Lecture notes by Professor Hiroshi Yamaguchi at Brown University in 1987-1988

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Chap. I Variations of domains in $\mathbb{C}$

§1 Domain of holomorphy

Let $\mathbb{C}^2$ be the complex two dimensional Euclidean space of variables $z = (x, y)$ with norm $\|z\|^2 = |x|^2 + |y|^2$.

We use the following notations:

$B(\zeta, r)$ = the ball of radius r and center of ζ in $\mathbb{C}^2$
$= \{ \|z - \zeta\| < r \}$

$V(\zeta, r)$ = the dicylinder of radius r and of center ζ
$= (|x - \xi| < r) \times (|y - \eta| < r)$
where $\zeta = (\xi, \eta)$

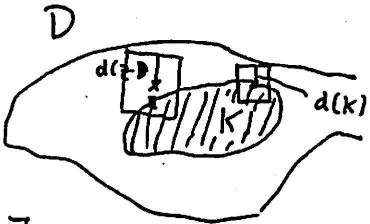
For $F \subset \mathbb{C}^2$, we denote by ∂F the boundary of F in $\mathbb{C}^2$.
If $F_1 \subset F_2 \subset \mathbb{C}^2$ and if F_1 is relatively compact in F_2, then we simply denote it by $F_1 \Subset F_2$.

"$f(x,y)$ is holomorphic at (a,b)" means that there exists a neighborhood of (a,b) in which $f(x,y)$ is holomorphic.

For any closed set K in $\mathbb{C}$, "$f(x,y)$ is holomorphic on K" means that there exists an open set $G \supset K$ in which f is holomorphic.

Let D be a domain of $\mathbb{C}^2$. For $z \in D$, we put

$$d(z) = \sup \{ r \mid V(z, r) \subset D \}$$



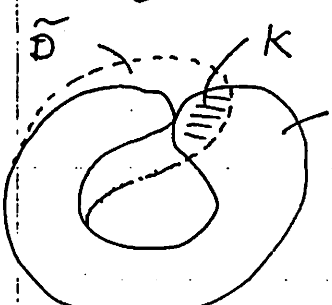
$d(z)$ is obviously continuous in D.

For given $K \Subset D$, we put

$$d(K) = \inf \{ d(z) \mid z \in K \}$$

It is clear that $d(K) > 0$, ~~and~~ $d(K) = d(K \cup \partial K)$ and ~~that~~ $V(a, d(K)) \subset D$ for all $a \in K$.

<u>Definition 1.1</u> The domain D is a domain of holomorphy in $\mathbb{C}^2$, if there exists a holomorphic function $f(z)$ in D such that f is not holomorphically extended to any unramified covering domain $\tilde{D}$ over $\mathbb{C}^2$ with $\tilde{D} \supsetneqq D$.



Say, for exemple, D and $\tilde{D}$ like those. Over the shaded part K, <u>we find</u> two parts K^+ and K^- of $\tilde{D}$. Even if we permit $f(z^+) \neq f(z^-)$ or $f(z^+) = f(z^-)$ for $z \in K$ (where $z^+ \in K^+$ and $z^- \in K^-$ are on z), we cannot find a holomorphic extension of f on $\tilde{D}$.

<u>Theorem 1.1</u> (Cartan-Thullen, 1932)

Let D be a domain of $\mathbb{C}^2$. Suppose that D is a domain of holomorphy in $\mathbb{C}^2$. Let $K \Subset D$. Then for any $\zeta \in K$ such that

$$d(\zeta) \leqq (1/3)\, d(K)$$

we find a holomorphic function $F(z)$ in D such that

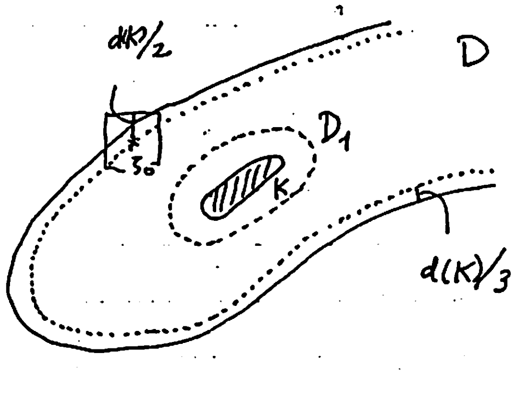
$$|F(\zeta)| \gneqq \sup_{z \in K} |F(z)|$$

<u>Proof.</u> First of all, since D is a domain of holomorphy we find a holomorphic function $f(z)$ in D which cannot be extended holomorphically at any boundary point of D.

Let us prove the theorem by contradiction.

Assume that there exists some point $\zeta_0 \in D$ with $d(\zeta_0) \leqq (1/3)\, d(K)$ such that <u>any</u> holomorphic function $F(z)$ in D satisfies

$$(1.1) \qquad |F(\zeta_0)| \leqq \sup_{z \in K} |F(z)|$$



We find at least a $\zeta_0^* \in \partial D$ such that $\zeta_0^* \in V(\zeta_0, d(K)/2)$.

We put $D_1 = \bigcup_{z \in K \cup \partial K} V(z, \frac{d(K)}{2})$. Since $d(K) = d(K \cup \partial K)$,

we have $K \Subset D_1 \Subset D$. Put

$$M = \sup_{z \in D_1} |f(z)| < +\infty .$$

Since $f(x,y)$ is holomorphic in $V(\zeta_0, d(\zeta_0)) \ (\subset D)$, we have

$$(1.2) \quad f(x,y) = \sum_{m,n=0}^{\infty} a_{mn} (x-\xi_0)^m (y-\eta_0)^n$$

in $V(\zeta_0, d(\zeta_0))$ where $\zeta_0 = (\xi_0, \eta_0)$ and $a_{mn} = \frac{1}{m!\,n!} \frac{\partial^{m+n} f}{\partial x^m \partial y^n}(\xi_0, \eta_0)$.

On the otherhand, $\partial^{m+n} f / \partial x^m \partial y^n$ is holomorphic in D, so the hypothesis of contradiction (1.1) leads us

$$(1.3) \quad \left| \frac{\partial^{m+n} f}{\partial x^m \partial y^n} (\xi_0, \eta_0) \right| \leqq \sup_{(x,y) \in K} \left| \frac{\partial^{m+n} f}{\partial x^m \partial y^n} (x,y) \right|$$

Take and fix $a \in K$. Since $V(a, d(K)/2) \subset D_1$,

$$|f(z)| \leqq M \quad \text{on } V(a, d(K)/2)$$

By Cauchy's formula, we obtain for $(x,y) \in V(a, d(K)/2)$

$$f(x,y) = \frac{-1}{4\pi^2} \int_{|\xi-a_1| = \frac{d(K)}{2}} \int_{|\eta-a_2| = \frac{d(K)}{2}} \frac{1}{\xi - x} \frac{1}{\eta - y} f(\xi, \eta)\, d\xi\, d\eta$$

where $a = (a_1, a_2)$. It follows that for any $m, n \geq 0$

$$\left| \frac{\partial^{m+n} f}{\partial x^m \partial y^n} (a) \right| \leqq m!\, n! \frac{M}{(d(K)/2)^{m+n}} \quad \text{— Cauchy's Inequality —}$$

The right hand side is independent of a in K, so that

$$\max_{a \in K} \left| \frac{\partial^{m+n} f}{\partial x^m \partial y^n} (a) \right| \leqq m!\, n! \frac{M}{(d(K)/2)^{m+n}}$$

By (1.3), we get

$$|a_{mn}| = \frac{1}{m!\, n!} \left| \frac{\partial^{m+n} f}{\partial x^m \partial y^n} (\xi_0, \eta_0) \right| \leqq \frac{M}{(d(K)/2)^{m+n}} \qquad (m, n = 0, 1, 2, \cdots)$$

Consequently, the power series of the right hand side of (1.2)

$$\sum_{m,n=0}^{\infty} a_{mn} \, (x-\zeta_0)^m (\eta-y_0)^n$$

uniformly converges in any compact set in $V(\zeta_0, d(K)/2)$, so that it represents a holomorphic function there. Since $\zeta_0^* \in (\partial D) \cap V(\zeta_0, d(K)/2$ it follows from (1.2) that $f(z)$ is holomorphically extended to a neighborhood of ζ_0^*. This contradicts that D is a domain of holomorphy. Hence Theorem 1.1 is valid.

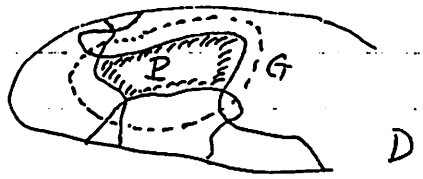
c.q.f.d.

<u>Definition 1.~~1~~ 2</u> Let D be a domain and $f_1, f_2, \dots, f_\nu$ $(1 \leq \nu < +\infty)$ be a holomorphic function in D. We put

$$P = \bigcap_{i=1}^{\nu} P_i \;;\quad P_i = \{z \in D \mid |f_i(z)| < 1\} \quad (i=1,\dots,\nu)$$

If there exists an open set $G \Subset D$ such that

$$P \cap G \Subset G,$$



then $P \cap G$ is called an analytic polyedral in D.

Therefore, if z_0 is ~~the~~ a boundary point of $P \cap G$, then $|f_j(z_0)| = 1$ for some j $(1 \leq j \leq \nu)$.

<u>Corollary 1.1</u> Let D be a domain of holomorphy. Then there exists a sequence of analytic polyedrals $P_k \cap G_k$ $(k=1,2,\dots)$ such that

$$P_1 \cap G_1 \Subset P_2 \cap G_2 \Subset \cdots \;:\; \bigcup_{k=1}^{\infty} P_k \cap G_k = D.$$

That is, a domain of holomorphy is exhausted from inside by analytic polyedrals.

<u>Proof</u>. It suffices to prove the following fact:
Let D be a domain of holomorphy. Let $K \Subset D$. Then we have an analytic polyedral $P \cap G$ such that

(1.4) $\quad K \Subset P \cap G \Subset G$.

In order to prove (1.4), take a sufficiently large $R > 0$ such that the ball (of radius R and of center 0): $B_R \Supset K$, and put

$$D_1^* = \{z \in D \mid d(z) > d(K)/3\} \cap B_R .$$

so that $\quad D \supset\supset D_1^* \supset\supset K.$

Take and fix $\zeta \in \partial D_1^*$. Since $d(z)$ is continuous in D, we have

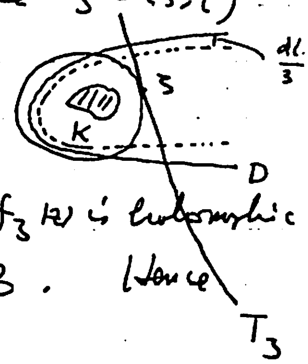
$$d(\zeta) = d(K)/3 \quad \text{or} \quad \zeta \in \partial B_R.$$

If $d(\zeta) = d(K)/3$, then by Theorem 1.1 we have a holomorphic function $f_\zeta(z)$ in D such that

$$|f_\zeta(\zeta)| \underset{*}{>} \sup_{z \in K} |f_\zeta(z)|.$$

If $\zeta \in \partial B_R$, draw a tangent plane T_ζ to ∂B_R at ζ from the outside. T_ζ can be written $\mathrm{Re}\,[\alpha(x-\xi)+\beta(y-\eta)] = 0$ where $\zeta = (\xi, \eta)$ and $^\exists \alpha, \beta \in \mathbb{C}$, and we have

$$K \subset\subset B_R \subset \{\underbrace{}_{(x,y) \in \mathbb{C}^2 \,|}\ \mathrm{Re}\,[\alpha(x-\xi)+\beta(y-\eta)] < 0\}.$$



Put $f_\zeta(z) = e^{\alpha(x-\xi)+\beta(y-\eta)}$. Then $f_\zeta(z)$ is holomorphic in $\mathbb{C}^2$ and $|f_\zeta(\zeta)| = 1$; $|f_\zeta(z)| < 1$ for any $z \in B$. Hence

$$|f_\zeta(\zeta)| = 1 > \sup_{z \in K} |f_\zeta(z)|$$

By considering the linear funct[ion], we may suppose that, for any $\zeta \in \partial D_1^*$, there exists a holomorphic function $f_\zeta(z)$ such that

$$|f_\zeta(\zeta)| = 2 > \sup_{z \in K}\{|f_\zeta(z)|\} = 1/2.$$

We put for any $\zeta \in \partial D_1^*$

$$Q_\zeta = \{z \in D \mid |f_\zeta(z)| > 1\}; \quad P_\zeta = \{z \in D \mid |f_\zeta(z)| < 1\}$$

Clearly,

$$Q_\zeta \ni \zeta \quad \text{and} \quad P_\zeta \supset\supset K.$$

Since $\bigcup_{\zeta \in \partial D_1^*} Q_\zeta \supset \partial D_1^*$ and ∂D_1^* is compact $\overbrace{}^{\text{in } D}$, Borel-Lebesgue Theorem shows that there exist a finite number of points $\zeta_1, \dots, \zeta_q \in \partial D_1^*$ such that

$$\bigcup_{i=1}^{q} Q_{\zeta_i} \supset \partial D_1^*.$$

Here we put

$$P = \bigcap_{i=1}^{q} P_{\zeta_i} \quad \text{and consider} \quad P \cap D_1^*.$$

It is clear that $K \Subset P \cap D_1^*$. To show $P \cap D_1^*$ is an analytic polygonal in D, it suffices to prove $P_1 \cap D_1^* \Subset D_1^*$.

We prove it by contradiction. If not, there is a sequence of points $\{z_j\}$ in $P \cap D_1^*$ such that

$$z_j \longrightarrow z_0 \in \partial D_1^* \quad (j \to \infty).$$

Because of $z_0 \in \partial D_1^* \subset \bigcup_{i=1}^{q} Q_i$, some $Q_{i_0} \ni z_0$ i.e., $|f_{\zeta_{i_0}}(z_0)| > 1$. On the other hand, each $z_j \in P_{\zeta_i}$ $(i=1,\dots,\nu)$ i.e., $|f_{\zeta_i}(z_j)| < 1$ $(j=1,2,\dots)$, so that, by letting $j \to \infty$, we have

$$|f_{\zeta_i}(z_0)| \leqq 1.$$

In particular, $|f_{i_0}(z_0)| \leqq 1$. This contradicts the above inequality. Therefore $P_1 \cap D_1^* \Subset D_1^*$. (1.4) is proved for $G = D_1^*$. c. q. f. d.

§2 Weierstrass Preparation Theorem

Let $f(x,y)$ be holomorphic at (a,b) in $\mathbb{C}^2$. This means that there exists a dicylinder $(|x-a|<r)\times(|y-b|<s)$ in which f is holomorphic. Assume that $f(a,b)=0$. Let us investigate the solution of the equation

$$f(x,y)=0$$

near (a,b).

If $(\partial f/\partial y)(a,b)\neq 0$, then we have only one solution $y=y(x)$ of $f(x,y)=0$ with $y(a)=b$ in a vicinity of (a,b). The function $y(x)$ becomes necessarily holomorphic at a. When we replace $\mathbb{C}^2$ and holomorphic by $\mathbb{R}^2$ and continuous differentiable, respectively, the same result remain valid. Hence there is no difference between for $(x,y)\in\mathbb{C}^2$ and for $(x,y)\in\mathbb{R}^2$. In both cases, we say them Implicit function theorem. A proof is obtained by the alternating method which is available for both cases.

On the contrary, if $(\partial f/\partial x)(a,b)=(\partial f/\partial y)(a,b)=0$, then there is an essential difference between in $\mathbb{C}^2$ and in $\mathbb{R}^2$. For example, consider the equation $f(x,y)=x^2+y^2=0$. Then, in $\mathbb{R}^2$ the solution is only one $(0,0)$, and hence is isolated in $\mathbb{R}^2$. In $\mathbb{C}^2$, the solution is $y=\pm ix$, and hence is not isolated.

In the case of $\mathbb{C}^2$, we have one of most important theorems in the theory of several complex variables due to Weierstrass: For the sake of convenience we assume $(a,b)=(0,0)$.

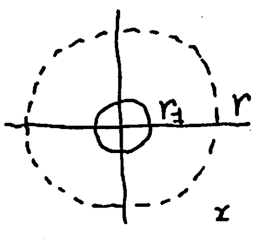
<u>Theorem 2.1</u> (Weierstrass Preparation Theorem)

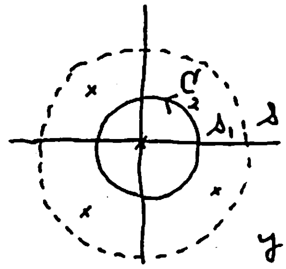
Let $f(x,y)$ be holomorphic at $(0,0)$ in $\mathbb{C}^2$. Suppose that $f(0,y)\not\equiv 0$. Then, in a neighborhood of $(0,0)$

$$(2.1)\quad f(x,y) = (y^n + a_1(x)y^{n-1}+\cdots+a_n(x))\,H(x,y)$$

where $n\geq 1$; each $a_i(x)$ is holomorphic at 0; $a_i(0)=0$ and $H(x,y)$ is holomorphic at $(0,0)$; $H(0,0)\neq 0$.

<u>Proof.</u> Assume that $f(x,y)$ is holomorphic in $(|x|<r)\times(|y|<s)$. Then $f(0,y)$ is a holomorphic function for y in $(|y|<s)$. Since $f(0,0)=0$ and $f(0,y)\not\equiv 0$, we find $0<s_1<s$ such that $f(0,y)\neq 0$ anywhere in $0<|y|\leq s_1$. Let $n(\geq 1)$ be the multiplicity of $f(0,y)=0$ at $y=0$.





Since $f(x,y)$ is continuous for $(x,y)\in(|x|<r)\times(|y|<s)$ and since $(0, |y|=s_1)$ is compact we find r_1 $(0<r_1<r)$ such that

$$f(x,y)\neq 0 \text{ anywhere on } (|x|\leq r_1)\times(|y|=s_1).$$

Put $C_2=(|y|=s_1)$, a circle of center $y=0$ whose direction is counterclockwise.

Let $x\in(|x|\leq r_1)$ and consider the integral

$$n(x)=\frac{1}{2\pi i}\int_{C_2}\frac{(\partial f/\partial y)(x,y)}{f(x,y)}\,dy \quad \left(=\frac{1}{2\pi i}\int_{C_2} d_y \log f(x,y)\right).$$

Then $n(x)$ represents the number of zeros : $f(x,y)=0$ (as function of y in $(|y|\leq s_1)$), so that $n(x)$ is integer. Since $n(x)$ is continuous for x in $(|x|\leq r_1)$, we see from $n(0)=n$ that $n(x)\equiv n$ in $(|x|\leq r_1)$. This means that $f(x,y)=0$ has just n zeros in $(|y|\leq s_1)$ counted its multiplicity. Say them

$$\{y_1(x), \ldots, y_n(x)\}.$$

Therefore, for any fixed x in $(|x|\leq r_1)$, if we put in $(|y|\leq s_1)-\{y_i(x)\}$

$$F(x,y)=\frac{f(x,y)}{(y-y_1(x))\cdots(y-y_n(x))}$$

then $F(x,y)$ becomes a holomorphic function in $(|y|\leq s_1)$ and $F(x,y)\neq 0$ anywhere in $(|y|\leq s_1)$.

Since x is arbitrary in $(|x|\leq r_1)$, it follows that $F(x,y)$ is certainly defined for all (x,y) in $(|x|\leq r_1)\times(|y|\leq s_1)$; $F(x,y)\neq 0$

anywhere in $(|x|\leqq r_1)\times(|y|\leqq s_1)$; $F(x,y)$ is holomorphic for y in $(|y|\leqq s_1)$ and

$$(2.2)\qquad f(x,y) = F(x,y)\,(y-y_1(x))\cdots(y-y_n(x))$$

for $(x,y)\in(|x|\leqq r_1)\times(|y|\leqq s_1)$. We remark that we don't know yet that $F(x,y)$ is holomorphic for x in $(|x|\leqq r_1)$.

Write
$$P(x,y) = (y-y_1(x))\cdots(y-y_n(x))$$
$$= y^n-(y_1(x)+\cdots+y_n(x))\,y^{n-1}+\cdots+(-1)^n y_1(x)\cdots y_n(x)$$
$$= y^n + a_1(x)\,y^{n-1}+\cdots+a_n(x)$$
so that $a_i(0)=0$ $(1\leqq i\leqq n)$.

Given $x\in(|x|\leqq r_1)$ and $j=1,\cdots,n$, consider the integral
$$S_j(x) \underset{\mathrm{def}}{=} \frac{1}{2\pi i}\int_{C_2} y^j\,\frac{(\partial f/\partial y)(x,y)}{f(x,y)}\,dy.$$
It is clear that $S_j(x)$ is holomorphic for x in $|x|\leqq r_1$. By the residue therem in the theory of function of one complex variable, we get
$$S_j(x) = y_1^j(x)+\cdots+y_n^j(x)$$
On the otherhand, it is well-known that $a_i(x)$ is represented by a polynomial of $\{S_1(x),\cdots,S_i(x)\}$. Hence $a_i(x)$ is a uniform holomorphic function of x in $(|x|\leqq r_1)$.

Since $P(x,y)\neq 0$ anywhere in $(|x|=r_1)\times(|y|=s_1)$ $(=C_1\times C_2)$, we form for any $(x,y)\in(|x|<r_1)\times(|y|<s_1)$
$$H(x,y) = \frac{1}{(2\pi i)^2}\iint_{C_1\times C_2}\frac{f(\xi,\eta)}{P(\xi,\eta)}\,\frac{1}{(\xi-x)(\eta-y)}\,d\xi\,d\eta,$$
so that $H(x,y)$ is holomorphic with respect to (x,y) in $(|x|<r_1)\times(|y|<s_1)$.

By Fubini's Theorem
$$H(x,y) = \frac{1}{2\pi i}\int_{C_2}\left\{\frac{1}{2\pi i}\int_{C_1}\frac{f(\xi,\eta)}{P(\xi,\eta)}\,\frac{1}{\xi-x}\,d\xi\right\}\frac{1}{\eta-y}\,d\eta.$$

Because $P(x,\eta)\neq 0$ for $|x|\leqq r_1$ and $\eta\in C_2$, $\dfrac{f(x,\eta)}{P(x,\eta)}$ is is holomorphic for x in $|x| \leq r_1$. Therefore by Cauchy's formula

$$H(x,y) = \frac{1}{2\pi i}\int_{C_2} \frac{f(x,\eta)}{P(x,\eta)} \frac{1}{\eta - y}\, d\eta$$

By $P(x,\eta) \neq 0$ for $(|x|<r_1) \times (|\eta| = s_1)$, (2.2) implies

$$H(x,y) = \frac{1}{2\pi i}\int_{C_2} \frac{F(x,\eta)}{\eta - y}\, d\eta$$

Since $F(x,y)$ was holomorphic with respect to y in $(|y| \leq s_1)$ for fixed x in $(|x| \leq r_1)$, it follows that

$$H(x,y) = F(x,y)$$

for $(x,y) \in (|x|<r_1) \times (|y|<s_1)$. Hence $H(x,y) \neq 0$ anywhere in $(|x|<r_1) \times (|y|<s_1)$ and $H(x,y) = f(x,y)/P(x,y)$ except $P(x,y)=0$.
Consequently, $f(x,y) = P(x,y)\,H(x,y)$ in $(|x|<r_1) \times (|y|<s_1)$.

C.Q.F.D.

Recalling the above proof, we easily have the following:

Remark 2.1

Let $f(x,y)$ be holomorphic at $(0,0)$ with $f(0,0)=0$ and $f(0,y) \neq 0$. If we write

$$f(0,y) = a_n y^n + a_{n+1} y^{n+1} + \cdots$$
where $a_n \neq 0$, $n \geq 1$

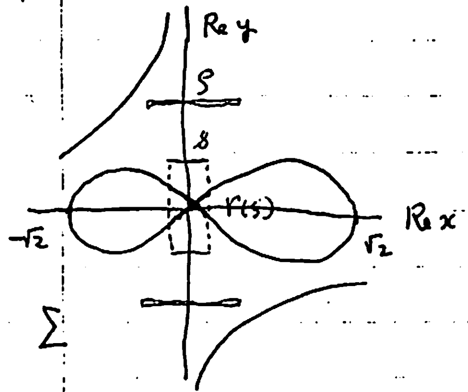
i.e., n = the multiplicity of zero of $f(0,y)$ at $y=0$, there exists a positive number $\rho > 0$ with the following property:

Let $0 < s \leq \rho$ be given. Then we find $r(s) > 0$ such that, for any $x \in (|x| < r(s))$, the equation of y: $f(x,y)=0$ has just n-solutions counted the multiplicity in $|y| < s$. $(\{y_1(x), \cdots, y_n(x)\})$

In particular, we get

(2.3) $\lim_{x \to 0} y_i(x) = 0 \quad (i=1,\cdots,n)$

For example, let $f(x,y) = x^2 + y^2 + xy^3$. Then
for $x \neq 0$, $f(x,y)=0$ has 3 solutions in $|y| < \infty$,
for $x = 0$, $f(x,y)=0$ has 2 solutions (double zero at $y=0$) in $|y|<\infty$



Hence, for $|x| \ll 1$, $f(x,y)=0$ has 2 solutions near $y=0$.

In general, a function $P(x,y)$ of the form

$$P(x,y) = a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$$

where $n \geq 1$, $\underbrace{a_i(x) \text{ is holomorphic}}_{(0 \leqq i \leqq n)}$ at 0 and $a_0(x) \not\equiv 0$, is called a pseudopolynomial of y of degree n at $(0,0)$

Moreover, if $a_0(x) \equiv 1$ and $a_i(0) = 0$ $(1 \leqq i \leqq n)$, then $P(x,y)$ is called a distinguished pseudopolynomial of y of degree n at $(0,0)$.

Thus, Weierstrass preparation theorem says that the zeros of a holomorphic function near $(0,0)$ are those of a distinguished pseudopolynomial near $(0,0)$.

<u>Lemma 2.1</u> Let $P(x,y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$ be a distinguished pseudopolynomial of degree $n (\geqq 2)$ at $(0,0)$.
Assume that

(2.4) $L(x)\, P(x,y) = H_1(x,y)\, H_2(x,y)$

where $H_1(x,y)$, $H_2(x,y)$ are $\underbrace{\text{pseudopolynomials}}_{\text{of } y \text{ at } (0,0)}$ of degree $\ell (\geqq 1)$ and $k (\geqq 1)$, respectively; $L(x)$ is holomorphic at 0 and $L(x) \not\equiv 0$.
Then we have

$$P(x,y) = P_1(x,y)\, P_2(x,y)$$

where $P_1(x,y)$, $P_2(x,y)$ are distinguished pseudopolynomials of y at $(0,0)$ degree l and k, respectively.

Proof. We put

$$H_1(x,y) = b_0(x)\,y^l + b_1(x)\,y^{l-1} + \cdots + b_l(x) \quad (l \geqq 1)$$
$$H_2(x,y) = c_0(x)\,y^k + c_1(x)\,y^{k-1} + \cdots + c_k(x) \quad (k \geqq 1)$$

where $b_i(x)$, $c_j(x)$ are all holomorphic at 0 ; $b_0(x), c_0(x) \not\equiv 0$.

By comparing the coefficient of y^n, it is clear that

$$n = l + k \quad \text{and} \quad L(x) = b_0(x)\, c_0(x).$$

Since $L(x) \not\equiv 0$, equation (2.4) yields

$$(2.5)\quad P(x,y) = (y^l + B_1(x)\,y^{l-1} + \cdots + B_l(x))\,(y^k + C_1(x)\,y^{k-1} + \cdots + C_k(x))$$
$$\underset{\text{def}}{=} P_1(x,y)\,P_2(x,y)$$

where $B_i(x) = b_i(x)/b_0(x)$, $C_j(x) = c_j(x)/c_0(x)$ are meromorphic at 0. It suffices for Lemma 2.1 to prove

(1) $B_i(x)$, $C_j(x)$ are holomorphic at 0 ;

(2) $B_i(0) = C_j(0) = 0 \quad (1 \leqq i \leqq l\ ;\ 1 \leqq j \leqq k)$.

Let us prove (1) by contradiction. Assume that (1) is not true. Then, one of $\{B_i(x)\}$ $(i=1,\dots,l)$, for example, is not holomorphic at 0. We can write, in a neighborhood of $x=0$,

$$B_i(x) = \beta_i(x) / x^{\alpha_i} \quad (1 \leqq i \leqq l)$$

where $\beta_i(x)$ is holomorphic at 0 ; $\beta_i(0) \neq 0$ and $\alpha_i \geqq 0$.

Put $\quad \alpha = \max\limits_{1 \leqq i \leqq n} \alpha_i$.

Then, by hypothesis, $\alpha \geqq 1$. We also put

$$j = \max\{\, i \mid 1 \leqq i \leqq n \text{ and } \alpha_i = \alpha \,\}.$$

It follows that

$$B_i(x) = \beta_i(x)\, x^{\alpha - \alpha_i} / x^\alpha$$
$$\underset{\text{def}}{=} \gamma_i(x) / x^\alpha$$

where $\gamma_i(x)$ is holomorphic at 0, and

$$\gamma_l(0) = \cdots = \gamma_{j+1}(0) = 0 \ ; \quad \gamma_j(0) \neq 0.$$

We thus have

$$P_1(x,y) = \frac{\gamma_0(x)\,y^l + \gamma_1(x)\,y^{l-1} + \cdots + \gamma_l(x)}{x^\alpha}$$

$$\overline{\overline{\text{def}}} \quad \frac{f_1(x,y)}{x^\alpha}$$

where $f_1(x,y)$ is holomorphic at $(0,0)$ and

$$f_1(0,y) = \gamma_j(0)\, y^{l-j} + \gamma_{j-1}(0)\, y^{l-j+1} + \cdots + \gamma_0(0)\, y^l$$

with $\gamma_j(0) \neq 0$.

If $j = l$, then $f_1(0,0) \neq 0$. The equation $f_1(x,y) = 0$ has no solution near $(0,0)$. If $1 \leq j \leq n-1$, then $f_1(0,0) = 0$ and $f_1(0,y) = 0$ has $y = 0$ of multiplicity $l-j$. By Remark 2.1, given $|x| \ll 1$, $f_1(x,y) = 0$ has just $l-j$ solutions near $y=0$, and so does $P_1(x,y) = 0$ for given $0 < |x| \ll 1$. Since, given $|x| \ll 1$, $P_2(x,y) = 0$ has at most k solutions in $\mathbb{C}$, we get, in any case, given $0 < |x| \ll 1$, $P_1(x,y)\, P_2(x,y) = 0$ has most $k + (l-j) = n - j$ $(\leq n-1)$ solutions near $y = 0$. On the other hand, By Remark 2.1, given $|x| \ll 1$, $P(x,y) = 0$ has just n solutions near $y = 0$. These are contradictions by (2,5). (1) is proved.

Assertion (2) is analogously proved as follows:

We have by (1)

$$\begin{aligned} P(x,y) &= P_1(x,y)\, P_2(x,y) \\ &= (y^l + B_1(x)\, y^{l-1} + \cdots + B_l(x))\,(y^k + C_1(x)\, y^{k-1} + \cdots + C_k(x)) \end{aligned}$$

where $B_i(x)$, $C_j(x)$ are all holomorphic at 0.

If (2) is not true, then, for example, some $B_i(0) \neq 0$. Put

$$j = \max\{\, i \mid 1 \leq i \leq l \text{ and } B_i(0) = 0 \,\}.$$

Then

$$P_1(0,y) = B_j(0)\, y^{l-j} + B_{j-1}(0)\, y^{l-j+1} + \cdots + y^l.$$

with $B_j(0) \neq 0$. It follows by Remark 2.1 that, given $|x| \ll 1$, $P_1(x,y) = 0$ has just $l-j$ solutions near $y=0$. Therefore, given $|x| \ll 1$, $P_1(x,y)\, P_2(x,y) = 0$ has at most $k + (l-j) = n - j$ $(\leq n-1)$ solutions near $y = 0$. This also contradicts that $P(x,y) = 0$ has just n solutions. (2) is proved. Lemma 2.1 is now proved.

Using Lemma 2.1, we prove the following

Theorem 2.2 Let $f(x,y)$ be holomorphic at $(0,0)$ with $f(0,0)=0$ and $f(0,y)\not\equiv 0$. Put

$$\Sigma = \{(x,y)\in\mathbb{C}^2 \mid f(x,y)=0\}.$$

Then we find a holomorphic mapping T of a disk $|\tau|<\rho$ into Σ of the form

$$T:\begin{cases} x=\tau^p \\ y=\varphi(\tau) \end{cases}$$

where $p\geqq 1$, $\varphi(\tau)$ is holomorphic in $|\tau|<\rho$ and $\varphi(0)=0$.

That is, if we put

$$\sigma = \{(\tau^p,\varphi(\tau))\in\mathbb{C}^2 \mid |\tau|<\rho\},$$

then $0\in\sigma\subset\Sigma$.

Proof. By Theorem 2.1 it suffices for Theorem 2.2 to prove it for the distinguished $\underbrace{\text{pseudopolynomial}}_{P(x,y)}$ with respect to y of degree $n\geqq 1$ at $(0,0)$:

$$P(x,y)=y^n+a_1(x)y^{n-1}+\cdots+a_n(x)$$

where $a_i(x)$ is holomorphic at 0 and $a_i(0)=0$ $(1\leqq i\leqq n)$.

Let us prove it by induction with respect to the degree n. If $n=1$, the assertion is trivial. In fact, take

$$T:\begin{cases} x=\tau \\ y=-a_1(\tau) \end{cases} \qquad (|\tau|\ll 1)$$

So we assume $n\geqq 2$ and the assertion is true for any distinguished pseudopolynomial of degree $k\leqq n-1$.

For the proof we need the Euclidean Algorithm:

In general, let

$$f(x,y)=a_0(x)y^n+a_1(x)y^{n-1}+\cdots+a_n(x)$$
$$g(x,y)=b_0(x)y^m+b_1(x)y^{m-1}+\cdots+b_m(x)$$

be pseudopolynomials at $(0,0)$ and $n>m$, namely, $a_i(x)$, $b_j(x)$ are holomorphic at 0 and $a_0(x), b_0(x)\not\equiv 0$.

Then we find an integer $\alpha\geqq 1$ and pseudopolynomials $Q(x,y)$, $R(x,y)$ at $(0,0)$ such that

$$(b_0(x))^\alpha f(x,y) = Q(x,y)\, g(x,y) + R(x,y)\ ;$$
$$\deg_y g(x,y) > \deg_y R(x,y) \geqq 0$$

It is easily proved by division. For the sake of convenience, we put $L(x) = (b_0(x))^\alpha$, i.e.,

(2.6) $L(x) f(x,y) = Q(x,y)\, g(x,y) + R(x,y)$

with $L(x) \not\equiv 0$ and $\deg_y g(x,y) > \deg_y R(x,y)$.

Let $P(x,y) = y^n + a_1(x) y^{n-1} + \cdots + a_n(x)$ be a pseudopolynomial at $(0,0)$ of degree n. We get

$$\frac{\partial P}{\partial y}(x,y) = n y^{n-1} + (n-1) a_1(x) y^{n-2} + \cdots + a_{n-1}(x).$$

By the Euclidean Algorithm (2.6), we successively have

(1) $L_{-1}(x) P(x,y) = Q_0(x,y) \frac{\partial P}{\partial y}(x,y) + R_1(x,y)$; $\deg_y R_1 < \deg \frac{\partial P}{\partial y}$

(2) $L_0(x) \frac{\partial P}{\partial y}(x,y) = Q_1(x,y) R_1(x,y) + R_2(x,y)$; $\deg_y R_2 < \deg_y R_1$

(3) $L_1(x)\, R_1(x,y) = Q_2(x,y) R_2(x,y) + R_3(x,y)$; $\deg_y R_3 < \deg_y R_2$

$\vdots$

(ρ-1) $L_{\rho-3}(x) R_{\rho-3}(x,y) = Q_{\rho-2}(x,y) R_{\rho-2}(x,y) + R_{\rho-1}(x,y)$; $\deg_y R_{\rho-1} < \deg_y R_{\rho-2}$

(ρ) $L_{\rho-2}(x) R_{\rho-2}(x,y) = Q_{\rho-1}(x,y) R_{\rho-1}(x,y) + R_\rho(x,y)$; $1 \leqq \deg_y R_\rho < \deg_y R_{\rho-1}$

(ρ+1) $L_{\rho-1}(x) R_{\rho-1}(x,y) = Q_\rho(x,y) R_\rho(x,y) + R_{\rho+1}(x)$

where $L_i(x)$ are holomorphic at 0 and $L_i(x) \not\equiv 0$ $(-1 \leqq i \leqq \rho-1)$; $Q_i(x,y)$, $R_i(x,y)$ are pseudopolynomial at $(0,0)$, while $R_{\rho+1}(x)$ is holomorphic at 0.

We remark $1 \leqq \deg_y R_\rho(x,y) < n$.

These equalities yield the following two expressions:

(2.7) $A(x,y) P(x,y) + B(x,y) \frac{\partial P}{\partial y}(x,y) = R_{\rho+1}(x)$

where $A(x,y)$, $B(x,y)$ are pseudopolynomials of y at $(0,0)$; and $A(x,y), B(x,y) \not\equiv 0$.

(2.8) If $R_{\rho+1}(x) \equiv 0$, then

$$L(x)P(x,y) = C(x,y)R_\rho(x,y)$$

where $L(x)$ is holomorphic at 0 and $L(x) \not\equiv 0$, and $C(x,y)$ is pseudopolynomial of y at $(0,0)$

In fact, multiply the formula (ρ) by $Q_\rho(x,y)$, then

$$Q_\rho L_{\rho-2} R_{\rho-2} = Q_\rho Q_{\rho-1} R_{\rho-1} + Q_\rho R_\rho$$
$$= Q_\rho Q_{\rho-1} R_{\rho-1} + L_{\rho-1} R_{\rho-1} - R_{\rho+1} \quad \text{by the formula } (\rho+1)$$

If we put $A_{\rho-2} = -Q_\rho L_{\rho-2}$ and $B_{\rho-1} = Q_\rho Q_{\rho-1} + L_{\rho-1}$, we have ($A_{\rho-2}, B_{\rho-1} \not\equiv 0$ and

$$A_{\rho-2} R_{\rho-2} = B_{\rho-1} R_{\rho-1} + R_{\rho+1}$$

Multiply the formula $(\rho-1)$ by $B_{\rho-1}$, then ....

$$\vdots \qquad \vdots$$

$$A_{-1}(x,y)P(x,y) = B_0(x,y)\frac{\partial P}{\partial y}(x,y) + R_{\rho+1}(x).$$

By construction, A_{-1}, $B_0 \not\equiv 0$. (2.7) is thus proved.

To prove (2.8), assume that $R_{\rho+1}(x) \equiv 0$. Then the formula $(\rho+1)$ becomes

$$L_{\rho-1} R_{\rho-1} = Q_\rho R_\rho$$

Multiply the formula (ρ) by $L_{\rho-1}$. Then

$$L_{\rho-1} L_{\rho-2} R_{\rho-2} = L_{\rho-1} Q_{\rho-1} R_{\rho-1} + L_{\rho-1} R_\rho$$
$$= (Q_\rho Q_{\rho-1} + L_{\rho-1}) R_\rho$$

If we put $C_{\rho-1} = Q_\rho Q_{\rho-1} + L_{\rho-1}$, then we get

$$L_{\rho-1} L_{\rho-2} R_{\rho-2} = C_{\rho-1} R_\rho.$$

Multiply the formula $(\rho-1)$ by $L_{\rho-1}(x) L_{\rho-2}(x)$, then

$$L_{\rho-1} L_{\rho-2} L_{\rho-3} R_{\rho-3} = Q_{\rho-2} \underline{L_{\rho-1} L_{\rho-2} R_{\rho-2}} + L_{\rho-2} \underline{L_{\rho-1} R_{\rho-1}}$$
$$= Q_{\rho-2} C_{\rho-1} R_\rho + L_{\rho-2} Q_\rho R_\rho$$

If we put $C_{\rho-2} = Q_{\rho-2} C_{\rho-1} + L_{\rho-2} Q_\rho$, then we get

$$L_{\rho-1} L_{\rho-2} L_{\rho-3} R_{\rho-3} = C_{\rho-2} R_\rho$$

$$\vdots \qquad\qquad \vdots$$

$$L_{\rho-1} L_{\rho-2} \cdots L_0 L_{-1} P(x,y) = C_0(x,y)\, R_\rho(x,y)$$

Hence, putting $L(x) = L_{\rho-1}(x) L_{\rho-2}(x) \cdots L_{-1}(x)$, we have $L(x) \not\equiv 0$ and

$$L(x)\, P(x,y) = C_0(x,y)\, R_\rho(x,y)$$

(2.8) is proved.

Now we return to our induction argument. Let

$$P(x,y) = y^n + a_1(x) y^{n-1} + \cdots + a_n(x) \quad (n \geqq 2)$$

be a distinguished pseudopolynomial of y at $(0,0)$. By (2.7) and (2.8) we obtain

(2.7′) $A(x,y)\, P(x,y) + B(x,y) \dfrac{\partial P}{\partial y}(x,y) = R_{\rho+1}(x)$;

(2.8′) $L(x)\, P(x,y) = C(x,y)\, R_\rho(x,y)$ if $R_{\rho+1}(x) \equiv 0$.

First assume that $R_{\rho+1}(x) \equiv 0$. Then we have (2.8′). Since $L(x) \not\equiv 0$, $L(x)$ is holomorphic at 0, and since $C(x,y)$, $R_\rho(x,y)$ are pseudopolynomials at $(0,0)$ with

$$1 \leqq \deg R_\rho \leqq n-1.$$

From Lemma 2.1, we get

$$P(x,y) = P_1(x,y)\, P_2(x,y)$$

where P_1, P_2 are distinguished pseudo-polynomials with

$$\deg P_1 = \deg C \quad \text{and} \quad \deg P_2 = \deg R_\rho.$$

Put $\Sigma_1 = \{(x,y) \in \mathbb{C}^2 \mid P_2(x,y) = 0\}$. Then by the hypothesis of induction, we get a set

$$\sigma_1 = \{(x,y) \in \mathbb{C}^2 \mid x = \tau^q \text{ and } y = \varphi(\tau),\ (|\tau| < \rho)\}$$

where $1 \leqq q \leqq n-1$; $\varphi(\tau)$ is holomorphic in $(|\tau| < \rho)$ and $\varphi(0) = 0$ such that

$$(0,0) \in \sigma_1 \subset \Sigma_1$$

Since $\Sigma_1 \subset \Sigma$, the Theorem 2.2 is proved in the case $R_{\rho+1}(x) \equiv 0$.

Next assume that $R_{\beta+1}(x) \not\equiv 0$.

continued to Sep. 23, '87

For Theorem 2.2, it rests to prove the following:

Let $P(x,y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$ be a distinguished pseudopolynomial with respect to y of degree $n \geqq 2$ at $(0,0)$, namely, $a_i(x)$ is holomorphic at 0 and $a_i(0) = 0$ $(1 \leqq i \leqq n)$.

Put $\Sigma = \{(x,y) \in \mathbb{C}^2 \mid P(x,y) = 0\}$.

By Euclidean Algorithm we got

$$(2.4) \quad A(x,y)P(x,y) + B(x,y)\frac{\partial P}{\partial y}(x,y) = R_{\rho+1}(x)$$

where $A(x,y)$, $B(x,y)$ are pseudopolynomials at $(0,0)$ and $R_{\rho+1}(x)$ is holomorphic at 0.

Assume that $R_{\rho+1}(x) \not\equiv 0$. Then at $(0,0)$ we will find a set σ in Σ of the form

$$\sigma : \quad (x,y) = (\tau^p, \varphi(\tau)), \quad (|\tau| < \rho)$$

where $1 \leqq p \leqq n$, $\varphi(\tau)$ is holomorphic in $(|\tau| < \rho)$ and $\varphi(0) = 0$, that is, $0 \in \sigma \subset \Sigma$.

Let us observe the behavior of Σ at $(0,0)$ and find such σ. Since $n \geqq 2$, (2.4) implies $R_{\rho+1}(0) = 0$. But $R_{\rho+1}(x)$ is holomorphic at 0 and $R_{\rho+1}(x) \not\equiv 0$, so we find $r > 0$ such that $R_{\rho+1}(x) \neq 0$ anywhere in the punctured disk

$$D^* = (0 < |x| < r).$$

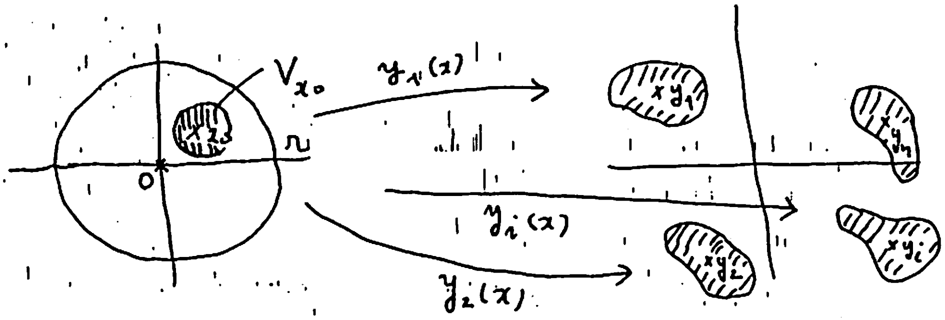
Let $x_0 \in D^*$ be given arbitrarily. By (2.4), we see that both $P(x_0, y)$ and $(\partial P/\partial y)(x_0, y)$ do not vanish simultaneously. Consequently, the equation of polynomial with respect to y, $P(x_0, y) = 0$ has n distinct solutions

$$\{y_1, y_2, \cdots, y_n\}.$$

We thus have $P(x_0, y_i) = 0$ and $\frac{\partial P}{\partial y}(x_0, y_i) \neq 0$.

By Implicit Function Theorem or $n=1$ in Weierstrass Preparation Theorem, for each i, $(1 \leqq i \leqq n)$, we find a disk V_{x_0} : $|x - x_0| < r(x_0)$, and a holomorphic function $y_i(x)$ in V_{x_0} such that

$$P(x, y_i(x)) = 0 \quad \text{and} \quad y_i(x_0) = y_i$$



Precisely speaking, to each $x_\alpha \in D^*$ we let correspond the double

$$\left(V_{x_\alpha},\ \{ y_1(x_\alpha, x), \cdots, y_n(x_\alpha, x) \} \right)$$

where V_{x_α} is a disk of center x_α in D^* and each $y_i(x_\alpha, x)$ $(1 \leqq i \leqq n)$ is holomorphic function in V_{x_α} such that

$$(2.6) \quad \begin{cases} y_i(x_\alpha, x) \neq y_j(x_\alpha, x) \ (i \neq j) & \text{for any } x \in V_{x_\alpha} \\ P(x, y_i(x)) \equiv 0 & \text{in } V_{x_\alpha} \end{cases}$$

Since $P(x, y)$ is of degree n w.r.t. y, we remark that

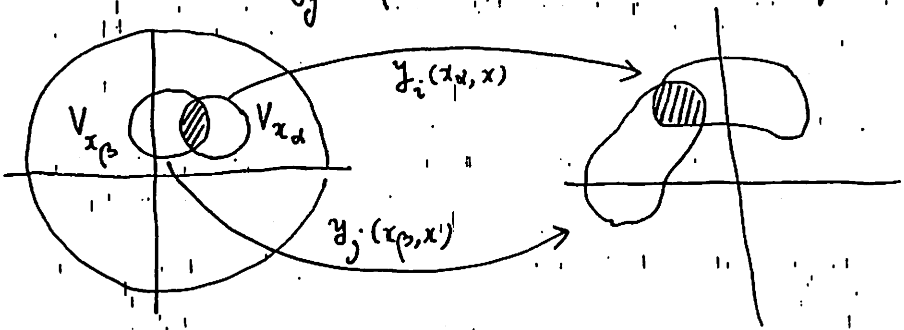
(2.7) given $x \in V_{x_\alpha}$, the equation of y : $P(x, y) = 0$ has no solution except for $\{ y_1(x_\alpha, x), \cdots, y_n(x_\alpha, x) \}$.

Now, let $V_{x_\alpha} \cap V_{x_\beta} \neq \phi$ where $x_\alpha, x_\beta \in D^*$. Fix one of $\{ y_1(x_\alpha, x), \cdots, y_n(x_\alpha, x) \}$, say $y_i(x_\alpha, x)$. Then by (2.6), (2.7) we can find j $(1 \leqq j \leqq n)$ such that

$$y_j(x_\beta, x) = y_i(x_\alpha, x) \quad \text{in } V_{x_\alpha} \cap V_{x_\beta}.$$

In other words

(2.8) $y_j(x_\beta, x)$ is the direct analytic continuation of $y_i(x_\alpha, x)$.



Fix $x_0 \in D^*$, and take $y_1(x_0, x)$ in V_{x_0}. Let

$$\gamma : t \in [0,1] \longrightarrow x = x(t) \in D^*$$

be any curve in D^* starting at x_0 and let x^* be the terminal point of γ, i.e., $x(1) = x^*$.

To each $t \in [0,1]$ we have the double

$$(V_{x(t)} , \{ y_1(x(t), x), \cdots, y_n(x(t), x) \})$$

with property (2.6). Since $x(t)$ is continuous for $t \in [0,1]$, given $t_0 \in [0,1]$, we find $\delta(t_0) > 0$ such that

$$\{ x(t) \in \mathbb{C} \mid t_0 - \delta(t_0) < t < t_0 + \delta(t_0) \} \subset V_{x(t_0)} .$$

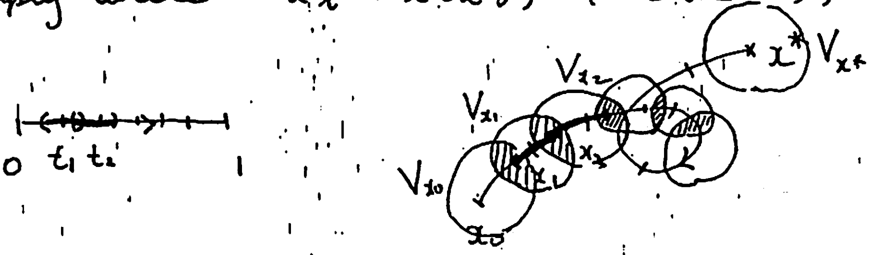
If we put $I(t_0) = (t_0 - \delta(t_0), t_0 + \delta(t_0))$ an open interval in $\mathbb{R}$, then

$$\bigcup_{t \in [0,1]} I(t) \supset [0,1] .$$

It follows from Borel-Lebesgue Theorem that there exists a finite number $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that

$$I(t_0) \cup I(t_1) \cup \cdots \cup I(t_n) \supset [0,1]$$

We simply write $x_i = x(t_i)$ $(0 \leq i \leq n)$, so that $x(t_n) = x^*$.



We get $V_{x_0}, V_{x_1}, \cdots, V_{x_{n-1}}, V_{x^*}$. Since $I(t_i) \cap I(t_{i+1}) \neq \phi$ $(0 \leq i \leq n-1)$, we have

$$V_{x_i} \cap V_{x_{i+1}} \neq \phi.$$

It follows by (2.8) that we successively find

$$y_1(x_0, x) \longrightarrow y_{j_1}(x_1, x) \longrightarrow \cdots\cdots \longrightarrow y_{j_{n-1}}(x_{n-1}, x) \longrightarrow y_{j_n}(x^*, x)$$

where $1 \leq j_k \leq n$ and

$y_{j_1}(x_1, x)$ is the direct continuation of $y_1(x_0, x)$

$\vdots \qquad\qquad \vdots$

$y_{j_n}(x^*, x)$ is the direct continuation of $y_{j_{n-1}}(x_{n-1}, x)$.

This means that

$y_{j_n}(x^*, x)$ is the analytic continuation of $y_1(x_0, x)$ along γ.

It turns out $y_1(x_0, x)$ can be analytically continued along any curve γ starting from x_0 in D^*. Moreover, we see that, if we put z^* the final point of γ, the final continuation function near z^* becomes one of

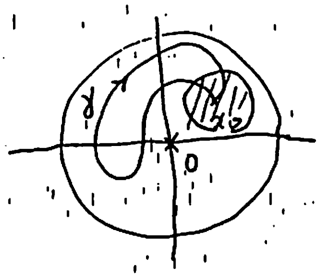
$$\{ y_1(x^*, x), \cdots, y_n(x^*, x) \}$$

which are already prepared in V_{x^*} at x^* with property (2.6).

Therefore $y_1(x_0, x)$ determines a Riemann surface $\mathcal{S}$ spread over the punctured disk D^* (without relative boundary over D^*), and a holomorphic function $y(x)$ on $\mathcal{S}$ such that $y(x) \equiv y_1(x_0, x)$ in V_{x_0} and $(\underline{x}, y(x)) \subset \Sigma$ for $x \in \mathcal{S}$, where $\underline{x}$ denotes the projection of x on x-plane.

Let us observe the Riemann surface $\mathcal{S}$.

First, let γ be a closed curve starting at x_0 in D^*, i.e. D^* which does not round the origin $x = 0$, like the figure,

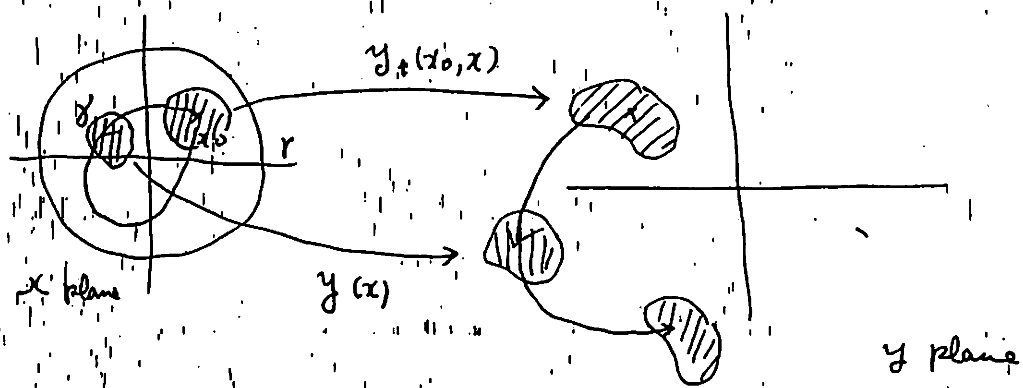


then, by Monodromy Theorem,

$$y_1(x_0, x) \xrightarrow[\gamma]{y(x)} y_1(x_0, x)$$

i.e., $y_1(x_0,x)$ returns to the original one.

Next, let γ be a closed loop arround 0 starting at x_0 in D^*.



Then $y_1(x_0, x) \xrightarrow[\gamma]{y(x)}$ one of $\{y_1(x_0,x), \cdots, y_n(x_0,x)\}$

Case 1 If the final continuation function is $y_1(x_0,x)$, then $y(x)$ becomes uniform on D^*, that is, $S = D^*$. Moreover, as already noted $\lim_{x\to 0} y_i(x) = 0$ (for $1 \leq i \leq n$, we see that $y(x)$ is a holomorphic function in the whole disk $D = (|x| < r)$ and $y(0) = 0$.

Hence, in this case, put $\sigma : (x, y) = (\tau, y(\tau))$, $\tau \in D$, then $0 \in \sigma \subset \Sigma$. Theorem is proved.

Case 2 If the final function is not $y_1(x_0,x)$, then it becomes $y_2(x_0,x)$, for exemple. We repeat the same procedure to $y_2(x_0,x)$ and the same loop γ as to $y_1(x_0,x)$ and γ.

$$y_1(x_0,x) \xrightarrow[\gamma]{} y_2(x_0,x) \xrightarrow[\gamma]{} \text{one of } \{y_1(x_0,x), y_2(x_0,x), \cdots, y_n(x_0,x)\}$$

$y_2(x_0,x)$ can be analytically continued along γ and becomes one of $\{y_1(x_0,x), y_2(x_0,x), \cdots, y_n(x_0,x)\}$. It is not $y_2(x_0,x)$. In fact, if it is $y_2(x_0,x)$, then by considering the inverse loop γ^{-1}, $y_2(x_0,x)$ varies along γ^{-1} and becomes $y_1(x_0,x)$ and $y_2(x_0,x)$, simultaneously. This is a contradiction.

Therefore, the final function is one of $\{y_1(x_0,x), y_3(x_0,x), \cdots, y_n(x_0,x)\}$.

If it is $y_1(x_0,x)$, then two functions $\{y_1(x_0,x), y_2(x_0,x)\}$ permutes each other when the variable x varies along the loop γ.

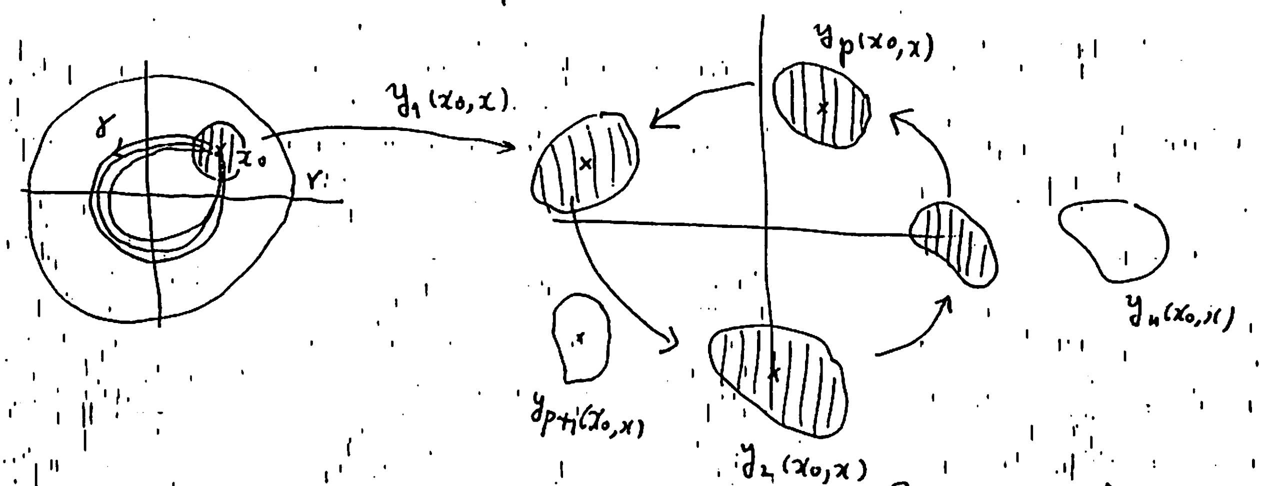
$$\begin{tikzcd} y_1(x_0,x) \arrow[r, bend left, "\gamma"] & y_2(x_0,x) \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

If it is not $y_1(x_0,x)$, say, it is $y_3(x_0,x)$. ....

....

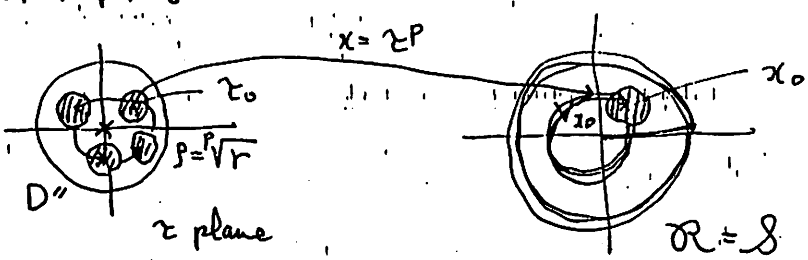
....

Finally we reach $p\ (\leqq n)$ functions $\{y_1(x_0,x), y_2(x_0,x), \cdots, y_p(x_0,x)\}$ in V_{x_0} permuting periodically, when the variable x draw the loop γ.



This means that the Riemann surface $\mathcal{S}$ of $y(x)$ becomes the Riemann surface of the function $\sqrt[p]{x}$ spread over D^*.

Let τ_0 be one of solutions: $\tau_0^p = x_0$, and let $D'' = \{0 < |\tau| < \sqrt[p]{r}\}$. Then



Then the function: $\tau \to x = \tau^p \underset{\mathrm{df}}{=} x(\tau)$ with $x(\tau_0) = x_0$ determines a holomorphic 1:1 function of D'' onto $\mathcal{R}$.

Hence, if we put

$$\varphi(\tau) \underset{\mathrm{df}}{=} y(x(\tau)),$$

then $\varphi(\tau)$ becomes a holomorphic function in D''. As already noted, we have $\lim_{x\to 0} y_i(x) = 0$, so $\lim_{\tau\to 0} \varphi(\tau) = 0$. That is, $\varphi(\tau)$ is a holomorphic function on the whole disk $(|\tau| < \sqrt[p]{r})$ and $\varphi(0) = 0$. By $(x, y(x)) \subset \Sigma$ for $x \in \mathcal{S} = \mathcal{R}$, we see that $(\tau, \varphi(\tau)) \in \Sigma$ for any $\tau \in D''$ when x is the projection of τ to D'' and $(0, \varphi(0)) = (0, 0)$. It follows that

$$\sigma: (x, y) = (\tau^p, \varphi(\tau)) \quad (|\tau| < \sqrt[p]{r})$$

is the desired set. Theorem 2.2 is now proved.

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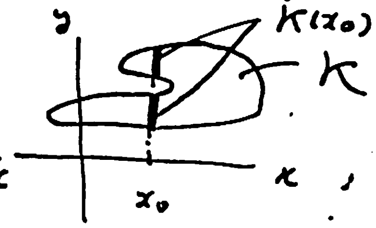
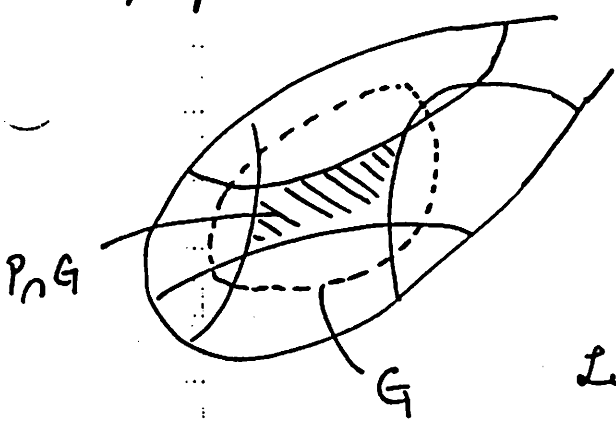
<u>Remark 2.1.</u> The set σ in the proof satisfies the equation $y^p + b_1(x) y^{p-1} + \cdots + b_p(x) = 0$ where $b_i(x)$ is holomorphic at 0 and $b_i(0)=0$.

<u>Notation 2.1</u> Let K be any set of $\mathbb{C}^2$. For each x_0 in $\mathbb{C}$ we put $K(x_0) = K \cap (\{x = x_0\} \times \mathbb{C})$ and call it the fiber of K at x_0. $K(x_0)$ is regarded as a subset of the complex y-plane.



<u>Corollary 2.2</u> Let $P \cap G$ be an analytic polyedral in a domain D in $\mathbb{C}^2$, that is



$$P = \bigcap_{j=1}^{n} \{|f_j(x,y)| < 1\}$$

$$P \cap G \subset\subset G \subset\subset D$$

where $f_j(x,y)$ is holomorphic for w.r.t (x,y) in D.

Let x_0 in $\mathbb{C}$ such that $(P \cap G)(x_0) \neq \phi$, and $y_0 \in \partial (P \cap G)(x_0)$ in $\mathbb{C}_y$.

Then we find holomorphic function of τ in $\Delta = (|\tau| < \rho)$ such that

$$\begin{cases} x = x(\tau) = x_0 + \tau^m \\ y = y(\tau) = y_0 + \varphi(\tau) \end{cases}$$

with $\varphi(0) = 0$ and

$$(x(\tau), y(\tau)) \in D - (P \cap G) \text{ for } \tau \in \Delta.$$

<u>Proof.</u> Since (x_0, y_0) becomes a boundary point of $P \cap G$, we have $|f(x_0, y_0)| = 1$ for some j $(1 \leq j \leq n)$

That is, $f_j(x_0, y_0) - e^{i\theta_0} = 0$. Put $f(x,y) = f_j - e^{i\theta_0}$. Then $f(x,y)$ is holo. at (x_0, y_0) and $f(x_0, y_0) = 0$. We also have $f(x_0, y) \not\equiv 0$ near $y = y_0$.

For, if $f(x_0, y) \equiv 0$ near $y = y_0$, then $|f_j(x_0, y)| \equiv 1$ there, so that $(x_0, y) \in \partial(P \cap G)(x_0)$ for $|y - y_0| \ll 1$. This contradicts $(x_0, y) \in \partial(P \cap G)(x_0)$.

It follows by Theorem 2.2 that we find an analytic set $\sigma : |\tau| < \rho \longrightarrow (x(\tau), y(\tau)) = (x_0 + \tau^m, \varphi(\tau))$ such that $\varphi(0) = 0$ and $\sigma \subset \{|f| = 1\} \subset \mathcal{D} - (P \cap G)$.

c. q. f. d.

§3 Green's function and Robin constant

Let W be a domain in the complex z-plane $\mathbb{C}$.

If $u(z)$ is a real-valued function of class C^2 and satisfies the Laplace equation

$$\Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(z) = 0 \qquad \text{where } z = x+iy$$

in W, then $u(z)$ is called a harmonic function in W.

For exemple,

$\log|z-a|$ where a is fixed, is harmonic in $\mathbb{C} - \{a\}$

const. is harmonic in $\mathbb{C}$

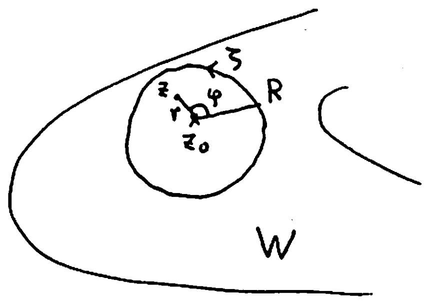
Harmonic functions have many properties. In this §, we need the following property for harmonic functions:

$u(z)$ is harmonic in W if and only if

for $\forall z_0 \in W$, $\forall R > 0$ $\ni$ $\{|z-z_0|<R\} \Subset W$, we have

$$(3.1) \qquad u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{|\zeta - z|^2} u(\zeta)\, d\theta$$

where $\zeta = z_0 + Re^{i\theta}$, $z = z_0 + re^{i\varphi}$ $(0 \leq r < R,\ 0 \leq \varphi \leq 2\pi)$.



(3.1) is called Poisson Integral Formula, and

$$P(\zeta, z) = \frac{R^2 - r^2}{|\zeta - z|^2} > 0$$

is called Poisson's Kernel.

$P(\zeta, z)$ is harmonic for z with fixed ζ.

In (3.1) we put $z = z_0$ and get

$$(3.2) \qquad u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})\, d\theta$$

This is called the Mean Value Property. This property easily yields the Maximum and Minimum principle for harmonic functions

(1) Non-constant harmonic function $u(z)$ never attains its local maximum (minimum) at a point in a domain W;

(2) Suppose $u(z)$ is harmonic in W. If $\varlimsup_{z \to \partial W} u(z) \leqq 0$ $(\geqq)$ (and $\varlimsup_{z \to \infty} u(z) \leqq 0$ in case W is unbounded), then $u(z) \leqq 0$ $(\geqq)$ in W.

Now, Let D be a domain in $\mathbb{C}$ with smooth boundary ∂G and let $\zeta \in D$. Then there exists a function $g(z)$ which is uniquely determined by the following conditions:

(i) $g(z)$ is harmonic in D except at ζ;

(ii) $g(z)$ continuously vanishes on ∂D;

(iii) $g(z)$ differs from $\log \frac{1}{|z-\zeta|}$ by a harmonic function in a neighborhood of ζ.

The uniqueness is clear by the maximum principle (2). Meanwhile we assume the existence of such function $g(z)$. Later we will need the concrete construction of $g(z)$. Anyway,

$g(z)$ is called the Green's function of D with pole at ζ, More simply, the Green's function for $(D, \{\zeta\})$.

By (iii), if we put $u(z) = g(z) - \log \frac{1}{|z-\zeta|}$, (then $u(z)$) is harmonic in the whole D. We put

$$\lambda = u(\zeta) = \lim_{z \to \zeta} (g(z) - \log 1/|z-\zeta|)$$

and λ is called the Robin constant for $(D, \{\zeta\})$.

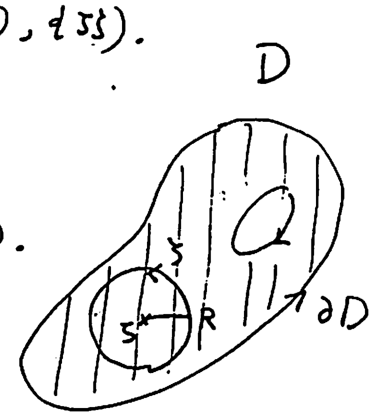
Hence we write

$$(3.3) \quad g(z) = \log \frac{1}{|z-\zeta|} + \lambda + h(z)$$

where $h(z)$ is harmonic in D and $h(\zeta) = 0$.

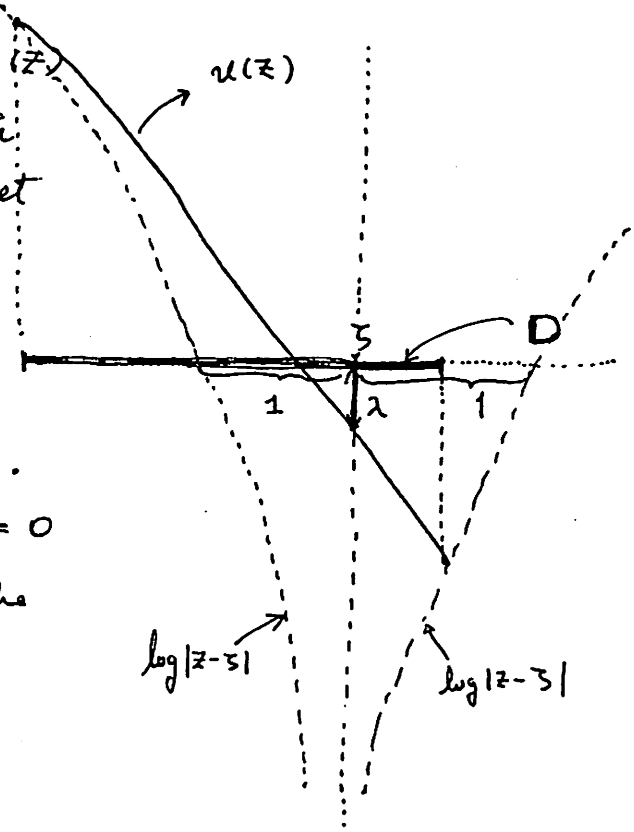
By (3.2), we have

$$(3.4) \quad \lambda = \frac{1}{2\pi} \int_0^{2\pi} g(\zeta + Re^{i\theta})\, d\theta + \log R.$$



Hence λ is determined by the values of $g(z)$ on a circle of center the pole ζ.

Since $u(\zeta)=\lambda$, and since $u(z)$ is a harmonic function in D with boundary value $\log|z-\zeta|$, we get the value λ in the following model, intuitively.



We get

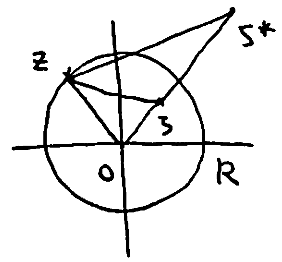
(3.4) $\quad 0 < g(z) < \infty \quad$ for $z \in D - \{\zeta\}$.

In fact, $\lim_{z\to\zeta} g(z) = +\infty$ and $\lim_{z\to\partial D} g(z) = 0$ imply $g(z) \geqq 0$ in $D-\{\zeta\}$ by the maximum principle (2). Moreover (1) induces $g(z) > 0$ in $D-\{\zeta\}$.

<u>Exemple 3.1</u> Let D be a disk : $|z| < R$ and let $\zeta \in D$

Then $\quad g(z) = \log \dfrac{|R^2 - \bar{\zeta} z|}{R|z-\zeta|} \quad ; \quad \lambda = \log \dfrac{R^2 - |\zeta|^2}{R}$

$\because$



Let ζ^* be the symmetric pt of ζ with respect to the circle $|z| = R$, that is,

$$\zeta^* = R^2/\bar{\zeta}$$

Then $\triangle O z \zeta^* \backsim \triangle O \zeta z$, so that

$$\frac{|z-\zeta^*|}{|z-\zeta|} = \frac{|\zeta^*|}{|z|} \quad \left(= \frac{R}{|\zeta|}\right),$$

or $\dfrac{|z-\zeta^*|}{|z-\zeta|}\dfrac{|\zeta|}{R} = 1$ for $z \in \partial D$. Let $W = \mathbb{C} - \{\zeta\} \cup \{\zeta^*\}$ and put

$$g(\zeta, z) = \log \frac{|z-\zeta^*|}{|z-\zeta|}\frac{|\zeta|}{R}$$

$$\left(= \log|z-\zeta^*| - \log|z-\zeta| + \log\frac{|\zeta|}{R}\right)$$

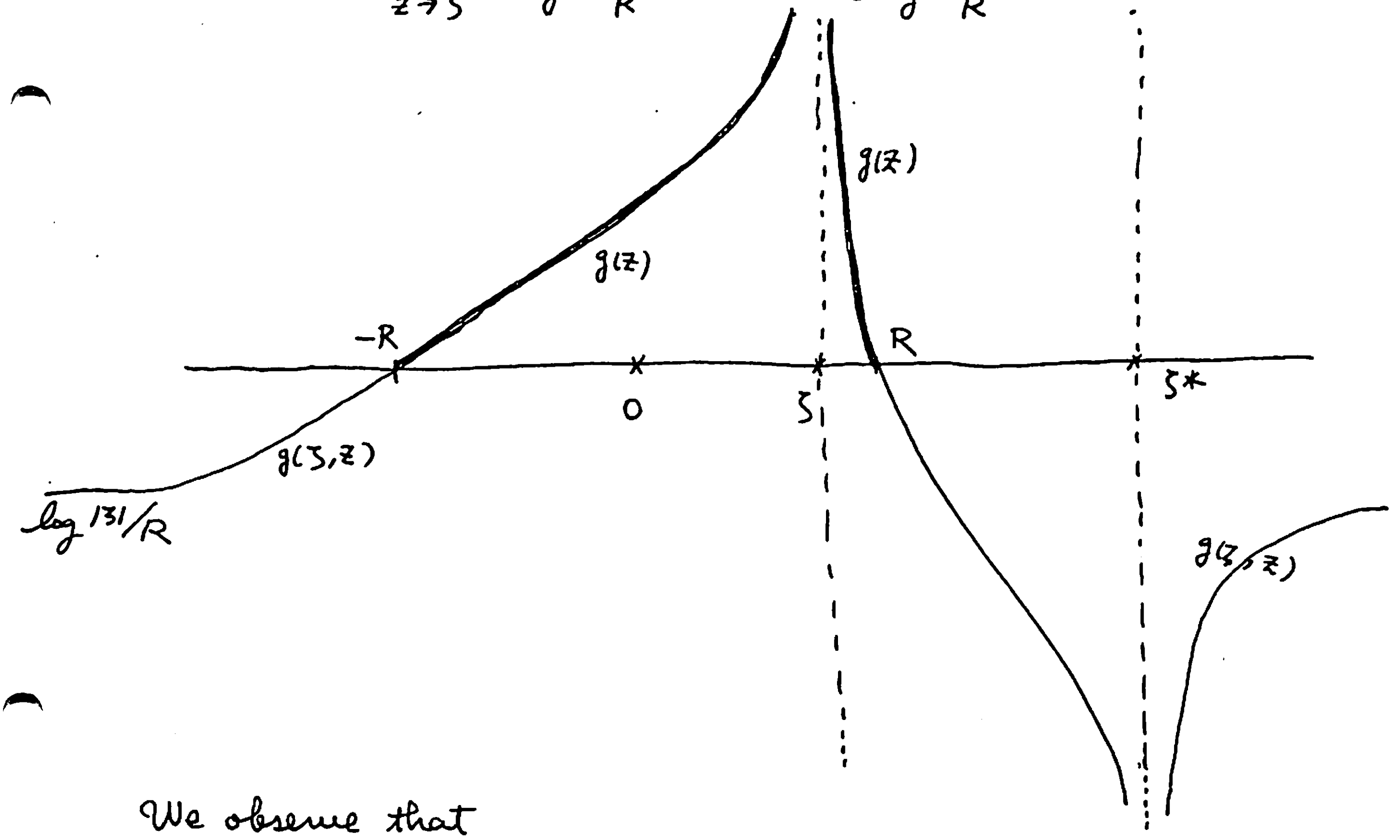
Then $g(\zeta, z)$ is harmonic for z in W and is $=0$ on ∂D. Moreover the singularity of $g(\zeta, z)$ at $z=\zeta$ and $z=\zeta^*$ are

$$\log 1/|z-\zeta| \quad \text{and} \quad \log |z-\zeta^*|$$

respectively. Hence $g(\zeta, z)$ restricted to D is not other than the Green's function for $(D, \{\zeta\})$:

$$g(z) = g(\zeta, z) = \log \frac{|R^2 - \bar{\zeta} z|}{R|z-\zeta|}, \quad \text{so that}$$

$$\lambda = \lim_{z \to \zeta} \log \frac{|R^2 - \bar{\zeta} z|}{R} = \log \frac{R^2 - |\zeta|^2}{R}$$



We observe that

if ζ is fixed in $\mathbb{C}$, then $\lim_{R \to \infty} \lambda = +\infty$;

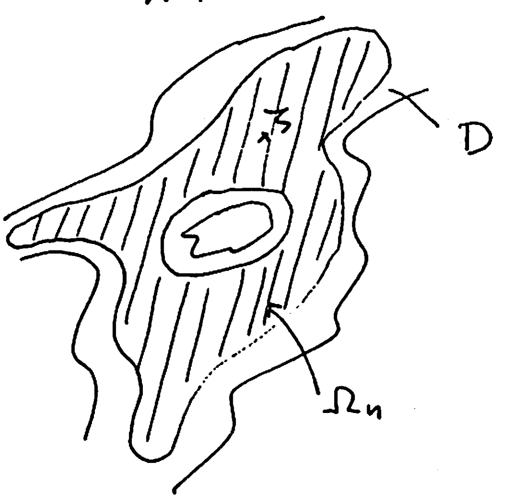
if R is fixed, then $\lim_{\zeta \to \partial D} \lambda = -\infty$.

