Free ribbon lemma for surface-link

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ABSTRACT

A free surface-link is a surface-link whose fundamental group is a free group not necessarily meridian-based. Free ribbon lemma says that every free sphere-link in the 4-sphere is a ribbon sphere-link. Four different proofs of Free ribbon lemma are explained. The first proof is done in an earlier paper. The second proof is done by showing that there is an O2-handle basis of a ribbon surface-link. The third proof is done by removing the commuter relations from a Wirtinger presentation of a free group, which a paper on another proof of Free ribbon lemma complements. The fourth proof is given by the special case of the proof of the result that every free surface-link is a ribbon surface-link which is a stabilization of a free ribbon sphere-link. As a consequence, it is shown that a surface-link is a sublink of a free surface-link if and only if it is a stabilization of a ribbon sphere-link.

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1. Introduction

A *surface link* is a closed, possibly disconnected, oriented surface F smoothly embedded in the 4-sphere S^4, and it is called a *surface knot* if F is connected. If F consists of 2-spheres $F_i\,(i = 1, 2, \dots, r)$, then F is called a *sphere-link* (or an S^2*-link*) of r components. It is shown that a surface-link F is a trivial surface-link (i.e., bounds disjoint handlebodies in S^4) if the fundamental group $\pi_1(S^4 \setminus F, x_0)$ is a meridian-based free group, [10], [11], [12]. A surface-link F is *ribbon* if F is obtained from a trivial S^2-link O in S^4 by surgery along a smoothly embedded disjoint 1-handle system h^O on O, [20], [29], [30], [32]. A surface-link F in the 4-sphere S^4 is *free* if the fundamental group $\pi_1(S^4 \setminus F, x_0)$ is a (not necessarily meridian-based) free group. In this paper, four different proofs of the following *Free ribbon lemma* and its generalization to a general free surface-link are explained.

Free ribbon lemma. Every free S^2-link in S^4 is a ribbon S^2-link.

Free ribbon lemma leads to the following conjectures: Poincaré conjecture, [15], [24], [25]. [26]. J. H. C. Whitehead asphericity conjecture for aspherical 2-complex, [5], [14], [17], [28]. Kervaire conjecture on group weight, [1], [16], [21], [23], [22]. The first proof is given [14]. For convenience, an outline of the first proof is explained here.

First proof of Free ribbon lemma. Let $L_i\,(i = 1, 2, \ldots, r)$ be the components of a free S^2-link L in S^4. By a base change of the free fundamental group $\pi_1(S^4 \setminus L, x_0)$, take a basis $x_i\,(i = 1, 2, \ldots, r)$ of $\pi_1(S^4 \setminus L, x_0)$ inducing a meridian basis of L in $H_1(S^4 \setminus L; Z)$, [23]. Let Y be the 4-manifold obtained from S^4 by surgery along L, which is diffeomorphic to the connected sum of r copies $S^1 \times S^3_i\,(i = 1, 2, \ldots, r)$ of $S^1 \times S^3$, [13], [14]. Under a canonical isomorphism $\pi_1(S^4 \setminus L, x_0) \to \pi_1(Y, x_0)$, the factors $S^1 \times p_i\,(i = 1, 2, \ldots, r)$ of $S^1 \times S^3_i\,(i = 1, 2, \ldots, r)$ with suitable paths to the base point x_0 represent the basis $x_i\,(i = 1, 2, \ldots, r)$. Let $k_i\,(i = 1, 2, \ldots, r)$ be the loop system in Y produced from the components $L_i\,(i = 1, 2, \ldots, r)$ by the surgery. By using the fact that any homotopy deformations of $k_i\,(i = 1, 2, \ldots, r)$ in Y do not change the link type of the surface-link L in S^4, the loop system $k_i\,(i = 1, 2, \ldots, r)$ is homotopically deformed in Y so that the surface-link L in S^4 obtained from the deformed loop system $k_i\,(i = 1, 2, \ldots, r)$ by back surgery is a ribbon surface-link in S^4, completing the proof of Free ribbon lemma.

