Ribbonness of a stable-ribbon surface-link, II. General case

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ABSTRACT

It is shown that any handle-irreducible summand of every stable-ribbon surface-link is a unique ribbon surface-link up to equivalences, so that every stable-ribbon surface-link is a ribbon surface-link. This is a generalization of a previously observed result for a stably trivial surface-link. Two applications are given. One application is an observation that a connected sum of two surface-links is a ribbon surface-link if and only if both the connected summands are ribbon surface-links. The other application is an observation that any sphere-link consisting of trivial components is a ribbon sphere-link, so that the fundamental group is a torsion-free group.

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1. Introduction

This paper generalizes the previous result that a stably trivial surface-link is a trivial surface-link to the result that a stable-ribbon surface-link is a ribbon surface-link, [10]. A *surface-link* is a closed oriented (possibly disconnected) surface F which is embedded in the 4-space $\mathbf{R}^4$ by a smooth embedding. When F is connected, it is also called a *surface-knot*. When a fixed (possibly disconnected) closed surface $\mathbf{F}$ is smoothly embedded into $\mathbf{R}^4$, it is also called an $\mathbf{F}$-*link*. If $\mathbf{F}$ is the disjoint union of some copies of the 2-sphere S^2, then it is also called an S^2-*link*. When $\mathbf{F}$ is connected, it is also called an $\mathbf{F}$-*knot*, and an S^2-*knot* for $\mathbf{F} = S^2$. Two surface-links F and F' are *equivalent* by an *equivalence* f if f is an orientation-preserving diffeomorphism $f : \mathbf{R}^4 \to \mathbf{R}^4$ sending orientation- preservingly F to F'. A *trivial* surface-link is a surface-link F which bounds disjoint handlebodies smoothly embedded in $\mathbf{R}^4$, where a handlebody is a 3-manifold which is a 3-ball, solid torus or a disk sum of some number of solid tori. A trivial surface-knot is also called an *unknotted* surface-knot. A trivial disconnected surface-link is also called an *unknotted-unlinked* surface-link. For any given closed oriented (possibly disconnected) surface **F**, a trivial **F**-link exists uniquely up to equivalences (see [6]). A *ribbon* surface-link is a surface-link F which is obtained from a trivial nS^2-link O for some n (where nS^2 denotes the disjoint union of n copies of the 2-sphere S^2) by surgery along an embedded 1-handle system, [**?**], [14, II], [25], [26], [27]. This object is an old concept in surface-knot theory, but in recent years it is considered as a chord diagram which is a relaxed version of a virtual graph diagram and a knotoid discussed in a plane diagram,[9], [8], [22], [23]. A *stabilization* of a surface-link F is a connected sum $\bar{F} = F\#_{k=1}^{s} T_k$ of F and a system of trivial torus-knots $T_k\,(k = 1, 2, \ldots, s)$. By granting $s = 0$, a surface-link F itself is regarded as a stabilization of F. The trivial torus-knot system T is called the *stabilizer* with *stabilizer components* $T_k\,(k = 1, 2, ..., s)$ on the stabilization $\bar{F}$ of F. A *stable-ribbon* surface-link is a surface-link F such that a stabilization $\bar{F}$ of F is a ribbon surface-link. Every surface-link F is equivalent to a stabilization of a surface-link F_* with minimal total genus. This surface-link F_* is called a *handle-irreducible summand* of F. The following result called *Stable-Ribbon Theorem* is our main theorem.

Theorem 1.1. Any handle-irreducible summand F_* of every stable-ribbon surface-link F is a ribbon surface-link which is determined uniquely from F up to equivalences and stabilizations.

Any stabilization of a ribbon surface-link is a ribbon surface-link. So, the following corollary is obtained from Theorem 1.

Corollary 1.2. Every stable-ribbon surface-link is a ribbon surface-link.

A *stably trivial* surface-link is a surface-link F such that a stabilization $\bar{F}$ of F is a trivial surface-link. Since a trivial surface-link is a ribbon surface-link, Theorem 1.1 also implies the following corollary, which is used to prove smooth unknotting conjecture for a surface-link, [10]. This result leads to 4D smooth and then classical Poincaré conjectures, [1], [11], [12], [18], [19], [20]. [21]

Corollary 1.3. Any handle-irreducible summand of every stably trivial surface-link is a trivial S^2-link, so that every stably trivial surface-link is a trivial surface-link.

The plan for the proof of Theorem 1.1 is to show the following two lemmas by a research of stabilization of a surface-link, [10].

Lemma I. Any handle-irreducible summand of any surface-link is unique up to equivalences and stabilizations.

Lemma II. Any stable-ribbon surface-link is a ribbon surface-link.

The proof of Theorem 1.1 is completed by these lemmas as follows:

Proof of Theorem 1.1 assuming Lemmas I, II. By Lemma II, any handle-irreducible summand of every stable-ribbon surface-link is a ribbon surface-link, which is unique up to equivalences and stabilizations by Lemma I. This completes the proof of Theorem 1.1.

An idea of the proof of Lemma I is to generalize the uniqueness result of an O2-handle pair on a surface-link earlier established to the case where the restriction on the attaching part is relaxed (see Theorem 2.2). An idea of the proof of Lemma II is to consider a semi-unknotted multi-punctured handlebody system, simply called a *SUPH system*, of a ribbon surface-link. Two applications of Theorem 1.1 are made. One application of Theorem 1.1 is the following theorem.

Theorem 1.4. A connected sum $F = F_1 \# F_2$ of surface-links $F_i\,(i = 1, 2)$ in S^4 is a ribbon surface-link if and only if both the surface-links $F_i\,(i = 1, 2)$ are ribbon surface-links.

