{tikzcd}$$

and proceed in the following steps:

(a) showing that $0 \to \operatorname{Ker}\varepsilon_0 \oplus P_b \xrightarrow{\iota_0'} P_{\mathrm{sc}(I)} \xrightarrow{\varepsilon_0'} V_I/\operatorname{soc} V_I \to 0$ is exact, where $\iota_0' := \begin{bmatrix} \iota_0 & \lambda \end{bmatrix}$; (b) defining an epimorphism $\tau_1' \colon P_{\mathrm{sc}_1(I)} \oplus P_b \to \operatorname{Ker}\varepsilon_0 \oplus P_b$; (c) showing that ε_1' factors through $\operatorname{Ker}\varepsilon_0 \oplus P_b$.

(a) We consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \operatorname{Ker}\iota_0' \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Ker}\varepsilon_0 \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[d, equal] & \operatorname{Ker}\varepsilon_0 \oplus P_b \arrow[r, "{(0,1)}"] \arrow[d, "\iota_0'"] & P_b \arrow[r] \arrow[d, "\varepsilon_0\circ\lambda"] \arrow[dl, "\lambda"'] & 0 \\
0 \arrow[r] & \operatorname{Ker}\varepsilon_0 \arrow[r, "\iota_0"] \arrow[d] & P_{\mathrm{sc}(I)} \arrow[r, "\varepsilon_0"] \arrow[d, "\operatorname{coker}\iota_0'"] & V_I \arrow[r] \arrow[d, "\pi"] & 0 \\
 & 0 \arrow[r] & \operatorname{Coker}\iota_0' \arrow[r, "\overline{\varepsilon_0}"] & V_I/\operatorname{soc}V_I \arrow[r] & 0
\end{tikzcd}$$

By applying the snake lemma to the second and third rows of short exact sequences, we obtain that $\operatorname{Ker}\iota_0' = 0$ and $\overline{\varepsilon_0}\colon \operatorname{Coker}\iota_0' \to V_I/\operatorname{soc}V_I$ is an isomorphism. Notice that $\overline{\varepsilon_0}\circ\operatorname{coker}\iota_0' = \pi\circ\varepsilon_0 = \varepsilon_0'$, it then follows that $0 \to \operatorname{Ker}\varepsilon_0 \oplus P_b \xrightarrow{\iota_0'} P_{\mathrm{sc}(I)} \xrightarrow{\varepsilon_0'} V_I/\operatorname{soc}V_I \to 0$ is exact.

(b) Set $\tau_1' := \begin{bmatrix}\tau_1 & 0\\ 0 & \mathbb{1}\end{bmatrix}$. τ_1' is surjective since τ_1 is.

(c) This is obvious since

$$\iota_0'\circ\tau_1' = \begin{bmatrix}\iota_0 & \lambda\end{bmatrix}\begin{bmatrix}\tau_1 & 0\\ 0 & \mathbb{1}\end{bmatrix} = \begin{bmatrix}\iota_0\tau_1 & \lambda\end{bmatrix} = \begin{bmatrix}\varepsilon_1 & \lambda\end{bmatrix} = \varepsilon_1'.$$

By Lemma 4.6, we have

$$\dim\operatorname{Hom}_{\Bbbk[I]}(V_I/\operatorname{soc}V_I, R(M)) = \sum_{i\in[n]}\dim M(a_i) - \operatorname{rank}\begin{bmatrix}\mathbf{M}\\ \beta\end{bmatrix}. \tag{5.18}$$

By equations (5.15), (5.16), (5.18) and Definition 5.21, we obtain (5.14).

□

Remark 5.8. The result in (5.14) is quite redundant because the projective presentations (5.6) and (5.17) are not minimal in general if there are order relations between pairwise joins in I. We provide the following lemma and corollary to explain this redundancy.

In (5.14), let $i, j \in [n]$ with $i \neq j$. For the next lemma, we note by definition that $a_{ij} = a_{ji}$. Hence even if $i > j$, we may say ij row of $\mathbf{M}$ to mean the ji row of $\mathbf{M}$. This convention allows us to consider the ij row of $\mathbf{M}$ without noticing the order relation between i and j.

Lemma 5.9. *We keep the setting of Theorem 5.7. Let $i, j, k \in [n]$ be pairwise distinct. For any distinct subsets S and T of $\{i, j, k\}$ of cardinality 2, the intersection $S \cap T$ has cardinality 1. We may set $S := \{i, j\}$ and $T := \{i, k\}$ with $S \cap T = \{i\}$. Keeping this in mind, consider a_{ij} and a_{ik}. Then the following are equivalent:*

(1) $a_{ij} \leq a_{ik}$;

(2) $a_j \leq a_{ik}$; *and*
(3) $a_i, a_j, a_k \leq a_{ik}$.

If one of the above holds, then the equation (5.14) *remains valid even if we replace* $\mathbf{M}$ *with the matrix obtained by deleting the* ik *row of* $\mathbf{M}$.

Proof. The equivalence of the three statements is trivial. Now assume that one of them holds. Then all of them hold. By (3), we have $a_{jk} \leq a_{ik}$. Thus there exist paths $p_{a_{ik},a_{ij}}$ and $p_{a_{ik},a_{jk}}$. The following row operations on $\mathbf{M}$ can be done keeping the ranks of both $\mathbf{M}$ and $\begin{bmatrix} \mathbf{M} \\ \beta \end{bmatrix}$ (to understand these operations easily, look at the $\mathbf{M}$ in Theorem 5.7 for $(i, j, k) = (1, 2, 3)$):

- To the ik row, add the row obtained from the ij row by the left multiplication with $-M_{a_{ik},a_{ij}}$.
- To the ik row, add the row obtained from the jk row by the left multiplication with $-M_{a_{ik},a_{jk}}$.

By these operations, the ik row of $\mathbf{M}$ becomes zero, and we can delete the ik row without changing the value of the right-hand side of (5.14). □

The next result for *2D-grid* (see Example 4.9) can be immediately obtained from Lemma 5.9.

Corollary 5.10 (Specialization to *2D-grids* for $(n, 1)$-type)**.** *Let* $\mathbf{P}$ *be a 2D-grid,* $I \in \mathbb{I}$, *and* $M \in \operatorname{mod} A$. *Without loss of generality, we assume that the first coordinate* (*i.e., the* x*-coordinate in* Example 4.9) *of* a_i *is strictly less than that of* a_{i+1} $(i \in [n-1])$. *Then using Notation 3.7, we have*

$$d_{R(M)}(V_I) = \operatorname{rank} \begin{bmatrix} \mathbf{M} \\ \beta \end{bmatrix} - \operatorname{rank} \mathbf{M}, \tag{5.19}$$

where

$$\mathbf{M} = \begin{array}{c} \\ 12 \\ 23 \\ \vdots \\ \text{n-2,n-1} \\ \text{n-1,n} \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \cdots & \text{n-1} & \text{n} \end{array} \\ \begin{bmatrix} M_{a_{12},a_1} & -M_{a_{12},a_2} & 0 & \cdots & 0 & 0 \\ 0 & M_{a_{23},a_2} & -M_{a_{23},a_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -M_{a_{n-2,n-1},a_{n-1}} & 0 \\ 0 & 0 & 0 & \cdots & M_{a_{n-1,n},a_{n-1}} & -M_{a_{n-1,n},a_n} \end{bmatrix} \end{array}$$

and

$$\beta = \begin{bmatrix} M_{b,a_1} & \underbrace{0\ 0\ \cdots\ 0\ 0}_{n-1} \end{bmatrix}.$$

Remark 5.11. As a suggestion for further generalization, we could discard the existence condition of pairwise joins in $\operatorname{sc}(I)$ and generalize the results by changing $P_{\operatorname{sc}_1(I)}$ in the projective presentation (5.6) to be the direct sum of all projective indecomposables at minimal elements of the upper bounds for each pair in $\operatorname{sc}(I)$.

