Ribbonness of a stable-ribbon surface-link, II. General case

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ABSTRACT

It is shown that any handle-irreducible summand of every stable-ribbon surface-link is a unique ribbon surface-link up to equivalences. This is a generalization of a previously observed result for a stably trivial surface-link. Two applications are given. One application is an observation that a connected sum of two surface-links is a ribbon surface-link if and only if both of the connected summands are ribbon surface-links. The other application is an observation that any sphere-link consisting of trivial components is a ribbon sphere-link.

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1. Introduction

This paper is a generalization of the result of the paper [10] with supplement [11] on a trivial surface-link to a result on a ribbon surface-link. A *surface-link* is a closed oriented (possibly disconnected) surface F embedded in the 4-space $\mathbf{R}^4$ by a smooth (or a piecewise-linear locally flat) embedding. When F is connected, it is also called a *surface-knot*. For a smooth embedding of a fixed (possibly disconnected) closed surface $\mathbf{F}$ into $\mathbf{R}^4$, it is also called an $\mathbf{F}$-*link*. If $\mathbf{F}$ is the disjoint union of some copies of the 2-sphere S^2, then it is also called an S^2-*link*. When $\mathbf{F}$ is connected, it is also called an $\mathbf{F}$-*knot*, and an S^2-*knot* for $\mathbf{F} = S^2$. Two surface-links F and F' are *equivalent* by an *equivalence* f if F is sent to F' orientation-preservingly by an orientation-preserving diffeomorphism (or piecewise-linear homeomorphism) $f : \mathbf{R}^4 \to \mathbf{R}^4$. A *trivial* surface-link is a surface-link F which bounds disjoint handlebodies smoothly embedded in $\mathbf{R}^4$, where a handlebody is a 3-manifold which is a 3-ball, solid torus or a disk sum of some number of solid tori. A trivial surface-knot is also called an *unknotted* surface-knot. A trivial disconnected surface-link is also called an *unknotted-unlinked* surface-link. For any given closed oriented (possibly disconnected) surface $\mathbf{F}$, a trivial $\mathbf{F}$-link exists uniquely up to equivalences (see [3]). A *ribbon* surface-link is a surface-link F which is obtained from a trivial nS^2-link O for some n (where nS^2 denotes the disjoint union of n copies of the 2-sphere S^2) by surgery along an embedded 1-handle system (see [4], [13, II]). A *stabilization* of a surface-link F is a connected sum $\bar{F} = F\#_{k=1}^{s}T_k$ of F and a system of trivial torus-knots $T_k\,(k = 1, 2, \dots, s)$. By granting $s = 0$, a surface-link F itself is regarded as a stabilization of F. The trivial torus-knot system T is called the *stabilizer* with *stabilizer components* $T_k\,(k = 1, 2, ..., s)$ on the stabilization $\bar{F}$ of F. A *stable-ribbon* surface-link is a surface-link F such that a stabilization $\bar{F}$ of F is a ribbon surface-link. For every surface-link F, there is a surface-link F^* with minimal total genus such that F is equivalent to a stabilization of F^*. The surface-link F^* is called a *handle-irreducible summand* of F. The following result called *Stable-Ribbon Theorem* is our main theorem.

Theorem 1.1. Any handle-irreducible summand F^* of every stable-ribbon surface-link F is a ribbon surface-link which is determined uniquely from F up to equivalences.

Since any stabilization of a ribbon surface-link is a ribbon surface-link, Theorem 1.1 implies the following corollary:

Corollary 1.2. Every stable-ribbon surface-link is a ribbon surface-link.

A *stably trivial* surface-link is a surface-link F such that a stabilization $\bar{F}$ of F is a trivial surface-link. Since a trivial surface-link is a ribbon surface-link, Theorem 1.1 also implies the following corollary, which is used to prove the unknotting conjecture for a surface-link in [10, 11, 12].

Corollary 1.3. Any handle-irreducible summand of every stably trivial surface-link is a trivial S^2-link, so that every stably trivial surface-link is a trivial surface-link.

The plan for the proof of Theorem 1.1 is to show the following two lemmas by an argument based on [10, 11].

Lemma I. Any two handle-irreducible summands of any surface-link are equivalent.

Lemma II. Any stable-ribbon surface-link is a ribbon surface-link.