This theorem contrasts with a behavior of classical ribbon knot, because every classical knot is a connected summand of a connected sum ribbon knot. In fact, for every knot k and the inversed mirror image $-k^*$ of k in the 3-sphere S^3, the connected sum $k\#(-k^*)$ is a ribbon knot in S^3, [2], [14, I]. A natural presentation of $k\#(-k^*)$ is seen in a chord diagram of the spun S^2-knot of k as a ribbon S^2-knot, [9]. The following theorem is the other application of Theorem 1.1.

Theorem 1.5. Every S^2-link L consisting of trivial components in S^4 is a ribbon S^2-link, so that the fundamental group $\pi_1(S^4 \setminus L, x_0)$ is a torsion-free group.

Since there are lots of classical non-ribbon links consisting of trivial components such as Hopf link, Borromean rings, etc., this theorem also contrasts with a behavior of classical ribbon link.

The proofs of Lemmas I and II are given in Sections 2 and 3, respectively. In Section 4, the proofs of Theorems 1.4 and 1.5 are given. For the proof of Theorem

1.5, *a characterization that a disjoint 2-handle system of a trivial surface-knot F in S^4 producing an S^2-knot by surgery extends to a disjoint O2-handle pair system on F in S^4* is used (See Lemma 4.1).

2. Proof of Lemma I

A 2-*handle* on a surface-link F in $\mathbf{R}^4$ is a 2-handle $D \times I$ on F with D a core disk embedded in $\mathbf{R}^4$ such that $D \times I \cap F = \partial D \times I$, where I denotes a closed interval containing 0 and $D \times 0$ is identified with D. Two 2-handles $D \times I$ and $E \times I$ on F are *equivalent* if there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from F to itself such that the restriction $f|_F : F \to F$ is the identity map and $f(D \times I) = E \times I$.

An *orthogonal 2-handle pair* (or simply, an *O2-handle pair*) on F is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I$, $D' \times I$ on F such that

$$D \times I \cap D' \times I = \partial D \times I \cap \partial D' \times I$$

and $\partial D \times I$ and $\partial D' \times I$ *meet orthogonally* on F, that is, the boundary circles ∂D and $\partial D'$ meet transversely at one point p so that the intersection $\partial D \times I \cap \partial D' \times I$ is homeomorphic to the square $Q = p \times I \times I$. Let $(D \times I, D' \times I)$ be an O2-handle pair on a surface-link F. Let $F(D \times I)$ and $F(D' \times I)$ be the surface-links obtained from F by the surgeries along $D \times I$ and $D' \times I$, respectively. Let $F(D \times I, D' \times I)$ be the surface-link which is the union $\delta \cup F^c_\delta$ of the plumbed disk

$$\delta = \delta_{D \times I, D' \times I} = D \times \partial I \cup Q \cup D' \times \partial I$$

and the surface

$$F^c_\delta = \mathrm{cl}(F \setminus (\partial D \times I \cup \partial D' \times I).$$

A once-punctured torus T^o in a 3-ball B is *trivial* if T^o is smoothly and properly embedded in B which splits B into two solid tori. A *bump* of a surface-link F is a 3-ball B in $\mathbf{R}^4$ with $F \cap B = T^o$ a trivial once-punctured torus in B. Let $F(B)$ be a surface-link $F^c_B \cup \delta_B$ which is the union of the surface $F^c_B = \mathrm{cl}(F \setminus T^o)$ and a disk δ_B in the 2-sphere ∂B with $\partial \delta_B = \partial T^o$.

A *cellular move* of a compact (possibly, bounded) surface P in $\mathbf{R}^4$ is a compact surface $\tilde{P}$ such that the intersection $P^o = P \cap \tilde{P}$ is a once-punctured compact surface of P and $\tilde{P}$ with $d = \mathrm{cl}(P \setminus P^o)$ and $\tilde{d} = (\tilde{P} \setminus P^o)$ disks in the interiors of P and $\tilde{P}$, respectively such that the union $d \cup \tilde{d}$ is a 2-sphere bounding a 3-ball smoothly embedded in $\mathbf{R}^4$ and not meeting the interior of P^o. Note that $F(B)$ is uniquely determined up to cellular moves on the disk δ_B keeping F^c_B fixed. For an O2-handle pair $(D \times I, D' \times I)$ on a surface-link F, let $\Delta = D \times I \cup D' \times I$ is a 3-ball in $\mathbf{R}^4$ called the *2-handle union*. Consider the 3-ball Δ as a Seifert hypersurface of the trivial S^2-knot $K = \partial\Delta$ in $\mathbf{R}^4$ to construct a 3-ball B_Δ obtained from Δ by adding an outer boundary collar. This 3-ball B_Δ is a bump of F, which we call the *associated bump* of the O2-handle pair $(D\times I, D'\times I)$. When the union of the 3-ball Δ and a boundary collar of F^c_δ are deformed into the 3-space $\mathbf{R}^3 \subset \mathbf{R}^4$, this associated bump B_Δ is also considered as a regular neighborhood of Δ in $\mathbf{R}^3$. It is observed that an O2-handle unordered pair $(D\times I, D'\times I)$ on a surface-link F is constructed uniquely from any given bump B of F in $\mathbf{R}^4$ with $F(D\times I, D'\times I) \cong F(B)$, [10]. Further, for any O2-handle pair $(D\times I, D'\times I)$ on any surface-link F and the associated bump B, there are equivalences

$$F(B) \cong F(D\times I, D'\times I) \cong F(D\times I) \cong F(D'\times I)$$

which are attained by cellular moves on the disk $\delta = \delta_{D\times I, D'\times I}$ keeping F^c_δ fixed. A once-punctured torus T^o in a 4-ball A is *trivial* if T^o is smoothly and properly embedded in A and there is a solid torus V in A with $\partial V = T^o \cup \delta_A$ for a disk δ_A in the 3-sphere ∂A. A *4D bump* of a surface-link F is a 4-ball A in $\mathbf{R}^4$ with $F \cap A = T^o$ a trivial once-punctured torus in A. A 4D bump A is obtained from a bump B of a surface-link F by taking a bi-collar $c(B\times[-1,1])$ of B in $\mathbf{R}^4$ with $c(B\times 0) = B$. The following lemma is proved by using a 4D bump A.

