LAGRANGIAN SUBMANIFOLDS SATISFYING MASLOV QUANTIZATION CONDITION

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Abstract. The main purpose of this paper is to construct compact Lagrangian submanifolds satisfying Maslov quantization condition in the cotangent bundle of the Cayley projective plane by making use of the explicit realization of its punctured cotangent bundle as a quadrics in the complex space $\mathbb{C}^{27}\backslash\{0\}$.

If the geodesic flow is completely integrable, then there are many Lagrangian submanifolds, which are tori. Our example is not a torus. For this purpose we explain Maslov class based on our earlier work of the Maslov index defined for arbitrary paths for the sake of the self-containedness, and based on this treatment of the Maslov index we determine the Lagrangian submanifolds satisfying Maslov quantization condition.

Contents

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1. Introduction

The role of Lagrangian submanifolds in the cotangent bundle is important in the classical mechanics from the point of quantization theory. In mathematics it is a topic of geometric quantization theory and Fourier integral operator theory in which compact Lagrangian submanifolds satisfing a condition called, "Maslov quantization condition" and conic Lagrangian submanifolds, are mutually related and play basic roles.

Also it is classical that if the geodesic flow is completely integrable, then there are Lagrangian submanifolds as the intersections of constant hypersurfaces defined by the maximal number of linearly independent first integrals. Such Lagrangian submanifolds are always tori by the Arnold's theorem.

As the image of the exterier derivative of a smooth function it is always a (exact) Lagrangian submanifold, however it will not be apparent of the existence of Lagrangian submanifolds included in the punctured cotangent bundle which are satisfying Maslov quantization condition. This condition can be seen as a final form of the N. Bohr's quantization condition.

Our main purpose is to present some class of compact Lagrangian submanifolds in the punctured cotangent bundle of the Cayley projective plane satisfying Maslov quantization condition.

In general, there are several method to construct Lagrangian submanifolds, such as using momentum maps under a group action, based on the complete integrability of the geodesic flow or using a functorial property of Lagrangian submanifolds under (Riemannian) submersions, etc.

In our earlier paper [8], in 4.7 we discussed a behavior of Lagrangian submanifolds in the cotangent bundle under Riemannian submersions (see also [22], [13], [4] for relating properties). So, in the cases of projective spaces other than the Calyley projective plane, the construction of Lagrangian submanifolds are reduced to the construction of them in the sphere cases.

The main arguments in this paper is to solve the defining equations of the puncture cotangent bundle of the Cayley projective plane realized in the space $\mathbb{C}^{27}\backslash\{0\}$ as a quadrics and find explicit solutions which are giving Lagrangian submanifolds. This realization was given in [6]. Here we determine Lagrangian submanifolds satisfying Maslov quantization condition based on the treatment of Maslov indeces explained in §2 ([5], [7], [20]).

The paper is organized as follows.

In §2, we recall the definition of the Maslov index for arbitrary paths and define Maslov class for a pair of Lagrangian subbundle in a symplectic vector bundle. As a special case it is defined for a Lagrangian submanifold in cotangent bundles.

In §3, we describe a property of the Maslov index of Lagrangian submanifolds in a cotangent bundle.

We recall here the Maslov quantization condition and state the Eigenvalue Theorem which guarantees the existence of a particular series of eigenvalues of the Laplacian under the assumption of the Lagrangian submanifold satisfying Maslov quantization condition.

We also recall typical examples of Lagrangian submanifolds relating with particular 1-forms which imply some class of Lagrangian submanifold for nilmanifolds and a case of contact manifolds.

In §4, we show a type of examples of Lagrangian submanifolds in the cotangent bundle of the sphere, which are not tori, and determine the Maslov class explicitly following our definition of Maslov index for arbitrary paths. The method here is elementary and gives a guide for the case of the Cayley projective plane.

In §5, we present Lagrangian submanifolds in the Cayley projective plane, which are not tori and determine the cases satisfying Maslov quantization condition.

All the examples of Lagrangian submanifolds in this paper are not exact (see for example [21] on the exact Lagrangian submanifolds).

2. Maslov index and Maslov class

We recall a definition of the Maslov class for symplectic vector bundles with two Lagrangian subbundles based on the Maslov index defined for arbitrary paths. The description given here is a summary given in the Appendix of [8] (see also [11], [20], [7], [8], [5]).

2.1. **Maslov index for paths.** We consider $\mathbb{C}^n$ as a typical symplectic vector space with the anti-symmetric and non-degenerate bilinear form $\omega^{(n)}(z,w)$

$$\omega^{(n)}(z,w) := \mathrm{Im}\left(\sum z_i\overline{w}_i\right) = \sum x_{n+i}y_i - x_iy_{n+i},$$

where $z = (z_1,\dots,z_n) = (x_1,x_2,\dots,x_n\,;x_{n+1},\dots,x_{2n})$, $z_i = x_i + x_{n+i}\sqrt{-1}$ and $w = (w_1,\dots,w_n) = (y_1,y_2,\dots,y_n\,;y_{n+1},\dots,y_{2n})$, $w_i = y_i + y_{n+i}\sqrt{-1}$.

For h a real subspace in $\mathbb{C}^n$, we denote by h° the (real) subspace defined by

$$h^\circ = \{z\in\mathbb{C}^n \mid \omega^{(n)}(z,v) = 0 \text{ for any } v\in h\}.$$

So, a subspace h is called isotropic if $h\subset h^\circ$ and h is a Lagrangian subspace if $h = h^\circ$.

The subspaces

$$\lambda_{\mathrm{Re}} := \left\{\ (x_1,\dots,x_n\,;0,\dots,0)\right\}$$

and

$$\lambda_{\mathrm{Im}} := \left\{\ (0,\dots,0\,;x_{n+1},\dots,x_{2n})\ \right\}$$

are typical Lagrangian subspaces and $\mathbb{C}^n\cong\mathbb{R}^n\oplus\mathbb{R}^n = \lambda_{\mathrm{Re}}\oplus\lambda_{\mathrm{Im}}$. We understand that the Euclidean inner product in $\mathbb{C}^n$ is the real part $< z,w >^E := \mathrm{Re}(\sum z_i\overline{w}_i) = \sum_{i=1}^{2n} x_iy_i$ and the Hermitian inner product on $\mathbb{C}^n$ is the sum $< z,w >^H = < z,w >^E + \sqrt{-1}\omega^n(z,w)$.

We denote the space of all Lagrangian subspaces in $\mathbb{C}^n$ by $\Lambda(n)$ (called the Lagrangian-Grassmanian) , which, as is well known, is isomorphic to the quotient space $U(n)/O(n)$ and denote the projection map by

$$\pi_\Lambda : U(n) \to \Lambda(n),\ U(n) \ni U \longmapsto U(\lambda_{\mathrm{Im}}).$$

Let $\lambda \in \Lambda(n)$ and denote by $\mathcal{P}_\lambda$ the orthogonal projection operator $\mathcal{P}_\lambda : \mathbb{C}^n \to \lambda \subset \mathbb{C}^n$. Then the operator $\tau_\lambda := 2\mathcal{P}_\lambda - Id$ is an involution with λ as the 1-eigenspace and the orthogonal complement $\lambda^\perp$ is the -1-eigenspace. Also for $U \in U(n)$ let's denote the operator $\tau_\lambda \circ U^* \circ \tau_\lambda$ by $\theta_\lambda(U)$. In particular, if $\lambda = \lambda_{\mathrm{Re}}$ and we express the matrix $U = \left(u_{ij}\right)$ with the standard orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbb{C}^n$ (in the sense of Hermitian inner product), then $\theta_{\lambda_{\mathrm{Re}}}(U) = \overline{U}$, that is $\overline{U} = \left(\overline{u}_{ij}\right)$.

For each $\lambda \in \Lambda(n)$, let $\mathcal{S}_\lambda : \Lambda(n) \to U(n)$ be a map, called *Souriou map*, defined by

$$\mathcal{S}_\lambda : \Lambda(n) \ni \mu \longmapsto U \circ \theta_\lambda(U) \in U(n),$$

where $\mu = U(\lambda_{\mathrm{Im}})$. In fact this does not depend on the operator U for $\mu = U(\lambda^\perp)$, since we have an expression

$$\mathcal{S}_\lambda(\mu) = -\tau_\mu \circ \tau_\lambda. \tag{2.1}$$

Let $U_{\mathfrak{M}}$ be a subset in $U(n)$ defined by

$$U_{\mathfrak{M}} = \{\, U \in U(n) \mid U + Id \text{ is not invertible}\,\}. \tag{2.2}$$

Then we call the subset defined by

$$\mathcal{M}_\lambda := \mathcal{S}_\lambda{}^{-1}(U_{\mathfrak{M}}) = \left\{\mu \in \Lambda(n) \;\middle|\; \mu \bigcap \lambda \neq \{0\}\right\} \tag{2.3}$$

the "*Maslov cycle*" passing through a Lagrangian subspace $\lambda \in \Lambda(n)$.

Let $\gamma : [0,\, 1] \to \Lambda(n)$ be a continuous curve. We define an intersection number of γ and $\mathcal{M}_\lambda$ in the following way (cf. [7]):

We can find a partition $\{0 = t_0 < t_1 < t_2 < \cdots < t_\ell = 1\}$ of the interval $[0,\, 1]$ and a set of small positive numbers $\{\, 0 < \varepsilon_j \ll 1\,\}_{j=0}^{\ell}$ satisfying the condition that for $j = 0, \ldots, \ell - 1$

$$\begin{cases} \text{the values } e^{\sqrt{-1}(\pi \pm \varepsilon_j)} \text{ are not eigenvalues of the operators} \\ \mathcal{S}_\lambda(\gamma(t)) \text{ for } t \in [t_j\,,\, t_{j+1}]. \end{cases} \tag{2.4}$$

This condition means that the eigenvalues of the operators $\{\mathcal{S}_\lambda(\gamma(t))\}_{t_j \le t \le t_{j+1}}$ included in the "*arc*" $\{e^{\sqrt{-1}s} \mid \pi - \varepsilon_j < s < \pi + \varepsilon_j\}$ stay there when the parameter $t \in [t_j\,,\, t_{j+1}]$. Then we define an integer $Mas(\{\gamma\}, \lambda)$, and call it Maslov index for path $\{\gamma\}$ with respect to the Maslov cycle $\mathcal{M}_\lambda$ by

Definition 2.1.

$$
\begin{aligned}
Mas(\{\gamma\},\,\lambda) : = \sum_{j=0}^{\ell-1} \; & \textit{the number of the eigenvalues of the operator } \mathcal{S}_\lambda(\gamma(t_{j+1})) \\
& \textit{in the arc } \left\{ e^{\sqrt{-1}s} \mid \pi \leq s < \pi + \varepsilon_j \right\} \\
& - \; \textit{the number of the eigenvalues of the operator } \mathcal{S}_\lambda(\gamma(t_j)) \\
& \textit{in the arc } \left\{ e^{\sqrt{-1}s} \mid \pi \leq s < \pi + \varepsilon_j \right\}.
\end{aligned}
$$

Then, this integer satisfies

M-ind(1) :The integer $Mas(\{\gamma\},\,\lambda)$ does not depend on the partition $\{t_j\}$ of the interval $[0,\,1]$ and the small positive numbers $\{\varepsilon_j\}$ satisfying the condition (2.4),

M-ind(2) :It is a homotopy invariant for the paths with the fixed end points,

M-ind(3) :It satisfies the additivity under catenations of paths.

We express the Maslov index $Mas(\{\gamma\},\,\lambda)$ for a special case of a continuously differentiable curve $\{\gamma(t)\}_{|t|\ll\epsilon}$ in $\Lambda(n)$ defined on a small interval $(-\epsilon,\,\epsilon)$ in terms of the number of eigenvalues of the derivative of the curve.

We may assume $\lambda = \lambda_{\mathrm{Re}}$ and for small $|t| < \epsilon$, $\gamma(t) \cap \lambda_{\mathrm{Im}} = \{0\}$. By this assmption there is a differential family of symmetric operator $A_t : \mathbb{R}^n = \lambda_{\mathrm{Re}} \to \mathbb{R}^n = \lambda_{\mathrm{Im}}$ such that

$$\gamma(t) = \{x \oplus A_t(x) \mid x \in \lambda_{\mathrm{Re}}\} := G(A_t).$$

Then,

Proposition 2.2. [20] *Assume* $\frac{d}{dt} A_{t_{|t=0}}$ *is non-singular on* $\operatorname{Ker} A_0$*, then*

(i) $Mas(\{\gamma(t)\}_{0\leq t\leq\epsilon'},\,\lambda)$

$$
\begin{aligned}
&= \textit{the number of the positive eigenvalues of } A_{\epsilon'} \\
&\qquad - \textit{number of the positive eigenvalues of } A_0 \\
&= \textit{the number of positive eigenvalues of } \frac{d}{dt} A_0{}_{|t=0} := \dot{A}_0 \textit{ on } \operatorname{Ker}(A_0)
\end{aligned}
$$

(ii) $Mas(\{\gamma(t)\}_{-\epsilon''\leq t\leq\epsilon'},\lambda)$

$$
\begin{aligned}
&= \textit{the number of the positive eigenvalues of } A_{\epsilon'} \\
&\qquad - \textit{the number of the positive eigenvalues of } A_{\epsilon''} \\
&= \textit{sign}\, \dot{A}_0 \textit{ on } \operatorname{Ker}(A_0)
\end{aligned}
$$

where $0 < \epsilon', \epsilon'' \ll \epsilon$.

This is proved based on the following Lemmas 2.3 and 2.4 and (ii) is a consequence of (i) by the additivity under catenation. The condition $0 < \epsilon', \epsilon'' \ll \epsilon$ requires that the point t at which $\gamma(t) \cap \lambda_{\rm Re} \neq \{0\}$ is only $t = 0$ in the interval $[-\epsilon'', \epsilon']$. The non-singularity assumption for $\dot{A}_0$ in Proposition above guarantees this property. Hence if $t = 0$ is the only such a point that $\gamma(t) \cap \lambda_{\rm Re} \neq \{0\}$ in the interval $[-\epsilon'', \epsilon']$, then the first equality in (i) (also the first equality in (ii)) in the above Proposition holds.

Lemma 2.3. *Let $\{A_t\}_{|t| \ll \epsilon}$ be a continuously differentialble family of $k \times k$ symmetric matrices defined for small t such that*

$$\text{the matrix } \dot{A}_0 \text{ is non-singular on } \mathrm{Ker}\,(A_0). \tag{2.5}$$

Then for sufficiently small $0 < t \ll \epsilon$, "the number of the positive eigenvalues of A_t" coincides with
"the number of the positive eigenvalues of $\dot{A}_0$ on Ker (A_0)*" + "the number of the positive eigenvalues of A_0 on the orthogonal complement of* Ker (A_0)*" and*
"the number of the negative eigenvalues of A_t" coincides with
"the number of the negative eigenvalues of $\dot{A}_0$ on Ker (A_0) + *"the number of the negative eigenvalues of A_0 on the orthogonal complement of* Ker (A_0)*".*
Also for sufficiently small $0 > t \gg -\epsilon$, it holds similar statements.

Under the same assumption in Propoistion 2.2 we have

Lemma 2.4. *For Lagrangian subspace $\gamma = \{x \oplus A(x) \mid x \in \lambda_{\rm Re}\}$*

$$\mathcal{S}_\lambda(\gamma) = (Id + A^2)^{-1}\big(A^2 - Id) - 2\sqrt{-1}A\big).$$

The proof of this is given by deternining the Souriou map $\mathcal{S}_\lambda(\mu)$ using the formula (2.1) for the case $\lambda = \lambda_{\rm Re}$ and $\mu = \{x \oplus A(x) \mid x \in \lambda_{\rm Re}\}$. Then we can see easily the behaviour of the eigenvalues of the unitary matrices $\mathcal{S}_{\lambda_{\rm Re}}(\gamma(t))$ from the form above, which gives the proof of Proposition 2.2.

