

Reduction of bielliptic hyperelliptic functions of genus 3

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Abstract

The present paper is devoted to the problem about the reduction of hyperelliptic functions of genus 3. Our research was motivated by applications to the theory of equations and dynamical systems integrable in hyperelliptic functions. In this paper, we consider a hyperelliptic curve of genus 3 which admits a morphism of degree 2 to an elliptic curve. We express the hyperelliptic functions associated with the curve of genus 3 in terms of the Weierstrass elliptic functions and hyperelliptic functions of genus 2.

1 Introduction

The elliptic sigma function, which is defined and studied by Weierstrass, is generalized to the sigma functions associated with hyperelliptic curves and many applications in integrable systems and mathematical physics are derived (cf. [14, 15, 16]). For example, it is well known that the hyperelliptic functions defined by the logarithmic derivatives of the hyperelliptic sigma functions satisfy the KdV hierarchy and the KP equation.

The problem whether the Jacobian variety of a hyperelliptic curve is isogenous to the direct product of the Jacobian varieties of hyperelliptic curves of lower genera is considered in many papers (e.g., [9, 12, 13, 20, 23, 24, 25, 28, 29]). This problem is naturally connected with the following well-known problem: Suppose that solutions of differential equations and dynamical systems are given in terms of hyperelliptic functions. Under conditions when a reduction of these functions to hyperelliptic functions of lower genera is possible, find an explicit form of these solutions in terms of the hyperelliptic functions of lower genera.

The reduction of the Riemann period matrices is studied by Weierstrass and Poincaré. This is summarized in [3, 4]. The reduction of the Riemann theta functions is studied

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in [3, 4, 30]. The reduction of hyperelliptic integrals to elliptic integrals is studied in many papers (e.g., [3, 4, 5, 10, 11]). In particular, O. Bolza derived many examples of the reduction of hyperelliptic integrals to elliptic integrals (cf. [10, 11]). This problem is closely related to that of a morphism from a hyperelliptic curve to an elliptic curve. In this paper, we consider the reduction of hyperelliptic functions. A curve is called *bielliptic* if it admits a morphism of degree 2 to an elliptic curve. In [19], the hyperelliptic functions associated with a bielliptic hyperelliptic curve of genus 2 are expressed in terms of the Jacobi elliptic functions. In [2], the hyperelliptic functions associated with a bielliptic hyperelliptic curve of genus 2 are expressed in terms of the Weierstrass elliptic functions. The purpose of this paper is to express the hyperelliptic functions associated with a bielliptic hyperelliptic curve of genus 3 in terms of the Weierstrass elliptic functions and hyperelliptic functions of genus 2.

For a hyperelliptic curve \mathcal{X} of genus g , let $\sigma_{\mathcal{X}}(u)$ with $u = {}^t(u_1, u_3, \dots, u_{2g-1}) \in \mathbb{C}^g$ be the sigma function associated with \mathcal{X} , which is a holomorphic function on \mathbb{C}^g (cf. [14, 15, 16]). Let $\wp_{i,j}^{\mathcal{X}} = -\partial_{u_j} \partial_{u_i} \log \sigma_{\mathcal{X}}$ and $\wp_{i,j,k}^{\mathcal{X}} = \partial_{u_k} \wp_{i,j}^{\mathcal{X}}$, where $\partial_{u_l} = \partial / \partial u_l$. Let $\text{Jac}(\mathcal{X})$ be the Jacobian variety of \mathcal{X} . The functions $\wp_{i,j}^{\mathcal{X}}$ and $\wp_{i,j,k}^{\mathcal{X}}$ are meromorphic functions on $\text{Jac}(\mathcal{X})$. Any meromorphic function on $\text{Jac}(\mathcal{X})$ has a rational representation in terms of $\wp_{1,i}^{\mathcal{X}}$ and $\wp_{1,1,i}^{\mathcal{X}}$ with $i = 1, 3, \dots, 2g-1$. It is well known that the function $\wp_{1,1}^{\mathcal{X}}$ gives a solution of the KdV hierarchy (cf. [14, 15, 16]). For $\alpha, \beta, \gamma \in \mathbb{C}$ satisfying $\beta\gamma \neq 0$, we consider the polynomial

$$f(x) = x(x-1)(x-\alpha^2)(x-\beta^2)(x-\gamma^2)(x-\alpha^2/\beta^2)(x-\alpha^2/\gamma^2).$$

We assume that $f(x)$ has no multiple roots. We consider the non-singular hyperelliptic curve of genus 3

$$V = \left\{ (x, y) \in \mathbb{C}^2 \mid y^2 = f(x) \right\}.$$

A hyperelliptic curve of genus 3 is bielliptic if and only if the curve is isomorphic to a curve in the form of V (see Section 3). Let

$$k_1 = \frac{\sqrt{-1}\beta}{\sqrt{(1-\beta^2)(\beta^2-\alpha^2)}}, \quad k_2 = \frac{4\sqrt{-1}\alpha\beta\gamma}{(1-\alpha)(\beta^2-\alpha)(\gamma^2-\alpha)}.$$

We consider the elliptic curve E and the hyperelliptic curve C of genus 2 defined by

$$E = \left\{ (X, Y) \in \mathbb{C}^2 \mid Y^2 = X(X-1)\left(X - k_1^2(1-1/\gamma^2)(\gamma^2-\alpha^2)\right) \right\},$$

$$C = \left\{ (X, Y) \in \mathbb{C}^2 \mid Y^2 = X(X-1)(X-a^2)(X-b^2)(X-c^2) \right\},$$

where

$$a = \frac{1+\alpha}{1-\alpha}, \quad b = \frac{\beta^2+\alpha}{\beta^2-\alpha}, \quad c = \frac{\gamma^2+\alpha}{\gamma^2-\alpha}.$$

Then we can define the morphisms of degree 2 (see Section 3)

$$\begin{aligned}\varphi_1: V &\rightarrow E, & (x, y) &\mapsto (X, Y) = \left(\frac{k_1^2(x-1)(x-\alpha^2)}{x}, \frac{k_1^3 y}{x^2} \right), \\ \varphi_2: V &\rightarrow C, & (x, y) &\mapsto (X, Y) = \left(\left(\frac{x+\alpha}{x-\alpha} \right)^2, \frac{4\alpha k_2(x+\alpha)y}{(x-\alpha)^5} \right).\end{aligned}$$

The maps φ_1 and φ_2 induce the homomorphisms of the Jacobian varieties

$$\varphi_1^*: \text{Jac}(E) \rightarrow \text{Jac}(V), \quad \varphi_2^*: \text{Jac}(C) \rightarrow \text{Jac}(V).$$

