Note on surface-link of trivial components

Akio Kawauchi

Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University

Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

kawauchi@omu..ac.jp

Abstract

As a previous result, it has shown that every sphere-link consisting of trivial components is a ribbon sphere-link. In this note, it is shown that for every closed oriented disconnected surface **F** with just one non-sphere component, every **F**-link consisting of trivial components is a ribbon surface-link. Further, it is shown that for every closed oriented disconnected surface **F** containing at least two non-sphere components, there exist a pair of a ribbon **F**-link and a non-ribbon **F**-link that consist of trivial components and have meridian-preservingly isomorphic fundamental groups.

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Let **F** be a (possibly disconnected) closed surface. An **F**-*link* in the 4-sphere S^4 is the image of a smooth embedding $\mathbf{F} \to S^4$. When **F** is connected, it is also called an **F**-*knot*. An **F**-link or **F**-knot for an **F** is called a *surface-link* or *surface-knot* in S^4, respectively. If **F** consists of some copies of the 2-sphere S^2, then it is also called an S^2-*link*. and an S^2-*knot* for $\mathbf{F} = S^2$. A *trivial surface-link* is a surface-link F which bounds disjoint handlebodies smoothly embedded in S^4. A *ribbon surface-link* is a surface-link F which is obtained from a trivial nS^2-link O for some n (where nS^2 denotes the disjoint union of n copies of S^2) by surgery along an embedded 1-handle system, [6], [7], [8], [9], [11]. It was shown that every S^2-link consisting of trivial components is a ribbon S^2-link, [10]. The following result is a generalization of this result.

Theorem 1. Let **F** be a closed oriented disconnected surface with at most one non-sphere component. Then every **F**-link L in S^4 consisting of trivial components is a ribbon **F**-link in S^4.

Proof of Theorem 1. The case that **F** consists of only S^2-components has been given, [10, Theorem 1.5]. This proof is done by a similar method. Let **F** have S^2-components and only one non-sphere component, and L an $\mathbf{F}$-link in S^4 consisting of trivial components. Let F be the trivial non-sphere component of L and $L' = L \setminus F$ the remaining S^2-link consists of trivial components. Since the second homology class $[F] = 0$ in $H_2(S^4 \setminus L'; Z) = 0$, there is a compact connected oriented 3-manifold V_F smoothly embedded in S^4 with $\partial V_F = F$ and $V_F \cap L' = \emptyset$. The S^2-link L' is a ribbon S^2-link in S^4, [10, Theorem 1.5]. Let W be a SUPH system in S^4 with $\partial W = L' \cup O$ for a trivial S^2-link O, [10]. Let α be an arc system in W spanning O such that the closed complement $\mathrm{cl}(W \setminus N(\alpha))$ is identified with a boundary collar $L' \times [0, 1]$ of L' in W with $L' \times 0 = L'$ where $N(\alpha)$ is a regular neighborhood of α in W which is written as a trivial disk fiber bundle $d \times \alpha$ over α. Since $V_F \cap L' = \emptyset$, it can be assumed that $(L' \times [0, 1]) \cap V_F = \emptyset$ and the interior of the disk fiber $d \times x$ for a point x of α meets V_F transversely with a simple proper arc system and a simple loop system. By deforming V_F, the interior of the disk fiber $d \times x$ meets V_F transversely only with a simple proper arc system β. As a result, the intersection $N(\alpha) \cap V_F$ is assumed to be a thickening $\beta \times [0, 1]$ of β. A regular neighborhood of $\beta \times [0, 1]$ in V_F is a 1-handl system $h(\beta)$ on F. By adding a disjoint 1-handle system h^+ on F embedded in V_F, the closed complement $H = \mathrm{cl}(V_F \setminus (h(\beta) \cup h^+))$ is a handlebody of genus, say n. Let $F^+ = \partial H$. Let H^0 be a once-punctured handlebody obtained from H by removing a 3-ball with $\partial H^0 = F^+ \cup O^H$. The union $W \cup H^0$ is a SUPH system for the surface-link $L' \cup F^+$ in S^4, so that $L' \cup F^+$ is a ribbon surface-link in S^4 by [10, Lemma 3.1]. The 1-handle system $h(\beta) \cup h^+$ on F is a 2-handle system $D_i \times I \, (i = 1, 2, \ldots, n)$ on F^+, where I denotes an interval containing 0 in the interior. Then there is a disjoint O2-handle pair system $(D_i \times I, D'_i \times I) \, (i = 1, 2, \ldots, n)$ on F^+ because any disjoint 1-handle system on the trivial surface-knot F is a disjoint trivial 1-handle system on F, [4], [10, Lemma 4.1]. Let $(O \cup O^H, \alpha \cup \alpha^H)$ be a chorded sphere system for the ribbon surface-link $L' \cup F^+$ constructed in the SUPH system $W \cup H^0$. Let $B(O) \cup B(O^H)$ be a disjoint 3-ball system bounded by the trivial S^2-link $O \cup O^H$ in S^4. The intersections $B(O) \cap D_i = B(O) \cap D'_i = \emptyset (i = 1, 2, \ldots, n)$ can be assumed by moving the 3-ball system $B(O) \cup B(O^H)$ in S^4. The intersections $(\alpha \cup \alpha^H) \cap D_i = (\alpha \cup \alpha^H) \cap D'_i = \emptyset \, (i = 1, 2, \ldots, n)$ can be also assumed by general position. Thus, the disjoint O2-handle pair system $(D_i \times I, D'_i \times I) \, (i = 1, 2, \ldots, n)$ on F is deformed into a disjoint O2-handle pair system on the ribbon surface-link $L' \cup F^+$ in S^4, whose surgery surface-link $L = L' \cup F$ is a ribbon surface-link, [10, Corollary 1.2, & Lemma 2.1]. This completes the proof of Theorem 1.