To explain the second and third proofs of Free ribbon lemma, the notion of an O2-handle basis of a surface-link is needed, [10], [19]. An *O2-handle pair* on a surface-link F in S^4 is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I$, $D' \times I$ on F in S^4 which intersect orthogonally only with the attaching parts $(\partial D) \times I$, $(\partial D') \times I$ to F, so that the intersection $Q = (\partial D) \times I \cap (\partial D') \times I$ is a square. Let $(D \times I, D' \times I)$ be an O2-handle pair on a surface-link F. Let $F(D \times I)$ and $F(D' \times I)$ be the surface-links obtained from F by the surgeries along $D \times I$ and $D' \times I$, respectively. Let $F(D \times I, D' \times I)$ be the surface-link which is the union $\delta \cup F^c_\delta$ of the plumbed disk

$$\delta = \delta_{D\times I, D'\times I} = D \times \partial I \cup Q \cup D' \times \partial I$$

and the surface $F^c_\delta = \mathrm{cl}(F \setminus (\partial D \times I \cup \partial D' \times I))$. The surface-links $F(D\times I), F(D'\times I)$ and $F(D\times I, D'\times I)$ are equivalent surface-links, [10]. An *O2-handle basis* of a surface-link F is a disjoint system of O2-handle pairs $(D_i \times I, D'_i \times I)\,(i = 1, 2, \ldots, r)$ on F in S^4 such that the boundary loop pair system $(\partial D_i, \partial D'_i)\,(i = 1, 2, \ldots, r)$ of the core disk system $(D_i, D'_i)\,(i = 1, 2, \ldots, r)$ of $(D_i \times I, D'_i \times I)\,(i = 1, 2, \ldots, r)$ is a spin loop basis for F in S^4, which is a system of a spin loop basis of every component F_i of F. Note that there is a spin loop basis for every surface-knot in F, [3]. In this paper, for simplicity, an O2-handle basis $(D_i \times I, D'_i \times I)\,(i = 1, 2, \ldots, r)$ for F is denoted by $(D \times I, D' \times I)$. The surgery surface-link of F by $(D_i \times I, D'_i \times I)\,(i = 1, 2, \ldots, r)$ is denoted by $F(D \times I, D' \times I)$. The following theorem is shown for the second and third proofs of Free ribbon lemma.

Theorem 1.1. For every free ribbon surface-link F in S^4, there is an O2-handle basis $(D \times I, D' \times I)$ on F in S^4 such that $D \times I$ belongs to the 1-handle system of the ribbon surface-link F.

The second proof of Free ribbon lemma is explained as follows.

Second proof of Free ribbon lemma. Let L be a free S^2-link. Then there is a ribbon surface-link F such that the fundamental group $\pi_1(S^4 \setminus F, x_0)$ is isomorphic to the free fundamental group $\pi_1(S^4 \setminus L, x_0)$ by a meridian-preserving isomorphism, [18]. By Theorem 1.1, the surgery surface-link $L' = F(D\times I, D'\times I)$ is a ribbon S^2-link, [10], [19]. Then there is a meridian-preserving isomorphism $\pi_1(S^4 \setminus L', x_0) \to \pi_1(S^4 \setminus L, x_0)$ on free groups, which implies that L' is equivalent to L, [14], [18]. Thus, L is a ribbon S^2-link, completing the proof of Free ribbon lemma.

The third proof of Free ribbon lemma is related to a Wirtinger presentation of a free group. A finite group presentation $(x_1, x_2, \ldots, x_n |\, R_1, R_2, \ldots, R_m)$ is a Wirtinger presentation if $R_j = W_j x_{s_j} W_j^{-1} x_{t_j}^{-1}$ for some indexes s_j, t_j in $\{1, 2, \ldots, n\}$ for every $j\,(j = 1, 2, \ldots, m)$. The relator R_j is a *commutator relation* if $x_{s_j} = x_{t_j}$. It is well-known that a Wirtinger presentation of a finitely presented group G with $H_1(G; Z) \cong Z^r$ is always equivalent (without changing the gerenating set) to a Wirtinger presentation P such that the Wirtinger presentation P' obtained by removing all the commutator relations from P has deficiency r. Such a Wirtinger presentation P is called a *normal* Wirtinger presentation. The following corollary is obtained from Theorem 1.1.

Corollary 1.2. If a free group G of rank r has a normal Wirtinger presentation P, then G has the Wirtinger presentation P' of deficiency r obtained from P by removing all the commutator relations.