The following is directly calculated

<u>Remark 3.1</u> Let $D = \{|z| < R\}$, $\zeta \in D$ and $z = Re^{i\theta} \in \partial D$. Then we get

$$\text{Poisson Kernel } P(z, \zeta) = R \cdot \left\{ \frac{\partial}{\partial r} g(\zeta, re^{i\theta}) \right\}_{r=R}$$

Next let D be a domain in $\mathbb{C}$ and let $\zeta \in D$.



We take a sequence of domains $\{\Omega_n\}$ $(n=1,2,\cdots)$ with smooth boundary such that

$\zeta \in \Omega_1 \Subset \Omega_2 \Subset \cdots$; $\bigcup_{n=1}^{\infty} \Omega_n = D$.

Each Ω_n carries the Green's function $g_n(z)$ and the Robin constant λ_n for $(\Omega_n, \{\zeta\})$, so that

$$(3.5) \qquad g_n(z) = \log \frac{1}{|z-\zeta|} + \lambda_n + h_n(z)$$

where $h_n(z)$ is harmonic in Ω_n and $h_n(\zeta) = 0$.

Since $g_{n+1}(z) - g_n(z)$ is harmonic in Ω_n and is >0 on $\partial\Omega_n$, it follows that $g_{n+1}(z) - g_n(z) > 0$ in Ω_n and, in particular, at ζ, $\lambda_{n+1} - \lambda_n > 0$. Hence $g_n(z) < g_{n+1}(z)$ in $\Omega_n - \{\zeta\}$ and $\lambda_n < \lambda_{n+1}$. We thus have

$$\lim_{n\to\infty} g_n(z) = g(z) \quad ; \quad \lim_{n\to\infty} \lambda_n = \lambda$$

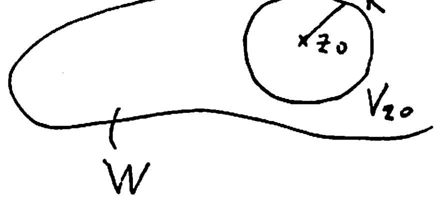
$g(z)$ and λ are called the Green's function and the Robin constant for $(D, \{\zeta\})$. It may happen to be $g(z) = +\infty$ or $\lambda = +\infty$.

Precisely speaking, we get

$$\exists z_0 \in D - \{\zeta\} \ni. \ g(z_0) < +\infty \overset{①}{\rightleftarrows} \forall z \in D - \{\zeta\}, \ g(z) < +\infty \overset{②}{\rightleftarrows} \lambda < +\infty$$

To prove these equivalences we need the powerful Harnack's principle for positive harmonic functions.

In general, let W be a region in $\mathbb{C}$ and $u(z)$ any positive harmonic function in W.



Let $z_0 \in W$ and $V_{z_0} = \{|z - z_0| < R\} \Subset W$.

By Poisson's Formula we have for $\forall z \in V_{z_0}$,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - |z-z_0|^2}{|\zeta - z|^2}\, u(\zeta)\, d\theta$$

Since $u > 0$, we have

$$u(z) \leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{R + |z-z_0|}{R - |z-z_0|}\, u(\zeta)\, d\theta \quad ; \ \zeta = z_0 + Re^{i\theta}$$

Assume that

$$|z - z_0| < \frac{R}{2} \quad \text{, or simply } \ z \in \tfrac{1}{2}V_{z_0}.$$

Then

$$u(z) \leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{R + \frac{R}{2}}{R - \frac{R}{2}}\, u(\zeta)\, d\theta$$

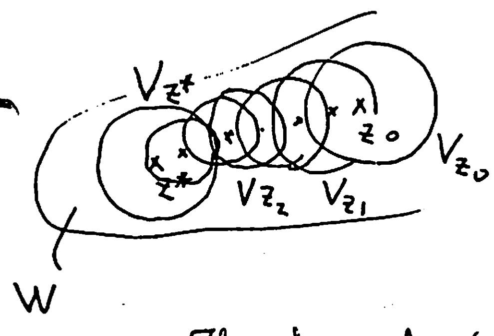
$$= \frac{3}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})\, d\theta$$

$$= 3\, u(z_0) \quad \text{(by Mean Value Property (3.2))}.$$

Consequently,

$$(3.6) \qquad u(z) \leqq 3\, u(z_0) \quad \text{for } z \in \tfrac{1}{2}V_{z_0}$$

Next, let $z^* \in W$. First we draw a disk $V_{z^*} : |z - z^*| < R$ $\subset\subset W$. Then we easily find a finite number of disks in W



$$V_{z_0},\ V_{z_1},\ \cdots,\ V_{z_{\nu-1}},\ V_{z^*}$$

such that $\quad z_{i+1} \in \frac{1}{2}V_{z_i} \quad (0 \leq i \leq \nu-1 \ ; \ z_\nu = z^*)$.

Therefore by (3.6)

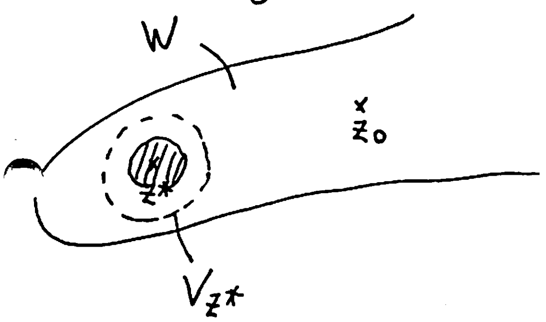
$$u(z_{i+1}) \leqq 3\, u(z_i) \quad (i = 0, 1, \cdots, \nu-1),$$

so that $u(z^*) \leqq 3^\nu u(z_0)$. It follows from (3.6) that

$$(3.7) \qquad u(z) \leqq 3^{\nu+1} u(z_0) \quad \text{for } z \in \tfrac{1}{2}V_{z^*}.$$

This induces the following

Theorem 3.1 Let W be a domain in $\mathbb{C}$ and let $\{u_n(z)\}$ be a sequence of harmonic fns in W such that $u_n(z) \leqq u_{n+1}(z)$ in W. Suppose that there exists a point z_0 in W such that $\lim_{n\to\infty} u(z_0)$ exists, i.e., $\{u_n(z_0)\}$ is bounded. Then $\{u_n(z)\}$ uniformly converges on any compact set in W, and the limiting function $u(z) = \lim_{n\to\infty} u_n(z)$ is harmonic in W.



Proof. Take any point z^* in W and draw a disk $V_{z^*} \subset\subset W$. Then by (3.7) there exists a constant $C = 2^{\nu+1} > 0$ such that, for any positive harmonic function $v(z)$

$$v(z) \leqq C\, v(z_0) \quad \text{in } \tfrac{1}{2} V_{z^*}.$$

Let $\varepsilon > 0$ be given. Since $\{u_n(z_0)\}$ converges, there exists $N (\geqq 1)$ such that

$$u_{n+p}(z_0) - u_n(z_0) < \varepsilon / C \quad \text{for } \forall n \geqq N \text{ and } p \geqq 1.$$

Since $u_{n+p}(z) - u_n(z)$ is positive harmonic function in W, it follows that

$$0 < u_{n+p}(z) - u_n(z) \leqq C\,(u_{n+p}(z_0) - u_n(z_0)) < \varepsilon \quad \text{in } \tfrac{1}{2} V_{z^*}.$$

We conclude that $\{u_n(z)\}$ uniformly converges in $(1/2)\, V_{z^*}$. Because z^* is any point of W, we see that $\{u_n(z)\}$ uniformly converges on any compact set in W. The latter assertion that the limiting function $u(z)$ is harmonic in W is easily proved by Poisson's formula (3.1). c.q.f.d.

Now we return to our Green's functions $g_n(z)$ and Robin constants λ_n for $(\Omega_n, \{\zeta\})$. Let us prove ① and ②: Assume that $\exists\, z_0 \in D - \{\zeta\}$ $\ni$ $g(z_0) < +\infty$.

Take any Ω_q ∋. $\Omega_q \ni z_0$, and put $u_n(z) = g_n(z) - g_q(z)$ for $z \in \Omega_q$. Then $\{u_n(z)\}$ is an increasing sequence of harmonic functions in the whole Ω_q with $u_n(\zeta) = \lambda_n - \lambda_q$. Since $\{u_n(z_0)\} = \{g_n(z_0) - g_q(z_0)\}$ converges, it follows from Theorem 3.1 that $\{u_n(z)\}$ uniformly converges on any compact set in Ω_q, and the limiting function $u(z)$ is harmonic in Ω_q. This means that $\{g_n(z)\}$ converges $g_q(z) + u(z)$ $(= g(z))$ in $\Omega_q - \{\zeta\}$ and $g(z)$ is harmonic in $\Omega_q - \{\zeta\}$. Moreover we get

$$\lambda = \lim_{n\to\infty} \lambda_n = u(\zeta) + \lambda_q$$

Since Ω_q is arbitrary we conclude that $g(z) < +\infty$ for any $z \in D - \{\zeta\}$ and that $\lambda < \infty$.

Conversely, assume that $\lambda < +\infty$. Then under the above situations, this implies $\{u_n(z)\}$ converges at ζ. It follows from Theorem 3.1 that $\{u_n(z)\}$ uniformly converges on any compact set in Ω_q. We conclude that $g(z) < +\infty$ in $z \in D - \{\zeta\}$.

By these argument we see that, for a domain D in $\mathbb{C}$, if $\lambda < +\infty$, then $g(z)$ is harmonic in $D - \{\zeta\}$ and we have

$$g(z) = \log \frac{1}{|z-\zeta|} + \lambda + h(z)$$

where $h(z)$ is harmonic in D and $h(\zeta) = 0$.
if $\lambda = +\infty$, then $g(z) \equiv +\infty$ in $D - \{\zeta\}$.

Anyhow, we defined the Green's function $g(z)$ and the Robin constant for any $(D, \{\zeta\})$. For definition, we need two certifications as follows:

(1) The limits $g(z) = \lim_{n\to\infty} g_n(z)$ and $\lambda = \lim_{n\to\infty} \lambda_n$ do not depend on the choice of $\{\Omega_n\}$;

(2) If D is a domain of $\mathbb{C}$ with smooth boundary, then the green's function $g(z)$ and the Robin constant λ for $(D, \{\zeta\})$ are identical with the limits $\lim_{n\to\infty} g_n(z)$ and $\lim_{n\to\infty} \lambda_n$, respectively where $g_n(z)$ and λ_n are the Green's function and the Robin constant for $(\Omega_n, \{\zeta\})$ such that

$$\zeta \in \Omega_1 \Subset \Omega_2 \Subset \cdots ; \quad \bigcup_{n=1}^{\infty} \Omega_n = D$$

Ω_n having smooth boundary.

In fact, (1) is easily proved by the maximum principle for harmonic functions. For the second, put

$$g^*(z) = \lim_{n\to\infty} g_n(z) \quad \text{and} \quad \lambda^* = \lim_{n\to\infty} \lambda_n .$$

Since $\Omega_n \Subset D$, we have $g^*(z) \leqq g(z)$ in $D - \{\zeta\}$ and $\lambda^* \leqq \lambda$. Let $\varepsilon > 0$ be given and put

$$D_\varepsilon = \{ z \in D \mid g(z) > \varepsilon \}.$$

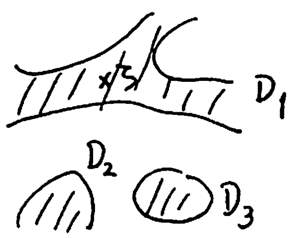
Then by (3.4) we have $D_\varepsilon \Subset D$, so that $\exists n$ s.t. $\Omega_n \Supset D_\varepsilon$. It follows from the maximum principle that

$$g_n(z) > g(z) - \varepsilon \quad \text{in } D_\varepsilon - \{\zeta\}$$

and hence $g^*(z) > g(z) - \varepsilon$ in $D_\varepsilon - \{\zeta\}$. Since $D_\varepsilon \searrow D$ by (ii) for Green's function, we conclude that

$$g^*(z) \geqq g(z) \text{ in } D - \{\zeta\} . \qquad \text{q.e.d}$$

Finally, let D be an open set in $\mathbb{C}$ and let $\zeta \in D$.



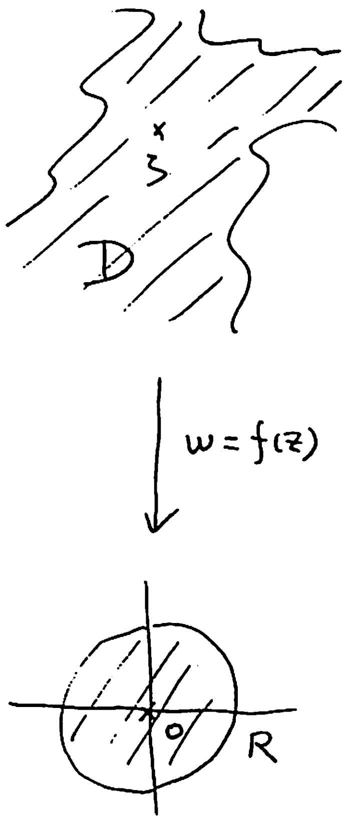
We write $D = \bigcup_{i=1}^{\infty} D_i$ (connected components). Say $\zeta \in D_1$. We thus have the Green's function $g_1(z)$ and the Robin constant λ for $(D_1, \{\zeta\})$.

We put $\quad g(z) = \begin{cases} g_1(z) & \text{in } D_1 \\ 0 & \text{in } D_2 \cup D_3 \cup \cdots \end{cases} \quad ; \lambda = \lambda_1 .$

and call $g(z)$ and λ the Green's function and the Robin constant for $(D, \{\zeta\})$, respectively.

Let us show some geometrical meanings of the Robin constant:

Exemple 3.2.



Suppose that D is a simply connected domain in $\mathbb{C}$. Let $\zeta \in D$. Then by Riemann's mapping theorem, we have an analytic function $f(z)$ in D such that

$f(\zeta) = 0 ; f'(\zeta) = 1 ; \quad f(D) = \{|w| < R\}$

where $^{\exists} R > 0$.

It is easily verified

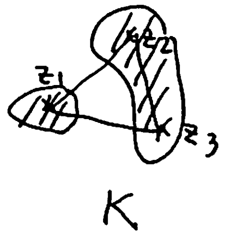
$$g(z) = \log \frac{R}{|f(z)|} \quad \text{in } D$$

where $g(z)$ denotes the Green's function for $(D, \{\zeta\})$. It follows that

$$\lambda = \log R, \quad \text{or} \quad R = e^{\lambda}.$$

To show another meaning, in general, let K be a compact set in $\mathbb{C}$, and $n \geqq 2$ an integer.

Define



$$V_n = \max_{z_1, \cdots, z_n \in K} \prod_{i \neq j} |z_i - z_j|$$

$$= \begin{array}{l} |z_1 - z_2| \cdots\cdots |z_1 - z_n| \\ \quad |z_2 - z_3| \cdots |z_2 - z_n| \\ \qquad \ddots \quad \vdots \\ \qquad\quad |z_{n-1} - z_n| \end{array}$$

That is, V_n is the modul of $n(n-1)/2$ products of differences.
Moreover put

$$d_n = \sqrt[\binom{n}{2}]{V_n} \qquad \text{where } \binom{n}{2} = n(n-1)/2$$

d_n is called the n^{th} diameter of K.

Lemma 3.1 (M. Fekete, 192)

$$d_{n+1} \geqq d_n \ (n=2,3,\cdots) \ ; d_\infty = \lim_{n\to\infty} d_n \text{ exists.}$$

Proof. For $n+1$, we find $z_1^0, \cdots, z_{n+1}^0$ in K such that

$$V_{n+1} = \prod_{\substack{i\neq j \\ i,j=1}}^{n+1} |z_i^0 - z_j^0|.$$

We have

$$V_{n+1} = |z_1^0 - z_2^0| \cdots |z_1^0 - z_{n+1}^0| \prod_{\substack{i\neq j \\ i,j=2}}^{n+1} |z_i^0 - z_j^0|$$

$$\leqq |z_1^0 - z_2^0| \cdots |z_1^0 - z_{n+1}^0| \cdot V_n$$

Analogously,

$$V_{n+1} \leqq |z_2^0 - z_1^0| \cdots |z_2^0 - z_{n+1}^0| \cdot V_n$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$V_{n+1} \leqq |z_{n+1}^0 - z_1^0| \cdots |z_{n+1}^0 - z_n^0| \cdot V_n$$

It follows that

$$(V_{n+1})^{n+1} \leqq (V_{n+1})^2 (V_n)^{n+1}$$

$$\therefore (V_{n+1})^{n-1} \leqq (V_n)^{n+1} \quad (\because V_{n+1}=0 \text{ implies } d_{n+1} \leqq d_n)$$

This means $d_{n+1} \leqq d_n$ c.q.f.d.

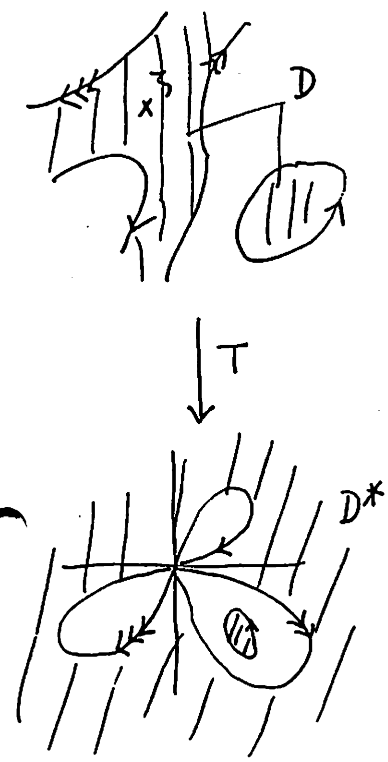
We call the limit d_∞ the transfinite diameter of K.

If K is n-distinct points, then $d_n > 0$ and $d_{n+1} = 0$.
Converse is also true.

「If K is countable, then $d_\infty = 0$」 will be proved later.

Now we return to the argument of the Robin constant.

Let D be an open set in $\mathbb{C}$ and $\zeta \in D$. Then we have the Green's function $g(z)$ and the Robin constant λ_D for $(D, \{\zeta\})$ (they may happen to be $+\infty$).



Consider the analytic mapping

$$T : w = \frac{1}{z-\zeta} : D \longrightarrow D^*$$

Put $K = \mathbb{C} - D^*$. Then K becomes a compact set in the complex w-plane $\mathbb{C}$, so that we have the transfinite diameter $d_\infty(K)$.

Under these notations, we have the following geometric meaning of Robin constant:

<u>Theorem</u> 3.2 (G. Szegö, 1924)

$$\lambda_D = \log \frac{1}{d_\infty(K)}$$

We shall prove it in §5. In the next §4, using this Theorem we show the main theorem in Chapter I.

§4 Main Theorem in Ch. I

Let B be a region in the complex t-plane, $\mathbb{C} = \{|z|<\infty\}$ the complex z-plane. In the product space $B\times\mathbb{C}$, consider a domain $\mathcal{D}$. For each $t\in B$, we put

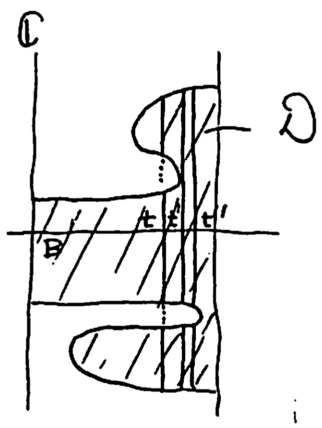
$$D(t) = \{ w\in\mathbb{C} \mid (t,w)\in\mathcal{D} \}$$

and call it the fiber of $\mathcal{D}$ at t. Therefore

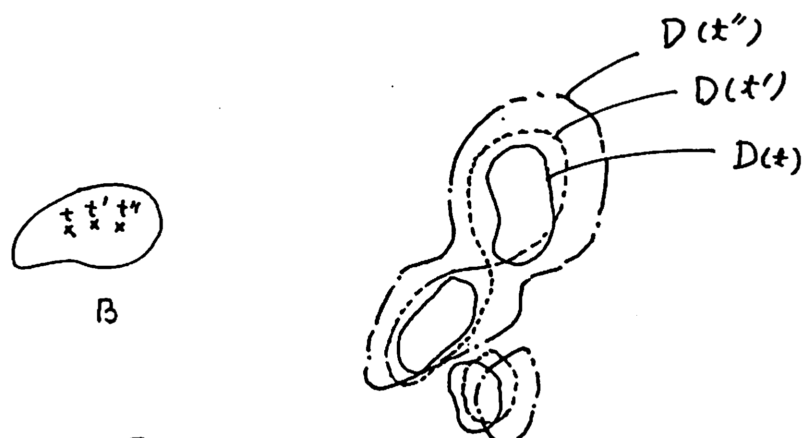
$$\mathcal{D} = \bigcup_{t\in B} (t, D(t))$$

We sometimes regard $\mathcal{D}$ as a variation of domain $D(t)$ of $\mathbb{C}$ with complex parameter t in B, and write it

$$\mathcal{D} : t \longrightarrow D(t) \quad (t\in B)$$



model



Reality

We pose the Assumption to $\mathcal{D}$:

(4.1) There exists a point $\zeta\in\mathbb{C}$ such that $B\times\{\zeta\}\subset\mathcal{D}$.

That is, $\zeta\in D(t)$ for all $t\in B$. We thus have the Green's function $g(t,z)$ and the Robin constant $\lambda(t)$ for $(D(t),\{\zeta\})$, respectively.

Theorem 4.1 (1971)

If $\mathcal{D}$ is a domain of holomorphy in $\mathbb{C}^2$, then $\lambda(t)$ is superharmonic in B.

That is, $\lambda(t) \equiv +\infty$ in B,

or $-\infty < \lambda(t) \leqq +\infty$ in B such that

$$\begin{cases} \text{(i) } \lambda(t) \text{ is lower semicontinuous in } B \text{, i.e.,} \\ \qquad \varliminf_{t \to t_0} \lambda(t) \geqq \lambda(t_0) \quad \text{for } \forall t_0 \in B \\ \text{(ii) } \quad \lambda(t_0) \geqq \frac{1}{2\pi}\int_0^{2\pi} \lambda(t_0 + re^{i\theta})\,d\theta \end{cases}$$

for $\forall t_0 \in B$, $0 < \forall r \ll 1$

Let $w = \varphi(z) = 1/(z-\zeta)$ and put for each $t \in B$

$$D^*(t) = \varphi(D(t)) \quad \text{and} \quad K(t) = \mathbb{C}_w - D^*(t).$$

Since $K(t)$ is compact, we have the transfinite diameter of $K(t)$ which we denote by $d_\infty(t)$. By Theorem 2.2

$$\lambda(t) = -\log d_\infty(t)$$

Hence Theorem 4.1 is equivalent to

<u>Theorem 4.1′</u> $\log d_\infty(t)$ is subharmonic in B.

In order to prove this, it suffices to prove

<u>Lemma 4.1</u> Let $n \geqq 2$. We denote by $d_n(t)$ the n^{th}-diameter of $K(t)$. Then $\log d_n(t)$ is subharmonic in B.

∵ Assume Lemma 4.1 was proved. Then $d_n(t) \searrow d_\infty(t)$ for all $t \in B$ decreasingly, and hence the subharmonicity of $\log d_n(t)$ in B is passed down to $\log d_\infty(t)$ in B.

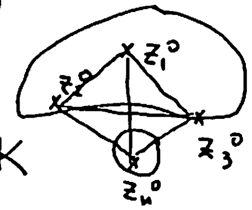
Before beginning to prove Lemma 4.1 we show two

<u>Remark 4.1</u> Let K be a compact set in $\mathbb{C}$. Take $z_1^0, \dots, z_n^0$ $(n \geqq 2)$ in K such that

$$d_n(K) = \prod_{i \neq j}^{n} |z_i^0 - z_j^0|.$$

Then each z_i^0 $(i = 1, \dots, n)$ belongs to ∂K.

∵ We prove it by contradiction. Say $z_1^0 \in K^0 = K - \partial K$. Consider the analytic fn:

K 

$$\varphi(z) = (z - z_2^0)\cdots(z - z_n^0)\prod_{\substack{i \neq j \\ i,j \geq 2}}^{n}(z_i^0 - z_j^0)$$

in K^0. By maximum principle we have

$z^* \in \partial K^0 \ni \cdot\ |\varphi(z^*)| \gneqq |\varphi(z_1^0)| = V_n(K)$, where $V_n(K) = d_n(K)^{\binom{n}{2}}$.

Since $\partial K^0 \subset K$, we have $V_n(K) \geq |\varphi(z^*)| > V_n(K)$, which is a contradiction.

<u>Remark 4.2</u> Let K_i $(i = 1, 2, \cdots)$ be compact, and $K_1 \supset K_2 \supset \cdots$. Put $K = \bigcap_{i=1}^{\infty} K_i$ and $n \geq 2$ an integer. Then

$$d_n(K_i) \searrow d_n(K)$$

, where $d_n(K_i)$ is the n^{th} diameter of K_i.

∵ By definition, we have $d_n(K_1) \geq d_n(K_2) \geq \cdots \geq d_n(K)$ so that we have $d_n^* = \lim_{\nu\to\infty} d_n(K_\nu) \geq d_n(K)$.

For the converse inequality, for each $\nu\ (=1, 2, \cdots)$ we take

$\exists\ z_1^{(\nu)}, \cdots, z_n^{(\nu)} \in K_\nu$ such that

$$V_n(K_\nu) = \prod_{i\neq j}^{n} |z_i^{(\nu)} - z_j^{(\nu)}|.$$

Since K_1 is compact, we have a subsequence of $(z_1^{(\nu)}, \cdots, z_n^{(\nu)})$: such that $(z_1^{(\nu_p)}, \cdots, z_n^{(\nu_p)}) \longrightarrow (z_1^*, \cdots, z_n^*)$ as $p \to \infty$.

It follows that each $z_i^* \in K$ $(i = 1, \cdots, n)$ and that

$$V_n(K_{\nu_p}) \longrightarrow \prod_{i\neq j} |z_i^* - z_j^*| \quad (\leqq V_n(K))$$

Consequently, $V_n(K) \geq (d_n^*)^{\binom{n}{2}}$ $\quad \therefore\ d_n(K) = d_n^*$.

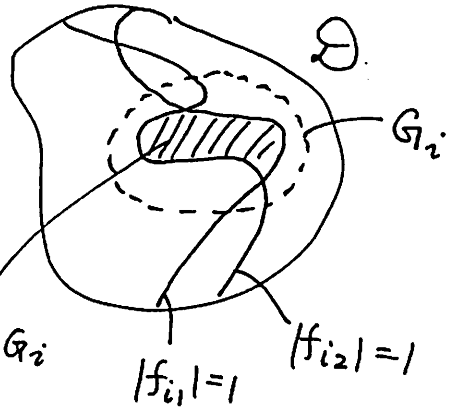
<u>Proof of Lemma 4.1</u> Let $B_0 \Subset B$ a region. Let us show that $\log d_n(t)$ is subharmonic in B_0.

Since $\mathcal{D}$ is a domain of holomorphy in $\mathbb{C}^2$, we find a sequence of analytic polyedral $\{P_i \cap G_i\}$ such that

$$B_0 \times \{\zeta\} \Subset P_1 \cap G_1 \Subset P_2 \cap G_2 \Subset \cdots\cdots ; \quad \bigcup_{i=1}^{\infty} P_i \cap G_i = \mathcal{D}$$

where

$$\begin{cases} P_i = \bigcap_{j=1}^{\nu_i} \{(t,z) \in \mathcal{D} \mid |f_{ij}(t,z)| < 1\} ; \\ f_{ij}(t,z) \quad (j=1,\cdots,\nu_i) \text{ is holo. fn in } \mathcal{D} ; \\ G_i \text{ is open in } \mathcal{D} ; \\ P_i \cap G_i \Subset G_i \end{cases}$$



Last condition means that for $\forall (t,z) \in \partial(P_i \cap G_i)$, there exists $1 \leq j \leq i_\nu$ $\ni$ $|f_{ij}(t,z)| = 1$

Let $T : \begin{cases} t = t \\ w = \varphi(z) = \dfrac{1}{z-\zeta} \end{cases}$

which is a analytic transformation of $B \times (\mathbb{C}_z \cup \{\infty\})$ to $B \times (\mathbb{C} \times \{\infty\})$

We write simply $\mathbb{C}_w \cup \{\infty\} = \overline{\mathbb{C}_w}$.

and $\mathcal{D}^* = T(\mathcal{D})$; $G_i^* = T(G_i)$; $F_{ij}(t,w) = f_{ij}(t,z)$ where $w = \varphi(z)$; $\mathcal{P}_i^* = T(\mathcal{P}_i)$.

Hence $$\mathcal{P}_i^* = \bigcap_{j=1}^{\nu_i} \{(t,z) \in \mathcal{D}^* \mid |F_{ij}(t,w)| < 1\} ;$$

$B_0 \times \{\infty\} \Subset \mathcal{P}_i^* \cap G_i^* \Subset G_i^*$;

For $\forall (t,w) \in \partial(P_i^* \cap G_i^*)$, we have $1 \leq {}^\exists j \leq \nu_i$ such that

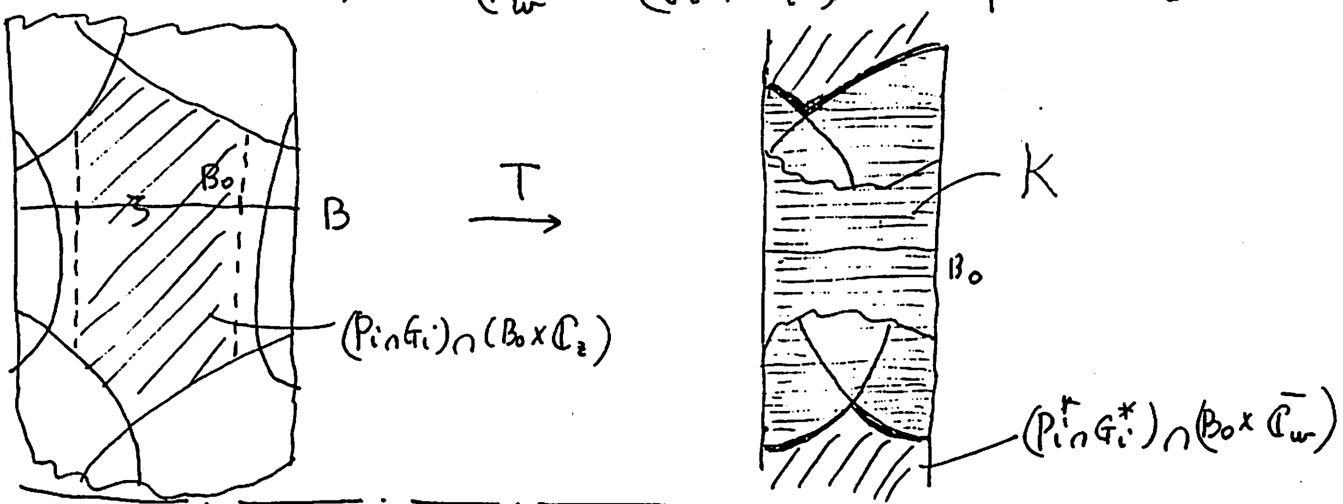
$$|F_{ij}(t,w)| = 1$$

We put

$$K_i = B_0 \times \overline{\mathbb{C}}_w - (\mathcal{P}_i^* \cap G_i^*) \qquad (i=1,2,\cdots)$$

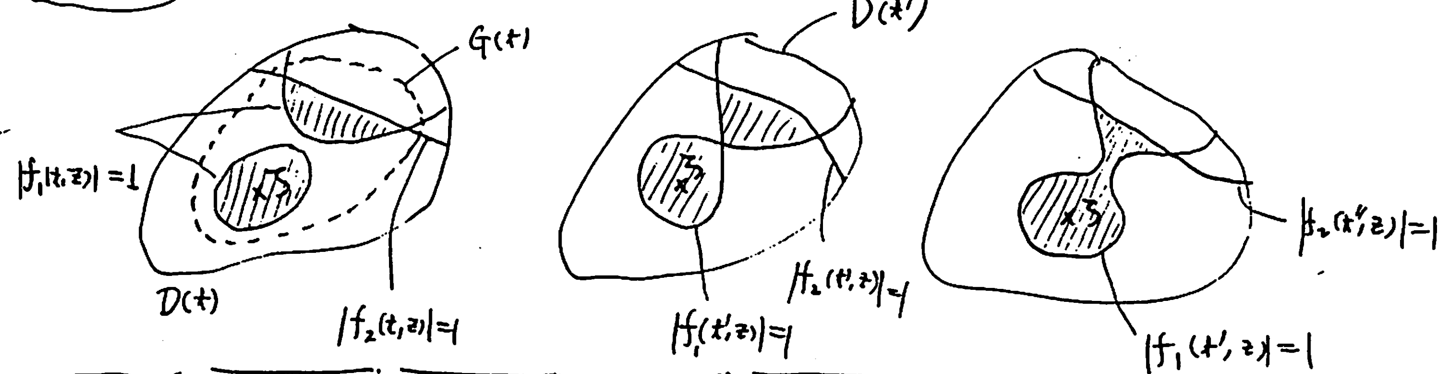
$\therefore K_i = \bigcup_{t \in B_0} (t, K(t))$

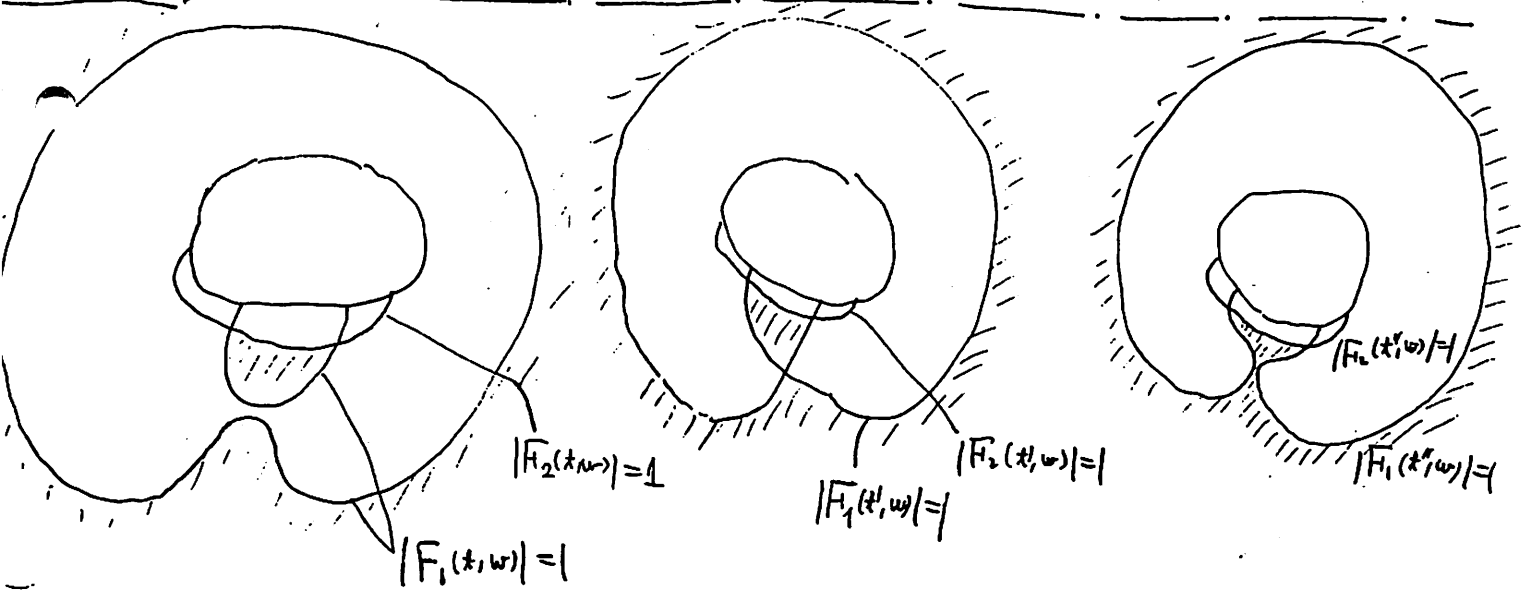
where $K(t) = \overline{\mathbb{C}_w} - (\mathcal{P}_i^* \cap G_i^*)(t)$ for $t \in B_0$



$t\ t'\ t''$ B_0

Variation: $t \to \mathcal{P}_i(t) \cap G_i(t)$ $(t \in B_0)$





Variation: $t \to \mathcal{P}_i^*(t) \cap G_i^*(t)$ $(t \in B_0)$

Since $(\mathcal{P}_i^* \cap G_i^*)(t) \nearrow D^*(t)$, we have

$$K_i(t) \longrightarrow K(t) = \overline{\mathbb{C}}_w - D^*(t).$$

By Remark 5.1, we thus have

$$d_n(K_i(t)) \longrightarrow d_n(K(t)) \quad (\nu \to \infty) \quad \text{for } {}^\forall t \in B_0.$$

Therefore, it suffices for Lemma 4.1 to prove

「 Let $n \geq 2$ and $i \geq 1$ be fixed. Then $\log d_n(K_i(t))$ is subharmonic in B_0 」

From now on, for the sake of convenience we change the notations as follows:

$$\begin{cases} F_{ij}(t,w) \ (j=1,\cdots,i_\nu) & \rightarrow \quad F_j(t,w) \ (j=1,\cdots,\nu) \\ K_i & \longrightarrow \quad K \\ G_i^* & \longrightarrow \quad G \\ \mathcal{P}_i^* & \longrightarrow \quad \mathcal{P} \\ B_0 & \longrightarrow \quad B \end{cases}$$

Dear

Oct. 7, '87 (14)

These are a precise description of Yesterday's Lecture. Thank you,

Hiroshi Yamaguchi

1

Let B be a domain of the complex t-plane $\mathbb{C}_t$, and $\overline{\mathbb{C}}_w = \{|w| < \infty\} \cup \{\infty\}$ Riemann sphere.

We use sometimes the following notation:

Given $\mathcal{E} \subset B \times \overline{\mathbb{C}}_w$, we put for each $t \in B$

$$E(t) = \{ w \in \overline{\mathbb{C}}_w \mid (t, w) \in \mathcal{E} \}$$

and call it the fiber of $\mathcal{E}$ at t. Hence

$$\mathcal{E} = \bigcup_{t \in B} (t, E(t)).$$

We are now on the following situation:

Let G be a region such that

$$B \times \{\infty\} \subset G \subset B \times \overline{\mathbb{C}}_w .$$

Let $F_j(t, w)$ $(j = 1, \dots, \nu)$ be holomorphic function in G and put

$$\mathcal{P} = \bigcap_{j=1}^{\nu} \{ (t, w) \in G \mid |F_j(t, w)| < 1 \} ;$$

$$\Sigma_j = \{ (t, w) \in G \mid |F_j(t, w)| = 1 \}.$$

Therefore

$$\mathcal{P} = \bigcup_{t \in B} (t, P(t)) ; \quad P(t) = \bigcap_{j=1}^{\nu} \{ w \in \overline{\mathbb{C}}_w \mid |F_j(t, w)| < 1 \}.$$

Hereafter, we pose the following

Assumptions (i) $\mathcal{P} \supset B \times \{\infty\}$;

(ii) $G(t) \supset\supset P(t)$ for $\forall t \in B$.

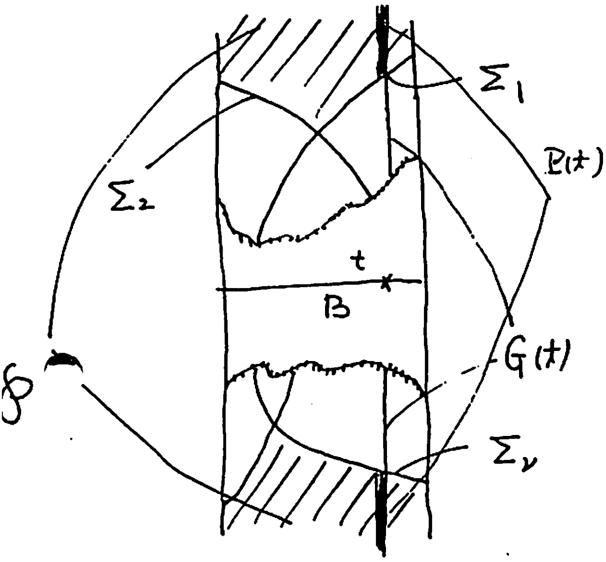
The assumption (ii) means that each $P(t)$ $(t \in B)$ is bounded by a finite number of analytic curves consisting of

[illegible]

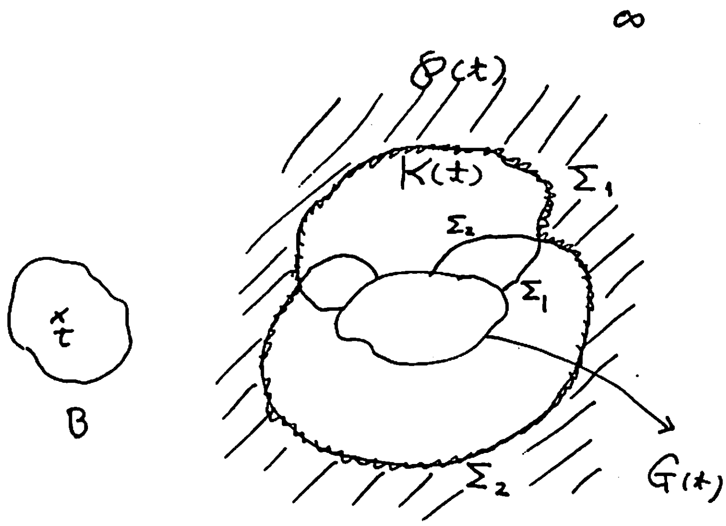
Ch. III でつかうように、書きかえること。とくに Remark については[illegible]。

parts of $\Sigma_j(t)$ $(j=1,\cdots,\nu)$. It follows that

$$(*)\quad \begin{cases} \text{for any } t_0 \in B \text{ and any } w_0 \in \partial P(t_0) \text{ we find} \\ \text{some } j \ (1 \leqq j \leqq \nu) \text{ such that } |F_j(t_0, w_0)| = 1 \end{cases}$$



Model



Moving pictures

Put $\quad \mathcal{K} = B \times \overline{\mathbb{C}_w} - \mathcal{P}$, so that

$$\mathcal{K} = \bigcup_{t \in B} (t, K(t)) \quad ; \quad K(t) = \overline{\mathbb{C}_w} - \mathcal{P}(t)$$

By the assumption (i), each $\mathcal{K}(t)$ is compact set in $\mathbb{C}_w$, and $\partial K(t) = \partial P(t)$ and $K \supset \Sigma_1 \cup \cdots \cup \Sigma_\nu$.

Fix $n (\geqq 2)$ an integer, we have the n^{th} diameter of $K(t)$:

$$d_n(t) = \sqrt[\binom{n}{2}]{V_n(t)} \quad ;$$

$$V_n(t) = \max_{w_i \in K_n(t)} \left(\prod_{i \neq j}^{n} |w_i - w_j| \right).$$

Under these circumstances, our aim is to show

(5.1) <u>$\log d_n(t)$ is subharmonic of t in B.</u>

Let us prove it by dividing two steps :

<u>1^{st} step.</u> $d_n(t)$ has no local maximum in B.

Precisely speaking, let $t_0 \in B$. Then there exists $\rho_0 > 0$ such that

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Either $\quad d_n(t_0) \leqq d_n(t) \quad$ for $\; |t - t_0| < \rho_0$,

Or $\quad d_n(t_0) \lneqq \underset{|t-t_0|=r}{\mathrm{Max}} \{d_n(t)\} \quad$ for $0 < {}^\forall r < \rho_0$

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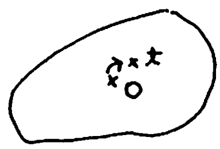
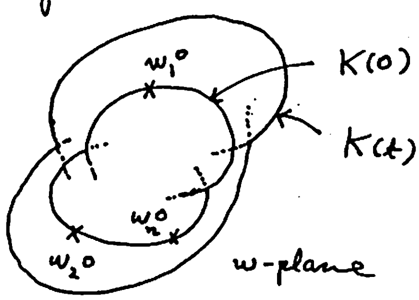
<u>Proof of 1^{st} step.</u> For the sake of convenience, we set $t_0 = 0$



B



$\mathcal{K} : t \longrightarrow K(t) \quad (t \in B)$

By the definition of $V_n(0)$, we find

$w_1^0, \dots, w_n^0 \in \partial K(0)$ such that $V_n(0) = \prod_{i \neq j}^{n} |w_i^0 - w_j^0|$

Since $\partial K(0) = \partial P(0)$ and each $w_i^0 \in \partial P(0)$, assertion(*) implies $\exists j \; (1 \leqq j \leqq \nu)$ such that

$$|F_j(0, w_0^i)| = 1.$$

By Corollary of Theorem 2.2, there exists

$$\sigma = \{(t, w) \in G \mid t = \tau^{p_i}, w = w_i(\tau), |\tau| < \rho_i\}$$

where $p_i \geqq 1$ integer, $w_i(\tau)$ is holomorphic there and $w_i(0) = w_i^0$

such that

$$(0, w_i^0) \in \sigma \subset \Sigma_j$$

In other words, we have an analytic set σ (of one dimensional) passing through the point $(0, w_i^0)$ in Σ_j

Since $\Sigma_j \subset \mathcal{K}$, we conclude that

$$w_i(\tau) \in K(\tau^{p_i}) \qquad (|\tau| < \rho_i)$$

Put $p = p_1 \cdots p_n$; $q = p/p_i$. Hence, setting $\tau = s^{q_i}$ for $|s| < \sqrt[q_i]{\rho_i}$ $(\underset{def}{=} \eta_i)$, we find

$$w_i(s^{q_i}) \in K(\underbrace{s^p}_{\text{independent of } i=1,2,\cdots,n}) \quad \text{for } \forall |s| < \eta_i$$

We put

$$\tilde{w}_i(s) = w_i(s^{q_i}) \quad \text{in } |s| < \eta_0 \left(\underset{def}{=} \min_{i=1,\cdots,n}\{\eta_i\}\right)$$

Then $\tilde{w}_i(s)$ is holomorphic in $|s| < \eta_0$
such that $\tilde{w}_i(0) = w_i^0$ & $\tilde{w}_i(s) \in K(s^p)$

Form the function

$$\psi(s) = \prod_{i \neq j}^{n} \left(\tilde{w}_i(s) - \tilde{w}_j(s)\right) \quad \text{in } |s| < \eta_0$$

which is holomorphic and $\psi(0) = \prod_{i\neq j}^{n} (w_i^0 - w_j^0)$.

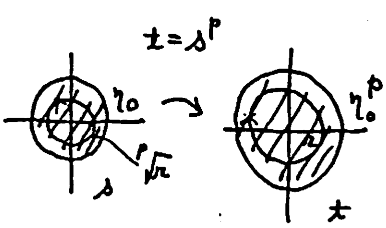
Let $0 < {}^\forall\eta' < \eta_0$. Then the Maximum principle for holomorphic functions, we see that

(a) If $\psi(s)$ is not constant in $|s| < \eta_0$, then

$$|\psi(0)| \lneqq \max_{|s|=\eta'} |\psi(s)|$$

By definition, $|\psi(0)| = V_n(0)$, and since $\tilde{w}_i(s) \in K(s^p)$ $(i=1,\cdots,n)$, we have $|\psi(s)| \leqq V_n(s^p)$

$$\therefore \quad V_n(0) \lneqq \max_{|s|=\eta'} V_n(s^p)$$



(b) If $\psi(s)$ is constant in $|s| < \eta_0$, then

$$|\psi(0)| = |\psi(s)| \text{ for all } |s| < \eta_0$$

$$\therefore \quad V_n(0) \leqq V_n(s^p) \text{ for all } |s| < \eta_0$$

It follows that, for any r $\ni$ $0 < r < \rho_0 (\underset{def}{=} \eta_0^p)$,

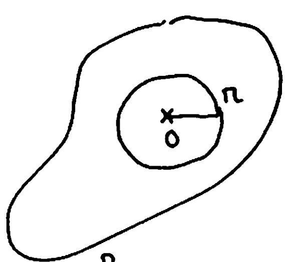
Either $\quad V_n(0) \leqq V_n(t)$ for $|t| < \rho_0$

or $\quad V_n(0) \lneqq \max_{|t|=r} \{V_n(t)\}$. $\qquad$ c.q.f.d of 1st st.

2nd step. $\log d_n(t)$ is subharmonic in B.

i.e., $\begin{cases} (1) & \log d_n(t) \text{ is uppersemicontinuous in } B; \\ (2) & (1/2\pi)\int_0^{2\pi} \log d_n(t_0 + re^{i\theta})d\theta \geqq \log d_n(t_0) \quad (0 < {}^\forall r \ll 1) \end{cases}$

For the sake of convenience, we put $t_0 = 0$.



B.

We can prove $\log d_n(t)$ is continuous at 0.

⌈ In fact, take $0 < r \ni (|t| < r) \Subset B$.

Then $K \cap [(|t| \leqq r) \times \mathbb{C}_w]$ is closed (compact) in $B \times \mathbb{C}_w$. Hence, by the definition of $V_n(t)$, we easily have

$$\overline{\lim_{t\to 0}}\, V_n(t) \leqq V_n(0).$$

Let us prove $\underline{\lim}_{t\to 0} V_n(t) \geqq V_n(0)$.

Take ${}^\exists w_1^0, \dots, w_n^0 \in \partial K(0) = \partial P(0)$ such that

$$V_n(0) = \prod_{i \neq j}^{n} |w_i^0 - w_j^0|$$

In each $1 \leqq i \leqq n$, we find $1 \leqq {}^\exists j \leqq \nu \ni |F_j(0, w_i^0)| = 1$, so that

$$\underset{\substack{\| \\ \text{an analytic set passing through } (0,w_i^0) \text{ in } \Sigma_j}}{\exists\, \sigma} : (0, w_i^0) \in \sigma \subset \Sigma_j \subset K$$

$\{(sp, \tilde{w}_i(s)) \mid |s| < \eta_0\}$ where $\tilde{w}_i(0) = w_i^0$

$\therefore \tilde{w}_i(s) \in K(sp)$ for ${}^\forall |s| < \eta_0$

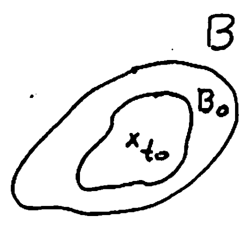
$$\therefore \underline{\lim_{t\to 0}}\, V_n(t) = \underline{\lim_{s\to 0}}\, V_n(sp) \geqq \lim_{s\to 0} \prod_{i\neq j}^{n} |\tilde{w}_i(s) - \tilde{w}_j(s)| = V_n(0) \;⌋$$

Since 0 is arbitrary, $\log d_n(t)$ is continuous in B. It follows from 1st step that

Remark 5.1.

Let $t_0 \in B$, and $t_0 \in B_0 \Subset B$. Then

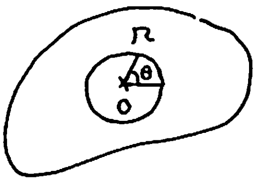
$$d_n(t_0) \leqq \max_{t \in \partial B_0} \{d_n(t)\}.$$



Let us prove (2) by following F. Hartogs.

Now assume that (2) is not true, i.e.,

$$\frac{1}{2\pi}\int_0^{2\pi} \log d_n(re^{i\theta})\,d\theta \lneqq \log d_n(0) \qquad (0 <^{\exists} r \ll 1).$$



Take $0 <^{\exists} \varepsilon < \log d_n(0) - \frac{1}{2\pi}\int_0^{2\pi} \log d_n(re^{i\theta})\,d\theta$.

Form the Poisson integral:

$$u(t) = \frac{1}{2\pi}\int_0^{2\pi} \log d_n(re^{i\theta})\, P(re^{i\theta}, t)\,d\theta$$

where $$P(re^{i\theta}, t) = \frac{r^2 - |t|^2}{|re^{i\theta} - t|^2} \qquad (0 \leq \theta \leq 2\pi)$$

Since $\log d_n(t)$ is continuous on $|t| = r$, it is known that

$$(5.2)\quad \begin{cases} u(t) \text{ is harmonic in } |t| \lneqq r \\ u(re^{i\theta}) = \log d_n(re^{i\theta}) \text{ continuously for } 0 \leq {}^{\forall}\theta \leq 2\pi. \\ u(0) = \frac{1}{2\pi}\int_0^{2\pi} \log d_n(re^{i\theta})\,d\theta < \log d_n(0) - \varepsilon \end{cases}$$

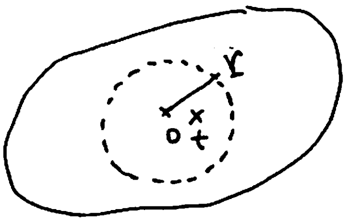
Consider the function

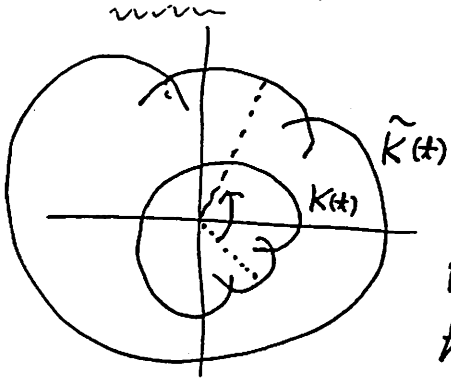
$$\varphi(t) = u(t) + i u^*(t) \quad \text{in } (|t| \lneqq r)$$

where $u^*(t)$ is a harmonic conjugate of $u(t)$ in $(|t| < r)$, so that $\varphi(t)$ is holomorphic in $(|t| < r)$.

Construct the transformation

$$T : \begin{cases} t = t \\ W = e^{-\varphi(t)} w \end{cases} : \underset{\sim\sim\sim}{(|t| < r) \times \mathbb{C}_w} \longrightarrow \underset{\sim\sim\sim}{(|t| < r) \times \mathbb{C}_W}$$





For each fixed t, Rotation and homothetic transformt.

Put $\tilde{K} = T(K \cap (|t|<r) \times \mathbb{C}_w)$

$\tilde{K}(t) = \{ e^{-\varphi(t)} w \mid w \in K(t) \}$ for $|t|<r$.

Since $\tilde{K}(t) \subset\subset \mathbb{C}_W$ (compact), we have the n^{th} diameter $\tilde{d}_n(t)$ of $\tilde{K}(t)$. Then we find easily

$$(5.3) \qquad \tilde{d}_n(t) = e^{-u(t)} d_n(t) \qquad (|t|<r)$$

where $e^{-u(t)} = |e^{-\varphi(t)}|$.

On the other hand,

$\tilde{K}$ in $(|t|<r) \times \mathbb{C}_W$ has the same property as K.

that is, given $\forall t_0 \in (|t|<r)$, $\forall W_0 \in \partial \tilde{K}(t_0)$

there exists an analytic set $\tilde{\sigma}$ passing through (t_0, W_0) in $\tilde{K}$,

$(t_0, W_0) \in \tilde{\sigma} \subset \tilde{K}$

It follows that $\tilde{d}_n(t)$ has the same property of 1st step as $d_n(t)$. Since $\tilde{d}_n(t)$ as well as $d_n(t)$ is continuous in $|t| \leq r$, we have by Remark 5.1

$$(5.4) \qquad \tilde{d}_n(0) \lneqq \max_{|t|=r'} \tilde{d}_n(t) \quad \text{for } 0 < \forall r' < r.$$

From (5.2) and (5.3) we get

$\log \tilde{d}_n(0) = -u(0) + \log d_n(0) > \varepsilon$

$\log \tilde{d}_n(re^{i\theta}) = -u(re^{i\theta}) + \log d_n(re^{i\theta}) = 0$ continuously.

It follows there exists r' $(0<r'<r)$ sufficiently close to r such that $\log \tilde{d}_n(r'e^{i\theta}) < \varepsilon/2$ $(0 \leq \forall\theta \leq 2\pi)$

We reach the following contradiction to (5.1):

$$\tilde{d}_n(0) > e^{\varepsilon} \; ; \quad \max_{|t|=r'} \tilde{d}_n(t) < e^{\varepsilon/2}$$

c.q.f.d.

Dear ________ (51) Oct. 19, '87

These are details of the lecture in last week.
Although they are lengthy, the contents are elementary.

Thank you, Hiroshi Yamaguchi

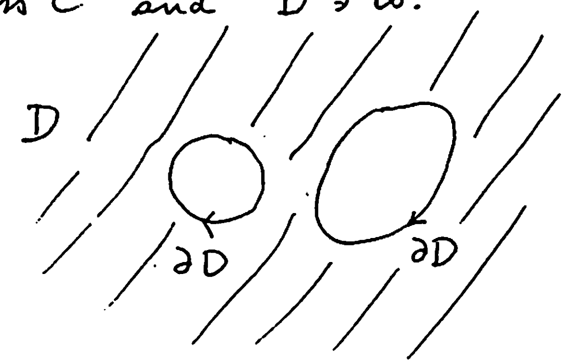
1

Next, we prove the converse inequality:

$$(5.1) \qquad \lambda \leqq -\log \rho$$

which is the most interesting part of Szegö's Theorem.
For this, he used one of the old formulas — Equilibrium Potential —

We always assume that the bdry of D is smooth of class C^∞ and $D \ni \infty$.



We put $K = \overline{\mathbb{C}}_z - D$, which is a compact set.

Let $\rho(\zeta) > 0$ be a continuous fn on ∂D with

$$\int_{\partial D} \rho(\zeta)\, ds_\zeta = 1$$

where ds_ζ is the arc length element of ∂D,

and form the logarithmic potential

$$N_\rho(z) = \int_{\partial D} \left(\log \frac{1}{|z-\zeta|}\right) \rho(\zeta)\, ds_\zeta \qquad \text{for } z \in \mathbb{C}_z$$

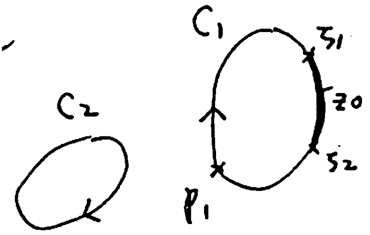
It is easily proved that

(1) $N_\rho(z)$ is continuous and superharmonic in $\mathbb{C}_z$;

(2) $N_\rho(z)$ is harmonic in D and K°

(3) $\lim_{z\to\infty} \left(N_\rho(z) - \log \frac{1}{|z|}\right) = 0$

For example, let us show the existence of $N_\rho(z_0)$ at $z_0 \in \partial D$:



Let $\partial D = C_1 \cup \dots \cup C_q$ (closed contours)

Say $z_0 \in C_1$. Take and fix $p_1 \in C_1$ with $p_1 \neq z_0$. We choose $\zeta_1, \zeta_2 \in C_1$ such that

$$C_1 \supset \widehat{\zeta_1 \zeta_2} \ni z_0 \quad \text{and}$$
$$|\zeta - z_0| < 1 \quad \text{for any } \zeta \in \widehat{\zeta_1 \zeta_2}$$

It suffices to prove

$$\int_{\zeta_1}^{\zeta_2} \left(\log \frac{1}{|\zeta - z_0|}\right) \rho(\zeta)\, ds < +\infty .$$

To prove this, put

$$x = f(\zeta) = \int_{p_1}^{\zeta} \rho(\zeta)\, ds_\zeta$$

so that $f(\zeta) : C_1 \longrightarrow [0, a_1)$

where $f(p_1) = 0$ and $a_1 = \int_{C_1} \rho(\zeta)\, ds_\zeta$

Since C_1 is smooth, $\rho(\zeta) > 0$ is continuous, this correspondance is one to one and of class C^1. It follows that, if we put

$x_1 = f(\zeta_1)$, $x_2 = f(\zeta_2)$, $x_0 = f(z_0)$ and $x = f(\zeta)$ for $\zeta \in \widehat{\zeta_1 \zeta_2}$, then we find $M \gg 1$ independent of $\zeta \in \widehat{\zeta_1 \zeta_2}$ such that

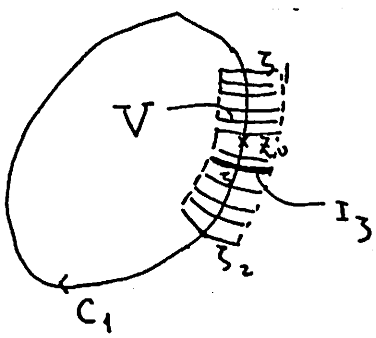
$$|\zeta - z_0| \geq \frac{1}{M} |x - x_0|$$

so that

$$\int_{\zeta_1}^{\zeta_2} \left(\log \frac{1}{|\zeta - z_0|}\right) \rho(\zeta)\, ds \leq \int_{x_1}^{x_2} \left[(\log M) + \log \frac{1}{|x - x_0|} \right] dx < +\infty .$$

Next, let us show the continuity at $z_0 \in \partial D$:

We use the same notations as above. For any $\zeta \in \widehat{\zeta_1 \zeta_2}$, draw an outer normal n_ζ at ζ to C_1 and put



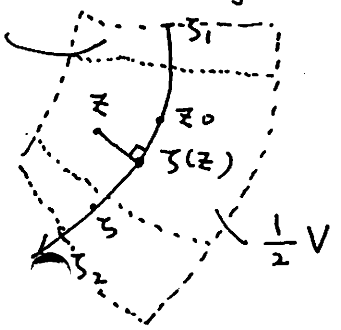
$$I_\zeta = (-\delta n_\zeta, \delta n_\zeta)$$

where $0 < \delta \ll 1$;

$$V = \bigcup_{\zeta \in \widehat{\zeta_1 \zeta_2}} I_\zeta$$

Then V becomes a neighborhood of z_0 in $\mathbb{C}$, which contains $\widehat{\zeta_1\zeta_2}$.

Let $\forall z \in V$. Then we find unique $\zeta(z) \in \widehat{\zeta_1\zeta_2}$ such that $I_{\zeta(z)}$ contains z. Since C_1 is smooth, we find a $m \gg 1$ such that



$$|z-\zeta| \geqq \frac{1}{m}|\zeta(z)-\zeta| \quad \text{for } z \in \tfrac{1}{2}V \text{ and } \zeta \in \widehat{\zeta_1\zeta_2}$$

where m is independent of z and ζ.

We already have $|\zeta'-\zeta| \geqq \frac{1}{M}|f(\zeta')-f(\zeta)|$.
Here M is independent of $\zeta', \zeta' \in \widehat{\zeta_1\zeta_2}$.

$$\therefore\ |z-\zeta| \geqq \frac{1}{mM}|x'-x|$$

where $x' = f(\zeta(z))$ and $x = f(\zeta)$, $x', x \in [x_1, x_2]$.

so that

$$\int_{\zeta_1}^{\zeta_2}\left(\log\frac{1}{|z-\zeta|}\right)\rho\, ds \leqq mM\int_{x_1}^{x_2}\left(\log\frac{1}{|x'-x|}\right)dx$$

$$\leqq |x_2-x'|\log\frac{1}{|x_2-x'|} + |x_1-x'|\log\frac{1}{|x_1-x'|} + (x_2-x_1).$$

$$\leqq (x_2-x_1)\log\frac{1}{(x_2-x_1)} + (x_2-x_1)$$

Therefore, if ζ_2 and ζ_1 are close to z_0, then the integral $\int_{\zeta_1}^{\zeta_2}(\log\frac{1}{|z-\zeta|})\rho(\zeta)ds_\zeta$ is small for $z \in (1/2)V$.

That is, given $\varepsilon > 0$, choose a neighborhood $(1/2)V$ as above such that

$$(0<)\int_{\zeta_1}^{\zeta_2}\left(\log\frac{1}{|z-\zeta|}\right)\rho\, ds < \varepsilon \quad \text{for all } z \in (1/2)V.$$

Since $\log\frac{1}{|z-\zeta|}$ is uniformly continuous for $(z,\zeta) \in ((1/2)V, \partial D - (\widehat{\zeta_1\zeta_2}))$, it follows that there exists a neighborhood $V^* \subset \frac{1}{2}V$

of z_0 such that

$$\left| \int_{\partial D - \widehat{\zeta_1 \zeta_2}} \left(\log \frac{1}{|z-\zeta|}\right) \rho \, ds - \int_{\partial D - \widehat{\zeta_1 \zeta_2}} \left(\log \frac{1}{|z_0-\zeta|}\right) \rho \, ds \right| < \varepsilon$$

for all $z \in V^*$. Consequently,

$$|N_\rho(z) - N_\rho(z_0)| = \left| \int_{\partial D} \log \frac{1}{|z-\zeta|} \rho \, ds - \int_{\partial D} \log \frac{1}{|z_0-\zeta|} \rho \, ds \right| < 2\varepsilon \quad \text{for any } z \in V^*,$$

which shows the continuity of $N_\rho(z)$ at $z_0 \in \partial D$.

c. q. f. d.

Equilibrium potential problem is to construct $\rho_0(\zeta) > 0$ continuous on ∂D with $\int_{\partial D} \rho_0 \, ds = 1$ such that

$$N_{\rho_0}(z) \equiv \text{const. } k \text{ on } K.$$

Then $\rho_0(\zeta) ds_\zeta$ and $N_{\rho_0}(z)$ are called equilibrium distribution and potential with respect to K.

Now suppose that such ρ_0 exists. Then it is clear that

$$k - N_{\rho_0}(z)$$

is not other than the $\overbrace{\text{green's function}}^{g(z)}$ for (D, ∞), and hence that k is equal to the Robin constant λ for (D, ∞).

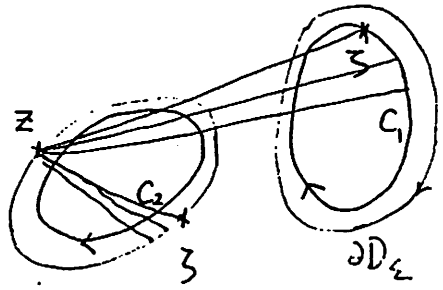
$$g(z) = k - N_{\rho_0}(z) \quad ; \quad \lambda = k$$

That is,

$$(5.2) \quad \boxed{\int_{\partial D} \left(\log \frac{1}{|z-\zeta|}\right) \rho_0(\zeta) \, ds_\zeta = \begin{cases} \lambda - g(z) & (z \in D) \\ \lambda & (z \in K) \end{cases}}$$

Using this formula, (5.1) is proved as follows:

Let $\varepsilon > 0$ be given apriori.



We put

$$D_\varepsilon = \{ z \in D \mid g(z) > \varepsilon \}$$
$$K_\varepsilon = \overline{\mathbb{C}}_z - D_\varepsilon$$

We write

$$\partial D = C_1 \cup \cdots \cup C_q \quad (\text{closed contours})$$

Take $M_1 \gg 1$ such that

$$\frac{1}{M_1} < |z - \zeta| < M_1 \quad \text{for } {}^\forall(z, \zeta) \in (\partial D_\varepsilon) \times (\partial D).$$

We thus have $\quad \left| \log \dfrac{1}{|z-\zeta|} \right| < \log M_1 \quad$ on $(\partial D_\varepsilon) \times (\partial D)$.

Let $\delta_1 = \varepsilon / M_1$ (<1). Then

For any $\zeta', \zeta'' \in C_k$ with $|\zeta' - \zeta''| < \delta_1$ and $z \in \partial D_\varepsilon$ we have

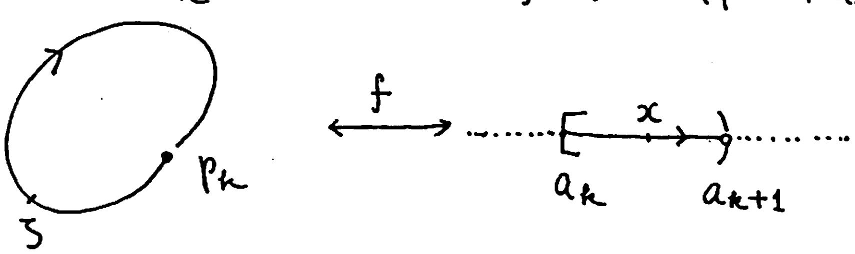
$$\left| \log \frac{1}{|z-\zeta'|} - \log \frac{1}{|z - \zeta''|} \right| = \left| \log \left| 1 + \frac{\zeta' - \zeta''}{z - \zeta'} \right| \right|$$
$$\leqq \frac{|\zeta' - \zeta''|}{|z - \zeta'|} < M_1 \cdot |\zeta' - \zeta''| < \varepsilon$$

Take and fix $p_k \in C_k$ and put $a_k = \int_{C_k} \rho_0(\zeta)\, ds_\zeta > 0$ $(k = 1, \ldots, q)$. Consider the transformation

$$\zeta \in C \longleftrightarrow x \in [0, 1]$$

in a way that

$$\zeta \in C_k \longleftrightarrow x = f(\zeta) = a_1 + \cdots + a_{k-1} + \int_{p_k}^{\zeta} \rho_0(\zeta)\, ds_\zeta$$



Since C_k is smooth and $\rho_0(\zeta)>0$ is continuous on C_k, we find $M_2>0$ s.t.

$$|\zeta'-\zeta''| < M_2 |x'-x''|$$

for $x' = f(\zeta')$, $x'' = f(\zeta'')$, $\forall \zeta', \zeta'' \in C_k$.

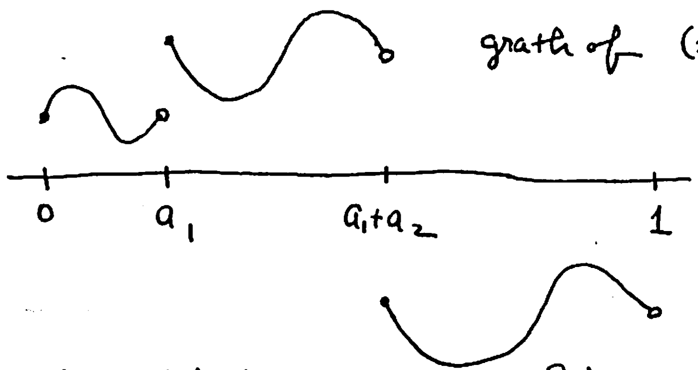
Consequently, it holds

$$(5.3)\begin{cases} \text{(i)} & \left|\log \dfrac{1}{|z-f^{-1}(x)|}\right| < \log M_1 \quad \text{on } (\partial D_\varepsilon)\times[0,1] \\[2ex] \text{(ii)} & \text{Putting } \delta_2 = \delta_1/M_2 \text{, we have} \\ & \left|\log \dfrac{1}{|z-f^{-1}(x')|} - \log \dfrac{1}{|z-f^{-1}(x'')|}\right| < \varepsilon \\ & \text{for } \forall z \in \partial D_\varepsilon,\ \forall x', x'' \in [a_k, a_{k+1}) \text{ with } |x'-x''|<\delta_2 \end{cases}$$

Now, let $z \in \partial D_\varepsilon$ and form

$$\int_{\partial D} \left(\log \frac{1}{|z-\zeta|}\right) \rho_0(\zeta)\, ds_\zeta = \sum_{k=1}^{q} \int_{C_k} \left(\log \frac{1}{|z-\zeta|}\right) \rho_0(\zeta)\, ds_\zeta$$

$$= \sum_{k=1}^{q} \int_{a_k}^{a_{k+1}} \left(\log \frac{1}{|z-f^{-1}(x)|}\right) dx$$



graph of $\left(x, \log \dfrac{1}{|z-f^{-1}(x)|}\right)$

It is clear that it becomes a Riemann Integral with finite $q-1$ gaps,

$$= \int_0^1 \log \frac{1}{|z-f^{-1}(x)|}\, dx$$

Under the conditions (i), (ii) of (5.3), we see by under-graduate Calculus that there exists N_ε $(\geqq 1)$ such that for any $n \geqq N_\varepsilon$ and any $z \in \partial D_\varepsilon$

$$\left| \int_0^1 \left(\log \frac{1}{|z - f^{-1}(x)|}\right) dx - \sum_{j=1}^{n} \left(\log \frac{1}{|z - \zeta_i|}\right) \frac{1}{n} \right| < \varepsilon$$

where $\zeta_i = f^{-1}(x_i) \in \partial D$

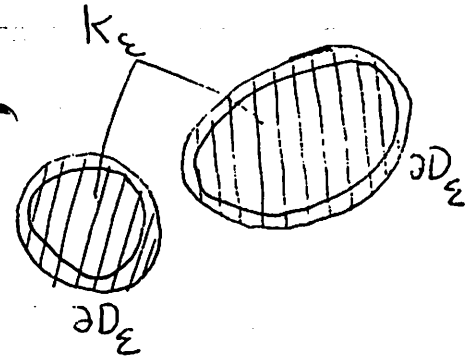
By (5.2) we have

$$\int_0^1 \left(\log \frac{1}{|z - f^{-1}(x)|}\right) dx = \int_{\partial D} \left(\log \frac{1}{|z - \zeta|}\right) \rho_0(\zeta)\, ds_\zeta = \lambda - \varepsilon$$

so that

$$\lambda - 2\varepsilon < \frac{1}{n} \sum_{j=1}^{n} \log \frac{1}{|z - \zeta_i|} \quad \text{for } \forall z \in \partial D_\varepsilon .$$

On the other hand, since $\frac{1}{n} \sum_{j=1}^{n} \log \frac{1}{|z-\zeta_i|}$ is superharmonic function in $\mathbb{C}_z$ and hence, in the compact domain $K_\varepsilon = \overline{\mathbb{C}}_z - D_\varepsilon$. It follows from the maximum principle that



$$\lambda - 2\varepsilon < \frac{1}{n} \sum_{j=1}^{n} \log \frac{1}{|z-\zeta_i|} \quad \text{on } K_\varepsilon \quad !!$$

and, in particular, on K. That is,

$$\lambda - 2\varepsilon < \log \frac{1}{\sqrt[n]{|(z-\zeta_1)\cdots(z-\zeta_n)|}} \quad \text{on } K$$

$$< \log \frac{1}{\sqrt[n]{m_n}} = \log \frac{1}{\rho_n}$$

for $\forall n \geqq N_\varepsilon$. Consequently, letting $n \to \infty$ we have

$$-\log \rho \geqq \lambda - 2\varepsilon, \text{ or } -\log \rho \geqq \lambda . \text{ (5.1) is proved.}$$

c.q.f.d.

As already remarked, the existence of the equilibrium dist. $\rho_0(\zeta)ds$ w.r.t. K induces that of the Green's fn $g(z)$ for (D,∞) where $K = \overline{\mathbb{C}_z} - D$. Let us show the converse. Before that, we will show the uniqueness of the equilibrium dist. w.r.t. K.

Say $\rho_1(\zeta)ds$ and $\rho_2(\zeta)ds$ be two equilibrium dist. w.r.t K, that is

$$N_{\rho_1}(z) \equiv k_1 \quad \text{and} \quad N_{\rho_2}(z) \equiv k_2 \qquad \text{fn } z \in K.$$

Then the function $N_{\rho_1}(z) - N_{\rho_2}(z)$ is harmonic on D even at ∞ such that

$$\lim_{z\to\infty} \left(N_{\rho_1}(z) - N_{\rho_2}(z) \right) = 0 \ ; \quad N_{\rho_1}(z) - N_{\rho_2}(z) = k_1 - k_2 \text{ on } \partial D.$$

It follows from the maximum principle that

$$N_{\rho_1}(z) - N_{\rho_2}(z) \equiv k_1 - k_2 = 0 \quad \text{on } D.$$

The uniqueness is proved.

Now, assume that

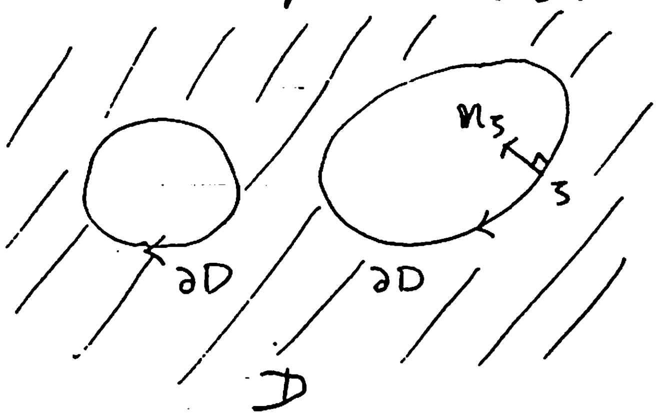
(1) there exists the Green's fn $g(z)$ for (D, ∞);

(2) $g(z)$ is extendable of class C^1 on ∂D.

Then

$$\rho_0(\zeta) = -\frac{1}{2\pi} \frac{\partial g(\zeta)}{\partial n_\zeta} \qquad \text{for } \zeta \in \partial D$$

where $\partial/\partial n_\zeta$ is the outer normal derivative at ζ with respect to ∂D. We have to show

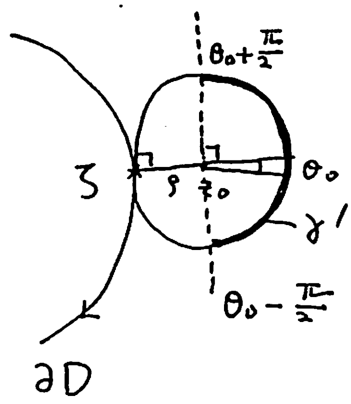


(i) $\rho_0(\zeta) > 0$;

(ii) $\int_{\partial D} \rho_0(\zeta)\, ds = 1$;

and the formula (5.2).

Let $\zeta \in \partial D$. Then we find a disk $E : |z - z_0| < \rho$ such that ∂E is tangent to ∂D at ζ, and $E \subset D$.



Let $\theta_0 = \arg(z_0 - \zeta)$ and

$$\gamma' = \partial E \cap \{\theta_0 - \tfrac{\pi}{2} \leqq \arg(z - z_0) \leqq \theta_0 + \tfrac{\pi}{2}\}.$$

If we put $m = \underset{z \in \gamma'}{\mathrm{Min}}\ g(z)$, then $m > 0$.