In this paper, we describe the maps φ_1^* and φ_2^* explicitly in Lemma 4.4. For simplicity, we denote $\wp_{1,1}^E$ by \wp . Let $\wp' = \frac{d}{du}\wp$. For $u \in \text{Jac}(E)$, the functions $\wp_{i,j}^V(\varphi_1^*(u))$ and $\wp_{i,j,k}^V(\varphi_1^*(u))$ are meromorphic functions on $\text{Jac}(E)$. In this paper, for $i = 1, 3, 5$, we describe the functions $\wp_{1,i}^V(\varphi_1^*(u))$ and $\wp_{1,1,i}^V(\varphi_1^*(u))$ in terms of $\wp(u)$ and $\wp'(u)$ explicitly in Theorem 5.1, i.e., we show that the restrictions of the hyperelliptic functions $\wp_{1,i}^V$ and $\wp_{1,1,i}^V$ to the appropriate subspace of dimension 1 in \mathbb{C}^3 are elliptic functions and describe them in terms of the Weierstrass elliptic functions associated with E . For $u \in \text{Jac}(C)$, the functions $\wp_{i,j}^V(\varphi_2^*(u))$ and $\wp_{i,j,k}^V(\varphi_2^*(u))$ are meromorphic functions on $\text{Jac}(C)$. In [7], the values of the hyperelliptic functions of genus g on the image of $g+1$ points with respect to the Abel-Jacobi map are expressed in terms of rational functions of the coordinates of these points. By using this formula, for $i = 1, 3, 5$, we describe the functions $\wp_{1,i}^V(\varphi_2^*(u))$ and $\wp_{1,1,i}^V(\varphi_2^*(u))$ in terms of $\wp_{1,j}^C(u)$ and $\wp_{1,1,j}^C(u)$ with $j = 1, 3$ explicitly in Theorem 5.2, i.e., we show that the restrictions of the hyperelliptic functions $\wp_{1,i}^V$ and $\wp_{1,1,i}^V$ to the appropriate subspace of dimension 2 in \mathbb{C}^3 are hyperelliptic functions of genus 2 and describe them in terms of the hyperelliptic functions associated with C . In [17], the addition formula for the hyperelliptic functions of genus g are given. By using this addition formula, for $i = 1, 3, 5$, we show that it is possible to express $\wp_{1,i}^V$ and $\wp_{1,1,i}^V$ on \mathbb{C}^3 in terms of the elliptic functions associated with E and the hyperelliptic functions of genus 2 associated with C explicitly in Corollary 5.4.

The reductions of the Riemann theta functions and the hyperelliptic functions are important in real physical problems (e.g., [1, 3, 4, 5, 18, 19, 21, 30, 31]). In [26], a graphical representation of the hyperelliptic functions of genus 2 is given. In [6], numerical computations for the hyperelliptic functions are studied. By using the reduction formulae for hyperelliptic functions, it is easier to compute the values of the hyperelliptic functions of higher genera. I believe that our results of this paper will contribute to numerical computations and graphical representations of the hyperelliptic functions.

The present paper is organized as follows. In Section 2, we review the definition and properties of the hyperelliptic sigma functions. In Section 3, we give a normal form of a bielliptic hyperelliptic curve of genus 3 and maps from the bielliptic hyperelliptic curve of genus 3 to an elliptic curve and a hyperelliptic curve of genus 2 explicitly. In Section 4, we describe the maps between the Jacobian varieties induced by the maps from the bielliptic hyperelliptic curve of genus 3 to the elliptic curve and the hyperelliptic curve of genus 2 explicitly. In Section 5, we express the hyperelliptic functions associated

with the bielliptic hyperelliptic curve of genus 3 in terms of the elliptic functions and the hyperelliptic functions of genus 2.

2 The hyperelliptic sigma functions

In this section, we review the definition of the hyperelliptic sigma functions and give facts about them which will be used later on. For details of the hyperelliptic sigma functions, see [14, 15, 16].

We set

$$N(x) = x^{2g+1} + \lambda_2 x^{2g} + \lambda_4 x^{2g-1} + \cdots + \lambda_{4g} x + \lambda_{4g+2}, \quad \lambda_i \in \mathbb{C}.$$

We assume that $N(x)$ has no multiple roots. We consider the non-singular hyperelliptic curve of genus g

$$\mathcal{X} = \left\{ (x, y) \in \mathbb{C}^2 \mid y^2 = N(x) \right\}.$$

A basis of the vector space consisting of holomorphic 1-forms on \mathcal{X} is given by

$$\omega_{2i-1}^{\mathcal{X}} = -\frac{x^{g-i}}{2y} dx, \quad 1 \leq i \leq g.$$

We set $\omega_{\mathcal{X}} = {}^t(\omega_1^{\mathcal{X}}, \omega_3^{\mathcal{X}}, \dots, \omega_{2g-1}^{\mathcal{X}})$. We consider the following meromorphic 1-forms on \mathcal{X} :

$$\eta_{2i-1}^{\mathcal{X}} = -\frac{1}{2y} \sum_{k=g-i+1}^{g+i-1} (k+i-g) \lambda_{2g+2i-2k-2} x^k dx, \quad 1 \leq i \leq g, \quad (2.1)$$

which are holomorphic at any point except ∞ . In (2.1), we set $\lambda_0 = 1$. For example, for $g = 1$ we have

$$\eta_1^{\mathcal{X}} = -\frac{x}{2y} dx,$$

for $g = 2$ we have

$$\eta_1^{\mathcal{X}} = -\frac{x^2}{2y} dx, \quad \eta_3^{\mathcal{X}} = -\frac{\lambda_4 x + 2\lambda_2 x^2 + 3x^3}{2y} dx,$$

and for $g = 3$ we have

$$\begin{aligned} \eta_1^{\mathcal{X}} &= -\frac{x^3}{2y} dx, & \eta_3^{\mathcal{X}} &= -\frac{\lambda_4 x^2 + 2\lambda_2 x^3 + 3x^4}{2y} dx, \\ \eta_5^{\mathcal{X}} &= -\frac{\lambda_8 x + 2\lambda_6 x^2 + 3\lambda_4 x^3 + 4\lambda_2 x^4 + 5x^5}{2y} dx. \end{aligned}$$

Let $\{\mathbf{a}_i, \mathbf{b}_i\}_{i=1}^g$ be a canonical basis in the one-dimensional homology group of the curve \mathcal{X} . We define the period matrices by

$$\begin{aligned} 2\omega' &= \left(\int_{\mathbf{a}_j} \omega_{2i-1}^{\mathcal{X}} \right), & 2\omega'' &= \left(\int_{\mathbf{b}_j} \omega_{2i-1}^{\mathcal{X}} \right), \\ -2\eta' &= \left(\int_{\mathbf{a}_j} \eta_{2i-1}^{\mathcal{X}} \right), & -2\eta'' &= \left(\int_{\mathbf{b}_j} \eta_{2i-1}^{\mathcal{X}} \right). \end{aligned}$$