Lemma 2.1. Let $(D\times I, D'\times I)$ be any O2-handle pair on any surface-link F in $\mathbf{R}^4$, and T a trivial torus-knot in $\mathbf{R}^4$ with any given spin loop basis (e, e'). Then there is an equivalence $f: \mathbf{R}^4 \to \mathbf{R}^4$ from the surface-link F to a connected sum $F(D\times I, D'\times I)\#T$ keeping F^c_δ fixed such that $f(\partial D) = e$ and $f(\partial D') = e'$.

Proof of Lemma 2.1. Let A be a 4D bump associated with the O2-handle pair $(D\times I, D'\times I)$ on F. Let δ_A be a disk in the 3-sphere ∂A such that there is a solid torus V in A whose boundary is the union of the trivial once-punctured torus $P = F\cap A$ and the disk δ_A. This solid torus V induces an equivalence $f': (\mathbf{R}^4, F) \to (\mathbf{R}^4, F(D\times I, D'\times I)\#T)$ sending P to the connected summand T^o of a connected sum $F(D\times I, D'\times I)\#T$ in A. Let $(\tilde{e}, \tilde{e}')$ be the spin loop basis of T^o which is the image of the spin loop pair $(\partial D, \partial D')$ on F under f'. There is an orientation-preserving diffeomorphism $g: \mathbf{R}^4 \to \mathbf{R}^4$ with $g|_{\mathrm{cl}(\mathbf{R}^4\setminus A)} = 1$ such that $g(\tilde{e}, \tilde{e}') = (e, e')$, [4], [5], [10]. The composition $f = gf'$ is a desired equivalence. This completes the proof of Lemma 2.1.

A surface-link F has *only unique O2-handle pair in the rigid sense* if for any O2-handle pairs $(D\times I, D'\times I)$ and $(E\times I, E'\times I)$ on F with $(\partial D)\times I = (\partial E)\times I$ and $(\partial D')\times I = (\partial E')\times I$, there is an equivalence $f: \mathbf{R}^4 \to \mathbf{R}^4$ from F to itself keeping F^c_δ fixed such that $f(D\times I) = E\times I$ and $f(D'\times I) = E'\times I$. It is proved that every surface-link F has only unique O2-handle pair in the rigid sense, [10]. A surface-link F has *only unique O2-handle pair in the soft sense* if for any O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F attached to the same connected component, say F_1 of F, there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from F to itself keeping $F^{(1)} = F \setminus F_1$ fixed such that $f(D \times I) = E \times I$ and $f(D' \times I) = E' \times I$. A surface-link not admitting any O2-handle pair is understood as a surface-link with only unique O2-handle pair in both the rigid and soft senses. The following uniqueness of an O2-handle pair in the soft sense is essentially a consequence of the uniqueness of an O2-handle pair in the rigid sense.

Theorem 2.2 (Uniqueness of an O2-handle pair in the soft sense). Every ribbon surface-link has only unique O2-handle pair in the soft sense.

Proof of Theorem 2.2. Let $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ be any two O2-handle pairs on a surface-link F attached to the same connected component F_1 of F. Let $F^{(1)} = F \setminus F_1$. By Lemma 2.1, there is an equivalences $f : \mathbf{R}^4 \to \mathbf{R}^4$ from F to the connected sum

$$F(D \times I, D' \times I) \# T = F^{(1)} \cup \tilde{F}_1 \# T$$

keeping $F^{(1)}$ fixed and sending F_1 to $\tilde{F}_1 \# T$, where $\tilde{F}_1 = F_1(D \times I, D' \times I)$ and T is a trivial torus-knot in $\mathbf{R}^4$. Similarly, there is an equivalences $f' : \mathbf{R}^4 \to \mathbf{R}^4$ from F to the connected sum

$$F(E \times I, E' \times I) \# T' = F^{(1)} \cup \tilde{F}'_1 \# T'$$

keeping $F^{(1)}$ fixed and sending F_1 to $\tilde{F}'_1 \# T'$, where $\tilde{F}'_1 = F_1(E \times I, E' \times I)$ and T' is a trivial torus-knot in $\mathbf{R}^4$. The diffeomorphism $g = f' f^{-1} : \mathbf{R}^4 \to \mathbf{R}^4$ is an equivalence from $F^{(1)} \cup \tilde{F}_1 \# T$ to $F^{(1)} \cup \tilde{F}'_1 \# T'$ keeping $F^{(1)}$ fixed. The connected sum $\tilde{F}_1 \# T$ is obtained from the split union $\tilde{F}_1 + T$ in $\mathbf{R}^4$ by surgery along an embedded 1-handle h connecting a disk $d_1 \subset \tilde{F}_1$ and a disk $d \subset T$, and the connected sum $\tilde{F}'_1 \# T'$ is obtained from the split union $\tilde{F}'_1 + T'$ in $\mathbf{R}^4$ by surgery along an embedded 1-handle h' connecting a disk $d'_1 \subset \tilde{F}'_1$ and a disk $d' \subset \tilde{T}$. Then there is a 4-ball A in $\mathbf{R}^4$ such that $T^o = A \cap (F^{(1)} \cup \tilde{F}_1 \# T)$ is a trivial once-punctured torus of T in A with d_1 a disk bounded by the trivial knot ∂T^o in the 3-sphere ∂A. Similarly, there is a 4-ball A' in $\mathbf{R}^4$ such that$(T')^o = A' \cap (F^{(1)} \cup \tilde{F}'_1 \# T')$ is a trivial once-punctured torus of T' in A' with d'_1 a disk bounded by the trivial knot $\partial (T')^o$ in the 3-sphere $\partial A'$. It may be assumed that $g(\partial d_1) = \partial d'_1$ by sliding the attaching loop $g(\partial d_1)$ in $g(\tilde{F}_1 \# T)$ and/or the attaching loop $\partial d'_1$ in $\tilde{F}'_1 \# T'$. Then it is assumed that $g(T^o) = (T')^o$ (For a special case that $g(T^0) = \mathrm{cl}(\tilde{F}'_1 \# T' \setminus (T')^o)$, there is a deformation from $g(A)$ into A' to obtain $g(T^0) = (T')^o$). Further, by Lemma 2.1, it is assumed that $g(f(\partial D), f(\partial D')) = (f'(\partial E), f'(\partial E'))$. Then

$$(f'(\partial D), f'(\partial D')) = g(f(\partial D), f(\partial D')) = (f'(\partial E), f'(\partial E')).$$