The proof of Theorem 1.1 is completed by these lemmas as follows:

Proof of Theorem 1.1. By Lemma II, any handle-irreducible summand of every stable-ribbon surface-link is a ribbon surface-link, which is unique up to equivalences by Lemma I. This completes the proof of Theorem 1.1.

An idea of the proof of Lemma I is to generalize the uniqueness result of an O2-handle pair on a surface-link to the case where the restriction on the attaching part is relaxed. An idea of the proof of Lemma II is to consider a semi-unknotted punctured handlebody system, simply called a *SUPH system*, of a ribbon surface-link. Two applications of Theorem 1.1 are given. One application is the following theorem.

Theorem 1.4. A connected sum $F = F_1 \# F_2$ of surface-links $F_i\,(i = 1, 2)$ in S^4 is a ribbon surface-link if and only if the surface-links $F_i\,(i = 1, 2)$ are both ribbon surface-links.

This theorem contrasts with a behavior of a classical ribbon knot, because every classical knot is a connected summand of a ribbon knot. In fact, for every knot k and the inversed mirror image $-k^*$ of k in the 3-sphere S^3, the connected sum $k\#(-k^*)$ is a ribbon knot in S^3 (see [13, I] and, for a recent interpretation, [9]). The other application is the following theorem.

Theorem 1.5. Every S^2-link consisting of trivial components in S^4 is a ribbon S^2-link.

The proofs of Lemmas I and II are given in Sections 2 and 3, respectively. In Section 4, the proofs of Theorems 1.4 and 1.5 are given.

2. Proof of Lemma I

A 2-*handle* on a surface-link F in $\mathbf{R}^4$ is an embedded 2-handle $D \times I$ on F with D a core disk such that $D \times I \cap F = \partial D \times I$, where I denotes a closed interval containing 0 and $D \times 0$ is identified with D. Two (possibly immersed) 2-handles $D \times I$ and $E \times I$ on F are *equivalent* if there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from F to itself such that the restriction $f|_F : F \to F$ is the identity map and $f(D \times I) = E \times I$. An *orthogonal 2-handle pair* (or simply, an *O2-handle pair*) on F is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I$, $D' \times I$ on F such that

$$D \times I \cap D' \times I = \partial D \times I \cap \partial D' \times I$$

and $\partial D \times I$ and $\partial D' \times I$ *meet orthogonally* on F, that is, the boundary circles ∂D and $\partial D'$ meet transversely at one point p and the intersection $\partial D \times I \cap \partial D' \times I$ is homeomorphic to the square $Q = p \times I \times I$. Let $(D \times I, D' \times I)$ be an O2-handle pair on a surface-link F. Let $F(D \times I)$ and $F(D' \times I)$ be the surface-links obtained from F by the surgeries along $D \times I$ and $D' \times I$, respectively. Let $F(D \times I, D' \times I)$ be the surface-link which is the union of the plumbed disk

$$\delta = \delta_{D \times I, D' \times I} = D \times \partial I \cup Q \cup D' \times partial I$$

and the surface

$$F^c_\delta = \mathrm{cl}(F \setminus (\partial D \times I \cup \partial D' \times I).$$

A once-punctured torus T^o in a 3-ball B is *trivial* if T^o is smoothly and properly embedded in B which splits B into two solid tori. A *bump* of a surface-link F is a 3-ball B in $\mathbf{R}^4$ with $F \cap B = T^o$ a trivial once-punctured torus in B. Let $F(B)$ be a surface-link $F^c_B \cup \delta_B$ for the surface $F^c_B = \mathrm{cl}(F \setminus T^o)$ and a disk δ_B in ∂B with $\partial \delta_B = \partial T^o$, where note that $F(B)$ is uniquely determined up to cellular moves on δ_B keeping F^c_B fixed. Here, a *cellular move* of a surface P in $\mathbf{R}^4$ is a surface $\tilde{P}$ in $\mathbf{R}^4$ such that the complements $d = \mathrm{cl}(P \setminus P_0)$ and $\tilde{d} = \mathrm{cl}(\tilde{P} \setminus P_0)$ of the intersection $P_0 = P \cap P'$ are disks in the interiors of P and $\tilde{P}$, respectively and the union $d \cup \tilde{d}$ is a 2-sphere bounding a 3-ball smoothly embedded in $\mathbf{R}^4$ and not meeting $P_0 \setminus \partial d = P_0 \setminus \partial \tilde{d}$. For an O2-handle pair $(D \times I, D' \times I)$ on a surface-link F, let $\Delta = D \times I \cup D' \times I$ is a 3-ball in $\mathbf{R}^4$ called the *2-handle union*. Consider the 3-ball Δ as a Seifert hypersurface of the trivial S^2-knot $K = \partial \Delta$ in $\mathbf{R}^4$ to construct a 3-ball B_Δ obtained from Δ by adding an outer boundary collar. This 3-ball B_Δ is a bump of F, which we call the *associated bump* of the O2-handle pair $(D \times I, D' \times I)$. When the 3-ball Δ and a boundary collar of F^c_δ are deformed into the 3-space $\mathbf{R}^3 \subset \mathbf{R}^4$, this associated bump B_Δ is also considered as a regular neighborhood of Δ in $\mathbf{R}^3$. By [10], it is observed that an O2-handle unordered pair $(D \times I, D' \times I)$ on a surface-link F is uniquely constructed from any given bump B of F in $\mathbf{R}^4$ with $F(D \times I, D' \times I) \cong F(B)$. Further, for any O2-handle pair $(D \times I, D' \times I)$ on any surface-link F and the associated bump B, there are equivalences