We define the period lattice $\Lambda = \{2\omega' m_1 + 2\omega'' m_2 \mid m_1, m_2 \in \mathbb{Z}^g\}$ and consider the Jacobian variety $\text{Jac}(\mathcal{X}) = \mathbb{C}^g / \Lambda$. Let $\pi_{\mathcal{X}}: \mathbb{C}^g \rightarrow \text{Jac}(\mathcal{X})$ be the natural projection. The normalized period matrix is given by $\tau = (\omega')^{-1} \omega''$. Let $\tau \delta' + \delta''$ with $\delta', \delta'' \in \mathbb{R}^g$ be the Riemann constant with respect to $(\{\mathbf{a}_i, \mathbf{b}_i\}_{i=1}^g, \infty)$. We denote the imaginary unit by \mathbf{i} . The sigma function $\sigma_{\mathcal{X}}(u)$ associated with the curve \mathcal{X} , $u = {}^t(u_1, u_3, \dots, u_{2g-1}) \in \mathbb{C}^g$, is defined by

$$\sigma_{\mathcal{X}}(u) = \varepsilon \exp\left(\frac{1}{2} {}^t u \eta' (\omega')^{-1} u\right) \theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} ((2\omega')^{-1} u, \tau),$$

where $\theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} (u, \tau)$ is the Riemann theta function with the characteristics $\begin{bmatrix} \delta' \\ \delta'' \end{bmatrix}$ defined by

$$\theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} (u, \tau) = \sum_{n \in \mathbb{Z}^g} \exp\{\pi \mathbf{i} {}^t(n + \delta') \tau (n + \delta') + 2\pi \mathbf{i} {}^t(n + \delta')(u + \delta'')\}$$

and ε is a non-zero constant.

Proposition 2.1 ([14, pp. 7–8]). *For $m_1, m_2 \in \mathbb{Z}^g$, let $\Omega = 2\omega' m_1 + 2\omega'' m_2$. Then, for $u \in \mathbb{C}^g$, we have*

$$\begin{aligned} & \sigma_{\mathcal{X}}(u + \Omega) / \sigma_{\mathcal{X}}(u) \\ &= (-1)^{2({}^t \delta' m_1 - {}^t \delta'' m_2) + {}^t m_1 m_2} \exp\{{}^t(2\eta' m_1 + 2\eta'' m_2)(u + \omega' m_1 + \omega'' m_2)\}. \end{aligned}$$

Let $\wp_{i,j}^{\mathcal{X}} = -\partial_{u_j} \partial_{u_i} \log \sigma_{\mathcal{X}}$ and $\wp_{i,j,k}^{\mathcal{X}} = \partial_{u_k} \wp_{i,j}^{\mathcal{X}}$, where $\partial_{u_l} = \partial / \partial u_l$.

Remark 2.2. The functions $\wp_{i,j}^{\mathcal{X}}$ and $\wp_{i,j,k}^{\mathcal{X}}$ are meromorphic functions on $\text{Jac}(\mathcal{X})$. Any meromorphic function on $\text{Jac}(\mathcal{X})$ has a rational representation in terms of $\wp_{1,i}^{\mathcal{X}}$ and $\wp_{1,1,i}^{\mathcal{X}}$ with $i = 1, 3, \dots, 2g - 1$.

Remark 2.3. In [14, Theorem 4.12], it is proved that the function $2\wp_{1,1}^{\mathcal{X}} + 2\lambda_2/3$ satisfies the KdV hierarchy.

3 Bielliptic hyperelliptic curves of genus 3

A curve is called *bielliptic* if it admits a morphism of degree 2 to an elliptic curve. For $a, b, c \in \mathbb{C}$, we consider the polynomial

$$F(s) = (s^2 - 1)(s^2 - a^2)(s^2 - b^2)(s^2 - c^2).$$

We assume that $F(s)$ has no multiple roots. We consider the non-singular hyperelliptic curve of genus 3

$$H = \left\{ (s, t) \in \mathbb{C}^2 \mid t^2 = F(s) \right\}.$$

It is known that a hyperelliptic curve of genus 3 is bielliptic if and only if the curve is isomorphic to a curve in the form of H (cf. [22, Lemma 2.2] and [24, Section 3]). We

consider the elliptic curve W and the hyperelliptic curve C of genus 2 defined by

$$\begin{aligned} W &= \left\{ (S, T) \in \mathbb{C}^2 \mid T^2 = (S-1)(S-a^2)(S-b^2)(S-c^2) \right\}, \\ C &= \left\{ (X, Y) \in \mathbb{C}^2 \mid Y^2 = X(X-1)(X-a^2)(X-b^2)(X-c^2) \right\}. \end{aligned}$$

Then we have the morphisms of degree 2

$$\begin{aligned} \phi_1: H &\rightarrow W, & (s, t) &\mapsto (S, T) = (s^2, t), \\ \phi_2: H &\rightarrow C, & (s, t) &\mapsto (X, Y) = (s^2, st) \end{aligned}$$

(cf. [22, Lemma 2.2] and [24, Section 3]). It is known that the Jacobian variety of H is isogenous to the direct product of the Jacobian varieties of W and C (cf. [28, Theorem 5] and [24, Section 3]). For $\alpha, \beta, \gamma \in \mathbb{C}$ satisfying $\beta\gamma \neq 0$, we consider the polynomial

$$f(x) = x(x-1)(x-\alpha^2)(x-\beta^2)(x-\gamma^2)(x-\alpha^2/\beta^2)(x-\alpha^2/\gamma^2). \quad (3.1)$$

We assume that $f(x)$ has no multiple roots. We consider the non-singular hyperelliptic curve of genus 3

$$V = \left\{ (x, y) \in \mathbb{C}^2 \mid y^2 = f(x) \right\}.$$

Let

$$k_1 = \frac{\sqrt{-1}\beta}{\sqrt{(1-\beta^2)(\beta^2-\alpha^2)}}, \quad k_2 = \frac{4\sqrt{-1}\alpha\beta\gamma}{(1-\alpha)(\beta^2-\alpha)(\gamma^2-\alpha)}.$$