Remark 1. *The definition of Maslov index for arbitrary paths given in* [20] *is described as the sum of signature of the matrices $\dot{A}_{t_i}$ under the assupmtion in Proposition 2.2 and includes artificial modification terms at the end points $t = 0, 1$ which are not necessary. However this formula is useful to calculate the Maslov index around the points satisfying condition in Proposition 2.2 for many of concrete cases. In* [11] *it was noticed for the first time that the Maslov index can be defined for any path without any modification term, and in* [5] *and* [7] *it was given based on the arguments in* [19] *including infinite dimensional symplectic Hilbert space cases.*

2.2. **Maslov class.** Let $\Psi : E \to X$ be a symplectic vector bundle over a space X with $\dim \Psi^{-1}(x) = \dim E_x = n$, where we assume X is path connected, locally simply path connected. We denote the anti-symmetric non-degenerate bilinear form on each fiber of E by ω^E, then we can install an inner product $< \cdot, \cdot >$ on

E "*compatible*" with the symplectic structure ω^E in such a sense that there exists an almost complex structure $J : E \to E$, $J^2 = -Id$, $\Psi \circ J = \Psi$ such that

$$\omega^E(u, v) =< J(u), v >, \ < J(u), J(v) >=< u, v >, \ u, \ v \in E_x.$$

We assume that there exist two Lagrangian sub-bundles λ and μ in E, that is the fiber at each point x is a Lagrangian subspace in E_x.

Let $\{\gamma(t)\}$ be a continuous curve, $\gamma : [0, 1] \to X$. We divide it into small segments $\left\{\{\gamma(t)\}_{t_i \le t \le t_{i+1}}\right\}$ in such a way that there exist a finite open covering $\{O_i\}_i$ around the curve $\{\gamma(t)\}$ and $\gamma([t_i, t_{i+1}]) \subset O_i$, such that the vector bundle E has local trivializations

$$\psi_i : O_i \times \mathbb{C}^n \cong \Psi^{-1}(O_i)$$

satisfying the property that by this trivialization for each $x \in O_i$, $(x, \lambda_{\mathrm{Im}})$ is mapped to $\psi_i(x, \lambda_{\mathrm{Im}}) = \lambda_x = \Psi^{-1}(x) \bigcap \lambda$. Then we can assign an integer $Mas_{(\lambda,\mu)}(\{\gamma(t)\})$ for an arbitrary continuous path $\gamma : [0, 1] \to X$ as the sum

$$Mas_{(\lambda,\mu)}(\{\gamma(t)\}) = \sum_i Mas(\{{\psi_i}^{-1}(\mu_{\gamma(t)})\}_{t_i \le t \le t_{i+1}}, \lambda_{\mathrm{Im}}). \tag{2.6}$$

This quantity can be defined for all paths and has the properties:

$$\begin{cases} \mathcal{M}(1) : \text{The definition does not depend on the partition of the interval } [0,\ 1], \\ \qquad\quad \text{nor the local trivializations of the symplectic vector bundle } E \\ \qquad\quad \text{satisfying conditions above nor does not depend on the inner} \\ \qquad\quad \text{product installed which satisfies the ``\textit{compatibility properties}''}, \\ \mathcal{M}(2) : \text{Homotopy invariance for paths with fixed end points,} \\ \mathcal{M}(3) : \text{Additivity under catenations.} \end{cases}$$

Hence, let $\pi : \tilde{X} \to X$ be the universal covering space of X consisting of homotopy classes of paths starting from a fixed initial point $x_0 \in X$. Then we can define a function

$$Mas_{(\lambda,\mu)} : \tilde{X} \longrightarrow \mathbb{Z}, \ \tilde{X} \ni \{\gamma\} \longmapsto Mas_{(\lambda,\mu)}(\{\gamma(t)\}). \tag{2.7}$$

Especially its restriction to the fiber $\pi^{-1}(x_0)$ defines a homomorphism:

$$Mas_{(\lambda,\mu)} : \pi^{-1}(x_0) \cong \pi_1(X) \to \mathbb{Z}.$$

Consequently, we have a cohomology class $\in H^1(X, \mathbb{Z})$, which we denote by $\mathfrak{m}_{(\lambda,\mu)}$ and is called the "*Maslov class*" of the pair of Lagrangian subbundles λ and μ. Note that $\mathfrak{m}_{(\lambda,\mu)} = -\mathfrak{m}_{(\mu,\lambda)}$.

Proposition 2.5. *It will be apparent if the intersection $\lambda \cap \mu$ on a curve $\{\gamma(t)\}$ is trivial bundle, then $Mas_{(\lambda,\mu)}(\{\gamma\}) = 0$*

Definition 2.6. *Let $\chi_{\pi/2}$ be the representation $\chi_{\pi/2} : \mathbb{Z} \to U(1)$, $n \mapsto e^{\pi/2\sqrt{-1}n}$ and we define an associated complex line bundle $L_{\mathfrak{m}_{(\lambda,\mu)}}$ on X to the principal bundle $\pi : \tilde{X} \to X$ through the representation $\pi_1(X) \overset{Mas_{(\lambda,\mu)}}{\longrightarrow} \mathbb{Z} \overset{\chi_{\pi/2}}{\longrightarrow} U(1)$. It is called Maslov line bundle.*

Let E be symplectic a vector bundle on a space X with two Lagrangian subbundle F and G. Let $\mathfrak{f} : Y \to X$ be a continuous map, then we can define the symplectic vector bundle $\mathfrak{f}^*(E)$ on Y with two Lagrangian subbundles $\mathfrak{f}^*(F)$ and $\mathfrak{f}^*(G)$. Let $\tilde{\mathfrak{f}} : \tilde{Y} \to \tilde{X}$ be the map between their universal covering spaces $\tilde{Y}$ and $\tilde{X}$. Then

$$Mas_{(F,G)} \circ \tilde{\mathfrak{f}} = Mas_{(\mathfrak{f}^*(F),\mathfrak{f}^*(G))}. \tag{2.8}$$

3. Lagrangian submanifolds

We treat Lagrangian submanifolds in the cotangent bundle, Maslov quantization condition and an existence crieterion which also will be useful to see the cases of nilmanifolds.

3.1. **Lagrangian submanifolds in cotangent bundles.** We consider a typical case of symplectic vector bundle with two Lagrangian subbundles. Namely, let L be a Lagrangian submanifold in a cotangent bundle $T^*(X)$. Then the restriction of the tangent bundle $T(T^*(X))$ to L is a symplectic vector bundle together with two Lagrangian subbundles, the tangent bundle $T(L)$ of L itself, and the restriction of $\mathrm{Ker}\,(d\pi^X)$ to L, the vertical subbundle with respect to the projection map $\pi^X : T^*(X) \to X$, which we denote by $\mathcal{V}^L$.

Hence we have a cohomology class $\mathfrak{m}_{\left(T(L),\,\mathrm{Ker}\,d\pi^X\right)}$ as a homomorphism

$$\mathfrak{m}_{\left(T(L),\,\mathrm{Ker}\,d\pi^X\right)} : \pi_1(L) \to \mathbb{Z},$$

which we will denote simply by $\mathfrak{m}_L$.

Proposition 3.1. *Let L be a compact Lagrangian submanifold in $T^*_0(X)$. Then for any positive real number $c_0 > 0$ and any closed curve $\{\gamma\}$ in L,*

$$< \mathfrak{m}_L,\, \gamma >=< \mathfrak{m}_{c_0\cdot L},\, c_0 \cdot \gamma > . \tag{3.1}$$

Proof. Since the Maslov index $< \mathfrak{m}_L, \gamma >$ for a path $\{\gamma\}$ is defined based on the data

$$\left\{\dim\left(T_{\gamma(t)}(L) \bigcap \left(\mathrm{Ker}\,d\pi^X\right)_{\gamma(t)}\right)\right\}_{t\in[0,\,1]}$$

and it holds that

$$\dim\left(T_{\gamma(t)}(\lambda) \bigcap \left(\mathrm{Ker}\,d\pi^X\right)_{\gamma(t)}\right) = \dim\left(T_{c_0\cdot\gamma(t)}(c_0\cdot\lambda) \bigcap \left(\mathrm{Ker}\,d\pi^X\right)_{c_0\cdot\gamma(t)}\right)$$

for any t, since the dilation $c_0\cdot : T^*_0(X) \longrightarrow T^*_0(X)$, $(x;\,\xi) \longmapsto c_0\cdot(x;\,\xi) = (x;\,c_0\cdot\xi)$, $c_0 > 0$, is a diffeomorphism. Hence (3.1) holds. □

Let M and N be two Riemannian manifolds and assume there exists a submersion $\varphi : M \to N$. Then we have the injective bundle map on M:

$$d\varphi^* : \varphi^*(T^*(N)) \to T^*(M),$$

and the principal symbol $\sigma_{\Delta^N} : T^*(N) \to \mathbb{R}$ can be seen as a function on $\varphi^*(T^*(N))$. If this submersion φ is Riemannian, then we have a relation

$$\sigma_{\Delta^M} \circ d\varphi^* = \sigma_{\Delta^N}.$$

In [8] we explained a behavior of Lagrangian submanifolds under Riemannian submersion. Here we state a Proposition which will cover the construction of Lagrangian submanifolds for the complex and quaternion projective spaces from those of spheres.

Proposition 3.2. *Let $L \subset T_0^*(N)$ be a Lagrangian submanifold, then $\varphi^*(L) \subset T_0^*(M)$ is also a Lagrangian submanifold.*

Hence the construction of Lagrangian submanifolds in $T_0^*(N)$ reduces to find a Lagrangian submanifold in $T_0^*(M)$ under suitable conditions. For example, if $\varphi : M \to N$ is a fiber bundle, then a Lagrangian submanifold in $T_0^*(M)$ invariant under the action of the structure group can be descented to $T_0^*(N)$.

3.2. **Maslov quantization condition and the Eigenvalue Theorem.** Let L be a Lagrangian submanifold in the cotangent bundle $T^*(X)$, where we always assume that X is a closed oriented Riemannian manifold without boundary. Then the Maslov quantization condition to a Lagrangian submanifold L is stated in Mas[1] $\sim$ Mas[3]:

$$(3.2)\quad \begin{cases} \text{Mas[1]: } \sigma_{\Delta^X}\big|_L \equiv E_L = \text{constant } (>0) \text{ on } L,\ \sigma_{\Delta^X} \text{ is the principal} \\ \qquad\qquad \text{symbol of the Laplacian } \Delta^X, \\ \text{Mas[2]: for any (smooth) closed curve } \{\gamma\} \text{ in } L, \\ \qquad\qquad \dfrac{1}{2\pi}\displaystyle\int_\gamma \theta^X - \frac{1}{4} < \mathfrak{m}_L, \gamma > \in \mathbb{Z}, \\ \qquad\qquad \text{where } \mathfrak{m}_L \text{ is a cohomology class} \in H^1(L,\mathbb{Z}), \text{ called Maslov} \\ \qquad\qquad \text{class of } L \text{ which was explained in §2 precisely,} \\ \text{Mas[3]: there exists a positive } \textit{invariant} \text{ measure } d\mu_L \text{ on } L, \text{ that} \\ \qquad\qquad \text{is the measure } d\mu_L \text{ is a nowhere vanishing highest degree} \\ \qquad\qquad \text{differential form invariant under the geodesic flow action} \\ \qquad\qquad \text{(we will treat only orientable } L). \end{cases}$$

Note that by the condition Mas[1], L itself is invariant under the geodesic flow action.

In the paper [23] it was proved that an existence theorem of eigenvalues of the Laplacian under the existence of a Lagrangian submanifold in the puncture cotangent bundle $T_0^*(X)$ satisfying all three conditions Mas[1] $\sim$ Mas[3] (we cite it as Eigenvalue Theorem):

Theorem 3.3. *We assume that there is a compact Lagrangian submanifold $L \subset T_0^*(X)$ satisfying the three conditions* Mas[1] $\sim$ Mas[3]. *Let d_L be the smallest integer in $\{1, 2, 4\}$ such that $\frac{d_L}{2\pi} \cdot L$ is "integral", that is the cohomology class of* *the restriction of the canonical one form to L is in the cohomology group* $H^1(L,\mathbb{Z})$*. Then there exists a sequence* $\{\lambda_k\}_{k=0}^{\infty}$ *of eigenvalues of the Laplacian* Δ^X *such that*

$$\left|\lambda_k - E_L(d_L k+1)^2\right| \leq C : \text{ bounded.}$$

The proof is given by constructing a *Fourier integral operator* $A : L_2(U(1)) \to L_2(X)$ ([24], [14],[3], [12], [23],[8]) (*quasi-*) commuting with the Laplacians on $U(1)$ and X. We noted in [8] that the theorem is also valid for the sub-Laplacian with an additional assumption that the Lagrangian submanifold does not intersect with the characteristic variety for the sub-Laplacian. Especially if the sub-Laplacian is compatible with the Laplacian through Riemannian submersion, then the existing eigenvalues correspond each other.

One main step for constructing the Fourier integral operator A is to construct a conic Lagrangian submanifold $\Lambda \subset T_0^*(U(1)) \times T_0^*(X)$ from the compact Lagrangian submanifold $L \subset T_0^*(X)$ assumed in the Eigenvalue Theorem.

In fact, the Malsov quatization condition implies that there is a constant $c_0 > 0$ such that $c_0 \cdot L$ is an integral Lagrangian submanifold in $T_0^*(X)$, that is the de Rham cohomology class $[c_0 \cdot \theta^X{}_{|L}] \in H^1(c_0 \cdot L, \mathbb{Z}) \cong H^1(L,\mathbb{Z})$, the restriction of the canonical one form θ^X ($d\theta^X = 0$ on L) to $c_0 \cdot L$ is integral. Now we may replace $c_0 \cdot L$ by just L. Then the local solutions $\{f_i\}$ of the equation $df_i = \theta^X$ defines an integral 1-cochain $c_{ij} := f_i - f_j$ on L. Then it defines a map $\varphi : L \longrightarrow U(1)$, $L \ni (x,\xi) \longmapsto e^{2\pi\sqrt{-1}f_i}$, which is a submersion, since df_i at $(x,\xi) \in L$ is equal to $\xi \neq 0$. Let $\Lambda = \{(x, \tau\cdot\xi, \overline{\varphi(x,\xi)}, \tau) \mid (x,\xi) \in L, \tau > 0\}$. Then this is a conic Lagrangian submanifold in $T_0^*(X) \times T_0^*(U(1))$ and determines the phase function of A(micro-locally). Such construction of a conic Lagrangian submanifold in $T_0^*(U(1)) \times T_0^*(X)$ from a compact Lagrangian submanifold in $T_0^*(X)$ connects Maslov theory of canonical operator and Hörmander theory of Fourier integral operators ([25], [3]). In this context, it will be interesting to find many concrete examples of Lagrangian submanifolds in the cotangent bundle of various famous (named) manifolds.

Remark 2. *This Lagrangian submanifold* Λ *is not a form of the normal bundle of a "incidence relation", which appears in the theory of Radon transformation* ([2], [9]).

3.3. **Closed 1-form and Lagrangian submanifold.** Let X be a closed manifold and $\varphi : X \to U(1)$ a submersion to $U(1) = \{e^{\sqrt{-1}s} \mid s \in \mathbb{R}\} \cong S^1$. Then the set of local solutions $\{f_i\}$, where each real valued function f_i is defined on an open set U_i and satisfying the equation $e^{2\pi\sqrt{-1}f_i} = \varphi$, defines an one-Čeck cochain $\{c_{ji} = f_j - f_i\}$ of the $\mathbb{Z}$-valued constant sheaf $\mathbb{Z}_X$ on X and globally defines a closed one-form η ($:= df_j$ on U_j, which also coincides with $\varphi^*(ds)$). The cohomology class $[\eta] \in H^1_{dR}(X)$ is integral.

Conversely, let $\alpha \in H^1_{dR}(X)$ and assume

(1) α is in $H^1(X,\mathbb{Z})$, that is α is an integral class, and

(2) there is a nowhere vanishing closed one-form η representating the class α.

Then, the set of local solutions $\{f_i\}$, $df_i = \eta$ on U_i where $\{U_i\}$ is an open covering of X, defines a submersion $\varphi : X \to U(1)$, $\varphi := e^{2\pi\sqrt{-1}\,f_i}$ on U_i, since by assumptions $f_j - f_i \in \mathbb{Z}$ and df_i does not vanish at any point.

Hence

Proposition 3.4. *Let X be a closed manifold. Then there is a submersion $\varphi : X \to U(1)$, if and only if there is a closed one-form η such that its cohomology class $[\eta]$ is integral and the one-form η never vanish.*

So, we assume that there is a closed one-form η satisfying the above conditions (1) and (2).

Then the image $\eta(X)$, $\eta : X \to T^*(X)$, is included in $T_0^*(X)$ and by the fact that $\eta^*(\theta^X) = \eta$, $\eta(X)$ is a Lagrangian submanifold and also coincides with the pul-back $\varphi^*(ds(S^1))$ of the Lagrangian submanifold $ds(S^1) = \{(e^{\sqrt{-1}s}, 1) \mid s \in \mathbb{R}\} \subset T^*(S^1) \cong S^1 \times \mathbb{R}$.