Proof of Corollary 1.2 assuming Theorem 1.1. For a free group G of rank r, let $P = (x_1, x_2, \dots, x_n | \, R_1, R_2, \dots, R_m)$ be a normal Wirtinger presentation of G such that the relators $R_j \, (n - r + 1 \leq j \leq m)$ are the commutator relations. Let O be a trivial S^2-link of n components in S^4 such that the meridian basis of the free fundamental group $\pi_1(S^4 \setminus O, x_0)$ are identified with $x_i \, (i = 1, 2, \dots, n)$. Let $h_j \, (1 \leq j \leq m)$ be the 1-handles on O indicated by the relators $R_j \, (1 \leq j \leq m)$. By the van Kampen theorem, the ribbon surface-link F in S^4 obtained by surgery along $h_j \, (1 \leq j \leq m)$ has the normal Wirtinger presentation P of the fundamental group $\pi_1(S^4 \setminus F, x_0)$ with the meridian generators set $\{x_1, x_2, \dots, x_n\}$, [7], [8]. Let L be the ribbon surface-link obtained from O by surgery along the 1-handles $h_j \, (1 \leq j \leq n-r)$, which is a ribbon S^2-link of r components. The fundamental group $\pi_1(S^4 \setminus L, x_0)$ has the Wirtinger presentation P' of deficiency r obtained from P by removing all the commutator relations. By Theorem 1.1, the 1-handles $h_j \, (n - r + 1 \leq j \leq m)$ on L are trivial 1-handles, so that $\pi_1(S^4 \setminus L, x_0)$ is isomorphic to $\pi_1(S^4 \setminus F, x_0)$ by a meridian-preserving isomorphism. This completes the proof of Corollary 1.2 assuming Theorem 1.1.

The author has published a paper on another proof of Free ribbon lemma, which this paper complements, [18]. The third proof of Free ribbon lemma is nothing but the proof of the paper except for adding to it the assertion of Corollary 1.5 which was missing from it. For convenience, an outline of the third proof is explained here.

Third proof of Free ribbon lemma. Let L be a free S^2-link of r components. Since the fundamental group $G = \pi_1(S^4 \setminus L, x_0)$ is a free group with $H_1(G; Z) = Z^r$ and $H_2(G; Z) = 0$, there is a normal Wirtinger presentation P of G whose generator set comes from meridians of L in S^4, [18], [31]. Note that there is also another method to find such a normal Wirtinger presentation P using a normal form of L in S^4, [6], [7], [8], [20]. Let L' be a ribbon S^2-link given by the Wirtinger presentation P' obtained from P by removing all the commutators. By Corollary 1.2, there is a meridian-preserving isomorphism $\pi_1(S^4 \setminus L', x_0) \to \pi_1(S^4 \setminus L, x_0)$, so that L' is equivalent to L. Thus, L is a ribbon S^2-link, completing the proof of Free ribbon lemma.

The fourth proof of Free ribbon lemma is given by a direct proof of the following theorem.

Theorem 1.3. Every free surface-link F in S^4 is a ribbon surface-link in S^4.

Fourth proof of Free ribbon lemma. It is obtained by restricting F to every free S^2-link, completing the proof of Free ribbon lemma.

Thus, after the proofs of Theorems 1.1 and 1.3, there are four different proofs of Free ribbon lemma.

To generalize the free ribbon lemma to a free surface-link, the notion of a stabilization of a surface-link is needed, [10], [19]. A *stabilization* of a surface-link L is a connected sum $F = L\#_{k=1}^{s} T_k$ of L and a system of trivial torus-knots T_k $(k = 1, 2, \ldots, s)$. By granting $s = 0$, a surface-link L itself is regarded as a stabilization of L. Free ribbon lemma is generalized to a general free surface-link as follows.

Corollary 1.4. Every free surface-link F in S^4 is a stabilization of a free ribbon S^2-link L in S^4.

Proof of Crollary 1.4 assuming Theorems 1.1 and 1.3. Theorem 1.1 implies that every free surface-link F is a stabilization of a free S^2-link L, [10]. By Free ribbon lemma, the free S^2-link L is a ribbon S^2-link. This completes the proof of Crollary 1.4 assuming Theorems 1.1 and 1.3.

It is shown that an S^2-link L is a sublink of a free S^2-link if and only if L is a ribbon S^2-link, [14]. The following corollary generalizes this property to a general surface-link.