Proposition 3.1. *Given $a, b, c \in \mathbb{C}$ such that $F(s)$ has no multiple roots, the curve H is isomorphic to the curve V with*

$$\alpha = \frac{a-1}{a+1}, \quad \beta = \sqrt{\frac{(a-1)(b+1)}{(a+1)(b-1)}}, \quad \gamma = \sqrt{\frac{(a-1)(c+1)}{(a+1)(c-1)}}$$

by the morphism

$$\zeta: H \rightarrow V, \quad (s, t) \mapsto (x, y)$$

with

$$(x, y) = \left(\frac{(a-1)(s+1)}{(a+1)(s-1)}, \frac{8(a-1)^3 t}{(a+1)^4 \sqrt{(1-b^2)(c^2-1)}(s-1)^4} \right).$$

Conversely, given $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\beta\gamma \neq 0$ and $f(x)$ has no multiple roots, the curve V is isomorphic to the curve H with

$$a = \frac{1+\alpha}{1-\alpha}, \quad b = \frac{\beta^2+\alpha}{\beta^2-\alpha}, \quad c = \frac{\gamma^2+\alpha}{\gamma^2-\alpha} \quad (3.2)$$

by the morphism

$$\tilde{\zeta}: V \rightarrow H, \quad (x, y) \mapsto (s, t) = \left(\frac{x+\alpha}{x-\alpha}, \frac{4\alpha k_2 y}{(x-\alpha)^4} \right).$$

Proof. By the direct calculations, we obtain the statement of the proposition. \square

The elliptic curve W with (3.2) is isomorphic to the elliptic curve in Legendre form

$$E = \left\{ (X, Y) \in \mathbb{C}^2 \mid Y^2 = X(X-1) \left(X - k_1^2(1 - 1/\gamma^2)(\gamma^2 - \alpha^2) \right) \right\}$$

by the morphism

$$\xi: W \rightarrow E, \quad (S, T) \mapsto (X, Y)$$

with

$$(X, Y) = \left(k_1^2 \frac{(\alpha + 1)^2 - (\alpha - 1)^2 S}{S - 1}, \frac{4\alpha k_1^3 T}{k_2(S - 1)^2} \right).$$

We consider the hyperelliptic curve C of genus 2 with (3.2) and the maps

$$\varphi_1 = \xi \circ \phi_1 \circ \tilde{\zeta}: V \rightarrow E, \quad \varphi_2 = \phi_2 \circ \tilde{\zeta}: V \rightarrow C.$$

Then the maps φ_i with $i = 1, 2$ are described by

$$\varphi_1: V \rightarrow E, \quad (x, y) \mapsto (X, Y) = \left(\frac{k_1^2(x-1)(x-\alpha^2)}{x}, \frac{k_1^3 y}{x^2} \right), \quad (3.3)$$

$$\varphi_2: V \rightarrow C, \quad (x, y) \mapsto (X, Y) = \left(\left(\frac{x+\alpha}{x-\alpha} \right)^2, \frac{4\alpha k_2(x+\alpha)y}{(x-\alpha)^5} \right). \quad (3.4)$$

4 The maps between the Jacobian varieties

Let $M_{m,n}(\mathbb{C})$ be the set of the $m \times n$ matrices such that all the components are complex numbers. We have $\varphi_1(\infty) = \infty$. We consider the maps

$$\begin{aligned} A: V &\rightarrow \text{Jac}(V), & P &\mapsto \int_{\infty}^P \omega_V, \\ A_1: E &\rightarrow \text{Jac}(E), & P &\mapsto \int_{\infty}^P \omega_E. \end{aligned}$$

It is known that we have the unique homomorphism

$$\varphi_{1,*}: \text{Jac}(V) \rightarrow \text{Jac}(E)$$

such that $A_1 \circ \varphi_1 = \varphi_{1,*} \circ A$ (cf. [27, p. 104, Proposition 6.1]).

$$\begin{array}{ccc} \text{Jac}(V) & \xrightarrow{\varphi_{1,*}} & \text{Jac}(E) \\ A \uparrow & & \uparrow A_1 \\ V & \xrightarrow{\varphi_1} & E. \end{array} \quad (4.1)$$

It is known that we have the unique linear map

$$\tilde{\varphi}_{1,*}: \mathbb{C}^3 \rightarrow \mathbb{C}, \quad u \mapsto K_1 u, \quad K_1 \in M_{1,3}(\mathbb{C})$$

such that $\varphi_{1,*} \circ \pi_V = \pi_E \circ \tilde{\varphi}_{1,*}$ (e.g., [8, Proposition 1.2.1]).

$$\begin{array}{ccc} \mathbb{C}^3 & \xrightarrow{\tilde{\varphi}_{1,*}} & \mathbb{C} \\ \pi_V \downarrow & & \pi_E \downarrow \\ \text{Jac}(V) & \xrightarrow{\varphi_{1,*}} & \text{Jac}(E). \end{array}$$

Lemma 4.1. *We have $K_1 = \frac{1}{k_1}(1, 0, -\alpha^2)$.*

Proof. Let $K_1 = (a_1, a_3, a_5)$. We consider the commutative diagram (4.1). The pullback of the holomorphic 1-form du_1 on $\text{Jac}(E)$ with respect to $\varphi_{1,*}$ is $a_1 du_1 + a_3 du_3 + a_5 du_5$. The pullback of $a_1 du_1 + a_3 du_3 + a_5 du_5$ with respect to A is $a_1 \omega_1^V + a_3 \omega_3^V + a_5 \omega_5^V$. On the other hand, the pullback of du_1 with respect to A_1 is ω_E . The pullback of ω_E with respect to φ_1 is $\frac{1}{k_1} \omega_1^V - \frac{\alpha^2}{k_1} \omega_5^V$. Since $\omega_1^V, \omega_3^V, \omega_5^V$ are linearly independent, we obtain the statement of the lemma. \square

We consider the map

$$A_2: C \rightarrow \text{Jac}(C), \quad P \mapsto \int_{(1,0)}^P \omega_C.$$

It is known that we have the unique homomorphism

$$\varphi_{2,*}: \text{Jac}(V) \rightarrow \text{Jac}(C)$$

such that $A_2 \circ \varphi_2 = \varphi_{2,*} \circ A$ (cf. [27, p. 104, Proposition 6.1]).

$$\begin{array}{ccc} \text{Jac}(V) & \xrightarrow{\varphi_{2,*}} & \text{Jac}(C) \\ A \uparrow & & A_2 \uparrow \\ V & \xrightarrow{\varphi_2} & C. \end{array} \tag{4.2}$$

It is known that we have the unique linear map

$$\tilde{\varphi}_{2,*}: \mathbb{C}^3 \rightarrow \mathbb{C}^2, \quad u \mapsto K_2 u, \quad K_2 \in M_{2,3}(\mathbb{C})$$

such that $\varphi_{2,*} \circ \pi_V = \pi_C \circ \tilde{\varphi}_{2,*}$ (e.g., [8, Proposition 1.2.1]).