Example 5.12. Let $\mathbf{P} := G_{5,2}$ as in Example 4.9, and I be the interval with $\mathrm{sc}(I) := \{(1,2),(2,1)\}$ and $\mathrm{sk}(I) := \{(3,2)\}$. Then ss_I does not essentially cover I. If we take $Q_I := H(\{(1,2),(2,2),(3,2),(2,1)\})$, the Hasse quiver of the full subposet $\{(1,2),(2,2),(3,2),(2,1)\}$ of I, and take $\xi_I \colon Q_I \to U(A)$ to be the inclusion, then ξ_I essentially covers I.

5.2. $(1,n)$**-type.** To compute the I^{op}-rank of a module M in $\operatorname{mod} A$ using the I-rank of M, we use the usual $\Bbbk$-duality $D := \operatorname{Hom}_{\Bbbk}(\text{-},\Bbbk) \colon \operatorname{mod}\Bbbk \to \operatorname{mod}\Bbbk$. We need the following three lemmas for this purpose. Here we denote by R^{op} the restriction functor $\operatorname{mod}\Bbbk[\mathbf{P}^{\mathrm{op}}] \to \operatorname{mod}\Bbbk[I^{\mathrm{op}}]$ defined by the inclusion functor $\Bbbk[I^{\mathrm{op}}] \to \Bbbk[\mathbf{P}^{\mathrm{op}}]$.

Lemma 5.13. *Let $M \in \operatorname{mod} A$ and $I \in \mathbb{I}$. Then we have*

$$d_{R^{\mathrm{op}}(DM)}(R^{\mathrm{op}}(V_{I^{\mathrm{op}}})) = d_{R(M)}(R(V_I)).$$

Proof. Denote the $\Bbbk$-duality $\operatorname{mod}\Bbbk[I] \to \operatorname{mod}\Bbbk[I^{\mathrm{op}}]$ by the same symbol D. Then it is easy to see that the following is a strict commutative diagram of functors and contravariant functors:

$$\begin{tikzcd} \operatorname{mod}\Bbbk[\mathbf{P}] \arrow[r, "D"] \arrow[d, "R"'] & \operatorname{mod}\Bbbk[\mathbf{P}^{\mathrm{op}}] \arrow[d, "R^{\mathrm{op}}"] \\ \operatorname{mod}\Bbbk[I] \arrow[r, "D"'] & \operatorname{mod}\Bbbk[I^{\mathrm{op}}] \end{tikzcd} .$$

Set $c := d_{R(M)}(R(V_I))$. Then we have $R(M) \cong R(V_I)^c \oplus N$ for some $N \in \operatorname{mod}\Bbbk[I]$ having no direct summand isomorphic to $R(V_I)$. By sending this isomorphism by D, we obtain

$$(D \circ R)(M) \cong (D \circ R)(V_I)^c \oplus DN,$$
$$R^{\mathrm{op}}(DM) \cong R^{\mathrm{op}}(DV_I)^c \oplus DN,$$

where DN does not have direct summand isomorphic to $D(R(V_I)) \cong R^{\mathrm{op}}(DV_I)$. Hence by noting that $DV_I \cong V_{I^{\mathrm{op}}} \cong R^{\mathrm{op}}(V_{I^{\mathrm{op}}})$, we have the conclusion that $d_{R^{\mathrm{op}}(DM)}(R^{\mathrm{op}}(V_{I^{\mathrm{op}}})) = c = d_{R(M)}(R(V_I))$. □

Lemma 5.14. *Let $f \colon V \to W$ be a linear map in* $\operatorname{mod}\Bbbk$. *Then* $\operatorname{rank} D(f) = \operatorname{rank} f$.

Proof. The linear map f is expressed as the composite $f = f_1 \circ f_2$ for some epimorphism $f_2 \colon V \to \operatorname{Im} f$ and some monomorphism $f_1 \colon \operatorname{Im} f \to W$. Then $D(f)$ is expressed as $D(f) = D(f_2) \circ D(f_1)$, where $D(f_1) \colon D(W) \to D(\operatorname{Im} f)$ is an epimorphism and $D(f_2) \colon D(\operatorname{Im} f) \to D(V)$ is a monomorphism. Hence we have $\operatorname{Im} D(f) \cong D(\operatorname{Im} f)$. Then the assertion follows from $\dim \operatorname{Im} f = \dim D(\operatorname{Im} f) = \dim \operatorname{Im} D(f)$. □

Lemma 5.15. *Let $f \colon V \to W$ be in* $\operatorname{mod}\Bbbk$ *and $V = \bigoplus_{i \in I} V_i$, $W = \bigoplus_{j \in J} W_j$ direct sum decompositions. If $f = [f_{j,i}]_{(j,i) \in J \times I}$ with $f_{ji} \colon V_i \to W_j$ is a matrix expression of f with respect to these direct sum decompositions, then $D(f)$ has a matrix expression $D(f) = [D(f_{j,i})]_{(i,j) \in I \times J}$ with $D(f_{j,i}) \colon D(W_j) \to D(V_i)$ with* *respect to the direct sum decompositions* $D(V) \cong \bigoplus_{i\in I} D(V_i)$ *and* $D(W) \cong \bigoplus_{j\in J} D(W_j)$. *Hence by Lemma 5.14, we have*

$$\operatorname{rank}\,[D(f_{j,i})]_{(i,j)\in I\times J} = \operatorname{rank}\,[f_{j,i}]_{(j,i)\in J\times I} = \operatorname{rank}{}^t[f_{i,j}]_{(i,j)\in I\times J},$$

where ${}^t(\text{-})$ *denotes the blockwise transpose.*

Proof. Let $(\sigma_i^V : V_i \to V)_{i\in I}$ be the family of canonical injections, and let $(\pi_j^W : W \to W_j)_{j\in J}$ be the family of canonical projections with respect to the decompositions of V, W above, respectively. Then $f_{j,i} = \pi_j^W \circ f \circ \sigma_i^V$ for all $i \in I, j \in J$. Now $(D(\sigma_i^V) : D(V) \to D(V_i))_{i\in I}$ forms the family of the canonical projections, and $(D(\pi_j^W) : D(W_j) \to D(W))_{j\in J}$ the canonical injections with respect to the decompositions $D(V) \cong \bigoplus_{i\in I} D(V_i)$ and $D(W) \cong \bigoplus_{j\in J} D(W_j)$, respectively. Hence $D(f)$ has the matrix expression $D(f) = [D(f)_{i,j}]_{(i,j)\in I\times J}$, where $D(f)_{j,i} = D(\sigma_i^V) \circ D(f) \circ D(\pi_j^W) = D(\pi_j^W \circ f \circ \sigma_i^V) = D(f_{j,i})$. ☐

These Lemmas give a formula for the intervals of $(1, n)$-type with $n \geq 2$ as follows.

Theorem 5.16. *Let* $M \in \operatorname{mod} A$, *and* I *be an interval in* $\mathbb{I}$ *of* $(1, n)$*-type with* $n \geq 2$, $\operatorname{sc}(I) = \{a\}$ *and* $\operatorname{sk}(I) = \{b_1, \dots, b_n\}$. *Assume that* I *satisfies the existence condition of pairwise meets of* $\operatorname{sk}(I)$, *and let* $b_{ij} := b_i \vee b_j$ *(in* $\mathbf{P}^{\mathrm{op}}$*)* $=$ $b_i \wedge b_j$ *(in* $\mathbf{P}$*) for all* i, j. *Then using Notation 3.7, we have*

$$d_{R(M)}(V_I) = \operatorname{rank}\begin{bmatrix}\gamma & \mathbf{N}\end{bmatrix} - \operatorname{rank}\mathbf{N}, \tag{5.20}$$

where $\mathbf{N}$ *is given by the following:*

$$\begin{array}{c|cccccccc}
 & 12 & 13 & \cdots & 1n & 23 & \cdots & 2n & \cdots & n-1,n \\
\hline
1 & M_{b_1,b_{12}} & M_{b_1,b_{13}} & \cdots & M_{b_n,b_{1n}} & 0 & \cdots & 0 & \cdots & 0 \\
2 & -M_{b_2,b_{12}} & 0 & \cdots & 0 & M_{b_2,b_{23}} & \cdots & M_{b_2,b_{2n}} & \cdots & 0 \\
3 & 0 & -M_{b_3,b_{13}} & \cdots & 0 & -M_{b_3,b_{23}} & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\
n-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & M_{b_{n-1},b_{n-1,n}} \\
n & 0 & 0 & \cdots & -M_{b_n,b_{1n}} & 0 & \cdots & -M_{b_n,b_{2n}} & \cdots & -M_{b_n,b_{n-1,n}}
\end{array}$$

and

$$\gamma = \begin{bmatrix} M_{b_1,a} \\ \left.\begin{matrix}\mathbf{0}\\ \vdots \\ \mathbf{0}\end{matrix}\right\} n-1 \end{bmatrix}.$$