Since every surface-link has only unique O2-handle pair in the rigid sense, there is an equivalence $g' : \mathbf{R}^4 \to \mathbf{R}^4$ from $F^{(1)} \cup \tilde{F}'_1 \# T'$ to itself keeping $F^{(1)}$ fixed such that $g'(f'(D) \times I, f'(D') \times I) = (f'(E) \times I, f'(E') \times I)$. The composite equivalence $g^* = (f')^{-1} g' g f : \mathbf{R}^4 \to \mathbf{R}^4$ is an equivalence from F to itself keeping $F^{(1)}$ fixed and sending $(D \times I, D' \times I)$ to $(E \times I, E' \times I)$. Thus, every surface-link F has only unique O2-handle pair in the soft sense. This completes the proof of Theorem 2.2.

The following corollary is obtained from the proof of Theorem 2.2.

Corollary 2.3. Let F and F' be surface-links with ordered components $F_i\,(i = 1, 2, \ldots, r)$ and $F'_i\,(i = 1, 2, \ldots, r)$, respectively. Assume that the stabilizations $\bar{F} = F \#_i T, \bar{F}' = F' \#_i T$ of F, F' with induced ordered components obtained by the connected sums $F_i \# T, F'_i \# T$ of the ith components F_i, F'_i and a trivial torus-knot T, respectively are equivalent by a component-order-preserving equivalence $\mathbf{R}^4 \to \mathbf{R}^4$. Then F is equivalent to F' by a component-order-preserving equivalence $\mathbf{R}^4 \to \mathbf{R}^4$.

Remark 2.4. For the case of ribbon surface-links F and F', Corollary 2.3 has a different proof, [9].

The proof of Lemma I is done as follows.

Proof of Lemma I. A surface-link F with r ordered components is kth *handle-reducible* if F is equivalent to a stabilization $F' \#_k n_k T$ of a surface-link F' for an integer $n_k > 0$, where $\#_k n_k T$ denotes the stabilizer components $n_k T$ attaching to the kth component of F'. Otherwise, the surface-link F is is said to be *kth handle-irreducible*. Note that if a surface-link G is equivalent to a kth handle-irreducible surface-link F by component-order-preserving equivalence, then G is also kth handle-irreducible. Let F and G be ribbon surface-links with components $F_i\,(i = 1, 2, \ldots, r)$ and $G_i\,(i = 1, 2, \ldots, r)$, respectively. Let $F_*$and G_* be handle-irreducible summands of F and G, respectively. Assume that there is an equivalence f from F to G. Then it is shown that F_* and G_* are equivalent as follows. Changing the indexes if necessary, assume that f sends F_i to G_i for every i.

Let

$$\begin{aligned} F &= F_* \#_1 n_1 T \#_2 n_2 T \#_3 \ldots \#_r n_r T, \\ G &= G_* \#_1 n'_1 T \#_2 n'_2 T \#_3 \ldots \#_r n'_r T. \end{aligned}$$

If necessary, by taking the inverse equivalence f^{-1} instead of f, assume that $n'_1 \geq n_1$. If $n'_1 > n_1$, then there is a component-order-preserving equivalence $f^{(1)}$ from the first-handle-irreducible surface-link

$$F_{(1)} = F_* \#_2 n_2 T \#_3 \ldots \#_r n_r T$$

to the first-handle-reducible surface-link

$$G_* \#_1 (n'_1 - n_1) T \#_2 n'_2 T \#_3 \ldots \#_r n'_r T,$$

by Corollary 2.3, which contradicts the first handle-irreducibility. Thus, $n'_1 = n_1$ and the first handle-irreducible surface-link $F_{(1)}$ is equivalent to the first-handle-irreducible ribbon surface-link

$$G_{(1)} = G_* \#_2 n'_2 T \#_3 \ldots \#_r n'_r T.$$

By continuing this process, it is shown that F_* is equivalent to G_*. This completes the proof of Lemma I.

3. Proof of Lemma II

A *chord graph* is a pair (o, α) of a trivial ink o and an arc system α attaching to o in the 3-space $\mathbf{R}^3$, where o and α are called a *based loop system* and a *chord system*, respectively. A *chord diagram* is a diagram $C(o, \alpha)$ in the plane $\mathbf{R}^2$ of a chord graph (o, α) as a spatial graph. Let D^+ be a proper disk system in the upper half-space $\mathbf{R}^4_+$ obtained from a disk system d^+ in $\mathbf{R}^3$ bounded by o by pushing the interior into $\mathbf{R}^4_+$. Similarly, let D^- be a proper disk system in the lower half-space $\mathbf{R}^4_-$ obtained from a disk system d^- in $\mathbf{R}^3$ bounded by o by pushing the interior into $\mathbf{R}^4_-$. Let O be the union of D^+ and D^- which is a trivial nS^2-link in the 4-space $\mathbf{R}^4$, where n is the number of components of o. The union $O \cup \alpha$ is called a *chorded sphere system* constructed from a chord graph (o, α). The chorded sphere system $O \cup \alpha$ up to orientation-preserving diffeomorphisms of $\mathbf{R}^4$ is independent of choices of d^+ and d^- and uniquely determined by the chord graph (o, α) by the Horibe-Yanagawa lemma, [14, I]. A ribbon surface-link $F = F(o, \alpha)$ is uniquely constructed from the chorded sphere system $O \cup \alpha$ so that F is obtained from O by surgery along a 1-handle system $N(\alpha)$ on O with core arc system α, where note that the surface-link F up to equivalences is unaffected by choices of the 1-handle system $N(\alpha)$, [6], [9]. A multi-punctured handlebody system V (smoothly embedded) in $\mathbf{R}^4$ is a *semi-unknotted multi-punctured handlebody system* (or simply a *SUPH system*) for a surface-link F in $\mathbf{R}^4$ if $\partial V = F \cup O$ for a trivial S^2-link O in $\mathbf{R}^4$. Note that the numbers of connected components of F and V are equal. The following lemma makes a characterization of a ribbon surface-link, [14, II], [27].