$$F(B) \cong F(D \times I, D' \times I) \cong F(D \times I) \cong F(D' \times I)$$

with equivalences attained by cellular moves keeping F^c_δ fixed. A punctured torus T^o in a 4-ball A is *trivial* if T^o is smoothly and properly embedded in A and there is a solid torus V in A with $\partial V = T^o \cup \delta_A$ for a disk δ_A in the 3-sphere ∂A. A *4D bump* of a surface-link F is a 4-ball A in $\mathbf{R}^4$ with $F \cap A = T^o$ a trivial punctured torus in A. A 4D bump A is obtained from a bump B of a surface-link F by taking a bi-collar $c(B \times [-1, 1])$ of B in $\mathbf{R}^4$ with $c(B \times 0) = B$. The following lemma is proved by using a 4D bump A.

Lemma 2.1. Let $(D \times I, D' \times I)$ be any O2-handle pair on any surface-link F in $\mathbf{R}^4$, and T a trivial torus-knot in $\mathbf{R}^4$ with a spin loop basis (ℓ, ℓ'). Then there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from the surface-link F to a connected sum $F(D \times I, D' \times I)\#T$ keeping F^c_δ fixed such that $f(\partial D) = \ell$ and $f(\partial D') = \ell'$.

Proof of Lemma 2.1. Let A be a 4D bump associated with the O2-handle pair $(D \times I, D' \times I)$ on F. Let δ_A be a disk in the 3-sphere ∂A such that the union of $\delta = \delta_{D\times I, D'\times I}$ and the trivial punctured torus $F \cap A = P$ bounds a solid torus V in A. Then there is an equivalence $f' : F \cong F(D \times I, D' \times I)\#T$ by deforming V in A so that P is isotopically deformed into the summand T^o of a connected sum $\delta\#T$ in A with the spin loop pair $(\partial D, \partial D')$ on F sent to a spin loop basis $(\tilde{\ell}, \tilde{\ell}')$ of T^o. By [2] (see [10, (2.4.2)]), there is an orientation-preserving diffeomorphism $g : \mathbf{R}^4 \to \mathbf{R}^4$ with $g|_{\mathrm{cl}(\mathbf{R}^4 \setminus A)} = 1$ such that

$$g(\tilde{\ell}, \tilde{\ell}) = (\ell, \ell').$$

By the composition gf', a desired equivalence f is obtained. This completes the proof of Lemma 2.1.

A surface-link F has *only unique O2-handle pair in the rigid sense* if for any O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F with $(\partial D) \times I = (\partial E) \times I$ and $(\partial D') \times I = (\partial E') \times I$, there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from F to F keeping F^c_δ fixed such that $f(D \times I) = E \times I$ and $f(D' \times I) = E' \times I$. It is proved in [10, 11] that every surface-link F has only unique O2-handle pair in the rigid sense. A surface-link F has *only unique O2-handle pair in the soft sense* if for any O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F attached to the same connected component of F, there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from F to F such that $f(D \times I) = E \times I$ and $f(D' \times I) = E' \times I$. A surface-link not admitting any O2-handle pair is understood as a surface-link with only unique O2-handle pair in both the rigid and soft senses. The following uniqueness of an O2-handle pair in the soft sense is essentially a consequence of the uniqueness of an O2-handle pair in the rigid sense.