From the Poisson Integral :

$$\omega(z) = m \cdot \frac{1}{2\pi} \int_{\theta_0 - \frac{\pi}{2}}^{\theta_0 + \frac{\pi}{2}} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\theta - \varphi) + r^2}\, d\varphi$$

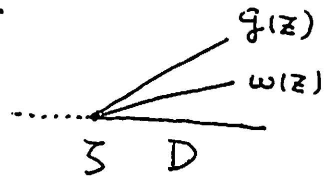
where $z = z_0 + r e^{i\theta} \in E$

Then $\omega(z)$ is bdd harmonic in E such that

$$\omega(z) = \begin{cases} m & \text{on } \gamma' \\ 0 & \text{on } E - \gamma' \end{cases}$$

It follows from the maximum principle that

$$g(z) \geqq \omega(z) \quad \text{in } E \cup \partial E$$



Since $g(\zeta) = \omega(\zeta) = 0$ and since ∂D and ∂E are tangent to each other, we have

$$\frac{\partial g}{\partial n_\zeta}(\zeta) = \lim_{r \to -0} \frac{g(\zeta + r(\zeta - z_0)) - g(\zeta)}{r}$$

$$\leqq \lim_{r \to -0} \frac{\omega(\zeta + r(\zeta - z_0)) - \omega(\zeta)}{r}$$

$$= \frac{\partial \omega}{\partial r}(\zeta) = \frac{m}{2\pi} \int^{\theta_0 + \frac{\pi}{2}} \left[\frac{\partial}{\partial r} \left(\frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\theta - \varphi) + r^2} \right) \right]_{\substack{r = \rho \\ \theta = \theta_0 + \pi}} d\varphi$$

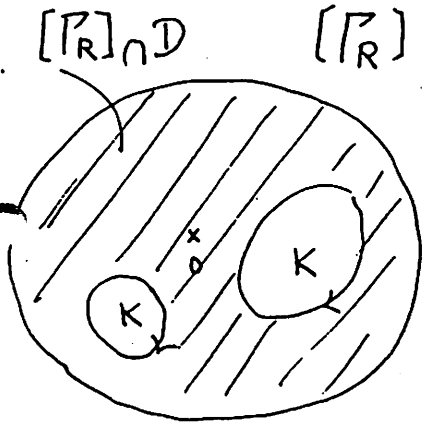
$$= \frac{m}{2\pi} \int_{\theta_0 - \frac{\pi}{2}}^{\theta_0 + \frac{\pi}{2}} \frac{d\varphi \qquad \theta_0 - \frac{\pi}{2}}{\rho(1 - \cos(\varphi - (\theta_0 + \pi)))}$$

$$= \frac{-m}{2\pi\rho} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\varphi}{1+\cos\varphi} < 0$$

$$\therefore \quad \rho_0(\zeta) \doteq \frac{-1}{2\pi} \frac{\partial g}{\partial n_\zeta}(\zeta) \geqq \frac{2m}{\rho} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{1+\cos\varphi} > 0.$$

assertion (i) is proved.

For assertion (ii), draw a large disk $[\Gamma_R] = \{|z| < R\}$ such that $[\Gamma_R] \gg K$. Apply the green's formula to $g(z)$ in $[\Gamma_R] \cap D$, and we have



$$\int_{\Gamma_R + \partial D} \frac{\partial g(\zeta)}{\partial n_\zeta} ds_\zeta = \iint_{[\Gamma_R] \cap D} (\Delta g(z))\, dx dy = 0$$

$$\therefore \quad \int_{\partial D} \frac{\partial g}{\partial n} ds = - \int_{\Gamma_R} \frac{\partial g}{\partial n} ds \quad \text{where } \Gamma_R = \{ Re^{i\theta} : 0 \leqq \theta \leqq 2\pi \}$$

$$= - \int_0^{2\pi} \left(\frac{\partial g}{\partial r}\right)(Re^{i\theta}) \cdot R\, d\theta$$

At ∞, we have the development

$$g(z) = \log|z| + \lambda + \mathrm{Re}\left\{\sum_{n=1}^{\infty} \frac{a_n}{z^n}\right\}$$

$$= \log r + \lambda + \mathrm{Re}\left\{\sum_{n=1}^{\infty} \frac{a_n}{r^n e^{in\theta}}\right\}$$

so that

$$\frac{\partial g}{\partial r} = \frac{1}{r} + \mathrm{Re}\left\{\sum_{n=1}^{\infty} \frac{-a_n}{n\, r^{n+1} e^{in\theta}}\right\}$$

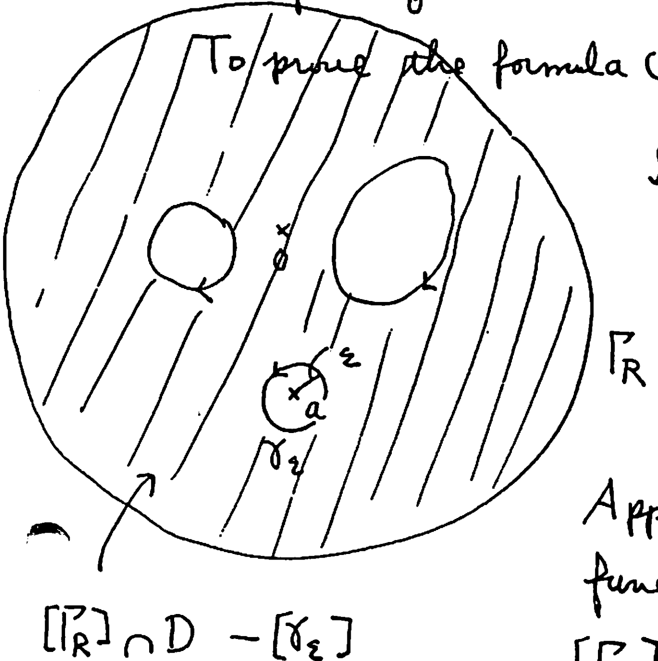
We thus get

$$\int_{\partial D} \frac{\partial g}{\partial n} ds = - \int_0^{2\pi} \left(1 + \mathrm{Re}\left\{\sum_{n=1}^{\infty} \frac{-a_n}{n R^n e^{in\theta}}\right\}\right) d\theta$$

$$\longrightarrow -2\pi \quad \text{as } R \to +\infty.$$

Consequently, $-\frac{1}{2\pi}\int_{\partial D} \frac{\partial g}{\partial n_\zeta} ds = 1$ i.e., (ii) is proved.

To prove the formula (5.2), let $a \in D$.



Draw a small disk $[\gamma_\varepsilon] = \{|z-a| \le \varepsilon\}$ of radius ε of center a.

Let $R \gg 1$ so large that

$$[\Gamma_R] \gg [\gamma_\varepsilon] \cup K.$$

Apply the Green's formula to harmonic functions $g(z)$ and $\log\frac{1}{|a-z|}$ in a domain $[\Gamma_R] \cap D - [\gamma_\varepsilon]$. Then we have

$$(5.4) \quad \int_{\Gamma_R - \gamma_\varepsilon + \partial D} g \frac{\partial}{\partial n_\zeta} \log\frac{1}{|a-\zeta|} ds_\zeta = \int_{\Gamma_R - \gamma_\varepsilon + \partial D} \log\frac{1}{|a-\zeta|} \frac{\partial g}{\partial n_\zeta} ds_\zeta$$

At a, we have

$$\int_{\gamma_\varepsilon} g(\zeta) \frac{\partial}{\partial n_\zeta} \log\frac{1}{|a-\zeta|} ds_\zeta = \int_0^{2\pi} g(a+\varepsilon e^{i\theta}) \left[\left(\frac{\partial}{\partial r} \log\frac{1}{r}\right)\right]_{r=\varepsilon} \cdot \varepsilon d\theta$$

$$= -\int_0^{2\pi} g(a+\varepsilon e^{i\theta}) d\theta \longrightarrow -g(a)\cdot 2\pi \quad (\varepsilon \to 0)$$

$$\int_{\gamma_\varepsilon} \log\frac{1}{|a-\zeta|} \frac{\partial g}{\partial n} ds = \varepsilon(\log\varepsilon) \int_0^{2\pi} \left[\frac{\partial g}{\partial r}(a+re^{i\theta})\right]_{r=\varepsilon} d\theta$$

$$= \varepsilon(\log\varepsilon)\cdot \int_0^{2\pi} \left[\frac{\partial g}{\partial x}\cos\theta + \frac{\partial g}{\partial y}\sin\theta\right]_{r=\varepsilon} d\theta$$

$$\longrightarrow 0\cdot\left(\frac{\partial g}{\partial x}(a)\cdot 0 + \frac{\partial g}{\partial y}(a)\cdot 0\right) = 0 \quad \text{as } \varepsilon \to 0$$

At ∞, we have

$$g(z) = \log|z| + \lambda + \mathrm{Re}\left\{\sum_{n=1}^{\infty} \frac{a_n}{z^n}\right\}$$

$$\log\frac{1}{|a-z|} = \log\frac{1}{|z|} + \mathrm{Re}\left\{\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cdot \frac{a^n}{z^n}\right\} \qquad (|z| \gg |a|)$$

$$\int_{\Gamma_R} \left(g\frac{\partial}{\partial n_\zeta}\log\frac{1}{|a-\zeta|} - \log\frac{1}{|a-\zeta|}\frac{\partial g}{\partial n_\zeta} \right) ds$$

$$= R\int_0^{2\pi} \Big[\Big(\log R + \lambda + \mathrm{Re}\Big\{\sum_{n=1}^{\infty} \frac{a_n}{R^n e^{in\theta}}\Big\}\Big) \Big[\frac{\partial}{\partial r}\Big(\log\frac{1}{r} + \mathrm{Re}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{a^n}{r^n e^{in\theta}}\Big)\Big]_{r=R}$$

$$- \Big(\log\frac{1}{R} + \mathrm{Re}\Big\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{1}{R^n e^{in\theta}}\Big\}\Big)\Big[\frac{\partial}{\partial r}\Big(\log r + \lambda + \mathrm{Re}\Big\{\sum_{n=1}^{\infty}\frac{a_n}{r^n e^{in\theta}}\Big\}\Big)\Big]_{r=R} \Big] d\theta$$

$$= \int_0^{2\pi} \Big[\Big(\log R + \lambda + \mathrm{Re}\Big\{\sum_{n=1}^{\infty} \frac{a_n}{R^n e^{in\theta}}\Big\}\Big) \Big(-1 + \mathrm{Re}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\frac{a^n}{R^n e^{in\theta}}\Big)$$

$$- \Big(\log\frac{1}{R} + \mathrm{Re}\Big\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{1}{R^n e^{in\theta}}\Big\}\Big)\Big(1 + \mathrm{Re}\sum_{n=1}^{\infty}\frac{-a_n}{nR^n e^{in\theta}}\Big) \Big] d\theta$$

$$= \int_0^{2\pi} \Big[(\log R)\cdot O\Big(\frac{1}{R}\Big) - \lambda + \lambda\cdot O\Big(\frac{1}{R}\Big) - O\Big(\frac{1}{R}\Big) + O\Big(\frac{1}{R^2}\Big)$$

$$- \Big\{\Big(\log\frac{1}{R}\Big) O\Big(\frac{1}{R}\Big) + O\Big(\frac{1}{R}\Big) + O\Big(\frac{1}{R^2}\Big)\Big\} \Big] d\theta$$

$$\longrightarrow -2\pi\lambda \quad \text{as } R \to \infty.$$

Since $g(\zeta) = 0$ on ∂D, (5.4) with $\varepsilon \to 0$ and $R \to +\infty$, induces

$$-2\pi\lambda + g(a)\cdot 2\pi = \int_{\partial D} \log\frac{1}{|a-\zeta|}\frac{\partial g}{\partial n_\zeta} ds_\zeta$$

$$\therefore\ -\frac{1}{2\pi}\int_{\partial D}\Big(\log\frac{1}{|a-\zeta|}\Big)\frac{\partial g}{\partial n_\zeta} ds_\zeta = \lambda - g(a) \qquad (a \in D)$$

By continuity, $\displaystyle -\frac{1}{2\pi}\int_{\partial D}\Big(\log\frac{1}{|a-\zeta|}\Big)\frac{\partial g}{\partial n_\zeta} ds_\zeta = \lambda \quad (a \in \partial D).$

Since the integral is harmonic in K^0 and is continuous on K, it follows that

$$-\frac{1}{2\pi}\int_{\partial D}\left(\log\frac{1}{|a-\zeta|}\right)\frac{\partial g}{\partial n_\zeta}\,ds_\zeta \equiv \lambda \quad \text{on } K.$$

The formula (5.2) is proved for $\rho_0(\zeta) = -\frac{1}{2\pi}\frac{\partial g}{\partial n_\zeta}\cdot \ (\zeta\in\partial D)$

c.q.f.d.

Remark. Please don't forget we assumed not only the existence of the Green's function $g(z)$ for (D, ∞) but also the of class C^1 on ∂D. We will prove it a little more complicated case.

Dear

These are details of last Thursday's lecture and a part of tomorrow's.

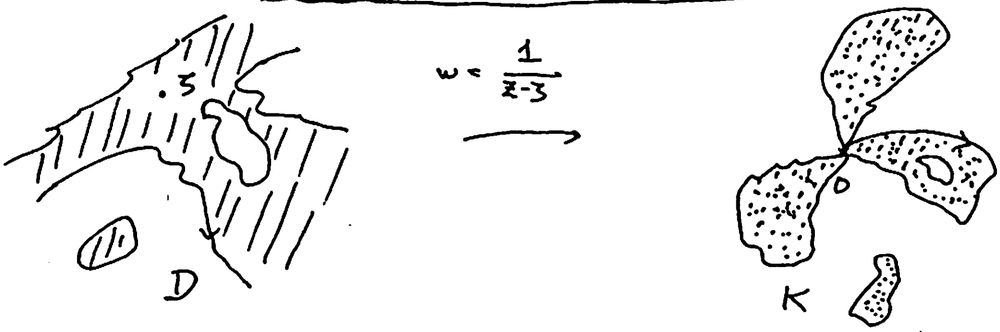
Thank you,

Hiroshi Yamaguchi

§ 5. Tschebyscheff's Polynomial

In this § we shall prove Theorem 3.2 (Szegö):

$$\boxed{\lambda_D = -\log d_\infty(K)}$$



To prove it we study the Tschebyscheff's polynomials.

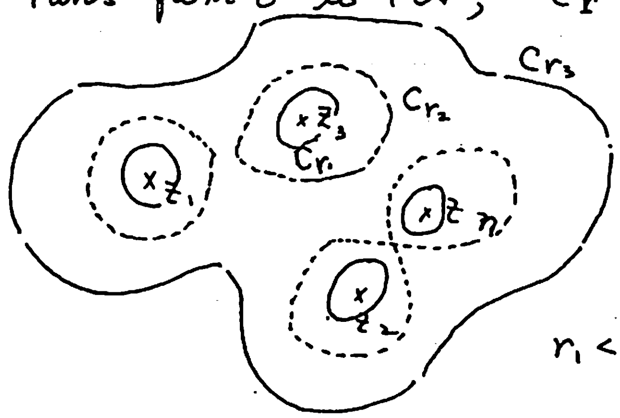
Let $n \geq 1$ and $p(z)$ be a monic polynomial of degree n:

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$
$$= (z-z_1)(z-z_2)\cdots(z-z_n)$$

For $r>0$, consider the level curve:

$$C_r : \quad |p(z)| = r^n$$

As r runs from 0 to $+\infty$, C_r moves as follows:



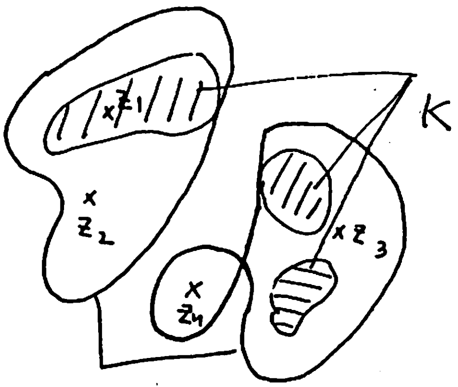
C_r is called the lemniscate of center $z_1, \cdots, z_n$ and of radius r

$r_1 < r_2 < r_3$

We put

$$\mathcal{P}_n = \{p(z)\} \begin{cases} = \{(z-z_1)\cdots(z-z_n)\} & z_i \in \mathbb{C} \\ = \{z^n + a_1 z^{n-1} + \cdots + a_n\} & a_i \in \mathbb{C}. \end{cases}$$

Now let K be a compact set in $\mathbb{C}$. We exclude the case when K is a finite point set.



$|p\ (z)| = m(p_n)$

Define

$$m(p\ ) = \max_{z \in K} \{|p\ (z)|$$

for each $p \in \mathcal{P}_n$;

and put

$$m_n = \inf_{p \in \mathcal{P}_n} \{m(p)\}$$

That is, $\sqrt[n]{m(p)}$ is the smallest radius of $|p(z)| = r^n$ such that $\{|p(z)| \leqq r^n\}$ contains K.

Theorem 5.1 (de la vallée Poussin 1 11)

There uniquely exists $t_n(z) \in \mathcal{P}_n$ such that $m_n = m(t_n)$.

$t_n(z)$ is called the n^{th}-Tschebyscheff's polynomial belonging to K and m_n the n^{th}-Tschebyscheff constant belonging to K.

Proof of Existence : Take and fix n distinct points $\{c_1, \cdots, c_n\}$ in K. We choose

$$\exists\ p_\nu(z) \in \mathcal{P}_n \quad (\nu = 1, 2, \cdots) \quad \text{such that} \quad m(p_\nu) \searrow m_n,$$

with $m(p_1) < m_n + 1$

$$p_\nu(z) = z^n + a_1^{(\nu)} z^{n-1} + \cdots + a_n^{(\nu)}$$

For each $\nu = 1, 2, \cdots$, form

$$\begin{cases} c_1^n + a_1^{(\nu)} c_1^{n-1} + \cdots + a_n^{(\nu)} = p_\nu(c_1) \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ c_n^n + a_1^{(\nu)} c_n^{n-1} + \cdots + a_n^{(\nu)} = p_\nu(c_n) \end{cases}$$

Since $|p_\nu(c_i)| \leqq m(p_\nu) < m_1 + 1$, it follows from Cramer's formula that

For each $j=1,\cdots,n$, the sequence $\{a_j^{(\nu)}\}_{\nu=1,2,\cdots}$ is bounded in $\mathbb{C}$. Hence we find a subsequence $\{a_j^{(\nu_k)}\}_{k=1,2,\cdots}$ which converges a_j^*. Put

$$T(z) = z^n + a_1^* z^{n-1} + \cdots + a_n^*,$$

then $T(z) \in \mathcal{P}_n$. Therefore we have

$$m(T) = \max_{z \in K} \{|T(z)|\} = |T(\exists z_0)| = \lim_{k\to\infty} |p_{\nu_k}(z_0)| \leqq \lim_{k\to\infty} m(p_{\nu_k}) = m_n$$
$$(z_0 \in K)$$

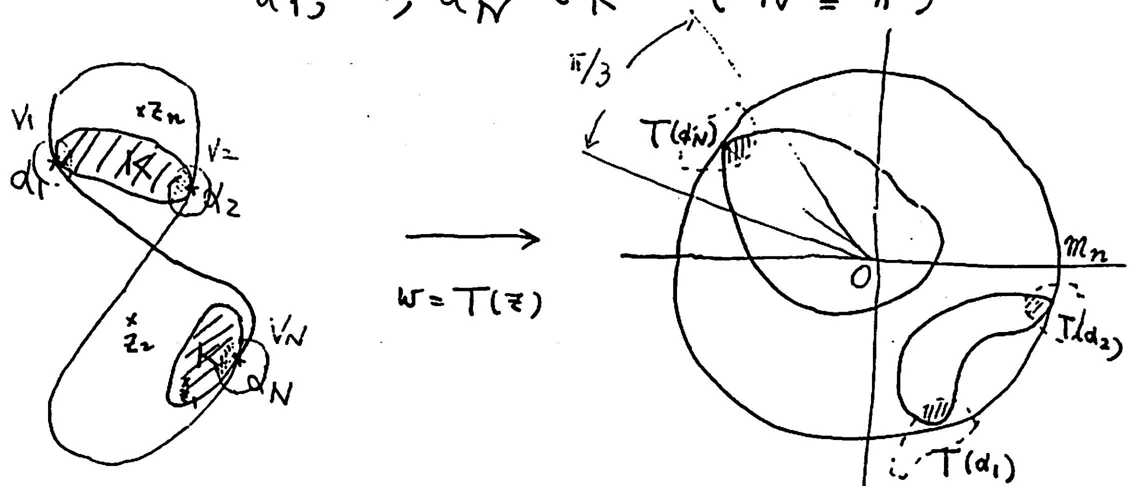
and hence $m(T) = m_n$. T is one of the desired functions.

<u>Proof of Uniqueness</u> To prove this we need the following

<u>Lemma 5.1</u> Let $T(z) \in \mathcal{P}_n$ such that $m(T) = m_n$. Then the equation $|T(z)| = m_n$ has at least $n+1$ distinct solutions in K.

<u>Proof</u>. Let us prove it by contradiction. Assume that $|T(z)| = m_n$ has at most n distinct solutions in K:

$$\alpha_1, \cdots, \alpha_N \in K \quad (N \leqq n)$$



We remark that $m_n > 0$. In fact, if $m_n = 0$, then $T(z) = 0$ in K. Since K is infinite, $T(z) \equiv 0$ which is a contradiction.

Because $N \leqq n$, there exists a polynomial $Q(z)$ (not necessarily monic) of degree $n-1$ such that

$$Q(\alpha_k) = T(\alpha_k) \quad (k=1,\cdots,N)$$

At each α_k, we find a neighborhood V_k of α_k such that

$|T(z)| > \frac{m_n}{2}$; $|\arg T(z) - \arg T(\alpha_k)| < \frac{\pi}{6}$; $|\arg Q(z) - \arg Q(\alpha_k)| < \frac{\pi}{6}$ in $\overline{V_k}$.

$$\therefore \ |\arg T(z) - \arg Q(z)| < \frac{\pi}{3} \text{ in } \overline{V_k}.$$

Since $\{\alpha_k\}_{k=1,\cdots,N}$ are all solutions of $|T(z)| = m_n$ and since $|T(z)| \leqq m_n$ in compact K, it follows that

$$|T(z)| \leqq {}^{\exists}m' < m_n \text{ for } \forall z \in K - \bigcup_{k=1}^{N} V_k$$

Hence for sufficiently small $\varepsilon > 0$ we have

$$|T(z) - \varepsilon Q(z)| \leqq \frac{m_n + m'}{2} < m_n \text{ for } \forall z \in K - \bigcup_{k=1}^{N} V_k$$

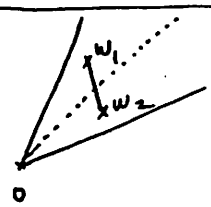
Moreover for the same ε we can find

$$|T(z)| > \frac{m_n}{2} > |\varepsilon Q(z)| \text{ for all } z \in \bigcup_{k=1}^{N} \overline{V_k}$$

On the other hand, it is easy to show that

> In general, for any $w_1, w_2 \ (\neq 0)$ such that $|\arg w_1 - \arg w_2| < \pi/3$, we have $|w_1 - w_2| \lneqq \operatorname{Max}\{|w_1|, |w_2|\}$

[(:)



We may suppose $w_1 = r_1 e^{i\theta}$, $w_2 = r_2 e^{-i\theta}$ $(r_1, r_2 > 0)$
Then where $0 < \theta < \pi/6$

$$\begin{aligned} |w_1 - w_2|^2 &= |r_1 e^{i\theta} - r_2 e^{-i\theta}|^2 \\ &= r_1^2 + r_2^2 - 2\,\mathrm{Re}\, r_1 r_2 \cos 2\theta \\ &< r_1^2 + r_2^2 - r_1 r_2 < \operatorname{Max}\{r_1^2, r_2^2\} \end{aligned}$$]

Since $\arg \varepsilon Q(z) = \arg Q(z)$, it follows that

$$|T(z) - \varepsilon Q(z)| \lneqq \operatorname{Max}\{|T(z)|, |\varepsilon Q(z)|\} = |T(z)|$$

for all $z \in \bigcup_{k=1}^{N} \overline{V_k}$. Because $T(z) \leqq m_n$ in K, we have

$$|T(z) - \varepsilon Q(z)| \lneqq m_n \text{ for } z \in K \cap \left(\bigcup_{k=1}^{N} \overline{V_k}\right).$$

Hence $|T(z) - \varepsilon Q(z)| < m_n$ for $z \in K$.

This is a contradiction to $T(z) - \varepsilon Q(z) \in \mathcal{P}_n$. Lemma 5.1 is proved.

Using Lemma 5.1, we easily prove Uniqueness as follows:

Let $T'(z), T''(z) \in \mathcal{P}_n$ such that $m(T') = m(T'') = m_n$.

Then $T(z) = \frac{1}{2}(T'(z) + T''(z)) \in \mathcal{P}_n$ and $m(T) \leqq \frac{1}{2}(m(T') + m(T''))$
$\leqq \frac{1}{2}(m_n + m_n) = m_n \quad \therefore \; m(T) = m_n.$

It follows from Lemma 5.1 that $|T(z)| = m_n$ has at least $(n+1)$ distinct solutions in K, say

$$\alpha_1, \alpha_2, \cdots, \alpha_N \quad (N \geqq n+1) \in K$$

$$\therefore m_n = |T(\alpha_k)| = \frac{1}{2}|T'(\alpha_k) + T''(\alpha_k)|$$
$$\leqq \frac{1}{2}\left(|T'(\alpha_k)| + |T''(\alpha_k)|\right) \leqq m_n$$

Therefore $|T'(\alpha_k) + T''(\alpha_k)| = |T'(\alpha_k)| + |T''(\alpha_k)|$ and $|T'(\alpha_k)| = |T''(\alpha_k)| = m_n \neq 0$. Hence

$$T'(\alpha_k) = T''(\alpha_k) \quad (k = 1, \cdots, n+1)$$

Since $T'(z), T''(z)$ are monic polynomial of degree n, we have $T'(z) \equiv T''(z)$. Uniqueness is proved. c.q.f.d

<u>Remark 5.1</u> Let $t_n(z)$ be the n^{th} Tschebyscheff's polynomial belonging to K. Then the solutions of $t_n(z) = 0$ are contained in the smallest convex set containing K.

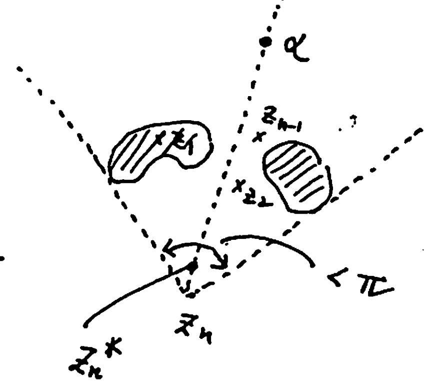
<u>Proof.</u> If non, there exists at least one solution z_n of $t_n(z) = 0$ such that

$$K \subseteqq \left\{ z \in \mathbb{C} \;\middle|\; \left|\arg \frac{z - z_n}{\alpha - z_n}\right| < A \right\}$$

where $\exists \alpha \in \mathbb{C}$, $0 \leqq \exists A < \pi/2$.

Since $z_n \notin K$ and K is compact, we put

$$0 < \exists m \leqq \left|\frac{\alpha - z_n}{z - z_n}\right| \leqq M < +\infty \text{ for } \forall z \in K.$$



Let $z_n^* = z_n + \varepsilon(\alpha - z_n)$ where $0 < \varepsilon \ll 1$.
Then, for any $z \in K$, we have

$$|z - z_n^*|^2 = |(z - z_n) - \varepsilon(\alpha - z_n)|^2$$
$$= |z - z_n|^2 \cdot \left|1 - \varepsilon \cdot \frac{\alpha - z_n}{z - z_n}\right|^2$$
$$= |z - z_n|^2 \left(1 - 2\varepsilon \left|\frac{\alpha - z_n}{z - z_n}\right| \cos\left(\underbrace{\arg\frac{\alpha - z_n}{z - z_n}}_{\equiv A}\right) + \varepsilon^2 \left|\frac{\alpha - z_n}{z - z_n}\right|^2\right).$$
$$\leqq |z - z_n|^2 (1 - 2\varepsilon m \cos A + \varepsilon^2 M^2)$$

Take $\varepsilon > 0$ so small that

$$2\varepsilon m \cos A \quad - \varepsilon^2 M > 0.$$

It is possible because $0 \leqq A < \pi/2$. Then

$$|z - z_n^*| \lneqq |z - z_n| \text{ for all } z \in K.$$

It follows that, if we put $t_n(z) = (z - z_1) \cdots (z - z_{n-1})(z - z_n)$, then $\max_{z \in K} |(z - z_1) \cdots (z - z_{n-1})(z - z_n^*)| \lneqq \underbrace{\max_{z \in K} |(z - z_1) \cdots (z - z_n)|}_{= m_n}$.

This is a contradiction to $(z - z_1) \cdots (z - z_n^*) \in \mathcal{P}_n$.

Now we put $\rho_n = \sqrt[n]{m_n}$ $(n = 1, 2, \cdots)$ where m_n is the n^{th} - Tschebyscheff's constant for a given compact K.

<u>Lemma 5.2</u> (Fekete, 1923)

$$\rho = \lim_{n \to \infty} \rho_n \text{ exists and } 0 \leqq \rho < +\infty.$$

<u>Proof.</u> It is easy by definition of m_n that

$$m_{p+q} \leqq m_p \cdot m_q$$

On the other hand, we have the following elementary

Let $\{a_n\}$ be a sequence such that $a_{p+q} \leqq a_p + a_q$ $(p,q=1,2,\cdots)$. Then the limit $A = \lim_{n\to\infty} \frac{a_n}{n}$ exists and $-\infty \leqq A < +\infty$.

Put $A = \varliminf_{n\to\infty} (a_n/n)$. First suppose $A > -\infty$. Given $\varepsilon > 0$, we find $n_0\ (\geqq 1)$ such that

$$\frac{a_{n_0}}{n_0} < A + \varepsilon.$$

We consider (first) $n_0 - 1$ numbers: $a_1, a_2, \cdots, a_{n_0-1}$, and take an integer K_0 so large that

$$\frac{1}{K_0} < \varepsilon \ ; \ |a_i| / K_0 n_0 < \varepsilon \quad \text{for } i = 1, 2, \cdots, n_0 - 1.$$

Let $^\forall n \geqq K_0 n_0$, so that

$$n = k n_0 + \nu \qquad (^\exists k \geqq K_0 \ ; \ 1 \leqq^\exists \nu \leqq n_0 - 1).$$

By assumption we have

$$a_n = a_{k n_0 + \nu} \leqq a_{k n_0} + a_\nu \leqq k a_{n_0} + a_\nu .$$

and hence

$$\frac{a_n}{n} \leqq \frac{k a_{n_0} + a_\nu}{k n_0 + \nu} \leqq \frac{a_{n_0}}{n_0} \frac{1}{\left(1 + \frac{\nu}{k n_0}\right)} + \left| \frac{a_\nu}{K_0 n_0} \right|$$

$$\leqq \frac{a_{n_0}}{n_0} + \frac{a_{n_0}}{n_0} \frac{\left(\frac{\nu}{k n_0}\right)}{1 + \left(\frac{\nu}{k n_0}\right)} + \varepsilon$$

$$\leqq (A + \varepsilon) + |A + \varepsilon| \left(\frac{1}{k}\right) + \varepsilon$$

$$\leqq A + \varepsilon\left(2 + |A + \varepsilon|\right).$$

Therefore $\lim_{n\to\infty} \frac{a_n}{n} = A$.

In the case $A = -\infty$, the same reasoning is available.

Theorem 5.2 $\rho = d_\infty$ for any compact set K.

Proof.

First, we shall show $\frac{V_{n+1}}{V_n} \geqq m_n$ $(n = 2, 3, \cdots)$.

We find $z_1^0, \cdots, z_n^0$ in K such that

$$V_n = \prod_{i \neq j}^{n} |z_i^\circ - z_j^\circ|$$

Consider the monic polynomial $p(z) = (z - z_1^\circ)\cdots(z - z_n^\circ)$ of degree n, and take $z_{n+1}^* \in K$ such that

$$m(p) = \max_{z \in K} |p(z)| = |p(z_{n+1}^*)| \geq m_n .$$

Then we have

$$|p(z_{n+1}^*)| = \frac{|(z_{n+1}^* - z_1^\circ)\cdots(z_{n+1}^* - z_n^\circ)\prod_{i\neq j}^{n}(z_i^\circ - z_j^\circ)|}{V_n} \leq \frac{V_{n+1}}{V_n}$$

$$\therefore \quad m_n \leq V_{n+1}/V_n .$$

Next we shall show $(n+1)\, m_n \geq V_{n+1}/V_n \quad (n = 2, 3, \cdots)$.
In fact, we find $z_1^\circ, z_2^\circ, \cdots, z_{n+1}^\circ \in K$ such that

$$V_{n+1} = \prod_{i\neq j}^{n+1} |z_i^\circ - z_j^\circ|$$

$$= \mathrm{mod} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1^\circ & z_2^\circ & \cdots & z_{n+1}^\circ \\ \vdots & \vdots & & \vdots \\ (z_1^\circ)^{n-1} & (z_2^\circ)^{n-1} & \cdots & (z_{n+1}^\circ)^{n-1} \\ (z_1^\circ)^n & (z_2^\circ)^n & \cdots & (z_{n+1})^n \end{vmatrix}$$

If we put $t_n(z) = z^n + a_1 z^{n-1} + \cdots + a_n$, then the above determinant becomes

$$= \mathrm{mod} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1^\circ & z_2^\circ & \cdots & z_{n+1}^\circ \\ \vdots & \vdots & & \vdots \\ (z_1^\circ)^{n-1} & (z_2^\circ)^{n-1} & \cdots & (z_{n+1}^\circ)^{n-1} \\ t_n(z_1^\circ) & t_n(z_2^\circ) & \cdots & t_n(z_{n+1}^\circ) \end{vmatrix}$$

$$= \mathrm{mod} \left\{ \sum_{k=1}^{n+1} (-1)^{n+k-1}\, t_n(z_k^\circ) \begin{vmatrix} 1 & \cdots & 1 \\ z_1^\circ & \cdots & z_{n+1}^\circ \\ \vdots & & \vdots \\ (z_1^\circ)^{n-1} & \cdots & (z_{n+1}^\circ)^{n-1} \end{vmatrix} \right\}$$

$$\leqq \sum_{k=1}^{n+1} m_n V_n = (n+1) m_n V_n . \rfloor$$

On the other hand, $m_1 \leqq V_2 \leqq 2m_1$

$\lceil$ ∵ Take $z_1^\circ, z_2^\circ \in K$ such that $V_2 = |z_1 - z_2^\circ|$, and put $p_1(z) = z - z_1^\circ$.

Then $m_1 \leqq \max_{z \in K} |p(z)| = |z_2^\circ - z_1^\circ| = V_2$. First inequality is true

Next, let $t_1(z)$ be the first T-polynomial. Then

$(z-\alpha) \quad K \subset \{|z-\alpha| \leqq m_1\}$ ∴ $|z_1 - z_2| \leqq 2m_1$ for any $z_1, z_2 \in K$

∴ $V_2 \leqq 2m_1 \rfloor$

It follows that

$$m_1 m_2 \cdots m_n \leqq \underbrace{V_2 \cdot \frac{V_3}{V_2} \cdots \frac{V_{n+1}}{V_n}}_{= V_{n+1}} \leqq 2\, 3 \cdots n+1 \cdot m_1 m_2 \cdots m_n$$

$$\therefore \sqrt[\frac{n(n+1)}{2}]{\rho_1 \rho_2^2 \cdots \rho_n^n} \leqq d_{n+1} \leqq \sqrt[\frac{n(n+1)}{2}]{(n+1)!}\ \sqrt[\frac{n(n+1)}{2}]{\rho_1 \rho_2^2 \cdots \rho_n^n}$$

Put
$$A_n = \frac{1}{\frac{n(n+1)}{2}} \cdot \log(\rho_1 \rho_2^2 \cdots \rho_n^n)$$

$$B_n = \frac{1}{\frac{n(n+1)}{2}} \log(n+1)!.$$

Then
$$A_n = \frac{\log \rho_1 + \log \rho_2 + \log \rho_2 + \cdots + \log \rho_n + \cdots + \log \rho_n}{n(n+1)/2},$$

$$0 \leqq B_n = \frac{2}{n} \frac{\log 2 + \cdots + \log(n+1)}{n+1} \leqq \frac{2}{n} \log(n+1) \longrightarrow 0 \text{ as } n \to \infty$$

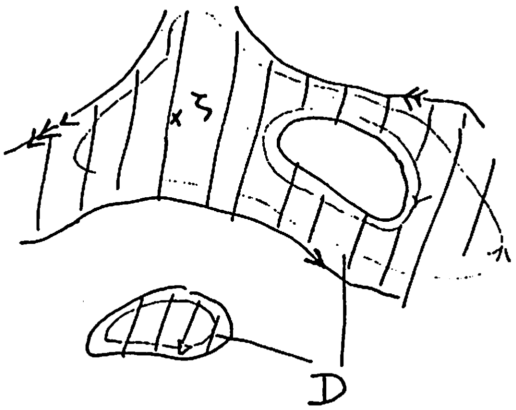
Since $\lim_{n\to\infty} \log \rho_n = \log \rho$, we have $A_n \longrightarrow \log \rho$

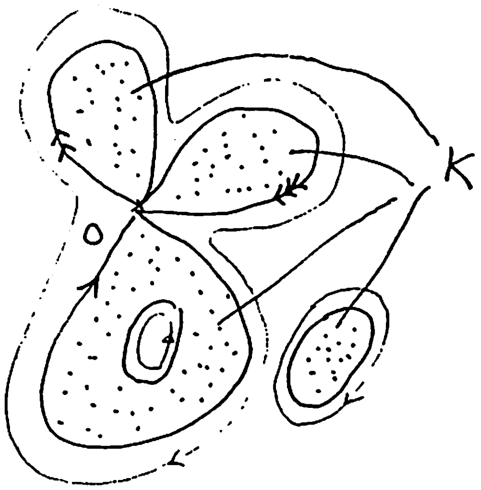
(∵ In general, $a_n \to \alpha \Rightarrow (a_1 + \cdots + a_n)/n \longrightarrow \alpha$)

$$\therefore \log \rho \leqq \underbrace{\lim_{n\to\infty} d_{n+1}}_{= \log d_\infty} \leqq \log \rho$$

∴ $d_\infty = \rho$. q.e.d.

Let us prove Theorem 3.2. $\lambda_D = -\log d_\infty(K)$





Let Ω_ν ($\nu=1,2,\cdots$) be an open set with smooth boundary such that $\zeta\in\Omega_1 \Subset \Omega_2 \Subset \cdots\cdots$; $\bigcup_{\nu=1}^{\infty}\Omega_\nu = D$.
Let $\Omega_\nu^{(1)}$ be the connected component of Ω_ν contaning ζ.
Put $\Omega_\nu^* = \varphi(\Omega_\nu)$; $\Omega_\nu^{(1)*} = \varphi(\Omega_\nu^{(1)})$; $K_\nu = \mathbb{C}_w - \Omega_\nu^*$; $K_\nu^{(1)} = \mathbb{C}_w - \Omega_\nu^{(1)*}$. Hence $K_\nu^{(1)} \geqq K_\nu$ and the outer contour of $K_\nu^{(1)}$ is equal to the outer contour of K_ν.
By definition, $\lambda_D = \lim_{\nu\to\infty}\lambda_{\Omega_\nu}$ and $\lambda_{\Omega_\nu} = \lambda_{\Omega_\nu^{(1)}}$ where λ_D, λ_{Ω_ν}, $\lambda_{\Omega_\nu^{(1)}}$ are Robin constants for $(D,\{\zeta\})$, $(\Omega_\nu,\{\zeta\})$, $(\Omega_\nu^{(1)},\{\zeta\})$, respectively.

On the other hand, we find $d_\infty(K) = \lim_{\nu\to\infty} d_\infty(K_\nu)$
In fact, since $\Omega_\nu \nearrow D$, it follows that $K_\nu \searrow K$. Hence $d_\infty(K_1) \geqq d_\infty(K_2) \geqq \cdots\cdots \geqq d_\infty(K)$. We thus put $d_\infty^* = \lim_{\nu\to\infty} d_\infty(K_\nu)$ and have $d_\infty^* \geqq d_\infty(K)$. Let $\varepsilon>0$ be given. Then there exists n_0 $\ni$ $d_{n_0}(K) < d_\infty(K)+\varepsilon$. As already noted, $\lim_{\nu\to\infty} d_{n_0}(K_\nu) = d_{n_0}(K)$, so that $\exists\, \nu_0$ $\ni$ $d_{n_0}(K_{\nu_0}) < d_{n_0}(K)+\varepsilon$.
Therefore $d_\infty^* \leqq d_\infty(K_{\nu_0}) \leqq d_{n_0}(K_{\nu_0}) < d_\infty(K)+2\varepsilon$, that is, $d_\infty^* = d_\infty(K)$. ⌟

Moreover, for each ν $(=1,2,\cdots)$ and each n $(=1,2,\cdots)$ we have

$$d_n(K_\nu) = d_n(K_\nu^{(1)})$$

In fact, let $z_1^\circ, \cdots, z_n^\circ \in K_\nu^{(1)}$ such that

$$V_n(K_\nu^{(1)}) = \prod_{i \neq j}^{n} |z_i^\circ - z_j^\circ|$$

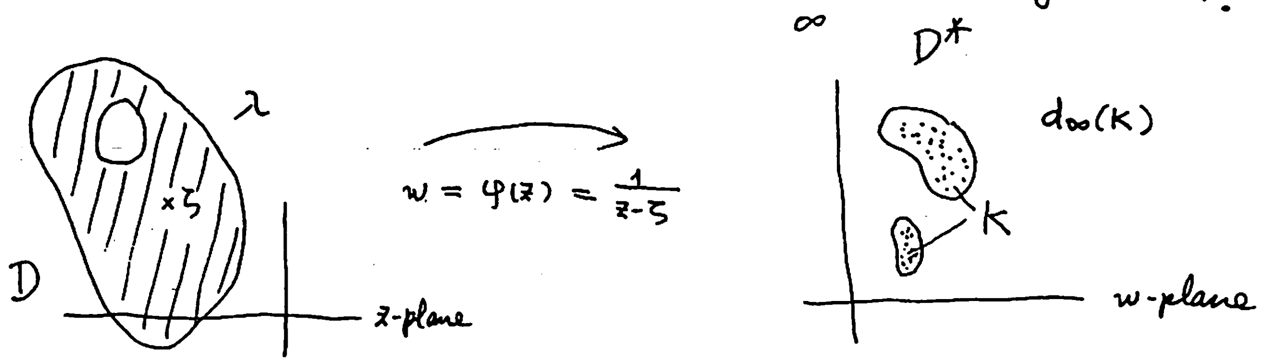
Then, the maximum principle for holomorphic functions implies that each z_i° $(i=1,\cdots,n)$ is contained in the outer contour of $K_\nu^{(1)}$, therefore in the outer contour of K_ν.

i.e. $z_i^\circ \in K_\nu$ $(i=1,\cdots,n)$, so that $V_n(K_\nu^{(1)}) \leq V_n(K_\nu)$.

Because of $K_\nu^{(1)} \supset K_\nu$, we have $V_n(K_\nu^{(1)}) = V_n(K_\nu)$.]

It suffice for $\lambda_D = -\log d_\infty(K)$ to prove $\lambda_{\Omega_\nu^{(1)}} = -\log d_\infty(K_\nu^{(1)})$ for $\nu = 1,2,\cdots$. Consequently, from the beginning, we may assume that

" D is a domain with smooth boundary in $\mathbb{C}_z$."



Let D be a domain with smooth boundary ∂D, and put $K^* = \overline{\mathbb{C}}_w - \varphi(D)$. We have the Green's function $g(z)$ and the Robin constant λ for $(D, \{\zeta\})$, so that

$$g(z) = \log \frac{1}{|z-\zeta|} + \lambda + h(z)$$

where $h(z)$ is harmonic in D and $h(\zeta) = 0$. Therefore

$$h(z) = \mathrm{Re}\left\{ a_1(z-\zeta) + a_2(z-\zeta)^2 + \cdots \right\}$$

Consider the function $G(w)$ on $D^* = \varphi(D)$:

$$G(w) = g(z) \quad \text{where} \quad w = \varphi(z) \text{ and } z \in D.$$

At $w = \infty$, we thus have

$$G(w) = \log|w| + \lambda + H(w)$$

where $H(w) = h(z) = \mathrm{Re}\left\{ a_1\left(\frac{1}{w}\right) + a_2\left(\frac{1}{w}\right)^2 + \cdots \right\}$

so that $H(\infty) = \lim_{w\to\infty} H(w) = 0$.

That is, D^* is a domain of $\overline{\mathbb{C}}_w$ such that $D^* \ni \infty$ and its boundary is smooth, and

$$\begin{cases} \text{(i) } G(w) \text{ is a harmonic function in } D^*; \\ \text{(ii) } G(w) \text{ vanishes on } \partial D^* \text{ continuously;} \\ \text{(iii) } G(w) = \log|w| + \lambda + H(w) \\ \qquad \text{where } H(w) \text{ is harmonic in } D^* \text{ and } H(\infty) = 0 \end{cases}$$

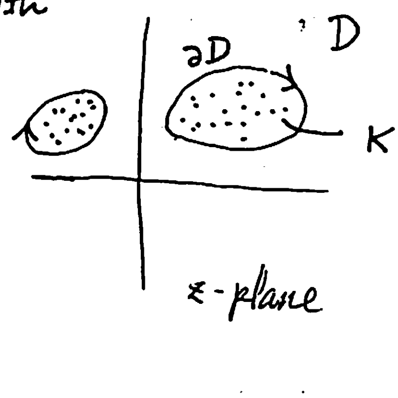
We say that $G(w)$ is the Green's function for $(D^*, \{\infty\})$, and λ is the Robin constant for $(D^*, \{\infty\})$.

Consequently, the Robin constant for $(D, \{\zeta\})$ is equal to that for $(D^*, \{\infty\})$.

Finally, in order to prove Theorem 3.2: $\lambda_D = -\log d_\infty(K)$, we consider it only on the same w-plane. So, changing the notations $\begin{pmatrix} \overset{w}{D^*} \\ G \end{pmatrix}$ to $\begin{pmatrix} \overset{z}{D} \\ g \end{pmatrix}$ anew, we are in the following situations:

Let D be a domain of $\overline{\mathbb{C}}_z$ with smooth boundary ∂D such that $D \ni \infty$.

Put $K = \overline{\mathbb{C}} - D$, so that K is a compact set. We denote by $g(z)$, λ the Green's function and the Robin constant



for $(D, \{\infty\})$, respectively. We denote by d_∞ the transfinite diameter of the compat K. Under these notations let us prove

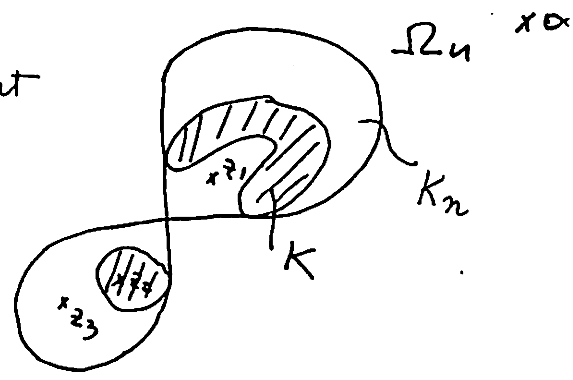
$$\lambda = -\log d_\infty .$$

If we put $\rho = \lim_{n\to\infty} \sqrt[n]{m_n}$; m_n being the n^{th}-Tschebyscheff constant belonging to K, then we proved in Theorem 5.2, $\rho = d_\infty$.
It suffices to prove

$$(5.3) \quad \lambda = -\log \rho$$

Proof. First we prove $\lambda \geqq -\log \rho$.

In fact, let $n \geqq 1$ and let $t_n(z) = (z-z_1)\cdots(z-z_n)$ be the n^{th}-T polynomial belonging to K.



Since $m_n = \max_{z\in K} |t_n(z)|$, we see that

$$K_n = \{z \in \mathbb{C} \mid |t_n(z)| \leqq m_n\}$$

is a compact set such that $K_n \supseteqq K$.
If we put $\Omega_n = \mathbb{C}_z - K_n$ and

$$u_n(z) = \frac{1}{n}\log|t_n(z)|,$$

then $u_n(z)$ is harmonic in Ω_n ; $u_n(z)$ vanishes on $\partial\Omega_n$ continuously ; at ∞ we have

$$u_n(z) = \frac{1}{n}\log\frac{|(z-z_1)\cdots(z-z_n)|}{m_n}$$

$$= \log|z| - \frac{1}{n}\log m_n + h_n(z)$$

where $h_n(z) = \sum_{k=1}^{n} \log\left|1 - \frac{z_k}{z}\right|$, so that $h_n(\infty) = 0$

This means that $u_n(z)$ is the green's function for $(\Omega_n, \{\infty\})$ and $-\log\sqrt[n]{m_n}$ is the Robin constant for $(\Omega_n, \{\infty\})$. Since $K_n \supseteq K$ i.e., $\Omega_n \subseteq D$, it follows from the maximum principle that

$$-\log\sqrt[n]{m_n} \leqq \lambda \quad \therefore \quad -\log\rho \leqq \lambda.$$

October 25, '87

Dear

We finished Chapter I. I thank you for your listening to my lectures with your great patience.

Last week's lectures seemed to be somewhat rough, so I wrote a note.

In the beginning of the proof of Theorem 7.2, I said
「..... $c(0) < e^{-M}$. Since $c(\Omega_n) \nearrow c(0)$, we have
$\exists N \ni \frac{1}{2}e^{-M} < c(\Omega_n) < e^{-M}$ for $^\forall n \geq N$.」
This is, as Mr. Xiang noted, fault. It must be fixed as in p.11
「 Put $c(0) = e^{-M^*}$, so that $M^* \geq M$. ...
.... $\frac{1}{2}e^{-M^*} < c(\Omega_n) < e^{-M^*}$ for $^\forall n \geq N$」

In the end of the proof of Theorem 7.1, I said
「 $g(z) - m > g_{n_j}(z)$ in D_{n_j}」. This is fault, because the domain D of definition of g does not always contain D_{n_j}.
I fixed it in p. 9. Please check it.

Thank you,

Hiroshi Yamaguchi.

§ 7 Uniformity

In order to show the uniformity of the titre, we need the notión of logarithmic capacity.

For a bounded open set O, we put

$$C(O) = \sup_{\substack{K \Subset O \\ K:\text{ compact}}} \{d_\infty(K)\}.$$

For a bonded set F, we t

$$C(F) = \inf_{\substack{F \subset O \\ O:\text{ open}}} \{C(O)\} \qquad (\text{not } F \Subset O)$$

For any set F, we put

$$C(F) = \lim_{R\to\infty} C(F \cap \{|z|<R\})$$

We call $C(F)$ the logarithmic capacity of F.

<u>Remark 7.1</u> If K is compact, then $C(K) = d_\infty(K)$

In fact, $C(K) \geqq d_\infty(K)$ is visible. For the converse, take a sequence of open sets:

$$\Omega_1 \gg \Omega_2 \gg \cdots \gg \Omega_n \gg \cdots \gg K \text{ s.t. } \overline{\Omega}_n \searrow K$$

For exemple, put $\Omega_n = \{z \in \mathbb{C} \mid d(z, K) < 1/n\}$.

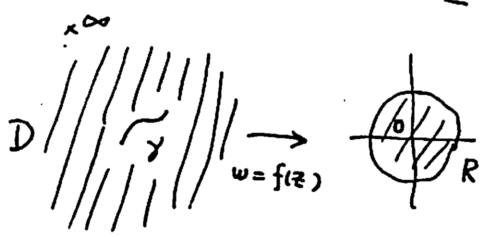
Then we already proved $\lim_{n\to\infty} d_\infty(\overline{\Omega}_n) = d_\infty(K)$

Since $d_\infty(\overline{\Omega}_n) \geqq C(\Omega_{n+1})$, it holds $d_\infty(K) \geqq C(K)$.

It is clear that $C(F_1) \geqq C(F_2)$ for $F_1 \supseteq F_2$.

Let F be a set which contains an arc γ. Then $C(F) \geqq C(\gamma) > 0$.

Indeed, put $D = \overline{\mathbb{C}_z} - \gamma$ and form the Green's



function $g(z)$ and the Robin constant λ for (D, ∞).
By Szegö's Theorem $d_\infty(\gamma) = e^{-\lambda}$. On the other hand, by Riemann's mapping Theorem, D is mapped conformally onto a disk $|w| < R$ of finite radius in such a way that

$$w = f(z) = \frac{1}{z} + \frac{a_2}{z^2} + \cdots \quad \text{at } \infty$$

Since $\log R/|f(z)| = g(z)$ in D, we get $\log R = \lambda < +\infty$. Hence $c(F) \geq c(\gamma) = d_\infty(\gamma) = 1/R > 0$.

Let us show a necessary and sufficient condition that a set F is of logarithmic capacity is zero:

<u>Theorem 7.1</u> Let $s(z) \not\equiv -\infty$ be a subharmonic function in a domain D in $\mathbb{C}$. Put $F = \{z \in D \mid s(z) = -\infty\}$. Then $c(F) = 0$.

<u>Theorem 7.2</u> Let F be a set in $\mathbb{C}$ suppose that $c(F) = 0$. Then there exists a subharmonic function $s(z)$ in $\mathbb{C}$ such that $s(z) = -\infty$ on F. —— Evans' potential.

In the case F is compact, the proofs of both Theorems are somewhat easier. For exemple, suppose F is compact and $c(F) = 0$. For Theorem 7.2, take $R > 0$ such that $(|z| < R) \supset\supset F$. It suffices to prove that, given $M > 1$, there exists a function $s_M(z)$ in $\mathbb{C}$ such that

(1) $s_M(z)$ is subharmonic in $\mathbb{C}$ and harmonic in $(|z| > R)$;
(2) $s_M(z) \leq -M$ on F;

(3) $s_M(z) < \log 3R + 1$ in $(|z| \leqq 3R)$

(4) $|s_M(z) - \log|z|| < 1$ in $(|z| \geqq 2R)$

In fact, for each $M = 2^n$ $(n=1,2,\cdots)$, construct the subharmonic fn $s_{2^n}(z)$ corresponding to 2^n. Then, if we put $s(z) \underset{def}{=} \sum_{n=1}^{\infty} (1/2^n)\, s_{2^n}(z)$, then $s(z)$ satisfies (1), (2′), (3), (4) where (2′) is 「$s(z) = -\infty$ on F」.

For, $p_n(z)$ $(n=1,2,\cdots)$ being subharmonic (resp. harmonic) and $\leqq 0$ in D, $\sum_{n=1}^{\infty} p_n(z)$ is subharmonic (resp. harmonic) and $\leqq 0$ in D, which may be $\equiv -\infty$ in D.

<u>Construction of $s_M(z)$</u>

Since $d_\infty(F) = 0$, we find $n \gg 1$ such that

$$\sqrt[n]{m_n} < e^{-M}$$

$$\|$$

$$\sup_{z\in F} \sqrt[n]{|t_n(z)|}$$ where $t_n(z) = (z-z_1)\cdots(z-z_n)$ is the T-polynomial belonging to F, so that $\{z_i\}_{i=1,\dots,n} \subset (|z| < R)$.

Put $s_M(z) = \frac{1}{n}\left(\log|z-z_1| + \cdots + \log|z-z_n|\right)$. Then $s_M(z)$ is subharmonic in $\mathbb{C}$ and harmonic in $(|z| > R)$ i.e., (1) is proved. It is clear that $s_M(z) < -M$ on F, which is (2). Moreover for $|z| < \infty$

$$s_M(z) = \log|z| + \frac{1}{n}\sum_{i=1}^{n} \log\left|1 - \frac{z_i}{z}\right|.$$

On $|z| = 3R$, we have $\left|\log\left|1-\frac{z_i}{z}\right|\right| \leqq \log\left|1 + \frac{R}{3R}\right| \leqq \frac{1}{3}$

$$\therefore\ s_M(z) \leqq \log 3R + 1 \quad \text{on } |z| = 3R.$$

Since $s_M(z)$ is subharmonic in $\mathbb{C}$, (3) follows

On $|z| = 2R$, we have

$$|s_M(z) - \log|z|| \leqq \left|\frac{1}{n}\sum_{i=1}^{n} \log\left|1-\frac{z_i}{z}\right|\right| = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2} = \frac{1}{2} < 1$$

Since $s_M(z) - \log|z|$ is harmonic in $(|z|>R) \cup \{\infty\}$, the maximum principle implies that $|s_M(z) - \log|z|| < 1$ in $(|z| > 2R)$. Assertion (4) is proved. Hence Theorem 7.2 in the case F is compact is proved.

<u>Lemma 7.1</u> Let F_i $(i = 1, 2, \cdots)$ be any set in $(|z| < \frac{1}{2})$. Then

$$\sum_{i=1}^{\infty} \frac{1}{\log \frac{1}{c(F_i)}} \geqq \frac{1}{\log \frac{1}{c(\bigcup_{i=1}^{\infty} F_i)}}$$

<u>Proof.</u> It suffices to prove

$$(7.1) \qquad \sum_{i=1}^{p} \frac{1}{\log \frac{1}{c(\bar{\Omega}_i)}} \geqq \frac{1}{\log \frac{1}{c(\bigcup_{i=1}^{p} \bar{\Omega}_i)}} \qquad (p : \text{finite})$$

where $\bar{\Omega}_i$ $(i = 1, \cdots, p)$ is a compact set with smooth boundary such that $\bar{\Omega}_i \subset (|z| < \frac{1}{2})$.

In fact, assume that (7.1) is true. First remark $c(|z| < \frac{1}{2}) = \frac{1}{2}$, so that $c(F_i) \leqq c(\bigcup_{i=1}^{\infty} F_i) \leqq \frac{1}{2}$.

Let $\varepsilon > 0$ be given. Choose an open O_i such that $O_i \supset F_i$ and

$$\frac{1}{\log \frac{1}{c(F_i)}} + \frac{\varepsilon}{2^i} > \frac{1}{\log \frac{1}{c(O_i)}} \qquad (i = 1, 2, \cdots)$$

$$\therefore \left(\sum_{i=1}^{\infty} \frac{1}{\log \frac{1}{c(F_i)}} \right) + \varepsilon > \sum_{i=1}^{\infty} \frac{1}{\log \frac{1}{c(O_i)}}$$

Moreover, choose a compact K such that $K \subset \bigcup_{i=1}^{\infty} O_i$ and

$$\frac{1}{\log \frac{1}{c(\bigcup_{i=1}^{\infty} F_i)}} - \varepsilon \leqq \frac{1}{\log \frac{1}{c(\bigcup_{i=1}^{\infty} O_i)}} - \varepsilon < \frac{1}{\log \frac{1}{c(K)}}$$

By Borel-Lebesgue Theorem, there exists a finite number of

$O_1, O_2, \cdots, O_p$ such that $\bigcup_{i=1}^{p} O_i \supset K$. Since K is compact, we easily find an open Ω_i with smooth boundary such that $\bar{\Omega}_i \subset O_i$ and $\bigcup_{i=1}^{p} \bar{\Omega}_i \supset K$. It follows from (7.1) that

$$\sum_{i=1}^{p} \frac{1}{\log \frac{1}{c(\bar{\Omega}_i)}} \geqq \frac{1}{\log \frac{1}{c(\bigcup_{i=1}^{p} \bar{\Omega}_i)}}$$

$$\begin{array}{ccc} \wedge\!\!\| & & \vee\!\!\| \\ \sum_{i=1}^{\infty} \frac{1}{\log \frac{1}{c(O_i)}} & & \frac{1}{\log \frac{1}{c(K)}} \end{array}$$

Hence

$$\left(\sum_{i=1}^{\infty} \frac{1}{\log \frac{1}{c(F_i)}}\right) + \varepsilon > \frac{1}{\log \frac{1}{c(\bigcup_{i=1}^{\infty} F_i)}} - \varepsilon$$

$\varepsilon > 0$ being arbitrary, Lemma 7.1 is proved.

Proof of (7.1). First, let us prove the following

Proposition Given $0 < \rho_1, \rho_2, \cdots, \rho_p < 1$. Put

$$r = \operatorname*{Inf}_{\left(\substack{\delta_1 + \cdots + \delta_p = 1 \\ \delta_i > 0}\right)} \{ \operatorname{Max}(\rho_1^{\delta_1}, \rho_2^{\delta_2}, \cdots, \rho_p^{\delta_p}) \}$$

Then

$$(7.2) \qquad \frac{1}{\log \frac{1}{r}} \leqq \frac{1}{\log \frac{1}{\delta_1}} + \cdots + \frac{1}{\log \frac{1}{\delta_p}}$$

Indeed, choose $0 < \delta_1, \cdots, \delta_p < 1$ such that

$$\rho_1^{\delta_1} = \rho_2^{\delta_2} = \cdots = \rho_p^{\delta_p} \quad \text{and} \quad \delta_1 + \delta_2 + \cdots + \delta_p = 1.$$

They certainly exist. Hence

$$r \leqq \rho_1^{\delta_1} < 1, \quad \cdots, \quad r \leqq \rho_p^{\delta_p} < 1$$

$$\therefore \quad \frac{1}{r} \geqq \left(\frac{1}{\rho_1}\right)^{\delta_1} > 1, \quad \cdots, \quad \frac{1}{r} \geqq \left(\frac{1}{\rho_p}\right)^{\delta_p} > 1$$

$$\therefore \quad (\log \tfrac{1}{r}) / (\log \tfrac{1}{\rho_1}) \geqq \delta_1, \quad \cdots, \quad (\log \tfrac{1}{r}) / (\log \tfrac{1}{\rho_p}) \geqq \delta_p.$$

Hence $\left(\log\frac{1}{r}\right)\left(\frac{1}{\log\frac{1}{\rho_1}}+\cdots+\frac{1}{\log\frac{1}{\rho_p}}\right) \geqq \delta_1+\cdots+\delta_p = 1$,

which proves the inequality (7.2).

Next, let us prove (7.1). Given $\varepsilon, \eta > 0$, we find an integer n_0 $\ni$ $\forall n \geqq n_0$ such that

$$(7.3) \quad \rho_n(\overline{\Omega}_i) = \sqrt[n]{m_n(\overline{\Omega}_i)} < \rho(\overline{\Omega}_i)+\varepsilon \quad ;$$

$$(7.4) \quad \rho\left(\bigcup_{i=1}^{p}\overline{\Omega}_i\right)-\eta < \sqrt[n]{m_n\left(\bigcup_{i=1}^{p}\overline{\Omega}_i\right)} = \rho_n\left(\bigcup_{i=1}^{p}\overline{\Omega}_i\right).$$

Therefore, $|t_n(\overline{\Omega}_i, z)| < (\rho(\overline{\Omega}_i)+\varepsilon)^n$ on $\overline{\Omega}_i$ $(i=1,\cdots,p)$

Let $m_1, m_2, \cdots, m_p$ be any integers $\geqq 1$. Then

$$|t_n(\overline{\Omega}_i, z)^{m_i}| < (\rho(\overline{\Omega}_i)+\varepsilon)^{nm_i} \quad \text{on } \overline{\Omega}_i \quad (i=1,\cdots,p)$$

Consider the monic polynomial of degree $nm_1+\cdots+nm_p$ such that

$$q(z) = t_n(\overline{\Omega}_1, z)^{m_1}\cdots\cdots t_n(\overline{\Omega}_p, z)^{m_p}.$$

By $\overline{\Omega}_i \subset (|z|<\frac{1}{2})$, we see that

$$\{\text{the solutions of } t_n(\overline{\Omega}_i, z)=0\} \subset \left(|z| < \frac{1}{2}\right)$$

and hence that $\quad |t_n(\overline{\Omega}_i, z)| < 1$ on $\overline{\Omega}_1 \cup\cdots\cup \overline{\Omega}_n$.

Consequently,

$$|q(z)| \leqq (\rho(\overline{\Omega}_i)+\varepsilon)^{nm_i} \quad \text{on } \overline{\Omega}_i \ (i=1,\cdots,p)$$

$$|q(z)| \leqq \max_{i=1,\cdots,p}\{(\rho(\overline{\Omega}_i)+\varepsilon)^{nm_i}\} \quad \text{on } \overline{\Omega}_1\cup\cdots\cup\overline{\Omega}_p.$$

By definition of T-constant, we have

$$m_{nm_1+\cdots+nm_n}(\overline{\Omega}_1\cup\cdots\cup\overline{\Omega}_p) \leqq \max_{i=1,\cdots,p}\left\{(\rho(\overline{\Omega}_i)+\varepsilon)^{nm_i}\right\}$$

Since $n \geqq n_0$, we have by (7.4)

$$m_{nm_1+\cdots+nm_n}(\overline{\Omega}_1\cup\cdots\cup\overline{\Omega}_p) \geqq \left(\rho(\overline{\Omega}_1\cup\cdots\cup\overline{\Omega}_p)-\eta\right)^{nm_1+\cdots+nm_p},$$

so that $\rho(\bigcup_{i=1}^{p} \overline{\Omega_i}) - \eta \leqq \operatorname{Max}_{i=1,\cdots,p} \left\{ \left(\rho(\overline{\Omega_i}) + \varepsilon \right)^{\frac{m_i}{m_1+\cdots+m_p}} \right\}$

Because $\varepsilon, \eta > 0$ are arbitrary, we have

$$\rho(\cup_{i=1}^{p} \overline{\Omega_i}) \leqq \operatorname{Max}_{i=1,\cdots,p} \left\{ \rho(\overline{\Omega_i})^{\frac{m_i}{m_1+\cdots+m_p}} \right\}$$

Since $m_1, \cdots, m_p \geqq 1$ are arbitrary integer, we see that

$$\rho(\cup_{i=1}^{p} \overline{\Omega_i}) \leqq \operatorname*{Inf}_{\substack{m_1,\cdots,m_p \geqq 1 \\ \text{integers}}} \left(\operatorname{Max}_{i=1,\cdots,p} \left\{ \rho(\overline{\Omega_i})^{\frac{m_i}{m_1+\cdots+m_p}} \right\} \right)$$

$$= \operatorname*{Inf}_{\substack{\delta_1+\cdots+\delta_p=1 \\ \delta_i>0,\ \text{rational}}} \left(\operatorname{Max} \left\{ \rho(\overline{\Omega_i})^{\delta_i} \right\} \right)$$

$$= \operatorname*{Inf}_{\substack{\delta_1+\cdots+\delta_p=1 \\ \delta_i>0,\ \text{real}}} \left(\operatorname{Max} \left\{ \rho(\overline{\Omega_i})^{\delta_i} \right\} \right) = r$$

It follows by Proposition that

$$\frac{1}{\log \frac{1}{\rho(\cup_{i=1}^{p} \overline{\Omega_i})}} \leqq \frac{1}{\log \frac{1}{r}} \leqq \sum_{i=1}^{p} \frac{1}{\log \frac{1}{\rho(\overline{\Omega_i})}}.$$

Since $\rho(K) = d_\infty(K) = C(K)$ for compact K, (7.1) is proved.

Corollary 7.1 If $c(F_i) = 0$ $(i=1,2,\cdots)$, then $c(\bigcup_{i=1}^{\infty} F_i) = 0$.

⌈ ∵ When each $F_i \subset (|z| < \frac{1}{2})$, Corollary 7.1 follows Lemma 7.1. In general case, observe that

$$c(\bigcup_{i=1}^{\infty} F_i) = \lim_{R\to\infty} c\left(\left(\bigcup_{i=1}^{\infty} F_i \right) \cap (|z|<R) \right) \text{ and } c(pF) = p \cdot c(F)$$

where $pF = \{ pz \mid z \in F \}$. We thus see that $c(F_i) = 0$ $(i=1,2,\cdots)$ yields $c(\bigcup_{i=1}^{\infty} F_i) = 0$. ⌋

<u>Proof of Theorem 7.1</u> By Corollary 7.1, it suffices to prove that, for any $z_0 \in F$, any $r > 0$ such that $V_{2r} = (|z - z_0| < 2r) \Subset D$, it holds $c(F_r) = 0$ where $F_r = F \cap V_r$.

If we put $O_n = \{z \in D \,/\, s(z) < -n\} \cap V_r$ $(n = 1, 2, \cdots)$, then O_n is open : $O_1 \supset O_2 \supset \cdots \supset O_n \supset \cdots$ and $\bigcap_{n=1}^{\infty} O_n = F_r$ ($\because$ $s(z)$ is uppersemiconti' in D). Let us prove

$$\lim_{n \to \infty} c(O_n) = 0$$

by contradiction. Assume that $\lim_{n \to \infty} c(O_n) = \delta > 0$.

$\therefore$ $\exists\, n_j$ $(j = 1, 2, \cdots)$ $\ni$ $c(O_{n_j}) > \frac{\delta}{2}$.

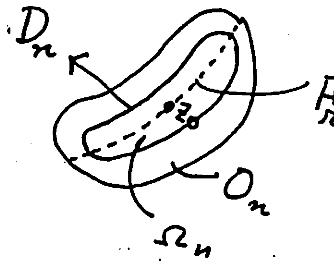
We write anew $n_j = n$. For each $n = 1, 2, \cdots$, we find a compact $\overline{\Omega_n}$ with smooth boundary such that

$$\overline{\Omega_n} \Subset O_n \quad \text{and} \quad c(\overline{\Omega_n}) > \frac{\delta}{2}.$$

Putting $D_n = \overline{\mathbb{C}_z} - \overline{\Omega_n}$, we form the Green's function $g_n(z)$ and the Robin constant for (D_n, ∞). By Szegö's Theorem

$$c(\overline{\Omega_n}) = d_\infty(\overline{\Omega_n}) = e^{-\lambda_n}$$

$\therefore$ $\lambda_n < \log \frac{2}{\delta}$ $(n = 1, 2, \cdots)$



On the other hand, we have the Equilibrium potential w.r.t. $\overline{\Omega_n}$:

$$N_{\rho_n}(z) = \begin{cases} \lambda_n - g_n(z) & \text{in } D_n \\ \lambda_n & \text{in } \overline{\Omega_n} \end{cases}$$

$$\| $$

$$\int_{\partial D_n} \log(1/|z - \zeta|)\, \rho_n(\zeta)\, ds_\zeta$$

$$\therefore\; g_n(z) = \lambda_n + \int_{\partial D_n} (\log|z - \zeta|)\, \rho_n(\zeta)\, ds_\zeta \quad \text{in } D_n$$

On ∂V_{2r}, we have

$$g_1(z) \leqq g_n(z) \leqq \log\frac{2}{\delta} + \log|z - z_0| + \int_{\partial D_n} \log\left|1 - \frac{\zeta - z_0}{z - z_0}\right| \rho_n(\zeta)\, ds_\zeta$$

$$\leqq \log\frac{2}{\delta} + \log 2r + \frac{1}{2}$$

Hence $g_n(z)$ is uniformly bdd in $V_{2r} \cap D_n$. Since $g_n(z)$ is harmonic, it follows that

$$\exists\, n_j \ni \cdot\ g_{n_j}(z) \longrightarrow g(z) \quad \text{in } V_{2r} \cap D$$

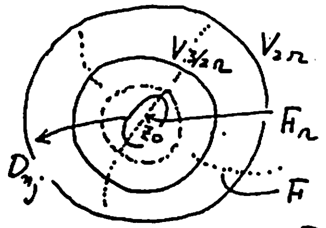
where $D = \left(\bigcap_{n=1}^{\infty} \bar{O}_n\right)^c \ (\subset F_r^c)$. On the other hand, $g_{n_j}(z)$ has the same singularity $\log|z|$ at ∞, it follows that $\lim_{j\to\infty} g_{n_j}(z) = g(z)$ exists in $D - V_{2r}$ and $g(z)$ has the singularity $\log|z|$ at ∞. Hence $g(z)$ becomes a non-constant positive harmonic function in D. If we put $m_1 = \min_{z\in\partial V_{2r}} g(z)$ and $m_2 = \min_{z\in\partial V_{\frac{3}{2}r}} g(z)$, then we thus have $m_1 \neq m_2$. Since $g_{n_j}(z) = 0$ on ∂D_{n_j} $(\subset\subset V_{\frac{3}{2}r})$, we see that

$$(7.5) \qquad m_1 \gneqq m_2$$

Given $\varepsilon > 0$, we find $J \geq 1 \ni \cdot\ g_{n_j}(z) \geq m_1 - \varepsilon$ for $\forall z \in \partial V_{2r}$ and $\forall j \geq J$. Putting $\alpha = \max_{z\in\partial V_{2r}} s(z)$, we consider the superharmonic function

$$u_j(z) = g_{n_j}(z) - (m_1 - \varepsilon)\left(1 + \frac{s(z)-\alpha}{n_j + \alpha}\right) \quad \text{in } V_{2r} \cap D_{n_j}.$$

On ∂V_{2r}, $u_j(z) \geq (m_1 - \varepsilon) - (m_1 - \varepsilon)\left(1 + \frac{\alpha - \alpha}{n_j + \alpha}\right)$
$= 0.$



Since $\partial D_{n_j} \subset O_{n_j} = \{z \in D \mid s(z) < -n_j\}$, we have, on $\partial V_{3/2 r}$

$$u_j(z) \geq 0 - (m_1 - \varepsilon)\left(1 + \frac{-n_j - \alpha}{n_j + \alpha}\right) = 0.$$

It follows that $u_j(z) \geq 0$ in $V_{2r} \cap D_{n_j}$, and hence

$$g(z) \geq (m_1 - \varepsilon) \lim_{j\to\infty}\left(1 + \frac{s(z)-\alpha}{n_j + \alpha}\right) \quad \text{in } V_{2r} \cap D.$$

Since $s(z) > -\infty$ on F^c, we have

$$g(z) \geq (m_1 - \varepsilon) \quad \text{in } (V_{2r} \cap D) - F$$

Because $g(z)$ is continuous $V_{2r} \cap D$ and F has no inner point,

$$g(z) \geq m_1 - \varepsilon \quad \text{in } V_{2r} \cap D \qquad (\because c(F) = 0)$$

$\varepsilon > 0$ being arbitrary, we have $g(z) \geq m_1$ in $V_{2r} \cap D$, so that $m_2 \geqq m_1$. This contradicts (7.5). Hence $\lim_{n\to\infty} c(O_n) = 0$ and $c(F) = 0$. Theorem 7.1 is proved. c.q.f.d.

Lemma 7.2 Let F be a bounded set in $\mathbb{C}$ and $c(F)=0$. Assume that $F \Subset (|z|<R)$. Then we find a function $s(z) \not\equiv -\infty$ such that

(1) $s(z)$ is subharmonic in $\mathbb{C}$ and is harmonic in $\mathbb{C} - (|z|<R)$;

(2) $s(z) = -\infty$ on F;

(3) $\log R \leqq s(z) \leqq \log 4R$ on $(2R<|z|<3R)$;

(4) $s(z) = \log|z| + \alpha + O(\frac{1}{|z|})$ at ∞

where $|\alpha| \leqq 1$ and $O(\frac{1}{|z|}) \leqq \log\frac{R}{|z|}$

Proof. We put $[\Gamma_R] = (|z|<R)$ and $\Gamma_R = \{|z|=R\}$, so that $F \Subset [\Gamma_R]$. It suffices for Lemma 7.2 to prove that

Given $M>1$, we have a function $s_M(z)$ in $\mathbb{C}$ such that

(1) $s_M(z)$ is subharmonic in $\mathbb{C}$ and harmonic in $\mathbb{C} - [\Gamma_R]$;

(2) $s_M \leqq -M$ on F;

(3) $\log R \leqq s_M(z) \leqq \log 4R$ on $(2R<|z|<3R)$;

(4) $s_M(z) = \log|z| + \alpha_M + O(\frac{1}{|z|})$ at ∞

where $|\alpha_M| \leqq 1$ and $O(\frac{1}{|z|}) \leqq \log\frac{R}{|z|}$.

In fact, if we construct $s_M(z)$ for given M, then put

$$s(z) \underset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} s_{2^n}(z) \quad \text{in } \mathbb{C}.$$

Then $s(z)$ easily satisfies conditions (1) ~ (4) in Lemma 7.2.

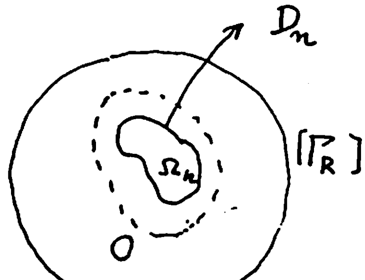
Construction of $s_M(z)$: Let $M>1$ be given.

By assumption: $c(F)=0$, we find an open O such that

$[\Gamma(R)] \supset O \supset F$ and $c(O) < e^{-M}$.

Put $c(O) = e^{-M^*}$, so that $M^* \geqq M$.
Choose a sequence of compacts $\overline{\Omega}_n$ with smooth boundary in O such that

$$\Omega_1 \Subset \Omega_2 \Subset \cdots \;;\; \bigcup_{n=1}^{\infty} \Omega_n = O$$



It follows that $c(\overline{\Omega}_n) \nearrow c(O)$.
Hence $\frac{1}{2} e^{-M^*} < c(\overline{\Omega}_n) < e^{-M^*}$ for sufficiently lage $n \geqq {}^\exists N$.
Put $D_n = \overline{\mathbb{C}}_z - \overline{\Omega}_n$ and form the Green's function $g_n(z)$ and the Robin constant λ_n for (D_n, ∞).
By Szegö's Theorem, we have $e^{-\lambda_n} = c(\overline{\Omega}_n)$, so that $-\log 2 - M^* < -\lambda_n < -M^*$, or, $\underline{0 < \lambda_n - M^* < \log 2}$ for $n \geqq N$.
Here, we put

$$G_n(z) = \begin{cases} g_n(z) - M^* & \text{on } D_n \\ -M^* & \text{on } \overline{\Omega}_n \end{cases}$$

By using the equilibrium potential, we write

$$G_n(z) = \begin{cases} \lambda_n - M^* + \displaystyle\int_{\partial D_n} (\log|z-\zeta|)\rho_n(\zeta)\,ds_\zeta & \text{on } D_n \\ -M^* & \text{on } \overline{\Omega}_n \end{cases}$$

It follows that

(1) $G_n(z)$ is subharmonic in $\mathbb{C}$ and harmonic in D_n $(\supset \mathbb{C} - [\Gamma_R])$.