$$\begin{array}{ccc} \mathbb{C}^3 & \xrightarrow{\tilde{\varphi}_{2,*}} & \mathbb{C}^2 \\ \pi_V \downarrow & & \pi_C \downarrow \\ \text{Jac}(V) & \xrightarrow{\varphi_{2,*}} & \text{Jac}(C). \end{array}$$

Lemma 4.2. *We have $K_2 = -\frac{1}{k_2} \begin{pmatrix} 1 & 2\alpha & \alpha^2 \\ 1 & -2\alpha & \alpha^2 \end{pmatrix}$.*

Proof. Let $K_2 = \begin{pmatrix} b_1 & b_3 & b_5 \\ c_1 & c_3 & c_5 \end{pmatrix}$. We consider the commutative diagram (4.2). The pullbacks of the holomorphic 1-forms du_1 and du_3 on $\text{Jac}(C)$ with respect to $\varphi_{2,*}$ are $b_1 du_1 + b_3 du_3 + b_5 du_5$ and $c_1 du_1 + c_3 du_3 + c_5 du_5$, respectively. The pullbacks of $b_1 du_1 + b_3 du_3 + b_5 du_5$ and $c_1 du_1 + c_3 du_3 + c_5 du_5$ with respect to A are $b_1 \omega_1^V + b_3 \omega_3^V + b_5 \omega_5^V$ and $c_1 \omega_1^V + c_3 \omega_3^V + c_5 \omega_5^V$, respectively. On the other hand, the pullbacks of du_1 and du_3 with respect to A_2 are ω_1^C and ω_3^C , respectively. The pullbacks of ω_1^C and ω_3^C with respect to φ_2 are $-\frac{1}{k_2} \omega_1^V - \frac{2\alpha}{k_2} \omega_3^V - \frac{\alpha^2}{k_2} \omega_5^V$ and $-\frac{1}{k_2} \omega_1^V + \frac{2\alpha}{k_2} \omega_3^V - \frac{\alpha^2}{k_2} \omega_5^V$, respectively. Since $\omega_1^V, \omega_3^V, \omega_5^V$ are linearly independent, we obtain the statement of the lemma. \square

We have $\varphi_1^{-1}(\infty) = \{\infty, (0, 0)\}$ and $\varphi_2^{-1}((1, 0)) = \{\infty, (0, 0)\}$. It is well known that the maps φ_i with $i = 1, 2$ induce the homomorphisms between the Jacobian varieties

$$\varphi_1^*: \text{Jac}(E) \rightarrow \text{Jac}(V), \quad \int_{\infty}^P \omega_E \mapsto \int_{\infty}^{P_1} \omega_V + \int_{(0,0)}^{P_2} \omega_V,$$

where $P \in E$ and $\varphi_1^{-1}(P) = \{P_1, P_2\}$,

$$\varphi_2^*: \text{Jac}(C) \rightarrow \text{Jac}(V), \quad \int_{(1,0)}^Q \omega_C \mapsto \int_{\infty}^{Q_1} \omega_V + \int_{(0,0)}^{Q_2} \omega_V,$$

where $Q \in C$ and $\varphi_2^{-1}(Q) = \{Q_1, Q_2\}$. It is known that we have the unique linear maps

$$\begin{aligned} \tilde{\varphi}_1^*: \mathbb{C} &\rightarrow \mathbb{C}^3, & u &\mapsto L_1 u, & L_1 &\in M_{3,1}(\mathbb{C}), \\ \tilde{\varphi}_2^*: \mathbb{C}^2 &\rightarrow \mathbb{C}^3, & u &\mapsto L_2 u, & L_2 &\in M_{3,2}(\mathbb{C}) \end{aligned}$$

such that $\varphi_1^* \circ \pi_E = \pi_V \circ \tilde{\varphi}_1^*$ and $\varphi_2^* \circ \pi_C = \pi_V \circ \tilde{\varphi}_2^*$ (e.g., [8, Proposition 1.2.1]).

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{\varphi}_1^*} & \mathbb{C}^3 \\ \pi_E \downarrow & & \downarrow \pi_V \\ \text{Jac}(E) & \xrightarrow{\varphi_1^*} & \text{Jac}(V) \end{array} \quad \begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\tilde{\varphi}_2^*} & \mathbb{C}^3 \\ \pi_C \downarrow & & \downarrow \pi_V \\ \text{Jac}(C) & \xrightarrow{\varphi_2^*} & \text{Jac}(V). \end{array}$$

Let

$$\begin{aligned} \varphi_*: \text{Jac}(V) &\rightarrow \text{Jac}(E) \times \text{Jac}(C), & u &\mapsto (\varphi_{1,*}(u), \varphi_{2,*}(u)), \\ \varphi^*: \text{Jac}(E) \times \text{Jac}(C) &\rightarrow \text{Jac}(V), & (u, v) &\mapsto \varphi_1^*(u) + \varphi_2^*(v). \end{aligned}$$

Lemma 4.3 ([24, Section 3]). *For $u \in \text{Jac}(V)$, we have*

$$\varphi^* \circ \varphi_*(u) = 2u.$$

Proof. For the sake to be complete and self-contained, we give a proof of this lemma. For $P = (x, y) \in V \setminus \{\infty\}$, let $\varphi_1^{-1}(\varphi_1(P)) = \{P, Q\}$ and $\varphi_2^{-1}(\varphi_2(P)) = \{P, R\}$. If $P \neq (0, 0)$, then we have $Q = \left(\frac{\alpha^2}{x}, \frac{\alpha^4 y}{x^4}\right)$ and $R = \left(\frac{\alpha^2}{x}, -\frac{\alpha^4 y}{x^4}\right)$. If $P = (0, 0)$, then we have $Q = R = \infty$. We have

$$\begin{aligned} \varphi^* \circ \varphi_* \left(\int_{\infty}^P \omega_V \right) &= \varphi^* \left(\int_{\infty}^{\varphi_1(P)} \omega_E, \int_{(1,0)}^{\varphi_2(P)} \omega_C \right) \\ &= \int_{\infty}^P \omega_V + \int_{(0,0)}^Q \omega_V + \int_{\infty}^P \omega_V + \int_{(0,0)}^R \omega_V \\ &= 2 \int_{\infty}^P \omega_V + \int_{\infty}^Q \omega_V + \int_{\infty}^R \omega_V = 2 \int_{\infty}^P \omega_V. \end{aligned}$$

For $u \in \text{Jac}(V)$, there exist $Z_1, Z_2, Z_3 \in V$ such that

$$\sum_{i=1}^3 \int_{\infty}^{Z_i} \omega_V = u.$$

Since the map $\varphi^* \circ \varphi_*$ is a homomorphism, we obtain the statement of the lemma. \square

Lemma 4.4. *We have $L_1 = k_1 \begin{pmatrix} 1 & \\ 0 & \\ -\alpha^{-2} & \end{pmatrix}$ and $L_2 = -\frac{k_2}{2} \begin{pmatrix} 1 & 1 \\ \alpha^{-1} & -\alpha^{-1} \\ \alpha^{-2} & \alpha^{-2} \end{pmatrix}$.*

Proof. Let K and L be the 3×3 matrices defined by $K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ and $L = (L_1 \ L_2)$.