Proof. (1) By Lemma 5.13, we have $d_{R(M)}(R(V_I)) = d_{R^{\mathrm{op}}(DM)}(R^{\mathrm{op}}(V_{I^{\mathrm{op}}}))$. Here, it is obvious that I^{op} is of $(n, 1)$-type with $\operatorname{sc}(I^{\mathrm{op}}) = \{b_1, \dots, b_n\}$ and $\operatorname{sk}(I^{\mathrm{op}}) = \{a\}$ and that I^{op} satisfies the existence condition of pairwise joins of $\operatorname{sc}(I^{\mathrm{op}})$. Hence to compute $d_{R^{\mathrm{op}}(DM)}(R^{\mathrm{op}}(V_{I^{\mathrm{op}}}))$, we can apply Proposition 5.7 to the following setting: poset $\mathbf{P}^{\mathrm{op}}$, module DM, the interval I^{op}, the quiver Q_I^{op}, and the quiver morphism $\xi_I^{\mathrm{op}} : Q_I^{\mathrm{op}} \to U(A^{\mathrm{op}})$ that is defined by $\xi_I^{\mathrm{op}}(x) := \xi_I(x)$ for all $x \in (Q_I^{\mathrm{op}})_0 = (Q_I)_0$ and $\xi_I^{\mathrm{op}}(\alpha) := \xi_I(\alpha) : \xi_I(y) \to \xi_I(x)$ for all arrows

$\alpha\colon x \to y$ in Q_I. Then we have the following.

$$d_{R^{\mathrm{op}}(DM)}(R^{\mathrm{op}}(V_{I^{\mathrm{op}}})) = \operatorname{rank}\begin{bmatrix}\mathbf{M}' \\ \beta'\end{bmatrix} - \operatorname{rank}\mathbf{M}',$$

where

$$\mathbf{M}' = \begin{array}{c} \\ 12 \\ 13 \\ \vdots \\ 1n \\ 23 \\ \vdots \\ 2n \\ \vdots \\ n\text{-}1,n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \cdots & n\text{-}1 & n \end{array} \\ \begin{bmatrix} (DM)_{b_{12},b_1} & -(DM)_{b_{12},b_2} & 0 & \cdots & 0 & 0 \\ (DM)_{b_{13},b_1} & 0 & -(DM)_{b_{13},b_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ (DM)_{b_{1n},b_1} & 0 & 0 & \cdots & 0 & -(DM)_{b_{1n},b_n} \\ 0 & (DM)_{b_{23},b_2} & -(DM)_{b_{23},b_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & (DM)_{b_{2n},b_2} & 0 & \cdots & 0 & -(DM)_{b_{2n},b_n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (DM)_{b_{n-1,n},b_{n-1}} & -(DM)_{b_{n-1,n},b_n} \end{bmatrix} \end{array}$$

and

$$\beta' = \big[(DM)_{a,b_1}\ \underbrace{0\ 0\ \cdots\ 0\ 0}_{n-1}\big].$$

Now for any $x, y \in \mathbf{P}^{\mathrm{op}}$ with $x \leq^{\mathrm{op}} y$ in $\mathbf{P}^{\mathrm{op}}$, let $p^{\mathrm{op}}_{y,x}$ be the unique morphism in $\mathbf{P}^{\mathrm{op}}(x, y)$. Then we have $y \leq x$ in $\mathbf{P}$, and $p_{x,y} = p^{\mathrm{op}}_{y,x}$, which is the unique morphism in $\mathbf{P}(y, x) = \mathbf{P}^{\mathrm{op}}(x, y)$. Hence we have

$$(DM)_{y,x} = (DM)(p^{\mathrm{op}}_{y,x}) = (DM)(p_{x,y}) = D(M(p_{x,y})) = D(M_{x,y}).$$

Then (5.20) follows by Lemma 5.15.

□

5.3. **(n, m)-type with $m, n \geq 2$.** Finally, we give a formula of $\operatorname{rank}^{\mathrm{tot}}_I M = d_{R(M)}(V_I)$ for any interval $I \in \mathbb{I}$ of (n, m)-type with $n, m \geq 2$.

Theorem 5.17. *Let $M \in \operatorname{mod} A$ and $I \in \mathbb{I}$ be of (n, m)-type with $m, n \geq 2$. Assume that I satisfies the existence conditions of pairwise joins in $\operatorname{sc}(I)$ and meets in $\operatorname{sk}(I)$. Obviously, for each $a \in \operatorname{sc}(I)$, there exists some $b \in \operatorname{sk}(I)$ such that $a \leq b$. Hence we may assume that $a_1 \leq b_1$ without loss of generality. Then We have*

$$d_{R(M)}(V_I) = \operatorname{rank}\begin{bmatrix} \mathbf{M} & 0 \\ \begin{bmatrix} M_{b_1,a_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & \mathbf{N} \end{bmatrix} - \operatorname{rank}\mathbf{M} - \operatorname{rank}\mathbf{N}, \tag{5.21}$$

where $\mathbf{M}, \mathbf{N}$ are defined in Theorems 5.7 and 5.16.

Proof. Since $m, n \geq 2$, note first that we can apply Propositions 5.4 and 5.5 and that V_I is not injective by $m \geq 2$. By the latter, there exists an almost split sequence

$$0 \to V_I \to E \to \tau^{-1}V_I \to 0 \tag{5.22}$$

starting from V_I. The value of $d_{R(M)}(V_I)$ can be computed from the three terms of this almost split sequence by using the formula of [4, Theorem 3] as follows:

$$\begin{aligned} d_{R(M)}(V_I) &= \dim \mathrm{Hom}_{\Bbbk[I]}(V_I, R(M)) - \dim \mathrm{Hom}_{\Bbbk[I]}(E, R(M)) \\ &\qquad + \dim \mathrm{Hom}_{\Bbbk[I]}(\tau^{-1}V_I, R(M)) \\ &= \dim \mathrm{Hom}_{\Bbbk[I]}(V_I, R(M)) - \dim \mathrm{Hom}_{\Bbbk[I]}(E \oplus P_2^t, R(M)) \\ &\qquad + \dim \mathrm{Hom}_{\Bbbk[I]}(\tau^{-1}V_I \oplus P_2^t, R(M)), \end{aligned} \tag{5.23}$$

where P_2 is a direct summand of $P_{\mathrm{sk}_1(I)}$ as in (5.10). Hence the assertion follows by the following proposition together with a projective presentation (5.6) of V_I, a projective presentation (5.13) of $\tau^{-1}V_I \oplus P_2^t$, and Lemma 4.6. □

Proposition 5.18. *Let $M \in \mathrm{mod}\, A$ and $I \in \mathbb{I}$ be of (n, m)-type with $m, n \geq 2$, E the middle term in (5.22), and P_2 a direct summand of $P_{\mathrm{sk}_1(I)}$ as in (5.10). Then the following is a projective presentation of $E \oplus P_2^t$:*

$$P_{\mathrm{sc}_1(I)} \oplus P_{\mathrm{sk}(I)} \xrightarrow{\mu_E} P_{\mathrm{sc}(I)} \oplus P_{\mathrm{sk}_1(I)} \xrightarrow{\varepsilon_E} E \oplus P_2^t \to 0,$$

where μ_E is given by

$$\mu_E := \begin{bmatrix} \varepsilon_1 & \begin{bmatrix} P_{b_1,a_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \mathbf{0} & {\varepsilon'_1}^t \end{bmatrix}.$$