(2) $G_n(z) = -M^* \leqq -M$ on $\overline{\Omega}_n$;

(3) In $(2R < |z| < 3R)$, we have

$$G_n(z) = \lambda_n - M^* + \log|z| + \int_{\partial D_n} \left(\log\left|1 - \frac{\zeta}{z}\right|\right) \rho_n(\zeta)\,ds_\zeta$$

and

$$\underline{\lambda_n - M^*} + \log 2R - \log\frac{3}{2} < G_n(z) < \underline{\lambda_n - M^*} + \log 3R + \log\frac{3}{2}.$$

$$\therefore \quad \log R < G_n(z) < \log 4R \quad \text{in } (2R < |z| < 3R)$$

(4) At ∞, we have

$$G_n(z) = \lambda_n - M^* + \log|z| + O\left(\frac{1}{|z|}\right)$$

where $|\lambda_n - M^*| < 1$ and $O(\frac{1}{|z|}) \leqq \frac{R}{|z|}$ ($\because \partial D_n \subset [\Gamma_R]$).

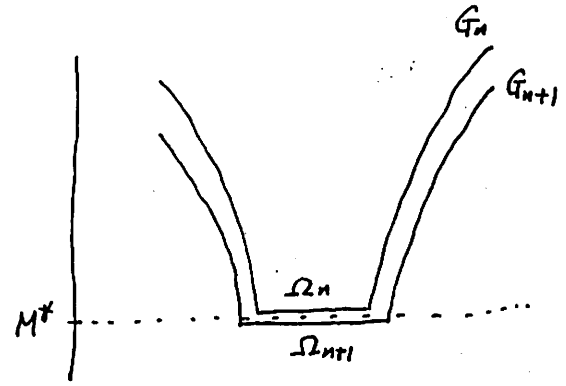
On the other hand, we get

$$G_n \geqq G_{n+1} \quad \text{in } \mathbb{C}$$

⌈ In fact, it is clear that

$$G_n(z) \geqq -M^* \text{ on } \mathbb{C}$$

$\therefore$ $G_n(z) \geqq G_{n+1}(z)$ on $\overline{\Omega_{n+1}}$.



Consider the function

$u_n(z) = G_n(z) - G_{n+1}(z)$

in D_{n+1} i.e., outside of $\overline{\Omega_{n+1}}^c$.

Then $u_n(z)$ is harmonic in D_{n+1}. Moreover, $u_n(z)$ is bounded at ∞ with $u_n(\infty) = \lambda_n - \lambda_{n+1}$, and on ∂D_{n+1}, $u_n(z) \geqq -M^* + M^* = 0$. It follows from the maximum principle that $u_n(z) \geqq 0$ in D_{n+1}, i.e., $G_n(z) \geqq G_{n+1}(z)$ in D_{n+1}.

Consequently, $G_n(z) \geqq G_{n+1}(z)$ in the whole $\mathbb{C}$. ⌋

We thus have the limit

$$G(z) = \lim_{n\to\infty} G_n(z) \quad \text{in } \mathbb{C},$$

which satisfies the following conditions:

(1) $G(z)$ is subharmonic in $\mathbb{C}$ and harmonic in $\overline{O}^c$ $(\supset \mathbb{C} - [\Gamma_R])$;

(2) $G(z) = -M^* \leqq -M$ on $\bigcup_{n=1}^{\infty} \overline{\Omega_n} = O$;

(3) In $(2R < |z| < 3R)$, $\log R < G(z) < \log 4R$;

(4) At ∞, we get

$$G(z) = \lambda - M^* + \log|z| + O\left(\frac{1}{|z|}\right)$$

where $\lambda = \lim_{n\to\infty} \lambda_n = \log 1/c(O)$, $|\lambda - M^*| < 1$ and $O(\frac{1}{|z|}) \leqq R/|z|$.

It follows that if we put $s_M(z) = G(z)$ in $\mathbb{C}$, then $s_M(z)$ is one of the desired functions. c. q. f. d.

Proof of Theorem 7.2. For each $n = 1, 2, \cdots$ consider the disk $[\Gamma_{\frac{1}{5}n}] = \{|z-n| < \frac{1}{5}n\}$ and put $F_n = F \cap [\Gamma_{\frac{1}{5}n}]$. By Lemma 7.2, we find a function $s_n(z)$ such that

(1) $s_n(z)$ is subharmonic in $\mathbb{C}$ and harmonic in $[\Gamma_{\frac{1}{5}n}]^c$;

(2) $s_n(z) = -\infty$ on F_n ;

(3) $\log\frac{n}{5} < s_n(z) < \log\frac{4}{5}n$ in $\left(\frac{2}{5}n < |z-n| < \frac{3}{5}n\right)$;

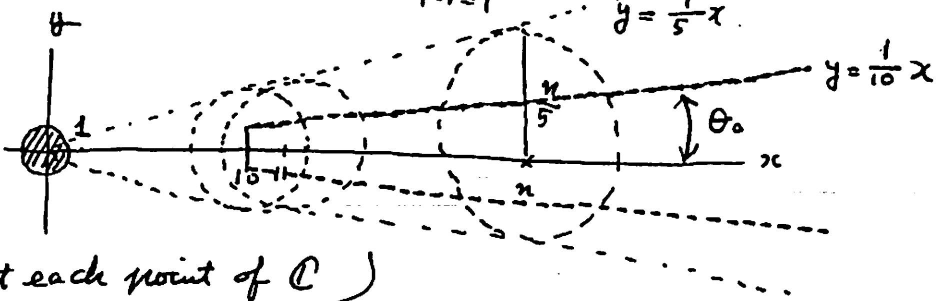
(4) $s_n(z) = \log|z-n| + \mu_n + O(\frac{1}{|z-n|})$ at ∞ where $|\mu_n| < 1$ and $O\left(\frac{1}{|z-n|}\right) \leqq \dfrac{\frac{1}{5}n}{|z-n|}$.

Let K be any compact set in $\mathbb{C}$. Since $\lim\limits_{n\to\infty} \operatorname{dist}(K, \Gamma_{\frac{1}{5}n}) = \infty$, the property (4) implies that

$\exists N_K$ (integer) $\ni$ $s_n(z) > 0$ for all $n \geqq N_K$ and $\forall z \in K$.

Put

$$\overset{(1)}{s}(z) = \sum_{n=10}^{\infty} \frac{1}{2^n} \frac{s_n(z)}{\max\limits_{|z|\leq 1}\{s_n(z)\}} \quad \text{in } \mathbb{C}$$



Then (at each point of $\mathbb{C}$) each term is positive harmonic except for finite terms which is subharmonic. It follows that $s(z)$ is subharmonic in $\mathbb{C}$ such that $\overset{(1)}{s}(z) = -\infty$ in $\bigcup\limits_{n=10}^{\infty} F_n$ and $0 < \overset{(1)}{s}(z) < 1$ in $(|z|<1)$.

We write $\theta_0 = \tan^{-1}\frac{1}{10} (>0)$. Then it is clear that

$$\bigcup_{n=10}^{\infty} F_n \supset F \cap \{|z| > 10\} \cap \{-\theta_0 < \arg z < \theta_0\}.$$

By the same argument we find subharmonic functions

$$s^{(i)}(z) \quad (\, i = 2, 3, \cdots, [\, 2\pi/\theta_0\,]\,)$$

in $\mathbb{C}$ such that

Gauss notation.

$$s^{(i)}(z) = -\infty \text{ in } F \cap \{|z|>10\} \cap \{ i\theta < \arg z < (i+1)\theta \}.$$

By Lemma 7.2, we have a subharmonic fn $s^{(0)}(z)$ in $\mathbb{C}$ such that $s^{(0)}(z) = -\infty$ in $F \cap \{|z|<11\}$. It follows that the sum

$$s(z) = s^{(0)}(z) + s^{(1)}(z) + \cdots + s^{l}(z) \quad \text{for } z \in \mathbb{C}$$

where $l = [2\pi/\theta_0]$, is a subharmonic fn in $\mathbb{C}$ such that $s(z) = -\infty$ in F. Theorem 7.2 is proved. q.e.d.

Now, let D be a domain of $\mathbb{C}$ and let $\zeta \in D$. We defined the Green's function $g(z)$ and the Robin constant λ for (D, ζ). As we noted, $g \equiv +\infty$ iff $\lambda = +\infty$. Moreover, whether $\lambda = +\infty$ or $< +\infty$ does not depend on the choice of ζ.

Definition 7.2 A domain D with $\lambda = +\infty$ (resp. $< +\infty$) is said to be parabolic (resp. hyperbolic).

Szegö's Theorem says that, if we put $K = \mathbb{C}_w - \frac{1}{\zeta - z}(D)$, a domain D is parabolic iff K is of logarithmic capacity 0. We remark that parabolicity is easily extended to the case of open Riemann surface.

We return to the discussion concerning the variation of domains in $\mathbb{C}$. Let $\mathcal{D}$ be a domain of $B \times \mathbb{C}_z$ where B is a region of the complex t-plane. Then we have the variation

$$\mathcal{D} : t \to D(t) \quad (t \in B)$$

where $D(t) = \{ z \in \mathbb{C} \mid (t, z) \in \mathcal{D} \}$. Put

$$F = \{ t \in B \mid D(t) \text{ is connected and parabolic} \}.$$

Under these notations, we obtain

<u>Theorem 7.3</u> (Uniformity).

Assume that $\mathcal{D}$ is a domain of holomorphy in $\mathbb{C}^2$. Then if $c(F) > 0$, then $F = B$, i.e., each fiber $D(t)$ is parabolic.

<u>Proof.</u> By Corollary 7.1, there exists a point $t_0 \in B$ such that $c(F \cap B_r) > 0$ for any $B_r = (\ |t - t_0| < r)$ where $r > 0$. Take $z_0 \in D(t_0)$ and choose $r_0 > 0$ such that $B_{r_0} \times \{z_0\} \subset \mathcal{D}$. We form the Robin constant $\lambda(t)$ for $(D(t), \zeta)$ where $t \in B_{r_0}$. Our main theorem says that $\lambda(t)$ is subharmonic in B_{r_0}. On the other hand, $\lambda(t) = +\infty$ for $t \in F \cap B_{r_0}$. Since $c(F \cap B_{r_0}) > 0$, it follows from Theorem 7.1 that $\lambda(t) \equiv +\infty$ in B_{r_0}. Hence $D(t)$ $(t \in B_{r_0})$ is connected and parabolic, and therefore $F \supset B_{r_0}$. If we repeat the same procedure at each boundary point t_1 of B_{r_0}, then we reach $F = B$.

c.q.f.d.

Dear

This is a note for last week's lecture, essentially based on H. E. Rauch (Comm. Pur. Appl. Math., 12 (1959) pp. 543–560

Thank you,
Hiroshi Yamaguchi

II. Variations of domains with moving branch points.

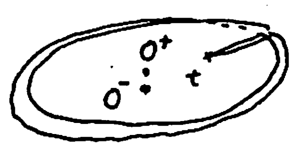
§1. Exemple.

In Chap. I, we discussed the variations of domains in the complex z-plane. In this Chapter we study special variations of ramified domains over the complex z-plane. Let us begin with a simple exemple.

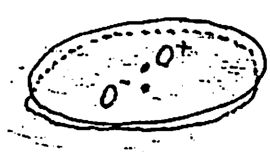
Put $B = \{t \in \mathbb{C} \mid 0 < |t| < 2\}$. Given $t \in B$, let $R(t)$ denote the Riemann surface over the complex z-plane determined by the function $\sqrt{z - t}$. Consider the part of $R(t)$ over the unit disk $|z| < 1$ and denote it by $D(t)$. Therefore,

If $t \in (0 < |t| < 1)$, then $D(t)$ is a twice-sheeted domain over the disk $|z| < 1$ whose branch point is found at $z = t$ and whose boundary C (independent of t) is a closed twice-rounding circle found on $|z| = 1$.

If $t \in (1 \leqq t < 2)$, then $D(t)$ consists of two separated unit disks $\{D^+, D^-\}$ over the disk $|z| < 1$.



for $t \in (0 < |t| < 1)$



for $2 > |t| \geqq 1$

We thus have the variation

$$\mathcal{D} : t \longrightarrow D(t) \quad (t \in B).$$

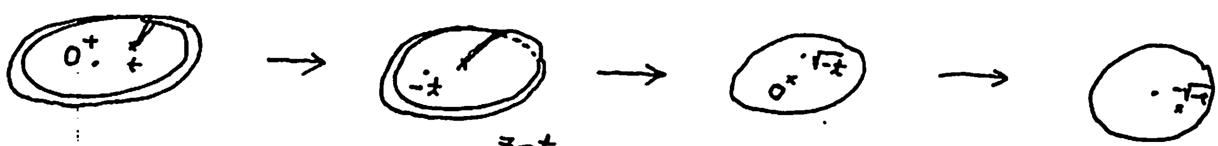
We write $\mathcal{D} = \bigcup_{t \in B} (t, D(t))$, which becomes a twice-sheeted ramified domain over $B \times \mathbb{C}$, i.e., $\mathcal{D}$ is the Riemann domain of the function $\sqrt{z - t}$ over $B \times (|z| < 1)$. It is clear that $\mathcal{D}$ is a domain of holomorphy over $\mathbb{C}^2$.

Each $D(t)$ $(t \in B)$ has two points $\{0^+, 0^-\}$ over the origin $z=0$. Then we have the Green's function $g(t,z)$ and the Robin constant $\lambda(t)$ for $(D(t), 0^+)$, so that in a neighborhood of 0^+

$$g(t,z) = \log\frac{1}{|z|} + \lambda(t) + h(t,z)$$

where $h(t,z)$ is harmonic for z and $h(t,0)=0$. Let us find the explicit form of $g(t,z)$ and $\lambda(t)$. We successively form the following analytic transformations:

Assume $0<|t|<1$. Then



$z \qquad z_1 = z_1(z) = \dfrac{z-t}{1-\bar{t}z} \qquad z_2 = z_2(z_1) = \sqrt{z_1} \qquad w = w(z_2) = \dfrac{z_2 - \sqrt{-t}}{1-\overline{\sqrt{-t}}\, z_2}$

By the combination : $w = w(z) = w \circ z_2 \circ z_1(z)$, the domain $D(t)$ is conformally transformed onto the unit disk $|w|<1$ such that $w(0^+)=0$. Hence

$$g(t,z) = \log\frac{1}{|w|}$$

where $w = w(z) = \left(\sqrt{(z-t)/(1-\bar{t}z)} - \sqrt{-t}\right) \Big/ \left(1 - \overline{\sqrt{-t}}\sqrt{(z-t)/(1-\bar{t}z)}\right)$,

$$\lambda(t) = -\log |w'(\sqrt{-t})|\cdot|z_2'(-t)|\cdot|z_1'(0^+)|$$

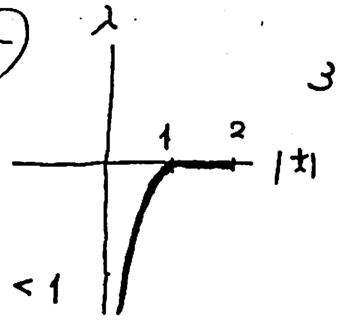
$$= -\log\left|\frac{1-|t|}{(1-|-t|)^2}\right|\left|\frac{1}{2}\frac{1}{\sqrt{-t}}\right|\cdot(1-|t|^2)$$

$$= -\log\frac{1+|t|}{2\sqrt{|t|}} \quad \text{where } 0<|t|<1.$$

In the case $2>|t|\geq 1$, we have

$$g(t,z) = \begin{cases} \log\frac{1}{|z|} & \text{on } D^+ \\ 0 & \text{on } D^- \end{cases}$$

$$\lambda(t) = 0$$



It follows that

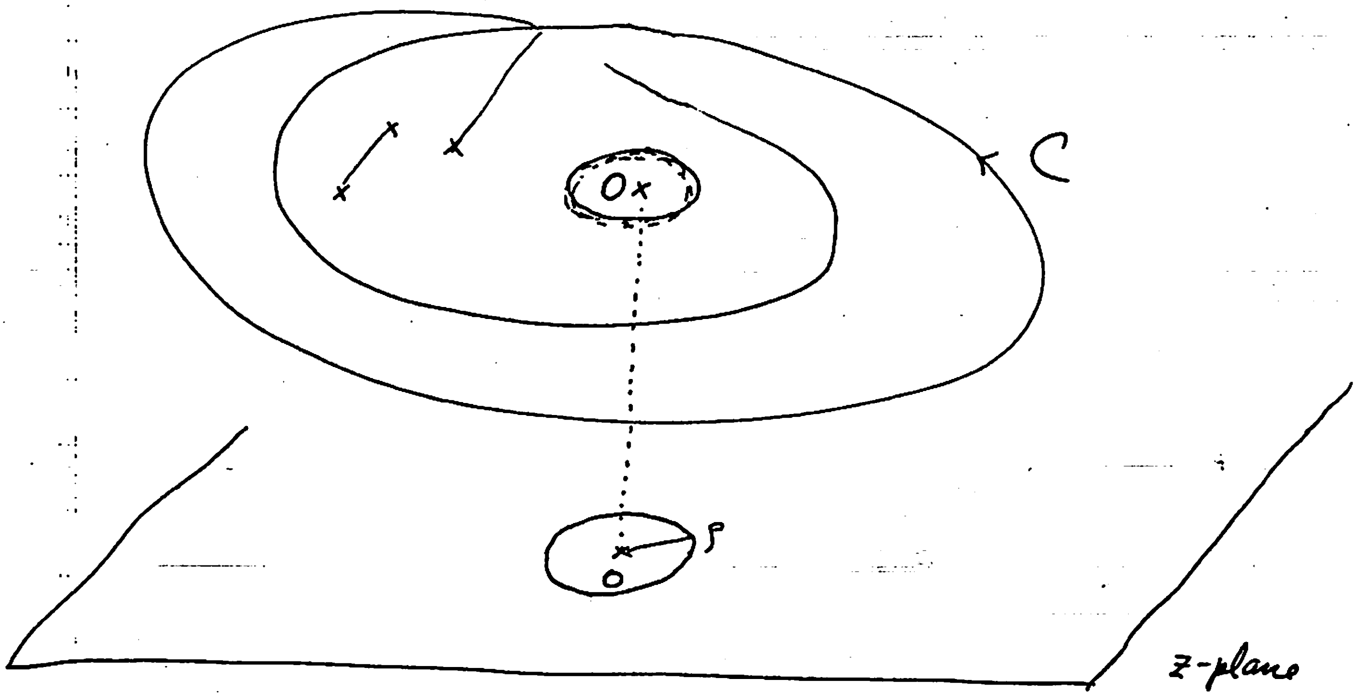
$$\frac{\partial\lambda}{\partial t} = \begin{cases} -\frac{1}{4}\frac{1}{t}\frac{|t|-1}{1+|t|} & \text{in } 0<|t|<1 \\ 0 & \text{in } 1 \leqq |t| < 2 \end{cases},$$

$$\frac{\partial^2\lambda}{\partial t \partial \bar t} = \begin{cases} -\frac{1}{4}\frac{1}{|t|(1+|t|)^2} < 0 & \text{in } 0<|t|<1 \\ 0 & \text{in } 1 \leqq t < 2 \end{cases}$$

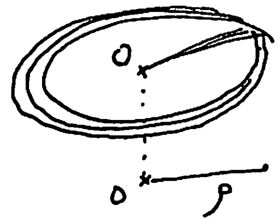
Consequently, $\lambda(t)$ is superharmonic on the whole $B = (0<|t|<$
and, moreover, $\lambda(t)$ is of class C^1 on B but not of class C^2 on $|t|=1$!!

§2. The case when one branch point is moving.

Let D be a non-compact Riemann surface over the complex z-plane $\mathbb{C}$ such that D is finitely sheeted over $\mathbb{C}$ and that the boundary of D consists of finite number of smooth closed curves C. We assume that D has some branch points:



Let O be one of the branch points of D and its order of ramification is $k-1$ $(k \geqq 2)$. Suppose that O is found on the origin $z=0$. We thus have a neighborhood U of O found over the disk $|z|<\rho$. U is realized as follows:

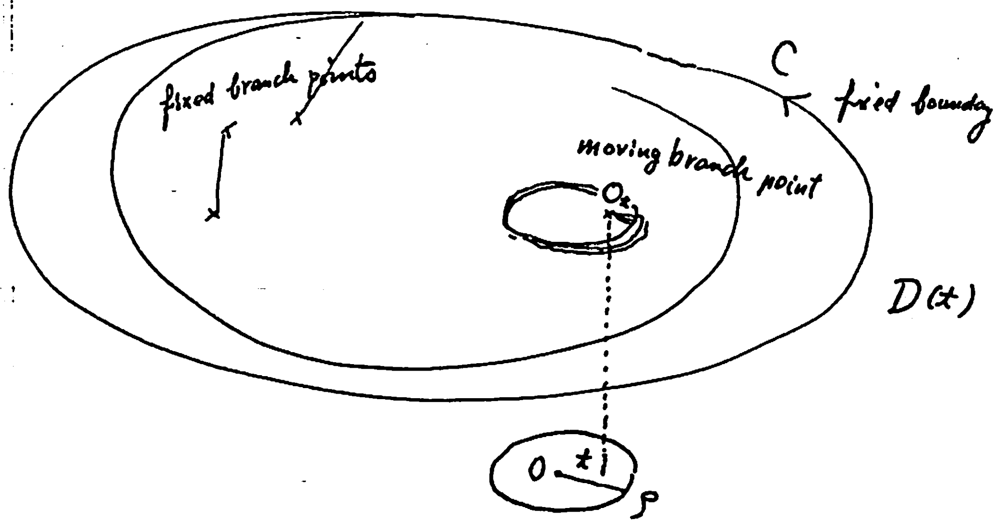


Consider a disk $B = \{ t \in \mathbb{C} \mid |t| < r \}$ where $0 < r < \rho$. For each $t \in B$, we form a new Riemann surface $D(t)$ such that $D(t)$ has the same boundary C as D and the same branch points as of D except at O, while, instead of the branch point O of D, $D(t)$ has the branch point O_t found on $z=t$ whose order of ramification is $k-1$, same as of O in D. Roughly speaking, the branch point O of the Riemann surface D varies with quantity t and we get a Riemann surface $D(t)$. Hence each $D(t)$ is diffeomorphically equivalent to the original D.

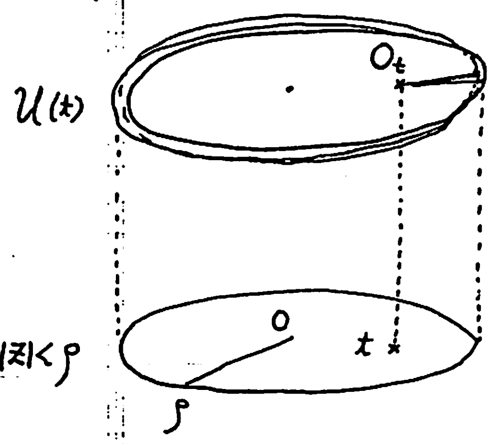
We thus have the variation

$$\mathcal{D} : t \longrightarrow D(t) \quad (t \in B)$$

where $D(0) = D$ such that $\mathcal{D}$ is diffeomorphically equivalent to the trivial variation $B \times D : t \rightarrow D \; (t \in B)$.



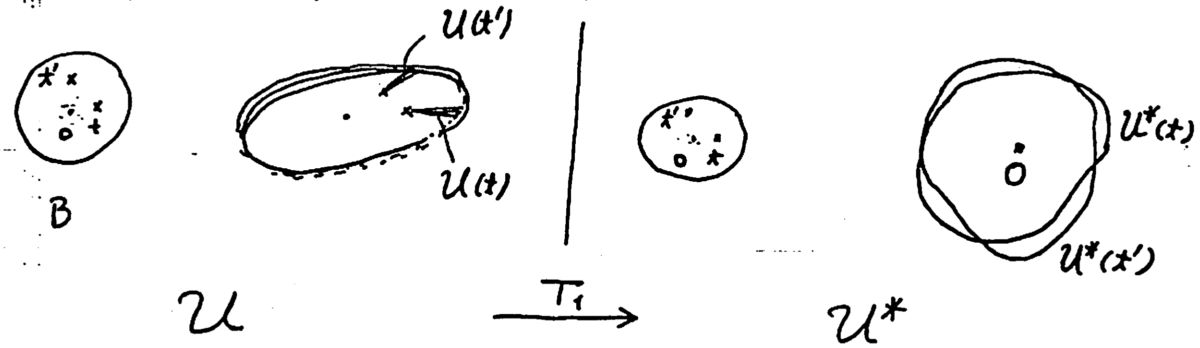
Put $\mathcal{D} = \bigcup_{t\in B}(t, D(t))$ and consider the connected subdomain $\mathcal{U}$ of $\mathcal{D}$ which is found over $B\times(|z|<\rho)$ containing $\bigcup_{t\in B}(t, O_t)$. If $\mathcal{U}(t)$ denotes the fiber of $\mathcal{U}$ at $t\,(\in B)$, then $\mathcal{U}(t)$ is just as



$\mathcal{D}$ is a two-dimensional complex manifold, and the most standard local parameter (t, W) of $\mathcal{U}$ is

$$(2.1)\qquad T_1 : \begin{cases} t = t \\ z = t + W^k \end{cases} \quad ; \quad \mathcal{U}^* = T_1(\mathcal{U})$$

where $t\in B$ and $W \in U^*(t)$. $U^*(t)$ corresponds to $\mathcal{U}(t)$ so that $U^*(t)$ is a univalent subregion in the complex W-plane bounded by a closed curve rounding 0. That is, two sheeted domain $\mathcal{U}$ over $B\times(|z|<\rho)$ is analytically transformed onto a subregion $\mathcal{U}^*$ of $B\times\mathbb{C}$ such that (t, O_t) corresponds to $(t, 0)$.

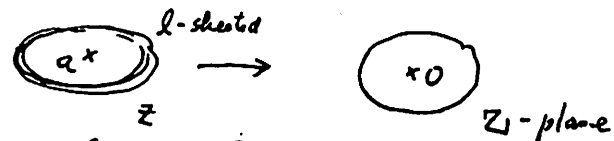


Now take and fix a point a in $D-U$ found on the coordinate $z=a$ in $\mathbb{C}$. Hence $a \in D(t)$ for all $t \in B$. Like as a domain in $\mathbb{C}$ each Riemann surface $D(t)$ carries the Green's function $g(t,z)$ with pole at a. If a is not a branch point of $D(t)$, then in a neighborhood of a we have

$$(2.2) \qquad g(t,z) = \log \frac{1}{|z-a|} + \lambda(t) + h(t,z)$$

where $h(t,z)$ is harmonic for z and $h(t,a)=0$. We call the constant term $\lambda(t)$ the Robin constant for $(D(t), a)$. If a is a branch point of order $l-1$ $(l \geqq 2)$, then, putting

$$z-a = Z^{l},$$



we have

$$g(t,z) = \log \frac{1}{|Z|} + \lambda(t) + h(t,Z)$$

where $h(t,Z)$ is harmonic for Z near 0 and $h(t,0)=0$. We call $\lambda(t)$ the Robin constant for $(D(t), a)$, precisely speaking, the Robin constant for $(D(t), a)$ w.r.t. the local parameter Z at a.

For the sake of convenience, we assume a is not a branch point, because all arguments will be available in the case of branch point a.

<u>Remark 2.1.</u> We restrict $g(t,z)$ to $\mathcal{U}$ (a neighborhood of $(0, O_0)$ in $\mathcal{D}$) and represent $g(t,z)$ by means of the local parameter $(t, W) \in \mathcal{U}^*$ where $\mathcal{U}^* = T_1(\mathcal{U})$, so that

$$G(t, W) = g(t,z) \qquad \text{where } z = t + W^{k}$$

$G(t,W)$ is harmonic for W in $\mathcal{U}^*(t)$. It is intuitively clear that

$G(t, W)$ is of class C^2 for (t, W) in $\mathcal{U}^*$, because our variation $\mathcal{D}: t \to D(t)$ $(t \in B)$ is diffeomorphically equivalent to the trivial.

As noted at the end of Chap. I, it is not easy to prove rigorously that $G(t, W)$ is of class C^2 w.r.t. (t, W) in $\mathcal{U}^*$. In this Chap. we also assume that fact.

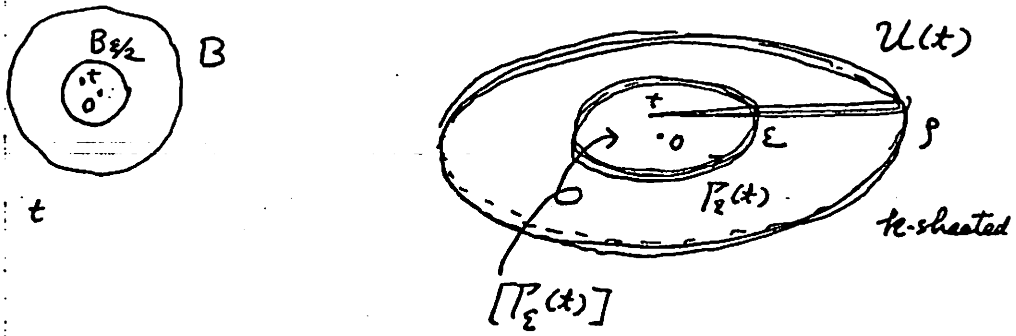
For the sake of convenience, we write $G(0, W) = G(W)$, for it is independent of t. Under these circumstances, we shall first prove

<u>Lemma 2.1</u> $$\frac{\partial \lambda}{\partial t}(0) = \frac{2}{k(k-2)!}\left\{\frac{\partial^{k-2}}{\partial W^{k-2}}\left(\frac{\partial G}{\partial W}\right)^2\right\}_{W=0}.$$

<u>Proof</u>. Let $0 < \varepsilon < \rho$ be given. Put $\gamma_\varepsilon = \{z \in \mathbb{C} \mid |z| = \varepsilon\}$ and $[\gamma_\varepsilon] = \{z \in \mathbb{C} \mid |z| \leq \varepsilon\}$. Moreover, put $B_{\varepsilon/2} = \{t \in B \mid |t| < \varepsilon/2\}$. We consider the following subset of $\mathcal{D}$:

Γ_ε = the part of $\mathcal{U}$ found on $B_{\varepsilon/2} \times \gamma_\varepsilon$

$[\Gamma_\varepsilon]$ = the part of $\mathcal{U}$ found on $B_{\varepsilon/2} \times [\gamma_\varepsilon]$



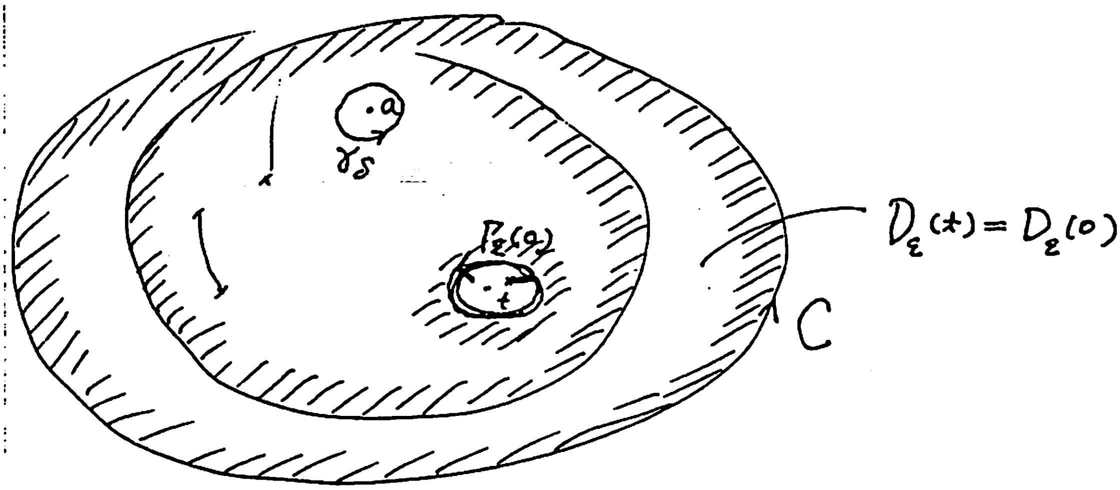
For each $t \in B_{\varepsilon/2}$, we put $[\Gamma_\varepsilon(t)]$ = the fiber of $[\Gamma_\varepsilon]$ at t and $\Gamma_\varepsilon(t)$ = the boundary of $[\Gamma_\varepsilon(t)]$, so that, as usual,

$$[\Gamma_\varepsilon] = \bigcup_{t \in B_{\varepsilon/2}} (t, [\Gamma_\varepsilon(t)]) \;;\quad \Gamma_\varepsilon = \bigcup_{t \in B_{\varepsilon/2}} (t, \Gamma_\varepsilon(t))$$

We write $D_\varepsilon(t) = D(t) - [\Gamma_\varepsilon(t)]$. Then each $D_\varepsilon(t)$ is quite same as $D_\varepsilon(0)$, which is obtained by excluding $[\Gamma_\varepsilon(0)]$ from $D(0)$. Consequently, each $g(t,z)$ ($t \in B_{\varepsilon/2}$) is harmonic for z in $D_\varepsilon(0) - \{a\}$.

0 ε

k-sheeted



We apply the Green's formula (this is applicable in the Riemann surface) to

$$g(t,z) - g(0,z) \quad \text{and} \quad g(0,z)$$

with respect to the domain $D_\varepsilon(0) - [\gamma_\delta]$, where $[\gamma_\delta] = \{ z \in D_\varepsilon(0) \mid |z-a| < \delta \}$, a being the pole of $g(0,z)$. Then we have

$$\int_{C-\gamma_\delta-\Gamma_\varepsilon(0)} \left(g(t,z) - g(0,z)\right) \frac{\partial g(0,z)}{\partial n_z} ds_z = \int_{C-\gamma_\delta-\Gamma_\varepsilon(0)} g(0,z) \frac{\partial}{\partial n_z}\left(g(t,z) - g(0,z)\right) ds_z$$

where $\partial/\partial n_z$ denotes the outer normal derivative, and ds_z the euclidean arc length. Since

$$g(t,z) = g(0,z) = 0 \quad \text{on } C,$$

$$g(t,z) - g(0,z) = \lambda(t) - \lambda(0) + h(t,z) - h(0,z) \quad \text{near } z=a\,;$$
$$h(t,a) - h(t,a) = 0\,;$$
$$g(0,z) = \log\frac{1}{|z-a|} + \lambda(0) + h(0,z) \quad \text{near } z = a,$$

It follows by the same reason as p.11 ~ p.12 at Oct. 19, that

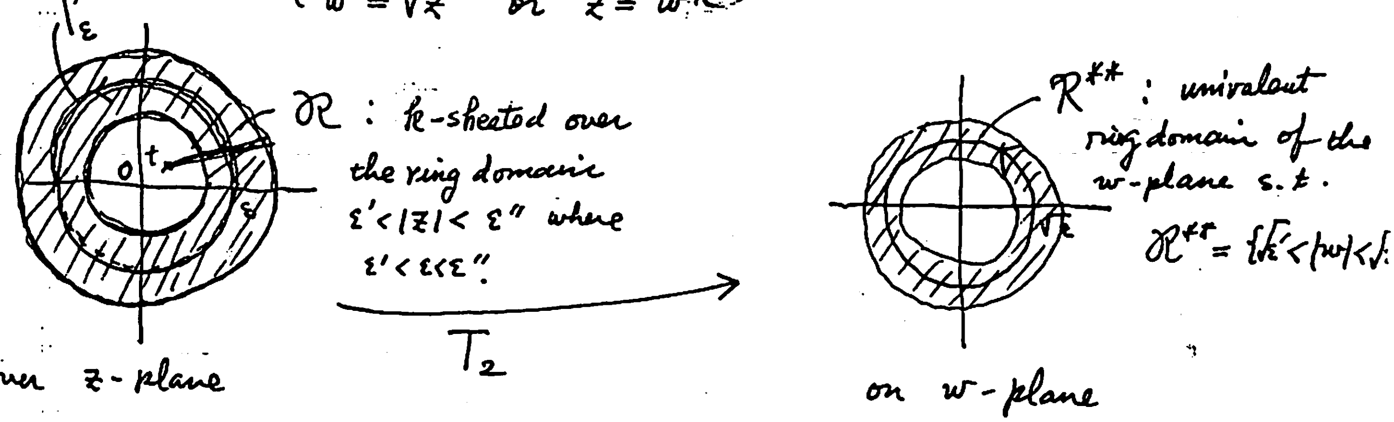
$$\int_C (g(t,z)-g(0,z))\frac{\partial g(0,z)}{\partial n_z}ds_z = \int_C g(0,z)\frac{\partial}{\partial n_z}(g(t,z)-g(0,z))ds_z = 0\,;$$
$$\lim_{\delta\to 0}\int_{\gamma_\delta}(g(t,z)-g(0,z))\frac{\partial g(0,z)}{\partial n_z}ds_z = -2\pi(\lambda(t)-\lambda(0))$$
$$\lim_{\delta\to 0}\int_{\gamma_\delta} g(0,z)\frac{\partial}{\partial n_z}(g(t,z)-g(0,z))ds_z = 0,$$

and hence that

$$(2.3) \qquad \lambda(t)-\lambda(0) = -\frac{1}{2\pi}\int_{\Gamma_\varepsilon(0)}\left(g(t,z)\frac{\partial g(0,z)}{\partial n_z} - g(0,z)\frac{\partial g(t,z)}{\partial n_z}\right)ds_z$$

for all $t\in B_{\varepsilon/2}$. We transform a neighborhood R of $T_\varepsilon = B_{\varepsilon/2}\times\Gamma_\varepsilon(0)$ by the following analytic mapping

$$T_2 : \begin{cases} t = t \\ w = \sqrt[k]{z} \quad \text{or} \quad z = w^k \end{cases}$$



We put $\mathcal{R}^{**} = T_2(\mathcal{R}) = B_{\varepsilon/2}\times(\sqrt{\varepsilon'}<|w|<\sqrt{\varepsilon''})$: product domain, and put $T_2(T_\varepsilon) = \Gamma_\varepsilon^{**} = B_{\varepsilon/2}\times(|w|=\sqrt{\varepsilon})$. Since

$$\partial/\partial n_z\, ds_z = \partial/\partial n_w\, ds_w \quad \text{where } w=\sqrt{z} \text{ on } \Gamma_\varepsilon(0)$$

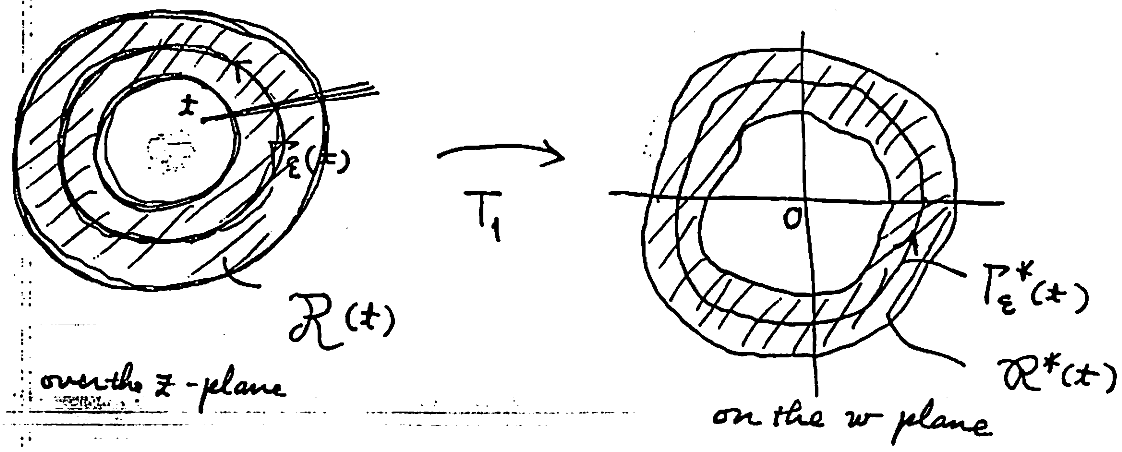
that is, $\frac{\partial}{\partial n} ds$ is an invariant form under the conformal transformations, the expression (2.3) is written in the form

$$(2.4)\quad \lambda(t)-\lambda(0) = -\frac{1}{2\pi}\int_{\Gamma_\varepsilon^{**}(0)}\left(g(t,w)\frac{\partial g(0,w)}{\partial n_w} - g(0,w)\frac{\partial g(t,w)}{\partial n_w}ds_w\right)$$

where $\Gamma_\varepsilon^{**}(0) = (|w| = \sqrt{\varepsilon})\ (\subset \mathbb{C}_w)$.

On the other hand, we transform the ring domain $\mathcal{R}$ in $\mathfrak{D}$ by the transformation T_1 defined in (2.1):

$$T_1 : \begin{cases} t = t \\ W = \sqrt[k]{z-t} \end{cases} \quad \text{or} \quad z = t + W^k$$



Put $\mathcal{R}^* = T_1(\mathcal{R}) = \bigcup_{t\in B_{\varepsilon/2}}(t, \mathcal{R}^*(t))$ and

$\Gamma_\varepsilon^* = T_1(\Gamma_\varepsilon) = \bigcup_{t\in B_{\varepsilon/2}}(t, \Gamma_\varepsilon^*(t))$.

At $t=0$, (we have) $T_1 = T_2$, so that $\mathcal{R}^*(0) = \mathcal{R}^{**}(0)$ and $\Gamma_\varepsilon^*(0) = \Gamma_\varepsilon^{**}(0) = \{|w| = \sqrt{\varepsilon}\}$, and $W = w$. Therefore, $\mathcal{R}^*$ and $\mathcal{R}^{**}$ are one to one analytically corresponding to each other by the transformation

$$T_2 \circ T_1^{-1} : \begin{cases} t = t \\ w = \sqrt[k]{W^k + t} \end{cases}$$

$$(\because\ z = w^k = t + W^k)$$

in such a way that $w = T_2 \circ T_1^{-1}(0, W) = W$, i.e., identical at $t = 0$.
Let $0 < \eta \ll \varepsilon$ and denote $B_\eta = \{t \in B \mid |t| < \eta\} (\subset\subset B_{\varepsilon/2})$. Then

$$w = \sqrt[k]{W^k + t} = W\left(1 + \frac{t}{W^k}\right)^{1/k}$$
$$= W\left(1 + \frac{1}{k}\frac{t}{W^k} + \frac{\frac{1}{k}(\frac{1}{k}-1)}{2!}\left(\frac{t}{W^k}\right)^2 + \cdots\right)$$
$$= W + \frac{t}{k}\frac{1}{W^{k-1}} - \frac{(k-1)t^2}{2k^2}\cdot\frac{1}{W^{2k-1}} + \varepsilon(t^3)$$

or
$$W = w - \frac{t}{k}\frac{1}{w^{k-1}} - \frac{(k-1)t^2}{2k^2}\frac{1}{w^{2k-1}} + \varepsilon(t^3)$$

where $|w| \fallingdotseq \sqrt{\varepsilon}$ and $t \in B_\eta$. We write

$$g(t, z) = g(t, w) = G(t, W)$$

where $z = w^k = W^k + t$ $(t \in B_\eta,\ |z| \fallingdotseq \varepsilon)$, so that

$$g(0, z) = g(0, w) = G(0, W) = G(W).$$

We remark that, although $g(t, w)$ is defined only near $B_{\varepsilon/2} \times (|w| = \sqrt{\varepsilon})$, $G(t, W)$ is defined in a neighborhood $\mathcal{U}^*$ of $(0,0)$ in (t, W) space as noted in Remark 2.1. Moreover $G(t, W)$ was of class C^2 w.r.t. (t, W) in $\mathcal{U}^*$.

Let us write the equality (2.4) by means of $G(t, W)$ as follows:

(2.5) $\lambda(t) - \lambda(0)$

$$= -\frac{1}{2\pi}\int_{\Gamma_\varepsilon^{**}(0)} G\left(t, w - \frac{t}{k}\frac{1}{w^{k-1}} - \frac{(k-1)t^2}{2k^2}\frac{1}{w^{2k-1}} + \varepsilon(t^3)\right)\frac{\partial G(0,w)}{\partial n_w}\,ds_w$$
$$- G(0,w)\frac{\partial}{\partial n_w} G\left(t, w - \frac{t}{k}\frac{1}{w^{k-1}} - \frac{(k-1)t^2}{2k^2}\frac{1}{w^{2k-1}} + \varepsilon(t^3)\right) ds_w$$

Since $G(t, W)$ can be devolopped into the Taylor series, we get

$$G\left(t, w - \frac{t}{k}\frac{1}{w^{k-1}} - \frac{(k-1)t^2}{2k^2}\frac{1}{w^{2k-1}} + \varepsilon(t^3)\right)$$
$$= G(0,w) + t\left\{\frac{\partial G}{\partial t} - \frac{1}{k w^{k-1}}\frac{\partial G}{\partial W}\right\}_{(0,w)} + \bar{t}\left\{\frac{\partial G}{\partial \bar{t}} - \frac{1}{k \bar{w}^{k-1}}\frac{\partial G}{\partial \bar{W}}\right\}_{(0,w)} + O(|t|$$

where $t \in B_\eta$ and $|w| = \sqrt{\varepsilon}$ i.e., $w \in \Gamma_\varepsilon^{**}(0)$

We have by (2.5)

(2.6) $\lambda(t)-\lambda(0)$

$$= \frac{-1}{2\pi}\int_{\Gamma_\varepsilon^{**}(0)} \Big[G(0,w) + t\Big\{\frac{\partial G}{\partial t} - \frac{1}{k w^{k-1}}\frac{\partial G}{\partial W}\Big\}_{(0,w)} + \bar t\Big\{\frac{\partial G}{\partial \bar t} - \frac{1}{k \bar w^{k-1}}\frac{\partial G}{\partial \bar W}\Big\}_{(0,w)} + O(|t|^2)\Big]\frac{\partial G(0,w)}{\partial n_w}ds_w$$

$$- G(0,w)\frac{\partial}{\partial n_w}\Big[G(0,w) + t\Big\{\frac{\partial G}{\partial t} - \frac{1}{k w^{k-1}}\frac{\partial G}{\partial W}\Big\}_{(0,w)} + \bar t\Big\{\frac{\partial G}{\partial \bar t} - \frac{1}{k \bar w^{k-1}}\frac{\partial G}{\partial \bar W}\Big\}_{(0,w)} + O(|t|^2)\Big]ds_w$$

(2.6) holds for all $t \in B_\eta$ ($\Subset B_{\varepsilon/2}$). So, take $\partial/\partial t$ of both sides of (2.6) and put $t=0$. Then we have

(2.7) $\frac{\partial \lambda}{\partial t}(0)$

$$= -\frac{1}{2\pi}\int_{\Gamma_\varepsilon^{**}(0)} \Big\{\frac{\partial G}{\partial t} - \frac{1}{k w^{k-1}}\frac{\partial G}{\partial W}\Big\}_{(0,w)}\frac{\partial G(0,w)}{\partial n_w}ds_w - G(0,w)\frac{\partial}{\partial n_w}\Big\{\frac{\partial G}{\partial t} - \frac{1}{k w^{k-1}}\frac{\partial G}{\partial W}\Big\}_{(0,w)} d$$

Since $\Gamma_\varepsilon^{**}(0) = \Gamma_\varepsilon^{*}(0) = \{W = \sqrt{\varepsilon}\}$ and $w = W$ at $t=0$, (2.7) is written as follows :

(2.8) $\frac{\partial \lambda}{\partial t}(0)$

$$= \frac{-1}{2\pi}\int_{|W|=\sqrt{\varepsilon}} \Big\{\frac{\partial G}{\partial t} - \frac{1}{k W^{k-1}}\frac{\partial G}{\partial W}\Big\}_{(0,W)}\frac{\partial G(0,W)}{\partial n_W}ds_W - G(0,W)\frac{\partial}{\partial n_W}\Big\{\frac{\partial G}{\partial t} - \frac{1}{k W^{k-1}}\frac{\partial G}{\partial W}\Big\}_{(0,W)} ds_W$$

This formula (2.8) holds for all $0<\varepsilon\ll 1$. By Remark 2.1, $G(t,W)$ is harmonic for $W \in \mathcal{U}^*(t)$ and is of class C^2 for $(t,W)\in\mathcal{U}^*$ where $\mathcal{U}^*$ is a nbd of $(0,0)$, so that $\mathcal{U}^* \supset (0, |W|\le\sqrt{\varepsilon})$, and hence

$$\lim_{\varepsilon\to 0}\int_{|W|=\sqrt{\varepsilon}} \Big\{\frac{\partial G}{\partial t}\frac{\partial G}{\partial n_W}\Big\}_{(0,W)} ds_W = 0 \ ;$$

$$\lim_{\varepsilon\to 0}\int_{|W|=\sqrt{\varepsilon}} G(0,W)\frac{\partial}{\partial n_W}\Big(\frac{\partial G}{\partial t}(0,W)\Big)ds_W = 0 .$$

By (2.8) we have

(2.9) $\frac{\partial\lambda}{\partial t}(0)$

$$= \frac{-1}{2\pi} \lim_{\varepsilon\to 0} \int_{|W|=\sqrt{\varepsilon}} \left(-\frac{1}{kW^{k-1}}\frac{\partial G}{\partial W}\right)_{(0,W)} \frac{\partial G(0,W)}{\partial n_W} ds_W - G(0,W)\frac{\partial}{\partial n_W}\left(-\frac{1}{kW^{k-1}}\frac{\partial G}{\partial W}\right)_{(0,W)} ds_W$$

Since the right hand side is determined by the Riemann surface $D(0)$, we have, by putting $G(0,W) = G(W)$, simply

$$\frac{\partial\lambda}{\partial t}(0) = \frac{1}{2\pi} \lim_{\varepsilon\to 0} \int_{|W|=\sqrt{\varepsilon}} \frac{1}{kW^{k-1}}\frac{\partial G}{\partial W}\frac{\partial G}{\partial n_W} ds_W - G(W)\frac{\partial}{\partial n_W}\left(\frac{1}{kW^{k-1}}\frac{\partial G}{\partial W}\right) ds_W.$$

In general,

$$\frac{\partial}{\partial n_W} ds_W = \frac{1}{i}\left(\frac{\partial}{\partial W}dW - \frac{\partial}{\partial \bar{W}}d\bar{W}\right)$$

$$d(\quad) = \partial/\partial W(\quad)\cdot dW + \partial/\partial\bar{W}(\quad)\cdot d\bar{W}$$

along the curve in $D(0)$, so that

$$\frac{\partial}{\partial n_W}\left(\frac{1}{kW^{k-1}}\frac{\partial G}{\partial W}\right) ds_W = \frac{1}{i}\left[\frac{\partial}{\partial W}\left(\frac{1}{kW^{k-1}}\frac{\partial G}{\partial W}\right)dW - \frac{\partial}{\partial \bar{W}}\left(\underbrace{\frac{1}{kW^{k-1}}\frac{\partial G}{\partial W}}\right)\cdot d\bar{W}\right.$$

this is holomorphic near $|W| = \sqrt{\varepsilon}$

$$\therefore \frac{\partial}{\partial\bar{W}}\left(\frac{1}{kW^{k-1}}\frac{\partial G}{\partial W}\right) \equiv 0$$

$$\therefore = \frac{1}{i}\left[\frac{\partial}{\partial W}\left(\frac{1}{k}\frac{1}{W^{k-1}}\frac{\partial G}{\partial W}\right)dW + \frac{\partial}{\partial\bar{W}}\left(\frac{1}{kW^{k-1}}\frac{\partial G}{\partial W}\right)\cdot d\bar{W}\right]$$

$$= \frac{1}{i}\, d\left(\frac{1}{k}\frac{1}{W^{k-1}}\frac{\partial G}{\partial W}\right) \quad !$$

Therefore, the integration by parts yields that

$$\int_{|W|=\sqrt{\varepsilon}} G(W)\frac{\partial}{\partial n_W}\left(\frac{1}{kW^{k-1}}\frac{\partial G}{\partial W}\right) ds_W = \frac{1}{i}\int_{|W|=\sqrt{\varepsilon}} \left(G\cdot\frac{1}{k}\frac{1}{W^{k-1}}\frac{\partial G}{\partial W}\right) - \frac{1}{kW^{k-1}}\frac{\partial G}{\partial W}dG$$

$$= \frac{-1}{i} \int_{|W|=\sqrt{\varepsilon}} \frac{1}{k\, W^{k-1}} \frac{\partial G}{\partial W} \cdot \left(\frac{\partial G}{\partial W} dW + \frac{\partial G}{\partial \bar{W}} d\bar{W} \right)$$

We thus have

$$\frac{\partial \lambda}{\partial t}(0) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{|W|=\sqrt{\varepsilon}} \frac{1}{kW^{k-1}} \frac{\partial G}{\partial W} \frac{1}{i} \left(\frac{\partial G}{\partial W} dW - \frac{\partial G}{\partial \bar{W}} d\bar{W} \right) + \frac{1}{i} \frac{1}{kW^{k-1}} \frac{\partial G}{\partial W} \left(\frac{\partial G}{\partial W} dW + \frac{\partial G}{\partial \bar{W}} d\bar{W} \right)$$

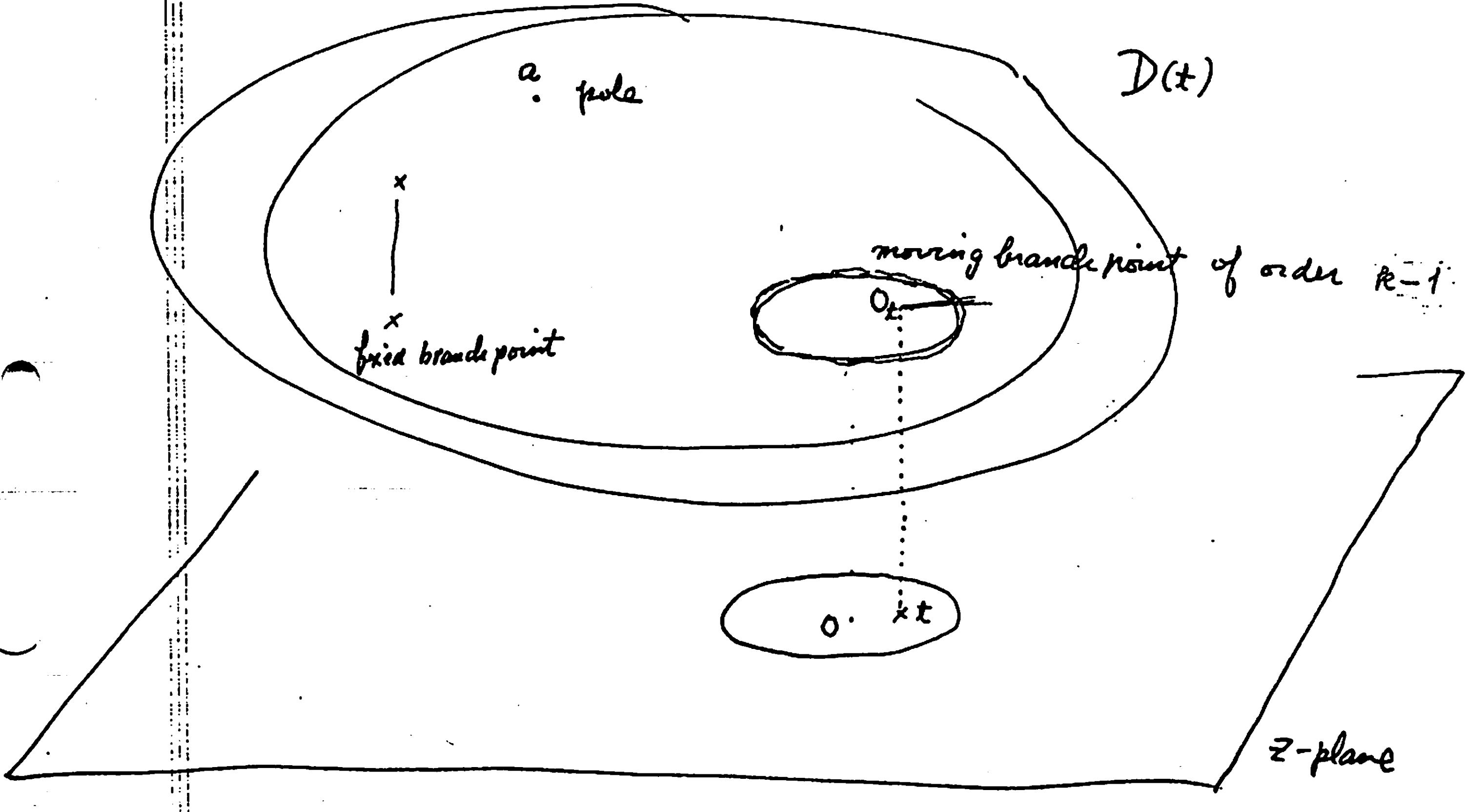
$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|W|=\sqrt{\varepsilon}} \frac{2}{kW^{k-1}} \left(\frac{\partial G}{\partial W} \right)^2 dW$$

$$= \lim_{\varepsilon \to 0} 2 \frac{1}{k(k-2)!} \left[\frac{\partial^{k-2}}{\partial W^{k-2}} \left(\frac{\partial G}{\partial W} \right)^2 \right]_{W=0} \quad (\because \text{Cauchy's formula})$$

$$= \frac{2}{k(k-2)!} \left[\frac{\partial^{k-2}}{\partial W^{k-2}} \left(\frac{\partial G}{\partial W} \right)^2 \right](0)$$

Lemma 2.1 is proved.

We are studying the special variation of ramified covering surface over $\mathbb{C}_z$.



We have the Green's function for $(D(t), a)$: $g(t, z)$
and the Robin constant for $(D(t), a)$: $\lambda(t)$
The most standard local parameter at O_t is

$$\begin{cases} t = t \\ z = t + W^k \end{cases} \quad \text{or} \quad W = \sqrt[k]{z - t}$$

so that O_t correspond to $W = 0$. With this local paramet (t, W), $g(t, z)$ is represented as follows:

$$g(t, z) = G(t, W)$$

where $t \in B^* = (|t| < \rho^*)$ and $W \in U^* = (|W| < r^*)$.
Then we know that

$G(t, W)$ is harmonic for $W \in U^*$ and of class C^2 for $(t, W) \in B^* \times U^*$.

We obtained in the last thursday

Lemma 2.1 $\dfrac{\partial \lambda}{\partial t}(t) = \dfrac{2}{k(k-2)!}\left\{\dfrac{\partial^{k-2}}{\partial W^{k-2}}\left(\dfrac{\partial G(t,W)}{\partial W}\right)^2\right\}_{W=0}$.

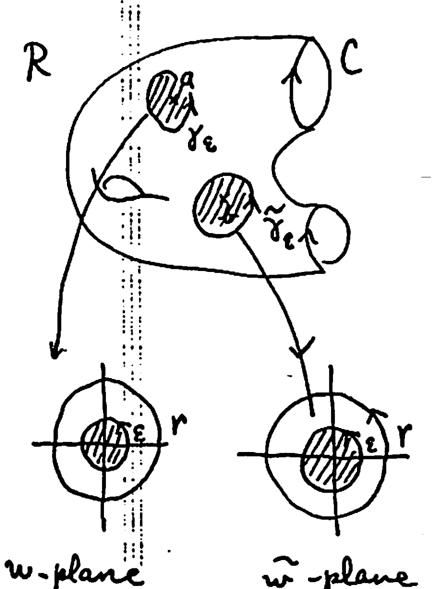
The right-hand side is a quantity determined by the Riemann surface $D(t)$ itself, i.e., not depending on the variation.

Let us differentiate both sides w.r.t. $\bar{t}$ and then put $t=0$. We have

$$(3.0)\quad \frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(0) = \frac{4}{k(k-2)!}\left\{\frac{\partial^{k-2}}{\partial W^{k-2}}\left(\frac{\partial G}{\partial W}\,\frac{\partial^2 G}{\partial \bar{t}\partial W}\right)\right\}_{t=0,\,W=0}$$

$$= \frac{4}{k(k-2)!}\sum_{i=1}^{k-2}\binom{k-2}{i}\left\{\frac{\partial^{i+1}G}{\partial W^{i+1}}\,\frac{\partial^{k-i}G}{\partial \bar{t}\,\partial W^{k-i-1}}\right\}_{t=0,\,W=0}$$

Our next aim is to describe $(\partial^{k-i}G/\partial\bar{t}\,\partial W^{k-i-1})(0,0)$ by means of the quantities determined by the Riemann surface $D(0)$ itself. To do this, we need some basic notations concerning Riemann surfaces:



Let R be a Riemann surface with smooth boundary C. Let $a, b \in R$ such that $a \neq b$. We consider the Green's function $g_a(z)$ and $g_b(z)$ for (R, a) and (R, b), respectively. Then we get the following symmetricity:

$$g_a(b) = g_b(a).$$

Proof. Let $|w| < r$ and $|\tilde{w}| < \tilde{r}$

$\sharp(\cdot\cdot\,\partial C$. (Corollary) $g_{z_1}(z)$ is harmonic w.r.t. z_1 in $R-\{z\}$.

be local parameters at the points a and b, such that $w=0$ and $\tilde{w}=0$ correspond to a and b, respectively. Therefore, in a neighborhood of a, we have

$$(3.1)\quad \begin{cases} g_a(z) = \log \dfrac{1}{|w|} + H(w) \\ g_b(z) = K(w) \end{cases}$$

where $H(w)$ and $K(w)$ are harmonic functions in $(|w|<r)$.
In a neighborhood of b, we have

$$(3.2)\quad \begin{cases} g_b(z) = \log \dfrac{1}{|\tilde{w}|} + \tilde{H}(\tilde{w}) \\ g_a(z) = \tilde{K}(\tilde{w}) \end{cases}$$

where $\tilde{H}(\tilde{w})$ and $\tilde{K}(\tilde{w})$ are harmonic fns in $(|\tilde{w}|<r)$.

Given $0<\varepsilon<r$, we draw circles $|w|=\varepsilon$ and $|\tilde{w}|=\varepsilon$ in the w and $\tilde{w}$ plane, respectively. We denote by γ_ε and $\tilde{\gamma}_\varepsilon$ the corresponding curves on R, which round a and b. Moreover we write $[\gamma_\varepsilon]$ and $[\tilde{\gamma}_\varepsilon]$ the domains of R bounded by γ_ε and $\tilde{\gamma}_\varepsilon$, respectively.

Since $g_a(z)$ and $g_b(z)$ are regular harmonic on $R-[\gamma_\varepsilon]-[\tilde{\gamma}_\varepsilon]$, it follows from Green's formula that

$$(3.3)\quad \int_{C-\gamma_\varepsilon-\tilde{\gamma}_\varepsilon} g_a(z)\frac{\partial g_b(z)}{\partial n_z}\,ds_z = \int_{C-\gamma_\varepsilon-\tilde{\gamma}_\varepsilon} g_b(z)\frac{\partial g_a(z)}{\partial n_z}\,ds_z .$$

Because of $g_a(z) = g_b(z) = 0$ on C ($\because$ by the definition of the Green's fn), the integrals along C are zero. By (3.1), we have by the standard method (which we often used)

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} g_a(z)\frac{\partial g_b(z)}{\partial n_z}ds_z = \lim_{\varepsilon\to 0}\int_0^{2\pi}\left(\log\frac{1}{|\varepsilon|}+H(\varepsilon e^{i\theta})\right)\left[\frac{\partial K(re^{i\theta})}{\partial r}\right]_{r=\varepsilon}\varepsilon\cdot d\theta$$

$$= \lim_{\varepsilon\to 0}\left[\left(\varepsilon\cdot\log\frac{1}{\varepsilon}\right)\cdot 2\pi + \varepsilon\, O(\varepsilon)\right] = 0 \;;$$

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} g_b(z)\frac{\partial g_a(z)}{\partial n_z}ds_z = \lim_{\varepsilon\to 0}\int_0^{2\pi} K(\varepsilon e^{i\theta})\left[\frac{\partial}{\partial r}\left(\log\frac{1}{r}+H(re^{i\theta})\right)\right]_{r=\varepsilon}\varepsilon\, d\theta$$

$$= \lim_{\varepsilon\to 0}\left\{\int_0^{2\pi} K(\varepsilon e^{i\theta})\cdot(-1) + \varepsilon\int_0^{2\pi} K(\varepsilon e^{i\theta})\left[\frac{\partial H(re^{i\theta})}{\partial r}\right]_{r=\varepsilon}d\theta\right\}$$

$$= -2\pi K(0) = -2\pi\, g_b(a).$$

Analogously,

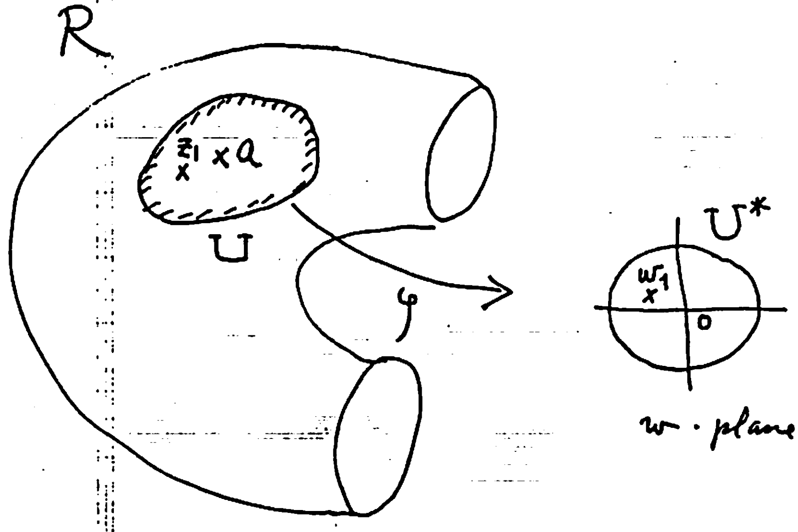
$$\lim_{\varepsilon\to 0}\int_{\tilde\gamma_\varepsilon} g_a(z)\frac{\partial g_b(z)}{\partial n_z}ds_z = -2\pi\tilde K(0) = -2\pi\, g_a(b)$$

$$\lim_{\varepsilon\to 0}\int_{\tilde\gamma_\varepsilon} g_b(z)\frac{\partial g_a(z)}{\partial n_z}ds_z = 0 .$$

Consequently, (3.3) yield by $\varepsilon\to 0$ that

$$2\pi\, g_a(b) = 2\pi\, g_b(a)$$

c. q. f. d.



Now, take $a\in R$ and choose a neighborhood U of a and a local parameter $U^*: |w|<r$

$$\varphi: U \longrightarrow U^*$$
$$z \longrightarrow w=\varphi(z)$$

such that

$$\varphi(a)=0.$$

Let $z_1 \in U$ and form the Green's fn $g_{z_1}(z)$ for (R, z_1). By use of the local parameter w, we write

$$g_{z_1}(z) = G_{w_1}(w)$$

where $w_1 = \varphi(z_1)$, $w = \varphi(z)$ and $z \in U$. Since $w - w_1$ is one of the local parameters at the point z_1, we have

$$(3.4) \quad G_{w_1}(w) = \log \frac{1}{|w - w_1|} + H(w_1, w)$$

where $H(w_1, w)$ is harmonic for w and $H(w_1, w_1)$ is finite. Therefore, $H(w_1, w)$ defines a finite valued fn on $U^* \times U^*$, while $G_{w_1}(w)$ is a function defined on $U^* \times U^* -$ (diagonal set). Under these circumstances, we want to show

$$\frac{\partial^2 G_{w_1}(w)}{\partial w_1 \partial \bar{w}} \text{ is real analytic w.r.t. } (w_1, w) \in U^* \times U^*.$$

Since $\dfrac{\partial^2}{\partial w_1 \partial \bar{w}} \log \dfrac{1}{|w - w_1|} \equiv 0$!, it suffices to prove that

$$\frac{\partial^2 H_{w_1}(w)}{\partial w_1 \partial \bar{w}} \text{ is real analytic w.r.t. } (w_1, w) \in U^* \times U^*.$$

Proof. First we remark from Symmetricity : $g_a(b) = g_b(a)$ for $\forall a, b \in R$ such that $a \neq b$ that
$g_{z_1}(z)$ is harmonic with respect to z_1 in $R - \{z\}$.

Precisely speaking, given $z_1 \in R$ we construct the Green's function $g_{z_1}(z)$ for (R, z_1). Hence, $g_{z_1}(z)$ defines a real valued function for $(z_1, z) \in R \times R - \Sigma$ where Σ is the diagonal set : $\{(z, z) \mid z \in R\}$. If z is fixed in R, then $g_{z_1}(z)$ becomes a function for z_1 in $R - \{z\}$. Since

$g_{z_1}(z) = g_z(z_1)$, $g_{z_1}(z)$ is harmonic for z_1 in $R - \{z\}$.

We return to the expression (3.4). Since $g_{z_1}(z) = G_{w_1}(w)$, $g_z(z_1) = G_w(w_1)$ and $g_{z_1}(z) = g_z(z_1)$, we have

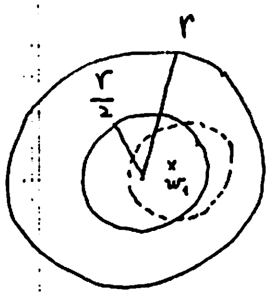
$$G_{w_1}(w) = G_w(w_1) \quad \text{for } (w_1, w) \in U^* \times U^* - \Sigma^*$$

where Σ^* is the diagonal set in $U^* \times U^*$. Since $\log 1/|w - w_1|$ has the same property, it follows from (3.4) that

$$H(w_1, w) = H(w, w_1) \quad \text{for } (w_1, w) \in U^* \times U^*.$$

Consequently, $H(w_1, w)$ is harmonic for both $w_1 \in U^*$ and $w \in U^*$.

Now, let $|w_1| < \frac{r}{2}$ i.e., $w_1 \in \frac{U^*}{2}$. Then by the maximum principle,



$$G_{w_1}(w) > \log \frac{\frac{r}{2}}{|w - w_1|} \quad \text{for } w \in U^*$$

so that

$$H(w_1, w) > \log \frac{r}{2} \quad \text{for } (w_1, w) \in \frac{U^*}{2} \times U^*.$$

Given $(w_1, w) \in \frac{U^*}{2} \times U^*$, we form the Poisson integral:

$$I(w_1, w) = \int_0^{2\pi}\int_0^{2\pi} \left\{ H(\zeta_1, \zeta) \frac{(\frac{r}{2})^2 - |w_1|^2}{|\zeta_1 - w_1|^2} \cdot \frac{r^2 - |w|^2}{|\zeta - w|^2} \right\} \frac{d\theta_1}{2\pi} \times \frac{d\theta}{2\pi}$$

where $\zeta_1 = \frac{r}{2} e^{i\theta_1}$ $(0 \leqq \theta_1 \leqq 2\pi)$ and $\zeta = re^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$.

$I(w_1, w)$ thus defines a real analytic function for $(w_1, w) \in (U^*/2) \times U^*$. Since $H(\zeta_1, \zeta)$ is bounded below in $(|\zeta_1| = \frac{r}{2})$ $(|\zeta| = r)$, we obtain from Fubini's Theorem

$$I(w_1, w) = \int_0^{2\pi} \left\{ \int_0^{2\pi} H(\zeta_1, \zeta) \frac{(\frac{r}{2})^2 - |w_1|^2}{|\zeta_1 - w_1|^2} \frac{d\theta_1}{2\pi} \right\} \frac{r^2 - |w|^2}{|\zeta - w|^2} \frac{d\theta}{2\pi}$$

$$= \int_0^{2\pi} H(w_1, \zeta) \frac{r^2 - |w|^2}{|\zeta - w|^2} \frac{d\theta}{2\pi}$$

($\because$ $H(w_1, \zeta)$ is harmonic for w_1 in $U^*/2$)

$= H(w_1, w)$ ($\because$ $H(w_1, w)$ is harmonic for w in U^*)

Hence $H(w_1, w)$ is real analytic for $(w_1, w) \in \frac{U^*}{2} \times U^*$.

By the same arguement, so is it for $(w_1, w) \in U^* \times U^*$.

c. q. f. d.

Now, we introduce the meaning of

$$\left\{ \frac{\partial^2 G_{w_1}(w)}{\partial \bar{w}_1 \partial w} \right\}_{w = w_1}$$

which was found by M. Schiffer in 1946.

Let Γ_a be the family of all holomorphic differentials on R with finite norm:

$$\Gamma_a = \left\{ a(z)dz \;\middle|\; a(z) \text{ is holo. and } \frac{-1}{2i}\iint_R a(z)\overline{a(z)}\, dz \wedge d\bar{z} < \infty \right\}$$

We regard $g_{z_1}(z)$ the function of z with parameter z.

Or equivalently, $g_{w_1}(z)$ is the function of z with parameter w_1 in U^*. Therefore, for each $w_1 \in U^*$

$\frac{\partial g_{w_1}(z)}{\partial \bar{w}_1}$ is a function of z on R

such that

(i) it vanishes on the boundary C;

(ii) using the local parameter w, we have

$$\frac{\partial g_{w_1}(z)}{\partial \bar{w}_1} = \frac{\partial G_{w_1}(w)}{\partial \bar{w}_1} = \frac{1}{2}\frac{1}{w - w_1} + \frac{\partial H(w_1, w)}{\partial \bar{w}_1}.$$

By (ii) we get

$$\frac{\partial g_{w_1}(z)}{\partial z \partial \bar{w}_1} dz = \frac{\partial G_{w_1}(w)}{\partial w \partial \bar{w}_1} dw = \frac{\partial^2 H}{\partial w \partial \bar{w}_1}(w_1, w)\, dw.$$

Therefore, the differential $\frac{\partial g_{w_1}(z)}{\partial z \partial \bar{w}_1} dz$ is regular at $z = z_1$, so that $\frac{\partial g_{w_1}(z)}{\partial z \partial \bar{w}_1} dz$ is belonging to Γ_a.

For given $a(z)dz \in \Gamma_a$, we calculate the inner

$$\langle a(z)dz, \frac{\partial^2 g_{w_1}(z)}{\partial z \partial \bar{w}_1} dz \rangle_R$$

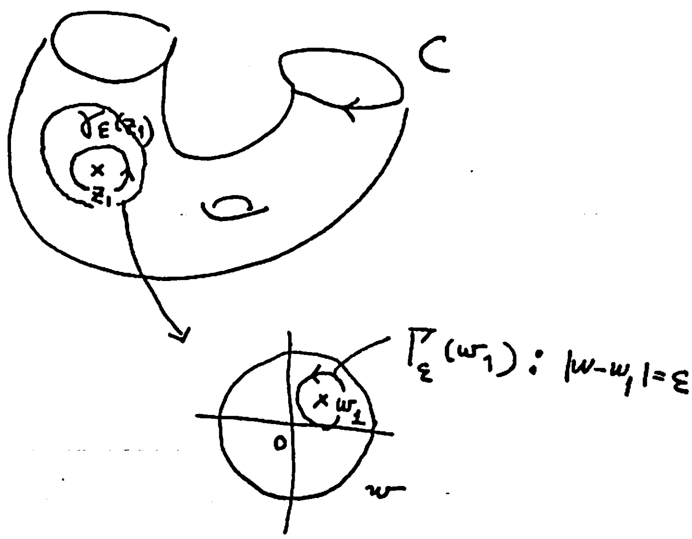
$$= \frac{-1}{2i} \iint_R a(z) \overline{\left(\frac{\partial g_{w_1}(z)}{\partial z \partial \bar{w}_1}\right)} \, dz \wedge d\bar{z}$$

$$= \frac{1}{2i} \iint_R d \left\{ \left(\frac{\partial g_{w_1}(z)}{\partial w_1}\right) \cdot a(z) dz \right\}$$

this is holo. differential on R

this is a function on R with pole at z_1

$$= \frac{1}{2i} \int_{C - \gamma_\varepsilon(z_1)} \left(\frac{\partial g_{w_1}(z)}{\partial w_1}\right) a(z) dz \qquad \text{by Stokes' Formula.}$$



$$= \frac{-1}{2i} \int_{\gamma_\varepsilon(z_1)} \left(\frac{\partial g_{w_1}(z)}{\partial w_1}\right) a(z) dz \qquad \text{by (i)}$$

$$= \frac{-1}{2i} \int_{|w-w_1|=\varepsilon} \frac{\partial G}{\partial w_1}(w_1, w) \, a(w) dw$$

$$= \frac{-1}{2i} \int_{|w-w_1|=\varepsilon} \left(\frac{1}{2} \frac{1}{w-w_1} + \frac{\partial H}{\partial w_1}(w_1, w) \right) a(w) dw \qquad \text{by (ii)}$$

$$\longrightarrow -\frac{\pi}{2} a(w_1) \quad \text{as } \varepsilon \to 0.$$

$$\therefore \quad \langle a(z)dz, -\frac{2}{\pi} \frac{\partial^2 g_{w_1}(z)}{\partial z \partial \bar{w}_1} dz \rangle_R = a(w_1)$$

This means, $-\frac{2}{\pi}\frac{\partial^2 g_{w_1}(z)}{\partial z \partial \bar{w}_1}dz \in \Gamma_a$ is the reproducing differential of the continuous linear functional

$$a(z)dz \in \Gamma_a \longrightarrow a(w_1) \in \mathbb{C}$$

In particular, if we choose $a(z)dz = -\frac{2}{\pi}\frac{\partial^2 g_{w_1}(z)}{\partial z \partial \bar{w}_1}dz$, then

$$(3.5) \quad -\frac{2}{\pi}\left\{\frac{\partial^2 G_{w_1}(w)}{\partial w \partial \bar{w}_1}\right\}_{w=w_1} = \left(\frac{2}{\pi}\right)^2 \iint_R \left|\frac{\partial^2 g_{w_1}(z)}{\partial z \partial \bar{w}_1}\right|^2 dxdy \quad \text{where } z = x+iy$$

We thus have

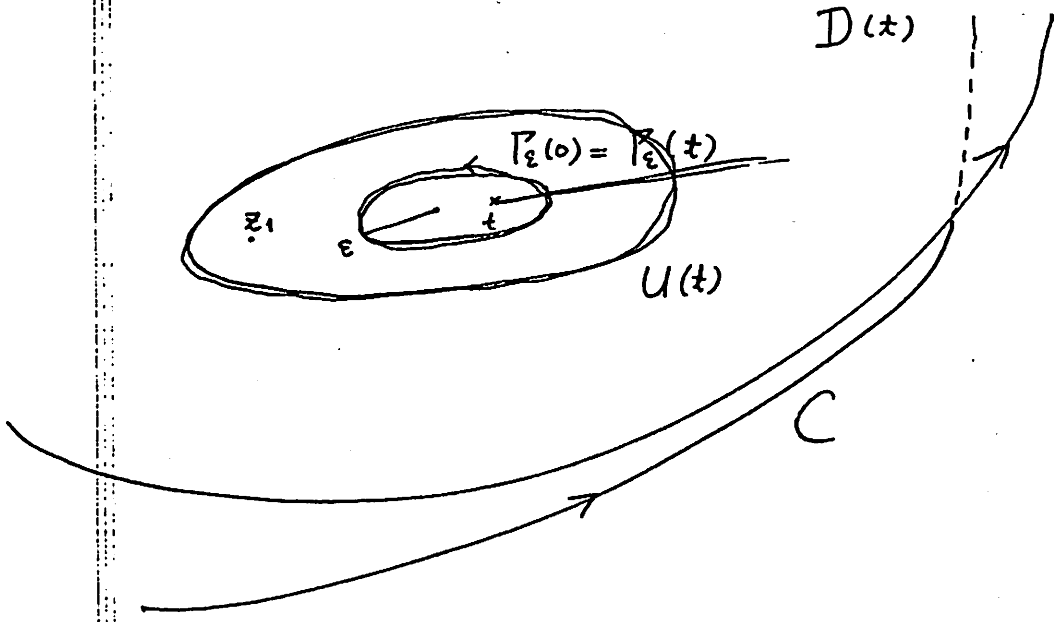
$$(3.6) \quad \left\{\frac{\partial^2 G_{w_1}(w)}{\partial w \partial \bar{w}_1}\right\}_{w=w_1} < 0$$

(∵) By (3.5), we have $\{\partial^2 G_{w_1}(w)/\partial w \partial \bar{w}_1\}_{w=w_1} \leqq 0$. Assume that it is equal to zero. Then (3.5) yields that $\partial^2 g_{w_1}(z)/\partial z \partial w_1 \equiv 0$ for $z \in R$, and hence $\partial g_{w_1}(z)/\partial w_1$ is holomorphic function for z on $R - \{w_1\}$. By property (i), we see that $\partial g_{w_1}(z)/\partial w_1 \equiv 0$ on $R - \{w_1\}$, which is a contradiction to (ii). Consequently, $\{\partial^2 G_{w_1}(w)/\partial w \partial \bar{w}_1\}_{w=w_1} < 0$.