Moreover we see that the cohomology class $[\theta^X{}_{|\eta(X)}]$ of the restriction of the Liouville one-form to $\eta(X)$ is in $H^1(\eta(X), \mathbb{Z}) \cong H^1(X, \mathbb{Z}) \subset H^1_{dR}(X)$.

In this case, since the tangent bundle $T(\eta(X))$ is transversal to the vertical subbundle $\mathcal{V}^{\eta(X)}$ at all the points in $\eta(X)$ $\big(\mathcal{V}^{\eta(X)} = \operatorname{Ker}(d\varphi)_{|\eta(X)}\big)$, the Maslov class $\mathfrak{m}_{\eta(X)}$ is zero. So if the dimension $\dim H^1(X) = 1$, then a constant multiple $c_0\eta(X)$ satisfys the condition Mas[2].

It will be seen that the above case is a special case of Proposition 3.2. Here we take L as $U(1) \times c_0 \subset T^*(U(1)) \cong U(1) \times \mathbb{R}$.

3.4. **Nilmanifolds.** In this subsection we treat a typical example satisfying the equivalent condition explained in Proposition 3.4.

Let $\mathbf{N}$ be a simply connected nilpotent Lie group having a lattice Γ. Then by Nomizu theorem (cf. [18]), the de Rham cohomology group $H^*_{dR}(\Gamma\backslash\mathbf{N})$ of the compact nilmanifold $\Gamma\backslash\mathbf{N}$ is isomorphic to the cohomology group of the corresponding Lie algebra $\mathfrak{n}$ through the induced map from the natural inclusion map of the subcomplex consisting of left invariant differential forms on $\mathbf{N}$ to the Γ-left action invariant differential forms, i.e., the de Rham complex on the nilmanifold $\Gamma\backslash\mathbf{N}$. In particular, $H^1_{dR}(\Gamma\backslash\mathbf{N}) \cong \{\eta \in \mathfrak{n}^* \mid \eta([X,Y]) = 0,\ X,\ Y \in \mathfrak{n}\}$.

So by Malcev theorem ([17]), let $\{X_i\}$ be a linear basis of the Lie algebra $\mathfrak{n}$ such that the structure constants $\{c^k_{ij}\}$, $[X_i,\ X_j] = \sum c^k_{ij}X_k$ are all rational numbers, then $\{\exp\ X_i\}$ generates a lattice. Let $\{\eta_i\}$ be the dual basis of the space $\mathfrak{n}^*$ and assume $\eta_1([\mathfrak{n},\mathfrak{n}]) = 0$. Then the space $\eta_1(\mathbf{N}) = \{(g,\eta_1) \mid g \in \mathbf{N}\} \subset \mathbf{N} \times \mathfrak{n}^* \cong T^*(\mathbf{N})$ is a Lagrangian submanifold. In this case if we consider a left invariant Riemannian metric on $\mathbf{N}$, then the energy function is constant on $\eta_1(\mathbf{N})$ and the transformed Haar measure on $\eta_1(\mathbf{N})$ by the map $\eta_1 : \mathbf{N} \to T^*(\mathbf{N})$ is invariant under the geodesic flow.

3.5. **Contact manifold and Lagrangian submanifold.** Let (M,α) be a compact contact manifold with a contact form α ($\dim M = 2n+1$) and denote by

$\Sigma_\alpha = \{\, t\alpha \mid t > 0\,\} \subset T_0^*(M)$, the cone bundle on M which is isomorphic to $M \times \mathbb{R}_+$. Then throught this isomorphism it holds

$$(\omega^M{}_{|\Sigma_\alpha})^{n+1} = (n+1)t^n \cdot dt \wedge \alpha \wedge (d\alpha)^n. \tag{3.3}$$

Hence the cone Σ_α is a symplectic manifold with the symplectic form $\omega^M{}_{|\Sigma_\alpha}$, the restriction of the natural symplectic form ω^M of $T^*(M)$ (and vise versa).

We assume that

$[\mathcal{RP}]$: *the action generated by the Reeb vector field $\mathcal{R}$ reduces to the $U(1)$-free action on M.*

The vector field $\mathcal{R}$ is uniquley determined by the conditions that $d\alpha(\mathcal{R}, \bullet) \equiv 0$ and $\alpha(\mathcal{R}) \equiv 1$. We may assume that the period is "2π".

Under this assumption the orbit space becomes a symplectic manifold in a natural way. In fact, let $\pi_\alpha : M \to M/U(1) =: \mathcal{O}$ be the projection map to the space of orbits, which is a $U(1)$-principal bundle and together with the Darboux's theorem for contact form, for any point $q \in \mathcal{O}$ we can find a local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$ defined on a small neighborhood $\overline{V} \ni q$ such that on which we have a local trivialization

$$V := \overline{V} \times U(1) \cong {\pi_\alpha}^{-1}(\overline{V}),\ (x_1, \ldots, x_n, y_1, \ldots, y_n\,;\, e^{\sqrt{-1}s}) \in \mathbb{R}^{2n} \times U(1)$$

and the contact form α is expressed as $\alpha = ds + \sum x_i dy_i$. The Reeb vector field $\mathcal{R}$ is expressed as $\partial/\partial s$ in terms of this coordinates and the projection is $\pi_\alpha(x_1, \ldots, x_n, y_1, \ldots, y_n\,;\, e^{\sqrt{-1}s}) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_n)$. The differential $d\alpha = \sum dx_i \wedge dy_i$ is invariant under the structure group action (= action generated by the Reeb vector field). Hence it defines a symplectic structure $\omega^{\mathcal{O}}$ on the orbit space $\mathcal{O}$.

Let $({x_1}', \ldots, {x_n}', {y_1}', \ldots, {y_n}')$ be another Darboux coodinates defined on $\overline{V}\,' \ni q$ and on which we have a local trivialization $\overline{V}\,' \times U(1) \cong {\pi_\alpha}^{-1}(\overline{V}\,') := V\,'$, then on $V \cap V\,'$ we have

$$\sum x_i\, dy_i - \sum {x_i}'\, {dy_i}' = ds - ds\,'.$$

Since $e^{\sqrt{-1}s} = g \cdot e^{\sqrt{-1}s\,'}$ with a transition function $g : \overline{V} \cap \overline{V}\,' \to U(1)$, also $e^{\sqrt{-1}s\,'} = h \cdot e^{\sqrt{-1}s\,''}$ on $\overline{V}\,' \cap \overline{V}\,''$ and so on, it holds that

$$e^{\sqrt{-1}s}\, e^{-\sqrt{-1}s'}\, e^{\sqrt{-1}s''} \;\equiv\; 1$$

on the intersection $\overline{V} \cap \overline{V}\,' \cap \overline{V}\,''$. This implies that the symplectic form $\frac{\omega^{\mathcal{O}}}{2\pi}$ is integral, i.e., the cohomology class $\frac{[\omega^{\mathcal{O}}]}{2\pi} \in H^2_{dR}(M)$ is in the image of the natural map $\check{H}^2(M, \mathbb{Z}) \to H^2_{dR}(M)$.

In this case, the maximal non-integrable subbundle $\mathrm{Ker}\,(\alpha) = \{Z \in T(M) \mid \alpha(Z) = 0\}$ defines a connection to the principal bundle $\pi_\alpha : M \to \mathcal{O}$ and is bracket generating so that it defines a 2-step sub-Riemannian structure on M.

Now let L be a Lagrangian submanifold in $\mathcal{O}$. Then from the expression (3.3)

Proposition 3.5. *The submanifold $\alpha({\pi_\alpha}^{-1}(L))$ is a $U(1)$-action invariant Lagrangian submanifold in Σ_α, where we regard $\alpha : M \to \Sigma_\alpha \subset T_0^*(M)$. Conversely if $\Lambda \subset M$ is $U(1)$-action invariant and $\alpha(\Lambda)$ is a Lagrangian submanifold in Σ_α, then $\pi_\alpha(\Lambda)$ is a Lagrangian submanifold.*

Let Z be a compact symplectic manifold with an integral symplectic form $\frac{1}{2\pi}\omega^Z$, that is $\frac{1}{2\pi}[\omega^Z]$ is in the image of $H^2(Z,\mathbb{Z}) \subset H^2_{dR}(Z)$, then we can construct a compact contact manifold such that the action generated by the Reeb vector field satisfies the condition $[\mathcal{RP}]$ and come back to $\mathcal{O} \cong Z$. These are explained precisely in [16].

Next we put one more strong assumption on the contact manifold M together with the assumption $[\mathcal{RP}]$:

$[\mathcal{PF}]$: *There exists a closed Riemannian manifold X with a Riemannnian metric $g(\cdot,\cdot)$ and its dual inner product Q^g on $T^*(X)$. When we realize the cotangent sphere bundle $S^*(X)$ as a submanifold $S^*(X) \cong \{(x,\xi) \in T^*(X) \mid Q^g(\xi,\,\xi) = 1\}$ with the contact form ${\theta^X}_{|S^*(X)}$, then we assume that there exists an isomorphism $\mathcal{C} : S^*(X) \to M$ keeping the contact structures, i.e., $\mathcal{C}^*(\alpha) = {\theta^X}_{|S^*(X)}$.*

Hence under these two assumptions $[\mathcal{RP}]$ and $[\mathcal{PF}]$ we may restate Prorposition 3.5

Proposition 3.6. *Let L be a Lagrangian submanifold in $\mathcal{O}$, then ${\pi_\alpha}^{-1}(L)$ is a Lagrangian submanifold in $T_0^*(X)$.*

These two assumptions $[\mathcal{RP}]$ and $[\mathcal{PF}]$ says that the manifold X must be a $SC_{2\pi}$-manifold and at the moment we may mention only spheres (including Zoll surfaces) and projective spaces as such manifolds.

4. Sphere case

In this section we consider the Lagrangian submanifolds for spheres. For this purpose we base on the defining equation of the (punctured) cotangent bundle $T_0^*(S^n)$ as a quadrics in $\mathbb{C}^{n+1}$. This gives us a similar method to deal with the Cayley projective plane case.

We can realize the cotangent bundle of the sphere $S^n \subset \mathbb{R}^{n+1}$ as

$$T^*(S^n) = \{(x,\xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \\ |x| = 1,\ < x, \xi >= 0\,\}$$

by identifying tangent and cotangent bundles using the standard Riemannian metric and we will denote hence force by $X^S := \{(x,\xi) \in T^*(S^n) \mid \xi \neq 0\,\}$, the punctured cotangent bundle $T_0^*(S^n)$.

By this realization of the cotangent bundle, the Liouville one-form $\theta^{S^n} =: \theta^S$ and the symplectic form $\omega^{S^n} = d\theta^S =: \omega^S$ are expressed as

$$\theta^S = \sum\ \xi_i\ dx_i,\ \omega^S = \sum d\xi_i \wedge dx_i,$$

that is these can be seen as restrictions of those for $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

Then by the map $\tau_S : X^S \longrightarrow \mathbb{C}^{n+1}$

$$\tau_S : (x,\xi) \longmapsto z = |\xi|x + \sqrt{-1}\xi$$

the punctured cotangent bundle $X^S = T_0^*(S^n)$ is identified with the quadrics

$$Q_2 = \left\{ z \in \mathbb{C}^{n+1}\backslash\{0\} \mid z^2 = \sum_{i=0}^{n} {z_i}^2 = 0 \right\}$$

and the symplectic form is expressed as

$$\omega^S = (\tau_S)^*\big(\sqrt{2}\sqrt{-1}\,\overline{\partial}\partial\,|z|\big),\ |z| = \sqrt{\sum |z_i|^2}, \tag{4.1}$$

which says that the space $T_0^*(S^n)$ has a Kähler manifold structure. By this realization of the space $T_0^*(S^n)$, the geodesic flow is expressed as the scalar multiplication of complex numbers of mudulus 1. Moreover let

$$\sigma = \frac{2}{|z|^2}\sum_j \overline{z}_j dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n,$$

then σ is a nowhere vanishing holomorphic n-form on $T_0^*(S^n) \overset{\tau_S}{\cong} Q_2$ and

$$\sigma \wedge \overline{\sigma} = \sqrt{-1}^{\,n} 2^{n/2+3}|z|^{n-2}\frac{(-1)^{n(n-1)/2}}{n!}(\omega^S)^n \tag{4.2}$$

through the identification by the map τ_S. These relations are found in [?].

We consider an $n+1$-dimensional submanifold Z in Q_2 defined by

$$Z = \{e^{\sqrt{-1}\tau}(s_0,\ldots,s_p,\sqrt{-1}t_{p+1},\ldots,\sqrt{-1}t_n) \mid s_i,\ t_j \in \mathbb{R}, \text{ and } \sum {s_i}^2 = \sum {t_j}^2 > 0\},$$

where we assume $p \geq 2$ and $n-p \geq 3$ (hence $n \geq 5$).

Let $\mathcal{H} : \mathbb{R}\times\mathbb{R}^{p+1}\times\mathbb{R}^{n-p}$ be a map

$$\mathcal{H} : \mathbb{R}\times\mathbb{R}^{p+1}\times\mathbb{R}^{n-p} \ni (\tau,s,t) \longmapsto e^{\sqrt{-1}\tau}(s,\sqrt{-1}t) \in \mathbb{C}^{n+1},$$

then the map $\mathcal{H}$ restricted to an $n+1$-dimensional submanifold $\mathbb{R}\times\{(s,t) \in \mathbb{R}^{p+1}\times\mathbb{R}^{n-p} \mid |s| = |t| > 0\} := \mathbb{R}\times H$ is a covering map to Z and can be descended to a double covering map from $U(1)\times H$ to Z, which we can see from the expession of the manifold Z. Then

(4.3)

$$\begin{aligned}
&{\tau_S}^{-1}(Z)\\
&= \Big\{\Big(\frac{s_0\cos\tau}{\sqrt{|s|^2\cos^2\tau+|t|^2\sin^2\tau}},\ldots,\frac{s_p\cos\tau}{\sqrt{|s|^2\cos^2\tau+|t|^2\sin^2\tau}},\\
&\qquad\frac{-t_{p+1}\sin\tau}{\sqrt{|s|^2\cos^2\tau+|t|^2\sin^2\tau}},\ldots,\frac{-t_n\sin\tau}{\sqrt{|s|^2\cos^2\tau+|t|^2\sin^2\tau}};\\
&\qquad s_0\sin\tau,\ldots,s_p\sin\tau,t_{p+1}\cos\tau,\ldots,t_n\cos\tau\Big)\Big\}.
\end{aligned}$$

We put $|s| = |t| = 1$ and denote $Z_1 := \mathcal{H}(U(1)\times S^p\times S^{n-p-1})$, then

Proposition 4.1. *$L_1 := {\tau_S}^{-1}(Z_1) \cong (U(1) \times S^p \times S^{n-p})/\mathbb{Z}_2$ is a geodesic flow action invariant Lagrangian submanifold in $T_0^*(S^n)$. The $\mathbb{Z}_2$-action on $U(1) \times S^p \times S^{n-p}$ is given by*

$$U(1) \times S^p \times S^{n-p} \ni (e^{\sqrt{-1}\tau}, s, t) \longmapsto$$
$$(e^{\sqrt{-1}(\tau+\pi)}, -s, -t) = -(e^{\sqrt{-1}\tau}, s, t) \in U(1) \times S^p \times S^{n-p}.$$

Proof. By definition it will be apparent of the geodesic flow invariance. Since,

(4.4)
$$\begin{aligned}&{\tau_S}^{-1}(Z_1) = L_1\\ &= \{(s_0\cos\tau, \dots, s_p\cos\tau, -t_{p+1}\sin\tau, \dots, -t_n\sin\tau;\\ &s_0\sin\tau, \dots, s_p\sin\tau, t_{p+1}\cos\tau, \dots .t_n\cos\tau) \mid \tau \in \mathbb{R}, \sum {s_i}^2 = \sum {t_{p+j}}^2 = 1\},\end{aligned}$$

(4.5)
The symplectic form is expressed as
$$\begin{aligned}\omega^S &= \sum d(s_i\sin\tau) \wedge d(s_i\cos\tau) - \sum d(t_{p+j}\cos\tau) \wedge d(t_{p+j}\sin\tau)\\ &= \sum s_i d\tau \wedge ds_i - \sum t_{p+j}dt_{p+j} \wedge d\tau = \frac{1}{2}\, d\tau \wedge d\left(\sum {s_i}^2 + \sum {t_{p+j}}^2\right) = 0,\end{aligned}$$
which shows that the submanifold L_1 is a Lagrangian submanifold. $\square$

The group $\pi_1(L_1) \cong \mathbb{Z}$ (this can be seen by the fact that the space $\mathbb{R} \times S^p \times S^{n-p-1}$ is the universal covering space of L_1 and the transformatin $\mathbb{R} \times S^p \times S^{n-p-1} \ni (\tau, x, y) \to (\tau+\pi, -x, -y) \in \mathbb{R} \times S^p \times S^{n-p-1}$ generates the covering transformation group) and the loop $\{c^0(\tau)\}_{0\leq\tau\leq 2\pi}$,

$$c^0(\tau) = (x^0(\tau), \xi^0(\tau)) = (\cos\tau, \underbrace{0, \dots, 0}_{n-1}, -\sin\tau; \sin\tau, \underbrace{0, \dots, 0}_{n-1}, \cos\tau) \in L_1, \tag{4.6}$$

is *twice* of the generator of $\pi_1(L_1)$.