Theorem 2.2 (Uniqueness of an O2-handle pair in the soft sense). Every ribbon surface-link has only unique O2-handle pair in the soft sense.

Proof of Theorem 2.2. Let $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ be any two O2-handle pairs on a surface-link F attached to the same connected component F_1 of F. Let $F^{(1)} = F \setminus F_1$. Let $F_1(h)$ be a *trivial* surface-knot obtained from F_1 by surgery along a system h of disjoint 1-handles h_j $(j = 1, 2, \dots, m)$ on F_1 where h may be disjoint from $(D \times I, D' \times I)$, $(E \times I, E' \times I)$ and $F^{(1)}$. Let (e_i, e'_i) $(i = 1, 2, \dots, g)$ be a spin loop basis for F_1 in with $(e_1, e'_1) = (\partial D, \partial D')$ in $\mathbf{R}^4$, and $(\tilde{e}_i, \tilde{e}'_i)$ $(i = 1, 2, \dots, g)$ be a spin loop basis for F_1 in with $(\tilde{e}_1, \tilde{e}'_1) = (\partial E, \partial E')$ in $\mathbf{R}^4$. Let $(m_j, \ell_j)\,(j = 1, 2, \dots, m)$ be a spin loop system on $F_1(h)$ such that $m_j = \partial d_i$ for a meridian disk (transverse disk) d_j of the 1-handle h_j and the systems $(e_i, e'_i)\,(i = 1, 2, \dots, g), (m_j, \ell_j)\,(j = 1, 2, \dots, m)$ and $(\tilde{e}_i, \tilde{e}'_i)\,(i = 1, 2, \dots, g), (m_j, \ell_j)\,(j = 1, 2, \dots, m)$ form loop bases of the trivial surface-kno$F_1(h)$. The loop system $\ell_j\,(j = 1, 2, \dots, m)$ bounds a smoothly embedded disjoint disk system $d'_j\,(j = 1, 2, \dots, n)$ in S^4 with $d'_j \cap F_1(h) = \ell_j\,(j = 1, 2, \dots, m)$ and disjoint from the interior system of the disk system $d_j\,(j = 1, 2, \dots, m)$ by [2]. The 1-handle h_j is deformed so that d_j is disjoint from $(D \times I, D' \times I)$, $(E \times I, E' \times I)$ and $F^{(1)}$. Since every surface-link has only unique O2-handle pair in the rigid sense, there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from $F(h)$ to itself keeping $F^{(1)}$ fixed such that $f(D\times I, D'\times I) = (E\times I, E'\times I)$ and $f(d_j\times I, d'_j\times I) = (d_j\times I, d'_j\times I)\,(j = 1, 2, \dots, m)$, which induces an equivalence $f' : \mathbf{R}^4 \to \mathbf{R}^4$ sending F to itself and $(D \times I, D' \times I)$ to $(E \times I, E' \times I)$. Thus, every surface-link F has only unique O2-handle pair in the soft sense. This completes the proof of Theorem 2.2.

The following corollary is obtained from the proof of Theorem 2.2.

Corollary 2.3. Let F and F' be surface-links with ordered components $F_i\,(i = 1, 2, \dots, r)$ and $F'_i\,(i = 1, 2, \dots, r)$, respectively. Assume that the stabilizations $\bar{F} = F\#_i T, \bar{F}' = F'\#_i T$ of F, F' with induced ordered components obtained by the connected sums $F_i\#T, F'_i\#T$ of the ith components F_i, F'_i and a trivial torus-knot T for some i, respectively are equivalent by a component-order-preserving equivalence $\mathbf{R}^4 \to \mathbf{R}^4$. Then F is equivalent to F' by a component-order-preserving equivalence $\mathbf{R}^4 \to \mathbf{R}^4$.

Remark 2.4. Corollary 2.3 for ribbon surface-links F and F' has a different proof using the result of [8].

The proof of Lemma I is done as follows.