Corollary 1.5. A surface-link L in S^4 is a sublink of a free surface-link F in S^4 if and only if L is a stabilization of a ribbon S^2-link in S^4.

Proof of Corollary 1.5 assuming Theorem 1.3. If L is a sublink of a free surface-link F, then L is a stabilization of a ribbon S^2-link since every free surface-link is a stabilization of a free ribbon S^2-link by Corollary 1.2. Conversely, if L is a stabilization of a ribbon S^2-link, then L is a sublink of a stabilization of a free ribbon S^2-link which is a free surface-link F since every ribbon S^2-link is a sublink of a free S^2-link. This completes the proof of Corollary 1.5 assuming Theorem 1.3.

2. Proofs of Theorems 1.1 and 1.3.

Let F be a *free* surface-link in S^4 with components F_i $(i = 1, 2, \ldots, r)$. Let $N(F) = \cup_{i=1}^{r} N(F_i)$ be a tubular neighborhood of $F = \cup_{i=1}^{r} F_i$ in S^4 which is a trivial normal disk bundle $F \times D^2$ over F, where D^2 denotes the unit disk of complex numbers of norm $\leqq 1$. Let $E = E(F) = \mathrm{cl}(S^4 \setminus N(F))$ be the exterior of F in S^4. The boundary $\partial E = \partial N(F) = \cup_{i=1}^{r} \partial N(F_i)$ of the exterior E is a trivial normal circle bundle over

$F = \cup_{i=1}^r F_i$. Identify $\partial N(F_i) = F_i \times S^1$ for $S^1 = \partial D^2$ such that the composite inclusion

$$F_i \times 1 \to \partial N(F_i) \to \mathrm{cl}(S^4 \setminus N(F_i))$$

induces the zero-map in the integral first homology. The following lemma uses the assumption that the fundamental group $\pi_1(E, x_0)$ is a free group of rank r and the fact that the first homology group $H_1(E; Z)$ is a free abelian group of rank r with meridian basis.

Lemma 2.1. The composite inclusion $F_i \times 1 \to \partial N(F_i) \to E$ is null-homotopic for all i.

Proof of Lemma 2.1. Since $\partial N(F_i) = F_i \times S^1$, the fundamental group elements between the factors $F_i \times 1$ and $q_i \times S^1$ are commutive. Let $a_i\,(i = 1, 2, \ldots, r)$ be embedded edges with common vertex x_0 in E such that $a_i \setminus \{x_0\}\,(i = 1, 2, \ldots, r)$ are mutually disjoint and $a_i \cap (\cup_{j=1}^r F_j \times 1) = p_i \times 1$ for a point p_i of $F_i \times 1$. The surface $F_i \times 1$ in $\partial N(F_i) = F_i \times S^1$ is chosen so that the inclusion $F_i \times 1 \to \mathrm{cl}(S^4 \setminus N(F_i))$ induces the zero-map in the integral first homology. Since $H_1(E; Z)$ is a free abelian group of rank r with meridian basis and $\pi_1(E, x_0)$ is a free group of rank r, the image of the homomorphism $\pi_1(a_i \cup F_i \times S^1, x_0) \to \pi_1(E, x_0)$ is an infinite cyclic group generated by the homotopy class $[a_i \cup p_i \times S^1]$. This implies that the inclusion $F_i \times 1 \to E$ is null-homotopic. This completes the proof of Lemma 2.1. □

By using the free group $\pi_1(E, x_0)$ of rank r, let

$$\Gamma = ((\cup_{i=1}^r a_i) \cup (\cup_{i=1}^r C_i)$$

be a connected graph in the interior $\mathrm{Int}(E)$ of E consisting of embedded edges $a_i\,(i = 1, 2, \ldots, r)$ with the common base point x_0 and disjoint embedded circles $C_i\,(i = 1, 2, \ldots, r)$ such that

(1) the half-open edges $a_i \setminus \{x_0\}\,(i = 1, 2, \ldots, r)$ are mutually disjoint and $a_i \cap (\cup_{j=1}^r C_j) = v_i$, a point in C_i for every i,

(2) the inclusion $i : (\Gamma, x_0) \to (E, x_0)$ induces an isomorphism $i_\# : \pi_1(K, x_0) \to \pi_1(E, x_0)$, and

(3) the homology class $[p_i \times S^1] = [C_i]$ in $H_1(E; Z)$ for all i.