Let us return to (3.0), and prove the following

<u>Lemma 2.2</u> $$\frac{\partial^{l+1} G}{\partial \bar{z} \partial W^l}(0,0) = \frac{2}{k(k-2)!}\left\{\frac{\partial^{k-2}}{\partial \bar{W}_1^{k-2}}\left(\frac{\partial G(0,W_1)}{\partial \bar{W}_1}\frac{\partial^{l+1} G_W(W_1)}{\partial \bar{W}_1 \partial W^l}\right)\right\}_{W_1=W=1}$$

where $G_{W_1}(W)$ is the representation of the Green's function $g_{z_1}(z)$ by means of the local parameter $W = \sqrt{z}$ for $z \in U^*$ such that $W_1 = \sqrt{z_1}$.



Proof: First, fix $z_1 (\neq O_0)$ near O_0 in $D(0)$. We form the Green's function $g_{z_1}(z)$ for $(D(0), z_1)$.

Let $0 < \varepsilon < |z_1|$. Draw the circle $\Gamma_\varepsilon(0)$ (which round O_0 k times) and the circle γ_δ of center z_1 with radius $\delta > 0$. We write $[\Gamma_\varepsilon(0)]$ and $[\gamma_\delta]$ their bounding domains of $D(0)$. As already noted, for each $|t| < \varepsilon$, the functions

$$g(t,z) - g(0,z), \quad g_{z_1}(z)$$

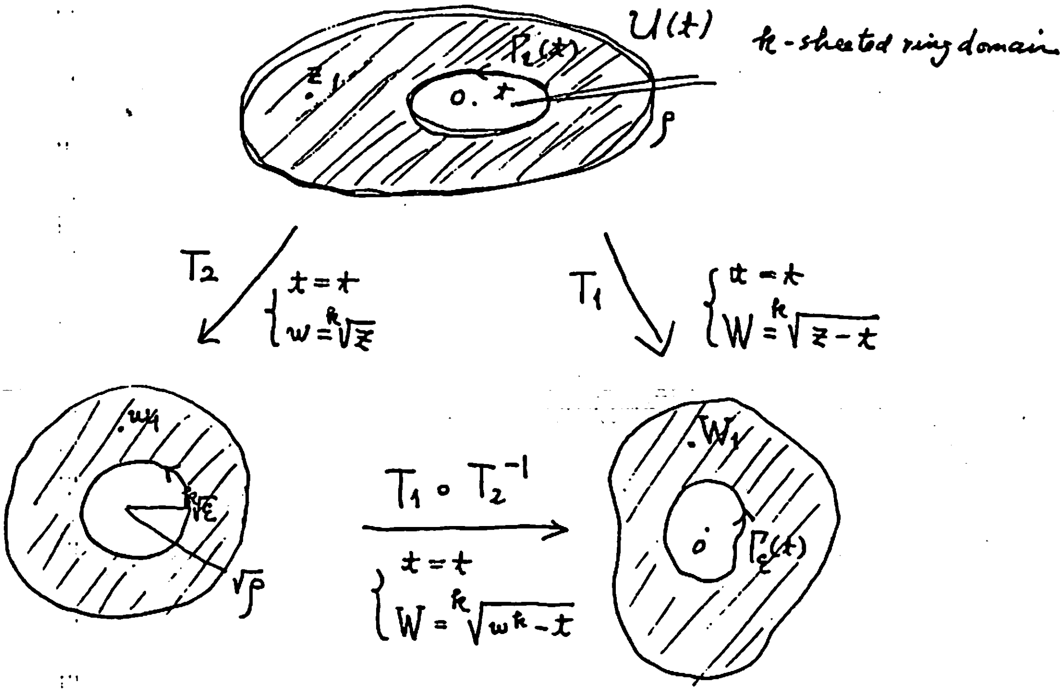
are harmonic in $D(0) - [\Gamma_\varepsilon(0)] - [\gamma_\delta]$ (even at a). It follows from Green's formula that

$$\int_{C - \gamma_\delta - \Gamma_\varepsilon(0)} \big(g(t,z) - g(0,z)\big) \frac{\partial g_{z_1}(z)}{\partial n_z}\, ds_z = \int_{C - \gamma_\delta - \Gamma_\varepsilon(0)} g_{z_1}(z) \frac{\partial}{\partial n_z}\big(g(t,z) - g(0,z)\big)\, ds_z$$

By the same argument as in p.9 at Nov. 2, we get

$$(3.7) \quad g(t,z_1) - g(0,z_1) = -\frac{1}{2\pi} \int_{\Gamma_\varepsilon(0)} \big(g(t,z) - g(0,z)\big) \frac{\partial g_{z_1}(z)}{\partial n_z}\, ds_z - g_{z_1}(z) \frac{\partial}{\partial n_z}\big(g(t,z) - g(0,z)\big)\, ds_z$$

In the right-hand side, function containing the variable t is separated from that containing z_1.



By means of the local parameter $w = \sqrt{z}$, (3.7) is written

$$(3.8)\quad g(t,w_1) - g(0,w_1) = -\frac{1}{2\pi}\int_{|w|=\sqrt[k]{\varepsilon}} \left(g(t,w)-g(0,w)\right)\frac{\partial g_{w_1}(w)}{\partial n_w}\,dw - g_{w_1}(w)\frac{\partial}{\partial n_w}\left(g(t,w)-g(0,w)\right)dw$$

where $w_1 = \sqrt[k]{z_1}$.

On the other hand, for $\rho > |z| \geqq \varepsilon$ and $\varepsilon \gg |t|$, we have by $T_1 \circ T_2^{-1}$

$$W = \sqrt[k]{w^k - t}$$
$$= w - \frac{t}{k}\frac{1}{w^{k-1}} - \frac{(k-1)t^2}{2k^2}\frac{1}{w^{2k-1}} + \cdots\cdots$$

In particular, at $t=0$, we get

$$W = w.$$

If we put $G(t,W) = g(t,z) = g(t,w)$ and $G_{W_1}(W) = g_{w_1}(w)$, then, for $|t| \ll \varepsilon$ and $\sqrt[k]{\rho} > |w| > \sqrt[k]{\varepsilon}$, we have

$$g(t,w) = G\left(t, w - \frac{t}{k}\frac{1}{w^{k-1}} - \frac{(k-1)t^2}{2k^2}\frac{1}{w^{2k-1}} + \cdots\right),$$

$$g(0,w) = G(0,w).$$

It follows from (3.8) that

$$(3.9) \quad G\left(t, w_1 - \frac{t}{k}\frac{1}{w_1^{k-1}} - \frac{(k-1)t^2}{2k^2}\frac{1}{w_1^{2k-1}} + \cdots\right) - G(0,w_1)$$

$$= \frac{-1}{2\pi}\int_{|w|=\sqrt[k]{\varepsilon}} \left[G\left(t, w - \frac{t}{k}\frac{1}{w^{k-1}} \cdots\right) - G(0,w)\right]\frac{\partial G_{w_1}(w)}{\partial n_w}\,ds_w - G_{w_1}(w)\frac{\partial}{\partial n_w}\left[G\left(t, w - \frac{t}{k}\frac{1}{w^{k-1}} \cdots\right) - G(0,w)\right]ds_w.$$

(3.9) holds for all $w_1 \neq 0$ near 0, $|w_1| > \sqrt[k]{\varepsilon} > 0$ and $\forall\, |t| < \sqrt[k]{\varepsilon}$. Since $G(t,W)$ is of class C^2 in $B^* \times U^*$ where $B^* = (|t| < r^*)$ and $U^* = (|W| < \rho^*)$, we differentiate both sides of (3.9) by $\bar{t}$ and then put $t = 0$. We obtain

$$\frac{\partial G}{\partial \bar{t}}(0,w_1) - \frac{1}{k}\frac{1}{\bar{w}_1^{k-1}}\frac{\partial G}{\partial \bar{W}}(0,w_1)$$

$$= \frac{-1}{2\pi}\int_{|w|=\sqrt[k]{\varepsilon}} \left[\frac{\partial G}{\partial \bar{t}}(0,w) - \frac{1}{k\bar{w}^{k-1}}\frac{\partial G}{\partial \bar{W}}(0,w)\right]\frac{\partial G_{w_1}(w)}{\partial n_w}\,ds_w - G_{w_1}(w)\frac{\partial}{\partial n_w}\left[\frac{\partial G}{\partial \bar{t}}(0,w) - \frac{1}{k\bar{w}^{k-1}}\frac{\partial G}{\partial \bar{W}}(0,w)\right]ds_w.$$

Observing $G(0,W)$ is harmonic w.r.t W, we differentiate both sides by w_1 l-times. We obtain

$$(3.10) \quad \frac{\partial^{l+1} G}{\partial \bar{t}\,\partial W^l}(0,w_1)$$

$$= \frac{-1}{2\pi}\int_{|w|=\sqrt[k]{\varepsilon}} \left[\frac{\partial G}{\partial \bar{t}}(0,w) - \frac{1}{k\bar{w}^{k-1}}\frac{\partial G}{\partial \bar{W}}(0,w)\right]\frac{\partial}{\partial n_w}\left(\frac{\partial^l G_{w_1}(w)}{\partial w_1^l}\right)ds_w - \frac{\partial^l G_{w_1}(w)}{\partial w_1^l}\frac{\partial}{\partial n_w}\left[\frac{\partial G}{\partial \bar{t}}(0,w) - \frac{1}{k\bar{w}^{k-1}}\frac{\partial G}{\partial \bar{W}}(0,w)\right]ds_w$$

Since (3.10) holds for $w_1 \neq 0$ and $|w_1| > \sqrt[k]{\varepsilon} > 0$, we let $\varepsilon \to 0$. $G(t, W)$ is of class in $B^* \times U^*$, so that

$$\lim_{\varepsilon\to 0} \int_{|w|=\sqrt[k]{\varepsilon}} \frac{\partial G}{\partial \bar{t}}(0,w) \frac{\partial}{\partial n_w}\left(\frac{\partial^l G_{w_1}(w)}{\partial w_1^{\,l}}\right) ds_w = 0 \ ;$$

$$\lim_{\varepsilon\to 0} \int_{|w|=\sqrt[k]{\varepsilon}} \frac{\partial^l G_{w_1}(w)}{\partial w_1^{\,l}} \frac{\partial}{\partial n_w}\left(\frac{\partial G}{\partial \bar{t}}(0,w)\right) ds_w = 0.$$

It follows that

$$\frac{\partial^{l+1} G}{\partial \bar{t}\, \partial W^l}(0, w_1) = \frac{1}{2k\pi} \lim_{\varepsilon\to 0} \int_{|w|=\sqrt[k]{\varepsilon}} \frac{1}{\bar{w}^{k-1}} \frac{\partial G}{\partial \bar{W}}(0,w) \frac{\partial}{\partial n_w}\left(\frac{\partial^l G_{w_1}(w)}{\partial w_1^{\,l}}\right) ds_w$$
$$- \frac{\partial^l G_{w_1}(w)}{\partial w_1^{\,l}} \frac{\partial}{\partial n_w}\left(\frac{1}{\bar{w}^{k-1}} \frac{\partial G}{\partial \bar{W}}(0,w)\right) ds_w.$$

All functions under the integral in the right-hand side are determined by the Riemann surface $D(0)$ itself. Since $\frac{\partial}{\partial n_w} ds_w = \frac{1}{i}\left(\frac{\partial}{\partial w} dw - \frac{\partial}{\partial \bar{w}} d\bar{w}\right)$ and $\partial^2 G/\partial w \partial \bar{w} = 0$, we have

$$\int_{|w|=\sqrt[k]{\varepsilon}} \frac{\partial^l G_{w_1}(w)}{\partial w_1^{\,l}} \frac{\partial}{\partial n_w}\left(\frac{1}{\bar{w}^{k-1}} \frac{\partial G}{\partial \bar{W}}(0,w)\right) ds_w$$

$$= \frac{1}{i} \int_{|w|=\sqrt[k]{\varepsilon}} \frac{\partial^l G_{w_1}(w)}{\partial w_1^{\,l}}\, d\left(\frac{1}{\bar{w}^{k-1}} \frac{\partial G}{\partial \bar{W}}(0,w)\right)$$

$$= \frac{1}{i} \int_{|w|=\sqrt[k]{\varepsilon}} d\left(\frac{\partial^l G_{w_1}(w)}{\partial w_1^{\,l}} \frac{1}{\bar{w}^{k-1}} \frac{\partial G}{\partial \bar{w}}(0,w)\right) - \frac{1}{\bar{w}^{k-1}} \frac{\partial G}{\partial \bar{W}}(0,w)\, d\left(\frac{\partial^l G_{w_1}(w)}{\partial w_1^{\,l}}\right.$$

$$= +\frac{1}{i} \int_{|w|=\sqrt[k]{\varepsilon}} \frac{1}{\bar{w}^{k-1}} \frac{\partial G}{\partial \bar{w}}(0,w) \left\{ \frac{\partial^{l+1} G_{w_1}(w)}{\partial w\, \partial w_1^{\,l}} dw + \frac{\partial^{l+1} G_{w_1}(w)}{\partial \bar{w}\, \partial w_1^{\,l}} d\bar{w} \right\}$$

Therefore $\quad \frac{\partial^{l+1} G}{\partial \bar{t}\, \partial W^l}(0, w_1)$

$$= \frac{1}{2k\pi} \lim_{\varepsilon\to 0} \int_{|w|=\sqrt[k]{\varepsilon}} \frac{1}{\bar{w}^{k-1}} \frac{\partial G}{\partial \bar{w}}(0,w)\ \frac{1}{i}\left\{\frac{\partial}{\partial w}\left(\frac{\partial^l G_{w_1}(w)}{\partial w_1^{\,l}}\right)dw - \frac{\partial}{\partial \bar{w}}\left(\frac{\partial^l G_{w_1}(w)}{\partial w_1^{\,l}}\right)d\bar{w}\right\}$$

$$-\frac{1}{i}\ \frac{1}{\bar{w}^{k-1}} \frac{\partial G}{\partial \bar{w}}(0,w)\left\{\frac{\partial^{l+1} G_{w_1}(w)}{\partial w\,\partial w_1^{\,l}}dw + \frac{\partial^{l+1} G_{w_1}(w)}{\partial \bar{w}\,\partial w_1^{\,l}}d\bar{w}\right\}$$

$$= \frac{-1}{k\pi i} \lim_{\varepsilon\to 0} \int_{|w|=\sqrt[k]{\varepsilon}} \frac{1}{\bar{w}^{k-1}} \frac{\partial G}{\partial \bar{w}}(0,w)\ \frac{\partial^{l+1} G_{w_1}(w)}{\partial \bar{w}\ \partial w_1^{\,l}}\,d\bar{w}$$

$$= \frac{2}{k} \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{|w|=\sqrt[k]{\varepsilon}} \frac{1}{w^{k-1}} \frac{\partial G}{\partial w}(0,w)\ \frac{\partial^{l+1} G_{w_1}(w)}{\partial w\,\partial \bar{w}_1^{\,l}}\,dw$$

$$= \frac{2}{k}\ \frac{1}{(k-2)!}\left\{\frac{\partial^{k-2}}{\partial w^{k-2}}\left(\frac{\partial G}{\partial w}(0,w)\ \frac{\partial^{l+1} G_{w_1}(w)}{\partial w\ \partial \bar{w}_1^{\,l}}\right)\right\}_{w=0} \qquad (\because \text{Cauchy's formula})$$

Putting $w_1 = 0$, we obtain

$$\frac{\partial^{l+1} G}{\partial \bar{t}\,\partial w^l}(0,0) = \frac{2}{k\,(k-2)!}\left\{\frac{\partial^{k-2}}{\partial \bar{w}^{k-2}}\left(\frac{\partial G}{\partial \bar{w}}(0,w)\ \frac{\partial^{l+1} G_{w_1}(w)}{\partial \bar{w}\,\partial w_1^{\,l}}\right)\right\}_{w=w_1=0}.$$

If we change the notations (w, w_1) to (w_1, w), then

$$\frac{\partial^{l+1} G}{\partial \bar{t}\,\partial W^l}(0,0) = \frac{2}{k\,(k-2)!}\left\{\frac{\partial^{k-2}}{\partial \bar{w}_1^{\,k-2}}\left(\frac{\partial G}{\partial \bar{w}_1}(0,w_1)\ \frac{\partial^{l+1} G_w(w_1)}{\partial \bar{w}_1\ \partial w^l}\right)\right\}_{w=w_1=}$$

Lemma 2.2 is now proved.

Let us substitute Lemma 2.2 for (3.0), and we obtain

$$\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(0) = \frac{8}{(k(k-2)!)^2} \sum_{i=1}^{k-2}\binom{k-2}{i}\left[\frac{\partial^{i+1} G}{\partial W^{i+1}}\cdot\left\{\frac{\partial^{k-2}}{\partial \bar{W}_1^{\,k-2}}\left(\frac{\partial G}{\partial \bar{W}_1}\cdot\frac{\partial^{k-i} G_W(W_1)}{\partial \bar{W}_1\,\partial W^{k-i-1}}\right)\right\}\right]_{W=W_1}$$

$$= \frac{8}{(k(k-2)!)^2}\left\{\frac{\partial^{k-2}}{\partial \bar{W}_1^{\,k-2}}\left[\frac{\partial G}{\partial \bar{W}_1}\left\{\sum_{i=1}^{k-2}\binom{k-2}{i}\frac{\partial^{i+1} G}{\partial W^{i+1}}\ \frac{\partial^{k-i} G_W(W_1)}{\partial W^{k-i-1}\ \partial \bar{W}_1}\right\}\right]\right\}_{W=W_1=}$$

$$= \frac{8}{(k(k-2)!)^2}\left\{\frac{\partial^{k-2}}{\partial \bar{W}_1^{\,k-2}}\left[\frac{\partial G}{\partial \bar{W}_1}\cdot\frac{\partial^{k-2}}{\partial W^{k-2}}\left(\frac{\partial G}{\partial W}\ \frac{\partial^2 G_W(W_1)}{\partial W\,\partial \bar{W}_1}\right)\right]\right\}_{W=W_1=c}$$

We conclude from $G_W(W_1) = G_{W_1}(W)$ that

$$(3.11)\quad \frac{\partial^2\lambda}{\partial t\partial\bar t}(0) = \frac{8}{(k(k-2)!)^2}\left\{\frac{\partial^{2k-4}}{\partial W^{k-2}\partial \bar W_1^{k-2}}\left(\frac{\partial G(W)}{\partial W}\,\frac{\partial G(W_1)}{\partial \bar W_1}\,\frac{\partial^2 G_W(W_1)}{\partial W\,\partial \bar W_1}\right)\right\}_{W=W_1=0}$$

$$= \frac{8}{(k(k-2)!)^2}\left\{\frac{\partial^{2k-4}}{\partial W^{k-2}\partial \bar W_1^{k-2}}\left(\frac{\partial G(W)}{\partial W}\,\frac{\partial G(W_1)}{\partial \bar W_1}\,\frac{\partial^2 G_{W_1}(W)}{\partial W\,\partial \bar W_1}\right)\right\}_{W=W_1=0}$$

where $G(W)$ and $G_{W_1}(W)$ are the representations of the Green's fns for $(D(0), a)$ and $(D(0), z_1)$, respectively, by means of the local parameter $W=\sqrt{z}$ at O, for $W_1 = \sqrt{z_1}$.

In particular, the case when $k=2$, (3.11) is

$$(3.11')\quad \frac{\partial^2\lambda}{\partial t\partial\bar t}(0) = 2\left|\frac{\partial G}{\partial W}(0)\right|^2 \left\{\frac{\partial^2 G_{W_1}(W)}{\partial W\,\partial \bar W_1}\right\}_{W=W_1=0}.$$

By (3.6) we see that

$$\frac{\partial^2\lambda}{\partial t\partial\bar t}(0) \leqq 0$$

and that the equality holds if and only if $\frac{\partial G}{\partial W}(0)=0$.

In the general case when $k \geqq 2$, we put, given $W_1 \in \Gamma^*(0)$,

$$\alpha_{W_1}(z)\,dz = \frac{\partial}{\partial z}\left(\frac{\partial^{k-2}}{\partial \bar W_1^{k-2}}\left\{\frac{\partial G(W_1)}{\partial \bar W_1}\,\frac{\partial g_{W_1}(z)}{\partial \bar W_1}\right\}\right)dz.$$

As already noted, $\alpha_{W_1}(z)\,dz$ is a holomorphic differential on $D(0)$ such that

(i) the fn. $\frac{\partial^{k-2}}{\partial \bar W_1^{k-2}}\left\{\frac{\partial G(W_1)}{\partial \bar W_1}\,\frac{\partial g_{W_1}(z)}{\partial \bar W_1}\right\} = 0$ on $\partial D(0)$;

(ii) At a neighborhood of $W=0$, we have

$$\frac{\partial g_{W_1}(W)}{\partial W_1} = \frac{1}{W-W_1} + (\text{regular}) \quad \text{where } W=\sqrt{z}.$$

Hence $\alpha_{W_1}(z)\,dz$ is with norm finite. Let $a(z)\,dz$ be any holomorphic differential with finite norm. Then

$\langle a(z)dz,\ \alpha_{w_1}(z)dz \rangle$

$$= -\frac{1}{2i} \iint_{D(0)} a(z) \frac{\partial^{k-2}}{\partial w_1^{k-2}} \left(\frac{\partial G(w_1)}{\partial w_1} \cdot \frac{\partial^2 g_{w_1}(z)}{\partial w_1 \partial \bar{z}} \right) dz \wedge d\bar{z}$$

$$= +\frac{1}{2i} \iint_{D(0)} d\Big[a(z) \underbrace{\frac{\partial^{k-2}}{\partial w_1^{k-2}} \left(\frac{\partial G(w_1)}{\partial w_1} \frac{\partial g_{w_1}(z)}{\partial w_1} \right)}_{\text{function on } D(0) \text{ with pole at } w_1} dz \Big]$$

$$= \frac{1}{2i} \lim_{\varepsilon \to 0} \int_{C - \gamma_\varepsilon(w_1)} \left[\frac{\partial^{k-2}}{\partial w_1^{k-2}} \left(\frac{\partial G(w_1)}{\partial w_1} \frac{\partial g_{w_1}(z)}{\partial w_1} \right) \right] a(z) dz$$

where $\gamma_\varepsilon(w_1)$ is the circle of center w_1 with radius $\varepsilon > 0$ on $U^*(0)$.

By (i) and (ii), it becomes

$$= \frac{-1}{4i} \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon(w_1)} \left[\frac{\partial^{k-2}}{\partial w_1^{k-2}} \left(\frac{\partial G(w_1)}{\partial w_1} \frac{1}{w - w_1} \right) \right] a(w) dw$$

$$= \frac{-1}{4i} \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon(w_1)} \sum_{i=0}^{k-2} \binom{k-2}{i} \left(\frac{\partial^i}{\partial w_1^i} \left(\frac{\partial G(w_1)}{\partial w_1} \right) \right) \cdot \frac{(k-2-i)!}{(w-w_1)^{k-2-i+1}} \cdot a(w) dw$$

$$= \frac{-1}{4i} \lim_{\varepsilon \to 0} \sum_{i=0}^{k-2} \binom{k-2}{i} \left(\frac{\partial^i}{\partial w_1^i} \left(\frac{\partial G(w_1)}{\partial w_1} \right) \right) \cdot \int_{\gamma_\varepsilon(w_1)} \frac{(k-2-i)!}{(w-w_1)^{k-2-i+1}} a(w) dw$$

$$= -\frac{\pi}{2} \lim_{\varepsilon \to 0} \sum_{i=0}^{k-2} \binom{k-2}{i} \left(\frac{\partial^i}{\partial w_1^i} \left(\frac{\partial G(w_1)}{\partial w_1} \right) \right) \cdot \left[\frac{\partial^{k-2-i}}{\partial w^{k-2-i}} a(w) \right]_{w=w_1} \quad \text{: by Cauchy's formula}$$

$$= -\frac{\pi}{2} \left[\frac{\partial^{k-2}}{\partial w^{k-2}} \left\{ \frac{\partial G(w)}{\partial w} \cdot a(w) \right\} \right]_{w=w_1}$$

In particular, we put $a(z)dz = \alpha_{w_1}(z)dz$. Then

$$\| \alpha_{w_1}(z) dz \|^2_{D(0)} = -\frac{\pi}{2} \left[\frac{\partial^{k-2}}{\partial w^{k-2}} \left\{ \frac{\partial G(w)}{\partial w} \cdot \frac{\partial^{k-2}}{\partial \bar{w}_1^{k-2}} \left(\frac{\partial G(w)}{\partial \bar{w}_1} \frac{\partial^2 G_{w_1}(w)}{\partial w \partial \bar{w}_1} \right) \right\} \right]_{w=w_1}$$

Hence

$$\left[\frac{\partial^{2k-4}}{\partial w^{k-2}\,\partial \bar{w}_1^{k-2}}\left\{\frac{\partial G(w)}{\partial w}\frac{\partial G(w_1)}{\partial w_1}\frac{\partial^2 G_{w_1}(w)}{\partial w\,\partial \bar{w}_1}\right\}\right]_{w=w_1}$$

$$= -\frac{2}{\pi}\iint_{D(0)}\left|\frac{\partial^{k-2}}{\partial \bar{w}_1^{k-2}}\left(\frac{\partial G(w_1)}{\partial \bar{w}_1}\frac{\partial^2 g_{w_1}(z)}{\partial \bar{w}_1\,\partial z}\right)\right|^2 dxdy \leqq 0$$

By (3.11), we obtain

<u>Theorem 2.1</u>

$$\frac{\partial^2 \lambda}{\partial t\partial \bar{t}}(0) = \frac{-16}{(k(k-2)!)^2\pi}\iint_{D(0)}\left|\frac{\partial^{k-2}}{\partial \bar{w}_1^{k-2}}\left(\frac{\partial G(w_1)}{\partial \bar{w}_1}\frac{\partial^2 g_{w_1}(z)}{\partial \bar{w}_1\,\partial z}\right)\right|^2_{w_1=0} dxdy \leqq 0$$

We study the case when $\frac{\partial^2\lambda}{\partial t\partial \bar{t}}(0)=0$. Then the function $F(z) = \left[\frac{\partial^{k-2}}{\partial w_1^{k-2}}\left(\frac{\partial G(w_1)}{\partial w_1}\frac{\partial g_{w_1}(z)}{\partial w_1}\right)\right]_{w_1=0}$ is holomorphic on $D(0)-\{0_0\}$. Since

$$F(z)=\sum_{i=1}^{k-2}\binom{k-2}{i}\left[\frac{\partial^i}{\partial w_1^i}\left(\frac{\partial G(w_1)}{\partial w_1}\right)\right]_{w_1=0}\cdot\left[\frac{\partial^{k-2-i}}{\partial w_1^{k-2-i}}\left(\frac{\partial g_{w_1}(z)}{\partial w_1}\right)\right]_{w_1=0}$$

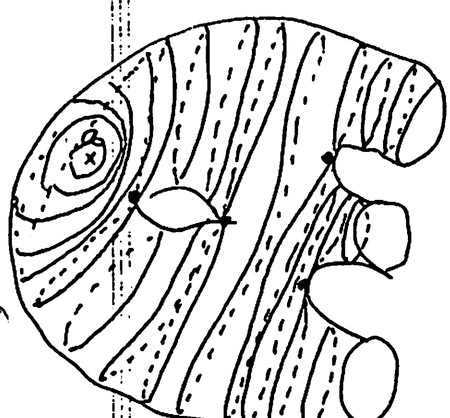
and $[(\partial^l/\partial w_1^l)(\partial g_{w_1}(z)/\partial w_1)]_{w=0}=0$ on $\partial D(0)$ by (i), we have $F(z)\equiv 0$ on $D(0)-\{0_0\}$. On the other hand, in a neighborhood of $w=w_1$, we have

$$\frac{\partial^l}{\partial w_1^l}\left(\frac{\partial g_{w_1}(w)}{\partial w_1}\right) = \frac{1}{2}\frac{l!}{(w-w_1)^{l+1}} + (\text{regular}) \qquad (l=0,1,\cdots),$$

which are independent each other. We conclude thus

$$(3.12)\quad \left[\frac{\partial^i}{\partial w_1^i}\left(\frac{\partial G(w_1)}{\partial w_1}\right)\right]_{w_1=0} = 0 \qquad (i=1,2,\cdots,k-2)$$

In general, let R be a Riemann surface with smooth boundary C. Assume that R is of genus g and with $n(\geq 1)$ boundary components. For $a \in R$, we form the Green's fn $g(z)$ for (R, a). Then $\frac{\partial g}{\partial z} dz$ defines a holomorphic differential on $R - \{a\}$. As by the left figure, the number of zero of $\frac{\partial g}{\partial z} dz$ (counted the multiplicity) is



$$(3.13) \qquad 2g - n - 1$$

in R. Precise proof is not difficult, but we won't show it here.

level curves of $g(z)$

Let us return to (3.12). Equalities (3.12) means that $\frac{\partial g}{\partial z} dz$ has zero at O_0 with multiplicity $k-1$, while $k-1$ is the order of the ramification of O_0. Therefore

<u>Lemma 2.3</u> $\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(0) = 0$ if and only if $\frac{\partial g(0,z)}{\partial z} dz$ has has zero at the branch point O_0 (of order $k-1$) with multiplicity $k-1$.

— This is a remark by Prof. J. Wermer.

Until now we consider the variation $\mathcal{D}: t \to D(t)$ $(|t|<\rho)$ such that $D(t)$ has only one moving branch point O_t of order $k-1$ $(k \geq 2)$ over the z-coordinate $z = t$. Instead of O_t, we assume that $D(t)$ has only one branch point $O(t)$ of same order $k-1$ over the z-coordinate $z = f(t)$ where $f(t)$ is a holomorphic fn on $B = (|t|<\rho)$.

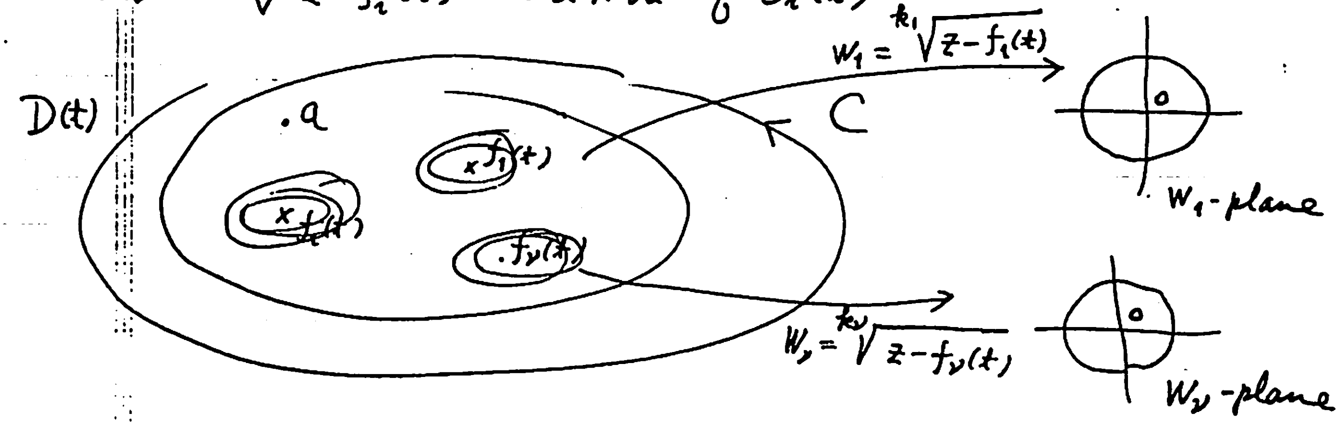
Assume there exists a such that $a \in D(t)$ for all $t \in B$. We consider the Green's fn $g(t,z)$ and the Robin constant $\lambda(t)$ for $(D(t), a)$. Since $f(t)$ is holomorphic on B, it follows from Lemma 2.1 and Lemma 2.2 that

$$(3.14)\quad \begin{cases} \dfrac{\partial \lambda}{\partial t}(t) = f'(t)\, \dfrac{2}{k(k-2)!} \left[\dfrac{\partial^{k-2}}{\partial W^{k-2}} \left(\dfrac{\partial G(t,W)}{\partial W} \right)^2 \right]_{W=0} ; \\[2ex] \dfrac{\partial^2 \lambda}{\partial \bar t \partial t}(t) = |f'(t)|^2 \dfrac{-16}{(k(k-2)!)^2} \displaystyle\iint_{D(t)} \left| \dfrac{\partial^{k-2}}{\partial \overline{W_1}^{k-2}} \left(\dfrac{\partial G(t,W_1)}{\partial \overline{W_1}} \dfrac{\partial^2 g_{W_1}(t,\bar z)}{\partial \overline{W_1} \partial \bar z} \right) \right|^2_{W_1=0} dx dy \end{cases}$$

where $G(t,W) = g(t,z)$ where $W = \sqrt[k]{z - f(t)}$ and $g_{W_1}(t,z)$ is the Green's fn for $(D(t), z_1)$ for $W_1 = \sqrt[k]{z_1 - f(t)}$.

§3. The case when the branch points are moving.

In §2, we studied the variation $\mathcal{D} : t \to D(t)$ $(t \in B)$ such that $D(t)$ has only one moving branch point and the fixed boundary C. In this §, we treat the case where $D(t)$ has some moving branch points $O_i(t)$ $(i = 1, \dots, \nu)$ and the fixed boundary C. We denote by $k_i - 1$ $(k_i \geqq 2)$ and $f_i(t)$ the order of ramification of $O_i(t)$ and the z-coordinate of $O_i(t)$, respectively. We assume that each $O_i(t)$ does not meet each other and does not cross the fixed boundary C. The most standard local parameter is $W_i = \sqrt[k_i]{z - f_i(t)}$ in a nbd of $O_i(t)$



Assume that $a \in D(t)$ for all $t \in B$. We construct the Green's fu $g(t,z)$ for $(D(t), a)$ and the Robin constant $\lambda(t)$ for $(D(t), a)$. Let $t_0 \in B$. Around each point $O_i(t_0)$, we draw a small circle $\Gamma_\varepsilon^{(i)}(O_i(t_0))$ with radius ε ($\ll 1$). Then, by the same argument as (2.3),

$$\lambda(t) - \lambda(t_0) = -\frac{1}{2\pi}\sum_{i=1}^{\nu} \int_{\Gamma_\varepsilon^{(i)}(O_i(t_0))} \left(g(t,z)\frac{\partial g(t_0,z)}{\partial n_z} - g(t_0,z)\frac{\partial g(t,z)}{\partial n_z}\right) ds_z$$

From this we have analogusly

Theorem 3.1

$$\frac{\partial \lambda}{\partial t}(t) = \sum_{i=1}^{\nu} \frac{2}{k_i(k_i-2)!} \cdot f_i'(t) \cdot \left[\frac{\partial^{k_i-2}}{\partial W_i^{k_i-2}}\left(\frac{\partial G_i(t,W_i)}{\partial W_i}\right)^2\right]_{W_i=0}$$

where $G_i(t, W_i)$ is the representation of the Green's fu $g(t,z)$ for $(D(t), a)$ by means of $W_i = \sqrt[k_i]{z - f_i(t)}$.

Moreover we get, by the same method which induces Lemma 2.2,

Theorem 3.2

$$\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(t) = \sum_{i=1}^{\nu} \frac{-16}{(k_i(k_i-2))^2} |f_i'(t)|^2 \, \|\alpha_i(t)\|^2_{D(t)} \; ;$$

where $$\|\alpha_i(t)\|^2_{D(t)} = \iint_{D(t)} \left| \frac{\partial^{k_i-2}}{\partial \bar{W}_1^{k-2}}\left(\frac{\partial G(t,W_1)}{\partial \bar{W}_1}\frac{\partial g_{W_1}(z)}{\partial \bar{W}_1 \partial z}\right)\right|^2_{W_1=0} dx dy \; ;$$

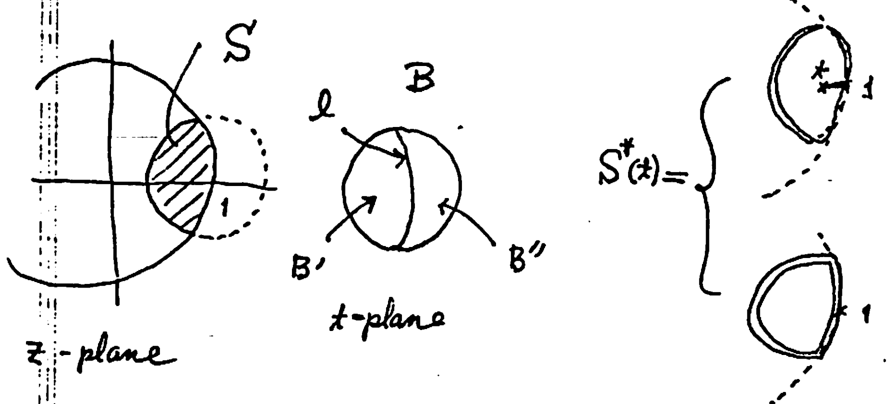
$G(t,W) = g(t,z)$ where $W = \sqrt[k_i]{z - f_i(t)}$ in a nbd of $O_i(t)$ and $g_{W_1}(z)$ is the Green's fu for $(D(t), z_1)$ for $W_1 = \sqrt[k_i]{z_1 - f_i(t)}$ (z_1 is near $O_i(t)$).

Corollary 3.1 Under the same situation as above, let g and n (≥ 1) be of genus and the number of boundary components of $D(t)$. If $k_1+k_2+\cdots+k_\nu-\nu \geq 2g+n$, and if $f_i(t)$ $(i=1,\dots,\nu)$ is not constant on B, then $\lambda(t)$ is not harmonic on B.

Proof. Assume that $\lambda(t)$ is harmonic on B, i.e., $\partial^2\lambda/\partial t\partial\bar{t}=0$. Theorem 3.2 yield that $\|\alpha_i(t)\|^2_{D(t)}=0$ $(i=1,\dots,\nu)$. By the same argument as Lemma 2.3, $\frac{\partial g(t,z)}{\partial z}dz$ has k_i-1 zero at each $O_i(t)$. Hence it has at least $k_1+\cdots+k_\nu-\nu$ zeros on R. This is a contradiction (3.14).

§4 The case when a moving branch point crosses the boundary C.

Let B be a disk on the complex t-plane: $B=(|t-1|<\rho)$ where $0<r<1$. Let $S=(|z|<1)\cap(|z-1|<r)$ in the complex z-plane. Consider the Riemann domain $\mathbb{R}$ of the function $\sqrt[k]{z-t}$ over $\mathbb{C}_t\times\mathbb{C}_z$. Denote $\mathcal{S}^*$ the part of $\mathbb{R}$ over $B\times S$. As usual, $S^*(t)$ denotes the fiber of $\mathcal{S}^*$ at $t\in B$, so that $\mathcal{S}^*=\bigcup_{t\in B}(t,S^*(t))$.

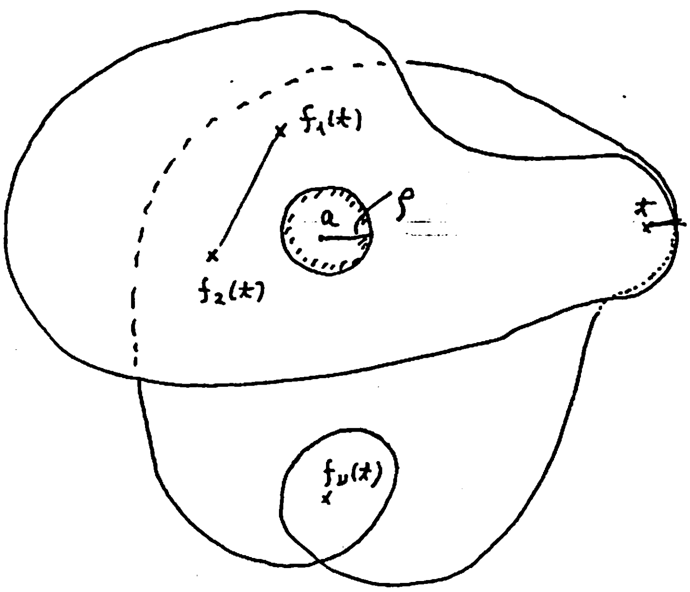


For $t\in B'$, $S^*(t)$ has one branch point at t.

For $t\in l\cup B''$, $S^*(t)$ consists of k-separated peaces over S

We write $B'=B\cap(|t|<1)$, $l=B\cap(|t|=1)$, $B''=B\cap(|t|>1)$. Using $\mathcal{S}^*$ we consider the variation $\mathcal{D}: t\to D(t)$ $(t\in B)$ such that

(1) The boundary $\partial D(t) = C$ does not move with $t \in B$;
(2) $D(t) \supset \mathcal{S}^*(t)$ for all $t \in B$;
(3) $D(t)$ has also ν moving points $f_i(t)$ ($i=1,\dots,\nu$) of order $k_i - 1$;
(4) We find a univalent domain $\mathcal{V}$ of $\mathcal{D}$ found over $B \times (|z-a|<\rho)$. where $(|z-a|<\rho) \cap \mathcal{S}' = \phi$.



$D(t)$
$(t \in B')$

Therefore $D(t)$ for $t \in B'$ has $\nu+1$ moving branch points, while $D(t)$ for $t \in \ell \cup B''$ has ν- moving branch points. By (4), each $D(t)$ carries the Green function $g(t,z)$ with pole at a and the Robin constant $\lambda(t)$, so that

$$(4.1) \qquad g(t,z) = \log \frac{1}{|z-a|} + \lambda(t) + h(t,z)$$

where $h(t,z)$ is harmonic in $|z-a|<\rho$ and $h(t,a)=0$.
The variation $\mathcal{D} : t \longrightarrow D(t)$ $(t \in B)$ is no longer diffeomorphically (incidently topologically) equivalent to the trivial one. Hence either $g(t,z)$ and $\lambda(t)$ is not of class C^2 on $\mathcal{D}$ or B, respectively (see §1). However we can prove the following differentiability of $\lambda(t)$ which is all we need:

<u>Lemma 4.1</u> $\lambda(t)$ is of class C^1 on B.

<u>Proof.</u> <u>1^{st} step</u>. $\lambda(t)$ is continuous on B.

There is no problem for $t \in B'$, for $\mathcal{D}|_{B'} : t \to D(t)$ $(t \in B')$ is (locally) diffeomorphically equivalent to the trivial one. Similar is true for $\mathcal{D}|_{B'' \cup \ell} : t \to D(t)$ $(t \in B'' \cup \ell)$. It rests for 1^{st} step to prove that, for a given $t_0 \in \ell$,

$$(4.2) \qquad \lim_{\substack{t \to t_0 \\ t \in B'}} \lambda(t) = \lambda(t_0).$$

Without loss of generality, we may assume $t_0 = 1$.

Since $D(t)$ is uniformly bounded w.r.t. $t \in B$, we find a large disk $K_R : |z-a| < R$ on the z-plane such that the projection of $D(t)$ to the z-plain is contained in K_R. Since $\log \frac{R}{|z-a|}$ becomes a superharmonic fn on $D(t)$, it follows from the maximum principle that

$$0 < g(t, z) < \log \frac{R}{|z-a|} \quad \text{on } D(t) \ (\forall t \in B)$$

By (4.1) we get

$$\left| g(t, z) - g(1, z) \right| < \log \frac{R}{\rho} \ \left(\underset{\text{def}}{=} M \right)$$

for $(|z-a| > \rho) \cap D(t) \cap D(1)$ and for $\forall t \in B$.

Now, let $\varepsilon > 0$ be given arbitrarily. Then we can find a small $\delta > 0$ such that the harmonic function

$$u_\delta(z) = \begin{cases} M & \text{on } (|z-1| = \delta) \cup \bigcup_{i=1}^{\nu} (|z - f_i(1)| = \delta) \\ 0 & \text{on } (|z-a| = R) \end{cases}$$

in $(|z-a| < R) - \left\{ (|z-1| < \delta) \cup \bigcup_{i=1}^{\nu} \left(|z - f_i(1)| < \delta \right) \right\} \ (\subset \mathbb{C}_z)$

satisfies $u_\delta(z) < \varepsilon$ for $|z-a| \leqq \rho$. This is possible

Since $u_\delta(z)$ is regarded as a harmonic fu on $D_\delta(1)$, it follows from the maximum principle that

$$|v(t,z)| \leqq u(t,z) \quad \text{for } z \in D_\delta(1)$$

where $|t-1|<\eta$. In particular,

$$|v(t,z)| < \varepsilon \quad \text{for } |z-a|<\rho$$

where $(|t-1|<\eta) \cap B'$. By (4.1), we thus have

$$|\lambda(t)-\lambda(1)|<\varepsilon \quad \text{for all } (|t-1|<\eta)\cap B'.$$

This means the equality (4.2) for $t_0=1$. 1st step is proved. We observe by the same argument that $g(t,z)$ is continuous for (t,z) in $\mathcal{D} - B\times\{a\}$.

<u>2nd step</u> $\lambda(t)$ is of class C^1 on B.

There is no problem for $t \in B' \cup B''$. It rests to prove it for $t_0 \in I$. Without loss of generality, we may assume $t_0 = 1$. Hence it is sufficient to show

$$(4.3) \quad \lim_{\substack{t \to 1 \\ t\in B'\cup B''}} \frac{\partial \lambda}{\partial t}(t) \text{ exists.}$$

From Theorem 3.1 we have for $t \in B' \cup B''$

$$\frac{\partial\lambda}{\partial t}(t) = \sum_{i=1}^{\nu} \frac{2}{k_i(k_i-2)!} \cdot f_i'(t) \left[\frac{\partial^{k_i-2}}{\partial W_i^{k_i-2}}\left(\frac{\partial G_i(t,W_i)}{\partial W_i}\right)^2\right]_{W_i=0} + \chi(t).$$

where

$$\chi(t) = \begin{cases} \dfrac{2}{k(k-2)!}\left[\dfrac{\partial^{k-2}}{\partial W^{k-2}}\left(\dfrac{\partial G(t,W)}{\partial W}\right)^2\right]_{W=0} & (t\in B') \\ 0 & (t \in B'') \end{cases}$$

At the branch point O_t of $D(t)$ $(t\in B')$, we took the local parameter $W = \sqrt[k]{z-t}$, and put because, if $\delta \searrow 0$, then $u_\delta(z) \searrow 0$ uniformly on any compact set in $(|z-a|<R) - \{1, f_1(1), \cdots, f_\nu(1)\}$.

There exists an $\eta > 0$ such that $\delta > \eta > 0$ and

$$|f_i(t) - f_i(1)| < \delta \text{ for all } |t-1| < \eta.$$

Put $\sigma_\delta = \{|z-1|<\delta\} \cap \{|z|<1\}$ and denote by $\sigma_\delta^*(t)$ the subset of $S^*(t)$ over σ_δ. Consequently, each $D(t)$ ($t \in \{|t-1|<\eta\}$) contains the domain

$$D_\delta(1) \underset{\text{def}}{=\!=} D(1) - \{\sigma_\delta^*(1) \cup \bigcup_{i=1}^{\nu} (|\tilde{z} - f_i(1)| < \delta)\},$$

where $(|\tilde{z} - f_i(1)| < \delta)$ means the nbd $U_{f_i(1)}$ of the branch point f_i of $D(1)$ formed over $(|z - f_i(1)| < \delta)$. We may assume

$$(\sigma_\delta^*(1) \cup \bigcup_{i=1}^{\nu} U_{f_i(1)}) \cap (|z-a|<\rho) = \phi \text{ in } D(1).$$

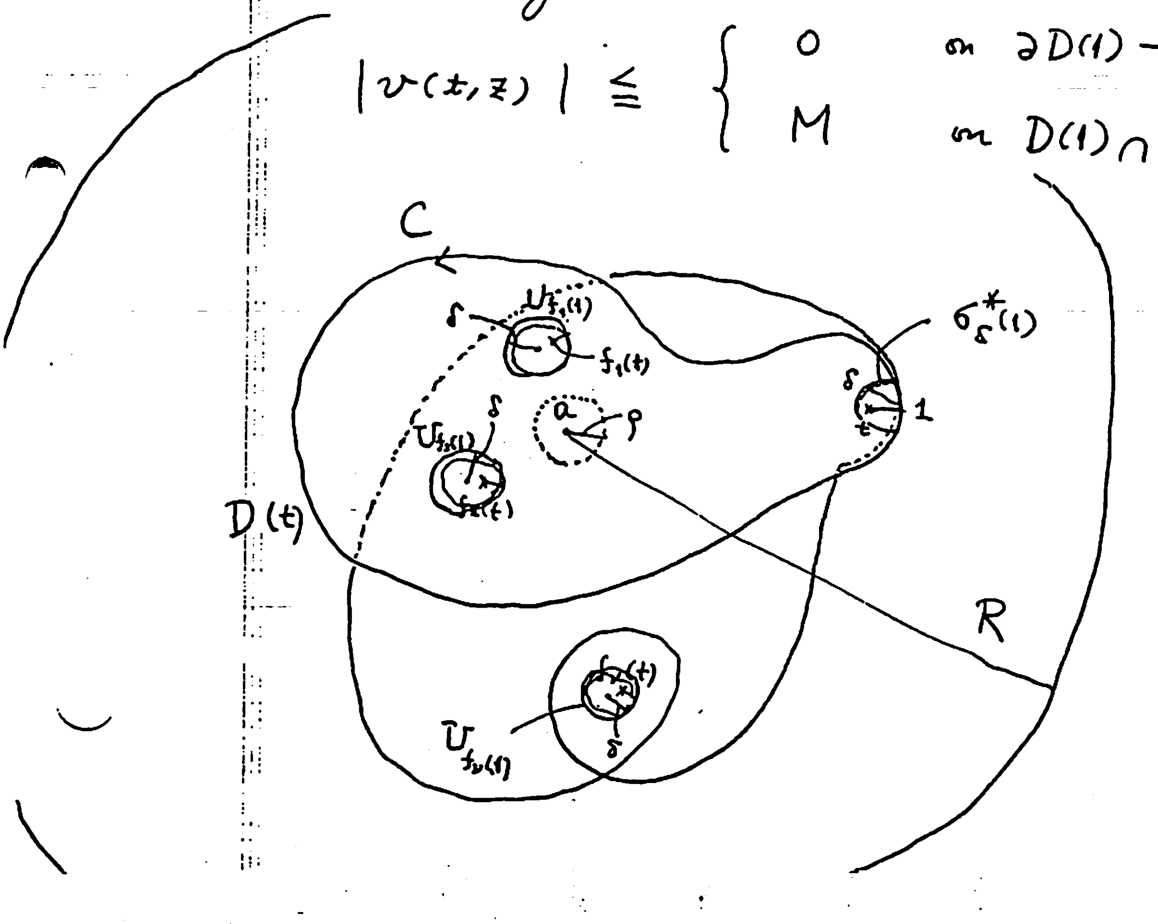
Under these circumstances, for each $t \in (|t-1|<\eta)$ ($\subset B$),

$$v(t,z) = g(t,z) - g(1,z)$$

becomes a harmonic function on $D_\delta(1)$.

On the boundary we have

$$|v(t,z)| \leq \begin{cases} 0 & \text{on } \partial D(1) - \sigma_\delta^*(1) \\ M & \text{on } D(1) \cap (\partial\sigma_\delta^*(1) \cup \partial U_{f_i(1)}) \end{cases}$$



$$G(t, W) = g(t, z)$$

where $|z - t| \ll 1$ and $|W| \ll 1$.

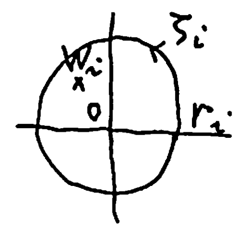
First, we show that

$$(4.4)\quad \lim_{\substack{t\to 1\\ t\in B'\cup B''}} \sum_{i=1}^{\nu} \frac{2}{k_i(k_i-2)!} f_i'(t) \left[\frac{\partial^{k_i-2}}{\partial W_i^{k_i-2}}\left(\frac{\partial G_i(t, W_i)}{\partial W_i}\right)^2\right]_{W_i=0}$$

exists.

In fact, as already noted, $g(t, z)$ is continuous for (t, z) in $\mathcal{D} - B\times\{a\}$. It follows that each $G_i(t, W_i)$ $(i=1,\cdots,\nu)$ is continuous in a product $(|t-1|<r)\times(|W_i|\leqq r_i)$. By Poisson's integral, we have, for $|W_i|<r_i$,

$$G_i(t, W_i) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r_i^2 - |W_i|^2}{|\zeta_i - W_i|^2} G_i(t, \zeta_i)\, d\theta$$



where $\zeta_i = r_i e^{\sqrt{-1}\theta}$ $(0\leqq\theta\leqq 2\pi)$

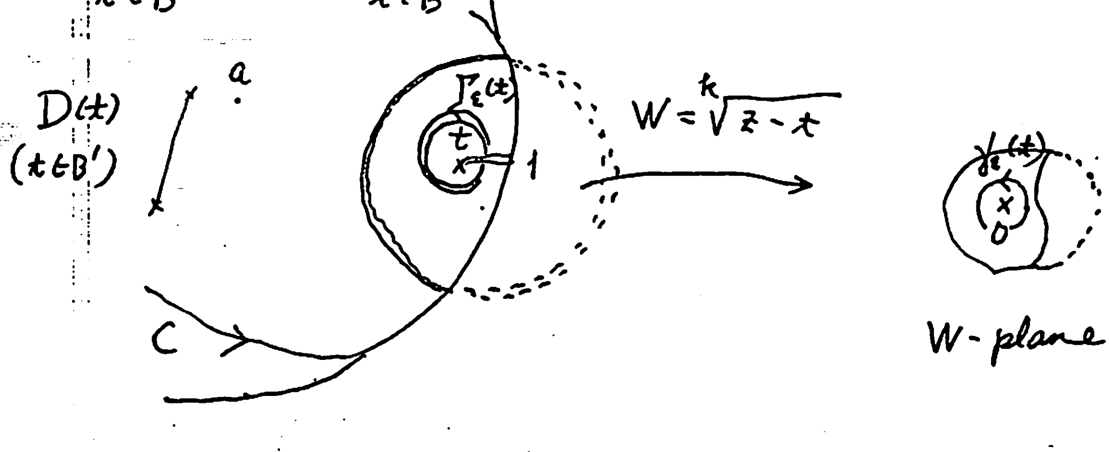
W_i-plane

$$\therefore\ \frac{\partial G_i}{\partial W_i}(t, 0) = -\frac{1}{2\pi}\int_0^{2\pi}\frac{G(t, \zeta_i)}{\zeta_i}\, d\theta$$

$$\to -\frac{1}{2\pi}\int_0^{2\pi}\frac{G(0, \zeta_i)}{\zeta_i}\, d\theta \quad (t\to 1)$$

Analogously, $\lim_{t\to 1}\frac{\partial^l G_i}{\partial W_i^l}(t, 0)$ exists for $l = 2, 3, \cdots$. It follows that (4.4) is valid. Therefore, to prove (4.3), we have to show

$$(4.5)\quad \lim_{\substack{t\to 1\\ t\in B'}} \chi(t) = \lim_{\substack{t\to 1\\ t\in B'}} \frac{2}{k(k-2)!}\left[\frac{\partial^{k-2}}{\partial W^{k-2}}\left(\frac{\partial G(t, W)}{\partial W}\right)^2\right]_{W=0} = 0.$$



On the W-plane we draw a small circle $\gamma_\varepsilon(t)$ around the origin O such that $\Gamma_\varepsilon(t)$ (the corresponding curve on $D(t)$) rounds the branch point O_t k times. Hence ε depends on $t \in$. Cauchy's Theorem implies

$$\left[\frac{\partial G(t,W)}{\partial W}\right]_{W=0} = \frac{1}{2\pi i} \int_{\gamma_\varepsilon(t)} \frac{\left(\frac{\partial G(t,W)}{\partial W}\right)}{W}\, dW$$

Since $(\partial/\partial W)\,dW = (\partial/\partial z)\,dz$, we have, from $W = \sqrt[k]{z-t}$,

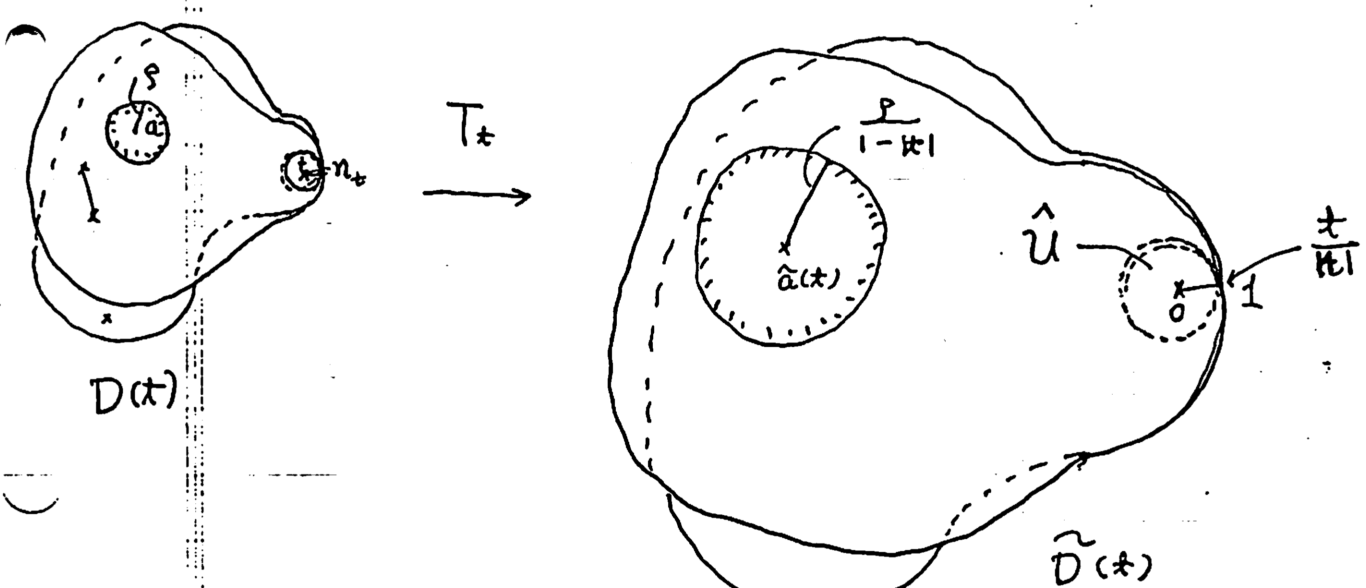
$$\left[\frac{\partial G(t,W)}{\partial W}\right]_{W=0} = \frac{1}{2\pi k i} \int_{\Gamma_\varepsilon(t)} \frac{\frac{\partial g(t,z)}{\partial z}\, dz}{\sqrt[k]{(z-t)}}$$

Given $t \in B'$, we consider the following transformation

$$T_t : z \longrightarrow \zeta = \frac{z-t}{1-|t|}$$

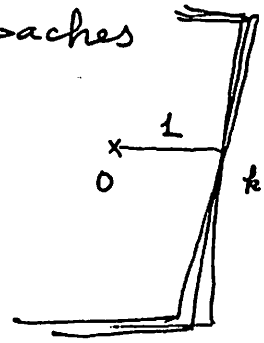
(translation with quantity t, real homothetic transformation with $\frac{1}{1-|t|}$)

from $\mathbb{C}_z$ onto $\mathbb{C}_\zeta$. Therefore T_t induces the transformation from $D(t)$ onto $\tilde{D}(t) = T_t(D(t))$.



$\tilde{D}(t) \backsim D(t)$ with similar ratio $\dfrac{1}{1-|t|}$.

independent of t! (134) 2;

$D(t)$ always contains k-sheeted disk $\hat{U}$ over $|\zeta|<1$. Moreover as $t(\in B') \to 1$, the domain $\tilde{D}(t)$ becomes larger and larger, and finally it approaches  Like (Imagine you are at its edge and walk directly to the wall. then the wall seems like half plane; k-sheeted.

Since the Green's fn is invariant under the transformation T_t, we have

$$\tilde{g}(t,\zeta) = g(t,z)$$

where $\tilde{g}(t,\zeta)$ is the Green's fn for $(\tilde{D}(t), \tilde{a}(t))$ where $\tilde{a}(t) = T_t(a)$. Because $z = t + (1-|t|)\zeta$, we have

$$\left[\frac{\partial G(t,W)}{\partial W}\right]_{W=0} = \frac{1}{2\pi k i}\frac{1}{\sqrt[k]{(1-|t|)}}\int_{\tilde{\Gamma}_\varepsilon(t)} \frac{\frac{\partial \tilde{g}(t,\zeta)}{\partial \zeta}d\zeta}{\sqrt[k]{\zeta}}$$

where $\tilde{\Gamma}_\varepsilon(t) = T_\zeta(\Gamma_\varepsilon(t))$. By Cauchy's Theorem, it becomes

$$(4.6)\quad \left[\frac{\partial G(t,W)}{\partial W}\right]_{W=0} = \frac{1}{2k\pi i}\frac{1}{\sqrt[k]{(1-|t|)}}\int_{|\zeta|=\frac{1}{2} \text{ in } \hat{U}} \frac{\frac{\partial \tilde{g}(t,\zeta)}{\partial \zeta}d\zeta}{\sqrt[k]{\zeta}}$$

the path is independent of $t(\in B')$.

As already noted in 1st step in the proof of Lemma 4.1, in p.23

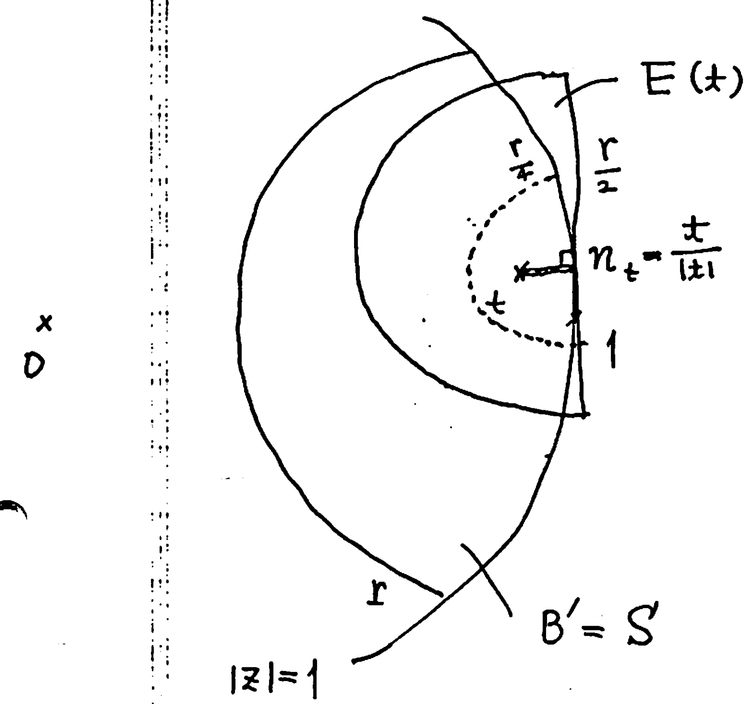
$$0 < g(t,z) < M \quad \text{on } (|z-a|>\rho)$$

$$\therefore\ 0 < \tilde{g}(t,\zeta) < M \quad \text{on } \left(|\zeta - \tilde{a}(t)| > \rho/(1-|t|)\right)$$

In particular, $0<\tilde{g}(t,\zeta)<M$ on $\tilde{S}^*(t)$ where $\tilde{S}^*(t) = T_t(S^*(t))$.

$$|z - n_t| = \frac{r}{2} \longrightarrow \left|\zeta - \frac{t}{|t|}\right| = \frac{r/2}{1-|t|}$$

We restrict t in $B'_{r/4} = B' \cap \left(|t-1| < \frac{r}{4}\right)$. and for each $t \in B'_{r/4}$

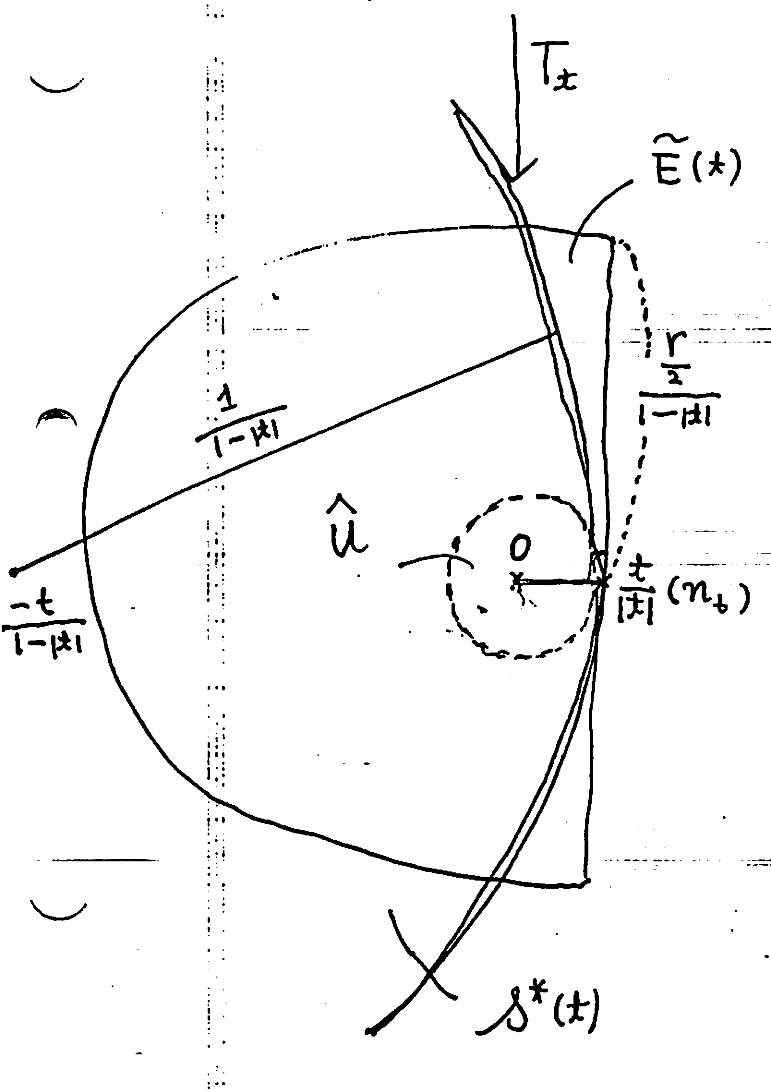


we consider the half disk $E(t)$ of center n_t (= the projection of t to $|z|=1$) and with radius $r/2$.
We put

$$\tilde{E}(t) = T_t(E(t)),$$

which becomes a half-disk of center $\frac{t}{|t|}$ with radius $\frac{r/2}{1-|t|}$.
We form the harmonic function $\omega_t(\zeta)$ on $\tilde{E}(t)$ whose boundary values are

$$\omega_t(\zeta) = \begin{cases} M & \text{on the half-circle} \\ 0 & \text{on the line.} \end{cases}$$



Then, it follows from the maximum principle that

$$0 < \tilde{g}(t, \zeta) \leqq \omega_t(\zeta)$$

for $S^*(t) \cap \tilde{E}(t)$.

On the other hand, $\omega_t(\zeta)$ is written in the explicit form

$$\omega_t(\zeta) = \frac{M}{2\pi}\int_{\frac{\pi}{2}+\eta(t)}^{\frac{3\pi}{2}+\eta(t)} \left\{\frac{\left(\frac{\frac{r}{2}}{1-|t|}\right)^2 - \left|\zeta - \frac{t}{|t|}\right|^2}{|\zeta - z|^2}\right\} d\theta - \frac{M}{2\pi}\int_{\frac{3\pi}{2}+\eta(t)}^{\frac{5\pi}{2}+\eta(t)} \left\{ \qquad \right\} d\theta$$

where $\eta(t) = \arg\left(\frac{t}{|t|} - \frac{-t}{1-|t|}\right) = \arg t$.

$$= \frac{M}{2\pi}\int_{\frac{\pi}{2}+\eta(t)}^{\frac{3\pi}{2}+\eta(t)} \left(\frac{1}{|\zeta - z|^2} - \frac{1}{|\zeta + z|^2}\right)\left\{\left(\frac{\frac{r}{2}}{1-|t|}\right)^2 - \left|\zeta - \frac{t}{|t|}\right|^2\right\} d\theta$$

$$= \frac{M}{2\pi}\int_{\frac{\pi}{2}+\eta(t)}^{\frac{3\pi}{2}+\eta(t)} \frac{4\,\mathrm{Re}(\zeta \bar{z})}{|\zeta^2 - z^2|^2}\left\{\left(\frac{\frac{r}{2}}{1-|t|}\right)^2 - \left|\zeta - \frac{t}{|t|}\right|^2\right\} d\theta$$

We restrict ζ in $\left|\zeta - \frac{t}{|t|}\right| < 3$. Then

$$\omega_t(\zeta) < \frac{4M}{2\pi}\int_{\frac{\pi}{2}+\eta(t)}^{\frac{3\pi}{2}+\eta(t)} \frac{3\cdot\left(\frac{\frac{r}{2}}{1-|t|}\right)}{\left(\left(\frac{r/2}{1-|t|}\right)^2 - 9\right)^2}\left(\frac{\frac{r}{2}}{1-|t|}\right)^2 d\theta$$

$$= 6M \cdot \frac{1}{\frac{(r/2)}{1-|t|}\left(1 - \frac{9}{(r/2)}(1-|t|)\right)^2}$$

If $|t-1| < \frac{r}{50}$, then we have

$$0 < \omega_t(\zeta) < \frac{32M}{r}\cdot(1-|t|) \quad \text{for } \left|\zeta - \frac{t}{|t|}\right| < 3.$$

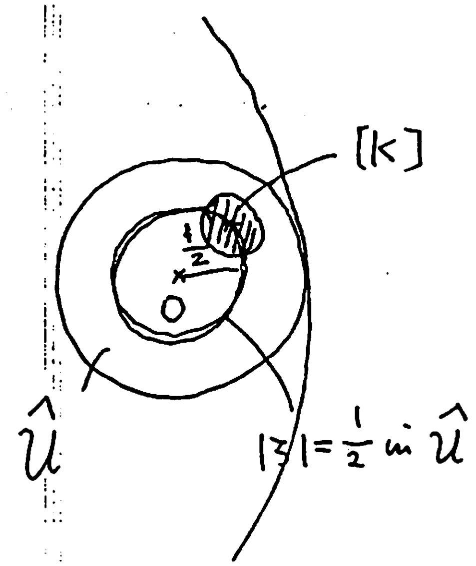
Moreover, for $|t-1| < r/50$ and $t \in B'$

$$S^*(t) \cap \left\{\left|\zeta - \frac{t}{|t|}\right| < 3\right\} \supset \hat{\mathcal{U}}.$$

It follows that

$$(4.7) \quad 0 < \tilde{g}(t,\zeta) < \frac{32M}{r}(1-|t|) \quad \text{on } \hat{\mathcal{U}}$$

for all $t \in B' \cap |t-1| < r/50$.



Take any point $\zeta_0 \in \hat{\mathcal{U}}$ found on $|\zeta_0| = \frac{1}{2}$. Draw a circle K_{ζ_0}: $|\zeta - \zeta_0| = \frac{r}{3}$. Then

$$\tilde{g}(t,\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{g}(t,\zeta) \frac{\left(\frac{r}{3}\right)^2 - |\zeta-\zeta_0|^2}{|\zeta - \zeta|^2} d\theta$$

in $\zeta \in [K_{\zeta_0}]$ ~~for $|\zeta - \zeta_0| < \frac{r}{3}$~~.

$$\therefore \left[\frac{\partial \tilde{g}(t,\zeta)}{\partial \zeta}\right]_{\zeta=\zeta_0} = -\frac{1}{2\pi}\int_0^{2\pi} \frac{\tilde{g}(t,\zeta)}{\zeta - \zeta_0} d\theta$$

By (4.7), it becomes

$$\left|\left[\frac{\partial \tilde{g}(t,\zeta)}{\partial \zeta}\right]_{\zeta=\zeta_0}\right| \leqq \frac{1}{2\pi} \cdot \frac{1}{\frac{r}{3}} \cdot 2\pi \cdot \frac{32M}{r}(1-|t|)$$

$$= \frac{94M}{r^2} \cdot (1-|t|)$$

Since ζ_0 is any point of $\hat{\mathcal{U}}$ over $|\zeta| = \frac{1}{2}$, it follows from (4.6) that, for any $|t-1| < \frac{1}{50}$ and $t \in B'$,

$$\left|\left[\frac{\partial G(t,W)}{\partial W}\right]_{W=0}\right| \leqq \frac{1}{2k\pi}(1-|t|)^{\frac{-1}{k}} \cdot 2^{\frac{1}{k}} \cdot 2k\pi \cdot \frac{94M}{r^2}(1-|t|)$$

$$= \left(\frac{188M}{r^2}\right)(1-|t|)^{\frac{k-1}{k}}$$

$$\therefore \lim_{\substack{t\to 1 \\ t\in B'}} \left[\frac{\partial G(t,W)}{\partial W}\right]_{W=0} = 0.$$

By the same method, we have

$$\lim_{\substack{t\to 1 \\ t\in B'}} \left[\frac{\partial^l G(t,W)}{\partial W^l}\right]_{W=0} = 0 \qquad (l = 1, 2, \cdots, k-1)$$

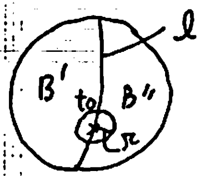
Hence (4.5) is proved. Lemma 4.1 is completely proved.

c.q.f.d.

Using this lemma we easily prove

<u>Theorem 4.1</u> $\lambda(t)$ is superharmonic on B.

<u>Proof.</u> $\lambda(t)$ is of class C^2 and superharmonic on $B' \cup B''$ by Theorem 2.1. We have shown that $\lambda(t)$ of class C^1 on the whole B by Lemma 4.1. It suffices to prove



that, for each $t_0 \in l$,

$$\frac{1}{2\pi}\int_0^{2\pi} \lambda(t_0 + re^{i\theta}) \leqq \lambda(t_0) \quad \text{for } 0 < {}^{\forall} r \ll 1.$$

Since $\lambda(t)$ is of class C^1 on B, it follows that

$$\int_{|t-t_0|=r} \frac{\partial\lambda}{\partial n_t}\, ds_t = \Big(\int + \int \Big) \frac{\partial\lambda}{\partial n_t}\, ds_t$$

$$= \Big(\iint + \iint \Big) (\Delta\lambda)\, dt_1 dt_2 \quad \text{where } t = t_1 + \sqrt{-1}\, t_2$$

($\because$ Stokes' Formula)

Since $\Delta\lambda \leqq 0$ on $B' \cup B''$, we see that the right-hand side is $\leqq 0$.

$$\therefore \quad \int_{|t-t_0|=r} \frac{\partial\lambda}{\partial n_t}\, ds_t \leqq 0$$

$$\int_0^{2\pi} r\, \frac{\partial\lambda}{\partial r}(t_0 + re^{i\theta})\, d\theta = r\, \frac{\partial}{\partial r}\Big(\int_0^{2\pi} \lambda(t_0 + re^{i\theta})\, d\theta\Big)$$

That means, $\int_0^{2\pi} \lambda(t_0 + re^{i\theta})d\theta$ is decreasing for for $0 < r \ll 1$.

Consequently, $\lambda(t_0) = \lim_{r\to 0} \int_0^{2\pi} \lambda(t_0 + re^{i\theta})d\theta \geqq \int_0^{2\pi} \lambda(t_0 + re^{i\theta})\, d\theta$. c.q.f.