Proof of Lemma I. A surface-link F with r ordered components is *kth-handle-reducible* if F is equivalent to a stabilization $F'\#_k n_k T$ of a surface-link F' for a positive integer n_k, where $\#_k n_k T$ denotes the stabilizer components $n_k T$ attaching to the kth component of F'. Otherwise, the surface-link F is *kth-handle-irreducible.* Note that if a kth-handle-irreducible surface-link F is component-order-preserving equivalent to a surface-link G, then G is also kth-handle-irreducible. Let F and G be ribbon surface-links with components $F_i\,(i = 1, 2, \dots, r)$ and $G_i\,(i = 1, 2, \dots, r)$, respectively. Let $F^*$and G^* be handle-irreducible summands of F and G, respectively. Assume that there is an equivalence f from F to G. Then it is shown that F^* and

G^* are equivalent as follows. Changing the indexes if necessary, we assume that f sends F_i to G_i for every i. Let

$$\begin{aligned} F &= F^*\#_1 n_1 T \#_2 n_2 T \#_3 \ldots \#_r n_r T, \\ G &= G^*\#_1 n_1' T \#_2 n_2' T \#_3 \ldots \#_r n_r' T. \end{aligned}$$

Taking the inverse equivalence f^{-1} instead of f if necessary, we may assume that $n_1' \geq n_1$. If $n_1' > n_1$, then by Corollary 2.3, there is an equivalence $f^{(1)}$ from the first-handle-irreducible surface-link

$$F^{(1)} = F^*\#_2 n_2 T \#_3 \ldots \#_r n_r T$$

to the first-handle-reducible surface-link

$$G^*\#_1 (n_1' - n_1) T \#_2 n_2' T \#_3 \ldots \#_r n_r' T,$$

which has a contradiction. Thus, $n_1' = n_1$ and the first-handle-irreducible surface-link $F^{(1)}$ is equivalent to the first-handle-irreducible ribbon surface-link

$$G^{(1)} = G^*\#_2 n_2' T \#_3 \ldots \#_r n_r' T.$$

By continuing this process, it is shown that F^* is equivalent to G^*. This completes the proof of Lemma I.

3. Proof of Lemma II

A *chord graph* is a pair (o, α) of a trivial ink o and an arc system α attaching to o in the 3-space $\mathbf{R}^3$, where o and α are called a *based loop system* and a *chord system*, respectively. A *chord diagram* is a diagram $C(o, \alpha)$ in the plane $\mathbf{R}^2$ of a chord graph (o, α) as a spatial graph. Let D^+ be a proper disk system in the upper half-space $\mathbf{R}^4_+$ obtained from a disk system d^+ in $\mathbf{R}^3$ bounded by o by pushing the interior into $\mathbf{R}^4_+$. Similarly, let D^- be a proper disk system in the lower half-space $\mathbf{R}^4_-$ obtained from a disk system d^- in $\mathbf{R}^3$ bounded by o by pushing the interior into $\mathbf{R}^4_-$. Let O be the union of D^+ and D^- which is a trivial nS^2-link in the 4-space $\mathbf{R}^4$, where n is the number of components of o. The union $O \cup \alpha$ is called a *chorded sphere system* constructed from a chord graph (o, α). By using the Horibe-Yanagawa lemma in [13, I], the chorded sphere system $O \cup \alpha$ up to orientation-preserving diffeomorphisms of $\mathbf{R}^4$ is independent of choices of d^+ and d^- and uniquely determined by the chord graph (o, α). A ribbon surface-link $F = F(o, \alpha)$ is uniquely constructed from the chorded sphere system $O \cup \alpha$ so that F isobtained from O by surgery along a 1-handle system $N(\alpha)$ on O with core arc system α (see [5, 6, 7, 8]), where note by [3] that the surface-link F up to equivalences is unaffected by choices of the 1-handle system $N(\alpha)$. A surface-link F in $\mathbf{R}^4$ admits a *semi-unknotted punctured handlebody system* (or simply a *SUPH system*) in $\mathbf{R}^4$ if there is a multi-punctured handlebody system V in $\mathbf{R}^4$ with $\partial V = F \cup O$ for a trivial S^2-link O. Note that the numbers of connected components of F and V are equal. The following lemma makes a characterization of a ribbon surface-link (cf. [13, II], Yanagawa [14]).

Lemma 3.1. A surface-link F is a ribbon surface-link if and only if F admits a SUPH system V in $\mathbf{R}^4$.