Proof. By [15, Sect. 3.6], an almost split sequence (5.22) can be obtained as a pushout of the sequence (5.12) along a morphism $\eta\colon P_{\mathrm{sk}(I)} \to V_I$ as follows:

$$\begin{tikzcd} P_{\mathrm{sk}(I)} \arrow[r, "{\varepsilon'_{11}}^t"] \arrow[d, "\eta"] & P_1^t \arrow[r] \arrow[d] & \tau^{-1}V_I \arrow[r] \arrow[d, equal] & 0 \\ V_I \arrow[r] & E \arrow[r] & \tau^{-1}V_I \arrow[r] & 0 \end{tikzcd}. \tag{5.24}$$

Here, η is the composite of morphisms

$$P_{\mathrm{sk}(I)} \xrightarrow{\text{can.}} \mathrm{top}\, P_{\mathrm{sk}(I)} \xrightarrow{\sim} \mathrm{soc}\, \nu P_{\mathrm{sk}(I)} \xrightarrow{\sim} \mathrm{soc}\, V_I \xrightarrow{\alpha} S \hookrightarrow \mathrm{soc}\, V_I \hookrightarrow V_I,$$

where ν is the Nakayama functor $\nu := D \circ \mathrm{Hom}_{\Bbbk[I]}(\text{-}, \Bbbk[I])$, S is any simple $\Bbbk[I]$-$\mathrm{End}_{\Bbbk[I]}(V_I)$-subbimodule of $\mathrm{soc}\, V_I$, and α is a retraction.

Here we claim that any simple $\Bbbk[I]$-submodule of $\mathrm{soc}\, V_I$ is automatically a simple $\Bbbk[I]$-$\mathrm{End}_{\Bbbk[I]}(V_I)$-subbimodule of $\mathrm{soc}\, V_I$. Indeed, this follows from the fact that $\mathrm{soc}\, V_I = \bigoplus_{i\in[m]} V_{\{b_i\}}$, where $V_{\{b_i\}}$ are mutually non-isomorphic simple $\Bbbk[I]$-modules. More precisely, it is enough to show that $f(S) \subseteq S$ for any $f \in \mathrm{End}_{\Bbbk[I]}(V_I)^{\mathrm{op}}$ because if this is shown, then S turns out to be a right $\mathrm{End}_{\Bbbk[I]}(V_I)$-submodule and a simple $\Bbbk[I]$-$\mathrm{End}_{\Bbbk[I]}(V_I)$-subbimodule of $\mathrm{soc}\, V_I$. Let T be any simple $\Bbbk[I]$-submodule of $\mathrm{soc}\, V_I$, then by the fact above $T \cong V_{\{b_i\}}$ for a unique $i \in [m]$, and hence $\mathrm{pr}_j(T) = 0$ for all $j \in [m] \setminus \{i\}$, where $\mathrm{pr}_j\colon \mathrm{soc}\, V_I \to V_{\{b_j\}}$ is the canonical projection. Thus $T \subseteq V_{\{b_i\}}$, which shows that $T = V_{\{b_i\}}$ because the both hand sides are simple. Now there exists a unique $i \in [m]$ such that

$S = V_{\{b_i\}}$. If $f = 0$, then $f(S) = 0 \subseteq S$; otherwise $f(S) \cong S$, and then $f(S) = V_{\{b_i\}} = S$ by the argument above. This proves our claim.

Therefore, we may take $S := V_{\{b_1\}}$, and

$$\eta := [\rho_{1_{b_1}}, 0, \dots, 0] \colon P_{\mathrm{sk}(I)} = P_{b_1} \oplus \cdots \oplus P_{b_m} \to V_I.$$

By assumption, $a_1 \leq b_1$ in I. Hence we have a commutative diagram

$$\begin{tikzcd} & P_{\mathrm{sc}(I)} = P_{a_1} \oplus \cdots \oplus P_{a_n} \arrow[d, "\varepsilon_0 = (\rho_{1_{a_1}}, \rho_{1_{a_2}}, \dots, \rho_{1_{a_n}})"] \\ P_{\mathrm{sk}(I)} \arrow[ur, "\eta' := \begin{bmatrix} P_{b_1,a_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}"] \arrow[r, "\eta"'] & V_I \end{tikzcd}$$

because for each $p \in P_{b_1}(= \Bbbk e_{b_1})$, we have

$$\rho_{1_{a_1}}(P_{b_1,a_1}(p)) = \rho_{1_{a_1}}(p \cdot p_{b_1,a_1}) = V_I(p \cdot p_{b_1,a_1})(1_{a_1}) = V_I(p)(1_{b_1}) = \rho_{1_{b_1}}(p).$$

The pushout diagram (5.24) yields the following exact sequences:

$$P_{\mathrm{sk}(I)} \xrightarrow{\begin{bmatrix} \eta \\ \varepsilon_{11}'^{\,t} \end{bmatrix}} V_I \oplus P_1^t \to E \to 0, \text{ and } P_{\mathrm{sk}(I)} \xrightarrow{\begin{bmatrix} \eta \\ \varepsilon_1'^{\,t} \end{bmatrix}} V_I \oplus P^t_{\mathrm{sk}_1(I)} \xrightarrow{\pi} E \oplus P_2^t \to 0.$$

The latter is extended to the following commutative diagram with the bottom row exact:

$$\begin{tikzcd} P_{\mathrm{sc}_1(I)} \oplus P_{\mathrm{sk}(I)} \arrow[r, "\mu_E := \begin{bmatrix} \varepsilon_1 & \eta' \\ 0 & \varepsilon_1'^{\,t} \end{bmatrix}"] \arrow[d, equal] & P_{\mathrm{sc}(I)} \oplus P_{\mathrm{sk}_1(I)} \arrow[r, "\varepsilon_E"] \arrow[d, "\begin{bmatrix} \varepsilon_0 & 0 \\ 0 & \mathbf{1} \end{bmatrix}"] & E \oplus P_2^t \arrow[r] \arrow[d, equal] & 0 \\ P_{\mathrm{sc}_1(I)} \oplus P_{\mathrm{sk}(I)} \arrow[r, "\begin{bmatrix} 0 & \eta \\ 0 & \varepsilon_1'^{\,t} \end{bmatrix}"'] & V_I \oplus P_{\mathrm{sk}_1(I)} \arrow[r, "\pi"'] & E \oplus P_2^t \arrow[r] & 0 \end{tikzcd},$$

where we set $\varepsilon_E := \pi \circ \begin{bmatrix} \varepsilon_0 & 0 \\ 0 & \mathbb{1} \end{bmatrix}$, which is an epimorphism as the composite of epimorphisms.

It remains to show that ε_E is a cokernel morphism of μ_E. By the commutativity of the diagram and the exactness of the bottom row, we see that $\varepsilon_E \mu_E = 0$. Let $(f, g) \colon P_{\mathrm{sc}(I)} \oplus P_{\mathrm{sk}_1(I)} \to X$ be a morphism with $(f, g)\mu_E = 0$. Then $f\varepsilon_1 = 0$. Since ε_0 is a cokernel morphism of ε_1, there exists some $f' \colon V_I \to X$ such that $f = f'\varepsilon_0$. Then we have $(f, g) = (f', g) \begin{bmatrix} \varepsilon_0 & 0 \\ 0 & \mathbb{1} \end{bmatrix}$. Now $(f', g) \begin{bmatrix} 0 & \eta \\ 0 & \varepsilon_1'^{\,t} \end{bmatrix} = (f', g) \begin{bmatrix} \varepsilon_0 & 0 \\ 0 & \mathbb{1} \end{bmatrix} \mu_E = (f, g)\mu_E = 0$. Hence (f', g) factors through π, that is, $(f', g) = h\pi$ for some $h \colon E \to X$. Therefore, we have

$(f,g) = h\pi \begin{bmatrix} \varepsilon_0 & 0 \\ 0 & \mathbb{1} \end{bmatrix} = h\,\varepsilon_E$. The uniqueness of h follows from the fact that ε_E is an epimorphism. As a consequence, ε_E is a cokernel morphism of μ_E. $\square$

In particular, when $(n,m) = (2,2)$ we have the following.