Proposition 4.2. *The action integral*
$$\frac{1}{2\pi}\int_{c^0(\tau)} \theta^S = -1.$$

Proof. By the explicit expression of the curve we have
$$\int_{c^0(\tau)} \theta^S = \int \sum \xi_i^0(\tau)\, dx_i^0(\tau) = \int_0^{2\pi} \sin\tau d(\cos\tau) - \cos\tau d\sin\tau = -\int_0^{2\pi} d\tau = -2\pi.$$
$\square$

Next, we determine the Maslov class $\mathfrak{m}_{L_1}$ of L_1. For this purpose, first we determine the points $(x^0(\tau), \xi^0(\tau)) = c^0(\tau)$ at which $T_{c^0(\tau)}(L_1) \cap {\mathcal{V}^{L_1}}_{c^0(\tau)} \neq \{0\}$ (see 3.1 for $\mathcal{V}^{L_1}$).

Let $\mathcal{F}$ be map $\mathcal{F} : \mathbb{R} \times \mathbb{R}^{p+1} \times \mathbb{R}^{n-p} \longrightarrow \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ by

(4.7)
$$\mathcal{F} : \mathbb{R} \times \mathbb{R}^{p+1} \times \mathbb{R}^{n-p} \ni (\tau, s, t) \longmapsto$$

$$(s_0\cos\tau,\dots,s_p\cos\tau,-t_{p+1}\sin\tau,\dots,-t_n\sin\tau;\\ s_0\sin\tau,\dots,s_p\sin\tau,t_{p+1}\cos\tau,\dots.t_n\cos\tau),$$

then $\mathcal{F}(\{(\tau,s,t)\mid |s|=|t|=1\})=L_1$. On the curve $\{p(\tau)\}_{\tau\in\mathbb{R}}$,

$$p(\tau)=(\tau,1,\underbrace{0,\dots,0}_{n-1},1)\in\mathbb{R}\times\mathbb{R}^{p+1}\times\mathbb{R}^{n-p},$$

the map $\mathcal{F}$ is periodic with the period 2π, $\mathcal{F}(p(\tau))=c^0(\tau)$ and

$$\begin{aligned} d\mathcal{F}_{p(\tau)}\left(\frac{\partial}{\partial\tau}\right) &= -\sum s_i\sin\tau\frac{\partial}{\partial x_i}-\sum t_{p+j}\cos\tau\frac{\partial}{\partial x_{p+j}}\\ &\qquad +\sum s_i\cos\tau\frac{\partial}{\partial\xi_i}-\sum t_{p+j}\sin\tau\frac{\partial}{\partial\xi_{p+j}}\\ &= -\sin\tau\frac{\partial}{\partial x_0}-\cos\tau\frac{\partial}{\partial x_n}+\cos\tau\frac{\partial}{\partial\xi_0}-\sin\tau\frac{\partial}{\partial\xi_n}\in T_{c^0(\tau)}(L_1),\\ d\mathcal{F}_{p(\tau)}\left(\frac{\partial}{\partial s_i}\right) &= \cos\tau\frac{\partial}{\partial x_i}+\sin\tau\frac{\partial}{\partial\xi_i},\ i=0,\dots,p,\\ d\mathcal{F}_{p(\tau)}\left(\frac{\partial}{\partial t_{p+j}}\right) &= -\sin\tau\frac{\partial}{\partial x_{p+j}}+\cos\tau\frac{\partial}{\partial\xi_{p+j}},\ j=1,\dots,n-p. \end{aligned}$$

Let α,β_i and $\delta_j\in\mathbb{R}$ with the conditions that $\sum_{i=0}^p\beta_is_i=0$ and $\sum_{j=1}^{n-p}\delta_jt_{p+j}=0$, that is we take

$$\alpha\frac{\partial}{\partial\tau}+\sum\beta_i\frac{\partial}{\partial s_i}+\sum\delta_j\frac{\partial}{\partial t_{p+j}}\in T_{p(\tau)}(\mathbb{R}\times S^p\times S^{n-p-1}),$$

where $\beta_0=0=\delta_{n-p}$, and α, β_i $(i\geq 1)$ and δ_j $(1\leq j\leq n-p-1)$ can be taken arbitrarily. The tangent space $T_{c^0(\tau)}(L_1)=d\mathcal{F}_{p(\tau)}(T_{p(\tau)}(U(1)\times S^p\times S^{n-p-1}))$ is expressed as

(4.8)

$$\begin{aligned} T_{c^0(\tau)}(L_1)=\Big\{\alpha\big(&-\sin\tau\frac{\partial}{\partial x_0}-\cos\tau\frac{\partial}{\partial x_n}+\cos\tau\frac{\partial}{\partial\xi_0}-\sin\tau\frac{\partial}{\partial\xi_n}\big)\\ &+\sum_{i=1}^p\beta_i\big(\cos\tau\frac{\partial}{\partial x_i}+\sin\tau\frac{\partial}{\partial\xi_i}\big)\\ &\quad+\sum_{j=1}^{n-p-1}\delta_j\big(-\sin\tau\frac{\partial}{\partial x_{p+j}}+\cos\tau\frac{\partial}{\partial\xi_{p+j}}\big)\ \Big|\ \alpha,\ \beta_i,\ \delta_j\ \in\mathbb{R}\Big\}. \end{aligned}$$

If a tangent vector

$$\alpha\frac{\partial}{\partial\tau}+\sum_{i=1}^p\beta_i\frac{\partial}{\partial s_i}+\sum_{j=1}^{n-p-1}\delta_j\frac{\partial}{\partial t_{p+j}}\in T_{p(\tau)}(U(1)\times S^p\times S^{n-p-1})$$

satisfies

$$d\pi^{S^n}_{c^0(\tau)}\circ d\mathcal{F}_{p(\tau)}\left(\alpha\frac{\partial}{\partial\tau}+\sum_{i=1}^p\beta_i\frac{\partial}{\partial s_i}+\sum_{j=1}^{n-p-1}\delta_j\frac{\partial}{\partial t_{p+j}}\right)$$

$$= -\alpha \sin\tau \frac{\partial}{\partial x_0} - \alpha \cos\tau \frac{\partial}{\partial x_n} + \sum_{i=1}^{p} \beta_i \cos\tau \frac{\partial}{\partial x_i} - \sum_{j=1}^{n-p-1} \delta_j \sin\tau \frac{\partial}{\partial x_{p+j}} = 0,$$

then $\alpha = 0$ and

(1) at the points $c^0(\tau)$ for $\tau \neq \pi/2$ nor $3\pi/2$, that is $\cos\tau \neq 0$ we have

$$\beta_i \cos\tau = 0 \ (i = 1, \ldots, p), \ \delta_j \sin\tau = 0 \ (j = 1, \ldots, n-p-1), \text{ and}$$

(2) at the points $c^0(\tau)$ for $\tau \neq 0$ nor π, that is $\sin\tau \neq 0$

$$\beta_i \cos\tau = 0 \ (i = 1, \ldots, p), \ \delta_j \sin\tau = 0 \ (j = 1, \ldots, n-p-1).$$

Hence except four points of $c^0(\tau)$ at $\tau = 0, \ \pi/2, \ \pi, \ 3\pi/2$, the intersection $T_{c^0(\tau)}(L_1) \cap \mathcal{V}^{L_1}{}_{c^0(\tau)} = \{0\}$ and non-trivial intersections are given as

Case 1: $\tau = 0$ or $\tau = \pi$,

$$T_{c^0(\tau)}(L_1) \cap \mathcal{V}^{L_1}{}_{c^0(\tau)} = \left\{ \sum_{1 \le j \le n-p-1} \delta_j \frac{\partial}{\partial \xi_{p+j}} \right\}$$

Case 2: $\tau = \pi/2$ or $3\pi/2$,

$$T_{c^0(\tau)}(L_1) \cap \mathcal{V}^{L_1}{}_{c^0(\tau)} = \left\{ \sum_{1 \le i \le p} \beta_i \frac{\partial}{\partial \xi_i} \right\}.$$

To determine the Malsov class of the Lagrangian submanifold L_1, it is enough to calculate the Maslov indeces on the small intervals including these four points. We follow our definition of the Malsov index (2.1) and determine the value by Proposition 2.2 and Lemma 2.3.

So, before the calculation we notice a Lemma whose proof will be apparent.

Lemma 4.3. *Let* $\mathbf{E}$ *be a symplectic vector space and* $\mathbf{F} \subset \mathbf{E}$ *a symplectic subspace. Let* λ *be a Lagrangian subspace of* $\mathbf{E}$ *and assume there is a continuous curve of Lagrangian subspaces* $\{\gamma(t)\}_{|t| \le \epsilon \ll 1}$ *of* $\mathbf{E}$*. These satisfy the conditions* (R1), (R2) *and* (R3) such that

(R1) $\lambda_F := \lambda \cap \mathbf{F}$ *is a Lagrangian subspace of* $\mathbf{F}$*,*

(R2) *the curve of the intersections* $\gamma_F(t) := \gamma(t) \cap \mathbf{F}$ *is a continuous family of Lagrangian subspaces of* $\mathbf{F}$ *and,*

(R3) *the intersection* $\lambda \cap \gamma(t) \subset \mathbf{F}$ *for each* t*.*

Then,

$$Mas(\{\gamma(t)\}_{|t| \le \epsilon}, \lambda) = Mas(\{\gamma_F(t)\}_{|t| \le \epsilon}, \lambda_F). \tag{4.9}$$

Let $\{\mathbf{e}_i, \mathbf{f}_i\}_{i=1}^n$ be the standard symplectic basis of the symplectic vector space $\mathbf{E} := \mathbb{R}^{2n}$ with the symplectic form ω^{2n}, that is they satisfy the conditions

$$\omega^{2n}(\mathbf{e}_i, \mathbf{e}_j) = \omega^{2n}(\mathbf{f}_i, \mathbf{f}_j) = 0, \omega^{2n}(\mathbf{e}_i, \mathbf{f}_j) = -\omega^{2n}(\mathbf{f}_j, \mathbf{e}_i) = \delta_{i\,j}.$$

Now we show that our cases can be proved by applying the Lemma 4.3 above. We must treat the two Cases 1 and 2 separately.

Case 1:

Let τ be around $\tau = 0$ or π, say $|\tau| \leq \epsilon \ll 1$, or $|\tau - \pi| \leq \epsilon \ll 1$. Then the tangent space $T_{c^0(\tau)}(X^S)$ at $c^0(\tau)$ are characterized as

(4.10)

$$T_{c^0(\tau)}(X^S)$$
$$= \Big\{ \sum_{i=0}^{n} a_i \frac{\partial}{\partial x_i} + \sum_{i=0}^{n} b_i \frac{\partial}{\partial \xi_i} \;\Big|\; a_0 \cos\tau = a_n \sin\tau, (a_n + b_0)\cos\tau = (b_n - a_0)\sin\tau \Big\}$$

(4.11)

$$= \Big\{ \sum_{i=1}^{n-1} a_i \frac{\partial}{\partial x_i} + a_n\big(\tan\tau \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_n} - \frac{1}{\cos^2\tau}\frac{\partial}{\partial \xi_0}\big) + \sum_{i=1}^{n-1} b_i \frac{\partial}{\partial \xi_i} + b_n\big(\tan\tau\frac{\partial}{\partial \xi_0} + \frac{\partial}{\partial \xi_n}\big) \Big\}$$

Based on these expressions we define symplectic isomorphisms $S_\tau : \mathbf{E} \to T_{c^0(\tau)}(X^S)$ by

$$\begin{cases} S_\tau : \mathbf{e}_i \longmapsto \frac{\partial}{\partial x_i}, \text{ for } i = 1, \dots, n-1, \\ S_\tau : \mathbf{e}_n \longmapsto \cos\tau\big\{\tan\tau\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_n} - \frac{1}{\cos^2\tau}\frac{\partial}{\partial \xi_0}\big\}, \\ S_\tau : \mathbf{f}_i \longmapsto \frac{\partial}{\partial \xi_i}, \text{ for } \ i = 1, \dots, n-1, \\ S_\tau : \mathbf{f}_n \longmapsto \cos\tau\big\{\tan\tau\frac{\partial}{\partial \xi_0} + \frac{\partial}{\partial \xi_n}\big\}. \end{cases}$$

Since $\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_j} \right\}_{i,j=0}^{n}$ are symplectic basis of the space $T(T^*(\mathbb{R}^{n+1}))$, it will be easily seen that these maps are symplectic, that is it leaves the symplectic forms.

Then the symplectic subspace $\mathbf{F}$ in $\mathbf{E}$ spanned by the basis vectors $\{\mathbf{e}_i, \mathbf{f}_i\}_{i\leq n-1}$ is maped to the subspace $S_\tau(\mathbf{F}) = F_{c^0(\tau)}$, where $F_{c^0(\tau)}$ is a symplectic subspace in $T_{c^0(\tau)}(X^S)$ spanned by the basis vectors $\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_i} \right\}_{i=1}^{n-1}$ for each $|\tau| \ll \pi/2$ or $|\tau - \pi| \ll \pi/2$.

Also by (4.11) the vertical subbundle $\mathcal{V}^{L_1}{}_{c^0(\tau)} = \mathrm{Ker}\,(d\pi^{S^n}{}_{c^0(\tau)})$, $\pi^{S^n} : T(X^S) \to S^n$, is characterized as

$$\mathcal{V}^{L_1}{}_{c^0(\tau)} = \Big\{ \sum_{i=1}^{n-1} b_i \frac{\partial}{\partial \xi_i} + b_n\big(\tan\tau\frac{\partial}{\partial \xi_0} + \frac{\partial}{\partial \xi_n}\big) \;\Big|\; b_i \in \mathbb{R}, \Big\}$$

Hence let λ_E be a Lagrangian subspace in $\mathbf{E}$ spanned by $\{\mathbf{f}_i\}_{i=1}^n$, then $\mathcal{V}^{L_1}{}_{c^0(\tau)} = S_\tau(\lambda_E)$.

For each τ, let $\gamma(\tau)$ be a subspace in $\mathbf{E}$ spanned by the vectors

$$\{\cos\tau\cdot\mathbf{e}_i + \sin\tau\cdot\mathbf{f}_i\ (i = 1, \dots, p), -\sin\tau\cdot\mathbf{e}_{p+j} + \cos\tau\cdot\mathbf{f}_{p+j}\ (j = 1, \dots, n-p-1), \mathbf{e}_n + \tan\tau\cdot\mathbf{f}_n\}$$

then $S_\tau(\gamma(\tau)) = T_{c^0(\tau)}(L_1)$.

Now we prove that these subspaces λ_E, $\mathbf{F}$ and $\{\gamma(\tau)\}_{|\tau|\le\epsilon\ll\pi/2}$ (also $\{\gamma(\tau)\}_{|\tau-\pi|\le\epsilon\ll\pi/2}$ satisfy the conditions (R1) $\sim$ (R3) in Lemma 4.3.

Proposition 4.4. (R1) *It will be apparent that the intersection $\mathbf{F}\cap\lambda_E =: \lambda_F$ is generated by $\{\mathbf{f}_1,\dots,\mathbf{f}_{n-1}\}$ and is a Lagrangian subspace in $\mathbf{F}$.*

(R2) $\gamma(\tau)\cap\mathbf{F} =: \gamma_F(\tau)$ *is a curve of Lagrangian subspace of* $\mathbf{F}$.

(R3) $\gamma(\tau)\cap\lambda_E \subset \mathbf{F}$.