Proof of Lemma 3.1. A SUPH system V for a ribbon surface-link F is constructed from a chorded sphere system $O \cup \alpha$ by taking the union of a thickening $O \times [0, 1]$ of O in $\mathbf{R}^4$ and the 1-handle system $N(\alpha)$ attaching only to $O \times 0$. Conversely, given a SUPH system V in $\mathbf{R}^4$ with $\partial V = F \cup O$ for a trivial S^2-link O, then take a chord system α in V attaching to O so that the frontier of the regular neighborhood of $O \cup \alpha$ in V is parallel to F in V. The chorded sphere system $O \cup \alpha$ shows that F is a ribbon surface-link. This completes the proof of Lemma 3.1.

Let F be a surface-link of components $F_i \, (i = 1, 2, \dots, r)$ in $\mathbf{R}^4$. Let $F\#T$ be the connected sum of F and a trivial torus-knot T in $\mathbf{R}^4$ consisting of the components $F_1\#T, F_i \, (i = 2, 3, \dots, r)$. Assume that $F\#T$ is a ribbon surface-link. By Lemma 3.1, let V be a SUPH system for $F\#T$ in $\mathbf{R}^4$. Let V_1 be the component of V for $F_1\#T$ and write $V_1 = U\#_\partial W$, a disk sum for a punctured 3-ball U and a handlebody W. The following lemma is needed to prove Lemma II.

Lemma 3.2. For a suitable spin loop basis (ℓ, ℓ') for T^o, there is a spin simple loop $\tilde{\ell}'$ in $F_1\#T$ with intersection number $\mathrm{Int}(\ell, \tilde{\ell}') \neq 0$ in $F_1\#T$ such that the loop $\tilde{\ell}'$ bounds a disk D' in W.

Proof of Lemma 3.2. Consider a disk sum decomposition of the handlebody W into into solid tori $S^1 \times D^2_j \, (j = 1, 2, \dots, g)$ pasting along mutually disjoint disks. Let (ℓ_j, m_j) be a longitude-meridian pair of the solid torus $S^1 \times D^2_j$ for all j. By [1] (see [10, (2.4.1)]), the loop basis (ℓ_j, m_j) for $S^1 \times D^2_j$ is chosen to be a spin loop basis in $\mathbf{R}^4$ for all j. By a choice of a spin loop basis (ℓ, ℓ') for T^o, the loop ℓ meets a meridian loop m_j with a non-zero intersection number in ∂W. The loop m_j is taken to be a loop $\tilde{\ell}'$ in $F_1\#T$ bounding a disk D' in W with $\mathrm{Int}(\ell, \tilde{\ell}') \neq 0$ since m_j bounds a meridian disk $1 \times D_j$ in $S^1 \times D^2_j \subset W$. This completes the proof of Lemma 3.2. .

The following lemma is obtained by using Lemma 3.2.

Lemma 3.3. There is a stabilization $\bar{F}$ of $F\#T$ in $\mathbf{R}^4$ consisting of the components $\bar{F}_1, F_i\,(i = 2, 3, \ldots, r)$ where $\bar{F}_1$ is the connected sum of $F_1\#T$ and trivial torus-knots $T_i\,(i = 1, 2, \ldots, m)$ for some $m \geq 0$ such that the surface-link $\bar{F}$ has the following conditions (i) and (ii).

(i) There is an O2-handle pair $(D \times I, D' \times I)$ on $\bar{F}$ attached to $\bar{F}_1$ such that the surface-link $\bar{F}(D' \times I)$ is a ribbon surface-link with trivial 1-handles $h'_i\,(i = 1, 2, \ldots, m)$ attached.

(ii) There is an O2-handle pair $(E \times I, E' \times I)$ on $\bar{F}$ attached to $\bar{F}_1$ such that the surface-link $\bar{F}(E' \times I)$ is F with trivial 1-handles $h''_i\,(i = 1, 2, \ldots, m)$ attached.