Example 5.19. Let $M \in \operatorname{mod} A$ and $I \in \mathbb{I}$ be of $(2,2)$-type with $\operatorname{sc}(I) = \{a_1, a_2\}$ and $\operatorname{sk}(I) = \{b_1, b_2\}$. Assume that both $x := a_1 \vee a_2$ and $y := b_1 \wedge b_2$ exist. Since $\operatorname{sk}(I) = \{b_1, b_2\}$, we have $a_1 \le b_1$ or $a_1 \le b_2$, and hence we may assume that $a_1 \le b_1$ without loss of generality. Then we have

$$\begin{aligned} d_{R(M)}(V_I) &= \operatorname{rank} \begin{bmatrix} M_{x,a_1} & -M_{x,a_2} & 0 \\ M_{b_1,a_1} & 0 & M_{b_1,y} \\ 0 & 0 & -M_{b_2,y} \end{bmatrix} - \operatorname{rank} \begin{bmatrix} M_{x,a_1}, -M_{x,a_2} \end{bmatrix} \\ &\qquad\qquad\qquad\qquad\qquad\qquad - \operatorname{rank} \begin{bmatrix} M_{b_1,y} \\ -M_{b_2,y} \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} M_{x,a_1} & M_{x,a_2} & 0 \\ M_{b_1,a_1} & 0 & M_{b_1,y} \\ 0 & 0 & M_{b_2,y} \end{bmatrix} - \operatorname{rank} \begin{bmatrix} M_{x,a_1}, M_{x,a_2} \end{bmatrix} - \operatorname{rank} \begin{bmatrix} M_{b_1,y} \\ M_{b_2,y} \end{bmatrix}. \end{aligned}$$

The formula in Theorem 5.17 itself covers all cases by using an empty matrix convention, namely, it is valid even if m or n is equal to 1. We summarize the result as follows.

Theorem 5.20. *Let $M \in \operatorname{mod} A$, and $I \in \mathbb{I}$ be of (n,m)-type $(m, n \ge 1)$. Assume that I satisfies the existence conditions of pairwise joins in $\operatorname{sc}(I)$ and meets in $\operatorname{sk}(I)$. Obviously, for each $a \in \operatorname{sc}(I)$, there exists some $b \in \operatorname{sk}(I)$ such that $a \le b$. Hence we may assume that $a_1 \le b_1$ without loss of generality. Then we have*

$$\operatorname{rank}^{\mathrm{tot}}_I M = \operatorname{rank} \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \begin{bmatrix} M_{b_1,a_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & \mathbf{N} \end{bmatrix} - \operatorname{rank} \mathbf{M} - \operatorname{rank} \mathbf{N}, \tag{5.25}$$

where if $m = 1$ (resp. $n = 1$), then $\mathbf{N}$ (resp. $\mathbf{M}$) is an empty matrix, and hence the formula has the form in Theorems 5.7, 5.16, or Proposition 4.7.

5.4. **Essential cover property.**

Definition 5.21. Let $I \in \mathbb{I}$. Then we say that ξ_I *essentially covers* I if there exists a formula of $d_{R(M)}(V_I)$ in terms of linear maps $M(p)$ $(p \in S)$ for some subset S of morphisms in I such that for each $p \in S$, there exist a path q in Q_I such that $\overline{\xi_I}(q) = p$.

For example, if $I = [x, y]$ is a segment of $\mathbf{P}$, then always ξ_I essentially covers I by our assumptions (2), (3) in Definition 3.1 and by the formula $d_{R(M)}(V_I) = \operatorname{rank} M_{y,x}$ given in Proposition 4.7.

Note that sometimes there exist several possibilities of the set S above, say $S_1, \dots, S_t$. Then ξ_I essentially covers I if for some $i \in [t]$, for each $p \in S_i$, there exists a path q in Q_I such that $\overline{\xi_I}(q) = p$.

Remark 5.22. Let $I \in \mathbb{I}$ be of (n,m)-type $(m,n \geq 1)$ satisfying the existence conditions of pairwise joins in $\mathrm{sc}(I)$ and meets in $\mathrm{sk}(I)$. Thanks to Theorem 5.20, we can explicitly state a sufficient condition for ξ_I to essentially cover I as follows: there exist paths $\mathbf{p}_{ij}^{+}, \mathbf{p}_{ij}^{-}, \mathbf{p}$ in Q_I $(i,j \in [n]$ with $i<j)$ and $\mathbf{q}_{ij}^{+}, \mathbf{q}_{ij}^{-}$ in Q_I $(i,j \in [m]$ with $i<j)$ such that $\overline{\xi_I}(\mathbf{p}_{ij}^{+}) = p_{a_{ij},a_i}$, $\overline{\xi_I}(\mathbf{p}_{ij}^{-}) = p_{a_{ij},a_j}$, $\overline{\xi_I}(\mathbf{q}_{ij}^{+}) = p_{b_i,b_{ij}}$, $\overline{\xi_I}(\mathbf{q}_{ij}^{-}) = p_{b_j,b_{ij}}$, and $\overline{\xi_I}(\mathbf{p}) \in \{p_{b_j,a_i} \mid a_i \leq b_j \text{ for some } i \in [n], j \in [m]\}$.

Example 5.23 (zz)**.** Assume that each interval $I \in \mathbb{I}$ satisfies the existence condition of pairwise joins in $\mathrm{sc}(I)$ and pairwise meets in $\mathrm{sk}(I)$. (For example, this assumption is satisfied if $\mathbf{P}$ is a 2D-grid.) Let $I \in \mathbb{I}$ be of (n,m)-type $(m,n \geq 1)$. Choose a pair $(a_I, b_I) \in \mathrm{sc}(I) \times \mathrm{sk}(I)$ such that $a_I \leq b_I$ in I. Define a quiver Q_I^{zz} as follows: $(Q_I^{\mathrm{zz}})_0 := \mathrm{sc}(I) \cup \mathrm{sc}_1(I) \cup \mathrm{sk}(I) \cup \mathrm{sk}_1(I)$, $(Q_I^{\mathrm{zz}})_1 := \{p_{a_{ij},a_i}, p_{a_{ij},a_j} \mid i,j \in [n], i<j\} \cup \{p_{b_i,b_{ij}}, p_{b_j,b_{ij}} \mid i,j \in [m], i<j\} \cup \{p_{b_I,a_I}\}$. For each $p_{y,x} \in (Q_I^{\mathrm{zz}})_1$, it is an arrow $p_x \to p_y$. Let $\mathrm{zz}_I \colon Q_I^{\mathrm{zz}} \to U(A)$ be the inclusion quiver morphism. $\mathrm{zz} := (\mathrm{zz}_I)_{I \in \mathbb{I}}$ is a compression system for A, which is called a *zigzag* compression system. By definition of zz and Theorem 5.20, zz_I essentially covers I for all zigzag zz and $I \in \mathbb{I}$.

In the case where $\mathbf{P}$ is a 2D-grid, we slightly change the quiver Q_I^{zz} for each $I \in \mathbb{I}$ according to the formula in Corollary 5.10 as follows:

Without loss of generality, we may assume that the x-coordinate of a_i (resp. b_i) is strictly less than that of a_{i+1} $(i \in [n-1])$ (resp. b_{i+1} $(i \in [m-1])$). Then we choose $a_I := a_1, b_I := b_1$, and set $(Q_I^{\mathrm{zz}})_0 := \mathrm{sc}(I) \cup \{a_{i,i+1} \mid i \in [n-1]\} \cup \mathrm{sk}(I) \cup \{b_{i,i+1} \mid i \in [m-1]\}$, and $(Q_I^{\mathrm{zz}})_1$ to be

$$\{p_{a_{i,i+1},a_i}, p_{a_{i,i+1},a_{i+1}} \mid i \in [n-1]\} \cup \{p_{b_i,b_{i,i+1}}, p_{b_{i+1},b_{i,i+1}} \mid i \in [m-1]\} \cup \{p_{b_I,a_I}\}.$$

Then this Q_I^{zz} also essentially covers I, and is a Dynkin quiver of type $\mathbb{A}$, which coincides with the support quiver of the so-called boundary cap (a zigzag path) defined in [11, Definition 3.7].