Chap. III Pseudoconvex Domains

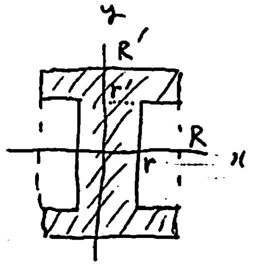
§1. Hartogs' Discovery.

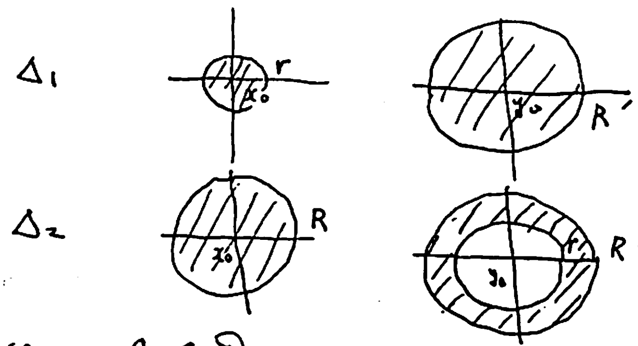
Let $\mathcal{D}$ be a domain of holomorphy in the space $\mathbb{C}^2$ of two complex variables x, y. Let

$$\Delta = (|x-x_0|<R) \times (|y-y_0|<R')$$

$$\Delta_1 = (|x-x_0|<r) \times (|y-y_0|<R') \quad \text{where } 0<r<R$$

$$\Delta_2 = (|x-x_0|<R) \times (r'<|y-y_0|<R') \quad \text{where } 0<r'<R'$$





Then, $\Delta_1 \cup \Delta_2 \subset \mathcal{D}$ induces $\Delta \subset \mathcal{D}$.

In fact, it suffices to show that

(*) Any holomorphic fn $f(x,y)$ in $\Delta_1 \cup \Delta_2$ is always extended to be holomorphic in Δ.

To prove this, let $0<\varepsilon \ll 1$ such that $r'<R-\varepsilon$ and put $[\Gamma_{1\varepsilon}] = \{|x-x_0|<R-\varepsilon\}$; $[\Gamma_{2\varepsilon}] = \{|y-y_0|<R'-\varepsilon\}$.

Consider, for any $(x,y) \in [\Gamma_{1\varepsilon}] \times [\Gamma_{2\varepsilon}]$,

$$F(x,y) = \frac{-1}{4\pi^2} \iint_{\Gamma_{1\varepsilon}\times\Gamma_{2\varepsilon}} \frac{f(\xi,\eta)}{(\xi-x)(\eta-y)} d\xi \times d\eta$$

where $\Gamma_{1\varepsilon} = \{|x-x_0|=R-\varepsilon\}$ and $\Gamma_{2\varepsilon} = \{|y-y_0|=R'-\varepsilon\}$. It is clear that $F(x,y)$ is holomorphic in $[\Gamma_{1\varepsilon}] \times [\Gamma_{2\varepsilon}]$. By Fubini's Theorem

$$F(x,y) = \frac{1}{2\pi i}\int_{\Gamma_{2\varepsilon}} \left\{\frac{1}{2\pi i}\int_{\Gamma_{1\varepsilon}} \frac{f(\xi,\eta)}{\xi-x}\, d\xi\right\} \frac{1}{\eta-y}\, d\eta .$$

Since $f(x,\eta)$ is holomorphic for x in $|x-x_0|<R$, we have

$$F(x,y) = \frac{1}{2\pi i}\int_{\Gamma_{2\varepsilon}} f(x,\eta)\,\frac{1}{\eta - y}\,d\eta$$

If we assume that $|x-x_0| < r$, then $f(x,y)$ is holomorphic for y in $|y-y_0| < R'$. Therefore

$$F(x,y) = f(x,y).$$

We thus get $F(x,y) = f(x,y)$ in $(|x-x_0|<r) \times (|y-y_0|<R'-\varepsilon)$. By the uniqueness theorem, $f(x,y)$ is extended to be holomorphic for (x,y) in $[\Gamma_{1\varepsilon}] \times [\Gamma_{2\varepsilon}]$. Since $\varepsilon > 0$ is arbitrary, it follows that $f(x,y)$ is holomorphically extended to Δ. c.q.f.d.

This is discovered by F. Hartogs in 1906. Holomorphicity is local property and is invariant under the biholomorphic mapping. Using these facts, he introduced the following definition:

If a domain $\mathcal{D}$ in $\mathbb{C}^2$ satisfies the following conditions (I_C) and (II_C), then $\mathcal{D}$ is called a pseudoconvex domain of type (C):

(I_C) Let (ζ,η) be any point of $\partial\mathcal{D}$. Then there exists a $\rho_0 > 0$ such that, for any $0<\rho<\rho_0$, the subset of $\mathcal{D}$

$\mathcal{D} \cap S_\rho$ where S_ρ is the ball of center (ζ,η) with radius ρ satisfies the following Continuity Theorem (C) at (ζ,η):

Consider arbitrary domains $\Delta, \Delta_1, \Delta_2$ in $\mathbb{C}^2$ such that

$$\Delta = (|x-x_0|<R) \times (|y-y_0|<R')$$
$$\Delta_1 = (|x-x_0|<r) \times (|y-y_0|<R')$$
$$\Delta_2 = (|x|<R) \times (r'<|y-y_0|<R')$$

Then, $\Delta_1 \cup \Delta_2 \subset \mathcal{D}\cap S_\rho$ induces $\Delta \subset \mathcal{D}\cap S_\rho$.

(II_C) Property (I_C) at (ζ,η) is invariant under any biholomorphic mapping of a neighborhood of (ζ,η) in $\mathbb{C}^2$.

Precisely speaking, let $T: (x,y) \to (x',y') = (u(x,y), v(x,y))$ be a biholomorphic transformation of a neighborhood U of (ξ,η) (in $\mathbb{C}^2$) onto the neighborhood V of (ξ',η') where $(\xi',\eta') = T(\xi,\eta)$. Then there exists a $\rho_0 > 0$ such that, for any $0 < \rho < \rho_0$, the domain

$T(U \cap \mathcal{D}) \cap S'_\rho$ where S'_ρ is the ball of center (ξ',η') with radius ρ satisfies the Continuity Theorem (C) at (ξ',η')

Remark 1.1 In the assertion $(*)$, we can replace "holomorphic" by "harmonic" as follows:

(**) Any real analytic fn $f(x,y)$ in $\Delta_1 \cup \Delta_2$ such that, for any fixed $x \ni \cdot\ |x-x_0| < r$, $f(x,y)$ is harmonic w.r.t. y in $|y-y_0| < R'$, is always extendable to be real analytic fn for (x,y) in Δ such that, for any fixed $x \ni \cdot\ |x| < R$, $f(x,y)$ is harmonic for y in $|y| < R'$.

In fact, let $0 < \varepsilon \ll 1 \ni \cdot\ r' < R' - \varepsilon$. Consider the Poisson Integral for $(x,y) \in [\Gamma_{1\varepsilon}] \times [\Gamma_{2\varepsilon}]$

$$H(x,y) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R'-\varepsilon)^2 - |y|^2}{|\eta - y|^2} f(x,\eta)\, d\theta$$

where $\eta = y_0 + (R'-\varepsilon)e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$. It is clear that $H(x,y)$ is real analytic fn (x,y) in $[\Gamma_{1\varepsilon}] \times [\Gamma_{2\varepsilon}]$ and is harmonic for y in $[\Gamma_{2\varepsilon}]$. For $|x| < r$, $f(x,y)$ is harmonic for y in $[\Gamma_{2\varepsilon}]$. We thus have

$$H(x,y) = f(x,y).$$

Hence $f(x,y)$ is extendable to be real analytic fn in $[\Gamma_{1\varepsilon}] \times [\Gamma_{2\varepsilon}]$ in such a way that $f(x,y)$ is harmonic for y in $[\Gamma_{2\varepsilon}]$. Since $\varepsilon >$ is arbitrary, the assertion $(**)$ is proved.

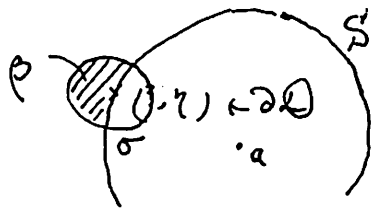
We note that this property (**) is not invariant under the biholomorphic transformation of a domain in $\mathbb{C}^2$.

§2. Equivalences.

Following K. Oka (1952), let us state three kinds of pseudoconvexities and their equivalences.

Let $\mathcal{D}$ be a domain of $\mathbb{C}^2$. If $\mathcal{D}$ satisfies the following conditions (I_B) and (II_B), then $\mathcal{D}$ is called a pseudoconvex domain of type (B):

(I_B) Let (ζ,η) be any boundary point of $\mathcal{D}$. Let S be any closed ball with center a $(\neq(\zeta,\eta))$ such that $\partial S \ni (\zeta,\eta)$, and σ any open ball with center (ζ,η). Put $\beta = S^c \cap \sigma$.



Then it never occurs $\beta \subset \mathcal{D}$.

In this case we simply say that $\mathcal{D}$ satisfies Continuity Theorem (B) at (ζ,η).

(II_B) Property (I_B) at (ζ,η) is invariant under any holomorphic transformation of a neighborhood of (ζ,η) in $\mathbb{C}^2$.

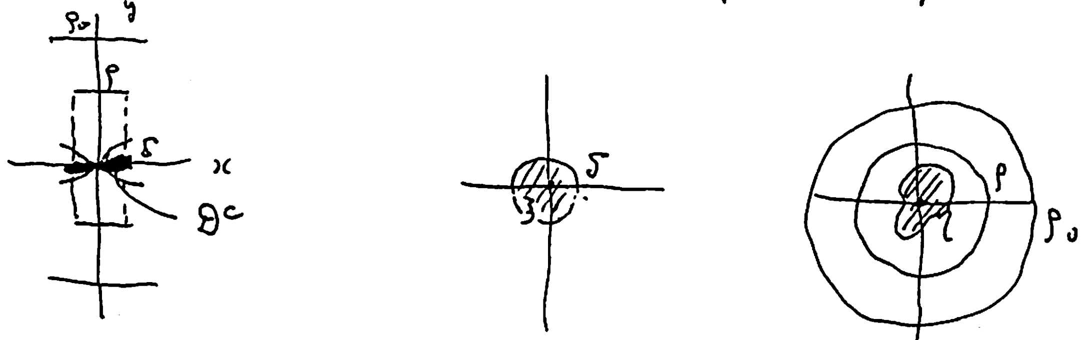
That means, let $T: (x,y)\to(x',y')=(u(x,y), v(x,y))$ be a biholomorphic mapping of a nbd U of (ζ,η) onto a nbd V of (ζ',η') where $(\zeta',\eta') = T(\zeta,\eta)$. Then $T(U\cap\mathcal{D})$ satisfies the Continuity Theorem (B) at (ζ',η')

Let $\mathcal{D}$ be a domain of $\mathbb{C}^2$. If $\mathcal{D}$ satisfies the following conditions (I_A) and (II_A), then $\mathcal{D}$ is called a pseudoconvex domain of type (A):

(I_A) Let (ζ,η) be any bdry point of $\mathcal{D}$. Assume that there exists a ρ_0 such that $(\zeta,\ 0<|y-\eta|<\rho_0)\subset\mathcal{D}$. Then, given $0<\rho<\rho_0$, we find a $\delta>0$ (depending on ρ) with the following property:

For any $0<|x-\zeta|<\delta$, there exists at least one point $y(x)\in\mathbb{C}_y$ such that $|y(x)-\eta|<\rho$ and $(x,y(x))\notin\mathcal{D}$.

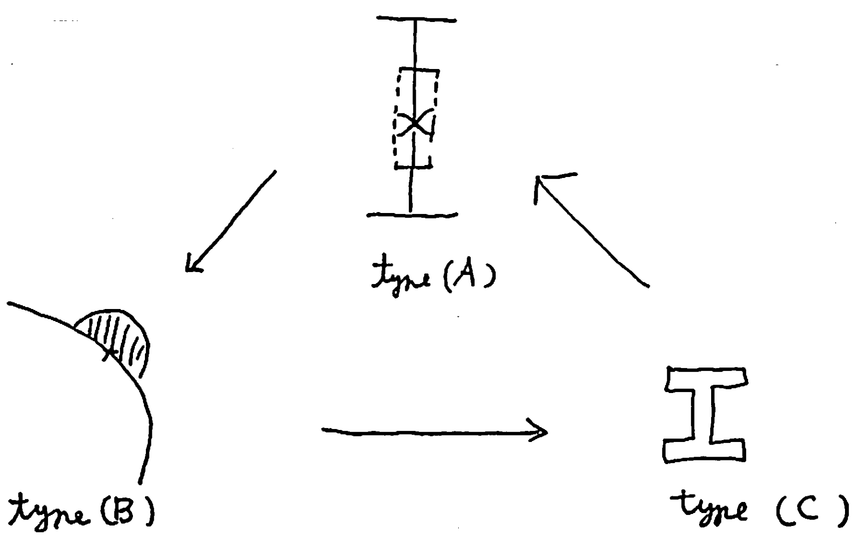
In this case, we simply say that $\mathcal{D}$ satisfies Continuity Theorem (A) at (ζ,η).



(II_A) Property (I_A) is invariant under any biholomorphic transformation of a nbd of (ζ,η) in $\mathbb{C}^2$.

That means, let $T:(x,y)\to(x',y')=(u(x,y),v(x,y))$ be any holomorphic mapping of a nbd U of (ζ,η) onto a nbd V of (ζ',η') when $(\zeta',\eta')=T(\zeta,\eta)$. Then the domain $T(U\cap\mathcal{D})$ satisfies the Continuity Theorem (A) at (ζ',η').

Let us prove that three types of pseudoconvex domains are equivalent as follows:



It is almost clear that the pseudoconvex domain $\mathcal{D}$ of type (C) is of type (A).

In fact, let $(\xi, \eta) \in \partial\mathcal{D}$. Since $\mathcal{D}$ satisfies (I_C), we find a $\rho^* > 0$ such that $\mathcal{D} \cap S_{\rho^*}$ satisfies the Continuity Theorem (C) at (ξ, η).

Assume that there exists $\rho_0 > 0$ such that

$$(\xi,\ 0 < |y - \eta| < \rho_0) \subset \mathcal{D}.$$

Take any $\rho \ni 0 < \rho < \min(\rho_0, \rho^*)$. Since $(\xi, |y-\eta| = \rho) \subset \mathcal{D} \cap S_\rho$ there exists $(|x-\xi| < \delta) \times ((\rho - \varepsilon) < |y - \eta| < \rho + \varepsilon) \subset \mathcal{D} \cap S_{\rho^*}$ where $0 < \delta \ll 1$ and $0 < \varepsilon \ll 1$. Our claim is that

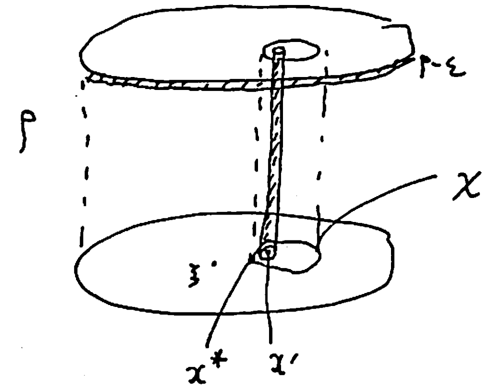
For any $|x - \xi| < \delta$, there exists at least one point $y(x)$ in $\mathbb{C}_y$ such that $|y(x) - \eta| < \rho$ and $(x, y(x)) \notin \mathcal{D}$.

We prove this by contradiction. Assume that we find a point x_0 with $|x_0 - \xi| < \delta$ and $(x_0, |y - \eta| < \rho) \subset \mathcal{D} \cap S_{\rho^*}$. Put

$$\mathcal{X} = \{ x \in \mathbb{C}_x \mid |x - \xi| < \delta \text{ and } (x, |y-\eta| < \rho) \subset \mathcal{D} \cap S_{\rho^*} \}$$

and $r = \inf_{x \in \mathcal{X}} |x - \xi|$.

It is clear that $\mathcal{X}$ is non void and open in $(|x - \xi| < \delta)$ and $0 \leqq r \leqq |x_0 - \xi| < \rho$. There exists $x^* \in \partial\mathcal{X}\ (\subset \mathcal{X}^c)$ such that $|x^* - \xi| = r$. We find an $x' \in \mathcal{X}$ so close to x^* that $|x' - x^*| < \delta - |x' - \xi|$. By Continuity Theorem (C),



we have $\{ x \in \mathbb{C}_x \mid |x - x'| < \delta - |x' - \xi| \} \times (|y - y_0| < \rho + \varepsilon) \subset \mathcal{D} \cap S_{\rho^*}$.

In particular, $(x^*, |y - \eta| < \rho) \subset \mathcal{D} \cap S_{\rho^*}$, so that $x^* \in \mathcal{X}$.

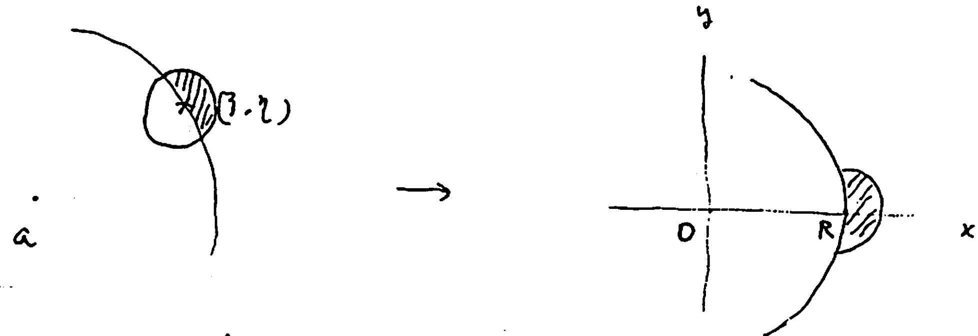
This is a contradiction. Hence $\mathcal{D} \cap S_{\rho *}$ satisfies Continuity Theorem (A). Since (II_C) induces (II_A), the pseudoconvex domain $\mathcal{D}$ of type (C) is of type (A).

Next, let us prove that the pseudoconvex domain $\mathcal{D}$ of type (A) is of type (B).

In fact, we prove it by contradiction. Let $(\xi, \eta) \in \partial \mathcal{D}$. Assume that there exists a closed ball S with $\partial S \ni (\xi, \eta)$. and an open ball σ of center (ξ, η) such that

$$\beta = \sigma \cap S^c \subset \mathcal{D}$$

Let $a = (a_1, a_2)$ be the center of S. After the translation : $(x, y) \to (x-\xi, y-\eta)$, we make the rotation which transforms the point $(a_1-\xi, a_2-\eta)$ to $(R, 0)$ where $R > 0$. Of course, the composed



transformation is holomorphic in $\mathbb{C}^2$. By the same notations $S, \sigma, \beta, \mathcal{D} \cdots$ we denote the images of $S, \sigma, \beta, \mathcal{D} \cdots$, respectively. Then $(R, 0) \in \partial \mathcal{D}$. By the hypothesis of contradiction,

$$(R,\ 0 < |y| < \rho) \subset \mathcal{D} \quad \text{where } 0 < \rho < r = \text{the radius of } \sigma.$$

Moreover, it is clear that, for any $\varepsilon > 0$ s.t. $0 < \varepsilon < \frac{r}{3}$,

$$\left(R+\varepsilon,\ |y| < \frac{r}{2}\right) \subset \mathcal{D}$$

This contradicts the Continuity Theorem (I_A) at $(R, 0)$.

The property (II_B) follows that of (II_A). Hence the domain $\mathcal{D}$ with (I_A) and (II_A) is of type (B).

Finally, let us prove that the pseudoconvex domain $\mathcal{D}$ of type (B) is of type (C).

We prove it by contradiction. Let $(\mathfrak{z}, \eta) \in \partial\mathcal{D}$. Assume that there exists a nbd U of $(\mathfrak{z}, \eta)$ such that $U \cap \mathcal{D}$ does not satisfies the Continuity Theorem (C). That is, there exist the following domain $\Delta, \Delta_1, \Delta_2$:

$$\Delta = (|x-\mathfrak{z}| < R) \times (|y-\eta| < R')$$

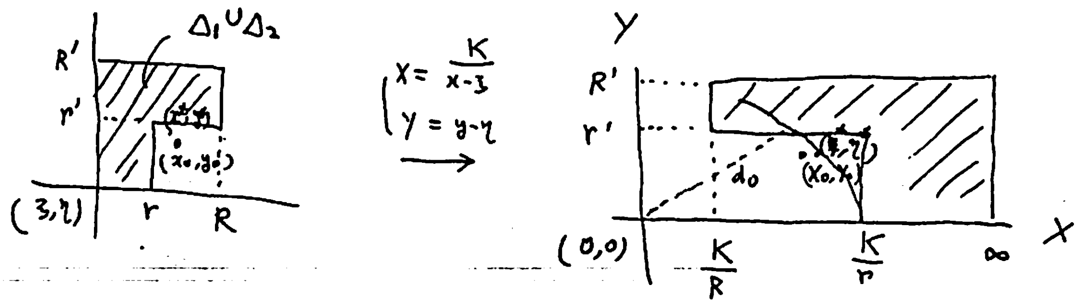
$$\Delta_1 = (|x-\mathfrak{z}| < r) \times (|y-\eta| < R') \qquad (0 < r < R)$$

$$\Delta_2 = (|x-\mathfrak{z}| < R) \times (r' < |y-\eta| < R') \qquad (0 < r' < R')$$

such that $\Delta_1 \cup \Delta_2 \subset \mathcal{D} \cap U$ but $\Delta \not\subset \mathcal{D} \cap U$.

Therefore we find a point $(x_0, y_0) \notin \mathcal{D} \cap U$ such that

$$r \leqq |x_0 - \mathfrak{z}| < R \quad \text{and} \quad |y_0 - \eta| < r'.$$



Take $K > 0$ so large that

$$\left(\frac{K}{|x_0 - \mathfrak{z}|}\right)^2 + |y_0 - \eta|^2 > \left(\frac{K}{R}\right)^2 + R'^2$$

which is possible, because $0 < |x_0 - \mathfrak{z}| < R$. Put

$$d(x, y) = \sqrt{\left|\frac{K}{x-\mathfrak{z}}\right|^2 + |y - \eta|^2} \quad \text{for } (x, y) \in \Delta$$

and consider

$$d_0 = \sup\{d(x, y) \mid (x, y) \in \Delta - \mathcal{D} \cap U\}.$$

Then there exists $(x', y') \in \Delta - \mathcal{D} \cap U$ such that

$$d_0 = d(x', y').$$

$\boxed{\because}$ The set $\Delta - \mathcal{D} \cap U$ is non void and closed in Δ.

and $\partial\Delta = A_1 \cup A_2$ where $A_1 = (|x-\xi| = R) \times (|y-\eta| < R')$ and $A_2 = (|x-\xi| < R) \times (|y-\eta| = R')$. By assumption, $A_2 \cap (\Delta - (\mathcal{D} \cap \Delta)) = \phi$ and, by the property of K,

$$d_0 \gneq d(x,y) \text{ for all } (x,y) \in A_2.$$

It follows that d_0 must be attained at some point of $\Delta - (\mathcal{D} \cap U)$.

Therefore we find a small nbd V of (x^*, y^*) in Δ such that any $(x,y) \in V$ with $d(x,y) > d(x^*, y^*)$ belongs to $\mathcal{D} \cap U$.

We form the biholomorphic mapping of V

$$T: (x,y) \to (X, Y) = \left(\frac{k}{x-\xi}, y-\eta\right).$$

Consider the closed ball S : $|X|^2 + |Y|^2 \leqq d_0^2$ and an open ball σ : $|X - \xi^*|^2 + |Y - \eta^*| < r'^2$ such that $\sigma \subset T(V)$ where $(\xi^*, \eta^*) = T(x^*, y^*)$. Since $(x', y') \in (\partial\mathcal{D}) \cap \Delta$, we get $(\xi^*, \eta^*) \in \partial T(\mathcal{D} \cap V)$. As noted above,

$$\text{any } (X, Y) \in \sigma \cap S^c \text{ belongs to } T(\mathcal{D} \cap V).$$

This is a contradiction to Continuity Theorem (B) at (ξ^*, η^*). Hence (I_B) and (II_B) induce (I_C). Since (II_C) follows (II_B), $\mathcal{D}$ with (I_B) and (II_B) is of type (C).

<u>Remark 2.1</u> The condition (I_C) and (II_C) are local. But the above proof of $((I_B)$ and $(II_B) \to I_C)$ teaches us that, if $\mathcal{D}$ is a pseudoconvex domain of type (C), then, for any large or small $\Delta, \Delta_1, \Delta_2$,

$$\Delta_1 \cup \Delta_2 \subset \mathcal{D} \text{ induces } \Delta \subset \mathcal{D}.$$

Instead of the Continuity Theorem (I_A), we put

$(I_{A'})$: Let (ξ, η) be any point of $\partial\mathcal{D}$. Assume that there exists $\rho > 0$ such that $(\xi, |y-\eta| = \rho) \subset \mathcal{D}$.

Then we find a $\delta > 0$ with

the following property:

Given $|x-\zeta| < \delta$, there exists at least one point $y(x)$ in $\mathbb{C}_y$ such that

$$|y(x)-\eta| < \rho \quad \text{and} \quad (x, y(x)) \notin \mathcal{D}.$$

Moreover we put the condition

$(\mathrm{II}_{A'})$: Property $(\mathrm{I}_{A'})$ is invariant under the biholomorphic transformation of a nbd of (ζ, η).

Then $(\mathrm{I}_{A'}) \cup (\mathrm{II}_{A'})$ is stronger than $(\mathrm{I}_{A}) \cup (\mathrm{II}_{A})$. By the same method of $((\mathrm{I}_{C}) \cup (\mathrm{II}_{C}) \longrightarrow (\mathrm{I}_{A}) \cup (\mathrm{II}_{A}))$, $(\mathrm{I}_{C}) \cup (\mathrm{II}_{C})$ induces $(\mathrm{I}_{A'}) \cup (\mathrm{II}_{A'})$. Therefore, they are equivalent to $(\mathrm{I}_{B'}) \cup (\mathrm{II}_{B'})$.

We say that the domain ($\mathcal{D}$ in $\mathbb{C}^2$) with one of these equivalence relations is pseudoconvex in $\mathbb{C}^2$.

Dear November 23, '87 149

This is a note for last week's lectures. In tomorrow's lecture, I want to show a generalization of Hartogs Th. 3.1 by K. Oka, if we have time. 7

Thank you,

Hiroshi Yamaguchi

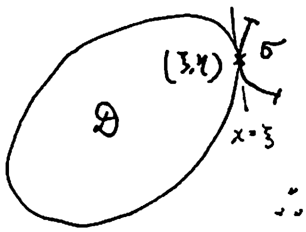
By definition, we see that

If D_1 and D_2 are pseudoconvex in $\mathbb{C}^2$, then $D_1 \cap D_2$ are pseudoconvex in $\mathbb{C}^2$.

<u>Remark 2.2</u> (Sufficient Condition)

Let D be a domain of $\mathbb{C}^2$. Assume that, for each $(\zeta, \eta) \in \partial D$, there exists at least an analytic set σ : $f(x,y)=0$ passing through (ζ,η) such that $\sigma \subset D^c$. Then D is pseudoconvex in $\mathbb{C}^2$.

<u>Proof.</u> Let $(\zeta,\eta) \in \partial D$ and assume $\exists\, \sigma : f(x,y)=0$ such that $\sigma \ni (\zeta,\eta)$ and $\sigma \subset D^c$, where $f(x,y)$ is holomorphic at (ζ,η).



Let us prove (ζ,η) satisfies Continuity Theorem (A). Assume that $\exists\, \rho > 0$ such that

$$(\zeta,\ 0<|y-\eta|<\rho\ ) \subset D$$

$\therefore f(\zeta,y) \not\equiv 0$ near $y=\eta$. We put $n =$ the order of $f(\zeta,y)=0$ at $y=\eta$. It follows from Weierstrass Preparation Theorem that there exists $\delta>0$ such that, for each x in $|x-\zeta|<\delta$, we find at least

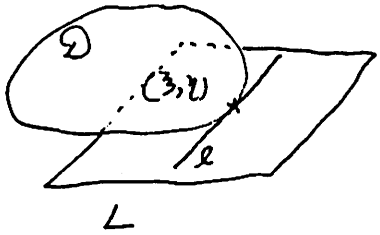
$$y = y_1(x), \dots, y_n(x) \quad \text{with} \quad |y_i(x)-\eta|<\rho \quad \text{and}$$
$$(x, y_i(x)) \subset \sigma \subset D^c \quad (i=1,\dots,n)$$

Hence Continuity Theorem (A) is satisfied at (ζ,η). Since analytic set is invariant under the biholomorphic mapping of a Nbd of (ζ,η), it follows that D is pseudoconvex.

c.q.f.d.

<u>Remark 2.3</u> The usual convex domain D in $\mathbb{C}^2 = \mathbb{R}^4$ is pseudoconvex in $\mathbb{C}^2$

<u>Proof.</u> Let $(\zeta,\eta) \in \partial D$. We draw the tangent plane L of real dim 3 to ∂D at (ζ,η). Then $L \subset D^c$.



L is written in the form

$$\mathrm{Re}\{\alpha(x-\zeta)+\beta(y-\eta)\}=0$$

Hence the analytic line l defined by

$$l:\ \alpha(x-\zeta)+\beta(y-\eta)$$

passes through (ζ,η) and lies on $\mathcal{D}^c$.

Hence (ζ,η) satisfies the condition of Remark 2.2, so $\mathcal{D}$ is pseudoconvex in $\mathbb{C}^2$.

Lemma 2.1 Let $\mathcal{D}$ be a domain in $\mathbb{C}^2$ with smooth boundary $\partial\mathcal{D}$. Precisely speaking, there exists a domain $\tilde{\mathcal{D}}$ and a C^∞-fn φ in $\tilde{\mathcal{D}}$ such that $\mathcal{D}\subset\subset\tilde{\mathcal{D}}$ and

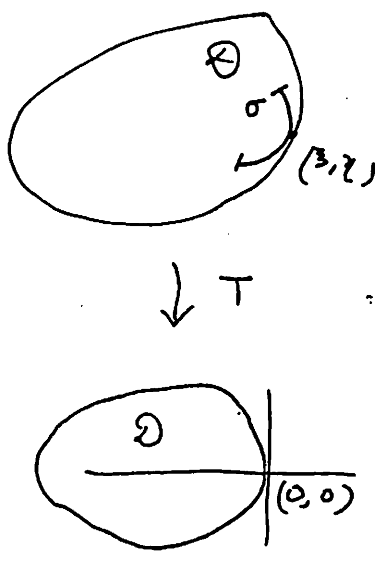
$$\mathcal{D}=\{(x,y)\in\tilde{\mathcal{D}}\mid \varphi(x,y)<0\}$$
$$\partial\mathcal{D}=\{(x,y)\in\tilde{\mathcal{D}}\mid \varphi(x,y)=0\}$$
$$\tilde{\mathcal{D}}-\mathcal{D}=\{(x,y)\in\tilde{\mathcal{D}}\mid \varphi(x,y)>0\}$$
$$\mathrm{Grad}\,\varphi=\left(\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y}\right)\neq 0 \text{ anywhere on } \partial\mathcal{D}.$$

Assume that $\mathcal{D}$ is pseudoconvex in $\mathbb{C}^2$. Then, for $(\zeta,\eta)\in\partial\mathcal{D}$, there is no regular analytic set σ such that

$$\sigma\ni(\zeta,\eta) \text{ and } \sigma-(\zeta,\eta)\subset\mathcal{D}$$

Proof. Assume that there exists a point (ζ,η) such that

$\exists\sigma$: regular analytic set such that $\sigma\ni(\zeta,\eta)$ and $\sigma-(\zeta,\eta)\subset\mathcal{D}$.



By some rotation and translation we may assume

$$(\zeta,\eta)=(0,0)$$
$$\mathrm{Grad}\,\varphi=(1,0) \text{ at } (0,0).$$

Moreover, smoothness does not depend on the choice of φ, we may assume

$$\varphi(x,y) = 2x_1 - u(x_2, y_1, y_2)$$

where $\frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial y_1} = \frac{\partial u}{\partial y_2} = 0$ at $(0,0,0)$.

$u(x_2, y_1, y_2)$ is C^∞-fn at $(0,0,0)$.

Since σ is regular, we write

$$\sigma : \begin{cases} x = a_1 \tau + a_2 \tau^2 + \cdots \\ y = b_1 \tau + b_2 \tau^2 + \cdots \end{cases} \qquad |\tau| \ll 1, \tau \in \mathbb{C}$$

where $(a_1, b_1) \neq (0,0)$.

Since $\sigma \subset D$ except at $(0,0)$, we see that

$$\begin{cases} \varphi(a_1\tau + a_2\tau^2 + \cdots, b_1\tau + b_2\tau^2 + \cdots) < 0 \text{ for } 0 < |\tau| \ll 1 \\ \varphi(0,0) = 0 \end{cases}$$

Hence $\varphi^*(\tau) \underset{\text{def}}{=} \varphi(a_1\tau + a_2\tau^2 + \cdots, b_1\tau + b_2\tau^2 + \cdots)$ as fn of $|\tau| \ll 1$, attains its local maximum at $\tau = 0$.

$$\therefore \left[\frac{\partial \varphi^*}{\partial \tau}\right]_{\tau=0} = 0 \ ; \ \left[\frac{\partial^2 \varphi^*}{\partial \bar\tau \partial \tau}\right]_{\tau=0} \leqq 0$$

By the first equality, we have

$$\underbrace{\frac{\partial \varphi}{\partial x}(0,0)}_{1} a_1 + \underbrace{\frac{\partial \varphi}{\partial y}(0,0)}_{0} b_1 = 0 \qquad \therefore a_1 = 0$$

Consequently, $b_1 \neq 0$, so that we write

$$\sigma : x = A_2 y^2 + A_3 y^3 + \cdots \qquad (|y| \ll 1)$$

Consider the biholomorphic transformation of a nbd U of $(0,0)$ such that

$$\begin{cases} X = x - (A_2 y^2 + A_3 y^3 + \cdots) \\ Y = y \end{cases}$$

or

$$\begin{cases} x = X + (A_2 Y^2 + A_3 Y^3 + \cdots) \\ y = Y \end{cases}$$

Put $\mathcal{D}' = T(\mathcal{D} \cap U)$. Then

$$(0,0) \in \partial\mathcal{D} \longrightarrow (0,0) \in \partial\mathcal{D}'$$

$$\sigma \longrightarrow \sigma' : X = 0$$

$$\varphi = x_1 - u(x_2, y_1, y_2) \longrightarrow \quad \psi = X_1 - v(X_2, Y_1, Y_2)$$

where $v(X_2, Y_1, Y_2)$ starts from the second orders

$$\because \quad \varphi(x,y) = x_1 - u(x_2, y_1, y_2)$$
$$= X_1 + \mathrm{Re}\{A_2Y^2 + A_3Y^3 + \cdots\} - u(X_2 + \mathrm{Im}(A_2Y^2 + A_3Y^3 + \cdots), Y$$
$$= X_1 - v(X_2, Y_1, Y_2)$$

Since $(0, 0 < |Y| < {}^\exists\rho) \subset \mathcal{D}'$, we have

$\psi(0, Y) < 0$ except for $Y = 0$ in $|Y| < \rho$

$\therefore \quad -v(0, Y_1, Y_2) < 0$ for $0 < |Y| < \rho$

Therefore, for any $0 < \varepsilon \ll 1$, we have

$$\underset{\displaystyle \psi(-\varepsilon, Y)}{\underset{\parallel}{-\varepsilon - v(0, Y_1, Y_2)}} < 0 \quad \text{for all } |Y| < \rho.$$

i.e., $(-\varepsilon, Y) \subset \mathcal{D}'$ for $|Y| < \rho$

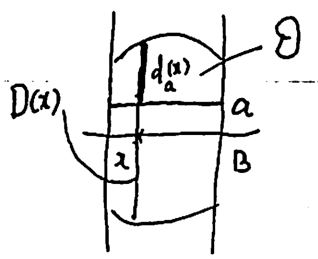
This contradict that $\mathcal{D}'$ satisfies Continuity Theorem (A) at $(0,0)$.

c.q.f.d.

Remark
① Pussing off
② If $\text{grad}\,\psi = 0$, then it is impossible.

§ 3. Radii of Hartogs.

Let $B = (|x| < \rho)$ and $\mathbb{C}_y = \mathbb{C} = (|y| < \infty)$. Let $\mathcal{D}$ be a domain in $B \times \mathbb{C}_y$. Assume that there exists $a \in \mathbb{C}_y$ such that $B \times \{a\} \subset \mathcal{D}$. We put, as usual,

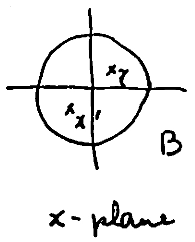


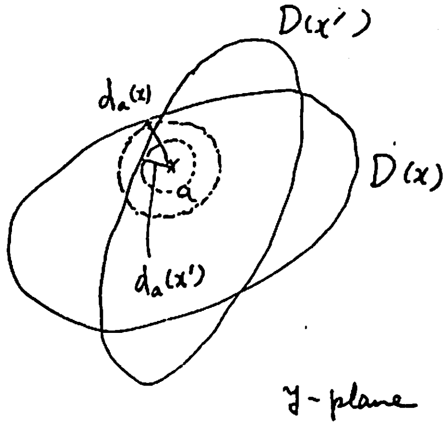
$$D(x) = \{ y \in \mathbb{C}_y \mid (x,y) \in \mathcal{D} \}$$

and call it the fiber of $\mathcal{D}$ at $x \in B$.

Moreover we define

$d_a(x)$ = the distance from a to the boundary $\partial D($





By assumption, $d_a(y) > 0$. We say that $d_a(x)$ is the radius of Hartogs of $D(x)$ w.r.t. the point a.

<u>Theorem 3.1</u> If $\mathcal{D}$ is pseudo convex in $\mathbb{C}^2$, then $\log d_a(x)$ is superharmonic for x in B.

<u>Proof:</u> (i) $d_a(x)$ is lower semi-continuous for x in B, i.e.,

$$\varliminf_{x \to x_0} d_a(x) = d_a(x_0) \qquad \text{for } x_0 \in B$$

($\because$ $\mathcal{D}$ is open in $B \times \mathbb{C}_y$)

(ii) $d_a(x)$ does not happen to attain its local strict minimum at any point in B. That is, Given $x_0 \in B$, take $r > 0$ $\ni$ $(|x - x_0| \leq r) \subset B$. Then it never occurs

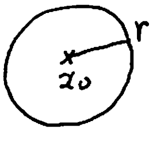
$$d_a(x_0) \lneqq d_a(x_0 + re^{i\theta}) \quad (0 \leq \theta \leq 2\pi)$$

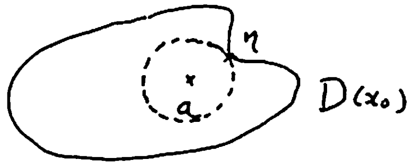
($\because$ Let us prove it by contradiction. Assume that such $x_0 \in B$ and $r_0 > 0$ exist:

$$d_a(x_0) < d_a(x_0 + r_0 e^{i\theta}) \quad (0 \leq \theta \leq 2\pi).$$

We can find $\eta \in \partial D(x_0)$ such that

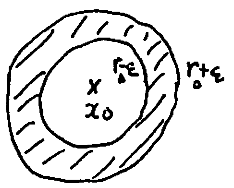
$$d_a(x_0) = |\eta - a|$$

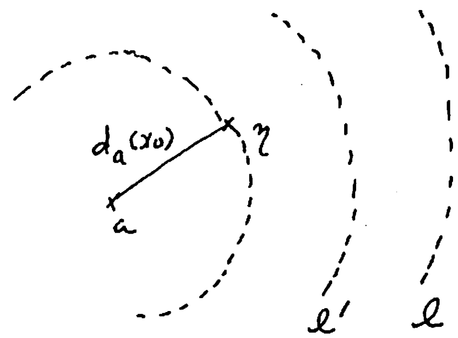




By (i), we have $\quad l = \min_{0 \leq \theta \leq 2\pi} d(x_0 + r_0 e^{i\theta}) \quad > d_a(x_0)$

$\therefore \; \exists\, \varepsilon > 0 \;\ni\; l' \underset{\text{def}}{=} \dfrac{l + d_a(x_0)}{2} < d_a(x_0 + re^{i\theta})$ for $\; r_0 - \varepsilon < r < r_0 + \varepsilon$, $\; 0 \leq \theta \leq 2\pi$

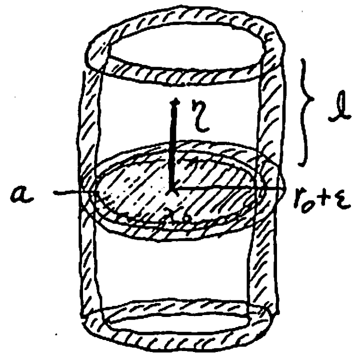




It follows that $\;(r_0 - \varepsilon \leq |x - x_0| \leq r_0 + \varepsilon, \; |y - a| < l') \subset \mathcal{D}$

On the other hand, we get

By assumption, $\quad (|x - x_0| \leq r_0 + \varepsilon, \; a) \Subset \mathcal{D}$



Consequently, Remark 2.1 yields that

$$(|x - x_0| < r_0 + \varepsilon) \times (|y - a| < l') \subset \mathcal{D}$$

so that $\quad \eta \in D(x_0)$. This is a contradiction to $\eta \in \partial D(x_0)$.

c.q.f.d

(iii) Let $\varphi(x)$ be holomorphic and $\neq 0$ anywhere in B_0 ($\subset B$). Then $|\varphi(x)|\, d_a(x)$ does not attain its local strict maximum at a point in B_0.

($\because$) Let φ be any such fn in B_0. Consider the holomorphic transformation, named Hartogs Transformation,

$$T : \begin{cases} x = x \\ Y = \varphi(x)(y - a) \end{cases}$$

and put $\quad \mathcal{D}' = T(\mathcal{D} \cap (B_0 \times \mathbb{C}_y))$;

$\quad D'(x) = \{ Y \in \mathbb{C} \mid (x, Y) \in \mathcal{D}' \}$ for $x \in B_0$.

If we form the radius of Hartogs $d_0'(x)$ of $D'(x)$ w.r.t the origin $Y=0$, then it is clear that

$$d_0'(x) = |\varphi(x)|\, d_a(x).$$

Since $\mathcal{D}'$ is pseudoconvex in $\mathbb{C}^2$, (ii) implies that $d_0'(x)$ does not attain its local strict minimum at any point x in B_0, and hence that so doesn't $|\varphi(x)|\, d_a(x)$.

c.q.f.d.

(iv) $\log d_a(x)$ is superharmonic in B.

($\because$) It suffices to prove that, for $x_0 \in B$ and $0 < r_0 \ll 1$,

$$\log d_a(x_0) \geqq \frac{1}{2\pi}\int_0^{2\pi} \log d_a(x_0 + re^{i\theta})\,d\theta.$$

Let us this inequality by contradiction. Assume that there exists $0 < r_0 \ll 1$ such that

$$\log d_a(x_0) < \frac{1}{2\pi}\int_0^{2\pi} \log d_a(x_0 + r_0 e^{i\theta})\,d\theta$$

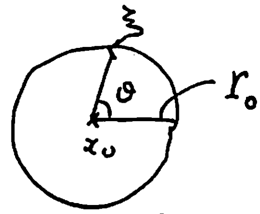
Since $\log d_a(x_0 + r_0 e^{i\theta})$ is lower semi continuous for $\theta \in [0, 2\pi]$, it follows that we have a sequence of continuous fns $h_n(\theta)$ s.t. $h_1(\theta) \leqq h_2(\theta) \leqq \cdots$; $\lim_{n\to\infty} h_n(\theta) = \log d_a(x_0 + r_0 e^{i\theta})$

Consequently, we find $n \gg 1$ such that

$$(3.1) \qquad \frac{1}{2\pi}\int_0^{2\pi} h_n(\theta)\,d\theta > \log d_a(x_0)$$

Construct the Poisson Integral

$$u(x) = \frac{1}{2\pi}\int_0^{2\pi} h_n(\theta)\, \frac{r_0^2 - |x|^2}{|\zeta - x|^2}\,d\theta.$$



so that $u(\zeta) = h_n(\theta) < \log d_a(\zeta)$ $\quad (\zeta = x_0 + r_0 e^{i\theta})$

and $u(x)$ is harmonic in $|x - x_0| < r_0$

Take $r_1 > r_0$ and define

$$\tilde{u}(x_0+re^{i\theta}) = u\left(x_0+\frac{r_0}{r_1}\cdot re^{i\theta}\right) \qquad (0 \leqq r < r_1)$$

so that $\tilde{u}(x_0+re^{i\theta})$ is harmonic in $|x-x_0|<r_1$ and, by (3.1),

$$\tilde{u}(x_0) = u(x_0) > \log d_a(x_0)$$

Since $u(x)$ is continuous on $|x-x_0| \leqq r_0$, we choose $r_1 > r_0$ so close to r_0 that

$$\tilde{u}(x_0+r_0e^{i\theta}) < h_u(\theta) + \frac{\tilde{u}(x_0)-\log d_a(x_0)}{2}. \quad (0 \leqq \theta \leqq 2\pi)$$

We form a harmonic conjugate $\tilde{u}^*(x)$ of $\tilde{u}(x)$ in $|x-x_0|<r_1$, and put

$$\varphi(x) = e^{-(\tilde{u}(x)+i\tilde{u}^*(x))} \quad \text{in } |x-x_0|<r_1.$$

By (ii),

$$d_a(x_0)|\varphi(x_0)| \lneqq d_a(x_0+r_0e^{i\theta})\,|\varphi(x_0+r_0e^{i\theta})| \quad (0 \leqq \theta \leqq 2\pi)$$

On the other hand,

$$\log d_a(x_0)|\varphi(x_0)| = \log d_a(x_0) - \tilde{u}(x_0) < 0;$$

$$\begin{aligned}\therefore \log d_a(x_0+r_0e^{i\theta})|\varphi(x_0+r_0e^{i\theta})| &= \log d_a(x_0+r_0e^{i\theta}) - \tilde{u}(x_0+re^{i\theta}) \\ &> h_u(\theta) - \tilde{u}(x_0+re^{i\theta}) > \frac{\log d_a(x_0)-\tilde{u}(x_0)}{2} \\ &> \log d_a(x_0) - \tilde{u}(x_0) = \log d_a(x_0)|\varphi(x_0)|,\end{aligned}$$

which is a contradiction. c.q.f.d.

<u>Theorem 3.1</u> (Hartogs)

Let $B = (|x|<\rho)$ and $\mathbb{C} = (|y|<\infty)$. Let $y=f(x)$ $(x \in B)$ is a bdd complex valued fn on B. Assume that $\mathcal{D} = B\times\mathbb{C} - \{(x,f(x)) \in B\times\mathbb{C} \mid x \in B\}$ is a pseudoconvex domain in $B\times\mathbb{C}$. Then the fn $f(x)$ is holomorphic in B.

Remark 3.1 In that case we say that the graph $y = f(x)$ $(x \in B)$ is pseudoconcave in $B \times \mathbb{C}$.

Proof. Consider the (local homeo) analytic transformation

$$T \quad \begin{cases} x = x \\ w = e^{y} \end{cases}$$

Then the graph $w = e^{f(x)}$ $(x \in B)$ is a pseudoconcave set in $B \times \mathbb{C}_w$, because pseudoconvexity is a local property. Put $D' = B \times \mathbb{C} - (w = e^{f(x)})$ and let $a = 0$ in Lemma 3.1. Then

$$d_0'(x) = \text{the radius of Hartogs of } D'(x) \text{ w.r.t. } 0$$
$$= |e^{f(x)}| = e^{\operatorname{Re} f(x)}$$

is logarithmic superharmonic in B, that is,

$\operatorname{Re} f(x)$ is superharmonic in B.

Next, consider the analytic transformation

$$T \quad \begin{cases} x = x \\ w = -y \end{cases}$$

Then we see that $\operatorname{Re}(-f(x))$ is superharmonic in B.

$\therefore$ $\operatorname{Re} f(x)$ is harmonic in B.

By considering the transformation

$$T \quad \begin{cases} x = x \\ w = -iy \end{cases}$$

we see analogously that $\operatorname{Im} f(x)$ is harmonic in B, and so is $f(x)$.

Finally, consider the analytic transformation

$$T \quad \begin{cases} x = x \\ w = (x-a)\,y \end{cases} \quad \text{where } a \notin B.$$

Then the graph $w = (x-a) f(x)$ is pseudoconcave in $B \times \mathbb{C}$.

$\therefore$ $(x-a)f(x)$ is harmonic in B.

We obtain

$$\frac{\partial^2}{\partial x \partial \bar{x}} (x-a) f(x) \equiv 0 \quad \text{in } B$$

$$\frac{\partial}{\partial x}\left((x-a)\frac{\partial f}{\partial \bar{x}}\right) = \frac{\partial f}{\partial \bar{x}} + \frac{\partial^2 f}{\partial x \partial \bar{x}}$$

$\frac{\partial^2 f}{\partial x \partial \bar{x}} = 0$ because f is harmonic.

$\therefore$ $\partial f/\partial \bar{x} \equiv 0$ in B.

That means, $f(x)$ is holomorphic in B. q.e.d.

§4. Levi's Condition.

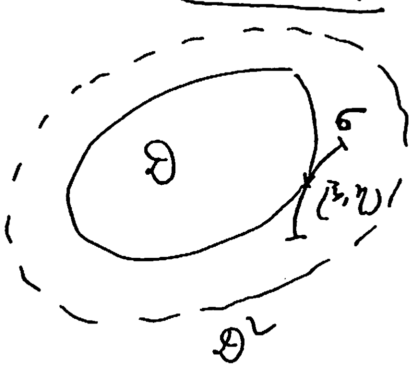
Let $\mathcal{D}$ be a domain in $\mathbb{C}^2$ with smooth boundary $\partial\mathcal{D}$, i.e., $\exists\, \tilde{\mathcal{D}} \supset\supset \mathcal{D}$ and C^∞-fn $\varphi(x)$ on $\tilde{\mathcal{D}}$ such that

$\mathcal{D} = \{(x,y) \in \tilde{\mathcal{D}} \mid \varphi(x,y) < 0\}$;

$\partial\mathcal{D} = \{(x,y) \in \tilde{\mathcal{D}} \mid \varphi(x,y) = 0\}$;

$\tilde{\mathcal{D}} - \mathcal{D} = \{(x,y) \in \tilde{\mathcal{D}} \mid \varphi(x,y) > 0\}$

& $\operatorname{Grad} \varphi = (\partial\varphi/\partial x, \partial\varphi/\partial y) \neq (0,0)$ anywhere on $\partial\mathcal{D}$.



__Lemma 4.1__ Let $\mathcal{D}$ be a domain in $\mathbb{C}^2$ with smooth boundary $\partial\mathcal{D}$. Assume that $\mathcal{D}$ satisfies the following condition: Given $(\zeta,\eta) \in \partial\mathcal{D}$, there exists a regular analytic set σ such that

$$\sigma \ni (\zeta,\eta) \quad \text{and} \quad \sigma \subset \mathcal{D}^c.$$

Then the Levi form $L\varphi$:

$$L\varphi = \frac{\partial^2\varphi}{\partial x \partial \bar{x}}\left|\frac{\partial\varphi}{\partial y}\right|^2 - 2\,\mathrm{Re}\left\{\frac{\partial^2\varphi}{\partial x \partial \bar{y}}\frac{\partial\varphi}{\partial \bar{x}}\frac{\partial\varphi}{\partial y}\right\} + \frac{\partial^2\varphi}{\partial y \partial \bar{y}}\left|\frac{\partial\varphi}{\partial x}\right|^2 \geqq 0$$

at (ζ,η).

__Proof.__ For the sake of convenience we put $(\zeta,\eta) = (0,0)$

and $(\partial\varphi/\partial y)(0,0) \neq 0$. By the same method of the proof of Lemma 2.1 σ can be written in the form

$$\sigma : \; y = a_1 x + a_2 x^2 + \cdots \qquad (|x| \ll 1)$$

Let

$$\varphi^*(x) = \varphi(x, a_1 x + a_2 x^2 + \cdots) \quad \text{i.e., restriction of } \varphi \text{ to } \sigma$$

for $|x| \ll 1$.

Then the assumptions imply that

$$\varphi^*(0) = 0 \quad \text{and} \quad \varphi^*(x) \geqq 0 \quad \text{for } |x| \ll 1$$

$\therefore$ $\varphi^*(x)$ attains its local minimum at $x=0$

As necessary conditions, we get

$$\frac{\partial \varphi^*}{\partial x}(0) = 0 \;; \qquad \frac{\partial^2 \varphi^*}{\partial x \partial \bar{x}}(0) \geqq 0 .$$

$$\therefore \quad \frac{\partial \varphi}{\partial x}(0,0) + \frac{\partial \varphi}{\partial y}(0,0)\, a_1 = 0 \qquad \therefore \quad a_1 = -\left(\frac{\partial\varphi}{\partial x} \Big/ \frac{\partial \varphi}{\partial y}\right)(0,0)$$

$$\frac{\partial^2\varphi}{\partial \bar{x} \partial x}(0,0) + 2\,\mathrm{Re}\left\{ \frac{\partial^2 \varphi}{\partial \bar{x} \partial y}(0,0)\, a_1 \right\} + \frac{\partial^2\varphi}{\partial y \partial \bar{y}}(0,0) \cdot |a_1|^2 \geqq 0$$

$$\therefore \quad \frac{\partial^2 \varphi}{\partial x \partial \bar{x}} + 2\,\mathrm{Re}\left\{ \frac{\partial^2\varphi}{\partial \bar{x} \partial y}\left(- \frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}} \right) \right\} + \frac{\partial^2\varphi}{\partial y \partial \bar{y}} \cdot \left| \frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial \varphi}{\partial y}} \right|^2 \geqq 0 \quad \text{at } (0,0)$$

$$\therefore \quad \frac{\partial^2\varphi}{\partial x \partial \bar{x}} \left| \frac{\partial \varphi}{\partial y} \right|^2 - 2\,\mathrm{Re}\left\{ \frac{\partial^2 \varphi}{\partial \bar{x} \partial y} \frac{\partial\varphi}{\partial x} \frac{\partial \varphi}{\partial \bar{y}} \right\} + \frac{\partial^2\varphi}{\partial y \partial \bar{y}} \left| \frac{\partial \varphi}{\partial x} \right|^2 \geqq 0 \quad \text{at } (0,0)$$

c.q.f.d.

<u>Lemma 4.2</u> (Invariances of $L\varphi$)

Let D be a domain in $\mathbb{C}^2$ with smooth boundary ∂D. Let $(\tilde{D}, \varphi)$ define the domain D. Then we get

(1) Let $(\tilde{\tilde{D}}, \psi)$ be another double which define D. Then

$$\frac{L\varphi}{\|\mathrm{Grad}\, \varphi\|^3} = \frac{L\psi}{\|\mathrm{Grad}\, \psi\|^3} \quad \text{on } \partial D$$

(2) Let $T : (x,y) \to (X,Y) = (u(x,y), v(x,y))$ be any local homeo. holomorphic mapping and put

$$\varphi^*(X,Y) = \varphi(x,y) \quad \text{where } (X,Y) = T(x,y).$$

Then
$$\left(L_{(X,Y)}\varphi^*\right)\cdot |J_T|^2 = L_{(x,y)}\varphi$$

where J_T = Jacobian of T = $\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}$

Proof of (1) Since Grad $\varphi \neq 0$ at $(z,\eta) \in \partial D$, we find

$\psi = \varphi \cdot \chi$ where χ is C^∞ function and $\chi > 0$ near (z,η). Hence Direct Calculation leads us

$$\frac{\partial \psi}{\partial x} = \frac{\partial \varphi}{\partial x}\chi + \varphi\frac{\partial \chi}{\partial x}$$

In particular, $\frac{\partial \psi}{\partial x}(z,\eta) = \frac{\partial \varphi}{\partial x}(z,\eta)\,\chi(z,\eta) \quad (\because \varphi(z,\eta)=0)$

$$\frac{\partial^2 \psi}{\partial \bar{x}\partial x} = \frac{\partial^2 \varphi}{\partial \bar{x}\partial x}\chi + \frac{\partial \varphi}{\partial x}\frac{\partial \chi}{\partial \bar{x}} + \frac{\partial \varphi}{\partial \bar{x}}\frac{\partial \chi}{\partial x} + \varphi\frac{\partial^2 \chi}{\partial x\partial \bar{x}}$$

In particular, $\frac{\partial^2 \psi}{\partial \bar{x}\partial x}(z,\eta) = \frac{\partial^2 \varphi}{\partial \bar{x}\partial x}\chi + 2\,\mathrm{Re}\left\{\frac{\partial \varphi}{\partial x}\frac{\partial \chi}{\partial \bar{x}}\right\}$ at (z,η)

$$\frac{\partial^2 \psi}{\partial \bar{x}\partial y} = \frac{\partial^2 \varphi}{\partial \bar{x}\partial y}\chi + \frac{\partial \varphi}{\partial \bar{x}}\frac{\partial \chi}{\partial y} + \frac{\partial \varphi}{\partial y}\frac{\partial \chi}{\partial \bar{x}} + \varphi\frac{\partial^2 \chi}{\partial \bar{x}\partial y}$$

In particular, $\frac{\partial^2 \psi}{\partial \bar{x}\partial y}(z,\eta) = \frac{\partial^2 \varphi}{\partial \bar{x}\partial y}\chi + \frac{\partial \varphi}{\partial \bar{x}}\frac{\partial \chi}{\partial y} + \frac{\partial \varphi}{\partial y}\frac{\partial \chi}{\partial \bar{x}}$ at (z,η)

$\vdots \quad \vdots$

$$\begin{aligned}\therefore L\psi = {} & \left|\frac{\partial \varphi}{\partial x}\cdot\chi\right|^2 \left\{\frac{\partial^2 \varphi}{\partial \bar{y}\partial y}\chi + 2\,\mathrm{Re}\left\{\frac{\partial \varphi}{\partial y}\frac{\partial \chi}{\partial \bar{y}}\right\}\right\} \\ & - 2\,\mathrm{Re}\left\{\frac{\partial \varphi}{\partial x}\chi\,\frac{\partial \varphi}{\partial \bar{y}}\chi\left(\frac{\partial^2 \varphi}{\partial \bar{x}\partial y}\chi + \frac{\partial \varphi}{\partial \bar{x}}\frac{\partial \chi}{\partial y} + \frac{\partial \varphi}{\partial y}\frac{\partial \chi}{\partial \bar{x}}\right)\right\} \\ & + \left|\frac{\partial \varphi}{\partial y}\chi\right|^2\left\{\frac{\partial^2 \varphi}{\partial \bar{x}\partial x}\chi + 2\,\mathrm{Re}\left\{\frac{\partial \varphi}{\partial x}\frac{\partial \chi}{\partial \bar{x}}\right\}\right\}\end{aligned}$$

Suitably cancelled!

$$= \chi^3 \cdot L\varphi$$

Since $\frac{\partial\psi}{\partial x} = \chi \frac{\partial\varphi}{\partial x}$, $\frac{\partial\psi}{\partial y} = \chi \frac{\partial\varphi}{\partial y}$ at (ζ,η) and $\chi > 0$, it follows that

$$\chi(\zeta,\eta) = \frac{\|\mathrm{Grad}\,\psi\|(\zeta,\eta)}{\|\mathrm{Grad}\,\varphi\|(\zeta,\eta)}$$

and hence $L\psi / \|\mathrm{Grad}\,\psi\|^3 = L\varphi / \|\mathrm{Grad}\,\varphi\|^3$ at (ζ,η) c.q.f.d.

Proof of (2) It is quite long and patient calculus.

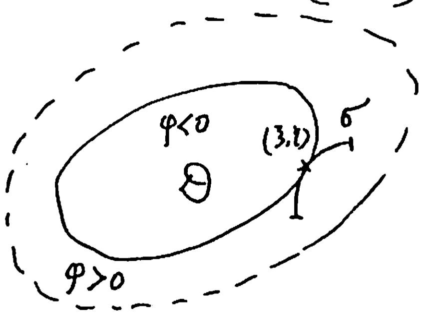
Remark 4.1

(1) $\frac{L\varphi}{\|\mathrm{Grad}\,\varphi\|^3}$ is determined by the figure of ∂D. Probably, it has some geometric meaning.

(2) $(L\varphi)\,dV$ is invariant under the analytic transformation. It will be no useful for quantity calculation.

(3) $L\varphi > 0$ or < 0 does not depend on the choice of φ and the local coordinates.

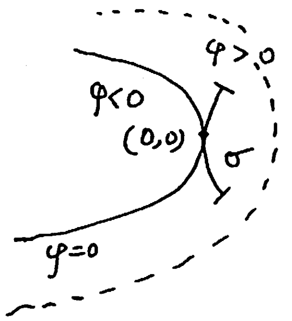
Lemma 4.3 Let D be a domain in $\mathbb{C}^2$ with smooth boundary ∂D. Let $(\tilde{D}, \varphi)$ define the domain D. Assume that $L\varphi > 0$ at $(\zeta,\eta) \in \partial D$. Then there exists a regular analytic set σ such that
$\sigma \ni (\zeta,\eta)$ and $\sigma \subset \tilde{D} - \bar{D} \cup \partial D$ except (ζ,η).



Proof. By Lemma 4.2, we may assume that $(\zeta,\eta) = (0,0)$ and $\mathrm{grad}\,\varphi = (1,0)$ at $(0,0)$.

$\therefore\ L\varphi = \frac{\partial^2\varphi}{\partial y \partial \bar{y}}$ at $(0,0)$

By assumption, $\dfrac{\partial^2 \varphi}{\partial y \partial \bar{y}}(0,0) > 0$.



Let us try to find σ : of the form

$\sigma : \quad x = A y^2$, A is constant.

Put

$$\varphi^*(y) = \varphi(A y^2, y) \qquad (|y| \ll 1)$$

Then

$$\left.\frac{\partial \varphi^*(y)}{\partial y}\right]_{y=0} = \left[\frac{\partial \varphi}{\partial x} \cdot 2Ay + \frac{\partial \varphi}{\partial y}\right]_{(0,0)} = 0$$

$$\left[\frac{\partial^2 \varphi^*(y)}{\partial \bar{y}\, \partial y}\right]_{y=0} = \frac{\partial^2 \varphi}{\partial \bar{y} \partial y}(0,0) > 0$$

$$\left[\frac{\partial^2 \varphi^*}{\partial y^2}\right]_{y=0} = 2A + \frac{\partial^2 \varphi}{\partial y \partial \bar{y}}(0,0).$$

By Taylor expansion, we have

$$\varphi^*(y) = \varphi^*(0) + 2\,\mathrm{Re}\left\{\frac{\partial \varphi^*}{\partial y}(0)\, y\right\}$$

$$+ 2\,\mathrm{Re}\left\{\frac{\partial^2 \varphi^*}{\partial y^2}(0)\, y^2\right\} + 2\frac{\partial^2 \varphi^*}{\partial \bar{y} \partial y}(0)|y|^2 + O(|y|^3)$$

$$= 2\,\mathrm{Re}\left\{\left(2A + \frac{\partial^2 \varphi}{\partial y^2}(0,0)\right) y^2\right\} + 2\frac{\partial^2 \varphi}{\partial \bar{y} \partial y}(0,0)|y|^2 + O(|y|^3)$$

Therefore, if we take $A = -\frac{1}{2}\dfrac{\partial^2 \varphi}{\partial y^2}(0,0)$, then

$$\varphi^*(y) = 2\frac{\partial^2 \varphi}{\partial y^2}(0,0)|y|^2 + O(|y|^3) \qquad (|y| \ll 1)$$

$\therefore \quad \varphi(Ay^2, y) > 0$ in $0 < |y| \ll 1$,

That is to say,

$(Ay^2, y) \subset \bar{D}^2 - D^2$ except for $(0,0)$.

Consequently, it is sufficient to put

$$\sigma : \quad x = -\frac{1}{2}\frac{\partial^2 \varphi}{\partial y^2}(0,0)\, y^2 \qquad (|y| \ll 1).$$

c. q. f. d.

<u>Theorem 4.1</u> Assume that $\mathcal{D}$ is a pseudoconvex domain in $\mathbb{C}^2$ with smooth boundary $\partial\mathcal{D}$. Let $(\tilde{\mathcal{D}}, \varphi)$ define the domain $\mathcal{D}$. Then

$$L\varphi \geqq 0 \quad \text{at } (z, \eta) \in \partial\mathcal{D}$$

<u>Proof.</u> We prove it by contradiction. Assume that $L\varphi\,(z,\eta) < 0$. Since

$$L(-\varphi) = -L\varphi,$$

we have $L(-\varphi)\,(z,\eta) > 0$ and $(-\varphi)(z,\eta) = 0$. It follows from Lemma 4.3 that

$\exists\, \sigma$: a regular analytic set such that

$(z,\eta) \in \sigma$ and $\sigma \subset \{-\varphi > 0\}$ except at (z,η)

i.e., $\sigma \subset \mathcal{D}$ except at (z,η)

This contradicts ~~the~~ Lemma 2.1.

c. q. f. d.

Dear

November 30, '87

This is a note for the lecture in Nov. 24. I did not prove the exceptional case (3) in p. 2. The proof was easy by using complex notations like this in p. 3. Please check it, Thank you,

Hiroshi Yamaguchi

§5. Plurisubharmonic functions

<u>Definition 5.1</u> Let $s(x,y)$ be a real-valued (which may attain $-\infty$ but not $+\infty$) function in a domain $\mathcal{D}$ in $\mathbb{C}^2$. If

(i) $s(x,y)$ is uppersemicontinuous in $\mathcal{D}$, i.e.,

$$\lim_{(x,y)\to(x_0,y_0)} s(x,y) = s(x_0,y_0) \quad \text{for } (x_0,y_0)\in\mathcal{D};$$

(ii) Let L be any 1-dimensional line in $\mathbb{C}^2$. Then the restriction of $s(x,y)$ to L is subharmonic on L, i.e., for any $(a,b)\in\mathcal{D}$ and $(\alpha,\beta)\in\mathbb{C}^2$ with $(\alpha,\beta)\neq(0,0)$, $s(a+\alpha\tau,\ b+\beta\tau)$ is subharmonic for τ in $|\tau|\ll 1$,

then $s(x,y)$ is said to be plurisubharmonic for (x,y) in $\mathcal{D}$.

The function $s(x,y)\equiv-\infty$ in $\mathcal{D}$ is thus plurisubharmonic in $\mathcal{D}$.

We consider the case when $s(x,y)$ is of class C^2 in $\mathcal{D}$.

(1) $s(x,y)$ is plurisubharmonic in $\mathcal{D}$, iff the complex Hessian of $s(x,y)$ is non-negative in $\mathcal{D}$, i.e.,

$$\begin{pmatrix} \frac{\partial^2 s}{\partial x\partial\bar{x}} & \frac{\partial^2 s}{\partial x\partial\bar{y}} \\ \frac{\partial^2 s}{\partial\bar{x}\partial y} & \frac{\partial^2 s}{\partial y\partial\bar{y}} \end{pmatrix} \geqq 0 \quad \text{at any } (x,y) \text{ in } \mathcal{D}.$$

<u>Proof.</u> $s(x,y)$ is pluri subharmonic in $\mathcal{D}$

$\rightleftarrows$ (ii) is satisfied

$$\rightleftarrows \left[\frac{\partial^2}{\partial\tau\partial\bar{\tau}} s(a+\alpha\tau,\ b+\beta\tau)\right]_{\tau=0} \geqq 0$$

for all $(a,b)\in\mathcal{D}$ and $(\alpha,\beta)\in\mathbb{C}^2$ with $(\alpha,\beta)\neq(0,0)$.

$$\rightleftarrows \frac{\partial^2 s}{\partial x\partial\bar{x}}(a,b)\,|\alpha|^2 + 2\,\mathrm{Re}\left\{\frac{\partial^2 s}{\partial x\partial\bar{y}}(a,b)\,\alpha\bar{\beta}\right\} + \frac{\partial^2 s}{\partial y\partial\bar{y}}(a,b)\,|\beta|^2 \geqq 0$$

for all $(a,b) \in \mathbb{C}^2$ and $(\alpha,\beta)\ (\neq (0,0)) \in \mathbb{C}^2$.

$\rightleftarrows$ $\dfrac{\partial^2 s}{\partial x \partial \bar{x}}(a,b) \geqq 0$, $\dfrac{\partial^2 s}{\partial y \partial \bar{y}}(a,b) \geqq 0$, $\dfrac{\partial^2 s}{\partial x \partial \bar{x}}(a,b)\dfrac{\partial^2 s}{\partial y \partial \bar{y}}(a,b) - \left|\dfrac{\partial^2 s}{\partial x \partial \bar{y}}\right|^2(a,b)$ $\geqq 0$ for any $(a,b) \in \mathcal{D}$; i.e., the complex Hessian of $s(x,y)$ is non-negative in $\mathcal{D}$.

c. q. f. d.

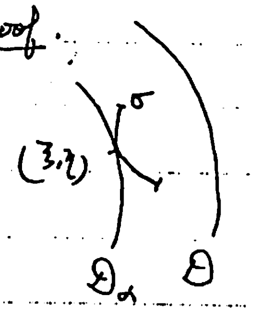
(2) Given $-\infty < \alpha < +\infty$, put $\mathcal{D}_\alpha = \{(x,y) \in \mathcal{D} \mid s(x,y) < \alpha\}$. Let $(\xi,\eta) \in (\partial \mathcal{D}_\alpha) \cap \mathcal{D}$ such that

(i) Grad $s(\xi,\eta) \neq 0$,

(ii) The complex Hessian of $s(x,y)$ is positive at (ξ,η), i.e., $\dfrac{\partial^2 s}{\partial x \partial \bar{x}} > 0$, $\dfrac{\partial^2 s}{\partial y \partial \bar{y}} > 0$, $\dfrac{\partial^2 s}{\partial x \partial \bar{x}}\dfrac{\partial^2 s}{\partial y \partial \bar{y}} - \left|\dfrac{\partial^2 s}{\partial x \partial \bar{y}}\right|^2 > 0$ at (ξ,η).

Then there exists a regular analytic set σ such that $(\xi,\eta) \in \sigma$; $\sigma \subset \mathcal{D} - (\mathcal{D}_\alpha \cup \partial \mathcal{D}_\alpha)$ except for (ξ,η).

<u>Proof</u>.



Consider the Levi form of $s - s(\xi,\eta)$ at $(\xi,\eta) \in \partial \mathcal{D}_\alpha$. Then we get

$$L(s - s(\xi,\eta)) = \left|\frac{\partial s}{\partial x}\right|^2 \frac{\partial^2 s}{\partial y \partial \bar{y}} - 2\,\mathrm{Re}\left\{\frac{\partial^2 s}{\partial x \partial \bar{y}}\frac{\partial s}{\partial \bar{x}}\frac{\partial s}{\partial y}\right\} + \left|\frac{\partial s}{\partial y}\right|^2 \frac{\partial^2 s}{\partial x \partial \bar{x}}$$

If we put $(\alpha,\beta) = \left(\dfrac{\partial s}{\partial y}, -\dfrac{\partial s}{\partial x}\right)(\xi,\eta) \neq (0,0)$, then (ii) implies that $L(s - s(\xi,\eta)) > 0$ at (ξ,η). By Lemma 4.2, we find a regular analytic set σ such that $\sigma \ni (\xi,\eta)$ and $\sigma \subset \{(x,y) \in \mathcal{D} \mid s(x,y) - s(\xi,\eta) > 0\}$ except for (ξ,η).

c. q. f. d.

(3) (Exceptional case) Under the same circumstances at (2) except (i), we find a regular analytic set σ such that $(\xi,\eta) \in \sigma$ and $\sigma \subset \mathcal{D} - (\mathcal{D}_\alpha \cup \partial \mathcal{D}_\alpha)$ except for (ξ,η).

<u>Proof.</u> For the sake of convenience we put

$(\xi, \eta) = (0,0)$ and $\alpha = 0$, i.e., $s(0,0) = 0$.
Since $S(0,0) = \begin{pmatrix} \frac{\partial^2 s}{\partial x \partial \bar{x}} & \frac{\partial^2 s}{\partial x \partial \bar{y}} \\ \frac{\partial^2 s}{\partial \bar{x} \partial y} & \frac{\partial^2 s}{\partial y \partial \bar{y}} \end{pmatrix}_{(0,0)} > 0$, there exists an Hermite non-singular matrix $M = \begin{pmatrix} a & b \\ -\bar{b} & c \end{pmatrix}$ such that

$$M^{-1} S(0,0) M = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where $\lambda_1, \lambda_2 > 0$. Consider the analytic transformation

$$\begin{cases} X = ax + by \\ Y = -\bar{b}x + \bar{a}y \end{cases}$$

and put $\hat{s}(X, Y) = s(x, y)$ near $(0,0)$. Since $s(0,0) = 0$ and $\operatorname{grad} s(0,0) = (0,0)$, it follows that

$$\begin{aligned} \hat{s}(X,Y) &= s(\bar{a}X - \bar{b}Y, bX + aY) \\ &= (X,Y) M^{-1} S(0,0) M \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \\ &\quad + 2\operatorname{Re}\{AX^2 + 2BXY + CY^2\} + O\left(\sqrt{|X|^2+|Y|^2}^{\,3}\right) \\ &= \lambda_1 |X|^2 + \lambda_2 |Y|^2 + 2\operatorname{Re}\{AX^2 + 2BXY + CY^2\} + O\left(\sqrt{|X|^2+|Y|^2}^{\,3}\right), \end{aligned}$$

where $A, B, C \in \mathbb{C}$.
We try to find $\hat{\sigma}$ of the form: $Y = kX$ $(k \in \mathbb{C})$.
Then $\hat{s}|_{\hat{\sigma}} = \hat{s}(X, kX)$

$$= (\lambda_1 + |k|^2 \lambda_2)|X|^2 + 2\operatorname{Re}\{(A + 2Bk + Ck^2)X^2\} + O(|X|^3)$$

We thus choose $k \in \mathbb{C}$ such that

$$k = (-A + \sqrt{B^2 - AC})/C \qquad (C \neq 0);$$
$$|k| \gg 1 \qquad (C = 0).$$

so that $\hat{s}(X, kX) > 0$ in $|X| \ll 1$ except for $X = 0$.
This means that

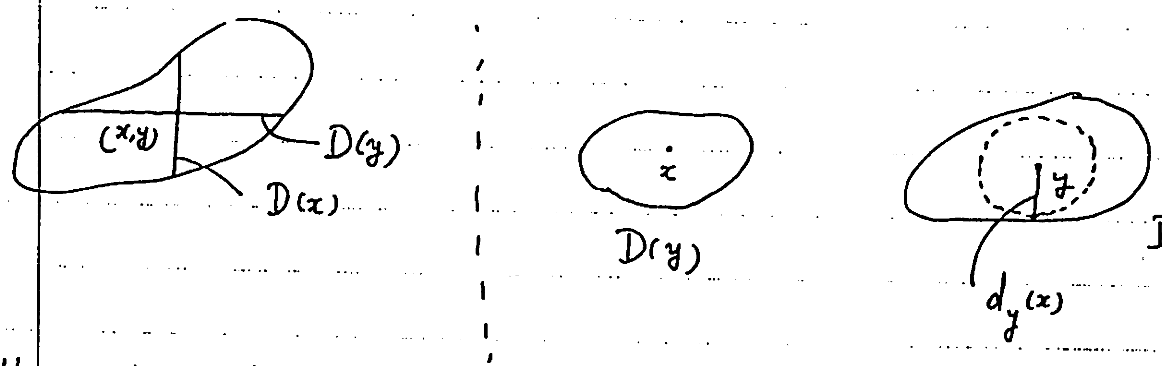
$$\sigma : -\bar{b}x + \bar{a}y = k(ax + by)$$

satisfies the conditions: $(0,0) \in \sigma$ and $\sigma \subset D - D_0 \cup \partial D_0$ except for $(0,0)$.

c.q.f.d.

We return to any pseudoconvex domain $\mathcal{D}$ in $\mathbb{C}^2$, and construct one of the typical plurisubharmonic function in $\mathcal{D}$.

Given $(x, y) \in \mathcal{D}$, we denote by $D(x)$ and $D(y)$ the fiber of $\mathcal{D}$ at x and y, respectively



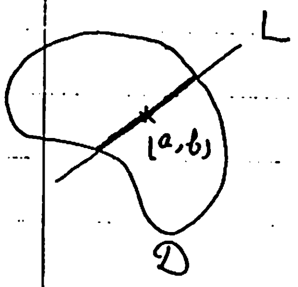
We put $d_y(x)$ = the radius of Hartogs of $D(x)$ w.r.t. y for any $y \in D(x)$. By Lemma 4.1, we see that $\log d_y(x)$ is superharmonic for x in $D(y)$. Moreover we shall show

<u>Lemma 4.3</u> $\log \dfrac{1}{d_y(x)}$ is plurisubharmonic for (x, y) in $\mathcal{D}$.

<u>Proof.</u> (i) $d_y(x)$ is lowersemicontinuous for (x, y) in $\mathcal{D}$, because $\mathcal{D}$ is open in the space $\mathbb{C}^2$.

(ii) First, consider the case where

$L : y = b + \alpha(x - a)$ where $(a, b) \in \mathcal{D}$ and $\alpha \in \mathbb{C}$.

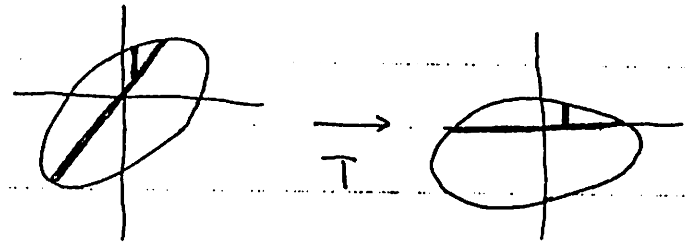


Let us show

$\log d_{b+\alpha(x-a)}(x)$ is superharmonic for $|x| \ll 1$.