Lemma 3.1. A surface-link F is a ribbon surface-link if and only if F admits a SUPH system V in $\mathbf{R}^4$.

Proof of Lemma 3.1. A SUPH system V for a ribbon surface-link F is constructed from a chorded sphere system $O \cup \alpha$ by taking the union of a thickening $O \times [0,1]$ of O in $\mathbf{R}^4$ and the 1-handle system $N(\alpha)$ attaching only to $O \times 0$. Conversely, given a SUPH system V in $\mathbf{R}^4$ with $\partial V = F \cup O$ for a trivial S^2-link O, then take a chord system α in V attaching to O so that the frontier of the regular neighborhood of $O \cup \alpha$ in V is parallel to F in V. The chorded sphere system $O \cup \alpha$ shows that F is a ribbon surface-link. This completes the proof of Lemma 3.1.

Let F be a surface-link of components $F_i\,(i = 1, 2, \ldots, r)$ in $\mathbf{R}^4$. Let $F\#T$ be the connected sum of F and a trivial torus-knot T in $\mathbf{R}^4$ consisting of the components $F_1\#T, F_i\,(i = 2, 3, \ldots, r)$. Assume that $F\#T$ is a ribbon surface-link. By Lemma 3.1, let V be a SUPH system for $F\#T$ in $\mathbf{R}^4$. Let V_1 be the component of V for $F_1\#T$ and write $V_1 = U\#_\partial W$, a disk sum for a multi-punctured 3-ball U and a handlebody W. The following lemma is needed to prove Lemma II.

Lemma 3.2. For a suitable spin loop basis (ℓ, ℓ') for T^o, there is a spin simple loop $\tilde{\ell}'$ in the ribbon-surface-link $F_1\#T$ with intersection number $\mathrm{Int}(\ell, \tilde{\ell}') \neq 0$ in $F_1\#T$ such that the loop $\tilde{\ell}'$ bounds a disk D' in the handlebody W.

Proof of Lemma 3.2. Consider a disk sum decomposition of the handlebody W into solid tori $S^1 \times D^2_j\,(j = 1, 2, \ldots, g)$ pasting along mutually disjoint disks. Let (ℓ_j, m_j) be a longitude-meridian pair of the solid torus $S^1 \times D^2_j$ for all j. The loop basis (ℓ_j, m_j) for $S^1 \times D^2_j$ is chosen to be a spin loop basis in $\mathbf{R}^4$ for all j, [4] , K21. By a choice of a spin loop basis (ℓ, ℓ') for T^o, the loop ℓ meets a meridian loop m_j with a non-zero intersection number in ∂W. The loop m_j is taken to be a loop $\tilde{\ell}'$ in $F_1\#T$ bounding a disk D' in W with intersection number $\mathrm{Int}(\ell, \tilde{\ell}') \neq 0$ since m_j bounds a meridian disk $1 \times D^2_j$ of the solid torus $S^1 \times D^2_j \subset W$. This completes the proof of Lemma 3.2.

The following lemma is obtained by using Lemma 3.2.

Lemma 3.3. There is a stabilization $\bar{F}$ of the ribbon surface-link $F\#T$ in $\mathbf{R}^4$ consisting of the components $\bar{F}_1, F_i\,(i = 2, 3, \ldots, r)$ where $\bar{F}_1$ is the connected sum of $F_1\#T$ and trivial torus-knots $T_i\,(i = 1, 2, \ldots, m)$ for some $m \geq 0$ such that the surface-link $\bar{F}$ has the following conditions (i) and (ii).

(i) There is an O2-handle pair $(D \times I, D' \times I)$ on $\bar{F}$ attached to $\bar{F}_1$ such that the surface-link $\bar{F}(D' \times I)$ is a ribbon surface-link with trivial 1-handles $h'_i\,(i = 1, 2, \ldots, m)$ attached.

(ii) There is an O2-handle pair $(E \times I, E' \times I)$ on $\bar{F}$ attached to $\bar{F}_1$ such that the surface-link $\bar{F}(E' \times I)$ is F with trivial 1-handles $h''_i\,(i = 1, 2, \dots, m)$ attached.