Lemma 5.24. *Let B be a linear category, W a B-module, and m,n positive integers. For each matrix $\boldsymbol{g} = [g_{ji}]_{(j,i) \in [n] \times [m]}$ with entries $g_{ji} \colon x_i \to y_j$ morphisms in B, we set*

$$W(\boldsymbol{g}) := [W(g_{ij})]_{j,i} \colon \bigoplus_{i \in [m]} W(x_i) \to \bigoplus_{j \in [n]} W(y_j)$$

to be the linear map expressed by this matrix. Assume that we have a direct sum decomposition $W \cong W_1 \oplus W_2$ of B-modules. Then we have an equivalence $W(\boldsymbol{g}) \cong W_1(\boldsymbol{g}) \oplus W_2(\boldsymbol{g})$ of linear maps. In particular, the equality

$$\operatorname{rank} W(\boldsymbol{g}) = \operatorname{rank} W_1(\boldsymbol{g}) + \operatorname{rank} W_2(\boldsymbol{g})$$

holds.

Proof. Let $f\colon W \to W_1 \oplus W_2$ be an isomorphism of B-modules. Then for any $i \in [m], j \in [n]$, we have a commutative diagram

$$\begin{tikzcd}
W(x_i) \arrow[rr, "W(g_{ji})"] \arrow[d, "f_{x_i}"'] & & W(y_j) \arrow[d, "f_{y_j}"] \\
W_1(x_i) \oplus W_2(x_i) \arrow[rr, "W_1(g_{ji}) \oplus W_2(g_{ji})"'] & & W_1(y_j) \oplus W_2(y_j)
\end{tikzcd}$$

with both f_{x_i} and f_{y_j} isomorphisms. This yields the commutative diagram

$$\begin{tikzcd}
\bigoplus_{i\in[m]} W(x_i) \arrow[rr, "{[W(g_{ji})]_{j,i}}", "{=W(\boldsymbol{g})}"'] \arrow[d, "\bigoplus_{i\in[m]} f_{x_i}"'] & & \bigoplus_{j\in[n]} W(y_j) \arrow[d, "\bigoplus_{j\in[n]} f_{y_j}"] \\
\bigoplus_{i\in[m]} (W_1(x_i) \oplus W_2(x_i)) \arrow[rr, "{[W_1(g_{ji}) \oplus W_2(g_{ji})]_{j,i}}"'] \arrow[d, "\sigma_x"'] & & \bigoplus_{j\in[n]} (W_1(y_j) \oplus W_2(y_j)) \arrow[d, "\sigma_y"] \\
(\bigoplus_{i\in[m]} W_1(x_i) \oplus (\bigoplus_{i\in[m]} W_2(x_i)) \arrow[rr, "{[W_1(g_{ji})]_{j,i} \oplus [W_2(g_{ji})]_{j,i}}", "{=W_1(\boldsymbol{g}) \oplus W_2(\boldsymbol{g})}"'] & & (\bigoplus_{j\in[n]} W_1(y_j)) \oplus (\bigoplus_{j\in[n]} W_2(y_j))
\end{tikzcd},$$

where σ_x, σ_y are given by the permutation matrices corresponding to the permutation σ_k (for $k = m, n$, respectively) of the set $[2k]$ defined by

$$\sigma_k(i) := \begin{cases} j & (i = 2j-1,\ \exists j \in [k]) \\ k+j & (j = 2j,\ \exists j \in [k]) \end{cases} \quad \text{for all } i \in [2k],$$

the nonzero entries of which are the identity maps. Then since all vertical maps above are isomorphisms, the assertion holds. $\square$

Theorem 5.25. *Let $M \in \operatorname{mod} A$, $I \in \mathbb{I}$, and ξ a compression system for A. If ξ_I essentially covers I, then we have*

$$\operatorname{rank}_I^{\xi} M = \operatorname{rank}_I^{\mathrm{tot}} M.$$

Proof. Let $\boldsymbol{g} = [g_{ji}]_{j,i}$ be a matrix with entries morphisms in B_I, and W a B_I-module. Then we set $W(\boldsymbol{g}) := [W(g_{ij})]_{j,i}$. By $\mathbf{C}$ we denote the matrix $\begin{bmatrix} M_{b_1,a_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ in (5.25). Then (5.25) can be rewritten as

$$\operatorname{rank}_I^{\mathrm{tot}} M = \operatorname{rank} \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{C} & \mathbf{N} \end{bmatrix} - \operatorname{rank} \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{bmatrix}. \tag{5.26}$$

Now assume that ξ_I essentially covers I. Without loss of generality, we may assume that $a_1 \le b_1$ and $F_I(g') = p_{b_1,a_1}$ for some path g' in Q_I. For any $u, v \in I$ with $u \le v$, and a path g in Q_I, if $F_I(g) = p_{v,u}$, then we have

$$M_{v,u} = M(p_{v,u}) = M(F_I(g)) = R_I(M)(g).$$

Define three matrices $\boldsymbol{g}_1$, $\boldsymbol{g}_2$, and $\boldsymbol{g}_3$ from $\mathbf{M}, \mathbf{N}, \mathbf{C}$ by replacing $M_{v,u}$ with $p_{v,u}$ $(u, v \in I)$, respectively. Then by Definition 5.21, we have $R_I(M)(\boldsymbol{g}_1) = \mathbf{M}$, $R_I(M)(\boldsymbol{g}_2) = \mathbf{N}$, and $R_I(M)(\boldsymbol{g}_3) = \mathbf{C}$.

Set $r := d_{R(M)}(V_I)$, $s := \operatorname{rank}_I^{\xi} M = d_{R_I(M)}(R'(V_I))$. Then it is enough to show that $r = s$. The former means that we have $R(M) \cong V_I^r \oplus N$ for some module N in $\operatorname{mod} \Bbbk[I]$, which shows that $R_I(M) = R'(R(M)) \cong R'(V_I)^r \oplus R'(N)$. Hence we have $r \leq s$. On the other hand, by the latter we have an isomorphism $R_I(M) \cong R_I(V_I)^s \oplus L$ for some module L in $\operatorname{mod} B_I$. Then by Lemma 5.24, we have the following equalities:

$$\begin{aligned}\operatorname{rank}\begin{bmatrix} R_I(M)(\boldsymbol{g}_1) & \mathbf{0} \\ R_I(M)(\boldsymbol{g}_3) & R_I(M)(\boldsymbol{g}_2)\end{bmatrix} &= s \operatorname{rank}\begin{bmatrix} R_I(V_I)(\boldsymbol{g}_1) & \mathbf{0} \\ R_I(V_I)(\boldsymbol{g}_3) & R_I(V_I)(\boldsymbol{g}_2)\end{bmatrix} \\ &\quad + \operatorname{rank}\begin{bmatrix} L(\boldsymbol{g}_1) & \mathbf{0} \\ L(\boldsymbol{g}_3) & L(\boldsymbol{g}_2)\end{bmatrix},\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}\operatorname{rank}\begin{bmatrix} R_I(M)(\boldsymbol{g}_1) & \mathbf{0} \\ \mathbf{0} & R_I(M)(\boldsymbol{g}_2)\end{bmatrix} &= s \operatorname{rank}\begin{bmatrix} R_I(V_I)(\boldsymbol{g}_1) & \mathbf{0} \\ \mathbf{0} & R_I(V_I)(\boldsymbol{g}_2)\end{bmatrix} \\ &\quad + \operatorname{rank}\begin{bmatrix} L(\boldsymbol{g}_1) & \mathbf{0} \\ \mathbf{0} & L(\boldsymbol{g}_2)\end{bmatrix}.\end{aligned} \tag{5.28}$$

By subtracting (5.28) from (5.27), the formula (5.21) shows the followng:

$$\begin{aligned} r = d_{R(M)}(V_I) &= s \cdot d_{R(V_I)}(V_I) + \operatorname{rank}\begin{bmatrix} L(\boldsymbol{g}_1) & \mathbf{0} \\ L(\boldsymbol{g}_3) & L(\boldsymbol{g}_2)\end{bmatrix} - \operatorname{rank}\begin{bmatrix} L(\boldsymbol{g}_1) & \mathbf{0} \\ \mathbf{0} & L(\boldsymbol{g}_2)\end{bmatrix} \\ &= s + \operatorname{rank}\begin{bmatrix} L(\boldsymbol{g}_1) & \mathbf{0} \\ L(\boldsymbol{g}_3) & L(\boldsymbol{g}_2)\end{bmatrix} - \operatorname{rank}\begin{bmatrix} L(\boldsymbol{g}_1) & \mathbf{0} \\ \mathbf{0} & L(\boldsymbol{g}_2)\end{bmatrix} \geq s. \end{aligned}$$

Hence we have $r = s$, and the theorem follows. □

The following is immediate from Theorem 5.25 and Example 5.23.