We consider the maps

$$\begin{aligned} \tilde{\varphi}_* &: \mathbb{C}^3 \rightarrow \mathbb{C}^3, & u &\mapsto Ku, \\ \tilde{\varphi}^* &: \mathbb{C}^3 \rightarrow \mathbb{C}^3, & u &\mapsto Lu, \\ \pi_{E,C} &: \mathbb{C}^3 \rightarrow \text{Jac}(E) \times \text{Jac}(C), & {}^t(u_1, u_3, u_5) &\mapsto \left(\pi_E(u_1), \pi_C({}^t(u_3, u_5)) \right). \end{aligned}$$

Then we have the following commutative diagrams:

$$\begin{array}{ccc} \mathbb{C}^3 & \xrightarrow{\tilde{\varphi}_*} & \mathbb{C}^3 \\ \pi_V \downarrow & & \pi_{E,C} \downarrow \\ \text{Jac}(V) & \xrightarrow{\varphi_*} & \text{Jac}(E) \times \text{Jac}(C), \end{array} \quad \begin{array}{ccc} \mathbb{C}^3 & \xrightarrow{\tilde{\varphi}^*} & \mathbb{C}^3 \\ \pi_{E,C} \downarrow & & \pi_V \downarrow \\ \text{Jac}(E) \times \text{Jac}(C) & \xrightarrow{\varphi^*} & \text{Jac}(V). \end{array}$$

Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^3 & \xrightarrow{\tilde{\varphi}^* \circ \tilde{\varphi}_*} & \mathbb{C}^3 \\ \pi_V \downarrow & & \pi_V \downarrow \\ \text{Jac}(V) & \xrightarrow{\varphi^* \circ \varphi_*} & \text{Jac}(V). \end{array}$$

We have

$$\tilde{\varphi}^* \circ \tilde{\varphi}_*: \mathbb{C}^3 \rightarrow \mathbb{C}^3, \quad u \mapsto LKu.$$

From Lemma 4.3, we obtain

$$LK = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

From Lemmas 4.1 and 4.2, we obtain the statement of the lemma. \square

5 Reduction of the hyperelliptic functions associated with V

For simplicity, we denote $\wp_{1,1}^E$ by \wp . Let $\wp' = \frac{d}{du}\wp$.

Theorem 5.1. *For $u \in \mathbb{C}$, the following relations hold:*

$$\begin{aligned} \wp_{1,1}^V(L_1u) &= \alpha^2 + 1 + \wp(u)/k_1^2, & \wp_{1,3}^V(L_1u) &= -\alpha^2, & \wp_{1,5}^V(L_1u) &= 0, \\ \wp_{1,1,1}^V(L_1u) &= \wp'(u)/k_1^3, & \wp_{1,1,3}^V(L_1u) &= 0, & \wp_{1,1,5}^V(L_1u) &= 0. \end{aligned}$$

Proof. Let U be the subset of E consisting of the elements $P \in E \setminus \{\infty\}$ such that $X \neq -k_1^2(\alpha \pm 1)^2$, where $P = (X, Y)$. Then U is an open set in E . Let $U' = A_1(U)$. Then U' is an open set in $\text{Jac}(E)$. We take a point $u \in U'$. Then there exists $P = (X, Y) \in U$ such that $u = A_1(P)$. We have

$$\varphi_1^*(u) = \int_{\infty}^{P_1} \omega_V + \int_{(0,0)}^{P_2} \omega_V = \int_{\infty}^{P_1} \omega_V + \int_{\infty}^{P_2} \omega_V + \int_{\infty}^{(0,0)} \omega_V,$$

where $\varphi_1^{-1}(P) = \{P_1, P_2\}$. From $P \neq \infty$, we have $P_1, P_2 \neq \infty, (0, 0)$. Let $P_i = (x_i, y_i)$ for $i = 1, 2$. From $P \in U$, we have $x_1 \neq x_2$. Therefore, from the well-known solution of the Jacobi inversion problem, we have

$$\begin{aligned} \wp_{1,1}^V(L_1u) &= \wp_{1,1}^V(\varphi_1^*(u)) = x_1 + x_2, & \wp_{1,3}^V(L_1u) &= \wp_{1,3}^V(\varphi_1^*(u)) = -x_1x_2, \\ \wp_{1,5}^V(L_1u) &= \wp_{1,5}^V(\varphi_1^*(u)) = 0, \\ \begin{pmatrix} 2y_1 \\ 2y_2 \\ 0 \end{pmatrix} &= - \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \wp_{1,1,1}^V(L_1u) \\ \wp_{1,1,3}^V(L_1u) \\ \wp_{1,1,5}^V(L_1u) \end{pmatrix}. \end{aligned}$$

From (3.3), $X = \wp(u)$, and $Y = -\wp'(u)/2$, the relations in Theorem 5.1 hold for $u \in \pi_E^{-1}(U')$. Since $\pi_E^{-1}(U')$ is an open set in \mathbb{C} , the relations in Theorem 5.1 hold on \mathbb{C} . \square

For $0 \leq i \leq 6$, we define μ_{2i} by

$$f(x) = \mu_0 x^7 + \mu_2 x^6 + \mu_4 x^5 + \mu_6 x^4 + \mu_8 x^3 + \mu_{10} x^2 + \mu_{12} x,$$

where $f(x)$ is defined by (3.1).