For the sake of convenience, we put $(a, b) = (0, 0)$. Consider the transformation :

$$T \begin{cases} X = x \\ Y = y - \alpha x \end{cases}$$

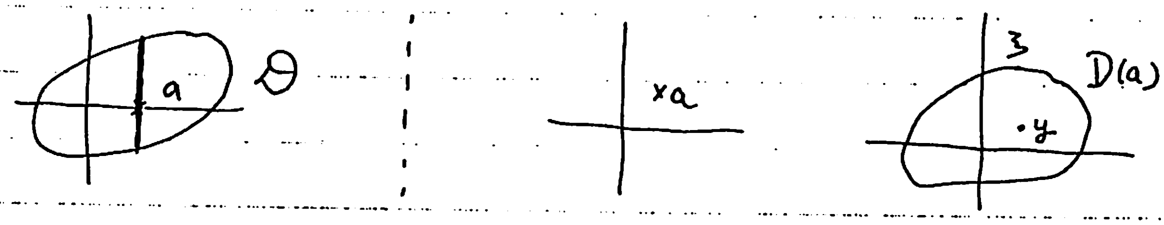


so that $y = \alpha x$ is transformed onto $Y = 0$. We put $\mathcal{D}^* = T(\mathcal{D})$ and $d_0^*(x)$ = the radius of Hartogs of $D^*(x)$ w.r.t. $Y = 0$. It is clear that

$$d_0^*(x) = d_{\alpha x}(x).$$

Since $\mathcal{D}^*$ is pseudoconvex in $\mathbb{C}^2$, $\log d_0^*(x)$ is superharmonic for x in $D^*(0)$. Consequently, $\log d_{\alpha x}(x)$ is superharmonic for x in $|x| \ll 1$.

Next, let $L : x = a$. Then $d_y(x)\big|_L = d_y(a)$. It suffices to prove that $\log d_y(a)$ is superharmonic for y in $D(a)$.



It is clear that

$$\log d_y(a) = \underset{\zeta \in \partial D(a)}{\mathrm{Min}} \{\log|y - \zeta|\}.$$

Since $d_y(a)$ is continuous for y in $D(a)$ and since each $\log|y-\zeta|$ is harmonic for y in $D(a)$, it follows that $\log d_y(a)$ is superharmonic for y in $D(a)$. Consequently, $\log d_y(x)$ is plurisubharmonic for (x,y) in $\mathcal{D}$.

c.q.f.d.

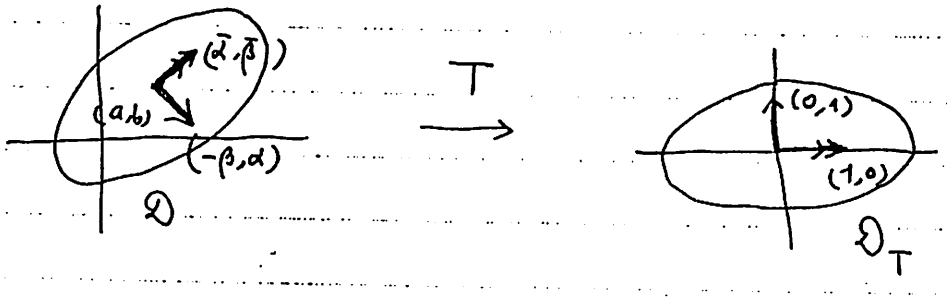
<u>Theorem 4.2</u> Let $\mathcal{D}$ be a pseudoconvex domain in $\mathbb{C}^2$ and put $d(x,y)$ = the Euclidean distance from $(x,y) \in \mathcal{D}$ to $\partial\mathcal{D}$. Then $\log \dfrac{1}{d(x,y)}$ is plurisubharmonic in $\mathcal{D}$.

Proof. (i) $d(x,y)$ is continuous for (x,y) in $\mathcal{D}$, because $|d(x,y) - d(x',y')| \leqq \sqrt{|x-x'|^2 + |y-y'|^2}$ for any (x,y), $(x',y') \in \mathcal{D}$.

(ii) Let T be any complex Euclidean motion such that

$$T : \begin{cases} X = \alpha(x-a) + \beta(y-b) \\ Y = -\bar{\beta}(x-a) + \bar{\alpha}(y-b) \end{cases}$$

where $(a,b) \in \mathcal{D}$ and $(\alpha,\beta) \in \mathbb{C}^2$ with $|\alpha|^2 + |\beta|^2 = 1$.



T is an analytic transformation of $\mathbb{C}^2$ so that $\mathcal{D}_T = T(\mathcal{D})$ is pseudoconvex in $\mathbb{C}^2$. It is clear that T preserves Euclidean length: $\|T(x,y) - T(x',y')\| = \|(x,y) - (x',y')\|$, and $T(-\beta,\alpha) = (1,0)$.

If we denote by $d^{(T)}_Y(X)$ the radius of Hartogs of $D_T(Y)$ w.r.t. X, then, by Lemma 4.3, $\log 1/d^{(T)}_Y(X)$ is plurisubharmonic fn in $\mathcal{D}_T$. Since T is linear, putting $d_T(x,y) = d^{(T)}_Y(X)$, we get $\log 1/d_T(x,y)$ is plurisubharmonic in $\mathcal{D}$.

On the other hand, it is clear that

$$d(x,y) = \inf_T \{d_T(x,y)\} \quad \text{for } (x,y) \in \mathcal{D}$$

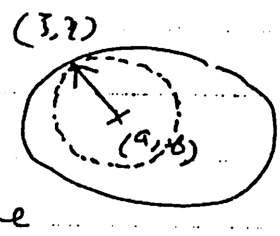
$\because$ For $\forall (a,b) \in \mathcal{D}$, we have

$d_T(a,b) = d^{(T)}_B(A) = |^{\exists}\xi - B|$

where $T(a,b) = (A,B)$ and $\xi \in \partial\mathcal{D}_T(A)$

If we put $T^{-1}(A, \zeta) = (a, \zeta^*)$, then we have $\zeta^* \in \partial D(a)$ and
$$|\zeta - B| = \|(A, \zeta) - (A, B)\|$$
$$= \|(a, \zeta^*) - (a, b)\|$$
$$\geqq d(a, b).$$

Conversely, given $(a, b) \in \mathcal{D}$, we find $(\zeta, \eta) \in \partial\mathcal{D}$ such that $d(a, b) = \|(\zeta, \eta) - (a, b)\|$. Consider the complex Euclidean motion



$$T: \begin{cases} X = \dfrac{(\overline{\eta - b})}{\sqrt{|\zeta - a|^2 + |\eta - b|^2}}(x - a) - \dfrac{(\overline{\zeta - a})}{\sqrt{\quad ''\quad}}(y - b) \\[2ex] Y = \dfrac{\overline{\zeta - a}}{\sqrt{\quad ''\quad}}(x - a) + \dfrac{\overline{\eta - b}}{\sqrt{\quad ''\quad}}(y - b) \end{cases}$$

so that $T(a, b) = (0, 0)$ and $T(\zeta, \eta) = (0, \sqrt{|\zeta - a|^2 + |\eta - b|^2})$.
It follows that $\sqrt{|\zeta - a|^2 + |\eta - b|^2} \in \partial\mathcal{D}^{(T)}(0)$ and hence
$$d(a, b) = \|(\zeta, \eta) - (a, b)\|$$
$$= \|T(\zeta, \eta) - T(a, b)\|$$
$$= \sqrt{|\zeta - a|^2 + |\eta - b|^2} \leqq d_0^{(T)}(0) = d_T(a, b).$$
We thus have $d(a, b) = \min_T d_T(a, b)$ for $(a, b) \in \mathcal{D}$. ⌋

Together with (i), we conclude that
$$\log \frac{1}{d(x, y)} = \sup_T \left\{ \log \frac{1}{d_T(x, y)} \right\}$$
is plurisubharmonic function for $(x, y) \in \mathcal{D}$. c.q.f.d.

By definition of $d(x, y)$ we get $\lim_{(x,y) \to \partial\mathcal{D}} d(x, y) = 0$. It follows that,

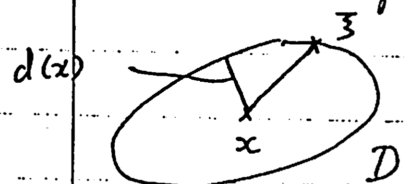
If $\mathcal{D}$ is a bounded pseudoconvex domain in $\mathcal{D}$, then the function $S(x, y) = \log \dfrac{1}{d(x, y)}$ is a continuous plurisubharmonic and exhaustion function in $\mathcal{D}$,

„exhaustion" means that, given $\alpha \in \mathbb{R}$,

$D_\alpha = \{(x,y) \in D \mid \delta(x,y) < \alpha\}$ is relatively compact in D

i.e., $D_\alpha \subset\subset D$.

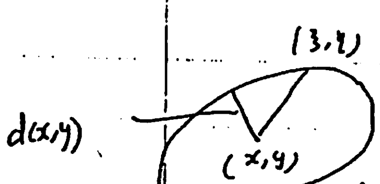
Here we reflect the fact that $\log \frac{1}{d(x,y)}$ is plurisubharmonic in a pseudoconvex domain D in $\mathbb{C}^2$. Following the above proof, the fact sound quite natural. But, when we see it from the other side, that is very rare event. Actually, in the case of $n = 1$, since



$$\log \frac{1}{d(x)} = \sup_{\xi \in \partial D} \{ \overbrace{\log \frac{1}{|x-\xi|}}^{\text{harmonic for } x \in} \},$$

we easily see that $\log 1/d(x)$ is subharmonic in D.

In the case of $n \geq 2$, we have similarly



$$\log \frac{1}{d(x,y)} = \sup_{(\xi,\eta) \in \partial D} \left\{ \log \frac{1}{\sqrt{|\xi - x|^2 + |\eta - y|^2}} \right\}.$$

Look at $\overbrace{}^{p(x,y)} = \log \sqrt{|\xi - x|^2 + |y - \eta|^2}$ for $\forall (x,y) \neq (\xi,\eta)$.
and calculate the Hessian of $p(x,y)$. Then we have

$$\begin{pmatrix} \frac{\partial^2 p}{\partial x \partial \bar{x}} & \frac{\partial^2 p}{\partial x \partial \bar{y}} \\ \frac{\partial^2 p}{\partial \bar{x} \partial y} & \frac{\partial^2 p}{\partial y \partial \bar{y}} \end{pmatrix} = \frac{1}{(|\xi - x|^2 + |y - \eta|^2)^2} \begin{pmatrix} |y-\eta|^2 & \overline{(x-\xi)}(y-\eta) \\ (x-\xi)\overline{y-\eta} & |x-\xi|^2 \end{pmatrix}$$

so that the eigenvalues are positive and zero,
hence $p(x,y)$ is plurisubharmonic fn in $(x,y) \neq (\xi,\eta)$.

Th. 4.2 $\therefore \log \frac{1}{d(x,y)} = \sup_{(\xi,\eta) \in \partial D} \{ \text{plurisuperharmonic fn } p(x,y) \}$

Consequently, Theorem 4.2 is not trivial fact!

Dear

December 7, '87

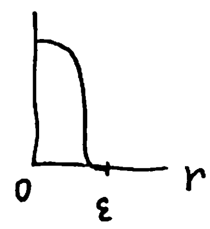
This note is for the lectures in last week. It contains one of Oka's Theorems in 1934 and the notion of Derived set. This week, we will study the variation of Riemann surfaces.

Hiroshi Yamaguchi

The Molifier $\varphi_\varepsilon(r)$.

Let $\varepsilon > 0$ be given. Consider the following function $\psi_\varepsilon(r)$ on $[0, \infty)$ such that

$$\psi_\varepsilon(r) = \begin{cases} e^{-1/(\varepsilon^2 - r^2)} & (0 \leqq r \leqq \varepsilon) \\ 0 & (r \geqq \varepsilon). \end{cases}$$



Then $\psi_\varepsilon(r)$ is of class C^∞ in $[0, \infty)$. We put

$$c_\varepsilon = \iint_{\mathbb{C}^2} \psi_\varepsilon\left(\sqrt{|x|^2 + |y|^2}\right) dV_{x,y}$$

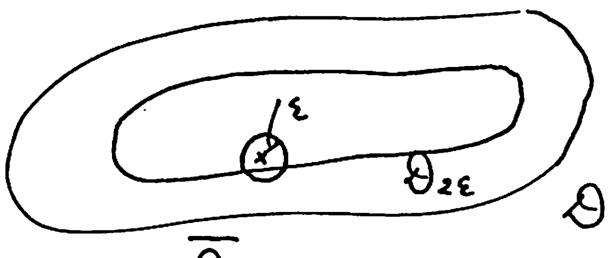
where $dV_{x,y}$ is the Euclidean volume element in $\mathbb{R}^4$, so that $\infty > c_\varepsilon > 0$. We define

$$\varphi_\varepsilon(r) = \frac{1}{c_\varepsilon} \cdot \psi_\varepsilon(r) \qquad (0 \leqq r \leqq \infty).$$

Assume that $s(x, y)$ is a continuous plurisubharmonic function in a domain $\mathcal{D}$ in $\mathbb{C}^2$. For any $\eta > 0$ we put

$$\mathcal{D}_\eta = \{ (x, y) \in \mathcal{D} \mid d(x, y) > \eta \}$$

where $d(x, y)$ is the Euclidean distance from (x, y) to $\partial \mathcal{D}$.



For $(x, y) \in \overline{\mathcal{D}}_{2\varepsilon}$, we consider the convolution:

$$s_\varepsilon(x, y) \underset{\text{def}}{=\!=} \varphi_\varepsilon * s(x, y)$$

$$= \iint_{\mathbb{C}^2} s(x + \xi, y + \eta)\, \varphi_\varepsilon\left(\sqrt{|\xi|^2 + |\eta|^2}\right) dV_{\xi,\eta}$$

It is easy to prove that

(i) $s_\varepsilon(x,y)$ is of class C^∞ in $\overline{D_{2\varepsilon}}$

($\because$ $\varphi_\varepsilon(r)$ is of class C^∞ in $\overline{[0,\infty)}$ and even fn of r)

(ii) $s_\varepsilon(x,y)$ is plurisubharmonic in $D_{2\varepsilon}$

($\because$ $a\varphi_1 + b\varphi_2$ is plurisubharmonic, if φ_1, φ_2 are subharmonic and $a, b > 0$)

(iii) On any compact set $K \subset\subset D$,

$$\lim_{\varepsilon \to 0} s_\varepsilon(x,y) = s(x,y) \text{ uniformly on } K.$$

($\because$ $s(x,y)$ is continuous on D and $\iint_{\mathbb{C}^2} \varphi_\varepsilon(\sqrt{|\zeta|^2+|\eta|^2})\, dV = 1$)

Using this smoothing operator, let us prove the following theorem:

Theorem 4.3. Let D be a pseudoconvex domain in $\mathbb{C}^2$. Then there exists a sequence of pseudoconvex domain D_ν such that

(i) $D_1 \subset\subset D_2 \subset\subset \cdots$: $\bigcup_{\nu=1}^{\infty} D_\nu = D$,

(ii) For any $(\zeta,\eta) \in \partial D_\nu$, we draw a regular analytic set σ such that $(\zeta,\eta) \in \sigma$ and $\sigma \subset D - (D_\nu \cup \partial D_\nu)$ except for (ζ,η).

Proof. It suffices to prove the following fact: Given $K \subset\subset D$, we find a subdomain D_K of D such that

(i') $K \subset\subset D_K \subset\subset D$

(ii') For any $(\zeta,\eta) \in \partial D_K$, there exists a regular analytic set σ such that $(\xi,\eta)\in\sigma$ and $\sigma\subset\mathcal{D}-(\mathcal{D}_K\cup\partial\mathcal{D}_K)$ except for (ξ,η).

In fact, let $K\subset\subset\mathcal{D}$. We have a large ball S_R: $|x|^2+|y|^2<R$ such that $K\subset\subset\mathcal{D}\cap S_R$. Since $\mathcal{D}\cap S_R$ is pseudoconvex in $\mathbb{C}^2$, we may assume that $\mathcal{D}$ is bounded pseudoconvex in $\mathbb{C}^2$. By Theorem 4.2, we get that

$$s(x,y)=\log\frac{1}{d(x,y)}$$

(where $d(x,y)$ is the distance of (x,y) to $\partial\mathcal{D}$) is a continuous, plurisubharmonic, exhaustion function in $\mathcal{D}$. Take $0<\varepsilon\ll 1$ so small that

$$K\subset\subset\mathcal{D}_{5\varepsilon}$$

and consider the convolution

$$s_\varepsilon(x,y)=\varphi_\varepsilon * s(x,y)\quad\text{for } (x,y)\in\overline{\mathcal{D}_{2\varepsilon}}.$$

$s_\varepsilon(x,y)$ becomes of class C^∞ and plurisubharmonic in $\overline{\mathcal{D}_{2\varepsilon}}$. Moreover, by construction of $s_\varepsilon(x,y)$, we have

$$s(x,y)<\log\frac{1}{4\varepsilon}\quad\text{in } \mathcal{D}_{5\varepsilon};$$
$$s(x,y)>\log\frac{1}{3\varepsilon}\quad\text{on } \partial\mathcal{D}_{2\varepsilon}$$

Since $\mathcal{D}$ is bounded in $\mathbb{C}^2$, we choose $\delta>0$ so small that, if we put $\tilde{s}(x,y)=s_\varepsilon(x,y)+\delta(|x|^2+|y|^2)$ in $\mathcal{D}_{2\varepsilon}$ then

$$\tilde{s}(x,y)<\log\frac{1}{3.5\varepsilon}\quad\text{in } \mathcal{D}_{5\varepsilon}:$$
$$\tilde{s}(x,y)>\log\frac{1}{3\varepsilon}\quad\text{on } \partial\mathcal{D}_{2\varepsilon}.$$

Hence $\tilde{\delta}(x,y)$ is of positive Hessian in $\mathcal{D}_{2\varepsilon}$ and, if we define

$$\mathcal{D}_K = \{(x,y)\in \mathcal{D}_{2\varepsilon} \mid \tilde{\delta}(x,y) < \log \frac{1}{3.5\varepsilon}\}$$

then $K \Subset \mathcal{D}_{5\varepsilon} \Subset \mathcal{D}_K \Subset \mathcal{D}_{2\varepsilon} \Subset \mathcal{D}$. It follows from (2),(3) at the beginning of this § that, for any $(\xi,\eta)\in \partial\mathcal{D}_K$, we can draw a regular analytic set σ passing through (ξ,η) such that $\sigma \subset \mathcal{D}_{2\varepsilon} - (\mathcal{D}_K \cup \partial\mathcal{D}_K)$ except for (ξ,η).

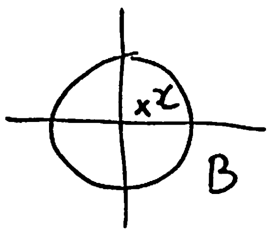
c.q.f.d.

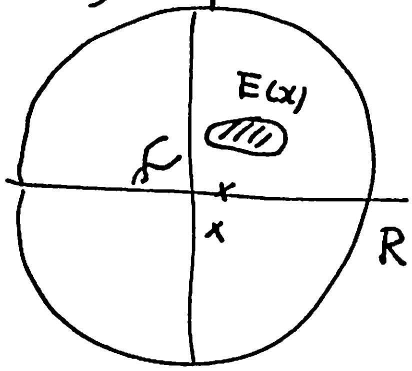
Let $B = (|x|<\rho)$, $\mathbb{C} = (|y|<\infty)$, and $\mathcal{E}$ a closed set in $B\times\mathbb{C}$. Assume that, for $\forall x\in B$, the fiber $E(x)$ of $\mathcal{E}$ is not empty, i.e.,

$$E(x) = \{y\in\mathbb{C} \mid (x,y)\in\mathcal{E}\} \neq \phi$$

and that there exists $R>0$ such that

$$E(x) \Subset (|y|<R) \quad \text{for all } x\in B.$$





Given $n \geq 2$, we defined the nth-diameter of $E(x)$ as follows:

$$d_n(x) = \operatorname*{Max}_{y_1,\dots,y_n\in E(x)} \sqrt[\binom{n}{2}]{\prod_{\substack{i\neq j\\ i,j=1}}^{n} |y_i - y_j|}\ .$$

Under these notations, we have proved in Chap. I, that

" Assume that, given a domain $B_0 \Subset B$, there exists a sequence of closed set $\mathcal{E}_\nu$ in $B_0 \times \mathbb{C}$ such that

(i) $\mathcal{E}_1(x) \Subset (|y|<R)$ for all $x \in B$;

(ii) $\mathcal{E}_1 \supset\supset \mathcal{E}_2 \supset\supset \cdots\cdots$; $\bigcap_{\nu=1}^{\infty} \mathcal{E}_\nu = \mathcal{E}$;

(iii) Each $(z, y) \in \partial \mathcal{E}_\nu$ admits a regular analytic set σ such that $(z, y) \in \sigma \subset \mathcal{E}_\nu$.

Then $\log d_x(x)$ is a subharmonic function in B."

In Ch. I, we construct such a sequence of closed set $\mathcal{E}_\nu$ by the assumption that $B \times \mathbb{C} - \mathcal{E}$ is a domain of holomorphy. By Oka's Theorem, a domain $\mathcal{D}$ in $\mathbb{C}^2$ is a domain of holomorphy if and only if $\mathcal{D}$ is a pseudoconvex domain in $\mathbb{C}^2$ (1942). Here we construct a sequence $\{\mathcal{E}_\nu\}$ by Theorem 4.3 (not using Oka's Theorem).

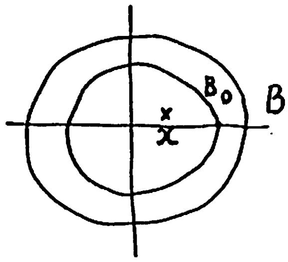
<u>Theorem 4.4</u> Let $\mathcal{E}$ be a closed set in $B \times \mathbb{C}$ such that $\mathcal{E}(x) \Subset (|y|<R)$ for all $x \in B$. Assume that $\mathcal{E}$ is pseudoconcave in $B \times \mathbb{C}$, i.e., $\mathcal{D} = B \times \mathbb{C} - \mathcal{E}$ is a pseudoconvex in $B \times \mathbb{C}$. Then $\log d_x(x)$ is subharmonic in B.

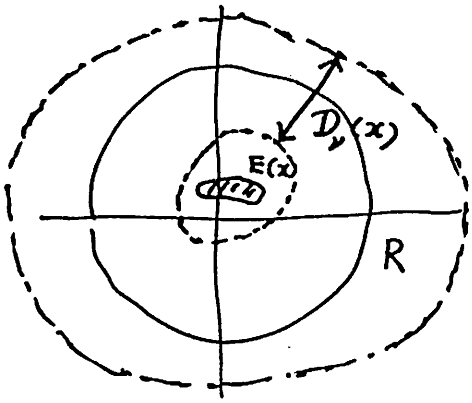
<u>Proof</u>. Let $B_0 \Subset B$. It suffices to construct a sequence $\{\mathcal{E}_\nu\}$ with (i), (ii), (iii). Since $\mathcal{D}$ is pseudoconvex, we have, by Theorem 4.4, a sequence of domains such that

(i') $B_0 \times (|y|=R) \Subset \mathcal{D}_1$;

(ii') $\mathcal{D}_1 \Subset \mathcal{D}_2 \Subset \cdots\cdots$; $\bigcup_{\nu=1}^{\infty} \mathcal{D}_\nu = \mathcal{D}$

(iii′) Each $(\mathfrak{z}, \eta) \in \partial\mathcal{D}_\nu$ admits a regular analytic set σ such that $(\mathfrak{z}, \eta) \in \sigma$ and $\sigma \subset \mathcal{D} - (\mathcal{D}_\nu \cup \partial\mathcal{D}_\nu)$ except for $(\mathfrak{z}, \eta)$.





Put $\hat{\mathcal{E}}_\nu = B_0 \times \mathbb{C} - \mathcal{D}_\nu$ and $\mathcal{E}_\nu = \hat{\mathcal{E}}_\nu \cap B_0 \times (|y| < R)$. Then (i′), (ii′) and (iii′) of $\mathcal{D}_\nu$ induces (i), (ii) and (iii) of $\mathcal{E}_\nu$, respectively. c.q.f.d.

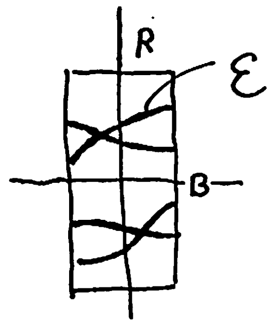
The following Corollary is a generalization of Hartogs' Theorem due to K. Oka (1934).

<u>Corollary 4.1</u> Under the same circumstances as in Theorem 4.4, let $K = \{x \in B \mid E(x) \text{ is a finite set}\}$. If K is of positive logarithmic capacity, then $\mathcal{E}$ must be a complete analytic set σ in $B \times (|y| < \infty)$, i.e.,

$$\mathcal{E} = \sigma = \{(x, y) \in B \times \mathbb{C} \mid y^n + a_1(x) y^{n-1} + \cdots + a_n(x)\}$$

where $1 \leq {}^\exists n < \infty$, $a_i(x)$ is holomorphic in B and "complete" means that

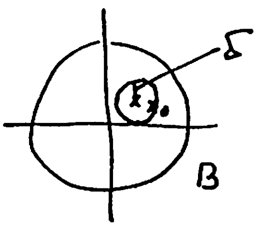
$$\sigma \cap [B \times (|y| \geq R)] = \phi.$$

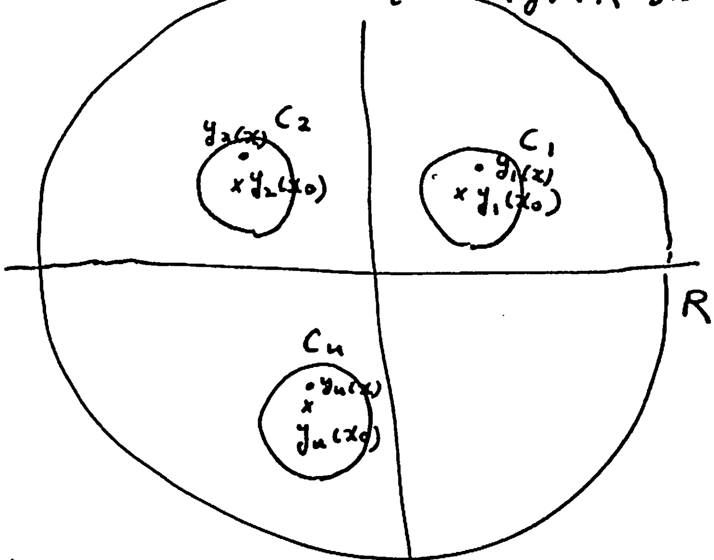


<u>Proof</u>. Let $K_n = \{x \in B \mid E(x) \text{ is at least } n \text{ points}\}$, and hence $K = \bigcup_{n=1}^{\infty} K_n$.

We thus find a unique integer $n \geq 1$ such that $c(K_n)$ (= the logarithmic capacity of K_n) is >0 and $c(K_{n-1})=0$. Since $d_{n+1}(x)=0$ for $x \in K_n$ and since $\log d_{n+1}(x)$ is subharmonic in B, it follows that $\log d_{n+1}(x) \equiv -\infty$ on B. Hence $d_{n+1}(x) \equiv 0$ on B, and $E_n = B$.

Put $E_n^* = B - E_{n-1}$. Let us prove that E_n^* is open in B, i.e., E_{n-1} is a closed set of logarithmic capacity zero. Fix $x_0 \in E_n^*$. Let $E_n^*(x_0) = \{ y_1(x_0), \cdots, y_n(x_0) \}$. About each $y_i(x_0)$ we draw a small circle C_i in $|y|<R$ such that $C_i \cap C_j = \phi \ (i \neq j)$.





Since $\mathcal{E}$ is closed, we find a $\delta > 0$ such that

$$(|x-x_0|<\delta) \times C_i \subset \mathcal{D} \quad (i=1, \cdots, n).$$

Since $\mathcal{D}$ is pseudoconvex in $\mathbb{C}^2$, the continuity theorem (A′) implies that, for $\forall x$ in $|x-x_0|<\delta$, there exists at least one $y_i(x)$ in the disk $[C_i]$ such that $(x, y_i(x)) \in \mathcal{E}$, i.e., $y_i(x) \in E(x)$ $(i=1, \cdots, n)$. Because $y_i(x) \neq y_j(x)$ $(i \neq j)$ and $E_n = B$, we get $E(x) = \{ y_1(x), \cdots, y_n(x) \}$ and hence $(|x-x_0|<\delta) \subset E_n^*$. E_n^* is thus open in B. Moreover, Hartogs Theorem 3.1 yields that each $y_i(x)$ is holomorphic in $|x-x_0|<\delta$.

For $x \in E_n^*$, we put $\mathcal{E}(x) = \{y_1(x), \ldots, y_n(x)\}$ and define

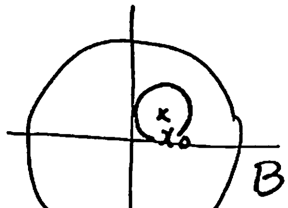
$$\begin{cases} a_1(x) = y_1(x) + \cdots + y_n(x) \\ \quad \vdots \qquad\qquad \vdots \\ a_n(x) = y_1(x) \cdots y_n(x) . \end{cases}$$

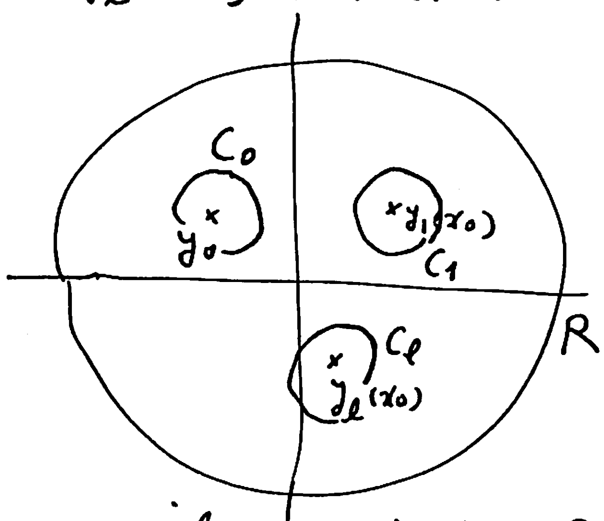
Then each $a_i(x)$ is a uniform holomorphic fn on the open $E_n^* = B - E_{n-1}$. Since $a_i(x)$ is bounded ($\because |y_j(x)| < R$) and since E_{n-1} is of logarithmic capacity zero, it follows that $a_i(x)$ is extended holomorphically on the whole B. Put

$$\mathcal{E}^* = \{(x,y) \in B \times \mathbb{C} \,/\, y^n + a_1(x) y^{n-1} + \cdots + a_n(x) = 0\}$$

Because $\mathcal{E}$ is closed in $B \times \mathbb{C}$, we have $\mathcal{E} \supset \mathcal{E}^*$. By construction of $\mathcal{E}^*$, we have $\mathcal{E}^*(x) = \mathcal{E}(x)$ for $x \in B - E_{n-1}$. It rests to prove $\mathcal{E}^*(x_0) \supseteq \mathcal{E}(x_0)$ for $^\forall x_0 \in E_{n-1}$. We shall prove this inclusion by contradiction.

Assume that there exists $y_0 \in \mathcal{E}(x_0) - \mathcal{E}^*(x_0)$. Let $\mathcal{E}^*(x_0) = \{y_1(x_0), \ldots, y_l(x_0)\}$ so that $l \leqq n-1$. About





each $y_i(x_0)$, we draw a circle C_i in $|y| < R$ such that

$C_i \cap C_j = \phi$ and that $y_0 \notin [C_1] \cup \cdots \cup [C_\ell]$, where $[C_j]$ denotes the disk bounded by C_j. If $\delta_0 > 0$ is very small, then, for any $x \in (|x - x_0| < \delta_0) - E_{n-1}$,

$$\mathcal{E}(x) = \mathcal{E}^*(x) \subset [C_1] \cup \cdots \cup [C_\ell];$$

$$\mathcal{E}(x) = \mathcal{E}^*(x) \text{ consists just } n \text{ distict points.}$$

We also draw a circle C_0 about y_0 in $B_0 - [C_1] \cup \cdots \cup [C_\ell]$. We apply the continuity theorem (A′) at $(x_0, y_0) \in \mathcal{E}$, and find a $\delta_1 > 0$ such that, for any $x \in (|x - x_0| < \delta_1)$, there exists at least one point $y(x)$ in $[C_0]$ such that $(x, y(x)) \in \mathcal{E}$. Therefore $\forall x \in (|x - x_0| < \min(\delta_0, \delta_1)) - E_{n-1}$ admits at least $n+1$ distincts points in $\mathcal{E}(x)$. This is a contradiction to $E_n = B$. c. q. f. d.

§5. Derived set.

Let $\mathcal{D}$ be a domain in $\mathbb{C}^2$, and $\mathcal{E}$ a closed set in $\mathcal{D}$. $\mathcal{E}$ is said to be pseudoconcave in $\mathcal{D}$, if, for any $(x_0, y_0) \in \partial\mathcal{E}$ in $\mathcal{D}$ and a ball S of center (x_0, y_0) such that $S \Subset \mathcal{D}$, $S - \mathcal{E}$ is pseudoconvex in $\mathbb{C}^2$.

Since (x_0, y_0) is arbitrary in $\partial\mathcal{E}$, it follows that for every pseudoconvex domain G in $\mathbb{C}^2$ such that $G \Subset \mathcal{D}$, $G - \mathcal{E}$ is pseudoconvesc in $\mathbb{C}^2$.

Remark 5.1 If any $(x_0, y_0) \in \partial\mathcal{E}$ admits an analytic set σ such that

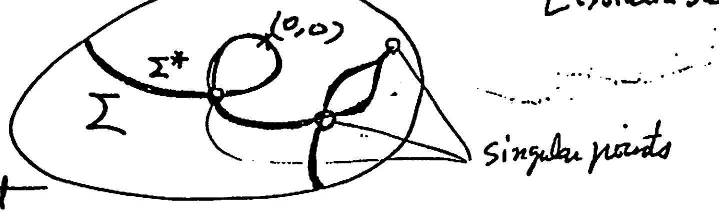
$$(x_0, y_0) \in \sigma \subset \mathcal{E},$$

then $\mathcal{E}$ is pseudoconcave in $\mathcal{D}$.

($\because$ By Remark 2.2 in a note of Nov. 23, '87)

<u>Remark 5.2</u> Let Σ be an irreducible analytic set in a domain $\mathcal{D}$, and assume that $\mathcal{E}$ is an non-empty pseudoconcave set in $\mathcal{D}$ such that $\mathcal{E} \subset \Sigma$. Then $\mathcal{E} = \Sigma$.

"irreducible" means that $\Sigma^* = \Sigma - \{\text{singular points}\}$ is connected. ↑ isolated set



<u>Proof.</u> Since $\mathcal{E}$ is closed in $\mathcal{D}$, $\mathcal{E} \cap \Sigma^*$ is closed in Σ^*. Next, let us show that $\mathcal{E} \cap \Sigma^*$ is open in Σ^*. In fact, let $(x_0, y_0) \in \mathcal{E} \cap \Sigma^*$. For the sake of convenience, we put $(x_0, y_0) = (0,0)$. Since Σ is regular at $(0,0)$, we may assume that

$$\Sigma : y = f(x) \quad \text{in } (|x| <^{\exists} \delta) \times (|y| <^{\exists} \rho) = \Delta \ (\Subset \mathcal{D})$$

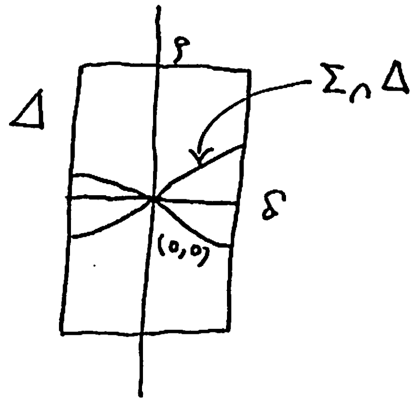
where $|f(x)| < \rho$ for $|x| < \delta$ and $f(x)$ is holomorphic in $|x| < \delta$. We have $\Sigma \cap \Delta = \Sigma^* \cap \Delta$. Since $\mathcal{E}$ is pseudoconcave in $\mathcal{D}$ and so in Δ, it follows from $\mathcal{E} \subset \Sigma$ that Continuity Theorem (A) implies

for $\forall x \in (|x| < \delta)$, there exists at least one $y(x)$ in $(|y| < \rho)$ such that $(x, y(x)) \in \mathcal{E}$.

Hence $y(x) \equiv f(x)$ for $(|x| < \delta)$, i.e., $\Sigma^* \cap \Delta \subset \mathcal{E} \cap \Delta$. Consequently, $\mathcal{E} \cap \Sigma^*$ is open in Σ^*.

Finally, we shall show $\mathcal{E} \cap \Sigma^* \neq \phi$. In fact, take $(x_0, y_0) \in \mathcal{E}$ in $\mathcal{D}$. We may assume $(x_0, y_0) = (0,0)$. By Weierstrass preparation theorem, there exists a bidisk $\Delta : (|x| < \delta) \times (|y| < \rho)$ such that

$$\Sigma \cap \Delta = \{(x,y) \in (|x| < \delta) \times \mathbb{C} \mid y^n + a_1(x) y^{n-1} + \cdots + a_n(x) = 0$$



Consequently, for any $x \in (|x|<\delta)$, we have

$\Sigma(x) \cap (|y|<\rho) = \{y_1(x), \dots, y_n(x)\}$;

$(0,0)$ is the only one singular point of Σ in Δ.

On the other hand, Continuity Theorem (A) for $\mathcal{E}$ at $(0,0)$ implies that there exists a $\delta_1>0$ such that, for any x with $|x|<\delta_1$, we find at least one $y(x)$ in $(|y|<\rho)$ with $(x, y(x)) \in \mathcal{E}$. Since $\mathcal{E} \subset \Sigma$, it follows that, for every $x \neq 0$ with $|x| < \min\{\delta, \delta_1\}$, $y(x) = y_i(x)$ for some $1 \leq i \leq n$. Consequently, $(x, y(x)) \in \mathcal{E} \cap \Sigma^*$, and $\mathcal{E} \cap \Sigma^* \neq \phi$.

Because Σ^* is connected, we see that $\mathcal{E} \cap \Sigma^* = \Sigma^*$, therefore $\mathcal{E} \supset \Sigma^*$. Since $\mathcal{E}$ is closed in $\mathcal{D}$, it follows that $\mathcal{E} \supset \Sigma$, and hence $\mathcal{E} = \Sigma$. c.q.f.d.

Let $\mathcal{E}$ be a closed set in a domain $\mathcal{D}$. A point (x_0, y_0) of $\mathcal{E}$ is said to be of the first kind if and only if there exist a neighborhood V of (x_0, y_0) and a holomorphic function $f(x,y)$ in V such that $f \not\equiv 0$, $f(x_0, y_0) = 0$ and $V \cap \mathcal{E} = \{(x,y) \in V \mid f(x,y) = 0\}$.

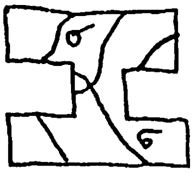
We denote by $\mathcal{E}_0$ the set of all the points of the first kind of $\mathcal{E}$, and put $\mathcal{E}' = \mathcal{E} - \mathcal{E}_0$. The set $\mathcal{E}'$ is called the derived set of $\mathcal{E}$. By definition $\mathcal{E}'$ is closed in $\mathcal{D}$.

<u>Theorem 5.1</u> (Oka, 1934)

If $\mathcal{E}$ is pseudoconcave in $\mathcal{D}$, then the derived set $\mathcal{E}'$

of $\mathcal{E}$ is pseudoconcave in $\mathcal{D}$.

Proof: (a) Let $(\zeta,\eta)\in\partial\mathcal{E}'$ in $\mathcal{D}$. Take a ball S about (ζ,η) such that $S\Subset\mathcal{D}$ and $S-\mathcal{E}$ is pseudoconvex in $\mathbb{C}$. For the sake of convenience, we write $(\zeta,\eta)=(0,0)$. Consider



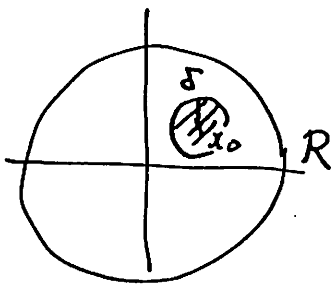
$$\Delta = (|x|<R)\times(|y|<R')\Subset S$$
$$\Delta_1 = (|x|<r)\times(|y|<R')$$
$$\Delta_2 = (|x|<R)\times(r'<|y|<R').$$

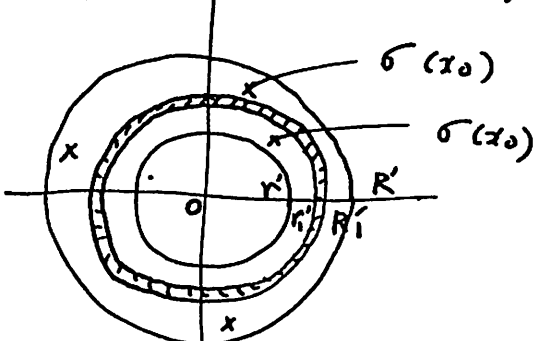
Assume that $\bar{\Delta}_1\cup\bar{\Delta}_2\subset S-\mathcal{E}'$. Then our claim is $\Delta\subset S-\mathcal{E}'$. By assumption, $(\Delta_1\cup\Delta_2)\cap\mathcal{E}$ is a finite number of analytic sets, which we denote by σ.

$$\sigma=(\Delta_1\cup\Delta_2)\cap\mathcal{E}.$$

We may assume that σ does not contain an analytic set of the form : $x=a$ (const.). For, if so, consider a linear transformation close to the identity mapping.

(i) Let $x_0\in(|x|<R)$. Then $\sigma(x_0)\cap(r'<|y|<R')$ consists of finite points, where $\sigma(x_0)=\{y\in\mathbb{C}\mid(x_0,y)\in\sigma\}$. Since σ is closed, there exist $\delta,\ r_1',\ R_1'>$



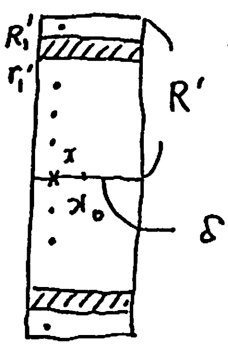


such that $(|x-x_0|<\delta)\times(r_1'<|y|<R_1')\cap\sigma=\phi$.

We can replace σ by $\mathcal{E}$ by definition of σ.

(ii) Put $F=\{x\in\mathbb{C}\mid |x|<R$ and $\mathcal{E}(x)\cap(|y|<R')$ is a finite set$\}$, and $O=\{x\in\mathbb{C}\mid |x|<R$ and, $(|x-x_0|<r)\cap F$ is of logarithmic capacity positive for every $r>0\}$

It is clear that O is closed in $|x|<R$. Moreover, $O \supset (|x|<r)$, because $\mathcal{E} \cap \bar{\Delta}_1$ is an analytic set in $\bar{\Delta}_1$. We shall prove that O is open in $|x|<R$. To prove this, fix $x_0 \in O$. By (i), we find $\delta, r_1', R_1' > 0$ (depending on x_0) such that

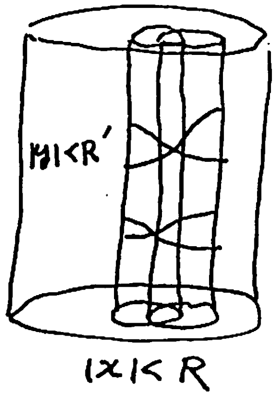


$$[(|x-x_0|<\delta) \times (r_1'<|y|<R_1')] \cap \mathcal{E} = \phi.$$

On the other hand, $\mathcal{E} \cap [(|x-x_0|<\delta) \times (|y|<R_1')]$ is pseudoconcave in $(|x-x_0|<\delta) \times (|y|<R')$. We put

$$K = \{ x \mid |x-x_0|<\delta \text{ and } \mathcal{E}(x) \cap (|y|<R_1') \text{ is a finite set} \}.$$

Then, $x_0 \in O$ implies that K is of logarithmic capacity positive. It follows from Corollary 4.1 that $\mathcal{E} \cap [(|x-x_0|<\delta) \times (|y|<R_1')]$ is a complete analytic set. Since $\mathcal{E} \cap [(|x|<R) \times (r'<|y|<R')]$ is analytic set and $r'<R_1'<R'$, it follows that $\mathcal{E} \cap [(|x-x_0|<\delta) \times (|y|<R')]$ is an analytic set in $(|x-x_0|<\delta) \times (|y|<R')$. Hence $O \supset (|x-x_0|<\delta)$, i.e., O is open in $|x|<R$.



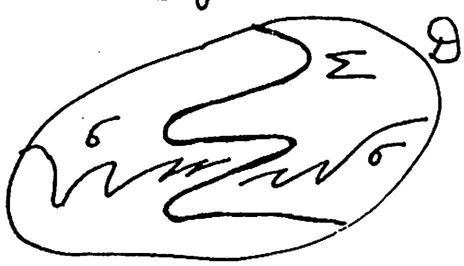
Consequently, $O = (|x|<R)$. It follows that Each $\mathcal{E}(x)$ for $x \in (|x|<R)$ is a finite set, and moreover that $\mathcal{E} \cap \Delta$ is an analytic set in Δ. We thus have $\Delta \subset \mathcal{S} - \mathcal{E}'$, that is, $(\zeta, \eta) \in \partial\mathcal{E}'$ satisfies Continuity Theorem (C).

(b) Consider any analytic transformation T of a nbd of $(\zeta, \eta) \in \partial\mathcal{E}'$ in $\mathcal{D}$: $V \longrightarrow \tilde{V}$. We put $(\tilde{\zeta}, \tilde{\eta}) = T(\zeta, \eta)$. Then it is clear that the set of the points of the first kind of $V \cap \mathcal{E}$ is transformed onto the set of the points of the first kind of $T(V \cap \mathcal{E})$ in $\tilde{V}$. It follows that $T(V \cap \mathcal{E}')$ satisfies Continuity Theorem (C) at $(\tilde{\zeta}, \tilde{\eta})$ by (a). Hence $\mathcal{E}'$ is pseudoconcave in $\mathcal{D}$. c.q.f.d.

The following Corollary is due to P. Thullen

<u>Corollary 5.1</u> Let Σ be an irreducible analytic set in $\mathcal{D}$. Assume that σ is an analytic set in $\mathcal{D}-\Sigma$. Put $\bar{\sigma}$ = the closure of σ in $\mathcal{D}$. Then either $\bar{\sigma}$ is an analytic set in $\mathcal{D}$ or $\bar{\sigma}-\sigma = \Sigma$.

<u>Proof.</u>



By Remark 5.1, $\Sigma \cup \bar{\sigma}$ is a pseudoconcave set in $\mathcal{D}$. Assume that $\bar{\sigma}$ is not analytic in $\mathcal{D}$. Then $(\Sigma \cup \bar{\sigma})$ is not analytic in $\mathcal{D}$, so that

$$(\Sigma \cup \bar{\sigma})' \neq \phi.$$

Moreover, since σ is analytic in $\mathcal{D}-\Sigma$, we have $(\Sigma \cup \bar{\sigma})' \subset \Sigma$.

It follows from Remark 5.2 that $(\Sigma \cup \bar{\sigma})' = \Sigma$. This easily yields that $\bar{\sigma} - \sigma = \Sigma$.

In fact, if not, we find a $(x_0, y_0) \in \Sigma$ but $(x_0, y_0) \notin \bar{\sigma} - \sigma$. Since $\bar{\sigma}$ is closed in Σ, there exists a nbd V of (x_0, y_0) in $\mathcal{D}$ such that $V \cap \bar{\sigma} = \phi$.

Thus, $V \cap (\Sigma \cup \bar{\sigma}) = V \cap \Sigma$: analytic set in V

$$\therefore \quad (\Sigma \cup \bar{\sigma})' \cap V = \phi$$

Consequently, $(\Sigma \cup \bar{\sigma})' \not\ni (x_0, y_0)$, which is a contradiction to $(\Sigma \cup \bar{\sigma})' = \Sigma$. c. q. f. d.

This proof was given by T. Nishino.

Dear

December 13, '87

This is a note for the lecture in Dec. 8. Lemma 2.2 is the principal result in the 1st semester.

Hiroshi Yamaguchi

Chap. IV Variations of Riemann surfaces

§1. Smooth variations.

Let B be a disk $|t|<\rho$ in the complex t-plane and $\mathbb{C}$ the complex z-plane. Consider an unramified covering domain $\mathcal{D}$ spreading over $B\times\mathbb{C}$. Precisely, we have the triple $(\mathcal{D}, \pi, B\times\mathbb{C})$ such that

$\mathcal{D}$: a connected Hausdorff Topological space

$\pi : \mathcal{D} \to B\times\mathbb{C}$ locally homeomorphic mapping (called „projection")

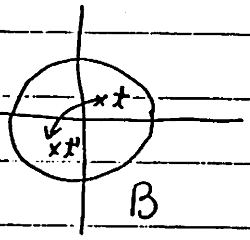
We denote by π_B and $\pi_{\mathbb{C}}$ the usual projection from $B\times\mathbb{C}$ to B and $\mathbb{C}$, respectively.

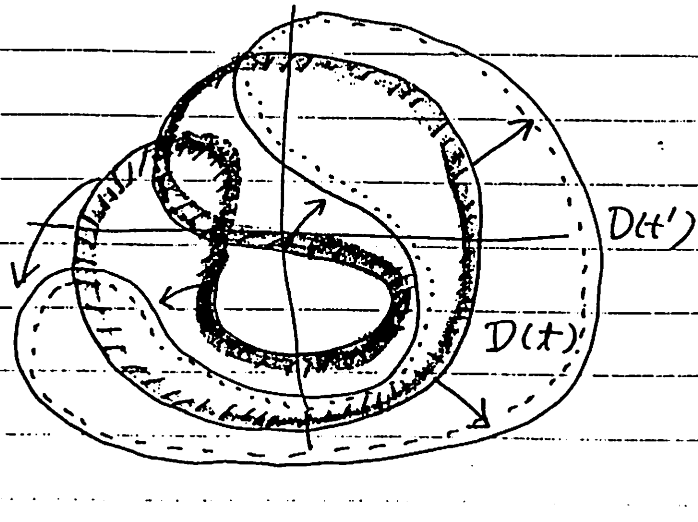
For each $t\in B$, we consider the set

$(\pi_B\circ\pi)^{-1}(t)$, which is called the fiber of $\mathcal{D}$ at t.

So $(\pi_B\circ\pi)^{-1}(t)$ is an unramified covering open set spreading over $\mathbb{C}$, precisely, each connected component of $((\pi_B\circ\pi)^{-1}(t), \pi_{\mathbb{C}}\circ\pi, \mathbb{C})$

determines the covering domain spreading over $\mathbb{C}$. For the sake of convenience, we write it $D(t)$.





$D(t)$ thus becomes a Riemann surface over $\mathbb{C}$ without branch points. As usual, we consider $\mathcal{D}$ as a variation of Riemann surfaces $D(t)$ with parameter $t \in B$.

$$\mathcal{D} : t \longrightarrow D(t) \quad (t \in B)$$

We sometimes use a convenient notation

$$\mathcal{D} = \bigcup_{t \in B} (t, D(t))$$

Assumption 1

There exists a point $\zeta \in \mathbb{C}$ such that $B \times \{\zeta\} \subset \mathcal{D}$, precisely there exists a constant holomorphic section $\zeta : B \to \mathcal{D}$ such that $\pi_B \circ \pi \circ \zeta(t) = t$ and $\pi_{\mathbb{C}} \circ \pi \circ \zeta(t) = \zeta$ (constant) for all $t \in B$. We simply note $(t, \zeta(t)) = (t, \zeta)$ for $t \in B$ and $\zeta(B) = B \times \{\zeta\}$.

Since $D(t) \ni \zeta$, we have the Green's function $g(t,z)$ and the Robin constant $\lambda(t)$ for $(D(t), \zeta)$. Hence in a neighborhood of ζ we have

$$(1.1) \quad g(t,z) = \log \frac{1}{|z-\zeta|} + \lambda(t) + h(t,z)$$

where $h(t,z)$ is harmonic for z, and

$$(1.2) \quad h(t,\zeta) = 0.$$

Assumption 2

There exist a domain $\tilde{\mathcal{D}}$ spreading over $B \times \mathbb{C}$ and a real-valued function $\varphi(t,z)$ of class C^∞ on $\tilde{\mathcal{D}}$ such that

(i) $\tilde{\mathcal{D}} \supset \mathcal{D}$, $\tilde{D}(t) \supset\supset D(t)$ for all $t \in B$.

We thus write $\partial \mathcal{D}$ the boundary of $\mathcal{D}$ in $\tilde{\mathcal{D}}$, and $\partial D(t)$ that of $D(t)$ in $\tilde{D}(t)$.

(ii) we have

$$\mathcal{D} = \{ (t,z) \in \tilde{\mathcal{D}} \mid \varphi(t,z) < 0 \};$$
$$\partial \mathcal{D} = \{ (t,z) \in \tilde{\mathcal{D}} \mid \varphi(t,z) = 0 \};$$
$$\tilde{\mathcal{D}} - \mathcal{D} = \{ (t,z) \in \tilde{\mathcal{D}} \mid \varphi(t,z) > 0 \}$$

and $\mathrm{grad}_{(t,z)}\, \varphi = \left(\frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial z} \right) \neq (0,0)$ anywhere on $\partial \mathcal{D}$

We say that $\partial \mathcal{D}$ is smooth in $\tilde{\mathcal{D}}$, and the double $(\tilde{\mathcal{D}}, \varphi)$ defines the domain $\mathcal{D}$.

Assumptión 3

For each $t \in B$, $\frac{\partial \varphi}{\partial z}(t,z) \neq 0$ anywhere on $\partial D(t)$.

We say that $\partial D(t)$ is smooth in $\tilde{D}(t)$, and the double $(\tilde{D}(t), \varphi(t,\cdot))$ defines the domain $D(t)$.

Therefore our variatión $\mathcal{D} : t \to D(t)$ $(t \in B)$ is a smooth variatión of smooth domains with parameter t.

Under these conditións, we see, without difficulty, that

(a) the Green's functión $g(t,z)$ is of class C^4 in $\mathcal{D} - (B \times \{\zeta\})$, and the difference $g(t,z) - \log \frac{1}{|z-\zeta|}$ $(= \lambda(t) + h(t,z))$ is of class C^4 in a neighborhood of $B \times \{\zeta\}$ in $\mathcal{D}$.

It follows that $\lambda(t)$ is of class C^4 in B and $h(t,z)$ is of class C^4 in a neighborhood of $B \times \{\zeta\}$ in $\mathcal{D}$. By (1.1) and (1.2) we get

$$\begin{cases} \dfrac{\partial g(t,z)}{\partial t} = \dfrac{\partial \lambda(t)}{\partial t} + \dfrac{\partial h(t,z)}{\partial t} ; \\ \dfrac{\partial h(t,\zeta)}{\partial t} = 0 \end{cases}$$

Hence, if we put

$$u_1(t,z) = \begin{cases} \dfrac{\partial g(t,z)}{\partial t} & \text{in } \mathcal{D} - B \times \{\zeta\} \\ \dfrac{\partial \lambda(t)}{\partial t} & \text{on } B \times \{\zeta\} \end{cases},$$

then $u_1(t,z)$ is of class C^3 in the whole domain $\mathcal{D}$, and, for each $t \in B$, $u_1(t,z)$ is a harmonic function for z on $D(t)$ such that $u_1(t,\zeta) = \partial\lambda(t)/\partial t$.

Analogously, if we put

$$u_2(t,z) = \begin{cases} \dfrac{\partial^2 g(t,z)}{\partial t \partial \bar{z}} & \text{in } \mathcal{D} - B \times \{\zeta\}, \\ \dfrac{\partial^2 \lambda(t)}{\partial t \partial \bar{z}} & \text{on } B \times \{\zeta\} \end{cases},$$

~~then $u_2(t,z)$ is of class C^2 in $\mathcal{D}$, and, for fixed $t \in B$, $u_2(t,z)$ is harmonic for z in $D(t)$ with $u_2(t,\zeta) = \dfrac{\partial^2 \lambda(\zeta)}{\partial t \partial \bar{z}}$.~~

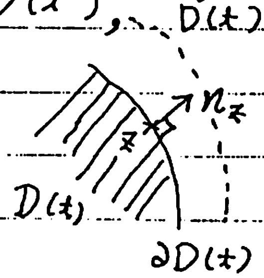
(b) $g(t,z)$ is extendable of class C^4 with respect to (t,z) beyond $\partial\mathcal{D}$.

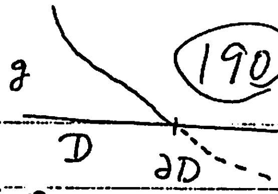
Hence, given $t \in B$, the functions $u_1(t,z)$ and $u_2(t,z)$ are continuous

As already noted in Chap. II, (b) implies that (for z on $D(t) \cup \partial D(t)$)

$$\frac{\partial g(t,z)}{\partial z} \neq 0 \text{ anywhere on } \partial D(t),$$

and the outer normal derivative of $g(t,z)$ at $z \in \partial D(t)$ is given





$$\frac{\partial g(t,z)}{\partial n_z} = -\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2} = -2\left|\frac{\partial g(t,z)}{\partial z}\right| \lneq 0 .$$

Consequently, we may assume that $g(t,z)$ is defined of class C^∞ in $\tilde{\mathcal{D}}$ except the pole $B\times\{\zeta\}$, and satisfies

$$\mathcal{D} = \{(t,z)\in\tilde{\mathcal{D}} \mid g(t,z)>0\} \ :$$
$$\partial\mathcal{D} = \{(t,z)\in\tilde{\mathcal{D}} \mid g(t,z)=0\}$$
$$\tilde{\mathcal{D}}-\mathcal{D} = \{(t,z)\in\tilde{\mathcal{D}} \mid g(t,z)<0\}$$

and $\dfrac{\partial g(t,z)}{\partial z} \neq 0$ anywhere on $\partial D(t)$ $(t\in B)$.

Under these circumstances we have

<u>Lemma 1.1</u> <u>(Hadamard's Variation Formulas)</u>

$$\frac{\partial\lambda(t)}{\partial t} = -\frac{1}{2\pi}\int_{\partial D(t)} \frac{\partial g(t,z)}{\partial t}\,\frac{\partial g(t,z)}{\partial n_z}\,ds_z \ ;$$

$$\frac{\partial^2\lambda(t)}{\partial t\partial\bar{t}} = -\frac{1}{2\pi}\int_{\partial D(t)} \frac{\partial^2 g(t,z)}{\partial t\partial\bar{t}}\,\frac{\partial g(t,z)}{\partial n_z}\,ds_z \ .$$

<u>Proof.</u> In general, let D be a domain over $\mathbb{C}$ with smooth boundary ∂D, and let $\zeta\in D$. Assume that $u(z)$ is a harmonic function in D and continuous on $D\cup\partial D$. Then

$$u(\zeta) = \frac{-1}{2\pi}\int_{\partial D} u(z)\,\frac{\partial g(z)}{\partial n_z}\,ds_z$$

where $g(z)$ denotes the Green's fn for (D,ζ). This is well known and was proved in Chap. II. Apply this equation for $D=D(t)$ and $u(z)=u_1(t,z)$ or $=u_2(t,z)$. Then we get two variation formulas in Lemma 1.1. c.q.f.d.

§2 Fundamental Formula

Under the same assumptions in §1, we got, in Chap I the following fact concerning the invariance of Levi's form.

Lemma 2.1

$$\frac{L\varphi}{\left|\frac{\partial\varphi}{\partial z}\right|^3} = \frac{L(-g)}{\left|\frac{\partial(-g)}{\partial z}\right|^3} \quad \text{on } \partial\mathcal{D}$$

where L = Levi's operator

$$= \frac{\partial^2}{\partial t\partial\bar{t}}\left|\frac{\partial}{\partial z}\right|^2 - 2\,\mathrm{Re}\left\{\frac{\partial^2}{\partial\bar{t}\partial z}\frac{\partial}{\partial t}\frac{\partial}{\partial\bar{z}}\right\} + \frac{\partial^2}{\partial z\partial\bar{z}}\left|\frac{\partial}{\partial t}\right|^2$$

We put

$$p(t,z) = \frac{L\varphi}{\left|\frac{\partial\varphi}{\partial z}\right|^3} \quad \text{for } (t,z)\in\partial\mathcal{D}.$$

This quantity which is real number is determined by the figure of the 3-dimensional surface $\partial\mathcal{D}$ at (t,z), i.e. it does not depend on the choice of the defining function $\varphi(t,z)$ of $\mathcal{D}$.

Lemma 2.2 (Fundamental Formula)

$$\frac{\partial^2\lambda(t)}{\partial t\partial\bar{t}} = -\frac{1}{\pi}\int_{\partial D(t)} p(t,z)\left|\frac{\partial g}{\partial z}\right|^2 ds_z - \frac{4}{\pi}\iint_{D(t)}\left|\frac{\partial^2 g}{\partial\bar{t}\partial z}\right|^2 dx\,dy.$$

Proof. Fix $t\in B$. By Lemma 2.1, we have

$$\frac{\frac{\partial^2 g}{\partial t\partial\bar{t}}\left|\frac{\partial g}{\partial z}\right|^2 - 2\,\mathrm{Re}\left\{\frac{\partial^2 g}{\partial\bar{t}\partial z}\frac{\partial g}{\partial\bar{z}}\frac{\partial g}{\partial t}\right\} + \frac{\partial^2 g}{\partial z\partial\bar{z}}\left|\frac{\partial g}{\partial t}\right|^2}{\left|\frac{\partial g}{\partial z}\right|^3} = -p(t,z) \quad \text{for } z\in\partial D(t)$$

Since $g(t,z)$ is harmonic for z in $D(t)$ and is of class C^4 beyond $\partial D(t)$, we have $\partial^2 g(t,z)/\partial z \partial \bar z = 0$ on $\partial D(t)$. Hence

$$\frac{\partial^2 g}{\partial t \partial \bar t} = \left(- p(t,z) \left|\frac{\partial g}{\partial z}\right|^3 + 2 \mathrm{Re} \left\{ \frac{\partial^2 g}{\partial \bar t \partial z} \frac{\partial g}{\partial \bar z} \frac{\partial g}{\partial t} \right\} \right) \Big/ \left|\frac{\partial g}{\partial z}\right|$$

$$= - p(t,z) \left|\frac{\partial g}{\partial z}\right| + 2 \mathrm{Re} \left\{ \frac{\frac{\partial^2 g}{\partial \bar t \partial z} \frac{\partial g}{\partial t}}{\frac{\partial g}{\partial z}} \right\} \qquad \text{for } z \in \partial D(t)$$

By Lemma 1.1, we thus get

$$\frac{\partial^2 \lambda(t)}{\partial t \partial \bar t} = \frac{-1}{2\pi} \int_{\partial D(t)} \frac{\partial^2 g}{\partial t \partial \bar t} \frac{\partial g}{\partial n_z} ds_z$$

$$= \frac{1}{2\pi} \int_{\partial D(t)} p(t,z) \left|\frac{\partial g}{\partial z}\right| \frac{\partial g}{\partial n_z} ds_z - \frac{1}{\pi} \mathrm{Re} \left\{ \int_{\partial D(t)} \frac{\frac{\partial^2 g}{\partial \bar t \partial z} \frac{\partial g}{\partial t}}{\frac{\partial g}{\partial z}} \frac{\partial g}{\partial n_z} ds_z \right\}$$

Since $\frac{\partial g}{\partial n_z} = -2 \left|\frac{\partial g}{\partial z}\right|$ on $\partial D(t)$ and $\frac{\partial g}{\partial n_z} ds_z = \frac{2}{i} \frac{\partial g}{\partial z} dz$

along $\partial D(t)$, it follows that

$$\frac{\partial^2 \lambda(t)}{\partial t \partial \bar t} = -\frac{1}{\pi} \int_{\partial D(t)} p(t,z) \left|\frac{\partial g}{\partial z}\right|^2 ds_z - \frac{2}{\pi} \mathrm{Im} \left\{ \int_{\partial D(t)} \frac{\partial^2 g}{\partial \bar t \partial z} \frac{\partial g}{\partial t} dz \right\}.$$

We look at the Integral : $\displaystyle\int_{\partial D(t)} \frac{\partial^2 g}{\partial \bar t \partial z} \frac{\partial g}{\partial t} dz$. Because $\frac{\partial g}{\partial t}$ is of class C^3 w.r.t z on $D(t) \cup \partial D(t)$, we have, by Stokes formula,

$$\int_{\partial D(t)} \frac{\partial^2 g}{\partial \bar t \partial z} \frac{\partial g}{\partial t} dz = \iint_{D(t)} d \left(\frac{\partial^2 g}{\partial \bar t \partial z} \frac{\partial g}{\partial t} dz \right)$$

$$= \iint_{D(t)} \left[\frac{\partial}{\partial \bar z} \left(\frac{\partial^2 g}{\partial \bar t \partial z} \frac{\partial g}{\partial t} \right) \right] d\bar z \wedge dz$$

$$= \iint_{D(t)} \left[\frac{\partial^3 g}{\partial \bar{t}\partial z \partial \bar{z}} \frac{\partial g}{\partial t} + \frac{\partial^2 g}{\partial \bar{t}\partial z} \frac{\partial^2 g}{\partial t \partial \bar{z}} \right] d\bar{z} \wedge dz$$

Since $g(t,z)$ is harmonic for $z \in D(t) - \{\zeta\}$, $\partial g/\partial t$ is harmonic for z in $D(t)$. Hence it becomes

$$= \iint_{D(t)} \left| \frac{\partial^2 g}{\partial \bar{t} \partial z} \right|^2 d\bar{z} \wedge dz = 2i \iint_{D(t)} \left| \frac{\partial^2 g}{\partial \bar{t}\partial z} \right|^2 dx\,dy$$

It turns out

$$\frac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} = \frac{-1}{\pi} \int_{\partial D(t)} p(t,z) \left| \frac{\partial g}{\partial z} \right|^2 ds_z - \frac{4}{\pi} \iint_{D(t)} \left| \frac{\partial^2 g}{\partial \bar{t} \partial z} \right|^2 dx\,dy.$$

c.q.f.d.

Here we impose the function-theoretic condition (C) on our variation $\mathcal{D}: t \to D(t)$ $(t \in B)$.

Condition (C) $\mathcal{D}$ is pseudoconvex over $B \times \mathbb{C}$. Precisely, For any $(\zeta, \eta) \in \partial\mathcal{D}$ and an open ball $S \Subset \tilde{\mathcal{D}}$ with center (ζ, η), $S \cap \mathcal{D}$ is pseudoconvex in $\mathbb{C}^2$.

This condition is well-defined and does not depend on the choice of S or of $\tilde{\mathcal{D}}$. For, the pseudoconvexity in $\mathbb{C}$ is a local property.

Theorem 1.1 If $\mathcal{D}$ is pseudoconvex over $B \times \mathbb{C}$, then

(1) $\lambda(t)$ is superharmonic for t in B;

(2) If $\lambda(t)$ is harmonic in B, then $\mathcal{D}$ is Levi flat i.e., $L\varphi \equiv 0$ on $\partial\mathcal{D}$ and $\frac{\partial g}{\partial z}(t,z)$ is holomorphic with respect to (t,z) in $\mathcal{D} - B \times \{\zeta\}$.

Proof. (1) If $\mathcal{D}$ is pseudoconvex in $B \times \mathbb{C}$, then $L\varphi \geq 0$ on $\partial\mathcal{D}$ (as proved in Ch. III) and hence $p(t,z) \geq 0$ on $\partial\mathcal{D}$. Lemma 2.2 automatically induces $\partial^2\lambda(t)/\partial t\partial\bar{t} \leq 0$ on B, i.e., $\lambda(t)$ is superharmonic in B.

(2) Assume that $\lambda(t)$ is harmonic for $t \in B$, i.e., $\partial^2\lambda/\partial t\partial\bar{t} \equiv 0$ on B. Since $p(t,z) \geq 0$ on $\partial\mathcal{D}$, Lemma 2.2 implies that, for each $t \in B$,

$$p(t,z) \equiv 0 \text{ on } \partial D(t);$$

$$\frac{\partial^2 g}{\partial\bar{t}\partial z}(t,z) \equiv 0 \text{ on } D(t)$$

Hence $p(t,z) \equiv 0$ on $\partial\mathcal{D}$ and $\frac{\partial^2 g}{\partial\bar{t}\partial z}(t,z) \equiv 0$ on $\mathcal{D}$. It follows that $L\varphi \equiv 0$ on $\partial\mathcal{D}$, and $(\partial g/\partial z)(t,z)$ is holomorphic with respect to (t,z) in $\mathcal{D} - B \times \{\zeta\}$ i.e., $\partial^2 g/\partial\bar{t}\partial z \equiv \partial^2 g/\partial\bar{z}\partial z \equiv 0$ in $\mathcal{D} - B \times \{\zeta\}$. c.q.f.d.

We show an application of Lemma 2.2. To do that, we recall the Bergman's kernel function for the domain D, which was studied in Ch. II.

Let D be a domain over $\mathbb{C}$ with smooth boundary ∂D. Let $\mathcal{L}^2 = \{ f(z) \mid$ holomorphic fn in D such that

$$\|f\|_D^2 = \iint_D |f|^2(z)\,dx\,dy < \infty \}.$$

Fix $\zeta \in D$. Then the linear functional on $\mathcal{L}^2$:

$$f \longrightarrow f(\zeta) \in \mathbb{C}$$

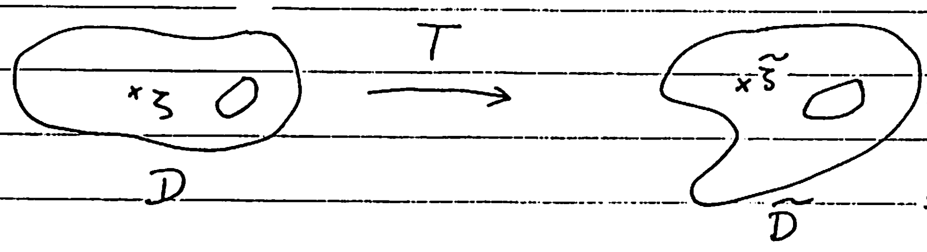
is continuous, so that there exists a unique $K(z,\zeta) \in \mathcal{L}^2$ such that

$$f(\zeta) = \iint_D f(z)\,\overline{K(z,\zeta)}\,dx\,dy.$$

$K(z,\zeta)$ is called the Bergman's kernel function for D. By definition we have

$$K(\zeta,\zeta) = \iint_D |K(z,\zeta)|^2\,dx\,dy \underset{\neq}{\geq} 0 \quad \text{for } \zeta \in D.$$

It is clear that $K(\zeta,\zeta)\,|d\zeta|^2$ is invariant under the holomorphic transformation of D, that is,

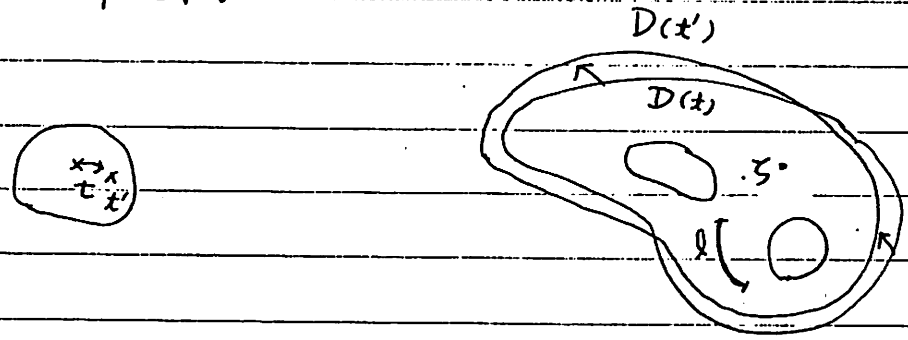


$\tilde{\zeta} = T\zeta$

$$K_D(\zeta,\zeta) = K_{\tilde{D}}(\tilde{\zeta},\tilde{\zeta}) \left| \frac{d\tilde{\zeta}}{d\zeta} \right|^2 \quad \text{where } \tilde{\zeta} = T\zeta \text{ in } D$$

$ds^2 = K(\zeta,\bar\zeta)|d\zeta|^2$ is called the Bergman metric on D.

We return to the variation $\mathcal{D}: t \to D(t)$ $(t \in B)$ which satisfies Assumptions 2 and 3 in §1, i.e., $\mathcal{D}$ is a smooth variation of smooth domains $D(t)$ with parameter. Each $D(t)$ thus defines the Bergman metric $ds_t^2 = K_t(\zeta,\bar\zeta)|d\zeta|^2$.



Corollary 2.1

If $\mathcal{D}$ is pseudoconvex in $B \times \mathbb{C}$, then $\log K_t(\zeta,\bar\zeta)$ is subharmonic for $t \in B$.

This Corollary easily yield the following interesting fact: Let l be a differentiable arc in $D(t)$. Put

$$L(t) = \int ds_t = \int_l \sqrt{K_t(\zeta,\bar\zeta)}\,|d\zeta|$$

Then $\log L(t)$ is l subharmonic for t.

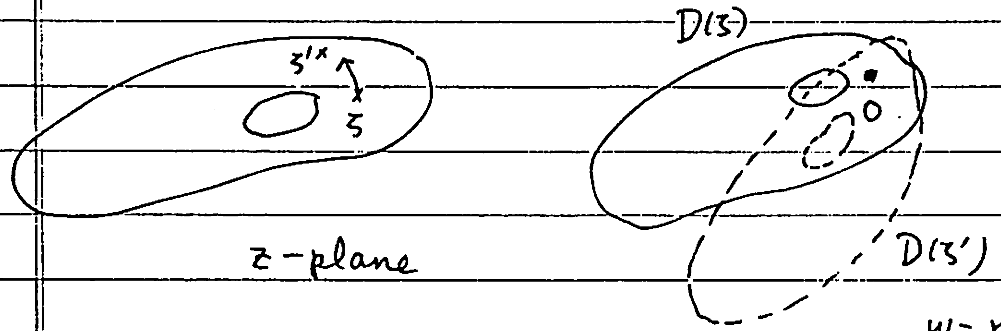
To prove Corollary 2.1, we recall the following formula due to M. Schiffer:

$$(2.1) \qquad K(z,\bar\zeta) = \frac{2i}{\pi}\,\frac{\partial^2 g(z,\zeta)}{\partial z\,\partial\bar\zeta}$$

where $g(z,\zeta)$ is the Green's function for (D,ζ).

$$(2.2)\quad K(\zeta,\zeta) = \frac{2i}{\pi}\left[\frac{\partial^2 g(z,\zeta)}{\partial z\,\partial\bar\zeta}\right]_{z=\zeta}$$
$$= \frac{4}{\pi^2}\iint_D \left|\frac{\partial^2 g(z,\zeta)}{\partial z\,\partial\bar\zeta}\right|^2 dx\,dy$$

Given $\zeta\in D$, we consider the translation $T_\zeta : z \longrightarrow w = z-\zeta$ from D onto $D(\zeta) = T_\zeta(D)$. Since $\zeta\in D$, each $D(\zeta)$ contains the origin O. We thus have the variation



$$\mathcal{D} : t \longrightarrow D(\zeta)\ (\zeta\in D).$$

Each $D(\zeta)$ carries the Green's fn $G(\zeta,w)$ and the Robin constant $\Lambda(\zeta)$ with respect to the origin O, so that

$$G(\zeta,w) = \log\frac{1}{|w|} + \Lambda(\zeta) + H(\zeta,w)$$

near the origin O. It is clear that

$$G(\zeta,w) = g(\zeta,z)\ ;\ H(\zeta,w) = h(z,\zeta)$$

where $w = z-\zeta$, and hence that

$$\Lambda(\zeta) = \lambda(\zeta).$$

Now assume that $\varphi(z)=0$ defines the boundary ∂D of D. Then $\varphi(w+\zeta)=0$ defines the boundary $\partial\mathcal{D}$ of $\mathcal{D} = \bigcup_{\zeta\in D}(\zeta, D(\zeta))$. We put $\psi(w,\zeta) = \varphi(w+\zeta)$ and calculate the Levi form $L\psi$ on $\partial\mathcal{D}$. Then we get

$$L\psi = \frac{\partial^2\varphi}{\partial z\partial\bar z}\left|\frac{\partial\varphi}{\partial z}\right|^2\{1-2+1\} \equiv 0$$

That is, $\mathcal{D}$ is Levi flat. Consequently, Lemma 2.2 implies that

$$\frac{\partial^2\Lambda(\zeta)}{\partial\zeta\partial\bar\zeta} = -\frac{4}{\pi}\iint_{D(\zeta)}\left|\frac{\partial^2 G(\zeta,w)}{\partial\bar\zeta\partial w}\right|^2 du\,dv$$

where $w = u+iv$.

and hence that

$$(2.3)\quad \frac{\partial^2\lambda(\zeta)}{\partial\zeta\partial\bar\zeta} = -\frac{4}{\pi}\iint_D\left|\frac{\partial^2 g(z,\zeta)}{\partial\bar\zeta\partial z}\right|^2 dx\,dy$$

$$= -\pi\, K(\zeta,\zeta) \qquad (\because (2.2))$$

This formula

$$(2.3')\quad K(\zeta,\zeta) = \frac{-1}{\pi}\frac{\partial^2\lambda(\zeta)}{\partial\zeta\partial\bar\zeta} \quad \text{for } \zeta\in D$$

was first introduced by N. Suita (1970).