Proof of Lemma 3.3. Let $p_i\,(i = 0, 1, \dots, m)$ be the intersection points of transversely meeting simple loops ℓ and $\tilde{\ell}'$ in $F_1\#T$ given by Lemma 3.2. For every $i > 0$, let α_i be an arc neighborhood of p_i in ℓ, and h_i a 1-handle on $F\#T$ with a core arc $\hat{\alpha}_i$ obtained by pushing the interior of α_i outside the SUPH system V. Let $\bar{F} = F\#T\#_{i=1}^m T_i$ be a stabilization of $F\#T$ with the component $\bar{F}_1 = F_1\#T\#_{i=1}^m T_i$ obtained from $F_1\#T$ by surgery along the disjoint trivial 1-handle system $h_i\,(i = 1, 2, \dots, m)$. Let $\alpha_i^+ = \alpha_i \cup (h_i \cap \ell)$ be the arc in ℓ extending α_i. Let $\tilde{\alpha}_i$ be a proper arc in the annulus $\mathrm{cl}(\partial h_i \setminus h_i \cap F\#T)$ which is parallel to the core arc $\hat{\alpha}_i$ of h_j in h_i with $\partial\tilde{\alpha}_i = \partial\alpha_i^+$. Let $\tilde{\ell}$ be a simple spin loop in $\bar{F}$ obtained from ℓ by replacing α_i^+ with $\tilde{\alpha}_i$ for every $i > 0$, which meets $\tilde{\ell}'$ transversely in just one point. Let $W^+(D')$ be the handlebody obtained from the handlebody $W^+ = W \cup_{i=1}^m h_i$ by removing a thickened disk $D' \times I$ of D'. The manifold $V^+(D')$ obtained from the SUPH system $V^+ = V \cup_{i=1}^m h_i$ for the ribbon surface-link $\bar{F}$ by replacing W^+ with $W^+(D')$ is a SUPH system for a surface-link $\bar{F}'$ in $\mathbf{R}^4$ consisting of $F_i\,(i = 2, 3, \dots, r)$ and a component $\bar{F}'_1$ with genus reduced by 1 from $\bar{F}_1$. By Lemma 3.1, $\bar{F}'$ is a ribbon-surface-link in $\mathbf{R}^4$. The SUPH system V^+ for $\bar{F}$ is a disk sum of $V^+(D')$ and a solid torus W_1 with the disk D' as a meridian disk and the loop $\tilde{\ell}$ as a longitude. Let $d_W = V^+(D') \cap W_1$ be the pasting disk between $V^+(D')$ and W_1, which is regarded as a 1-handle h_W joining $V^+(D')$ and W_1. Let $(E \times I, E' \times I)$ be an O2-handle pair on $F\#T$ in $\mathbf{R}^4$ attached to T^o with $(\partial E, \partial E') = (\ell, \ell')$ in Lemma 3.2. Let A be a 4D bump of the associated bump B of $(E \times I, E' \times I)$. In the case of (i), since there is no need to worry about the intersection of A with $E \cup E'$, the 4D ball A is deformed so that $V^+ \cap A = W_1 \cup h_W$ by observing that $V \setminus V_1$ and U are disjoint from A by construction of A and by taking spine graphs of $W^+(D')$, W_1 and h_W. Then the loop $\tilde{\ell}$ bounds a disk D in A not meeting the interior of W_1 and h_W. This means that there is an O2-handle pair $(D \times I, D' \times I)$ on the surface-link $\bar{F}$ such that $\bar{F}(D' \times I)$ is a ribbon surface-link with trivial 1-handles $h'_i\,(i = 1, 2, \dots, m)$ attached, showing (i). For the case of (ii), note that the 1-handles $h_i\,(i = 1, 2, \dots, m)$ on $F\#T$ are deformed isotopically in A into 1-handles $h''_i\,(i = 1, 2, \dots, m)$ on $F\#T$ disjoint from the disk pair (E, E') because the core arcs of the 1-handles $h_i\,(i = 1, 2, \dots, m)$ are deformed to be disjoint from the disk pair (E, E') in A. The surface-link $\bar{F}(E' \times I)$ which is equivalent to $\bar{F}(E \times I, E' \times I)$ is the surface-link F with the trivial 1-handles $h''_i\,(i = 1, 2, \dots, m)$ attached, showing (ii). Thus, the proof of Lemma 3.3 is completed.

The following lemma is a combination of Lemma 3.3 and the uniqueness of an O2-handle pair in the soft sense (Theorem 2.2).

Lemma 3.4. If a connected sum $F\#T$ of a surface-link F and a trivial torus-knot T in $\mathbf{R}^4$ is a ribbon surface-link, then F is a ribbon surface-link.

Proof of Lemma 3.4. Let $F\#T = F_1\#T \cup F_2 \cup \cdots \cup F_r$ be a ribbon surface-link for a trivial torus-knot T. By Lemma 3.3 (i), the surface-link $F'' = \bar{F}(D \times I, D' \times I)$ equivalent to $\bar{F}(D \times I)$ is a ribbon surface-link and further the surface-link F^* obtained from F'' by the surgery on O2-handle pairs of all the trivial 1-handles $h'_i\,(i = 1, 2, \ldots, m)$ is a ribbon surface-link. By Lemma 3.3 (ii), the surface-link $\bar{F}(E \times I, E' \times I)$ equivalent to $\bar{F}(E \times I)$ is the surface-link F with the 1-handles $h''_i\,(i = 1, 2, \ldots, m)$ trivially attached. By an inductive use of Theorem 2.2 (Uniqueness of an O2-handle pair in the soft sense), the surface-link F is equivalent to the ribbon surface-link F^*. Thus, F is a ribbon surface-link and the proof of Lemma 3.4 is completed.

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Lemma II is a direct consequence of Lemma 3.4 as follows.

Proof of Lemma II. If a stabilization $\bar{F}$ of a surface-link F is a ribbon surface-link, then F is a ribbon surface-link by an inductive use of Lemma 3.4. This completes the proof of Lemma II.

4. Proofs of Theorems 1.4 and 1.5

The proof of Theorem 1.4 is done as follows.

Proof of Theorem 1.4. The 'if' part of Theorem 1.4 is seen from the definition of a ribbon surface-link. The proof of the 'only if' part of Theorem 1.4 uses the fact that every surface-link is made a trivial surface-knot by surgery along a finite number of possibly non-trivial 1-handles, [6]. The connected summand F_2 of $F_1\#F_2$ is made a trivial surface-knot by surgery along 1-handles within the 4-ball defining the connected summand F_2, so that the surface-link F changes into a new ribbon surface-link and hence F_1 is a stable-ribbon surface-link. By Corollary 1.2, F_1 is a ribbon surface-link. By interchanging the roles of F_1 and F_2, the connected summand F_2 is also a ribbon surface-link. This completes the proof of Theorem 1.4.