Proof of Lemma 3.3. Let $p_i\,(i = 0, 1, \ldots, m)$ be the intersection points of transversely meeting simple loops ℓ and $\tilde{\ell}'$ in $F_1\#T$ given by Lemma 3.2. For every $i > 0$, let α_i be an arc neighborhood of p_i in ℓ, and h_i a 1-handle on $F\#T$ with a core arc $\hat{\alpha}_i$ obtained by pushing the interior of α_i outside V. Let $\tilde{\alpha}_i$ be a proper arc in the annulus $\dot{\partial} h_i = \mathrm{cl}(\partial h_i \setminus h_i \cap F\#T)$ parallel in h_i to the core arc $\hat{\alpha}_i$ of the 1-handle h_j. Let $\partial\tilde{\alpha}_i = \partial\alpha_i$. Let $\bar{F} = F\#T\#_{i=1}^m T_i$ be a stabilization of $F\#T$ with the component $bar F_1$ obtained from $F_1\#T$ by surgery along the disjoint trivial 1-handle system $h_i\,(i = 1, 2, \ldots, m)$. Let $\tilde{\ell}$ be a simple loop obtained from ℓ by replacing α_i with $\tilde{\alpha}_i$ for every $i > 0$. The loop $\tilde{\ell}$ is a spin loop in $\bar{F}$ meeting $\tilde{\ell}'$ transversely in just one point. Let $W^+(D')$ be the handlebody obtained from the handlebody $W^+ = W \cup_{i=1}^m h_i$ by removing a thickened disk $D' \times I$ of D'. The manifold $V^+(D')$ obtained from $V^+ = V \cup_{i=1}^m h_i$ by replacing W^+ with $W^+(D')$ is a SUPH system for a surface-link $\bar{F}'$ in $\mathbf{R}^4$ consisting a component $\bar{F}'_1$ with genus reduced by 1 from $\bar{F}_1$ and the components $F_i\,(i = 2, 3, \ldots, r)$. By Lemma 3.1, $\bar{F}'$ is a ribbon-surface-link in $\mathbf{R}^4$. The SUPH system V^+ is ambient isotopic in $\mathbf{R}^4$ by an ambient isotopy keeping $V \setminus V_1$ fixed to a SUPH system $\tilde{V}^+$ by connecting a solid torus W_1 with a deformed disk $\tilde{D}'$ of D' as a meridian disk and the loop $\tilde{\ell}$ as a longitude with $V^+(D')$ along a 1-handle h_W. Let A be a 4D bump of the associated bump B of an O2-handle pair $(E \times I, E' \times I)$ on $F\#T$ in $\mathbf{R}^4$ attached to T^o with $(\ell, \ell') = (\partial E, \partial E')$. Since in the case of (i), there is no need to worry about the intersection of A with E, E', A is deformed so that $\tilde{V}^+$ meets A with W_1 and h_W. (In fact, $V \setminus V_1$ and U are deformed disjoint from A. Then the conclusion is made by taking spine graphs of $V^+(D')$, W_1 and h_W.) Then the loop $\tilde{\ell}$ bounds a disk $\tilde{D}$ in A not meeting the interior of W_1 and h_W. This means that there is an O2-handle pair $(D \times I, D' \times I)$ on the surface-link $\bar{F}$ such that $\bar{F}(D' \times I)$ is a ribbon surface-link with trivial 1-handles $h'_i\,(i = 1, 2, \ldots, m)$ attached, showing (i).

For the case of (ii), note that the 1-handles $h_i\,(i = 1, 2, \ldots, m)$ on $F\#T$ are isotopically deformed in A into 1-handles $h''_i\,(i = 1, 2, \ldots, m)$ on $F\#T$ disjoint from the disk pair (E, E') because the core arcs of the 1-handles $h_i\,(i = 1, 2, \ldots, m)$ are deformed to be disjoint from the disk pair (E,E')in A. Hence the surface-link $\bar{F}(E'\times I)\cong\bar{F}(E\times I,E'\times I)$ is the surface-link F with the trivial 1-handles $h''_i\,(i=1,2,\dots,m)$ attached, showing (ii). Thus, the proof of Lemma 3.3 is completed.

The following lemma is a combination of Lemma 3.3 and the uniqueness of an O2-handle pair in the soft sense (Lemma 2.2).

Lemma 3.4. If a connected sum $F\#T$ of a surface-link F and a trivial torus-knot T in $\mathbf{R}^4$ is a ribbon surface-link, then F is a ribbon surface-link.