In fact, by (2), the homotopy classes $[a_i \cup C_i]\,(i = 1, 2, \ldots, r)$ form a basis of the free group $\pi_1(E, q_0)$. (3) is obtained by a base change of the free group $\pi_1(E, x_0)$, [23]. Since Γ is a $K(\pi, 1)$-space, there is a piecewise-linear map $f : (E, x_0) \to (\Gamma, x_0)$

inducing the inverse isomorphism $f_\# = (i_\#)^{-1} : \pi_1(E, q_0) \to \pi_1(\Gamma, q_0)$, and by the homotopy extension property, the restriction of f to Γ is the identity map, [27]. The restriction of f to ∂E is homotopic to the composite map

$$g : \partial E = F \times S^1 \to \cup_{i=1}^r q_i \times S^1 \to \Gamma$$

such that the first map $F \times S^1 \to \cup_{i=1}^r q_i \times S^1$ is induced from the constant map $F \to \cup_{i=1}^r \{q_i\}$ and the second map $\cup_{i=1}^r q_i \times S^1 \to \Gamma$ is defined by the map f. By using a boundary collar of ∂E in E, assume that the piecewise-linear map $f : (E, x_0) \to (\Gamma, x_0)$ defines the map $g : \partial E \to \Gamma$. For a non-vertex point p_i of C_i, the preimage $V_i = (f)^{-1}(p_i)$ is a bi-collard compact oriented proper piecewise-linear 3-manifold in E. Take the compact 4-manifold E' obtained from E by splitting along $V = \cup_{i=1}^r V_i$ to be connected. Then join the components in each V_i with 1-handles in E'. By these modifications, V_i is assumed to be connected for all i within a homotopic deformation of f. The boundary ∂V_i is the disjoint union $P_i(F)$ of m_{ij} parallel copies $m_{ij}F_j$ of $F_j \times 1$ for all $j\,(j = 1, 2, \ldots, r)$ in S^4, where m_{ii} is an odd integer and m_{ij} for distinct i, j is an even integer. Let $P(F) = \cup_{i=1}^r P_i(F)$ be the surface-link in S^4. Let h_i be a disjoint 1-handle system on $P_i(F)$ embedded in V_i such that the surface $P_i(F; h_i)$ obtained from $P_i(F)$ by surgery along h_i is connected and the genus of $P_i(F; h_i)$ is equal to the total genus of $P_i(F)$. Assume that one copy of the parallel $m_{ii}F_i$ of F_i is identified with F_i and just one 1-handle h_i^F of h_i attaches to F_i. Let $P(F; h) = \cup_{i=1}^r P_i(F; h_i)$ be a surface-link in S^4. By further taking a disjoint 1-handle system h'_i on $P_i(F, h_i)$ embedded in V_i, the closed surface $P_i(F; h_i, h'_i)$ obtained from $P_i(F; h_i)$ by surgery along h'_i bounds a handlebody V'_i in V_i, so that the surface-link $P(F; h, h') = \cup_{i=1}^r P_i(F; h_i, h'_i)$ is a trivial surface-link in S^4. Because of the isomorphism $f_\#$, the compact 4-manifold E' is simply connected, so that the 1-handle system $h' = \cup_{i=1}^r h'_i$ is a trivial 1-handle system on the surface-link $P(F; h)$ in S^4, [4], [12]. Thus, the surface-link $P(F; h)$ is a trivial surface-link in S^4, [10], [11]. The proof of Theorem 1.1 is done as follows.

Proof of Theorem 1.1. A ribbon surface-link F is obtained from a trivial S^2-link O in S^4 by surgery along a disjoint 1-handle system h^O on O, so that the surface-link $P(F)$ of a free ribbon surface-link F is a ribbon surface-link obtained from a trivial S^2-link $P(O)$ in S^4 by surgery along a disjoint 1-handle system $P(h^O)$ on $P(O)$. Let $VP(F)$ is a SUPH system for the ribbon surface-link $P(F)$ in S^4, namely a multi-punctured handlebody system $VP(F)$ in S^4 such that $\partial VP(F) = P(F) \cup P(O)$, [19]. Actually, consider the SUPH system $VP(F)$ obtained from the collar $P(O) \times [0, 1]$ of O in S^4 by attaching the 1-handle system $P(h^O)$ on $P(O) \times 0 = P(O)$. The 1-handle system $h = \cup_{i=1}^r h_i$ on $P(F)$ and the SUPH system $VP(F)$ construct a SUPH system $VP(F) \cup h$ for the trivial surface-link $P(F; h)$ with $\partial(VP(F) \cup h) = P(F; h) \cup P(O)$.