Theorem 5.2. Let $u \in \mathbb{C}^2$. For simplicity, we set $p_2 = \wp_{1,1}^C(u)$, $p_3 = \wp_{1,1,1}^C(u)$, $p_4 = \wp_{1,3}^C(u)$, and $p_5 = \wp_{1,1,3}^C(u)$. The following relations hold:

$$\begin{aligned}
\wp_{1,1}^V(L_2u) &= \frac{D_1^2}{16k_2^2p_4^2(1-p_2-p_4)^2} - \mu_2 - h_1, \\
\wp_{1,3}^V(L_2u) &= \frac{\alpha D_1 D_2}{8k_2^2p_4^2(1-p_2-p_4)^3} - \mu_4 - \mu_2 h_1 - h_2, \\
\wp_{1,5}^V(L_2u) &= \frac{\alpha^2(D_3p_3^2p_4^2 - 2D_4p_3p_4p_5 + D_5p_5^2)}{16k_2^2p_4^2(1-p_2-p_4)^4} - \mu_6 - \mu_4 h_1 - \mu_2 h_2 - h_3, \\
\wp_{1,1,1}^V(L_2u) &= \frac{\alpha(p_2 - 3p_4 + 3)(p_2p_5 - p_5 - p_3p_4)}{2k_2p_4(p_2 + p_4 - 1)} + \wp_{1,1}^V(L_2u)D_6, \\
\wp_{1,1,3}^V(L_2u) &= \frac{\alpha^2(3p_2p_4p_5 - p_4p_5 + p_2^2p_5 + 2p_2p_5 - 3p_5 - 3p_3p_4^2 - p_2p_3p_4 - p_3p_4)}{2k_2p_4(p_2 + p_4 - 1)} \\
&\quad + \wp_{1,3}^V(L_2u)D_6, \\
\wp_{1,1,5}^V(L_2u) &= \frac{\alpha^3(p_3p_4 - p_2p_5 + p_5)}{2k_2p_4} + \wp_{1,5}^V(L_2u)D_6,
\end{aligned}$$

where

$$\begin{aligned}
D_1 &= p_2p_4p_5 - 3p_4p_5 - p_2^2p_5 + 2p_2p_5 - p_5 - p_3p_4^2 + p_2p_3p_4 - 3p_3p_4, \\
D_2 &= p_2p_4^2p_5 + 3p_4^2p_5 + 4p_2^2p_4p_5 - 14p_2p_4p_5 + 14p_4p_5 - p_2^3p_5 + p_2^2p_5 \\
&\quad + p_2p_5 - p_5 - p_3p_4^3 - 4p_2p_3p_4^2 + 14p_3p_4^2 + p_2^2p_3p_4 + 3p_3p_4, \\
D_3 &= p_4^4 + 4(p_2 + 7)p_4^3 + 2(19p_2^2 - 118p_2 + 243)p_4^2 \\
&\quad - 4(7p_2^3 - 37p_2^2 + 77p_2 - 119)p_4 + p_2^4 - 4p_2^3 + 6p_2^2 + 28p_2 + 33, \\
D_4 &= (p_2 - 1)p_4^4 + 4(p_2^2 + 14p_2 - 47)p_4^3 + 2(19p_2^3 - 137p_2^2 + 329p_2 - 323)p_4^2 \\
&\quad - 4(7p_2^4 - 36p_2^3 + 74p_2^2 - 76p_2 + 47)p_4 + (p_2 - 1)^5, \\
D_5 &= (p_2^2 - 2p_2 + 33)p_4^4 + 4(p_2^3 + 21p_2^2 - 101p_2 + 119)p_4^3 \\
&\quad + 2(19p_2^4 - 156p_2^3 + 418p_2^2 - 492p_2 + 243)p_4^2 - 28(p_2 - 1)^5p_4 + (p_2 - 1)^6, \\
D_6 &= \frac{p_2p_4p_5 - 3p_4p_5 - p_2^2p_5 + 2p_2p_5 - p_5 - p_3p_4^2 + p_2p_3p_4 - 3p_3p_4}{2k_2p_4(p_2 + p_4 - 1)}, \\
h_1 &= \frac{4\alpha(p_4 + 1)}{p_2 + p_4 - 1}, \quad h_2 = \frac{2\alpha^2(5p_4^2 - 2p_2p_4 + 22p_4 + p_2^2 + 2p_2 + 5)}{(p_2 + p_4 - 1)^2}, \\
h_3 &= \frac{4\alpha^3(p_4 + 1)(5p_4^2 - 6p_2p_4 + 54p_4 + 5p_2^2 + 6p_2 + 5)}{(p_2 + p_4 - 1)^3}.
\end{aligned}$$

Proof. Let $\text{Sym}^2(C)$ be the 2-fold symmetric product of C . Then $\text{Sym}^2(C)$ is a complex manifold of dimension 2. We consider the holomorphic map

$$J: \text{Sym}^2(C) \rightarrow \text{Jac}(C), \quad (P_1, P_2) \mapsto \sum_{i=1}^2 \int_{(1,0)}^{P_i} \omega_C.$$

Let U be the subset of $\text{Sym}^2(C)$ consisting of the elements (P_1, P_2) such that $P_1, P_2 \in C \setminus \{\infty\}$, $X_1, X_2 \neq 1$, $X_1 \neq X_2$, and $X_1 X_2 \neq 0$, where $P_i = (X_i, Y_i)$ for $i = 1, 2$. Then U is an open set in $\text{Sym}^2(C)$. Let $U' = J(U)$. Then U' is an open set in $\text{Jac}(C)$. We take a point $u \in U'$. Then there exists $(P_1, P_2) \in U$ such that $u = J((P_1, P_2))$. We have

$$\begin{aligned}\varphi_2^*(u) &= \int_{\infty}^{S_1} \omega_V + \int_{(0,0)}^{S_2} \omega_V + \int_{\infty}^{S_3} \omega_V + \int_{(0,0)}^{S_4} \omega_V \\ &= \int_{\infty}^{S_1} \omega_V + \int_{\infty}^{S_2} \omega_V + \int_{\infty}^{S_3} \omega_V + \int_{\infty}^{S_4} \omega_V,\end{aligned}$$

where $\varphi_2^{-1}(P_1) = \{S_1, S_2\}$ and $\varphi_2^{-1}(P_2) = \{S_3, S_4\}$. From $X_1, X_2 \neq 1$, we have $S_i \neq \infty$ for $1 \leq i \leq 4$. Let $S_i = (x_i, y_i)$ for $1 \leq i \leq 4$. From $(P_1, P_2) \in U$, we have $x_i \neq x_j$ for any i, j satisfying $i \neq j$. From [7, Theorem 3.1], we have

$$\begin{aligned}\wp_{1,i}^V(L_2 u) &= \wp_{1,i}^V(\varphi_2^*(u)) = \mathcal{H}_{i+1}, & i = 1, 3, 5, \\ \wp_{1,1,i}^V(L_2 u) &= \wp_{1,1,i}^V(\varphi_2^*(u)) = 2\mathcal{L}_{i+2}, & i = 1, 3, 5,\end{aligned}$$

where $\mathcal{H}(x) = x^3 - \mathcal{H}_2 x^2 - \mathcal{H}_4 x - \mathcal{H}_6$ with

$$\mathcal{H}(x) = - \sum_{1 \leq k < l \leq 4} \frac{(y_k - y_l)^2}{(x_k - x_l)^2} \prod_{i=1, i \neq k, l}^4 \frac{x - x_i}{(x_k - x_i)(x_l - x_i)} + \sum_{i=0}^3 x^i \sum_{j=0}^{3-i} \mu_{6-2i-2j} \tilde{h}_j$$

and $\mathcal{L}(x) = \mathcal{L}_3 x^2 + \mathcal{L}_5 x + \mathcal{L}_7$ with

$$\mathcal{L}(x) = \sum_{i=1}^4 y_i \frac{\prod_{j=1, j \neq i}^4 (x - x_j) - \mathcal{H}(x)}{\prod_{j=1, j \neq i}^4 (x_i - x_j)}.$$