Proof. Since the intersection $\gamma(\tau)\cap\mathbf{F}$ is spanned by the vectors

$$\{\cos\tau\cdot\mathbf{e}_i+\sin\tau\cdot\mathbf{f}_i\ (i=1,\dots,p),\ -\sin\tau\cdot\mathbf{e}_{p+j}+\cos\tau\cdot\mathbf{f}_{p+j}\ (j=1,\dots,n-p-1)\},$$

we know that $\gamma(\tau)\cap\mathbf{F}$ is a family of Lagrangian subspace in $\mathbf{F}$, which shows (R2) condition.

(R3) condition will be seen by the expression that

$$\gamma(\tau)\cap\lambda_E = \{0\} \text{ or a subspace spanned by } \{\mathbf{f}_{p+j}\}_{j=1}^{n-p-1} \text{ for } \tau=0 \text{ or } \pi,$$

which is a subspace in $\mathbf{F}$. ☐

Then

Proposition 4.5. *The curve $\{\gamma(\tau)\}_{|\tau|\le\epsilon}$ and a fixed Lagrangian subspace λ_E in $\mathbf{E}$ are mapped to the curves of Lagrangian subspaces $\{T_{c^0(\tau)}(L_1)\}_{|\tau|\le\epsilon}$ and $\{\mathcal{V}^{L_1}{}_{c^0(\tau)}\}$ in $T_{c^0(\tau)}(X^S)$. Hence by α-construction*

$$Mas(\{T_{c^0(\tau)}(L_1)\}_{|\tau|\le\epsilon},\ \{\mathcal{V}^{L_1}{}_{c^0(\tau)}\}_{|\tau|\le\epsilon}) = Mas(\{\gamma_E(\tau)\}_{|\tau|\le\epsilon},\lambda_E).$$

Also by applying Lemma 4.3

$$Mas(\{\gamma_E(\tau)\}_{|\tau|\le\epsilon},\lambda_E) = Mas(\{\gamma_F(\tau)\}_{|\tau|\le\epsilon},\lambda_F).$$

The explicit determination of the value $Mas(\{\gamma_F(\tau)\}_{|\tau|\le\epsilon},\lambda_F)$ is done as follows:

Let μ be the Lagrangian subspace of $\mathbf{F}$ spanned by the basis vectors $\{\mathbf{e}_i+\mathbf{f}_i\}_{i=1}^{n-1}$, then μ and λ_F are transversal, and also μ and $\gamma_F(\tau)$ are transversal when $|\tau|\le\epsilon\ll\pi/2$ and the subspace $\gamma_F(\tau)$ is spanned by the basis vectors

$$\cos\tau\cdot\mathbf{e}_i+\sin\tau\cdot\mathbf{f}_i\ (i=1,\dots,p), -\sin\tau\cdot\mathbf{e}_{p+j}+\cos\tau\cdot\mathbf{f}_{p+j}\ (j=1,\dots,n-p-1).$$

For each τ we define a map

$$\begin{aligned}
&A_\tau : \lambda_F \to \mu,\\
&A_\tau(\mathbf{f}_i) = \frac{\cos\tau}{\sin\tau-\cos\tau}(\mathbf{e}_i+\mathbf{f}_i) \text{ for } i=1,\dots,p,\\
&A_\tau(\mathbf{f}_{p+j}) = \frac{-\sin\tau}{\sin\tau+\cos\tau}(\mathbf{e}_{p+j}+\mathbf{f}_{p+j}) \text{ for } j=1,\dots,n-p-1.
\end{aligned}$$

Then the space spanned by vectors $\{\mathbf{f}_i+A_\tau(\mathbf{f}_i)\}_{i=1}^{n-1}$ coincides with the subspace $\gamma_F(\tau)$ and the map A_τ can be seen as a symmetric matrix

$$A_\tau(\tau) = \begin{pmatrix} \frac{\cos\tau}{\sin\tau-\cos\tau}I_p & \mathcal{O} \\ \mathcal{O} & \frac{-\sin\tau}{\sin\tau+\cos\tau}I_{n-p-1}, \end{pmatrix}$$

where I_k denotes the identity matrix of size k. Then by Lemma 2.3

Proposition 4.6.

$$Mas(\{\gamma_F(\tau)\}_{|\tau|\leq\epsilon},\lambda_F)=\mathrm{sign}\,(\dot{A}_0)\ \text{on}\ \mathrm{Ker}\,(A_0)=1+p-n,$$

and

$$Mas(\{\gamma_F(\tau)\}_{|\tau-\pi|\leq\epsilon},\lambda_F)=\mathrm{sign}\,(\dot{A}_\pi)\ \text{on}\ \mathrm{Ker}\,(A_\pi)=1+p-n.$$

The determination of the Maslov indeces around the points $c^0(\pi/2)$ and $c^0(3\pi/2)$ can be carried out by the same way. We list the necessary data here. Assume $|\tau-\pi/2|\leq\epsilon\ll\pi/2$ or $|\tau-3\pi/2|\leq\epsilon\ll\pi/2$. Then

(4.12)

$$T_{c^0(\tau)}(X^S)$$
$$=\Big\{a_0\Big(\frac{\partial}{\partial x_0}+\cot\tau\frac{\partial}{\partial x_n}+\frac{1}{\sin^2\tau}\frac{\partial}{\partial\xi_n}\Big)+\sum_{i=1}^{n-1}a_i\frac{\partial}{\partial x_i}+\sum_{i=1}^{n-1}b_i\frac{\partial}{\partial\xi_i}+b_0\Big(\frac{\partial}{\partial\xi_0}+\cot\tau\frac{\partial}{\partial\xi_n}\Big)\Big\}$$

The symplectic isomorphism $U_\tau:\mathbf{E}\to T_{c^0(\tau)}(X^S)$ is defined as

$$\begin{cases} U_\tau:\mathbf{e}_i\longmapsto\frac{\partial}{\partial x_i},\ \text{for}\ i=1,\dots,n-1,\\[2ex] U_\tau:\mathbf{e}_n\longmapsto\sin\tau\Big(\frac{\partial}{\partial x_0}+\cot\tau\frac{\partial}{\partial x_n}+\frac{1}{\sin^2\tau}\frac{\partial}{\partial\xi_n}\Big)\\[2ex] U_\tau:\mathbf{f}_i\longmapsto\frac{\partial}{\partial\xi_i},\ \text{for}\ \ i=1,\dots,n-1,\\[2ex] U_\tau:\mathbf{f}_n\longmapsto\sin\tau\Big(\frac{\partial}{\partial\xi_0}+\cot\tau\frac{\partial}{\partial\xi_n}\Big).\end{cases}$$

The vertical subbundle $\mathcal{V}^{L_1}{}_{c^0(\tau)}$ is

$$\mathcal{V}^{L_1}{}_{c^0(\tau)}=\Big\{\sum_{i=1}^{n-1}b_i\frac{\partial}{\partial\xi_i}+b_0\Big(\frac{\partial}{\partial\xi_0}+\cot\tau\frac{\partial}{\partial\xi_n}\Big)\Big\}\tag{4.13}$$

The Lagrangian subspace $\mathbf{F}$ and λ_E are the same spaces with the first case. Then

$$U_\tau(\mathbf{F})=\Big\{\sum_{i=1}^{n-1}a_i\frac{\partial}{\partial x_i}+\sum_{i=1}^{n-1}b_i\frac{\partial}{\partial\xi_i}\Big\}.$$

For each τ, $|\tau-\pi/2|\leq\epsilon$ (or $|\tau-3\pi/2|\leq\epsilon$), let $\varphi(\tau)$ be a family of Lagrangian subspaces of $\mathbf{E}$ spanned by the vectors

$$\{\cos\tau\cdot\mathbf{e}_i+\sin\tau\cdot\mathbf{f}_i\ (i=1,\dots,p),$$
$$-\sin\tau\cdot\mathbf{e}_{p+j}+\cos\tau\cdot\mathbf{f}_{p+j}\ (j=1,\dots,n-p-1),\ \mathbf{e}_n-\cot\tau\cdot\mathbf{f_n}\},$$

then $U_\tau(\varphi(\tau))=T_{c^0(\tau)}(L_1)$.

We can take the same Lagrangian subspace μ in $\mathbf{F}$ which is transversal to $\varphi(\tau)\cap\mathbf{F}$ and when we express the space $\varphi(\tau)\cap\mathbf{F}$ as the graph of a map $B_\tau:\lambda_F\to\mu$ the operator has the same expression of A_τ, so that finnaly we have

Proposition 4.7.

$$Mas(\{T_{c^0(\tau)}(L_1)\}_{|\tau-\pi/2|\le\epsilon}, \{\mathcal{V}^{L_1}{}_{c^0(\tau)}\}_{|\tau-\pi/2|\le\epsilon}) = Mas(\{\varphi(\tau)\}\}_{|\tau-\pi/2|\le\epsilon} \cap \mathbf{F}, \lambda_F)$$
$$= \mathrm{sign}\,(\dot{A}_{\pi/2}) \ \mathit{on}\ \mathrm{Ker}\,(A_{\pi/2}) = -p,$$

and

$$Mas(\{T_{c^\tau}(L_1)\}_{|\tau-\pi/2|\le\epsilon}, \{\mathcal{V}^{L_1}{}_{c^0(\tau)}\}_{|\tau-3\pi/2|\le\epsilon}) = Mas(\{\varphi(\tau)\}_{|\tau-3\pi/2|\le\epsilon} \cap \mathbf{F}, \lambda_F)$$
$$= \mathrm{sign}\,(\dot{A}_{3\pi/2}) \ \mathit{on}\ \mathrm{Ker}\,(A_{3\pi/2}) = -p.$$

Summing up these calculation we have

Proposition 4.8.

$$\mathfrak{m}_{L_1} : \pi_1(L_1) \cong \mathbb{Z} \longrightarrow \mathbb{Z}, 1 \longmapsto (1-n).$$

Corollary 4.9. *For $n = 4k+3$, then L_1 satisfies the condition* Mas[2].
For $n = 4k+2$, then $1/2 \cdot L_1$ satisfies the condition Mas[2].
For $n = 4k+1$, then $2 \cdot L_1$ satisfies the condition Mas[2].
For $n = 4k$, then $3/2 \cdot L_1$ satisfies the condition Mas[2].

It will be clear that on L_1 the principal symbol of the Laplacian is constant $= 1$ (Condition Mas[1]).

As for the condition Mas[3], there is a way to construct a measure on any geodesic flow invariant Lagrangian submanifold in $T_0^*(S^n) \cong Q_2$ based on the Kähler structure and the properties (4.1) and (4.2).

In fact, the property (4.2) says that $|\sigma|$ is a nowhere vanishing half density on the whole space Q_2. If Λ is a $U(1)$-invariant Lagrangian subspace, then by the characterization of Q_2 and the relation (4.2) we can regard that the complexification $T^*(\Lambda) \otimes \mathbb{C}$ is isomorphic to $T^*(Q_2)_{|\Lambda}$ considered as a complex vector bundle, or it is the same thing that it is isomorphic to the restriction to Λ of the holomorphic part $T^{*\prime}(Q_2)$ of the complexification $T^*(Q_2) \otimes \mathbb{C} = T^{*\prime}(Q_2) \oplus T^{*\prime\prime}(Q_2)$, hence

$$\bigwedge^n (T^*(\Lambda) \otimes \mathbb{C}) = \Big(\bigwedge^n T^*(\Lambda)\Big) \otimes \mathbb{C} \cong \bigwedge^n T^{*\prime}(Q_2)_{|\Lambda}.$$

Then we can define a half density on Λ by restricting the half density $|\sigma|$ to Λ.

5. Cayley projective plane

In this section we construct Lagrangian submanifolds in the cotangent bundle of the Cayley projective plane satisfying Maslov quantization condition based on our earlier works [7], [6],[8] and [1].

5.1. **Octanion number field and Cayley projective plane.** Let $\mathbf{e}_0, \ldots, \mathbf{e}_7$ be the standard basis of the octanion number field $\mathbb{O}$, where $\mathbf{e}_0$ is the basis of the center with respect to the octanionic multiplication law. Here we only recall some properties of the octanion numbers what we need in the following sections(for details see [24] and [7]).

We define for $x = \sum_{i=0}^{7} x_i \mathbf{e}_i \in \mathbb{O}(x_i \in \mathbb{R})$ its conjugation $\theta(x)$ by

$$\theta : x \longmapsto \theta(x) = x_0 - \sum_{i=1}^{7} x_i \mathbf{e}_i \in \mathbb{O}.$$

This operation satisfies the relation

$$\theta(yz) = \theta(z)\theta(y). \tag{5.1}$$

We denote by $\mathcal{J}(3)$

$$\mathcal{J}(3) := \left\{ A = \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \;\Big|\; z,\ y,\ x \in \mathbb{O}, \xi_1, \xi_3, \xi_3 \in \mathbb{R} \right\}, \tag{5.2}$$

that is $\mathcal{J}(3)$ is a 3×3 "symmetric" octanion matrices with the "Jordan product":

$$\mathcal{J}(3) \ni A, B \longmapsto A \circ B := \frac{AB + BA}{2} \in \mathcal{J}(3).$$

This is called a Jordan algebra (over $\mathbb{R}$).

When we consider the complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$ of the octanion number field, the elements in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$ is understood as $\sum z_i \mathbf{e}_i$ with the complex coefficients z_i.

We will also use the expression of the (complex) octanion number as $z = \sum_{i=0}^{7} \{z\}_i \mathbf{e}_i$, that is $\{z\}_i$ indicates the coefficient of $z \in \mathbb{O}$ (or $z \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$) of the basis $\mathbf{e}_i$.

We should disitinguish the conjugation $\theta(z)$ and complex conjugation $\overline{z} := \sum \overline{z}_i \mathbf{e}_i$. These two satisfy the relation:

$$\theta(\overline{z}) = \overline{\theta(z)}.$$

The inner product in $\mathbb{O}$ is defined as the Euclidean inner product:

$$< x, y >^{\mathbb{O}} := \sum_{i=0}^{7} x_i y_i \tag{5.3}$$

for $x = \sum_{i=0}^{7} x_i \mathbf{e}_i$ and $y = \sum_{i=0}^{7} y_i \mathbf{e}_i$. This inner product is multiplicative:

$$< xy, xy > = < x, x >< y, y > . \tag{5.4}$$

Also the inner product $< \cdot, \cdot > = < \cdot, \cdot >^{\mathcal{J}(3)}$ in $\mathcal{J}(3)$ is defined by

$$< A, B > := \mathrm{tr}(A \circ B) = \sum_{i=1}^{3} \xi_i \eta_i + 2(< z, z' >^{\mathbb{O}} + < y, y' >^{\mathbb{O}} + < x, x' >^{\mathbb{O}}) \tag{5.5}$$

for $A = \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix}$ and $B = \begin{pmatrix} \eta_1 & z' & \theta(y') \\ \theta(z') & \eta_2 & x' \\ y' & \theta(x') & \eta_3 \end{pmatrix}$.

We also consider the complexification of this Jordan algebra, $\mathcal{J}(3)^{\mathbb{C}} := \mathbb{C} \otimes \mathcal{J}(3)$, with the elements in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$. The inner products $< \cdot, \cdot >^{\mathbb{O}}$ and $< \cdot, \cdot >^{\mathcal{J}(3)}$ are naturally extended to the complex bilinear form on $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$ and $\mathcal{J}(3)^{\mathbb{C}}$ respectively and we denote them with the same notations. Then the Hermitian inner products are given as

$$z = \sum z_i \mathbf{e}_i,\ z' = \sum {z_i}' \mathbf{e}_i \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{O} \longmapsto < z, \overline{z}' > = \sum z_i \overline{z}_i{}', \tag{5.6}$$

$$A, B \in \mathcal{J}(3)^{\mathbb{C}} \longmapsto < A, \overline{B} > \in \mathbb{C}, \tag{5.7}$$

where $\overline{B}$ is the matrix with the complex conjugate elements:

$$\overline{B} = \begin{pmatrix} \overline{\eta_1} & \overline{z'} & \overline{\theta(y')} \\ \overline{\theta(z')} & \overline{\eta_2} & \overline{x'} \\ \overline{y'} & \overline{\theta(x')} & \overline{\eta_3} \end{pmatrix}.$$

We denote by $|z| = \sqrt{< z, z >^{\mathbb{O}}}$ for $z \in \mathbb{O}$, or for $z \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$ by $|z| = \sqrt{< z, \overline{z} >}$, and $||A|| = \sqrt{\mathrm{tr}\,(A \circ A)}$ for $A \in \mathcal{J}(3)$ or $||A|| = \sqrt{\mathrm{tr}\,(A \circ \overline{A})}$ for $A \in \mathcal{J}(3)^{\mathbb{C}}$ for their norms, respectively.