Corollary 5.26. *Let* zz *be any zigzag compression system for A. Then we have*

$$\operatorname{rank}^{\mathrm{zz}}(M) = \operatorname{rank}^{\mathrm{tot}}(M)$$

for all $M \in \operatorname{mod} A$.

Remark 5.27. Corollary 5.26 above gives an alternative proof of Theorem in [11, Theorem 3.12] by Dey–Kim–Mémoli for the case where $\mathbf{P}$ is a 2D-grid because $\operatorname{rank}_I^{\mathrm{tot}}(M)$ coincides with their generalized rank invariant M. The latter statement follows by [10, Lemma 3.1], but the description of the proof was imprecise, and in the process of making it accurate we found a small gap in the proof. Therefore, we give a precise proof of it by filling the gap below.

First, we recall the definition of the generalized rank invariant of a module.

Definition 5.28. Let I be a finite connected poset, and $M \in \operatorname{mod} \Bbbk[I]$. Since I is finite and $M \in \operatorname{mod} \Bbbk[I]$, both $\varprojlim M$ and $\varinjlim M$ are easily constructed in

$\operatorname{mod}\Bbbk[I]$. By definition, we have a commutative diagram

$$\begin{tikzcd}
 & M(x) \arrow[dd, "M(p_{y,x})"] \arrow[dr, "\sigma_x"] & \\
\varprojlim M \arrow[ur, "\pi_x"] \arrow[dr, "\pi_y"'] & & \varinjlim M \\
 & M(y) \arrow[ur, "\sigma_y"'] &
\end{tikzcd}$$

for any $(x, y) \in [\leq]_I$, which shows that for any $x, y \in I$, we have $\sigma_x \pi_x = \sigma_y \pi_y$ if x and y are in the same connected component of I. But since I is connected, the equality above holds for all $x, y \in I$. The common linear map is denoted by $\mu_M \colon \varprojlim M \to \varinjlim M$.

Now, for a (locally finite) poset $\mathbf{P}$, a finite interval subposet I of $\mathbf{P}$, and $M \in \operatorname{mod}\Bbbk[\mathbf{P}]$, the rank of the linear map $\mu_{R(M)}$ for the module $R(M) \in \operatorname{mod}\Bbbk[I]$ is called the *generalized rank invariant* of M at I.

We now give a proof of [10, Lemma 3.1] below.

Lemma 5.29. *Let I be a connected poset, and $M \in \operatorname{mod}\Bbbk[I]$. Then M has a direct sum decomposition*

$$M \cong V_I^d \oplus N$$

as $\Bbbk[I]$-modules for some N, where $\operatorname{rank}\mu_M = d$, $\operatorname{rank}\mu_N = 0$. *Hence in particular, we have* $d_M(V_I) = \operatorname{rank}\mu_M$.

Proof. There exist some subspaces $P \leq \varprojlim M$ and $T \leq \varinjlim M$ such that $P \oplus \operatorname{Ker}\mu_M = \varprojlim M$ and $\operatorname{Im}\mu_M \oplus T = \varinjlim M$. Let $\sigma \colon P \to \varinjlim M$ be the inclusion and $\pi \colon \varinjlim M \to \operatorname{Im}\mu_M$ the projection with respect to this decomposition. We set $\phi_x := \pi_x \sigma$, $\rho_x := \pi\sigma_x$ for all $x \in I$. Then we have the following commutative diagram:

$$\begin{tikzcd}
 & & M(x) \arrow[dd, "M(p_{y,x})"] \arrow[dr, "\sigma_x"] \arrow[drr, bend left, "\rho_x"] & & \\
P \arrow[r, "\sigma"] \arrow[urr, bend left, "\phi_x"] \arrow[drr, bend right, "\phi_y"'] & \varprojlim M \arrow[ur, "\pi_x"] \arrow[dr, "\pi_y"'] \arrow[rr, "\mu_M"] & & \varinjlim M \arrow[r, "\pi"] & \operatorname{Im}\mu_M \\
 & & M(y) \arrow[ur, "\sigma_y"'] \arrow[urr, bend right, "\rho_y"'] & &
\end{tikzcd} .$$

Since μ_M restricts to an isomorphism $\mu' \colon P \to \operatorname{Im}\mu_M$, we have $\mu' = \pi\mu_M\sigma$. Thus $\dim P = \dim \operatorname{Im}\mu_M = \operatorname{rank}\mu_M$. Set $d := \operatorname{rank}\mu_M$, the common value. Then there exists an isomorphism $\alpha \colon \Bbbk^d \to P$, which gives an isomorphism $\beta := (\mu'\alpha)^{-1} \colon \operatorname{Im}\mu_M \to \Bbbk^d$. By setting $\phi'_x := \phi_x\alpha \colon \Bbbk^d \to M(x)$ and $\rho'_x := \beta\rho_x \colon M(x) \to \Bbbk^d$, we have the following two commutative diagrams with exact rows:

$$
\begin{gathered}
\begin{tikzcd}
0 \arrow[r] & \Bbbk^d \arrow[r, "\phi'_x"] \arrow[d, equal] & M(x) \arrow[r, "\psi_x"] \arrow[d, "M(p_{y,x})"] & \operatorname{Coker}\phi_x \arrow[r] \arrow[d, dashed, "f_{y,x}"] & 0 \\
0 \arrow[r] & \Bbbk^d \arrow[r, "\phi'_y"'] & M(y) \arrow[r, "\psi_y"'] & \operatorname{Coker}\phi_y \arrow[r] & 0
\end{tikzcd}, \text{ and} \\
\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}\rho_x \arrow[r, "\tau_x"] \arrow[d, dashed, "g_{y,x}"] & M(x) \arrow[r, "\rho'_x"] \arrow[d, "M(p_{y,x})"] & \Bbbk^d \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Ker}\rho_y \arrow[r, "\tau_y"'] & M(y) \arrow[r, "\rho'_y"'] & \Bbbk^d \arrow[r] & 0
\end{tikzcd},
\end{gathered}
\tag{5.29}
$$

where $f_{y,x}$ (resp. $g_{y,x}$) is the unique linear map making the diagram commutative, and $\tau_z\colon \operatorname{Ker}\rho_z \to M(z)$ is the inclusion map for all $z \in I$. The uniqueness of $f_{y,x}$ (resp. $g_{y,x}$) for all $(x, y) \in [\leq]_I$ defines a $\Bbbk[I]$-module C (resp. K) by setting $C(x) := \operatorname{Coker}\phi_x$, $C(p_{y,x}) := f_{y,x}$ (resp. $K(x) := \operatorname{Ker}\rho_x$, $K(p_{y,x}) := g_{y,x}$) for all $x \in I$, $(x, y) \in [\leq]_I$. Set $\alpha := (\alpha_x)_{x\in I}$ for all $\alpha \in \{\phi', \psi, \tau, \rho'\}$. Then the commutative diagrams above show that ϕ', ψ, τ, ρ' are morphisms of $\Bbbk[I]$-modules, and give us the following short exact sequences of $\Bbbk[I]$-modules:

$$
0 \to V_I^d \xrightarrow{\phi'} M \xrightarrow{\psi} C \to 0 \text{ and } 0 \to K \xrightarrow{\tau} M \xrightarrow{\rho'} V_I^d \to 0. \tag{5.30}
$$

We here have $\rho'\phi' = \mathbb{1}_{V_I}$. Indeed, for each $x \in I$, we have $\rho'_x\phi'_x = \beta\rho_x\phi_x\alpha = \beta\mu'\alpha = \mathbb{1}_{\Bbbk^d}$. As a consequence, the short exact sequences above split, and hence M has direct sum decompositions $V_I^d \oplus C \cong M\ (\cong K \oplus V_I^d)$ as $\Bbbk[I]$-modules. By the additivity of both $\varprojlim$ and $\varinjlim$, we have $\operatorname{rank}\mu_M = d \operatorname{rank}\mu_{V_I} + \operatorname{rank}\mu_C$. Here note that μ_{V_I} is given by the identity $\mathbb{1}_\Bbbk\colon \Bbbk \to \Bbbk$, thus $\operatorname{rank}\mu_{V_I} = 1$, which together with $\operatorname{rank}\mu_M = d$ shows that $\operatorname{rank}\mu_C = 0$. Therefore the assertion holds for $N := C$.