For the proof of Theorem 1.5, the following lemma is used.

Lemma 4.1. Let K be an S^2-knot in S^4 obtained from a trivial surface-knot F of genus n in S^4 by surgery along disjoint 2-handles $D_i \times I\,(i = 1, 2, \ldots, n)$. Then there is a disjoint O2-handle pair system $(D_i \times I, D'_i \times I)\,(i = 1, 2, \ldots, n)$ on F in S^4 if and only if K is a trivial S^2-knot. in S^4.

Proof of Lemma 4.1. If there is a disjoint O2-handle pair system $(D_i \times I, D'_i \times I)\,(i = 1, 2, \ldots, n)$ on F in S^4, then K is a trivial S^2-knot by Corollary 1.3. Note that the 2-handle system $D_i \times I\,(i = 1, 2, \ldots, n)$ on F is a 1-handle system on K. If K is a trivial S^2-knot, then the 1-handle system $D_i \times I\,(i = 1, 2, \ldots, n)$ on the trivial S^2-knot K is always a trivial 1-handle system on K, [6]. Hence there is a disjoint O2-handle pair system $(D_i \times I, D'_i \times I)\,(i = 1, 2, \ldots, n)$ on F in S^4. This completes the proof of Lemma 4.1.

The proof of Theorem 1.5 is done as follows.

Proof of Theorem 1.5. Let L be an S^2-link of r trivial components L_i, $(i = 1, 2, \ldots, r)$. For $r = 1$, L is a trivial S^2-knot and there is nothing to prove. Assume that the sublink L' consisting of the trivial components L_i, $(i = 1, 2, \ldots, r-1)$ is a ribbon S^2-link. Let W be a SUPH system in $\mathbf{R}^4 \subset S^4$ with $\partial W = L' \cup O$ for a trivial S^2-link O. Since the connected components of L' are 2-spheres, there is a compact connected oriented 3-manifold V_r in S^4 with $\partial V_r = L_r$ and $V_r \cap L' = \emptyset$. Let α be an arc system in W spanning O such that the closed complement $\mathrm{cl}(W \setminus N(\alpha))$ is identified with a boundary collar $L' \times [0,1]$ of L' in W with $L' \times 0 = L'$ where $N(\alpha)$ is a regular neighborhood of α in W which is written as a trivial disk fiber bundle $d \times \alpha$ over α. Since $V_r \cap L' = \emptyset$, it can be assumed that $(L' \times [0,1]) \cap V_r = \emptyset$ and the interior of the disk fiber $d \times x$ for a point x of α meets V_r transversely with a simple proper arc system and a simple loop system. By deforming V_r, the interior of the disk fiber $d \times x$ meets V_r transversely only with a simple proper arc system β. As a result, the intersection $N(\alpha) \cap V_r$ is assumed to be a thickening $\beta \times [0,1]$ of β. A regular neighborhood of $\beta \times [0,1]$ in V_r is a 1-handl system $h(\beta)$ on L_r. By adding a disjoint 1-handle system h^+ on L_r embedded in V_r, the closed complement $H = \mathrm{cl}(V_r \setminus (h(\beta) \cup h^+))$ is a handlebody of genus, say n. Let $F = \partial H$. Let H^0 be a oncei-punctured handlebody obtained from H by removing a 3-ball with $\partial H^0 = F \cup O^H$. The union $W \cup H^0$ is a SUPH system for the surface-link $L' \cup F$ in S^4, so that $L' \cup F$ is a ribbon surface-link in S^4 by Lemma 3.1. A meridian disk system $D_i\,(i = 1, 2, \ldots, n)$ of the handlebody H disjoint from the 2-sphere O^H produces a 2-handle system $D_i \times I\,(i = 1, 2, \ldots, n)$ on F whose surgery produces the trivial S^2-knot L_r. By Lemma 4.1, there is a disjoint O2-handle pair system $(D_i \times I, D'_i \times I)\,(i = 1, 2, \ldots, n)$ on F. Let $B(O) \cup B(O^H)$ be a disjoint 3-ball system bounded by the trivial S^2-link $O \cup O^H$. Consider that the SUPH system $W \cup H^0$ is given by $(O \times [0,1] \cup N(\alpha)) \cup (O^H \times [0,1] \cup N(\alpha^H))$ for a chorded sphere system $(O \cup O^H, \alpha \cup \alpha^H)$ with a regular neighborhood $N(\alpha) \cup N(\alpha^H)$ of the chord system $\alpha \cup \alpha^H$ in $W \cup H^0$. Then the disk system $D_i\,(i = 1, 2, \ldots, n)$

is a meridian disk system of $N(\alpha^H)$ of the chord system α^H. By construction, the intersections $B(O) \cap D_i = \alpha \cap D_i \, (i = 1, 2, \ldots, n)$ are taken to be $\emptyset$. Further, it may be also assumed that the intersections $B(O) \cap D'_i = \emptyset \, (i = 1, 2, \ldots, n)$ by moving the 3-ball system $B(O)$ in S^4. By general position, $\alpha \cap D'_i = \emptyset \, (i = 1, 2, \ldots, n)$. Then the disjoint O2-handle pair system $(D_i \times I, D'_i \times I) \, (i = 1, 2, \ldots, n)$ on F is a disjoint O2-handle pair system on the ribbon surface-link $L' \cup F$ in S^4. By Corollary 1.2, $L = L' \cup L_r$ is a ribbon S^2-link. Then the fundamental group $\pi_1(S^4 \setminus L, x_0)$ is known to be a torsion-free group, [13]. By induction on r, the proof of Theorem 1.5 is completed.