The Calyley projective plane $P^2\mathbb{O}$ is realized in $\mathcal{J}(3)$ as

$$P^2\mathbb{O} = \left\{ X \in \mathcal{J}(3) \mid X^2 = X, \mathrm{tr}\,(X) = 1 \right\}. \tag{5.8}$$

The tangent bundle $T(P^2\mathbb{O})$ is given as

$$T(P^2\mathbb{O}) = \left\{ (X, Y) \in \mathcal{J}(3) \times \mathcal{J}(3) \;\middle|\; X \in P^2\mathbb{O}, X \circ Y = 1/2 \cdot Y \right\}. \tag{5.9}$$

The tangent bundle $T(P^2\mathbb{O})$ and the cotangent bundle $T^*(P^2\mathbb{O})$ are identfied by the inner product $< \cdot, \cdot >^{\mathcal{J}(3)}$ following the identification $T(\mathcal{J}(3)) \cong T^*(\mathcal{J}(3))$, that is we have an inclusion map $T^*(P^2\mathbb{O}) \longrightarrow T^*(\mathcal{J}(3))$:

$$T^*(P^2\mathbb{O}) \longrightarrow T(P^2\mathbb{O}) \longrightarrow T(\mathcal{J}(3)) \longrightarrow T^*(\mathcal{J}(3)).$$

Through this inclusion map the natural symplectic form on $T^*(\mathcal{J}(3))$ is pull-backed to the natural symplectic form on $T^*(P^2\mathbb{O})$, so that we work on the tangent bundle $T(P^2\mathbb{O})$.

Under this isomorphism, we may express the canonical one form $\theta^{P^2\mathbb{O}}$ on $P^2\mathbb{O}$ in terms of the inner product $< \cdot\,, \cdot >^{\mathcal{J}(3)}$ as

$$\theta^{P^2\mathbb{O}} = \sum \eta_i d\xi_i + 2 \sum a_i dx_i + b_i dy_i + c_i dz_i =: < Y, dX >^{\mathcal{J}(3)}, \tag{5.10}$$

where we express $(X, Y) \in T^*(P^2\mathbb{O})$ as

$$X = \begin{pmatrix} \xi_1 & z & \theta(y) \\ y & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \text{ and } Y = \begin{pmatrix} \eta_1 & c & \theta(b) \\ \theta(c) & \eta_2 & a \\ b & \theta(a) & \eta_3 \end{pmatrix}.$$

Theorem 5.1 ([7])**.** *The puncture tangent bundle $T_0(P^2\mathbb{O}) = T(P^2\mathbb{O}) \backslash \{0\}$ (hence the punctured cotangent bundle $T_0^*(P^2\mathbb{O})$ through the above identification) is realized as a quadrics $\mathbb{X}_{\mathbb{O}}$ in the complexfied Jordan algebra $\mathcal{J}(3)^{\mathbb{C}} \cong \mathbb{C}^{27}$ (as a vector space):*

$$\mathbb{X}_{\mathbb{O}} := \left\{ A \in \mathcal{J}(3)^{\mathbb{C}} \mid A^2 = 0, A \neq 0 \right\} \tag{5.11}$$

by the map

$$\tau_{\mathbb{O}} : T^*(P^2\mathbb{O}) \longmapsto \mathbb{X}_{\mathbb{O}} \subset \mathcal{J}(3)^{\mathbb{C}},$$

$$\tau_{\mathbb{O}} : (X, Y) \longmapsto ||Y||^2 X - Y^2 + \sqrt{-1} \otimes \frac{||Y||}{\sqrt{2}} Y. \tag{5.12}$$

Moreover by this map

Theorem 5.2 ([7]).

$$\tau_{\mathbb{O}}{}^*(\sqrt{-2}\partial\overline{\partial}||A||^{1/2}) = \omega^{P^2\mathbb{O}} = d\theta^{P^2\mathbb{O}}, \tag{5.13}$$

where $\omega^{P^2\mathbb{O}}$ is the natural symplectic form of the cotangent bundle $T^*(P^2\mathbb{O})$ under the identification $T^*(P^2\mathbb{O}) \cong T(P^2\mathbb{O})$.

The inverse $\tau_{\mathbb{O}}{}^{-1}(A) = (X(A), Y(A)) \in \mathcal{J}(3) \times \mathcal{J}(3)$ is given by

$$X(A) = \frac{A + \overline{A}}{2||A||} + \frac{A \circ \overline{A}}{||A||^2}, \tag{5.14}$$

$$Y(A) = -\frac{\sqrt{-1}(A - \overline{A})}{\sqrt{2||A||}}. \tag{5.15}$$

5.2. **Lagrangian submanifold in $T_0^* P^2\mathbb{O}$.** Let $A = \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \in \mathcal{J}(3)^{\mathbb{C}}$. Then the condition for $A \in \mathbb{X}_{\mathbb{O}}$ is expressed in terms of each component as

(5.16)
$$(\xi_3 + \xi_2)x + \theta(yz) = 0,\ (\xi_1 + \xi_3)y + \theta(zx) = 0,\ (\xi_2 + \xi_1)z + \theta(xy) = 0,$$

(5.17)
$${\xi_1}^2 + z\theta(z) + \theta(y)y = 0,\ {\xi_2}^2 + \theta(z)z + x\theta(x) = 0,\ {\xi_3}^2 + \theta(x)x + y\theta(y) = 0.$$

Also we have one condition (see Proposition 4.1 in [1] or [7]):

$$\operatorname{tr} A = \xi_1 + \xi_2 + \xi_3 = 0. \tag{5.18}$$

Assume z and y are in $\mathbb{O}$, that is z and y have real coefficients in the expression

$$z = \sum z_i \mathbf{e}_i,\ y = \sum y_i \mathbf{e}_i$$

and we assume

$$\theta(z)z + y\theta(y) = |z|^2 + |y|^2 \equiv r^2,\ r > 0. \tag{5.19}$$

Then we can solve the equations (5.16) and (5.17) in terms of (z, y) in two ways:

(1) from the first equation of (5.17) we have $\xi_1 = \pm\sqrt{-1}r$, then

(2) from the first equation of (5.16) we have $x = \mp\sqrt{-1}\dfrac{\theta(yz)}{r}$.

Then the second equation in (5.17) is

$${\xi_2}^2 + |z|^2 - \frac{|z|^2|y|^2}{r^2} = {\xi_2}^2 + |z|^2\left(\frac{r^2 - |y|^2}{r^2}\right) = {\xi_2}^2 + \frac{|z|^4}{r^2} = 0.$$

Hence $\xi_2 = \pm\sqrt{-1}\frac{|z|^2}{r}$ and similarly $\xi_3 = \pm\sqrt{-1}\frac{|y|^2}{r}$. If we take a solution $\xi_1 = \sqrt{-1}r$, then the condition (5.18) requires that

$$\xi_2 = -\sqrt{-1}\frac{|z|^2}{r} \text{ and } \xi_3 = -\sqrt{-1}\frac{|y|^2}{r}, \text{ and } x = -\sqrt{-1}\frac{\theta(yz)}{r}.$$

Other set of solutions is given by

$$\xi_1 = -\sqrt{-1}r, \xi_2 = \sqrt{-1}\frac{|z|^2}{r},\ \xi_3 = \sqrt{-1}\frac{|y|^2}{r}, \text{ and } x = \sqrt{-1}\frac{\theta(yz)}{r}, \tag{5.20}$$

and a relation with the first solutions is explained in Remark 3.

By using the first set of solutions of the equations (5.16) and (5.17), that is

$$\xi_1 = \sqrt{-1}r, \xi_2 = -\sqrt{-1}\frac{|z|^2}{r}, \xi_3 - \sqrt{-1}\frac{|y|^2}{r} =, x = -\sqrt{-1}\frac{\theta(yz)}{r} \tag{5.21}$$

we consider a submanifold $\mathbb{L}$ defined by

$$\mathbb{L} = \left\{ \begin{pmatrix} \sqrt{-1}\sqrt{|z|^2+|y|^2} & z & \theta(y) \\ \theta(z) & \frac{-\sqrt{-1}|z|^2}{\sqrt{|z|^2+|y|^2}} & \frac{-\sqrt{-1}\theta(yz)}{\sqrt{|z|^2+|y|^2}} \\ y & \frac{-1\sqrt{-1}yz}{\sqrt{|z|^2+|y|^2}} & \frac{-\sqrt{-1}|y|^2}{\sqrt{|z|^2+|y|^2}} \end{pmatrix} \;\middle|\; (z,y) \in \mathbb{O}^2\backslash\{0\}, \right\}. \tag{5.22}$$

Then $\mathbb{L}$ is a conic (i.e., invariant under the $\mathbb{R}_+$-action) submanifold in $\mathbb{X}_{\mathbb{O}}$ isomorphic to $\mathbb{R}^{16}\backslash\{0\}$ and we have

Proposition 5.3.

$$\tau_{\mathbb{O}}{}^{-1}(\mathbb{L}) := \Lambda$$

is a conic Lagrangian submanifold.

Proof. If $A \in \mathbb{L}$, then

$$||A||^2 = r^2 + \frac{|z|^4}{r^2} + \frac{|y|^4}{r^2} + 2\left(|z|^2 + |y|^2 + \frac{|y|^2|z|^2}{r^2}\right) = 4r^2.$$

By the formulas (5.14) and (5.15), for $A \in \mathbb{L}$ we have

$$X(A) = \frac{A+\overline{A}}{2||A||} + \frac{A\circ\overline{A}}{||A||^2} = \frac{1}{4r}\begin{pmatrix} 0 & 2z & 2\theta(y) \\ 2\theta(z) & 0 & 0 \\ 2y & 0 & 0 \end{pmatrix} + \frac{1}{4r^2}\begin{pmatrix} 2r^2 & 0 & 0 \\ 0 & 2|z|^2 & 2\theta(yz) \\ 0 & 2yz & 2|y|^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{z}{2r} & \frac{\theta(y)}{2r} \\ \frac{\theta(z)}{2r} & \frac{|z|^2}{2r^2} & \frac{\theta(yz)}{2r^2} \\ \frac{y}{2r} & \frac{yz}{2r^2} & \frac{|y|^2}{2r^2} \end{pmatrix}, \tag{5.23}$$

(5.24)

$$Y(A) = -\sqrt{-1}\frac{A-\overline{A}}{\sqrt{2||A||}} = \frac{1}{2\sqrt{r}}\begin{pmatrix} 2r & 0 & 0 \\ 0 & -\frac{2|z|^2}{r} & -\frac{2\theta(yz)}{r} \\ 0 & -\frac{2yz}{r} & -\frac{2|y|^2}{r} \end{pmatrix} = \frac{1}{\sqrt{r}}\begin{pmatrix} r & 0 & 0 \\ 0 & -\frac{|z|^2}{r} & -\frac{\theta(yz)}{r} \\ 0 & -\frac{yz}{r} & -\frac{|y|^2}{r} \end{pmatrix}.$$

Since a closed submanifold of the same dimension with the base manifold in a punctured cotangent bundle is a conic Lagrangian submanifold, if and only if the canonical one-form $\theta^{P^2\mathbb{O}}$ vanishes on it ([15]).

So by using the above formulas (5.23) and (5.24), the canonical one form $\theta^{P^2\mathbb{O}}$ is expressed on Λ in terms of the coordinates (z, y) as

$$\begin{aligned}\theta^{P^2\mathbb{O}}{}_{|\Lambda} &= < Y(A), dX(A) >^{\mathcal{J}(3)} \\ &= -\frac{|z|^2}{r\sqrt{r}} d\left(\frac{|z|^2}{2r^2}\right) - \frac{|y|^2}{r\sqrt{r}} d\left(\frac{|y|^2}{2r^2}\right) - 2\sum_{i=0}^{7} \frac{\{yz\}_i}{r\sqrt{r}} d\left(\frac{\{yz\}_i}{2r^2}\right) \quad \text{(see (5.10))}.\end{aligned}$$

Then

$$\begin{aligned}-2\theta^{P^2\mathbb{O}}{}_{|\Lambda} =& \frac{|z|^2}{r\sqrt{r}}\left(\frac{r^2 d|z|^2 - |z|^2 dr^2}{r^4}\right) + \frac{|y|^2}{r\sqrt{r}}\left(\frac{r^2 d|y|^2 - |y|^2 dr^2}{r^4}\right) \\ &+ 2\sum_{i=0}^{7} \frac{\{yz\}_i}{r\sqrt{r}}\left(\frac{r^2 d\{yz\}_i - \{yz\}_i dr^2}{r^4}\right) \\ =& \frac{1}{2r^3\sqrt{r}} \cdot dr^4 - \frac{1}{r\sqrt{r}} dr^2 = \frac{2}{\sqrt{r}} dr - \frac{2}{\sqrt{r}} dr = 0,\end{aligned}$$

where we used the multiplicative property of the norm $|yz| = |y||z|$ and replace all the term r^2 with $r^2 = |z|^2 + |y|^2$ (see (5.4)). $\square$

Put

$$\mathbb{S}_r := \{A \in \mathbb{L} \mid ||A|| = 2r = 2\sqrt{|z|^2 + |y|^2} = \text{constant} > 0\}. \tag{5.25}$$

By definition this is isomorphic to 15-dimensional sphere S^{15} and $\mathbb{S}_r$ is an isotropic submanifold. Also $\mathbb{L} \cong \mathbb{S}_r \times \mathbb{R}_+$.

Let

$$\varphi : \mathbb{R} \times \mathbb{S}_r \ni (t, A) \longmapsto e^{2\pi\sqrt{-1}t} \cdot A \in \mathbb{X}_{\mathbb{O}}. \tag{5.26}$$

For $\varphi(t, A) = e^{2\pi\sqrt{-1}t} \cdot A$,

$$\begin{aligned}X(e^{2\pi\sqrt{-1}t} \cdot A) &= \frac{e^{2\pi\sqrt{-1}t} \cdot A + e^{-2\pi\sqrt{-1}t} \cdot \overline{A}}{2||A||} + \frac{A \circ \overline{A}}{||A||^2} \\ &= \begin{pmatrix} -\frac{\sin 2\pi t}{2} & \frac{\cos 2\pi t \cdot z}{2r} & \frac{\cos 2\pi t \cdot \theta(y)}{2r} \\ \frac{\cos 2\pi t \cdot \theta(z)}{2r} & \frac{|z|^2 \sin 2\pi t}{2r^2} & \frac{\sin 2\pi t \cdot \theta(yz)}{2r^2} \\ \frac{\cos 2\pi t \cdot y}{2r} & \frac{\sin 2\pi t \cdot yz}{2r^2} & \frac{|y|^2 \sin 2\pi t}{2r^2} \end{pmatrix} + \frac{1}{4r^2}\begin{pmatrix} 2r^2 & 0 & 0 \\ 0 & 2|z|^2 & 2\theta(yz) \\ 0 & 2yz & 2|y|^2 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 - \sin 2\pi t & \frac{\cos 2\pi t \cdot z}{r} & \frac{\cos 2\pi t \cdot \theta(y)}{r} \\ \frac{\cos 2\pi t \cdot \theta(z)}{r} & \frac{(1+\sin 2\pi t)|z|^2}{r^2} & \frac{(1+\sin 2\pi t) \cdot \theta(yz)}{r^2} \\ \frac{\cos 2\pi t \cdot y}{r} & \frac{(1+\sin 2\pi t) \cdot yz}{r^2} & \frac{(1+\sin 2\pi t)|y|^2}{r^2} \end{pmatrix}, \qquad (5.27)\end{aligned}$$

$$Y(e^{2\pi\sqrt{-1}t} \cdot A) = -\sqrt{-1} \cdot \frac{e^{2\pi\sqrt{-1}t} A - e^{-2\pi\sqrt{-1}t} \overline{A}}{\sqrt{2||A||}}$$

$$= \frac{1}{\sqrt{r}} \begin{pmatrix} r\cos 2\pi t & \sin 2\pi t \cdot z & \sin 2\pi t \cdot \theta(y) \\ \sin 2\pi t \cdot \theta(z) & -\frac{|z|^2 \cos 2\pi t}{r} & -\frac{\cos 2\pi t}{r}\theta(yz) \\ \sin 2\pi t \cdot y & -\frac{\cos 2\pi t}{r} \cdot yz & -\frac{|y|^2 \cos 2\pi t}{r} \end{pmatrix}. \tag{5.28}$$

Then we can see from the expressions (5.27) and (5.28),

Proposition 5.4. *The space $\mathbb{R} \times \mathbb{S}_r$ is the universal covering of the space*

$$\varphi(\mathbb{R} \times \mathbb{S}_r) \cong U(1) \times \mathbb{S}_r.$$

Since the multiplications by $\{e^{2\pi\sqrt{-1}t}\}$ is the geodesic flow action of the space $P^2\mathbb{O}$ expressed via the map $\tau_{\mathbb{O}}$, the Hamiltonian ($=$ the metric function $= \mathrm{tr}\,(A \circ \overline{A})$) is constant $||A||^2 \equiv 4r^2$ on the space $\varphi(\mathbb{R} \times \mathbb{S}_r)$, so that the submanifold ${\tau_{\mathbb{O}}}^{-1}(\varphi(\mathbb{R} \times \mathbb{S}_r))$ is a geodesic flow invariant compact Lagrangian submanifold.