Note that N does not have direct summand isomorphic to V_I because $\operatorname{rank}\mu_N = 0$. Hence we have $d_M(V_I) = d = \operatorname{rank}\mu_M$. □

Remark 5.30. In the proof of [10, Lemma 3.1], the authors said that the decomposition $M(x) \cong P \oplus \operatorname{Coker}\phi_x$ is preserved by $M(p_{y,x})$, which is equivalent to the existence of the commutative diagram with exact rows with a unique morphism $f_{y,x}$ in (5.29). They continued to say that this fact establishes a direct sum $M \cong V_I^d \oplus C$. This assertion is obvious as vector spaces, but as $\Bbbk[I]$-modules it is not clear. This fact was not proved in the paper. Namely, the missing part is to show that the exact sequence in (5.30) on the left splits over $\Bbbk[I]$. For this, we need one more exact sequence in (5.30) on the right that serves us the necessary retraction ρ' for ϕ'.

6. Examples

Although the interval rank invariant of a persistence module M with respect to a compression system ξ captures more information than the rank invariant, it can still not retrieve all the information contained in M in general. Namely, it is possible to construct ξ and two objects $M, N \in \operatorname{mod} A$ not isomorphic to each other such that $\delta^{\xi}_{M}(I) = \delta^{\xi}_{N}(I)$ for all $I \in \mathbb{I}$. We now give such examples.

Example 6.1. (1) Define a quiver Q and its representations $M(\theta)$ by

$$Q := \begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \\ 3 & 4 \arrow[l] \arrow[u] \end{tikzcd}, \quad M(\theta) := \begin{tikzcd} \mathbb{R} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \arrow[l, "1"] \arrow[u, "\theta"'] \end{tikzcd}$$

for $\theta \in \mathbb{R}\backslash\{0,1\}$. We take $\xi := \mathrm{tot}$ (see Example 3.3). Let $\theta_1, \theta_2 \in \mathbb{R}\backslash\{0,1\}$ such that $\theta_1 \neq \theta_2$. Then $M(\theta_1)$ and $M(\theta_2)$ are clearly not isomorphic to each other but they have the same interval replacement. One can compute the interval replacement of $M(\theta)$ for $\theta = \theta_1, \theta_2$ by using Remark 3.21:

TABLE 1. Computation of $\delta^{\xi}_{M(\theta)}(I)$ for $\theta \in \mathbb{R} \setminus \{0,1\}$.

Interval I	I**-rank** $\operatorname{rank}^{\xi}_{I} M(\theta)$	**Signed interval multiplicity** $\delta^{\xi}_{M(\theta)}(I)$
$\{1,2,3,4\}$	0	0
$\{1,2,3\}$	1	1
$\{1,2,4\}$	1	1
$\{1,3,4\}$	1	1
$\{2,3,4\}$	1	1
$\{1,2\}$	1	-1
$\{1,3\}$	1	-1
$\{2,4\}$	1	-1
$\{3,4\}$	1	-1
$\{1\}$	1	0
$\{2\}$	1	0
$\{3\}$	1	0
$\{4\}$	1	0

(2) Define a quiver Q_2 and its representations $N(\theta)$ by

$$Q_2 := \begin{tikzcd} & 7 \arrow[rr] & & 8 \\ 3 \arrow[ur] \arrow[rr] & & 4 \arrow[ur] & \\ & 5 \arrow[uu] \arrow[rr] & & 6 \arrow[uu] \\ 1 \arrow[uu] \arrow[ur] \arrow[rr] & & 2 \arrow[uu] \arrow[ur] & \end{tikzcd}, \quad N(\theta) := \begin{tikzcd} & \mathbb{R} \arrow[rr] & & 0 \\ \mathbb{R} \arrow[ur, "1"] \arrow[rr, "1"] & & \mathbb{R} \arrow[ur] & \\ & \mathbb{R} \arrow[uu, "1"] \arrow[rr, "1"] & & \mathbb{R} \arrow[uu] \\ 0 \arrow[uu] \arrow[ur] \arrow[rr] & & \mathbb{R} \arrow[uu, "1"] \arrow[ur, "\theta"'] & \end{tikzcd}$$

for $\theta \in \mathbb{R} \setminus \{0,1\}$.

(3) Define a quiver Q_3 and its representations $L(\theta)$ by

$$Q_3 := \begin{tikzcd} 1 & 2 \arrow[l] \arrow[r] & 3 \\ 4 \arrow[u] \arrow[d] & 5 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & 6 \arrow[u] \arrow[d] \\ 7 & 8 \arrow[l] \arrow[r] & 9 \end{tikzcd}, \quad L(\theta) := \begin{tikzcd} \mathbb{R} & \mathbb{R} \arrow[l, "1"'] \arrow[r, "1"] & \mathbb{R} \\ \mathbb{R} \arrow[u, "1"] \arrow[d, "1"'] & 0 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathbb{R} \arrow[u, "1"'] \arrow[d, "1"] \\ \mathbb{R} & \mathbb{R} \arrow[l, "1"] \arrow[r, "\theta"'] & \mathbb{R} \end{tikzcd}$$

for $\theta \in \mathbb{R} \setminus \{0, 1\}$.

We give another example satisfying commutativity relations non-trivially that shows the incompleteness of the interval rank invariant with respect to the compression system tot.

Example 6.2. Let $\lambda \in \Bbbk$ and M_λ be the following representation of $\mathbf{P} := G_{5,2}$:

$$\begin{tikzcd} \Bbbk \arrow[r, "\binom{1}{0}"] & \Bbbk^2 \arrow[r, "\mathbf{1}"] & \Bbbk^2 \arrow[r, "(\lambda,-1)"] & \Bbbk \arrow[r] & 0 \\ 0 \arrow[r] \arrow[u] & \Bbbk \arrow[r, "\binom{0}{1}"'] \arrow[u, "\binom{0}{1}"'] & \Bbbk^2 \arrow[r, "\mathbf{1}"'] \arrow[u, "\mathbf{1}"'] & \Bbbk^2 \arrow[r, "(1,-1)"'] \arrow[u, "(\lambda,-1)"'] & \Bbbk \arrow[u] \end{tikzcd}.$$

Then it is easy to see that the endomorphism algebra of M_λ is isomorphic to $\Bbbk$, and hence M_λ is indecomposable, and if $\lambda \neq \mu$ in $\Bbbk$, then $\mathrm{Hom}_{\mathbf{P}}(M_\lambda, M_\mu) = 0$. Thus $M_\lambda \cong M_\mu$ if and only if $\lambda = \mu$. Let $\lambda \neq \mu$ in $\Bbbk \setminus \{0, 1\}$. By Theorem 5.20 for $m, n \leq 2$, it is easy to check that for any interval $I \in \mathbb{I}$, $\mathrm{rank}_I^{\mathrm{tot}} M_\lambda = \mathrm{rank}_I^{\mathrm{tot}} M_\mu$.

The dimension vector of M_λ is taken from [18, A2. The frames of the tame concealed algebras] for $\tilde{E}_7$, and the representation M_λ is constructed by modifying a homogeneous representation of $\tilde{D}_4$ in [13, Ch.6 Tables].

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