In the case that $\mathbf{F}$ has at least two non-sphere components, the following result is obtained.

Theorem 2. Let $\mathbf{F}$ be any closed oriented disconnected surface with at least two non-sphere components. Then there exist a pair of a ribbon $\mathbf{F}$-link L and a non-ribbon $\mathbf{F}$-link L' in S^4 that consist of trivial components and have meridian-preservingly isomorphic fundamental groups.

Proof of Theorem 2. Let $k \cup k'$ be a non-splitable link in the interior of a 3-ball B such that

k and k' are trivial knots. For the boundary 2-sphere $S = \partial B$ and the disk D^2 with the boundary circle S^1, let L be the torus-link consisting of the torus-components $T = k \times S^1$ and $T' = k' \times S^1$ in the 4-sphere S^4 with $S^4 = B \times S^1 \cup S \times D^2$, which is a ribbon torus-link in S^4, [6]. Since k and k' are trivial knots in B, the torus-knots T and T' are trivial torus-knots in S^4 by construction. Since $k \cup k'$ is non-splitable in B, there is a simple loop $t(k)$ in T coming from the longitude of k in B such that $t(k)$ does not bound any disk not meeting T' in S^4, meaning that there is a simple loop c in T unique up to isotopies of T which bound a disk d in S^4 not meeting T', where c and d are given by $c = p \times S^1$ and $d = a \times S^1 \cup q \times D^2$ for a simple arc a in B joining a point p of k to a point q in S with $a \cap (k \cup k') = \{p\}$ and $a \cap S = \{q\}$. Regard the 3-ball B as the product $B = B_1 \times [0, 1]$ for a disk B_1. Let τ_1 is a diffeomorphism of the solid torus $B_1 \times S^1$ given by one full-twist along the meridian disk B_1, and $\tau = \tau_1 \times 1$ the product diffeomorphism of $(B_1 \times S^1) \times [0, 1] = B \times S^1$. Let $\partial\tau$ be the diffeomorphism of the boundary $S \times S^1$ of $B \times S^1$ obtained from τ by restricting to the boundary, and the 4-manifold M obtained from $B \times S^1$ and $S \times D^2$ by pasting the boundaries $\partial(B \times S^1) = S \times S^1$ and $\partial(S \times D^2) = S \times S^1$ by the diffeomorphism $\partial\tau$. Since the diffeomorphism $\partial\tau$ of $S \times S^1$ extends to the diffeomorphism τ of $B \times S^1$, the 4-manifold M is diffeomorphic to S^4. Let $L_M = T_M \cup T'_M$ be the torus-link in the 4-sphere M arising from $L = T \cup T'$ in $B \times S^1$. The fundamental groups $\pi_1(S^4 \setminus L, x)$ and $\pi_1(M \setminus L_M, x)$ are meridian-preservingly isomorphic by van Kampen theorem. The loop $t(k)$ in T_M does not bound any disk not meeting T'_M in M, so that the loop c in T_M is a unique simple loop up to isotopies of T_M which bounds a disk $d_M = a \times S^1 \cup D^2_M$ in M not meeting T'_M, where D^2_M denotes a proper disk in $S \times D^2$ bounded by the loop $\partial\tau(q \times S^1)$. An important observation is that the self-intersection number $\mathrm{Int}(d_M, d_M)$ in M with respect to the surface-framing on L_M is ± 1. This means that the loop c in T_M is a non-spin loop and thus, the torus-link L_M in M is not any ribbon torus-link, [3], [5]. Let $(S^4, L') = (M, L_M)$. If $\mathbf{F}$ consists of two tori, then the pair (L, L') forms a desired pair. If $\mathbf{F}$ is any surface consisting of two non-sphere components, then a desired $\mathbf{F}$-link pair is obtained from the pair (L, L') by taking connected sums of some trivial surface-knots, because every stabilization of a ribbon surface-link is a ribbon surface-link and every stable-ribbon surface-link is a ribbon surface-link, [10]. If $\mathbf{F}$ has some other surface $\mathbf{F}_1$ in addition to a surface $\mathbf{F}_0$ of two non-sphere components, then a desired $\mathbf{F}$-link pair is obtained from a desired $\mathbf{F}_0$-link pair by adding the trivial $\mathbf{F}_1$-link as a split sum. Thus, a desired $\mathbf{F}$-link pair (L, L') is obtained. This completes the proof of Theorem 2.

In the proof of Theorem 2, the diffeomorphism $\partial\tau$ of $S \times S^1$ coincides with Gluck's non-spin diffeomorphism of $S^2 \times S^1$, [2]. The torus-link (M, T_M) called a *turned torus-link* of a link $k \cup k'$ in B is an analogy of a turned torus-knot of a knot in B, [1]. There is an invariant of a surface-knot used to confirm a non-ribbon surface-knot, [5]. This invariant is easily generalized to an invariant of a surface-link, which can be also applied to confirm that L' is a non-ribbon surface-link.

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