Here, \tilde{h}_j denotes the complete symmetric polynomial of degree j in $\{x_i\}_{i=1}^4$. From (3.4) and

$$X_1 + X_2 = \wp_{1,1}^C(u), \quad X_1 X_2 = -\wp_{1,3}^C(u), \quad 2Y_i = -\wp_{1,1,1}^C(u)X_i - \wp_{1,1,3}^C(u), \quad i = 1, 2,$$

the relations in Theorem 5.2 hold for $u \in \pi_C^{-1}(U')$. Since $\pi_C^{-1}(U')$ is an open set in \mathbb{C}^2 , the relations in Theorem 5.2 hold on \mathbb{C}^2 . \square

For $u \in \mathbb{C}^3$, let $\mathbf{c}_i(u)$ with $1 \leq i \leq 4$ be the 3-dimensional vectors defined by

$$\mathbf{c}_1(u) = \begin{pmatrix} \wp_{1,5}^V(u) \\ \wp_{1,3}^V(u) \\ \wp_{1,1}^V(u) \end{pmatrix}, \quad \mathbf{c}_2(u) = \begin{pmatrix} \wp_{1,1,5}^V(u)/2 \\ \wp_{1,1,3}^V(u)/2 \\ \wp_{1,1,1}^V(u)/2 \end{pmatrix}, \quad \mathbf{c}_{i+2}(u) = G(u)\mathbf{c}_i(u), \quad 1 \leq i \leq 2,$$

where $G(u) = \begin{pmatrix} 0 & 0 & \wp_{1,5}^V(u) \\ 1 & 0 & \wp_{1,3}^V(u) \\ 0 & 1 & \wp_{1,1}^V(u) \end{pmatrix}$. Let us define the 3×3 matrix $B(u) = (\mathbf{c}_1(u), \mathbf{c}_2(u), \mathbf{c}_3(u))$.

For a square matrix A , we denote the determinant of A by $|A|$. For $u, v \in \mathbb{C}^3$, let

$$R(x, y) = \begin{vmatrix} 1 & x & x^2 & x^3 & y & x^4 & xy \\ I_3 & & B(u) & & \mathbf{c}_4(u) & & \\ I_3 & & B(v) & & \mathbf{c}_4(v) & & \end{vmatrix} / |B(v) - B(u)|,$$

where I_3 is the identity matrix of size 3. There exists a polynomial $\Phi(w_1, w_2)$ in variables w_1 and w_2 such that $\Phi(x, y^2) = -R(x, y)R(x, -y)$. Let us represent $R(x, y)$ as a linear combination of monomials:

$$R(x, y) = (x + \mathfrak{d}_2)y + x^3(\mathfrak{d}_1x + \mathfrak{d}_3) + \mathfrak{d}_5x^2 + \mathfrak{d}_7x + \mathfrak{d}_9.$$

Let $\mathfrak{d} = {}^t(\mathfrak{d}_9, \mathfrak{d}_7, \mathfrak{d}_5)$.

Theorem 5.3 ([17, Corollary 3.2]). (i) For $u, v \in \mathbb{C}^3$, as a polynomial in x , we have the following relation

$$(x^3 - z\mathfrak{c}_1(u + v))(x^3 - z\mathfrak{c}_1(u))(x^3 - z\mathfrak{c}_1(v)) = \Phi(x, f(x)),$$

where $z = (1, x, x^2)$.

(ii) For $u, v \in \mathbb{C}^3$, we have

$$\mathfrak{c}_2(u + v) = (G(u + v) + \mathfrak{d}_2I_3)^{-1}(\mathfrak{d}_1\mathfrak{c}_3(u + v) + \mathfrak{d}_3\mathfrak{c}_1(u + v) + \mathfrak{d}).$$

We consider the 3×3 matrix L defined in the proof of Lemma 4.4.

Corollary 5.4. Let $u = {}^t(u_1, u_3, u_5) \in \mathbb{C}^3$. For $i = 1, 3, 5$, we can express $\wp_{1,i}^V(Lu)$ and $\wp_{1,1,i}^V(Lu)$ in terms of $\wp(u_1)$, $\wp'(u_1)$, $\wp_{1,j}^C(u_3, u_5)$, and $\wp_{1,1,j}^C(u_3, u_5)$ with $j = 1, 3$ explicitly.

Proof. We have $Lu = L_1u_1 + L_2{}^t(u_3, u_5)$. From Theorem 5.3, for $i = 1, 3, 5$, we can express $\wp_{1,i}^V(Lu)$ and $\wp_{1,1,i}^V(Lu)$ in terms of $\wp_{1,j}^V(L_1u_1)$, $\wp_{1,1,j}^V(L_1u_1)$, $\wp_{1,j}^V(L_2{}^t(u_3, u_5))$, and $\wp_{1,1,j}^V(L_2{}^t(u_3, u_5))$ with $j = 1, 3, 5$. From Theorem 5.1, for $j = 1, 3, 5$, we can express $\wp_{1,j}^V(L_1u_1)$ and $\wp_{1,1,j}^V(L_1u_1)$ in terms of $\wp(u_1)$ and $\wp'(u_1)$. From Theorem 5.2, for $j = 1, 3, 5$, we can express $\wp_{1,j}^V(L_2{}^t(u_3, u_5))$ and $\wp_{1,1,j}^V(L_2{}^t(u_3, u_5))$ in terms of $\wp_{1,k}^C(u_3, u_5)$ and $\wp_{1,1,k}^C(u_3, u_5)$ with $k = 1, 3$. Therefore, we obtain the statement of the corollary. \square

Remark 5.5. We assume that one of the following three conditions is satisfied:

$$a^2 = b^2c^2, \quad b^2 = a^2c^2, \quad c^2 = a^2b^2,$$

where a , b , and c are defined by (3.2). Then the functions $\wp_{1,j}^C$ and $\wp_{1,1,j}^C$ with $j = 1, 3$ are expressed in terms of elliptic functions (cf. [2, Theorem 5.9]). Therefore, in this case, we can express $\wp_{1,i}^V$ and $\wp_{1,1,i}^V$ with $i = 1, 3, 5$ in terms of elliptic functions.

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