Proof of Lemma 3.4. Let $F\#T=F_1\#T\cup F_2\cup\cdots\cup F_r$ be a ribbon surface-link for a trivial torus-knot T. By Lemma 3.3 (i), the surface-link $F''=\bar{F}(D\times I,D'\times I)\cong\bar{F}(D\times I)$ is a ribbon surface-link and further the surface-link F^* obtained from F'' by the surgery on O2-handle pairs of all the trivial 1-handles $h'_i\,(i=1,2,\dots,m)$ is equivalent to F'' and hence a ribbon surface-link. By Lemma 3.3 (ii), the surface-link $\bar{F}(E\times I,E'\times I)\cong\bar{F}(E\times I)$ is the surface-link F with the 1-handles $h''_i\,(i=1,2,\dots,m)$ trivially attached. By an inductive use of Theorem 2.2 (Uniqueness of an O2-handle pair in the soft sense), the surface-link F is equivalent to the ribbon surface-link F^*. Thus, F is a ribbon surface-link and the proof of Lemma 3.4 is completed.

Lemma II is a direct consequence of Lemma 3.4 as follows.

Proof of Lemma II. If a stabilization $\bar{F}$ of a surface-link F is a ribbon surface-link, then F is a ribbon surface-link by an inductive use of Lemma 3.4. This completes the proof of Lemma II.

4. Proofs of Theorems 1.4 and 1.5

The proof of Theorem 1.4 is done as follows.

Proof of Theorem 1.4. The 'if' part of Theorem 1.4 is seen from the definition of a ribbon surface-link. The proof of the 'only if' part of Theorem 1.4 uses an argument of [3] showing the fact that every surface-link is made a trivial surface-knot by the surgery along a finite number of (possibly non-trivial) 1-handles. The connected summand F_2 is made a trivial surface-knot by the surgery along 1-handles within the 4-ball defining the connected sum, so that the surface-link F changes into a new ribbon surface-link and hence F_1 is a stable-ribbon surface-link. By Corollary 1.2, F_1 is a ribbon surface-link. By interchanging the roles of F_1 and F_2, F_2 is also a ribbon surface-link. This completes the proof of Theorem 1.4.

The proof of Theorem 1.5 is done as follows.

Proof of Theorem 1.5. Let L be an S^2-link of r trivial components L_i, ($i = 1, 2, \ldots, r$). For $r = 1$, L is a trivial S^2-knot and there is nothing to prove. Assume that the sublink L' consisting of the trivial components L_i, ($i = 1, 2, \ldots, r-1$) is a ribbon S^2-link. Then there is a 3-manifold V in S^4 such that $\partial V = L_n$ and $V \cap L' = \emptyset$. There is a disjoint 1-handle system h_j ($j = 1, 2, \ldots, m$) on L_n which is embedded in V so that $V_0 = \mathrm{cl}(V \setminus \cup_{j=1}^{m} h_j)$ is a handlebody. Let $F = \partial V_0$ be a trivial surface-knot in $\mathbf{R}^4$ by [3]. Since $V_0 \cap L' = \emptyset$ and L' is a ribbon surface-link, the surface-link $L' \cup F$ is a ribbon surface-link in $\mathbf{R}^4$. Let (B_*, α_*) be a pair system of a 3-ball system B_* and a chord system α_* for L' given by the ribbonness of $L' \cup F$. The surface F is also obtained from L_n by surgery along h_j ($j = 1, 2, \ldots, m$). Let (m_j, ℓ_j) ($j = 1, 2, \ldots, m$) be a spin loop basis of the surface F such that $m_j = \partial d_i$ for a meridian disk (transverse disk) d_j of the 1-handle h_j. The loop system ℓ_j ($j = 1, 2, \ldots, m$) bounds a smoothly embedded disjoint disk system δ_j ($j = 1, 2, \ldots, n$) in S^4 with $\delta_j \cap F = \ell_j$ ($j = 1, 2, \ldots, m$) and disjoint from the interior system of the disk system d_j ($j = 1, 2, \ldots, m$) by [2]. By moving B_* in S^4, it may be also assumed that $B_* \cap \delta_j = \emptyset$ ($j = 1, 2, \ldots, m$). By general position, $\alpha_* \cap \delta_j = \emptyset$ ($j = 1, 2, \ldots, m$). Thus, (d_j, δ_j) ($j = 1, 2, \ldots, m$) are the core system of an O2-handle part system $(d_j \times I, \delta_j \times I)$ ($j = 1, 2, \ldots, m$) on the ribbon surface-link $L' \cup F$ in S^4. By Lemma 2.1 and Theorem 1.1, $L = L' \cup L_r$ is a ribbon S^2-link. By induction on r, the proof of Theorem 1.5 is completed.

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