A spin loop basis (ℓ, ℓ') for $P(F)$ is the system consisting of a spin loop basis of every component of $P(F)$ where the spin loop system ℓ belongs to a meridian system of the 1-handle system h^O. This system (ℓ, ℓ') is a spin loop basis of the trivial ribbon surface-link $P(F;h)$. Equivalent ribbon surface-links are faithfully equivalent and they are moved into each other by the moves M_0, M_1, M_2, [9]. This means that there is an orientation-preserving diffeomorphism f of S^4 sending the SUPH system $VP(F) \cup h$ for $P(F;h)$ to a standard multi-punctured handlebody system W in S^4. By a choice of f, the system $(f(\ell), f(\ell'))$ is a meridian-longitude pair system of the standard multi-punctured handlebody system W in S^4, [2], [10]. The loop system $f(\ell')$ bounds a disjoint disk system δ' in S^4 with $\delta' \cap W = f(\ell')$, so that the loop system ℓ' bounds a disjoint disk system $D' = f^{-1}(\delta')$ in S^4 with $D' \cap (VP(F) \cup h) = \ell'$. The loop system ℓ belongs to a meridian system of the 1-handle system $P(h^O)$ and hence bounds a sub-system D of the meridian disk system $P(h^O)$. Thus, it is shown that there is an O2-handle basis $(D \times I, D' \times I)$ on $P(F)$ in S^4, whose sub-system to F gives an O2-handle basis on F in S^4. This completes the proof of Theorem 1.1.

The proof of Theorem 1.3 is done as follows.

Proof of Theorem 1.3. An *anti-parallel* of F_j in S^4 is the boundary surface-link of a normal line bundle $F_j \times I$ of F_j in S^4. Note that the boundary surface-knot $\partial(F_j^{(0)} \times I)$ for a compact once-punctured surface $F_j^{(0)}$ of F_j is a trivial surface-knot in S^4 since $F_j^{(0)} \times I$ is a handlebody. Let $n_{ii} = (m_{ii} - 1)/2$ and $n_{ij} = m_{ij}/2$ for distinct i, j. The surface-link $P_i(F)$ consists of F_i and n_{ij} copies of the anti-parallel of F_j for all j. Since E' is simply connected, the 1-handle system h_i can be chosen so that every anti-parallel of F_j in $P_i(F)$ produces a trivial surface-knot F_{ij}^t by surgery along a 1-handle in h_i. Let $h(1)_i$ be the system of 1-handles in h_i other than the system $h(0)_i$ of 1-handles in h_i used for these surgeries in $P_i(F)$. The trivial surface-knot $P_i(F;h_i)$ is obtained from the surface-link $P_i(F;h(0)_i)$ consisting of F_i and a system of n_{ij} trivial surface-knots as F_{ij}^t for all j by surgery along the 1-handle system $h(1)_i$. Let $h(1) = \cup_{i=1}^r h(1)_i$. Let $P(F;h(0)) = \cup_{i=1}^r P_i(F;h(0)_i)$. The surface-link $P(F;h,h')$ bounds a handlebody system $V' = \cup_{i=1}^r V_i'$ in V. Note that h' is a trivial 1-handle system on $P(F;h)$. Then there is a disjoint handlebody system U in S^4 with $\partial U = P(F;h)$ extending the handlebody system V' by adding a 2-handle system $e \times I$ which makes an O2-handle system together with a thickened meridian disk system of h', [10], [11]. Let $U_i \, (i = 1, 2, \dots, r)$ be the components of U with $\partial U_i = P_i(F;h_i) \, (i = 1, 2, \dots, r)$. Let d_i be a meridian disk system of the 1-handle system $h(1)_i$, and $d = \cup_{i=1}^r d_i$ a meridian disk system of $h(1)$. In general, the disk system d meets the core disk system e of $e \times I$ transversely in finite points in S^4, but the interior of d is deformed so to have $d \cap U = \partial d$ by Finger Move Canceling