We put

$$\Lambda_r := {\tau_{\mathbb{O}}}^{-1}(\varphi(\mathbb{R} \times \mathbb{S}_r)) \tag{5.29}$$

Remark 3. *Let* $\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ *and define a map on $\mathcal{J}(3)$ (also on $\mathcal{J}(3)^{\mathbb{C}}$) by*

$$\tilde{\sigma} : \mathcal{J}(3) \ni A \longmapsto \sigma A \sigma, \tag{5.30}$$

then $\tilde{\sigma}$ is an automorphism of the Jordan algebra $\mathcal{J}(3)$ (also of the complexfied Jordan algebra $\mathcal{J}(3)^{\mathbb{C}}$) and the space $\mathbb{X}_{\mathbb{O}}$ is invariant, $\tilde{\sigma}(\mathbb{X}_{\mathbb{O}}) = \mathbb{X}_{\mathbb{O}}$. Then for $A \in \mathbb{S}_r$, $-\sigma A \sigma$ is the matrix constructed from the solutions (5.20).

The space $\Lambda_{-r} := -{\tau_{\mathbb{O}}}^{-1}\left(\tilde{\sigma}(\varphi(\mathbb{R} \times \mathbb{S}_r))\right)$ is also a Lagrangian submanifold and $\Lambda_r \cap \Lambda_{-r} = \emptyset$.

Although there are several choices of the variables instead of (y, z), from now on we only deal with the manifold Λ_r.

Let $A_0 = \begin{pmatrix} \sqrt{-1}r & r\mathbf{e}_0 & 0 \\ r\mathbf{e}_0 & -\sqrt{-1}r & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{S}_r$, and define a loop $\{\ell_1(t)\}$ in Λ_r by

$$\ell_1 : [0,1] \ni t \longmapsto {\tau_{\mathbb{O}}}^{-1}\left(e^{2\pi\sqrt{-1}t} \cdot \begin{pmatrix} \sqrt{-1}r & r & 0 \\ r & -\sqrt{-1}r & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$= \left(X(e^{2\pi\sqrt{-1}t} \cdot A_0),\ Y(e^{2\pi\sqrt{-1}t} \cdot A_0)\right)$$

$$= \left(\frac{1}{2} \cdot \begin{pmatrix} 1 - \sin 2\pi t & \cos 2\pi t & 0 \\ \cos 2\pi t & 1 + \sin 2\pi t & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \frac{1}{\sqrt{r}} \cdot \begin{pmatrix} r\cos 2\pi t & r\sin 2\pi t & 0 \\ r\sin 2\pi t & -r\cos 2\pi t & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \in \Lambda_r. \tag{5.31}$$

Then,

Lemma 5.5.

$$ {\ell_1}^*(\theta^{P^2\mathbb{O}})(t) = -2\pi\sqrt{r}dt. \tag{5.32}$$

This can be seen by the expression (5.31) as

$$\begin{aligned}\theta^{P^2\mathbb{O}}{}_{|\ell_1(t)} &= \sqrt{r}\cos 2\pi t\cdot(-\pi\cos 2\pi t)dt - \sqrt{r}\cos 2\pi t\cdot(\pi\cos 2\pi t)dt \\ &\qquad + 2\sqrt{r}\sin 2\pi t\cdot(-\pi\sin 2\pi t)dt \\ &= -2\pi\sqrt{r}dt.\end{aligned}$$

Proposition 5.6. *The action integral*

$$\int_{\ell_1}\theta^{P^2\mathbb{O}} = \int_0^1 -2\pi\sqrt{r}dt = -2\pi\sqrt{r}. \tag{5.33}$$

Let $\mathbf{m}_{\Lambda_r}$ be the Maslov class of the Lagrangian submanifold Λ_r.

Proposition 5.7.

$$< \mathbf{m}_{\Lambda_r}, \ell_1 >= -22. \tag{5.34}$$

We consider the projection map $\mathbf{q} : \Lambda_r \longrightarrow P^2\mathbb{O}$ on the loop $\{\ell_1(t)\}_{t\in[0,1]}$, where

$$\mathbf{q}(\ell_1(t)) = X(e^{2\pi\sqrt{-1}t}\cdot A_0) = \frac{1}{2}\cdot\begin{pmatrix}1-\sin 2\pi t & \cos 2\pi t & 0\\ \cos 2\pi t & 1+\sin 2\pi t & 0\\ 0 & 0 & 0\end{pmatrix},$$

and determine the caustics on the loop $\{\ell_1(t)\}$ in the Lagrangian submanifold Λ_r.

By the expressions (5.22) and (5.27), the projection map $\mathbf{q}_{|\Lambda_r}$ does not degenerate at $t \neq 1/4, 3/4 \bmod \mathbb{Z}$. In fact for such t the function $1-\sin 2\pi t$ is monotone and the function $\cos 2\pi t$ does not vanish. Hence we can see that the map

$$\begin{aligned}&\{(t,z,y) \mid t \neq 1/4, 3/4 \bmod \mathbb{Z}, |z|^2+|y|^2 = r^2\} \ni (t,z,y) \longmapsto \\ &\mathbf{q}\circ\tau_{\mathbb{O}}^{-1}(e^{2\sqrt{-1}\pi t}\cdot A) = X(e^{2\sqrt{-1}\pi t}\cdot A) \\ &= \frac{1}{2}\begin{pmatrix}1-\sin 2\pi t & \frac{\cos 2\pi t\cdot z}{r} & \frac{\cos 2\pi t\cdot\theta(y)}{r}\\ \frac{\cos 2\pi t\cdot\theta(z)}{r} & \frac{(1+\sin 2\pi t)|z|^2}{r^2} & \frac{(1+\sin 2\pi t)\cdot\theta(yz)}{r^2}\\ \frac{\cos 2\pi t\cdot y}{r} & \frac{(1+\sin 2\pi t)\cdot yz}{r^2} & \frac{(1+\sin 2\pi t)|y|^2}{r^2}\end{pmatrix}\end{aligned}$$

is locally diffeomorphic.

Hence, we consider a neighborhood of the points

$$\ell_1(1/4) = \left(\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, \sqrt{r}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\right), \text{ and}$$

$$\ell_1(3/4) = \left(\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \sqrt{r}\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&0\end{pmatrix}\right),$$

for the determination of the Malsov index of the loop $\{\ell_1(t)\}$. The condition for $X = \begin{pmatrix} s_1 & c & \theta(b) \\ \theta(c) & s_2 & a \\ b & \theta(a) & s_3 \end{pmatrix} \in P^2\mathbb{O}$ is given in terms of the components as

$$
\left\{
\begin{array}{l}
(s_3 + s_2)a + \theta(bc) = a, \\
(s_1 + s_3)b + \theta(ca) = b, \\
(s_2 + s_1)c + \theta(ab) = c, \\
{s_1}^2 + c\theta(c) + \theta(b)b = s_1, \\
{s_2}^2 + \theta(c)c + a\theta(a) = s_2, \\
{s_3}^2 + \theta(a)a + b\theta(b) = s_3, \\
\operatorname{tr} X \ = \ s_1 + s_2 + s_3 = 1.
\end{array}
\right.
\tag{5.35}
$$

where $a, b, c \in \mathbb{O}, s_i \in \mathbb{R}$. They are rewritten as

$$
\left\{
\begin{array}{l}
s_1 a = \theta(bc), \quad s_2 b = \theta(ca), \quad s_3 c = \theta(ab), \\
(s_1 - 1/2)^2 + c\theta(c) + \theta(b)b = (s_1 - 1/2)^2 + |c|^2 + |b|^2 = 1/4, \\
(s_2 - 1/2)^2 + \theta(c)c + a\theta(a) = (s_2 - 1/2)^2 + |c|^2 + |a|^2 = 1/4, \\
(s_3 - 1/2)^2 + \theta(a)a + b\theta(b) = (s_3 - 1/2)^2 + |a|^2 + |b|^2 = 1/4.
\end{array}
\right.
\tag{5.36}
$$

We consider local coordinates around the point

$$
X(e^{2\sqrt{-1}\pi t} \cdot A) = \frac{1}{2} \begin{pmatrix} 1 - \sin 2\pi t & \frac{\cos 2\pi t \cdot z}{r} & \frac{\cos 2\pi t \cdot \theta(y)}{r} \\ \frac{\cos 2\pi t \cdot \theta(z)}{r} & \frac{(1+\sin 2\pi t)|z|^2}{r^2} & \frac{(1+\sin 2\pi t)\cdot \theta(yz)}{r^2} \\ \frac{\cos 2\pi t \cdot y}{r} & \frac{(1+\sin 2\pi t)\cdot yz}{r^2} & \frac{(1+\sin 2\pi t)|y|^2}{r^2} \end{pmatrix}
$$

with $|t - 1/4| \ll 1$ and $|t - 3/4| \ll 1$.

$\boxed{\text{[I] } t = 1/4 \text{ case.}}$

So we solve the equations (5.36) in terms of a and c with $|a| \ll 1, |c| \ll 1$. The components b, s_1, s_2, s_3 are

$$
\begin{aligned}
& s_2(a, c) = 1/2 + \sqrt{1/4 - |a|^2 - |c|^2} > 1/2, \\
& b(a, c) = \frac{\theta(ca)}{s_2}, \\
& s_1(a, c) = 1/2 - \sqrt{1/4 - |b|^2 - |c|^2} = 1/2 - \sqrt{1/4 - |\frac{|c|^2|a|^2}{{s_2}^2} - |c|^2} < 1/2, \\
& s_3(a, c) = 1 - s_1 - s_2.
\end{aligned}
$$

Put $W_1 = \{(a, c) \in \mathbb{O} \times \mathbb{O} \mid |a|^2 + |c|^2 \ll 1\}$ and with these solutions (b, s_1, s_2, s_3) we consider the map $\mathcal{G}_1 : W_1 \to P^2\mathbb{O}$

$$
(a, c) \longmapsto X = \begin{pmatrix} s_1(a, c) & c & \theta(b(a, c)) \\ \theta(c) & s_2(a, c) & a \\ b(a, c) & \theta(a) & s_3(a, c) \end{pmatrix}
$$

and put $\tilde{W}_1 = \mathcal{G}_1(W_1)$, where $\mathcal{G}_1(0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Let $(\alpha, \gamma) \in \mathbb{O} \times \mathbb{O}$ be the dual coordinates of (a, c), then

$$d\mathcal{G}_1 : W_1 \times \mathbb{O}^2 \cong \mathbf{q}^{-1}(\tilde{W}_1),$$

which is understood as

$$(a, c, \alpha, \gamma) \longmapsto (X, Y) = \left(\begin{pmatrix} s_1(a,c) & c & \theta(b(a,c)) \\ \theta(c) & s_2(a,c) & a \\ b(a,c) & \theta(a) & s_3(a,c) \end{pmatrix}, \begin{pmatrix} t_1 & \gamma & \theta(\beta) \\ \theta(\gamma) & t_2 & \alpha \\ \beta & \theta(\alpha) & t_3 \end{pmatrix} \right)$$

where the components (t_1, t_2, t_3, β) are given by solving the equation

$$XY + YX = Y$$

with the variables (a, c, α, γ).

This equation are expressed in terms of the components as

$$\begin{cases} s_1 t_1 + c\theta(\gamma) + \theta(b)\beta + t_1 s_1 + \gamma\theta(c) + \theta(\beta) b = t_1, \\ s_1\gamma + c t_2 + \theta(b)\theta(\alpha) + t_1 c + \gamma s_2 + \theta(\beta)\theta(a) = \gamma, \\ s_1\theta(\beta) + c\alpha + \theta(b) t_3 + t_1\theta(b) + \gamma a + \theta(\beta) s_3 = \theta(\beta), \\ \theta(c)\gamma + s_2 t_2 + a\theta(\alpha) + \theta(\gamma) c + t_2 s_2 + \alpha\theta(a) = t_2, \\ \theta(c)\theta(\beta) + s_2\alpha + a t_3 + \theta(\gamma)\theta(b) + t_2 a + s_3\alpha = \alpha, \\ b\theta(\beta) + \theta(a)\alpha + s_3 t_3 + \beta\theta(b) + \theta(\alpha) a + t_3 s_3 = t_3. \end{cases} \tag{5.37}$$

Although the solutions β, t_1, t_2, t_3 in terms of the variables (a, c, α, γ) are obtained by differentiate the corresponding variables $b = b(a,c), s_1 = s_1(a,c), s_2 = s_2(a,c), s_3(a,c)$, for example we calculate $\frac{d\, b(a(\delta), c(\delta))}{d\delta}\Big|_{\delta=0}$ where a and c are functions of a temporary variable δ, $(|\delta| \ll 1)$ and by replacing $\dot{a}(0) = \alpha$, and $\dot{c}(0) = \gamma$. Also we can solve these algebraic equations by the following order

$$\begin{aligned} t_2 &= \frac{\theta(c)\gamma + \theta(\gamma)c + a\theta(\alpha) + \alpha\theta(a)}{1 - 2s_2}, \\ \beta &= \frac{\theta(c\alpha + \gamma a) - t_2 b}{s_2}, \\ t_1 &= \frac{c\theta(\gamma) + \gamma\theta(c) + \theta(b)\beta + \theta(\beta)b}{1 - 2s_1}, \\ t_3 &= t_1 - t_2. \end{aligned}$$

Of course, by the definition of the matrix Y these two solutions coincide.

By the map $d\mathcal{G}_1$ the subbundle at the point $(0, 0, \alpha, \gamma) \in W_1 \times \mathbb{O}^2 \in \mathbf{q}^{-1}(\tilde{W}_1)$ and the vertical subbundle $\mathcal{V}_{(0,0,\alpha,\gamma)}(\Lambda_r) \subset T(\mathbf{q}^{-1}(\tilde{W}_1))$ are isomorphic, for the determination of the Maslov index of the curve $\{\ell_1(t)\}$ when it across the caustics, it is enough to calculate in the space $W_1 \times \mathbb{O}^2$.