5. Conclusion

Ribbonness of a stable-ribbon surface-link shown in Theorem 1.1 is applied to determine Ribbonness of some classes of surface-links as they are shown in Theorems 1.4 and 1.5. For this last theorem, it would be an interesting problem to consider when a general surface-link with trivial components is a ribbon surface-link. Ribbonness of a surface-link relates not only to smooth unknotting conjecture for a surface-link leading to classical and 4D smooth Poincaré conjectures, but also to J. H. C. Whitehead asphericity conjecture for aspherical 2-complex and Kervaire conjecture on group weight, [3], [7], [13], [15], [16], [17], [24]. In another direction, it may be an interesting problem to observe a canonical relationship between a ribbon surface-knot and a knot diagram, [12]. In conclusion, ribbon surface-knot theory will be a tool to studies of low dimensional topology.

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References

[1] R. H. Bing, Necessary and sufficient conditions that a 3-manifold be S3, Ann. of Math. 68 (1958), 17-37.

[2] R. H. Fox and J. W. Milnor, Singularities of 2-spheres in 4-space and cobordism of knots, Osaka J. Math. 3 (1966), 257-267.

[3] F. González Acuña and A. Ramírez, A knot-theoretic equivalent of the Kervaire conjecture, J. Knot Theory Ramifications, 15 (2006), 471-478.

[4] J. A. Hillman and A. Kawauchi, Unknotting orientable surfaces in the 4-sphere, J. Knot Theory Ramifications 4 (1995), 213-224.

[5] S. Hirose, On diffeomorphisms over surfaces trivially embedded in the 4-sphere, Algebraic and Geometric Topology 2 (2002), 791-824.

[6] F. Hosokawa and A. Kawauchi, Proposals for unknotted surfaces in four-space, Osaka J. Math. 16 (1979), 233-248.

[7] J. Howie, Some remarks on a problem of J. H. C. Whitehead, Topology 22 (1983), 475-485.

[8] L. H. Kauffman, Virtual knot theory, European J. Combin. 20 (1999), 663-690.

[9] A. Kawauchi, A chord diagram of a ribbon surface-link, J. Knot Theory Ramifications 24 (2015), 1540002 (24 pages); Supplement to a chord diagram of a ribbon surface-link, J. Knot Theory Ramifications 26 (2017), 1750033 (5 pages); A chord graph constructed from a ribbon surface-link, Contemporary Mathematics (AMS) 689 (2017), 125-136. Amer. Math. Soc., Providence, RI, USA; Faithful equivalence of equivalent ribbon surface-links, J. Knot Theory Ramifications 27 (2018),1843003 (23 pages); Knotting probability of an arc diagram, J. Knot Theory Ramifications 29 (2020) 2042004 (22 pages).

[10] A. Kawauchi, Ribbonness of a stable-ribbon surface-link, I. A stably trivial surface-link, Topology and its Applications 301(2021), 107522 (16pages); Uniqueness of an orthogonal 2-handle pair on a surface-link, Contemporary Mathematics (UWP) 4 (2023), 182-188; Triviality of a surface-link with meridian-based free fundamental group, Transnational Journal of Mathematical Analysis and Applications 11 (2023), 19-27.

[11] A. Kawauchi, Smooth homotopy 4-sphere, WSEAS Transactions on Mathematics 22 (2023), 690-701.

[12] A. Kawauchi, Classical Poincaré Conjecture via 4D topology, J Math Techniques Comput Math 3(4) (2024), pp.1-7; Another proof of free ribbon lemma, J Math Techniques Comput Math 3(9) (2024), 01-03 (For notes added after publication, see https://sites.google.com/view/kawauchiwriting).

[13] A. Kawauchi, Ribbonness of Kervaire's sphere-link in homotopy 4-sphere and its consequences to 2-complexes, J Math Techniques Comput Math 3(4) (2024), 01-08; Kervaire conjecture on weight of group via fundamental group of ribbon sphere-link, J Math Techniques Comput Math 3(4) (2024), 1-3; Whitehead aspherical conjecture via ribbon sphere-link, J Math Techniques Comput Math 3(5) (2024), 01-10.

[14] A. Kawauchi, T. Shibuya and S. Suzuki, Descriptions on surfaces in four-space, I : Normal forms, Math. Sem. Notes, Kobe Univ. 10(1982), 75-125; II: Singularities and cross-sectional links, Math. Sem. Notes, Kobe Univ. 11(1983), 31-69.

[15] M. A. Kervaire, On higher dimensional knots, Differential and combinatorial topology 27 (1965), 105-119, Princeton Univ. Press.

[16] Ant. A. Klyachko, A funny property of a sphere and equations over groups, Comm. Algebra 21 (1993), 2555-2575.

[17] W. Magnus, A. Karrass and D. Solitar, Combinatorial group theory: Presentations of groups in terms of generators and relations, Interscience Publisheres (1966).

[18] J. Milnor, Towards the Poincaré conjecture and the classification of 3-manifolds, Notices AMS 50 (2003), 1226-1233.

[19] G. Perelman, Ricci flow with surgery on three-manifolds. arXiv: math. DG/0303109 v1, 10 Mar 2003.

[20] H. Poincaré, Second complément à l'Analysis Sitis, Proc. London Math. Soc. 32 (1900), 277-308.

[21] H. Poincaré, Cinquième complément à l'Analysis Sitis, Rend. Circ. Mat. Palermo 18 (1904), 45-110.

[22] S. Satoh, Virtual knot presentation of ribbon torus-knots, J. Knot Theory Ramifications, 9 (2000), 531- 542.

[23] V. Turaev, Knotoids, Osaka J. Math. 49 (2012), 195-223.

[24] J. H. C. Whitehead, On adding relations to homotopy groups, Ann. Math. 42 (1941), 409-428.

[25] T. Yajima, On the fundamental groups of knotted 2-manifolds in the 4-space, J. Math. Osaka City Univ., 13(1962), 63-71.

[26] T. Yajima, On simply knotted spheres in R4, Osaka J. Math, 1(1964), 133-152.

[27] T. Yanagawa, On ribbon 2-knots; the 3-manifold bounded by the 2-knot, Osaka J. Math., 6 (1969), 447-164.