Another proof of free ribbon lemma

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ABSTRACT

Free ribbon lemma that every free sphere-link in the 4-sphere is a ribbon sphere-link is shown in an earlier paper by the author. In this paper, another proof of this lemma is given.

Keywords: Free ribbon lemma, Wirtinger presentation, Ribbon sphere-link.
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1. Introduction

A *surface-link* is a closed oriented (possibly, disconnected) surface L smoothly embedded in the 4-sphere S^4. When L is connected, L is called a *surface-knot*. If L consists of 2-spheres $L_i\,(i = 1, 2, \dots, n)$, then L is called a *sphere-link* (or an S^2-*link*) of n components. It is shown that every surface-link L is a trivial surface-link (i.e., bounds disjoint handlebodies in S^4 if $\pi_1(S^4 \setminus L, x_0)$ is a meridian-based free group (see [4, 5, 6]). A surface-link L is *ribbon* if L is obtained from a trivial S^2-link O in S^4 by surgery along smoothly embedded disjoint 1-handles on O. A surface-link L in the 4-sphere S^4 is *free* if the fundamental group $\pi_1(S^4 \setminus L, x_0)$ is a (not necessarily meridian-based) free group. The *free ribbon lemma* is the following theorem.

Theorem. Every free S^2-link is a ribbon S^2-link.

This theorem is a basic result concerning Whitehead aspherical conjecture [8] [10] and classical Poincaré conjecture [9], and the proof is done in [8] as an appendix.

At present, it appears unknown whether or not every free surface-link is a ribbon surface-link. In this paper, another proof of this theorem is given as follows.

Proof of Theorem. Let $L_i\,(i=1,2,\dots,n)$ be the components of a free S^2-link L. Let x_i, $(i=1,2,\dots,n)$ be a basis of the free fundamental group $G=\pi_1(S^4\setminus L,x_0)$. Let y_i be a meridian element of L_i in G, so that $y_i\,(i=1,2,\dots,n)$ are a meridian system of G. By Nielsen transformations, y_i is equal to x_i modulo the commutator subgroup $[G,G]$ of G. It is known that the group G is isomorphic to a group G^P with Wirtinger presentation

$$P=< y_{ij}\,(1\leq j\leq m_i,\,1\leq i\leq n)|\,r_{ij}\,(2\leq j\leq m_i+s_i,\,1\leq i\leq n)>$$

such that $y_{i1}=y_i\,(i=1,2,\dots,n)$ and the relators $r_{ij}\,(j=2,3,\dots,m_i+s_i,\,i=1,2,\dots,n)$ are given by $r_{ij}:\,y_{ij}=w_{ij}y_{i1}w_{ij}^{-1}$ for j with $2\leq j\leq m_i$, $1\leq i\leq n$, and $r_{ij}:\,y_{i1}=w_{ij}y_{i1}w_{ij}^{-1}$ for j with $m_i+1\leq j\leq m_i+s_i$, $1\leq i\leq n$, where $w_{ij}\,(j=2,3,\dots,m_i+s_i,\,i=1,2,\dots,n)$ are words in the letters $y_{ij}\,(j=1,2,\dots,m_i,\,i=1,2,\dots,n)$. This result is obtained from Yajima [13] because G has a weight system $y_i\,(i=1,2,\dots,n)$, $H_1(G;Z)\cong Z^n$ and $H_2(G;Z)=0$. It is observed that this result can be also obtained by an alternative geometric method using a normal form of a surface-link in R^4 [11]. In fact, put the S^2-link L in a normal form of in the 4-space R^4 with $L[0]=L\cap R^3[0]$ a middle cross-sectional link and calculate the fundamental groups $\pi_1(R^3[0,+\infty)\setminus L\cap R^3[0,+\infty),x_0)$ and $\pi_1(R^3(-\infty,0]\setminus L\cap R^3(-\infty,0],x_0)$ with Wirtinger presentations starting from the fundamental group $\pi_1(R^3[0]\setminus L[,0],x_0)$ with a Wirtinger presentation to obtain the group G with a Wirtinger presentation by van Kampen theorem. See [2, 3] for this construction and [1] for a generalization. By fixing an isomorphism $G^P\to G$, regard the generators $y_{ij}\,(j=1,2,\dots,m_i,\,i=1,2,\dots,n)$ of P as fixed words in the basis x_i, $(i=1,2,\dots,n)$ of G. Then the relator $y_{i1}=w_{ij}y_{i1}w_{ij}^{-1}$ for every i and j with $m_i+1\leq j\leq m_i+s_i$ can be written as $y_{i1}=a_{ij}^{u(i,j)}$ and $w_{ij}=a_{ij}^{v(i,j)}$ for a deduced word a_{ij} in x_i, $(i=1,2,\dots,n)$ and some integers $u(i,j),v(i,j)$ by Dehn's solution of the word problem of the free group $< x_1,x_2,\dots,x_n >$. The elements $y_i=y_{i1}\,(i=1,2,\dots,n)$ form the same abelian basis as $x_i\,(i=1,2,\dots,n)$ in the free abelian group $G/[G,G]$, so that $u(i,j)=\pm1$ for every i and j. Thus, $w_{ij}=y_{i1}^{u(i,j)v(i,j)}$ for every i and j with $m_i+1\leq j\leq m_i+s_i$, which means that the relators $r_{ij:}\,y_{i1}=w_{ij}y_{i1}w_{ij}^{-1}\,(m_i+1\leq j\leq m_i+s_i)$ are identity relations in the free group $< y_{ij}\,(1\leq j\leq m_i,\,1\leq i\leq n)>$. Thus, the Wirtinger presentation P is equivalent to the Wirtinger presentation

$$R=< y_{ij}\,(1\leq j\leq m_i,\,1\leq i\leq n)|\,r_{ij}\,(2\leq j\leq m_i,\,1\leq i\leq n)>$$

with $y_{i1}=y_i\,(i=1,2,\dots,n)$ and the relators $r_{ij}\,(2\leq j\leq m_i,\,1\leq i\leq n)$ given by $r_{ij}:\,y_{ij}=w_{ij}y_{i1}w_{ij}^{-1}\,(1\leq j\leq m_i,\,1\leq i\leq n)$. By Yajima's construction [12] (see also