Let $z = (z_0, \dots, z_7) = (z_0, z') \cong \sum z_i \mathbf{e}_i \in \mathbb{R}^8 \cong \mathbb{O}$ and $y = (y_0, \dots, y_7) \cong \sum y_i \mathbf{e}_i \in \mathbb{R}^8 \cong \mathbb{O}$, where we put $z_0 = \sqrt{r^2 - |z'|^2 - |y|^2}$ and define a map $\mathcal{R} : (t, z', y) \longrightarrow (a, c, \alpha, \gamma)$ by

$$\begin{aligned}\mathcal{R}(t, z', y) &= (a_0, \dots, a_7, c_0, \dots, c_7, \alpha_0, \cdots, \alpha_7, \gamma_0, \cdots, \gamma_7) \\ &= \left(\frac{(1 + \sin 2\pi t)}{2r^2} \cdot \theta(yz), \frac{\cos 2\pi t}{2r} \cdot z, \frac{-\cos 2\pi t}{r\sqrt{r}} \cdot \theta(yz), \frac{\sin 2\pi t}{\sqrt{r}} \cdot z \right).\end{aligned}$$

Then $\tau_{\mathbb{O}} \circ \mathcal{G}_1 \circ \mathcal{R}(t, 0, 0) = e^{2\pi\sqrt{-1}t} \cdot A_0$ and its differential at (t, z', y) are

$$\begin{aligned} d\mathcal{R}_{(t,z',y)} \left(\frac{\partial}{\partial t} \right) &= \frac{\pi \cos 2\pi t}{r^2} \sum_{i=0}^{7} \{\theta(yz)\}_i \left(\frac{\partial}{\partial a_i} \right) - \frac{\pi \sin 2\pi t}{r} \sum_{i=0}^{7} \{z\}_i \left(\frac{\partial}{\partial c_i} \right) \\ &\quad - \frac{2\pi \sin 2\pi t}{r\sqrt{r}} \sum_{i=0}^{7} \{\theta(yz)\}_i \left(\frac{\partial}{\partial \alpha_i} \right) + \frac{2\pi \cos 2\pi t}{\sqrt{r}} \sum_{i=0}^{7} \{z\}_i \left(\frac{\partial}{\partial \gamma_i} \right), \end{aligned}$$

$$\begin{aligned} d\mathcal{R}_{(t,z',y)} \left(\frac{\partial}{\partial z_i} \right) &= \frac{1 + \sin 2\pi t}{2r^2} \sum_{j=0}^{7} \frac{\partial \{\theta(yz)\}_j}{\partial z_i} \left(\frac{\partial}{\partial a_j} \right) + \frac{\cos 2\pi t}{2r} \left(\frac{\partial z_0}{\partial z_i} \left(\frac{\partial}{\partial c_0} \right) + \left(\frac{\partial}{\partial c_i} \right) \right) \\ &\quad - \frac{\cos 2\pi t}{r\sqrt{r}} \sum_{j=0}^{7} \frac{\partial \{\theta(yz)\}_j}{\partial z_i} \left(\frac{\partial}{\partial \alpha_j} \right) + \frac{\sin 2\pi t}{\sqrt{r}} \left(\frac{\partial z_0}{\partial z_i} \left(\frac{\partial}{\partial \gamma_0} \right) + \left(\frac{\partial}{\partial \gamma_i} \right) \right), \\ d\mathcal{R}_{(t,z',y)} \left(\frac{\partial}{\partial y_i} \right) &= \frac{1 + \sin 2\pi t}{2r^2} \sum_{j=0}^{7} \frac{\partial \{\theta(yz)\}_j}{\partial y_i} \left(\frac{\partial}{\partial a_j} \right) - \frac{\cos 2\pi t}{r\sqrt{r}} \sum_{j=0}^{7} \frac{\partial \{\theta(yz)\}_j}{\partial y_i} \left(\frac{\partial}{\partial \alpha_j} \right). \end{aligned}$$

Here note that $\left.\dfrac{\partial z_0}{\partial z_i}\right|_{z'=0} = 0$, so that especially we have

$$\begin{aligned} d\mathcal{R}_{(t,0,0)} \left(\frac{\partial}{\partial t} \right) &= -\pi \sin 2\pi t \left(\frac{\partial}{\partial c_0} \right) + 2\pi\sqrt{r} \cos 2\pi t \left(\frac{\partial}{\partial \gamma_0} \right), \\ d\mathcal{R}_{(t,0,0)} \left(\frac{\partial}{\partial z_i} \right) &= \frac{\cos 2\pi t}{2r} \left(\frac{\partial}{\partial c_i} \right) + \frac{\sin 2\pi t}{\sqrt{r}} \left(\frac{\partial}{\partial \gamma_i} \right), \ i = 1, \dots, 7, \\ d\mathcal{R}_{(t,0,0)} \left(\frac{\partial}{\partial y_0} \right) &= \frac{(1 + \sin 2\pi t)}{2r} \left(\frac{\partial}{\partial a_0} \right) - \frac{\cos 2\pi t}{\sqrt{r}} \left(\frac{\partial}{\partial \alpha_0} \right), \\ d\mathcal{R}_{(t,0,0)} \left(\frac{\partial}{\partial y_i} \right) &= -\frac{(1 + \sin 2\pi t)}{2r} \left(\frac{\partial}{\partial a_i} \right) + \frac{\cos 2\pi t}{\sqrt{r}} \left(\frac{\partial}{\partial \alpha_i} \right), i = 1, \dots, 7. \end{aligned}$$

The Maslov index of the line segment, when it acrosses the vertical subspace $\mathcal{V}_{\ell_1(1/4)}$ at $(a, c) = (0, 0)$ is the sum of each such value in the symplectic subspaces spanned by the symplectic bases

$$\begin{aligned} H_0 &= \left\{ \left(\frac{\partial}{\partial c_0} \right), \left(\frac{\partial}{\partial \gamma_0} \right) \right\}, \\ H_i &= \left\{ \left(\frac{\partial}{\partial c_i} \right), \left(\frac{\partial}{\partial \gamma_i} \right) \right\}, \ i = 1, \dots 7, \end{aligned}$$

$$H_{7+i} = \left\{ \left(\frac{\partial}{\partial a_i} \right), \left(\frac{\partial}{\partial \alpha_i} \right) \right\}, \; i = 0, \dots 7.$$

Since the vectors $\mathcal{R}_{(1/4,0,0)}\left(\frac{\partial}{\partial t}\right)$ and $\mathcal{R}_{(1/4,0,0)}\left(\frac{\partial}{\partial y_i}\right)$ $(i = 0, \dots 7)$ are transversal to the vertical space $\mathcal{V}_{\ell_1(1/4)}$ and the derivative $\frac{d \frac{\cos 2\pi t}{\sin 2\pi t}}{dt}\Big|_{t=1/4} < 0$

Proposition 5.8. *The Maslov index of the line segment $\{\ell_1(t)\}_{|t-1/4| \ll 1}$ with respect to the vertical subbundle $\mathcal{V}_{\ell_1(1/4)}$, when t varies from $1/4 - \epsilon$ to $1/4 + \epsilon$ is -7.*

$\boxed{\text{[II] } t = 3/4 \text{ case}}$

In this case $\cos 2\pi t \cdot z$ and $\cos 2\pi t \cdot y$ both vanish when $t = 3/4$, and also $s_1 = \frac{1 - \sin 2\pi t}{2} > 1/2$ when t is close to $3/4$, we consider the local coordinates

$$\mathcal{P}_2 : P^2\mathbb{O} \ni X = \begin{pmatrix} s_1 & c & \theta(b) \\ \theta(c) & s_2 & a \\ b & \theta(a) & s_3 \end{pmatrix} \longmapsto (b, c) \in \mathbb{O}^2 \cong \mathbb{R}^{16}.$$

In fact, as the same way for the case of $t = 1/4$ we have explicit solutions s_1, a and s_2 (hence s_3) in terms of the coordinates (c, b), when $|b|^2 + |c|^2 \ll 1$:

$$\begin{aligned} s_1 &= 1/2 + \sqrt{1/4 - |b|^2 - |c|^2} > 1/2, \\ a &= \frac{\theta(bc)}{s_1} = \frac{\theta(bc)}{1/2 + \sqrt{1/4 - |b|^2 - |c|^2}}, \\ s_2 &= 1/2 - \sqrt{1/4 - |a|^2 - |c|^2} < 1/2, \text{ and} \\ s_3 &= 1 - s_1 - s_2. \end{aligned}$$

Let $W_2 \subset \mathbb{O}^2$ be an small open subset around the point $(0, 0)$ on which we have the above solutions (s_1, a, s_2, s_3) in terms of $(c, b) \in W_2$ and define a map $\mathcal{G}_2$: $W_2 \ni (c, b) \longmapsto \begin{pmatrix} s_1 & c & \theta(b) \\ \theta(c) & s_2 & a \\ b & \theta(a) & s_3 \end{pmatrix}$ with these solutions in the matrix elements. We denote $\mathcal{G}_2(W_2) := \tilde{W}_2 \subset P^2\mathbb{O}$ and satisfies

$$\mathcal{G}_2 \circ \mathcal{P}_2 = Id \text{ on } \tilde{W}_2.$$

The point $\ell_1(3/4) \in \mathbf{q}^{-1}(\tilde{W}_2)$, hence points $\{\ell_1(t)\}_{|t-3/4| \ll 1} \subset \mathbf{q}^{-1}(\tilde{W}_2)$.

As before we identify

$$\mathbf{q}^{-1}(\tilde{W}_2) \cong T^*(\tilde{W}_2) \cong W_2 \times \mathbb{R}^{16} \cong T(W_2)$$

and denote (γ, β) the dual coordinates of the local coordinates (c, b). The Lagrangian submanifold $\Lambda_r \cap \mathbf{q}^{-1}(\tilde{W}_2)$ on this coordinates neighborhood $\tilde{W}_2$ is given

$${\tau_{\mathbb{O}}}^{-1}\left(\{\varphi(t, A) \mid |t - 3/4| \ll 1, \; A \in \mathbb{S}_r\}\right).$$

Then it is expressed on $W_2 \times \mathbb{O}^2 = W_2 \times \mathbb{R}^8 \times \mathbb{R}^8$ as

$$\left\{(c,b,\gamma,\beta) = \left(\frac{\cos 2\pi t}{2r}z, \frac{\cos 2\pi t}{2r}y, \frac{\sin 2\pi t}{\sqrt{r}}z, \frac{\sin 2\pi t}{\sqrt{r}}y\right) \;\Big|\; |t-3/4| \ll 1, |z|^2+|y|^2 = r^2\right\}.$$

Now we can work in this expression of the Lagrangian submanifold Λ_r on $\tilde{W}_2$.

Then the curve ℓ_1 is expressed as

$$\ell_1(t) = \left(\frac{\cos 2\pi t}{2r}r\mathbf{e}_0, 0, \frac{\sin 2\pi t}{\sqrt{r}}r\mathbf{e}_0, 0\right) = \left(\frac{\cos 2\pi t}{2}, 0, \ldots, 0, \sqrt{r}\sin 2\pi t, 0, \ldots, 0\right).$$

Hence gain as before the case of [I], we work on the coordinates $(t, z_1, \ldots, z_7, y_0, \ldots, y_7)$ by expressing $z_0 = \sqrt{r^2 - \sum_{i=1}^7 {z_i}^2 - |y|^2}$ and denote the map

$$\mathcal{T} : (t, z', y) = (t, z_1, \ldots, z_7, y_0, \ldots, y_7) \mapsto \mathcal{P}_2 \circ \varphi(t, z_0, z_1, \ldots, z_7, y_0, \ldots, y_7)$$

When we identify the tangent space $T_{\varphi(t,z,y)}(\Lambda_r)$ with the space $W_2 \times \mathbb{O}^2 \cong W_2 \times \mathbb{R}^{16}$ through the map (differential of the graph map $\mathcal{G}_2$)

$$\mathcal{S} : (t, z', y) \mapsto (c,b,\gamma,\beta) = \left(\frac{\cos 2\pi t}{2r}z, \frac{\cos 2\pi t}{2r}y, \frac{\sin 2\pi t}{\sqrt{r}}z, \frac{\sin 2\pi t}{\sqrt{r}}y\right),$$

where we put $z_0 = \sqrt{r^2 - |z'|^2 - |y|^2}$ is spanned by the basis vectors

$$\begin{aligned}
d\mathcal{S}_{(t,z',y)}\left(\frac{\partial}{\partial t}\right) &= -\frac{\pi \sin 2\pi t}{r}\sum_{i=0}^{7} z_i\left(\frac{\partial}{\partial c_i}\right) - \frac{\pi \sin 2\pi t}{r}\sum_{i=0}^{7} y_i\left(\frac{\partial}{\partial b_i}\right) \\
&\quad + \frac{2\pi \cos 2\pi t}{\sqrt{r}}\sum_{i=0}^{7} z_i\left(\frac{\partial}{\partial \gamma_i}\right) + \frac{2\pi \cos 2\pi t}{\sqrt{r}}\sum_{i=0}^{7} y_i\left(\frac{\partial}{\partial \beta_i}\right) \\
d\mathcal{S}_{(t,z',y)}\left(\frac{\partial}{\partial z_i}\right) &= \frac{\cos 2\pi t}{2r}\left(\frac{\partial z_0}{\partial z_i}\left(\frac{\partial}{\partial c_0}\right) + \left(\frac{\partial}{\partial c_i}\right)\right) \\
&\quad + \frac{\sin 2\pi t}{\sqrt{r}}\left(\frac{\partial z_0}{\partial z_i}\left(\frac{\partial}{\partial \gamma_0}\right) + \left(\frac{\partial}{\partial \gamma_i}\right)\right), (i = 1, \ldots, 7), \\
d\mathcal{S}_{(t,z',y)}\left(\frac{\partial}{\partial y_i}\right) &= \frac{\cos 2\pi t}{2r}\left(\frac{\partial}{\partial b_i}\right) + \frac{\sin 2\pi t}{\sqrt{r}}\left(\frac{\partial}{\partial \beta_i}\right), (i = 0, \ldots, 7).
\end{aligned}$$

Especially on the curve ℓ_1 around $t = 3/4$, they are spanned by basis vectors

$$\begin{aligned}
d\mathcal{S}_{(t,0,0)}\left(\frac{\partial}{\partial t}\right) &= -\pi \sin 2\pi t\left(\frac{\partial}{\partial c_0}\right) + 2\pi\sqrt{r}\cos 2\pi t\left(\frac{\partial}{\partial \gamma_0}\right), \\
d\mathcal{S}_{(t,0,0)}\left(\frac{\partial}{\partial z_i}\right) &= \frac{\cos 2\pi t}{2r}\left(\frac{\partial}{\partial c_i}\right) + \frac{\sin 2\pi t}{\sqrt{r}}\left(\frac{\partial}{\partial \gamma_i}\right), \\
d\mathcal{S}_{(t,0,0)}\left(\frac{\partial}{\partial y_i}\right) &= \frac{\cos 2\pi t}{2r}\left(\frac{\partial}{\partial b_i}\right) + \frac{\sin 2\pi t}{\sqrt{r}}\left(\frac{\partial}{\partial \beta_i}\right).
\end{aligned}$$

The spaces $G_i, G_{7+i}, (i = 0, \ldots, 7)$, each spanned by symplectic basis vectors $\left\{\left(\frac{\partial}{\partial c_i}\right), \left(\frac{\partial}{\partial \gamma_i}\right)\right\}$ and $\left\{\left(\frac{\partial}{\partial b_i}\right), \left(\frac{\partial}{\partial \beta_i}\right)\right\}$ are 2-dimensional symplectic subspaces and give the decompositions of the tangent space $T_{(t,0,0)}(W_2 \times \mathbb{O}^2) \cong \mathbb{R}^{32}$ invariant along the curve $\{\ell_t\}_{3/4-\epsilon<t<3/4+\epsilon}$.

In the space G_0 the tangent space

$$T_{(t,0,0)}(\mathcal{S}((3/4-\epsilon, 3/4+\epsilon) \times \{(z', y) \mid |z'|^2 + |y|^2 \ll 1\})$$

is transversal to the vertical space spanned by $\frac{\partial}{\partial \beta_0}$ so that the Maslov index of the line segment $\{\ell_t\}_{3/4-\epsilon<t<3/4+\epsilon}$ is zero.

From the expression of the curve $\{\ell_t\}_{3/4-\epsilon<t<3/4+\epsilon}$ in each subspace $G_i, i \geq 1$ and $G_{7+j}, j \geq 0$ we know that the Maslov index are all -1, so that totally the Maslov index of the curve $\{\ell_t\}_{1/4-\epsilon<t<1/4+\epsilon}$ is -15.

Finally we have

Proposition 5.9. *The Maslov index of the loop $\{\ell_1\}$ with respect to the vertical subbundle of the projection map $\mathbf{q} : T^*P^2\mathbb{O} \to P^2\mathbb{O}$ is -22.*

Then the Maslov quantization condition requires that

$$\frac{1}{2\pi}\int_{\ell_1} \theta^{P^2\mathbb{O}} - \frac{1}{4} < \mathbf{m}_{\Lambda_r}, \ell_1 > \in \mathbb{Z},$$

which in our case of Λ_r is

$$-\sqrt{r} + \frac{22}{4} \in \mathbb{Z}.$$

Hence

Theorem 5.10. *When $r = \left(\frac{11+2k}{2}\right)^2$ for $k \in \mathbb{Z}$, the Lagrangian submanifold satisfies the Maslov quantization condition.*

Proof. The existence of the invariant measure is shown by two ways:

(1) The geodesic flow action is periodic, so that we can prove the existence of geodesic flow invariant measure by integrating any measure.

(2) Since the space $\mathbb{X}_{\mathbb{O}}$ has a Calabi-Yau structure, that is the canonical line bundle is holomorphically trivial by a nowhere vanishing holomorphic 16-form, the absolute section of this holomorphic 16 form restricted to any Lagrangian submanifold defines a half density on it. □

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