Operation, [11]. For $h(1) = d \times I$, the union $U \cup h(1)$ is a compact oriented 3-manifold with boundary $P(F; h(0))$ obtained from U by adding the 2-handle system $h(1)$ to $P(F; h)$ since $h(1) \cap U = (\partial d) \times I$. Let (ℓ, ℓ') be a spin loop basis of $P(F; h)$ given for $P(F; h(0))$ such that when restricted to every trivial surface-knot F^t_{ij} it becomes a standard spin loop basis. Since $P(F; h)$ is a trivial surface-link, there is an orientation-preserving diffeomorphism w of S^4 sending $P(F; h)$ to the boundary ∂W of a standard handlebody system W in S^4 such that the spin loop basis $(w(\ell), w(\ell'))$ of ∂W is a meridian-longitude pair system of W, [2], [10]. Let $U(W) = w^{-1}(W)$ be the handlebody system in S^4 with $\partial U(W) = P(F; h)$. The spin loop basis $(w(\ell), w(\ell'))$ of W bounds a core disk-pair system (δ, δ') of an O2-handle basis $(\delta \times I, \delta' \times I)$ of the trivial surface-link ∂W in S^4, where δ denotes a meridian disk system of W. Hence the spin loop basis (ℓ, ℓ') of $P(F; h)$ bounds the core disk pair system (D, D') of the O2-handle basis $(D \times I, D' \times I) = (w^{-1}(\delta) \times I, w^{-1}(\delta') \times I)$ on $P(F; h)$ in S^4 with $D \subset U(W)$, so that $d \cap D = h(1) \cap D = \emptyset$. The handlebody system $U(W)$ is isotopically deformed to be $U(W) = U$ in S^4. Consider the 3-manifold $U' \cup h(1)$ obtained from the 3-manifold $U \cup h(1)$ by splitting along the disk system D, which is a multi-punctured 3-sphere system consisting of the multi-punctured 3-spheres $U'_i \cup h(1)_i \, (i = 1, 2, \ldots, r)$ obtained from the 3-manifolds $U_i \cup h(1)_i \, (i = 1, 2, \ldots, r)$. Since $\partial(U_i \cup h(1)_i) = P_i(F; h(0)_i)$, the boundary $\partial(U'_i \cup h(1)_i)$ consists of an S^2-knot S_i obtained from F_i and a system of n_{ij} S^2-knots as S^t_{ij} obtained from the system of n_{ij} trivial surface-knots as F^t_{ij} for all j. Every S^2-knot as S^t_{ij} is shown to be a trivial S^2-knot in S^4. To see this, note that there is a 2-handle system $h_D(1)$ on $P(F; h)$ embedded in U such that there is a diffeomorphism of S^4 sending $P(F; h) \cup h_D(1)$ to $P(F; h) \cup h(1)$ with the spin loop basis (ℓ, ℓ') and the disk system D preserved. The O2-handle subsystem of the O2-handle basis $(D \times I, D' \times I)$ on $P(F; h)$ is disjoint from $h_D(1)$. On the other hand, every trivial surface-knot F^t_{ij} admits a standard O2-handle basis $(D(F^t_{ij}) \times I, D'(F^t_{ij}) \times I)$ on F^t_{ij} not meeting $h(1)$ with the spin loop basis $(\partial D(F^t_{ij}), \partial D'(F^t_{ij}))$ on F^t_{ij} a subsystem of (ℓ, ℓ'). Then the O2-handle system on every trivial surface-knot F^t_{ij} restricted from the O2-handle basis $(D \times I, D' \times I)$ on $P(F; h)$ is considered to be a standard O2-handle basis by the uniqueness of an O2-handle pair, [11]. Thus, every S^2-knot as S^t_{ij} is seen to be a trivial S^2-knot, as desired. This means that the S^2-link $S = \cup_{i=1}^r S_i$ is a ribbon S^2-link in S^4, [14]. Thus, the surface-link F is a ribbon surface-link in S^4 because F is obtained from the ribbon S^2-link S by surgery along 1-handles. This completes the proof of Theorem 1.3.

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