[2, 3]), there is a ribbon S^2-link L^R with the fundamental group $G^R = \pi_1(S^4 \setminus L^R, x_0)$ of the Wirtinger presentation R which is isomorphic to G by an isomorphism $G^R \to G$ sending a meridian element y_i^R of the ith component L_i^R of L^R to the meridian element y_i of L_i in G for every $i\,(i = 1, 2, \ldots, n)$ and a basis x_i^R, $(i = 1, 2, \ldots, n)$ of G^R to the basis x_i, $(i = 1, 2, \ldots, n)$ of G. Let Y^R and Y be the 4D manifolds (both diffeomorphic to the n-fold connected sum of $S^1 \times S^3$) obtained from S^4 by surgeries along L^R and L, respectively, and ℓ_i^R, $(i = 1, 2, \ldots, n)$ and ℓ_i, $(i = 1, 2, \ldots, n)$ the loop systems obtained from $L_i^R\,(i = 1, 2, \ldots, n)$ and $L_i\,(i = 1, 2, \ldots, n)$, respectively. By [8], there is an orientation-preserving diffeomorphism $f : Y^R \to Y$ sending the loop system $\ell_i^R\,(i = 1, 2, \ldots, n)$ to the loop system $\ell_i\,(i = 1, 2, \ldots, n)$. Note that this result is obtained from the smooth unknotting conjecture for S^2-knots [4, 5, 6] and the 4D smooth Poincaré conjecture [7]. By the back surgeries from Y^R to S^4 along $\ell_i^R\,(i = 1, 2, \ldots, n)$ and from Y to S^4 along $\ell_i\,(i = 1, 2, \ldots, n)$, this diffeomorphism f induces an orientation-preserving diffeomorphism $f' : S^4 \to S^4$ sending L^R to L. Thus, the S^2-link L is a ribbon S^2-link. This completes the proof of Theorem.

In the proof of Theorem, the ribbon S^2-link L^R is called a *ribbon presentation* of the free S^2-link L. The following corollary is obtained from the proof of Theorem.

Corollary. Let L be a free S^2-link in the 4-sphere S^4 containing a free S^2-link K as a sublink. For any ribbon presentation of K^R of K, there is a ribbon presentation L^R of L containing K^R as a sublink.

Proof of Corollary. The ribbon presentation of K^R of K is in a normal form. Thus, the result is obtained from the observation that a normal form of L is taken to contain K^R as a sublink (see [11]).

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References

[1] S. Kamada, Wirtinger presentations for higher dimensional manifold knots obtained from diagrams, Fund. Math.,168(2001), 105-112.

[2] A. Kawauchi, On the surface-link groups, Intelligence of low dimensional topology 2006, Series on knots and everything 40 (2007), 157-164, World Sci. publ.

[3] A. Kawauchi, The first Alexander Z[Z]-modules of surface-links and of virtual links, Geometry & Topology Monographs 14 (2008), 353-371.

[4] A. Kawauchi, Ribbonness of a stable-ribbon surface-link, I. A stably trivial surface-link, Topology and its Applications 301(2021), 107522 (16pages).

[5] A. Kawauchi, Uniqueness of an orthogonal 2-handle pair on a surface-link, Contemporary Mathema tics (UWP) 4 (2023), 182-188.

[6] A. Kawauchi, Triviality of a surface-link with meridian-based free fundamental group, Transnational Journal of Mathematical Analysis and Applications 11 (2023), 19-27.

[7] A. Kawauchi, Smooth homotopy 4-sphere, WSEAS Transactions on Mathematics 22 (2023), 690-701.

[8] A. Kawauchi, Ribbonness of Kervaire's sphere-link in homotopy 4-sphere and its consequences to 2-complexes. Journal of Mathematical Techniques and Computational Mathematics, 3(4) (2024), 01-08 (online).

[9] A. Kawauchi, Classical Poincaré conjecture via 4D topology. Journal of Mathematical Techniques and Computational Mathematics, 3(4) (2024), 1-7 (online).

[10] A. Kawauchi, Whitehead aspherical conjecture via ribbon sphere-link. Journal of Mathematical Techniques and Computational Mathematics, 3(5) (2024), 01-10 (online).

[11] A. Kawauchi, T. Shibuya, S. Suzuki, Descriptions on surfaces in four-space I : Normal forms, Mathematics Seminar Notes, Kobe University, 10 (1982), 75-125; II: Singularities and cross-sectional links, Mathematics Seminar Notes, Kobe University 11 (1983), 31-69. Available from: https://sites.google.com/view/kawauchiwriting

[12] T. Yajima, On the fundamental groups of knotted 2-manifolds in the 4-space, J. Math. Osaka City Univ., 13(1962), 63-71.

[13] T. Yajima, Wirtinger presentations of knot groups, Proc. Japan Acad., 46(1970), 997-1000.