<u>Proof of Corollary 2.1</u>. For each $t\in B$ and $\zeta\in D(t)$, we denote by $g(t,z,\zeta)$ and $\lambda(t,\zeta)$ the Green's function for $(D(t),\zeta)$ and the Robin constant for $(D(t),\zeta)$ respectively. It suffices to prove that

$$\frac{\partial^2}{\partial t\partial\bar t}\log K_t(\zeta,\zeta) \geq 0 \quad \text{for } t\in B \text{ and } \zeta\in D(t)$$

Or, by (2.3'), it suffices to prove

$$(2.4)\quad \frac{\partial^4\lambda(t,\zeta)}{\partial\zeta\partial\bar\zeta\partial t\partial\bar t}\cdot\frac{\partial^2\lambda(t,\zeta)}{\partial\zeta\partial\bar\zeta} \geq \left|\frac{\partial^3\lambda(t,\zeta)}{\partial\zeta\partial\bar\zeta\partial t}\right|^2$$

for $(t,\zeta)\in\mathcal{D}$.

By Fundamental formula in Lemma 2.2 we have

$$\frac{\partial^2\lambda(t,\zeta)}{\partial t\partial\bar t} = -\frac{1}{\pi}\int_{\partial D(t)} p(t,z)\left|\frac{\partial g(t,z,\zeta)}{\partial z}\right|^2 ds_z - \frac{4}{\pi}\iint_{D(t)}\left|\frac{\partial^2 g(t,z,\zeta)}{\partial z\partial\bar t}\right|^2 dx\,dy$$

Let us differentiate both sides by ζ and $\bar{\zeta}$.

$$\frac{\partial^3\lambda(t,\zeta)}{\partial\zeta\partial t\partial\bar{t}} = -\frac{1}{\pi}\int_{\partial D(t)} p(t,z)\left(\frac{\partial^2 g}{\partial\zeta\partial z}\frac{\partial g}{\partial\bar{z}} + \frac{\partial g}{\partial z}\frac{\partial^2 g}{\partial\zeta\partial\bar{z}}\right)ds_z$$

$$-\frac{4}{\pi}\iint_{D(t)}\left(\frac{\partial^3 g}{\partial\zeta\partial z\partial\bar{t}}\frac{\partial^2 g}{\partial t\partial\bar{z}} + \frac{\partial^2 g}{\partial z\partial\bar{t}}\frac{\partial^3 g}{\partial\zeta\partial t\partial\bar{z}}\right)dxdy$$

Observing $g(t,z,\zeta)$ is harmonic w.r.t both z and ζ (as noted in Chap. II), we get

$$\frac{\partial^4\lambda(t,\zeta)}{\partial\bar{\zeta}\partial\zeta\partial t\partial\bar{t}} = -\frac{1}{\pi}\int_{\partial D(t)} p(t,z)\left(\left|\frac{\partial^2 g}{\partial\zeta\partial z}\right|^2 + \left|\frac{\partial^2 g}{\partial\zeta\partial\bar{z}}\right|^2\right)ds_z$$

$$-\frac{4}{\pi}\iint_{D(t)}\left(\left|\frac{\partial^3 g}{\partial\zeta\partial z\partial\bar{t}}\right|^2 + \left|\frac{\partial^3 g}{\partial\zeta\partial t\partial\bar{z}}\right|^2\right)dxdy$$

Since $\mathcal{D}$ is pseudoconvex, $p(t,z) \geq 0$ on $\partial\mathcal{D}$, so that

$$(2.5)\quad \frac{\partial^4\lambda(t,\zeta)}{\partial\bar{\zeta}\partial\zeta\partial t\partial\bar{t}} \leq \frac{-4}{\pi}\iint_{D(t)}\left(\left|\frac{\partial^3 g}{\partial\zeta\partial z\partial\bar{t}}\right|^2 + \left|\frac{\partial^3 g}{\partial\zeta\partial t\partial\bar{z}}\right|^2\right)dxdy$$

By Lemma 1.1 (Hadamard's variation formula), we have

$$\frac{\partial\lambda(t,\zeta)}{\partial t} = \frac{-1}{2\pi}\int_{\partial D(t)}\frac{\partial g(t,z,\zeta)}{\partial t}\frac{\partial g(t,z,\zeta)}{\partial n_z}ds_z$$

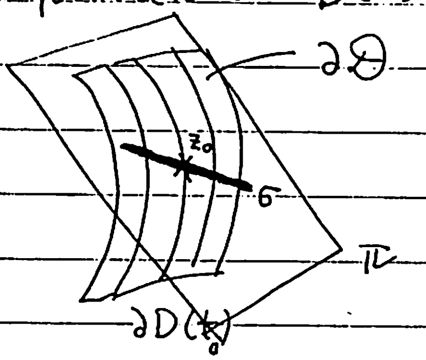
On $\partial D(t)$, we have $(\partial g/\partial n_z)ds_z = \frac{2}{i}\frac{\partial g}{\partial z}dz$, and

$$\frac{\partial\lambda(t,\zeta)}{\partial t} = \frac{-1}{\pi i}\int_{\partial D(t)}\frac{\partial g(t,z,\zeta)}{\partial t}\frac{\partial g(t,z,\zeta)}{\partial z}dz.$$

If we differentiate both sides with respect to ζ and $\bar{\zeta}$, then we get

$$(2.6)\quad \frac{\partial^3 \lambda(t,\zeta)}{\partial\bar{\zeta}\,\partial\zeta\,\partial t} = \frac{-1}{\pi i}\int_{\partial D(t)} \left\{ \frac{\partial^2 g(t,z,\zeta)}{\partial t\partial\zeta}\,\frac{\partial^2 g(t,z,\zeta)}{\partial\bar{\zeta}\partial z} + \frac{\partial^2 g(t,z,\zeta)}{\partial t\partial\bar{\zeta}}\,\frac{\partial^2 g(t,z,\zeta)}{\partial\zeta\partial z} \right\} dz$$

In the integrand, the function $\partial^2 g/\partial\zeta\partial z$ has the singularity $1/(z-\zeta)^3$ at $z=\zeta$. Therefore we cannot use the Green's formula in $D(t)$. We find the following technic:



Take $(t_0, z_0) \in \partial\mathcal{D}$. By definition of green's function

$$g(t, z, \zeta) \equiv 0$$

for all $t \in B$, $\zeta \in D(t)$ and $z \in \partial D(t)$.

$$\therefore\ \frac{\partial g}{\partial\zeta}(t, z, \zeta) = 0$$

for all $t \in B$, $\zeta \in D(t)$ and $z \in \partial D(t)$.

That means $\dfrac{\partial g}{\partial\zeta}(t,z,\zeta) \equiv 0$ on $\partial\mathcal{D}$.

Draw the tangent plane π to $\partial\mathcal{D}$ at (t_0, z_0), which contains only one analytic line σ : $z = z(t)$ passing through (t_0, z_0). Then, for any fixed $\zeta \in D(t_0)$

$$\frac{\partial g}{\partial\zeta}(t, z(t), \zeta) = O(|t-t_0|^2)$$

$$\therefore\ \left[\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial\zeta}(t, z(t), \zeta)\right)\right]_{t=t_0} = 0$$

$$\therefore\ \frac{\partial^2 g}{\partial t\partial\zeta}(t_0, z_0, \zeta) + \frac{\partial^2 g}{\partial z\partial\zeta}(t_0, z_0, \zeta)\cdot z'(t_0) = 0.$$

Analogously, we apply the same calculation to the function $\dfrac{\partial g}{\partial\bar{\zeta}}(t, z, \zeta)$ which is $\equiv 0$ on $\partial\mathcal{D}$, and have

$$\frac{\partial^2 g}{\partial t\partial\bar\zeta}(t_0,z_0,\zeta) + \frac{\partial^2 g}{\partial z\partial\bar\zeta}(t_0,z_0,\zeta)\, z'(t_0) = 0$$

It follows that

$$\frac{\partial^2 g}{\partial t\partial\bar\zeta}\,\frac{\partial^2 g}{\partial z\partial\zeta} = \frac{\partial^2 g}{\partial t\partial\zeta}\,\frac{\partial^2 g}{\partial z\partial\bar\zeta} \quad \text{at } (t_0,z_0,\zeta)$$

where $\forall (t_0,z_0)\in\partial\mathcal{D}$ and $\zeta\in D(t_0)$. Since t_0 is arbitrary in B, we substitute this equality for (2.6), and get

$$\frac{\partial^3\lambda(t,\zeta)}{\partial\bar\zeta\partial\zeta\partial t} = \frac{-2}{\pi i}\int_{\partial D(t)} \frac{\partial^2 g}{\partial t\partial\zeta}(t,z,\zeta)\cdot\frac{\partial^2 g}{\partial z\partial\bar\zeta}(t,z,\zeta)\,dz$$

Since the integrand is of class C^2 w.r.t z in $D(t)$, it follows from Green's formula that

$$= \frac{-2}{\pi i}\iint_{D(t)} \frac{\partial}{\partial\bar z}\left(\frac{\partial^2 g}{\partial t\partial\zeta}\cdot\frac{\partial^2 g}{\partial z\partial\bar\zeta}\right) d\bar z\wedge dz$$

Observing $\partial^3 g/\partial\bar z\partial z\partial\bar\zeta \equiv 0$ on $D(t)$, we see that

$$= \frac{-2}{\pi i}\iint_{D(t)}\left(\frac{\partial^3 g}{\partial t\partial\bar z\partial\zeta}\,\frac{\partial^2 g}{\partial z\partial\bar\zeta}\right)2i\,dxdy$$

$$\therefore\quad \frac{\partial^3\lambda(t,\zeta)}{\partial\bar\zeta\partial\zeta\partial t} = \frac{-4}{\pi}\iint_{D(t)}\frac{\partial^3 g}{\partial t\partial\bar z\partial\zeta}\,\frac{\partial^2 g}{\partial z\partial\bar\zeta}\,dxdy.$$

By (2.3), we have $\dfrac{\partial^2\lambda(t,\zeta)}{\partial\zeta\partial\bar\zeta} = -\dfrac{4}{\pi}\displaystyle\iint_{D(t)}\left|\frac{\partial^2 g(t,z,\zeta)}{\partial z\partial\bar\zeta}\right|^2 dxdy$

By Schwarz Inequality, we get from (2.5)

$$\left|\frac{\partial^3\lambda(t,\zeta)}{\partial\bar\zeta\partial\zeta\partial t}\right|^2 \le \frac{16}{\pi^2}\iint_{D(t)}\left|\frac{\partial^3 g}{\partial t\partial\bar z\partial\zeta}\right|^2dxdy\cdot\iint_{D(t)}\left|\frac{\partial^2 g}{\partial z\partial\bar\zeta}\right|^2dxdy$$

$$\le \left(\frac{\partial^4\lambda(t,\zeta)}{\partial\zeta\partial\bar\zeta\partial t\partial\bar t}\right)\cdot\left(\frac{\partial^2\lambda(t,\zeta)}{\partial\zeta\partial\bar\zeta}\right)$$

c. q. f. d.

Chapter V. Variations of domains over $\mathbb{C}^n$ $(n \geq 2)$

§1. Robin constants.

Let $\mathbb{R}^m$ be the real m-dimensional Euclidean space whose points are given by m real coordinates $x=(x_1, \cdots, x_m)$ with norm $\|x\|^2 = |x_1|^2 + \cdots + |x_m|^2$. We assume $m \geq 3$.

Let D be a domain of $\mathbb{R}^m$. Let $u(x)$ be a complex-valued function in D is said to be harmonic if u is of class C^2 and satisfies Laplace's Equation

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_m^2} = 0.$$

If $\Delta u \geq 0$ (resp. ≤ 0), u is said to be subharmonic (resp. superharmonic). It satisfies the minimum (resp. maximum) principle.

The most important harmonic function is

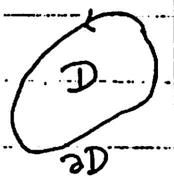
$$\frac{1}{\|x-a\|^{m-2}} \quad \text{in } \mathbb{R}^m - \{a\}.$$

Definition 1.1 A domain D in $\mathbb{R}^m$ is said to have smooth boundary, if there exist a domain $\tilde{D}$ of $\mathbb{R}^m$ and a real-valued function $\psi(x)$ of class C^∞ in $\tilde{D}$ such that

(a) $\tilde{D} \supset\supset D$ i.e., D is relatively compact in $\tilde{D}$;

(b) if we denote by ∂D the boundary of D in $\tilde{D}$ then

$$D = \{x \in \tilde{D} \mid \psi(x) < 0\};$$
$$\partial D = \{x \in \tilde{D} \mid \psi(x) = 0\}$$



(c) $\mathrm{Grad}_{(x)} \psi = \left(\frac{\partial \psi}{\partial x_1}, \cdots, \frac{\partial \psi}{\partial x_m}\right) \neq (0, \cdots, 0)$ for $x \in \partial D$.

∂D becomes a union of $(m-1)$ dimensional smooth surfaces. ∂D is positively oriented w.r.t. the domain D.

Now let ξ be a fixed point of D. According to potential theory, D carries the Green's function $g(x)$ with pole at ξ

which is uniquely determined by the following conditions:

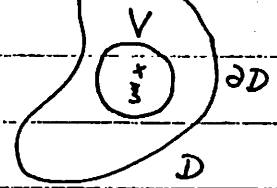
(i) $g(x)$ is harmonic in D except at ξ;

(ii) $g(x)$ is continuous up to ∂D and $g(x)=0$ on ∂D;

(iii) in a neighborhood of ξ, $g(x)$ differs from $1/\|x-\xi\|^{n-2}$ by a harmonic function.

From (iii) we write, for x in a neighborhood of ξ,

(1.1) $$g(x) = \frac{1}{\|x-\xi\|^{n-2}} + \lambda + h(x)$$



where λ is a constant, $h(x)$ is harmonic and $h(\xi)=0$. The constant term λ is called the Robin constant for (D, ξ).

By the maximum principle for harmonic functions we have

(1.2) $$\begin{cases} 0 < g(x) < \dfrac{1}{\|x-\xi\|^{n-2}} & \text{for } x \in D \\ -\infty < \lambda < 0 \end{cases}$$

∵ Consider the function $u(x) = \dfrac{1}{\|x-\xi\|^{n-2}} - g(x)$ in $D \cup \partial D$. Then $u(x)$ is harmonic in D, because (1.1) implies that $u(x)$ has no singularity at ξ. Moreover, $u(\xi) = -\lambda$. By (ii) $u(x) > 0$ on ∂D, it follows from the maximum principle that $u(x) > 0$ on D. Hence $1/\|x-\xi\|^{n-2} > g(x)$ on D. Moreover, $u(\xi) > 0$, or $\lambda < 0$. c.q.f.d.

Take a ball $V: \|x-\xi\| = r$ such that $V \subset\subset D$. Integrate both sides of (1.1) over the sphere $\partial V: \|x-\xi\| = r$, an we have

$$\int_{\partial V} g(x)\, dS_x = \int_{\partial V} \frac{1}{r^{n-2}}\, dS_x + \lambda \int_{\partial V} dS_x + \int_{\partial V} h(x)\, dS_x$$

where dS_x is the surface area element of ∂D at x. If we denote by ω_m the surface area of $(m-1)$-dimensional unit surface in R^m, then, since $\int_{\partial V} h(x)\, dS_x = r^{m-1}\omega_m h(\zeta) = 0$, we have

$$\int_{\partial V} g\, dS_x = \frac{1}{r^{m-2}} r^{m-1}\omega_m + \lambda r^{m-1}\omega_m$$

$$(1.3) \qquad \lambda = -\frac{1}{r^{m-2}} + \frac{1}{r^{m-1}\omega_m}\int_{\partial V} g(x)\, dS_x ,$$

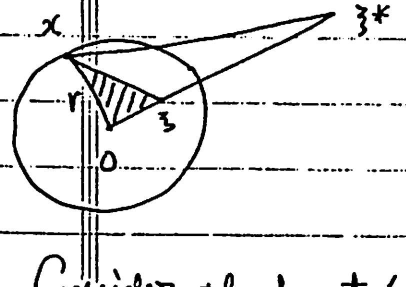
That is, λ is determined by the values of $g(x)$ on a surface centered at the pole ζ.

Example 1.1 Let V_r be the ball $\|x\| < r$ in R^m and let $\zeta \in V_r$. Then the Green's function $g_r(\zeta, x)$ and the Robin constant for (V_r, ζ) are given

$$(1.4) \qquad \begin{cases} g_r(\zeta, x) = \dfrac{1}{\|x-\zeta\|^{m-2}} - \left(\dfrac{r}{\|\zeta\|}\right)^{m-2} \dfrac{1}{\|x-\zeta^*\|^{m-2}} \\ \lambda_r(\zeta) = -\left(\dfrac{r}{r^2 - \|\zeta\|^2}\right)^{m-2} \end{cases}$$

where $\zeta^* = \left(\frac{r}{\|\zeta\|}\right)^2 \cdot \zeta$ is the symmetric point of ζ with respect to the sphere $\partial V_r : \|x\| = r$.

Proof: Since $\|\zeta\|\,\|\zeta^*\| = r^2$, we see, for $x \in \partial V_r$,



$$\Delta O\zeta x \backsim \Delta O x \zeta^*$$

$$\therefore \frac{\|\zeta\|}{\|x-\zeta\|} = \frac{\|x\|}{\|\zeta^* - x\|} \quad \left(= \frac{r}{\|\zeta^* - x\|}\right)$$

$$\therefore \frac{1}{\|x-\zeta\|} = \frac{r}{\|\zeta\|}\frac{1}{\|x-\zeta^*\|} \quad \text{for } x \in \partial V_r .$$

Consider the function $u(x)$ of x in V_r:

$$u(x) = \frac{1}{\|x-\zeta\|^{m-2}} - \left(\frac{r}{\|\zeta\|}\right)^{m-2} \frac{1}{\|x-\zeta^*\|^{m-2}} .$$

Then we have

(i) $u(x)$ is harmonic in $V_r - \{\zeta\}$; (ii) $u(x) = 0$ on ∂V_r as mentioned above; (iii) $u(x) - \frac{1}{\|x-\zeta\|^{m-2}} = -\left(\frac{r}{\|\zeta\|}\right)^{m-2} \frac{1}{\|x-\zeta^*\|^{m-2}}$ is regular at ζ. It follows that $u(x)$ is the green's fn $g_r(\zeta, x)$ for (V_r, ζ), i.e.,

$$g_r(\zeta, x) = \frac{1}{\|x-\zeta\|^{m-2}} - \left(\frac{r}{\|\zeta\|}\right)^{m-2} \frac{1}{\|x-\zeta^*\|^{m-2}}$$

Hence

$$\lambda_r(\zeta) = -\left(\frac{r}{\|\zeta\|}\right)^{m-2} \frac{1}{\|\zeta-\zeta^*\|^{m-2}} = -\left(\frac{r}{\|\zeta\|}\right)^{m-2} \frac{1}{\|\zeta - \left(\frac{r}{\|\zeta\|}\right)^2 \zeta\|^{m-2}}$$

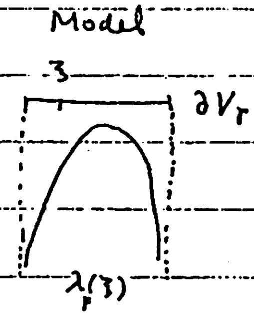
$$= -\left(\frac{r}{r^2 - \|\zeta\|^2}\right)^{m-2}$$

c.q.f.d.

It is clear from (1.4) that

For a fixed $\zeta \in V_r$, $\lim_{r\to\infty} \lambda_r(\zeta) = 0$;

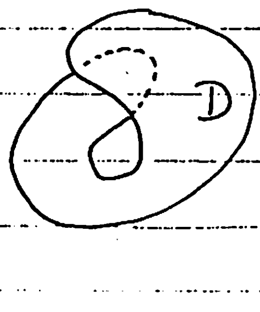
For a fixed $r < \infty$, $\lim_{\zeta\to\partial D} \lambda_r(\zeta) = -\infty$



We show two remarks here:

(1) As you know, λ depends on both the pole ζ and the domain D. Our purpose of this lectures is to study how λ varies when ζ and D vary. And we find that λ is deeply concerned with the theory of functions of several complex variables, especialy with pseudoconvex domains.

(2) For the sake of simplicity of description, we restrict ourselves to the case where D is a subdomain in $\mathbb{R}^m$. But our argument is available for the domain over $\mathbb{R}^m$, or over $\mathbb{C}^n$ without branch set, i.e., unramified covering domain over $\mathbb{R}^m$, or over $\mathbb{C}^n$



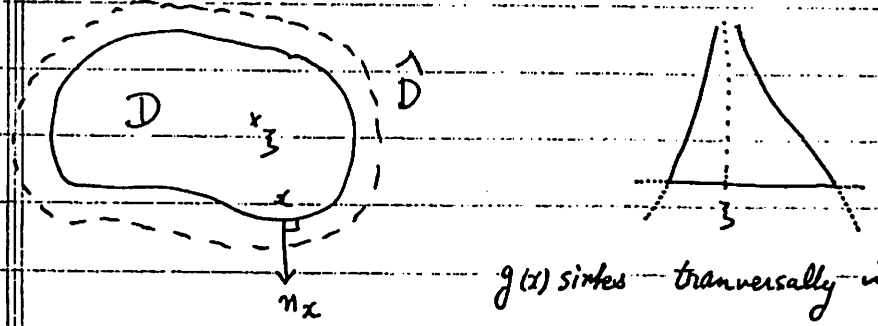
We state, without proofs, the elementary property of the Green's function which shall need.

Preliminary 1.1 Let D be a domain of R^m with smooth boundary ∂D. For $\zeta \in D$ we denote by g $(= g(\zeta, x))$ the Green's function for (D, ζ). Then g can be extended beyond ∂D to be a function of class C^4 in a domain $\hat{D}$ $(\supset D)$ such that, for every $x \in \partial D$,

$$(1.5) \qquad \frac{\partial g}{\partial n_x}(x) = -\sqrt{\left(\frac{\partial g}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial g}{\partial x_m}\right)^2} < 0,$$

where n_x denotes the unit outernormal vector to ∂D at x, and such that

$$(1.6) \qquad \begin{cases} D = \{x \in \hat{D} \mid g(x) > 0\}; \\ \partial D = \{x \in \hat{D} \mid g(x) = 0\} \end{cases}$$



$g(x)$ sinks tranversally in the ∂D like this.

This Preliminary is intuitively clear but a regorous proof need some argument.

Preliminary 1.2 Let D be a domain in R^m with smooth boundary ∂D. Then any harmonic function $u(x)$ in D continuous up to ∂D can be written in the form

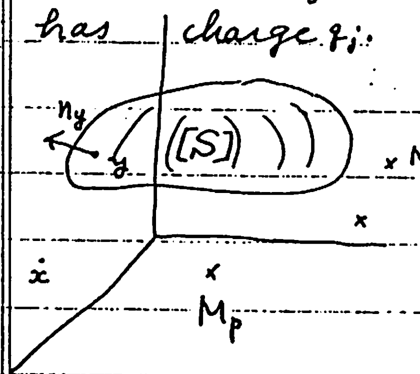
$$(1.7) \qquad u(\zeta) = \frac{-1}{(m-2)\,\omega_m} \int_{\partial D} u(x) \frac{\partial g(\zeta, x)}{\partial n_x} \, ds_x \quad \text{for } \zeta \in D.$$

For the special case where D is a ball V_r: $\|x - a\| < r$ we have

Poisson formula

$$(1.8) \qquad u(\zeta) = \frac{1}{r\omega_m} \int_{\partial V_r} u(x) \frac{r^2 - \|\zeta - a\|^2}{\|x - \zeta\|^{m-2}} dS_x .$$

Here let me explain briefly the Robin's paper (1886) and the meaning of Robin's constant. Let (S) be an electric conductor bounded by a smooth surface S and let $\{M_j\}_{j=1,\dots,p}$ be a finite number of charged particles. Let (S) be placed in R^3 and let M_j be placed outside of (S), x^j being the position coordinate of M_j. Assume that (S) has total charge q and that each M_j has charge q_j.



Since (S) is a conductor, the charge on (S) will be continuously redistributed on the surface S as a charge distribution

$$\mu : x \to \mu(x) \quad (x \in S)$$

in such a way that the electric field $e(x) = (e_1(x), e_2(x), e_3(x))$ in R^3 induced by μ is identically zero on $(S) - S$. If we construct the Newton potential

$$v(x) = \int_S \mu(y) \frac{1}{\|x - y\|} dS_y + \sum_{j=1}^{p} \frac{q_j}{\|x - x^j\|},$$

then Coulomb's law implies that

$$e(x) = \operatorname{Grad} v(x) \qquad \text{for } x \in R^3 - S \cup \{x^j\}_{j=1,\dots,p}$$

It follows that $v(x)$ is continuous in R^m and

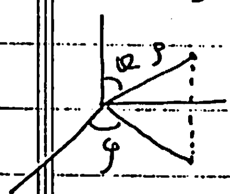
$$v(x) \equiv \text{constant, say } k, \text{ on } (S).$$

μ and $v(x)$ are called equilibrium distribution and potential.

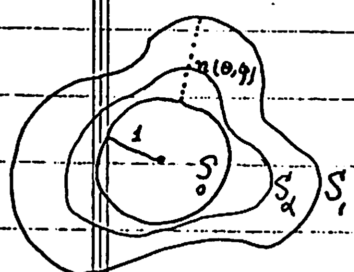
In the classical electromagnetism, this phenomena was known experimentally. It was a problem proved by Poisson (1811) to represent μ mathematically. Robin (1866) showed that μ must satisfy the following integral equation:

$$(1.9) \quad \mu(x) = \frac{1}{2\pi}\int_S \mu(y)\frac{\langle x-y, n_y\rangle}{\|x-y\|^2}\,dS_y + \frac{1}{2\pi}\sum_{j=1}^{p}\frac{q_j\langle x-x^j, n_x\rangle}{\|x-x^j\|^2}$$

for $x \in S$, where n_y is the unit outer normal vector to S at y, and $\langle a, b\rangle$ is the cosine of the angle between vectors a and b. Further, he showed that (1.9) could be solved in the following case: In terms of polar coordinates (ρ, θ, φ) of $\mathbb{R}^3$, consider a sphere: $\rho = 1$ and a surface: $\rho = 1 + n(\theta, \varphi)$ where $n(\theta,\varphi)$ is a real analytic function of (θ, φ).



Given $\alpha \geqq 0$, denote by S_α the surface $\rho = 1 + \alpha\, n(\theta, \varphi)$. Then there exists an $\alpha_0 > 0$ such that (1.9) is solvable for all S_α such that $0 \leqq \alpha \leqq \alpha_0$.



It is known that the solution of (1.9) for general S need the theory of Fredholm's integral equation developed in 1906.

From now on we restrict ourselves to the case where $q = 1$ and there are no particles M_j.

We thus have

$$(1.10) \quad v(x) = \int_S \mu(y)\frac{1}{\|x-y\|}\,dS_y \quad \text{and} \quad \int_S \mu(y)\,dS_y = 1.$$

We take it for granted that μ and $v(x)$ exist. $v(x)$ becomes a continuous function in $\mathbb{R}^3$ such that

(i) $v(x)$ reduces to a constant $k > 0$ on (S) ;

(ii) $v(x)$ is harmonic in $\mathbb{R}^3$ except for the surface S ;

(iii) $v(x) - \dfrac{1}{\|x\|} = v_1(x)$ where $|v_1(x)| \leq {}^{\exists}C \dfrac{1}{\|x\|^2}$ near $x = \infty$, where $C > 0$ is a constant depending on (S).

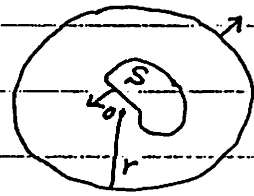
We set

$$\|e\|^2 = \iint_{\mathbb{R}^3} \|e(x)\|^2 dV_x = \iint_{\mathbb{R}^3} \left[\left(\frac{\partial v}{\partial x_1}\right)^2 + \left(\frac{\partial v}{\partial x_2}\right)^2 + \left(\frac{\partial v}{\partial x_3}\right)^2\right] dV$$

which is called the total energy of the electric field $e(x)$.

By Green's formula, we have

$$\|e\|^2 = k\,\omega_3$$



$$\|e\|^2 = \lim_{r \to +\infty} \int_{\|x\| = r} v \frac{\partial v}{\partial n_x} dS_x - \int_S v \frac{\partial v}{\partial n_x} dS_x$$

$$= \lim_{r \to \infty} \int_{\|x\| = r} \underbrace{\left(\frac{1}{r} + v_1(x)\right)}_{\approx\ \frac{1}{r}} \underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r} + v_1(x)\right)}_{\approx\ \frac{-1}{r^2}} r^2 d\theta d\varphi - k \int_S \frac{\partial v}{\partial n_x} ds_x$$

$$= -k \int_S \frac{\partial v}{\partial n_x} dS_x = -k \lim_{r \to \infty} \int_{\|x\| = r} \frac{\partial v}{\partial n_x} dS_x$$

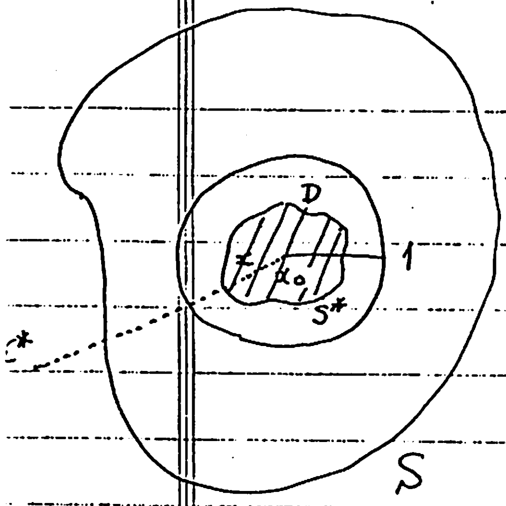
$$= -k \lim_{r \to \infty} \int_{\|x\| = r} \frac{\partial}{\partial r}\left(\frac{1}{r} + v_1(x)\right) \cdot r^2 d\theta d\varphi = k\,\omega_3 \quad \rfloor$$

In the electromagnetism, the reciprocal of k is an important quantity called the capacity c of the conductor (S):

$$c = \frac{1}{k} = \frac{\omega_3}{\|e\|^2}.$$

For $x_0 \in (S) - S$, we let π denote the sphere $\|x - x_0\| = 1$.

For $x \in \mathbb{R}^3 \cup \{\infty\}$, the symmetric point x^* of x with respect to π is



$$\begin{cases} x^* = x_0 + \dfrac{x - x_0}{\|x - x_0\|^2} & (x \neq x_0, \infty); \\ x_0^* = \infty \quad \text{and} \quad \infty^* = x_0 \end{cases}$$

Put $S^* = \{x^* \mid x \in S\}$;

$$D = \{x^* \mid x \in \mathbb{R}^3 \cup \{\infty\} - (S)\},$$

that is, D is the banded domain surrounded by the surface S^*.

Since $D \ni x_0 (= \infty^*)$, we have the green's function $g(x)$ and the Robin constant λ for (D, x_0). Under this notation, we have

Proposition 1.1 $\quad c = -\lambda$ and $\|e\|^2 = \dfrac{\omega_3}{-\lambda}$

Proof Let $v(x)$ be the equilibrium potential defined by (1.10), so that

$$v(x) \equiv k \ (>0) \text{ on } (S).$$

We restrict $v(x) - k$ (defined in $\mathbb{R}^3$) to the domain $\mathbb{R}^3 - (S)$ and consider the Kelvin's transformation of $v(x) - k$ with respect to the sphere π as follows:

$$K(x) = \frac{1}{\|x - x_0\|}(v(x^*) - k) \quad \text{for } x \in D.$$

Then $K(x)$ is harmonic! in $D - \{x_0\}$, continuous up to $\partial D = S^*$ and vanishes on ∂D by (i) and (ii). Moreover from (iii), we get near x_0

$$K(x) = \frac{1}{\|x - x_0\|}\left(\frac{1}{\|x^*\|} + v_1(x^*) - k\right)$$

Since $x^* = x_0 + \dfrac{x - x_0}{\|x - x_0\|^2}$, we have $\|x^*\| \cdot \|x - x_0\| \to 1$ as $x \to x_0$. It follows that, if we put $H(x) = \dfrac{1}{\|x - x_0\|} \cdot v_1(x^*)$, then

$$K(x) = 1 + H(x) - \frac{k}{\|x - x_0\|}$$

where $|H(x_0)| = \lim\limits_{x \to x_0} \left|\dfrac{1}{\|x - x_0\|} v_1(x^*)\right| \leq \lim\limits_{x \to x_0} C \dfrac{1}{\|x - x_0\| \, \|x^*\|^2} = 0.$

Or equivalently

$$\frac{K(x)}{-k} = \frac{1}{\|x - x_0\|} - \frac{1}{k} + H_1(x)$$

where $H_1(x_0) = H(x_0)/(-k) = 0$. It follows that

$$g(x) = K(x)/(-k)$$

and that $\lambda = -1/k$. Hence $\lambda = -1/(\|e\|^2/\omega_3) = -\omega_3/\|e\|^2$. Proposition 1.1 is proved.

We conclude that the Robin constant means minus of the capacity of the conductor !! Although $g(x)$ and hence λ are defined with respect to $x_0 \in (S) - S$, Proposition 1.1 says that the Robin constant λ does not depend on the choice of $x_0 \in (S) - S$.

§2. Smooth variations of smooth domains.

Let $\mathbb{C}^n$ be the complex n-dimensional Euclidean space whose points are given by n complex variables $z=(z_1,\cdots,z_n)$ with norm $\|z\|^2 = |z_1|^2+\cdots+|z_n|^2$.

We assume $n \geqq 2$.

We put $z_\alpha = x_{2\alpha-1} + i x_{2\alpha}$ $(1 \leqq \alpha \leqq n)$

$x_{2\alpha-1}$, $x_{2\alpha}$ are real.

$\therefore \mathbb{C}^n = \mathbb{R}^{2n}$.

Notation 2.1

$$\frac{\partial}{\partial z_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2\alpha-1}} - i\frac{\partial}{\partial x_{2\alpha}}\right); \quad \frac{\partial}{\partial \bar{z}_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2\alpha-1}} + i\frac{\partial}{\partial x_{2\alpha}}\right)$$

$$\mathrm{Grad}_{(z)} = \left(\frac{\partial}{\partial z_1},\cdots,\frac{\partial}{\partial z_\alpha}\right); \quad \overline{\mathrm{Grad}}_{(z)} = \left(\frac{\partial}{\partial \bar{z}_1},\cdots,\frac{\partial}{\partial \bar{z}_n}\right)$$

$$\mathrm{Grad}_{(x)} = \left(\frac{\partial}{\partial x_1},\cdots\cdots,\frac{\partial}{\partial x_{2n}}\right)$$

$$\Delta_{(z)} = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \cdots + \frac{\partial^2}{\partial z_n \partial \bar{z}_n}$$

$$\Delta_{(x)} = \frac{\partial^2}{\partial x_1^2} + \cdots\cdots + \frac{\partial^2}{\partial x_{2n}}$$

We note $\overline{\mathrm{Grad}}_{(z)} = \frac{1}{2}\mathrm{Grad}_{(x)}$:

$$\Delta_{(z)} = \frac{1}{4}\Delta_{(x)}.$$

Let B be a region in the complex t plane. Consider a domain $\mathcal{D}$ in the product space $B \times \mathbb{C}^n$. Given $t \in B$ we set

$$D(t) = \{ z \in \mathbb{C}^n \,/\, (t,z) \in \mathcal{D} \}$$

We call $D(t)$ the fiber of $\mathcal{D}$ at t. As usual we regard the domain $\mathcal{D}$ in $B \times \mathbb{C}^n$ as a variation of open sets $D(t)$ of $\mathbb{C}^n$ with complex parameter $t \in B$:

$$\mathcal{D} : t \to D(t) \quad (t \in B)$$

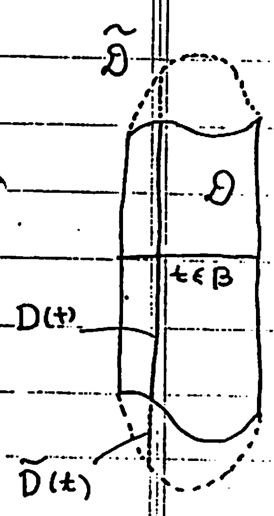
In this section we impose the following conditions on $\mathcal{D}$.

Condition 2.1 There exist another domain $\tilde{\mathcal{D}}$ in $B \times \mathbb{C}^n$ and a real-valued function $\psi(t,z)$ of class C^∞ in $\tilde{\mathcal{D}}$ which satisfy

(a) $\tilde{\mathcal{D}} \supset \mathcal{D}$ and $\tilde{D}(t) \supset\supset D(t) \neq \phi$ for each $t \in B$.
We denote by $\partial\mathcal{D}$ the boundary of $\mathcal{D}$ in $\tilde{\mathcal{D}}$, and $\partial D(t)$ the boundary of $D(t)$ in $\tilde{D}(t)$.

(b) $\mathcal{D} = \{(t,z) \in \tilde{\mathcal{D}} \mid \psi(t,z) < 0\}$;
$\partial\mathcal{D} = \{(t,z) \in \tilde{\mathcal{D}} \mid \psi(t,z) = 0\}$

(c) $\left(\frac{\partial\psi}{\partial t}, \frac{\partial\psi}{\partial z_1}, \dots, \frac{\partial\psi}{\partial z_n}\right)(t,z) \neq 0$ for all $(t,z) \in \partial\mathcal{D}$.

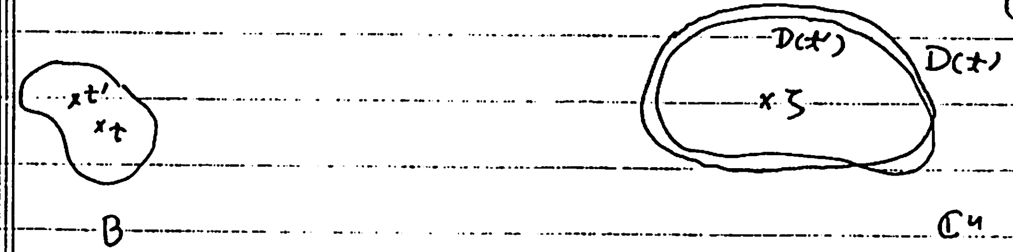


We say that $\mathcal{D}$ has smooth boundary in $B \times \mathbb{C}^n$, and that the double $(\tilde{\mathcal{D}}, \psi)$ defines the domain $\mathcal{D}$ with smooth boundary.

Condition 2.2 For each $t \in B$, the fiber $D(t)$ is connected and the double $(\tilde{D}(t), \psi(t,z))$ defines the domain $D(t)$ in the sense of Definition 1.1, namely,

$$\operatorname{Grad}_{(z)} \psi(t,z) = \left(\frac{\partial\psi}{\partial z_1}, \dots, \frac{\partial\psi}{\partial z_n}\right)(t,z) \neq (0,\dots,0)$$

for all $z \in \partial D(t)$.



We simply say that, if a domain $\mathcal{D}$ in $B \times \mathbb{C}^n$ satisfies conditions 2.1 and 2.2, then the variation $\mathcal{D}: t \to D(t)$ $(t \in B)$ is the smooth variation of smooth domains.

Condition 2.3: $\mathcal{D}$ has a constant section defined on B, namely, there exists a point $\zeta \in \mathbb{C}^n$ such that $B \times \{\zeta\} \subset \mathcal{D}$.

By condition 2.3, each $D(t)$ $(t \in B)$ carries the Green's function $g(t,z)$ with pole at ζ and the Robin constant $\lambda(t)$ for $(D(t), \zeta)$. Thus $g(t,z)$ can be written in a neighborhood of ζ in the form

$$(2.1) \quad g(t,z) = \frac{1}{\|z-\zeta\|^{2n-2}} + \lambda(t) + h(t,z)$$

where $h(t,z)$ is harmonic with respect to z, and

$$(2.2) \quad h(t,\zeta) = 0$$

Under condition 2.2, we see from Preliminary 1.1 that each $g(t,z)$ is extended to be of class C^4 beyond $\partial D(t)$. Under condition 2.1, the following Proposition is intuitively clear, but it needs some arguments for the precise. For a while we take it for granted.

Proposition 2.1 Given $t_0 \in B$, we can find a disk B_0 of center t_0 in B with the following property: putting

$$\mathcal{D}_{B_0} = \bigcup_{t \in B_0} (t, D(t)) \quad (= \mathcal{D} \cap (B_0 \times \mathbb{C}^n)),$$

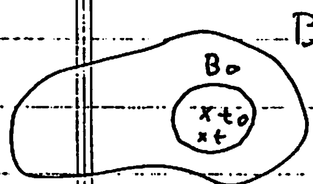
we have a domain $\hat{\mathcal{D}}$ of $B_0 \times \mathbb{C}^n$ and a function $\hat{g}(t,z)$ defined on $\hat{\mathcal{D}}$ which satisfy

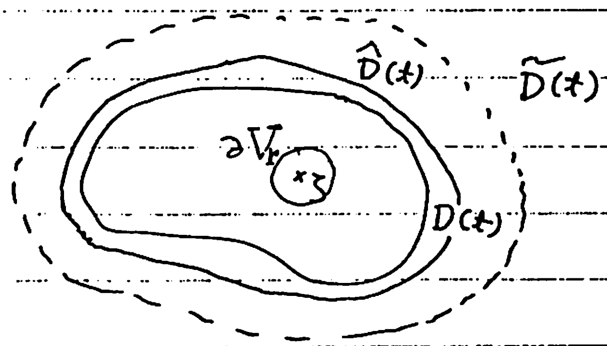
(1) $\mathcal{D}_{B_0} \subset \hat{\mathcal{D}} \subset \tilde{\mathcal{D}}_{B_0}$ and $D(t) \subset\subset \hat{D}(t) \subset \tilde{D}(t)$ $(t \in B)$;

(2) $\hat{g}(t,z)$ is of class C^4 with respect to (t,z) in $\hat{\mathcal{D}}$ except at the pole $B_0 \times \{\zeta\}$, and $\hat{g}(t,z) = g(t,z)$ in $\mathcal{D}_{B_0}$;

(3) for each $t \in B_0$, $\mathrm{Grad}_{(z)}\, \hat{g}(t,z) \neq 0$ for all $z \in \partial D(t)$;

(4) $\mathcal{D}_{B_0} = \{(t,z) \in \hat{\mathcal{D}} \mid \hat{g}(t,z) > 0\}$;
$\partial\mathcal{D}_{B_0} = \{(t,z) \in \hat{\mathcal{D}} \mid \hat{g}(t,z) = 0\}$





We will write $g(t,z)$ for $\hat{g}(t,z)$ in $\hat{\mathcal{D}}$.

We draw a sphere $\partial V_r : \|z - \zeta\| = r$ in $D(t)$ $(t \in B_0)$.

By (1.3) we have

$$\lambda(t) = -\frac{1}{r^{2n-2}} + \frac{1}{r^{2n-1}\,\omega_{2n}} \int_{\partial V_r} g(t,z)\, ds_z .$$

Proposition 2.1 (2) thus implies that

(2.3) $\lambda(t)$ is of class C^4 on B.

Since $h(t,z) = \dfrac{1}{r\,\omega_{2n}} \displaystyle\int_{\partial V_r} \left\{-\lambda(t) - \frac{1}{r^{2n-2}} + g(t,\xi)\right\} \frac{r^2 - \|z-\zeta\|^2}{\|\xi - \zeta\|^{2n-2}}\, ds_\xi$,

$h(t,z)$ is also of class C^4 with respect to (t,z). We can thus differentiate both sides of (2.1) and (2.2), and have

$$\frac{\partial g}{\partial t}(t,z) = \frac{\partial \lambda}{\partial t}(t) + \frac{\partial h}{\partial t}(t,z) \qquad (z \neq \zeta)$$

$$\frac{\partial h}{\partial t}(t,\zeta) = 0$$

If we put

$$u(t,z) \underset{\text{def}}{=} \begin{cases} \dfrac{\partial g}{\partial t}(t,z) & (z \neq \zeta) \\[2mm] \dfrac{\partial \lambda}{\partial t}(t) & (z = \zeta), \end{cases}$$

then, for each $t \in B$, $u(t,z)$ is a harmonic function of z in the whole domain $D(t)$ (although $g(t,z)$ has a singularity at ζ) and is continuous on $D(t) \cup \partial D(t)$. It follows from Proposition 1.2 that

$$u(t,\zeta) = \frac{-1}{(2n-2)\,\omega_{2n}} \int_{\partial D(t)} u(t,z)\, \frac{\partial g(t,z)}{\partial n_z}\, ds_z ,$$

namely

$$(2.4)\qquad \frac{\partial \lambda}{\partial t}(t) = \frac{-1}{(2n-2)\,\omega_{2n}} \int_{\partial D(t)} \frac{\partial g}{\partial t}(t,z)\, \frac{\partial g(t,z)}{\partial n_z}\, ds_z .$$

By the same method, we have the analogous representation of $\dfrac{\partial^2 \lambda}{\partial t \partial \bar t}$, $\dfrac{\partial^2 \lambda}{\partial t^2}$, ----

$$(2.5)\qquad \frac{\partial^2 \lambda}{\partial t \partial \bar t}(t) = \frac{-1}{(2n-2)\,\omega_{2n}} \int_{\partial D(t)} \frac{\partial^2 g(t,z)}{\partial t \partial \bar t}\, \frac{\partial g(t,z)}{\partial n_z}\, ds_z .$$

We thus get

<u>Proposition 2.2</u> If a domain $\mathcal{D}$ in $B \times \mathbb{C}^4$ satisfies

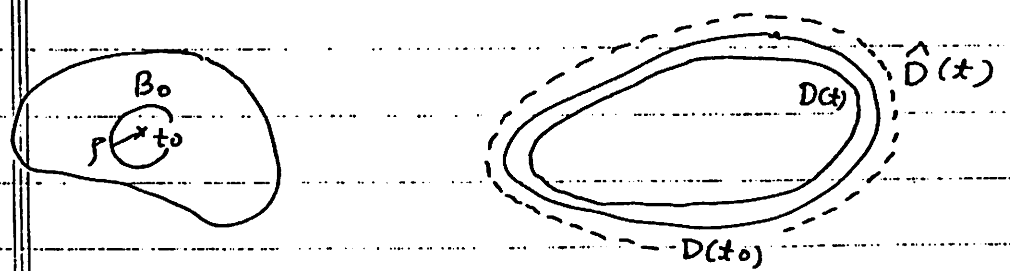
conditions 2.1, 2.2 and 2.3, then formulas (2.4) and (2.5) are valid.

Expression (2.4) is called Hadamard's variation formula.

Here let me explain two remarks about Hadamard's variation formulas, which is not directly concerned with later argument.

(i) The above proof of (2.4) seems somewhat artificial. In order to understand (2.4) better, we consider a more restrictive case where $\psi(t,z)$ of Condition 2.1 is real analytic with respect to (t,z) in $\tilde{D}$, and give an intuitive proof following Hadamard:

Let $t_0 \in B$. Since $\partial D(t)$ $(t \in B)$ is real analytic, $g(t,z)$ can be extended beyond $\partial D(t)$ to be real analytic and hence to be harmonic for z in $\hat{D}(t) - \{\zeta\}$ where $\hat{D}(t) \supset\supset D(t)$. Moreover, we can find a disk B_0



$: |t - t_0| < \rho$ such that $\hat{D}(t) \supset\supset D(t_0)$ for each $t \in B_0$. For each $t \in B_0$, we set

$$u(t,z) = g(t,z) - g(t_0,z) \quad \text{for } z \in \hat{D}(t_0) \cap \hat{D}(t).$$

Then, from (2.1), $u(t,z)$ is regular at ζ and assumes the value $\lambda(t) - \lambda(t_0)$ at ζ. Consequently, $u(t,z)$ is a harmonic function for z in a neighborhood of $D(t) \cup \partial D(t_0)$ with $u(\zeta) = \lambda(t) - \lambda(t_0)$. Applying Proposition 1.1 to $u(t,z)$

in $D(t_0)$, we obtain easily

$$\lambda(t)-\lambda(t_0) = \frac{-1}{2(n-1)\omega_{2n}} \int_{\partial D(t_0)} g(t,z) \frac{\partial g(t_0,z)}{\partial n_z} ds_z$$

for all $t \in B_0$. Differentiate both sides with respect to t and then put $t=t_0$. We get formula (2.4). Moreover differentiate both sides with respect to t and $\bar{t}$ and then put $t=t_0$. We get formula (2.5).

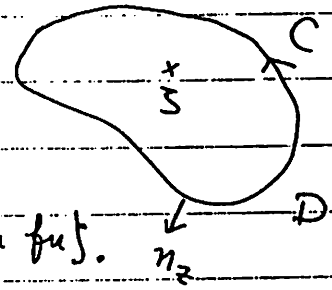
(ii) We briefly introduce the paper of Hadamard where his formula appeared:

title "Problème d'analyse relatif à l'équilibre des plaques élastiques encastrées" (1907)

We consider the biharmonic function $u(z)=u(x,y)$, that is, where $z=x+iy$.

$$\Delta\Delta u(x,y)=0 \text{ on } D$$

Given $\zeta \in D$, we have the Green's function $\Gamma_\zeta^D(z)$ of order 2 which is uniquely determined by the following conditions:



(a) $\Gamma_\zeta^D(z)$ is biharmonic except $D-\{\zeta\}$;

(b) $\Gamma_\zeta^D(z) = \frac{\partial\Gamma_\zeta(z)}{\partial n_z} = 0$ for $z\in\partial D$;

(c) $\Gamma_\zeta^D(z) = |z-\zeta|^2\log|z-\zeta| + \{\text{regular fn}\}$.

In the study of $\Delta\Delta u=0$, the following "Stoke's Formula" is fundamental: Let u, v be of class C^4 on $D\cup\partial D$. Then

$$\iint_D u(\Delta\Delta v - v\Delta\Delta u)dxdy = -\int_{\partial D} \left\{ \begin{array}{l} u\dfrac{\partial(\Delta v)}{\partial n} - \dfrac{\partial u}{\partial n}\Delta v \\ -v\dfrac{\partial(\Delta u)}{\partial n} + \dfrac{\partial v}{\partial n}\Delta u \end{array} \right\} ds.$$

Boundary value problem is solved:

Given $f_0(z)$ and $f_1(z)$ on $C = \partial D$, there exists a biharmonic function $u(z)$ on D such that

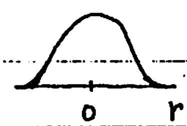
$$u = f_0 \quad \text{and} \quad \frac{\partial u}{\partial n_z} = f_1 \quad \text{on } \partial D.$$

Moreover u is given by

$$8\pi \; u(z) = \int_C \left(f_1(\zeta)\, \Delta \Gamma_z^D(\zeta) - f_0(\zeta) \frac{\partial \Delta \Gamma_z^D(\zeta)}{\partial n_\zeta} \right) ds_\zeta .$$

$\Gamma_\zeta^D(z)$ is called "the flexion" with respect to (D, ζ). Hadamard studied how $\Gamma^D(\zeta)$ varies when ζ and D vary. In the circle $D: |z| < r$, we have

$$\Gamma_\zeta^D(\zeta) = \frac{(r^2 - |\zeta|^2)^2}{2r^2}$$



He solved the following maximum problem:

Given the length l and the point ζ, find the domain D_0 with boundary length l and with $D_0 \ni \zeta$ such that

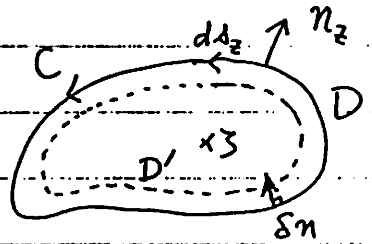
$$\Gamma_\zeta^{D_0}(\zeta) \geqq \Gamma_\zeta^D(\zeta)$$

for all D with boundary length l and with $D \ni \zeta$. He said that D_0 is a circle with center ζ.

To prove this, the variation formula

$$2\pi\, \delta \Gamma_\zeta^D(\zeta) = -\int_C (\Delta \Gamma_\zeta^D(z))^2 \,\delta n \, ds$$

plays an essential rôle.



For exemple, take $\delta n > 0$. Then D' is smaller than D
$D' \subset D$. The above formula is

$$2\pi\left(\Gamma_\zeta^{D'}(\zeta) - \Gamma_\zeta^{D}(\zeta)\right) = -\int_C \frac{\left(\Delta\Gamma_\zeta^{D}(\zeta)\right)^2}{V_0}\,\frac{\delta n}{V_0}\,ds$$

$\therefore$ this is negative!

Therefore the flexion $\Gamma_\zeta^D(\zeta)$ is an increasing function for the domain D. It is not trivial, because biharmonic function no longer satisfy the max- or min-imum principle.

§3. Levi form with respect to t and $z=(z_1,\cdots,z_n)$

Let $B\times\Omega$ be a bidisk in $\mathbb{C}^2$ of two complex variables t and z. Let $\psi(t,z)$ be a real valued function of class C^2 in $B\times\Omega$. We defined in Chap. IV

$$L_{(t,z)}\psi = \frac{\partial^2\psi}{\partial t\partial\bar t}\left|\frac{\partial\psi}{\partial z}\right|^2 - 2\,\mathrm{Re}\left\{\frac{\partial^2\psi}{\partial t\partial\bar z}\frac{\partial\psi}{\partial \bar t}\frac{\partial\psi}{\partial z}\right\} + \left|\frac{\partial\psi}{\partial t}\right|^2\frac{\partial^2\psi}{\partial z\partial\bar z}$$

in $B\times\Omega$, and call $L_{(t,z)}\psi$ the Levi form of ψ with respect to t and z.

Let $\varphi(t,z)$ be another real-valued function of class C^2 in $B\times\Omega$ such that

$$\varphi(t,z) = f(t,z)\,\psi(t,z)$$

where $f(t,z)$ is a real-valued function of class C^2 in $B\times\Omega$. Then the direct calcul lends us

$$(3.1)\qquad \frac{\partial\varphi}{\partial t} = f\,\frac{\partial\psi}{\partial t};$$

$$(3.2)\qquad L_{(t,z)}\varphi = f^3\,L_{(t,z)}\psi$$

for all $(t,z)\in B\times\Omega$ such that $\psi(t,z)=0$.

Next, let $B\times\Omega_1\times\cdots\times\Omega_n$ be a polydisk in the space $\mathbb{C}^{n+1}$ of $(n+1)$ complex variables $t, z_1,\cdots,z_n$. For a real-valued function $\psi(t,z_1,\cdots,z_n)$ of class C^2 in $B\times\Omega_1\times\cdots\times\Omega_n$, we define

$$\mathcal{L}\psi = L_{(t,z_1)}\psi + \cdots + L_{(t,z_n)}\psi$$

$$= \frac{\partial^2 \psi}{\partial t \partial \bar{t}} \|\mathrm{Grad}_{(z)} \psi\|^2 - 2\,\mathrm{Re}\left\{ \frac{\partial \psi}{\partial t} \sum_{\alpha=1}^{n} \frac{\partial^2 \psi}{\partial \bar{t} \partial z_\alpha} \frac{\partial \psi}{\partial \bar{z}_\alpha} \right\} + \left|\frac{\partial \psi}{\partial t}\right|^2 \Delta_{(z)} \psi$$

in $B \times \Omega_1 \times \cdots \times \Omega_n$, and call $\mathcal{L}\psi$ the Levi form of ψ with respect to t and $(z_1, \ldots, z_n)$.

As usual we simply set $z = (z_1, \ldots, z_n)$ and $\Omega = \Omega_1 \times \cdots \times \Omega_n$. Let $\varphi(t,z)$ be another real-valued function of class C^2 in $B \times \Omega$ such that

$$\varphi(t,z) = f(t,z)\, \psi(t,z)$$

where $f(t,z)$ is a real-valued function of class C^2 in $B \times \Omega$. Then it follows from (3.1) and (3.2) that

$$(3.3) \qquad \frac{\partial \varphi}{\partial t} = f \frac{\partial \psi}{\partial t};$$

$$(3.4) \qquad \mathcal{L}\varphi = f^3 \mathcal{L}\psi$$

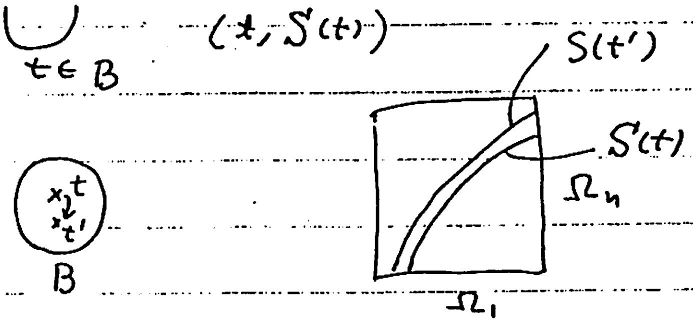
on $\psi = 0$ in $B \times \Omega$.

Let $\mathcal{S} = \{ (t,z) \in B \times \Omega \mid \psi(t,z) = 0 \}$;

$$S(t) = \{ z \in \Omega \mid (t,z) \in \mathcal{S} \} \text{ for } t \in B.$$

We call $S(t)$ the fiber of $\mathcal{S}$ at t. We thus have

$$\mathcal{S} = \bigcup_{t \in B} (t, S(t))$$



Assume that

$$(3.5) \qquad \mathrm{Grad}_{(z)} \psi \neq 0 \text{ for all } (t,z) \in \mathcal{S},$$

namely, each $S(t)$ is a $(2n-1)$-dimensional smooth surface in Ω. Moreover assume that

(3.6) $f(t,z) > 0$ on S.

Under these conditions, we have

Lemma 3.1 It holds, on S,

$$\frac{\frac{\partial\varphi}{\partial t}}{\|\mathrm{Grad}_{(z)}\varphi\|} = \frac{\frac{\partial\psi}{\partial t}}{\|\mathrm{Grad}_{(z)}\psi\|};$$

$$\frac{\mathcal{L}\varphi}{\|\mathrm{Grad}_{(z)}\varphi\|^3} = \frac{\mathcal{L}\psi}{\|\mathrm{Grad}_{(z)}\psi\|^3}$$

Proof. Since $\varphi(t,z) = f(t,z)\,\psi(t,z)$ in $B\times\Omega$, it follows that

$$\left(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial z_1}, \dots, \frac{\partial\varphi}{\partial z_n}\right) = f\left(\frac{\partial\psi}{\partial t}, \frac{\partial\psi}{\partial z_1}, \dots, \frac{\partial\psi}{\partial z_n}\right) \text{ on } S.$$

Hence $\|\mathrm{Grad}_{(z)}\varphi\| = |f|\,\|\mathrm{Grad}_{(z)}\psi\|$ on S

By assumptions (3.5) and (3.6) we have

$$f = \frac{\|\mathrm{Grad}_{(z)}\varphi\|}{\|\mathrm{Grad}_{(z)}\psi\|} \quad \text{on } S$$

It follows from (3.3) and (3.4) that we have, on S,

$$\frac{\partial\varphi}{\partial t}\Big/\|\mathrm{Grad}_{(z)}\varphi\| = \frac{\partial\psi}{\partial t}\Big/\|\mathrm{Grad}_{(z)}\psi\|;$$

$$\mathcal{L}\varphi/\|\mathrm{Grad}_{(z)}\varphi\|^3 = \mathcal{L}\psi/\|\mathrm{Grad}_{(z)}\psi\|^3. \qquad \text{c.q.f.d.}$$

From Lemma 3.1, the quantities $(\partial\psi/\partial t)/\|\mathrm{Grad}_{(z)}\psi\|$ and $\mathcal{L}\psi/\|\mathrm{Grad}_{(z)}\psi\|^3$ do not depend on the choice of $\psi(t,z)$, i.e. they are determined by the figure of the smooth surface S in $B\times\Omega$. We thus set, on S,

$$(3.7)\qquad k_1(t,z) = \frac{\frac{\partial\psi}{\partial t}}{\|\mathrm{Grad}_{(z)}\psi\|}$$

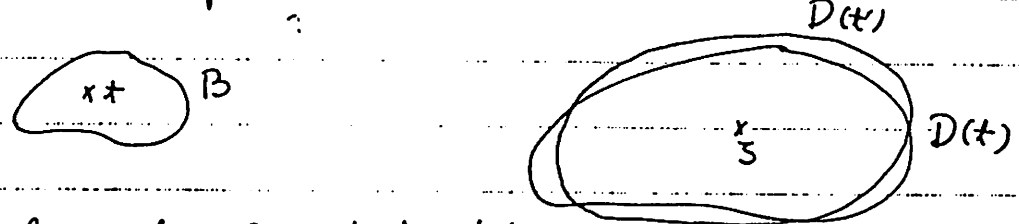
$$(3.8)\qquad k_2(t,z) = \frac{\mathcal{L}\psi}{\|\mathrm{Grad}_{(z)}\psi\|^3}.$$

§4 Variation formulas for smooth variations

Let $\mathcal{D}$ be a domain in $B \times \mathbb{C}^n$ which satisfies conditions 3.1, 3.2 and 3.3, i.e., the variation

$$\mathcal{D} : t \to D(t) \quad (t \in B)$$

is a smooth variation of smooth domains $D(t)$ with parameter $t \in B$ such that there exists a point $\zeta \in \mathbb{C}^n$ with $D(t) \ni \zeta$ for all $t \in B$.



We have the Green's function $g(t, z)$ and the Robin constant $\lambda(t)$ for $(D(t), \zeta)$. Under these conditions we get

Theorem 4.1 (Fundamental formula)

$$(4.1) \quad \frac{\partial \lambda(t)}{\partial t} = \frac{-1}{(n-1)\omega_{2n}} \int_{\partial D(t)} k_1(t,z) \, \| \mathrm{Grad}_{(z)} \, g(t,z) \|^2 \, ds_z$$

$$(4.2) \quad \frac{\partial^2 \lambda(t)}{\partial \bar{t} \partial t} = \frac{-1}{(n-1)\omega_{2n}} \int_{\partial D(t)} k_2(t,z) \, \| \mathrm{Grad}_{(z)} \, g(t,z) \|^2 \, ds_z - \frac{4}{(n-1)\omega_{2n}} \iint_{D(t)} \left\{ \sum_{\alpha=1}^{n} \left| \frac{\partial^2 g(t,z)}{\partial t \partial \bar{z}_\alpha} \right|^2 \right\} dV.$$

where dV is the Euclidean volume element.

Proof. We get by (2.4) and (2.5)

$$(4.3)\quad \frac{\partial \lambda(t)}{\partial t} = \frac{-1}{(2n-2)\omega_{2n}} \int_{\partial D(t)} \frac{\partial g(t,z)}{\partial t} \frac{\partial g(t,z)}{\partial n_z} ds_z$$

$$(4.4)\quad \frac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} = \frac{-1}{(2n-2)\omega_{2n}} \int_{\partial D(t)} \frac{\partial^2 g(t,z)}{\partial t \partial \bar{t}} \frac{\partial g(t,z)}{\partial n_z} ds_z .$$

Let $(\tilde{\mathcal{D}}, \Psi)$ be a double which defines the domain $\mathcal{D}$ with smooth boundary $\partial\mathcal{D}$ in $B \times \mathbb{C}^4$. Since $g = 0$ on $\partial\mathcal{D}$, it follows from Proposition 2.1 that

$$-g(t,z) = f(t,z)\,\Psi(t,z) \quad \text{near } \partial\mathcal{D} \text{ in } \tilde{\mathcal{D}}$$

where $f(t,z)$ is a positive real-valued function of class C^4 near $\partial\mathcal{D}$ in $\tilde{\mathcal{D}}$. By Lemma 3.1 we get

$$k_1(t,z) = \frac{-\partial g(t,z)/\partial t}{\|\mathrm{Grad}_{(z)}\, g\|} ;$$

$$k_2(t,z) = \frac{-\mathcal{L}\, g(t,z)}{\|\mathrm{Grad}_{(z)}\, g\|^3}$$

for $(t,z) \in \partial\mathcal{D}$. Since $g(t,z)$ is harmonic for z in $D(t)$ except at ζ, and is continuous up to $\partial D(t)$, we have

$$\mathcal{L}\, g(t,z) = \frac{\partial^2 g}{\partial t \partial \bar{t}} \|\mathrm{Grad}_{(z)}\, g\|^2 - 2\,\mathrm{Re}\left\{ \frac{\partial g}{\partial t} \sum_{\alpha=1}^{n} \frac{\partial^2 g}{\partial \bar{t} \partial z_\alpha} \frac{\partial g}{\partial \bar{z}_\alpha} \right\}$$

on $\partial\mathcal{D}$. It follows that

$$\frac{\partial g}{\partial t} = -k_1 \|\mathrm{Grad}_{(z)}\, g\| ;$$

$$\frac{\partial^2 g}{\partial t \partial \bar{t}} = -k_2 \|\mathrm{Grad}_{(z)}\, g\| + \frac{2\,\mathrm{Re}\left\{ \frac{\partial g}{\partial t} \sum_{\alpha=1}^{n} \frac{\partial^2 g}{\partial \bar{t} \partial z_\alpha} \frac{\partial g}{\partial \bar{z}_\alpha} \right\}}{\|\mathrm{Grad}_{(z)}\, g\|^2}$$

on $\partial\mathcal{D}$. On the other hand, we have from Preliminary 1.1

$$\frac{\partial g(t,z)}{\partial n_z} = -2\,\|\mathrm{Grad}_{(z)}\, g(t,z)\| \quad \text{on } \partial D(t).$$

We substitute those into (4.3) and (4.4), and obtain

$$\frac{\partial \lambda(t)}{\partial t} = \frac{-1}{(n-1)\,\omega_{2n}} \int_{\partial D(t)} k_1(t,z)\,\|\mathrm{Grad}_{(z)}\, g(t,z)\|^2 \, ds_z \,;$$

$$\frac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} = \frac{-1}{(n-1)\,\omega_{2n}} \int_{\partial D(t)} k_2(t,z)\,\|\mathrm{Grad}_{(z)}\, g(t,z)\|^2 \, ds_z$$

$$+ \frac{2}{(n-1)\,\omega_{2n}} \mathrm{Re}\left\{ \sum_{\alpha=1}^{n} \int_{\partial D(t)} \frac{\frac{\partial g}{\partial t}\frac{\partial^2 g}{\partial \bar{t}\partial z_\alpha}\frac{\partial g}{\partial \bar{z}_\alpha}}{\|\mathrm{Grad}_{(z)}\, g\|} \, ds_z \right\}.$$

On the other hand, as we show in the next Proposition, it holds

(4.5) $$\frac{\partial g}{\partial \bar{z}_\alpha}\, dS_z = \frac{i^n}{2^{n-1}}\,\|\mathrm{Grad}_{(z)}\, g\|\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_\alpha \wedge \widehat{d\bar{z}_\alpha} \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

on $\partial D(t)$, where $\hat{\square}$ denotes the absence of $\square$. It follows that

$$I = \sum_{\alpha=1}^{n} \int_{\partial D(t)} \frac{\frac{\partial g}{\partial t}\frac{\partial^2 g}{\partial \bar{t}\partial z_\alpha}\frac{\partial g}{\partial \bar{z}_\alpha}\, ds_z}{\|\mathrm{Grad}_{(z)}\, g\|}$$

$$= \sum_{\alpha=1}^{n} \int_{\partial D(t)} \frac{\partial g}{\partial t}\frac{\partial^2 g}{\partial \bar{t}\partial z_\alpha} \left(\frac{i}{2^{n-1}}\right) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_\alpha \wedge \widehat{d\bar{z}_\alpha} \wedge \cdots \wedge dz_n \wedge d\bar{z}_n .$$

Since $(\partial g/\partial t)$, $(\partial^2 g/\partial \bar{t}\partial z_\alpha)$ are of class C^2 for z on $D(t) \cup \partial D(t)$, we have by Stokes formula

$$I = \frac{i^n}{2^{n-1}} \sum_{\alpha=1}^{n} \iint_{D(t)} d\left(\frac{\partial g}{\partial t}\frac{\partial^2 g}{\partial \bar{t}\partial z_\alpha} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_\alpha \wedge \widehat{d\bar{z}_\alpha} \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \right)$$

$$= \frac{-i^n}{2^{n-1}} \sum_{\alpha=1}^{n} \iint_{D(t)} \left(\left|\frac{\partial^2 g}{\partial \bar{t}\partial z_\alpha}\right|^2 + \frac{\partial g}{\partial t}\frac{\partial^3 g}{\partial \bar{t}\partial z_\alpha \partial \bar{z}_\alpha} \right) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

Since $\frac{\partial g}{\partial \bar{t}}$ is harmonic for z in $D(t)$, we have $\sum_{\alpha=1}^{n} \frac{\partial^3 g}{\partial \bar{t}\partial z_\alpha \partial \bar{z}_\alpha} = 0$ for $z \in D(t)$. It follows from $dz_\alpha \wedge d\bar{z}_\alpha = -2i\,dx dy$ that

$$I = \frac{-i^n}{2^{n-1}}(-2i)^n \iint_{D(t)} \Big(\sum_{\alpha=1}^{n} \Big|\frac{\partial^2 g}{\partial \bar{t}\partial z_\alpha}\Big|^2\Big) dV$$

$$= -2 \iint_{D(t)} \Big(\sum_{\alpha=1}^{n} \Big|\frac{\partial^2 g}{\partial \bar{t}\partial z_\alpha}\Big|^2\Big) dV.$$

It turns out

$$\frac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} = \frac{-1}{(n-1)\,\omega_{2n}} \int_{\partial D(t)} k_2(t,z)\,\|\mathrm{Grad}_{(z)}\, g(t,z)\|^2 dS_z$$

$$- \frac{4}{(n-1)\,\omega_{2n}} \iint_{D(t)} \Big(\sum_{\alpha=1}^{n} \Big|\frac{\partial^2 g}{\partial \bar{t}\partial z_\alpha}\Big|^2\Big) dV. \qquad \text{c.q.f.d.}$$

<u>Proposition 4.1</u> It holds

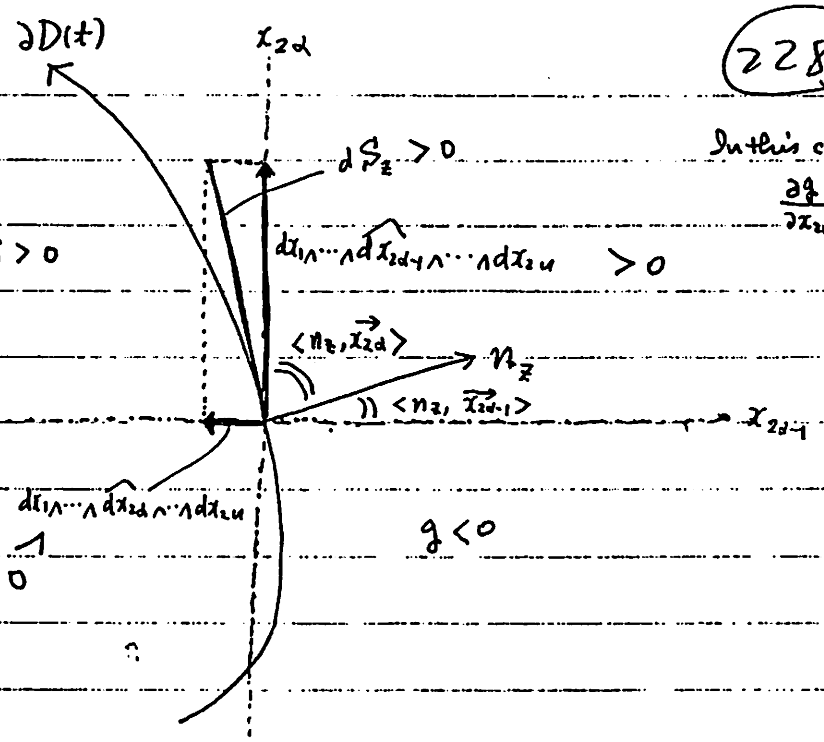
$$\frac{\partial g}{\partial \bar{z}_\alpha} dS_z = \frac{i^n}{2^{n-1}} \|\mathrm{Grad}_{(z)}\, g\|\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{dz_\alpha \wedge d\bar{z}_\alpha} \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

for all $z \in \partial D(t)$, $t \in B$.

<u>Proof:</u> Let $t \in B$ and $z \in \partial D(t)$. Draw the outer normal n_z to the real $(2n-1)$-dimensional surface $\partial D(t)$ at the point z, and denote by $\langle \overrightarrow{x_{2\alpha-1}}, n_z \rangle$ the cosine of the angle between n_z and the real positive $x_{2\alpha-1}$-axis. Since $n_z = -\mathrm{Grad}_{(x)}\, g(t,x) / \|\mathrm{Grad}_{(x)}\, g(t,x)\|$, it follows that, along $\partial D(t)$

$$\underbrace{dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_{2\alpha-1}} \wedge \cdots \wedge dx_{2n}}_{\substack{\text{the projection of } dS_z \text{ to the} \\ \text{space } (x_1, \ldots, \widehat{x_{2\alpha-1}}, \ldots, x_{2n})}} = (dS_z)\,\langle \overrightarrow{x_{2\alpha-1}}, n_z \rangle$$

$$= (dS_z)\Big(-\frac{\partial g}{\partial x_{2\alpha-1}}\Big) / \|\mathrm{Grad}_{(x)}\, g\|$$



In this case:
$$\frac{\partial g}{\partial x_{2\alpha-1}} < 0 ; \quad \frac{\partial g}{\partial x_{2\alpha}} > 0$$

or equivalently

$$\frac{\partial g}{\partial x_{2\alpha-1}} dS_z = -\|\mathrm{Grad}_{(x)}\, g\|\, dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_{2\alpha-1}} \wedge \cdots \wedge dx_{2n}$$

By observing the orientation of $\partial D(t)$, we similarly have

$$\frac{\partial g}{\partial x_{2\alpha}} dS_z = \|\mathrm{Grad}_{(x)}\, g\|\, dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_{2\alpha}} \wedge \cdots \wedge dx_{2n}$$

It follows that

$$\frac{\partial g}{\partial \bar{z}_\alpha} dS_z = -\frac{\|\mathrm{Grad}_{(x)}\, g\|}{2} \left\{ dx_1 \wedge dx_2 \wedge \cdots \wedge (dx_{2\alpha} - i\, dx_{2\alpha-1}) \wedge \cdots \wedge dx_{2n} \right\}$$

$$= \frac{i^n}{2^{n-1}} \|\mathrm{Grad}_{(z)}\, g\|\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_\alpha} \wedge \cdots \wedge dz_n \wedge d\bar{z}_n .$$

q.e.d.

By Theorem 4.1, we have

Theorem 4.2 If $k_2(t,z) \geqq 0$ on $\partial\mathcal{D}$, then $-\lambda(t)$ is a subharmonic function on B.

As we shall show in the next section, if $\mathcal{D}$ is pseudoconvex in $B \times \mathbb{C}^n$, then $k_2(t,z) \geqq 0$ on $\partial\mathcal{D}$. Of course it is not necessary condition. For example, assume that $\mathcal{D}$ is a pseudoconvex domain in $B \times \mathbb{C}^n$ such that $n \geqq 2$ and that $\mathcal{D} \supset B \times [S]$ where $[S]$ is a ball in $\mathbb{C}^n$. Then $\mathcal{D}^* = \mathcal{D} - B \times [S]$ is not pseudoconvex in $B \times \mathbb{C}^n$ but it satisfies the condition: $k_2(t,z) \geqq$ on $\partial\mathcal{D}^*$ $(= \partial\mathcal{D} \cup (B \times S)$ where S is the sphere$)$. For, $k_2(t,z) =$ on $B \times S$.

As another exemple, we consider the domain

$$\mathcal{D} = \left\{(t, x, y) \,/\, |t| < \tfrac{1}{20} \text{ and } \psi(t,z) = \tfrac{1}{3} + |x|^2 - \tfrac{1}{1+|y|^2} + t + \bar{t} < \right. \cap \mathbb{C}^3$$

Then $L_{(t,x)}\psi = 1$, $L_{(t,y)}\psi = \dfrac{1-|y|^2}{(1+|y|^2)^3}$, $\mathcal{L}\psi \geqq (1+2|y|^2)/(1+|y|^2)^3 > 0$. $\mathrm{Grad}_{(z)}\psi = \left(2\bar{x}, \dfrac{\bar{y}}{1+|y|^2}\right.$ $\neq (0,0)$ for all $z \in \partial\mathcal{D}$. Since $-\frac{1}{10} < t + \bar{t} < \frac{1}{10}$, we have $\dfrac{1}{\frac{13}{30} + |x|^2} < 1 + |y|^2 < \dfrac{1}{\frac{7}{30} + |x|^2}$, so that $\{y\}$ are bdd and hence $\{x\}$ are bdd, i.e., each $D(t)$ is bdd smooth domain. If $x = 0$, then $1 < \frac{17}{13} < |y|^2 < \frac{23}{7}$. Hence each $D(t)$ has a bound point (x,y) where $|x| < \rho \ll 1$ such that $L_{(t,y)}\psi < 0$. It follow that The restriction of $\mathcal{D}$ to (t, x, y_0) is 2-dimensional pseudo convex but the restriction of $\mathcal{D}$ to (t, x_0, y) where $|x_0| < \rho$ is not 2-dimensional pseudoconvex domain.

Professor A. Browder gave us the following Remark, which will be useful when we extend our argument to the variation of domains in the complex manifold.

Remark 4.1 Let $\mathcal{D}: t \to D(t)$ $(t \in B)$ be a smooth variation. Consider a transformation

$$T: \begin{cases} t = t \\ w = cAz + b \end{cases}$$

where $c \in \mathbb{C}$ with $c \neq 0$, A is a unitary $n \times n$ matrix i.e., ${}^t\bar{A}A = E_n$ and $b \in \mathbb{C}^n$. Denoting by $\mathcal{D}^* = T(\mathcal{D})$, we have a smooth variation $\mathcal{D}^*: t \to D^*(t)$ $(t \in B)$. Then

$$k_2(t, w) = |c|\, k_2(t, z).$$

Hence either $k_2(t, z) \geq 0$ or < 0 does not depend on the choice of the Euclidean coordinates.

Proof. Let $\psi(t, z)$ define the domain $\mathcal{D}$. Then $\psi^*(t, w) = \psi(t, z)$ where $w = cAz + b$ defines the domain $\mathcal{D}^*$. By direct calculation, we get

$$\frac{\partial \psi}{\partial t} = \frac{\partial \psi^*}{\partial t}; \quad \frac{\partial^2 \psi}{\partial t \partial \bar{t}} = \frac{\partial^2 \psi^*}{\partial t \partial \bar{t}}$$

$$\frac{\partial \psi}{\partial z_\alpha} = \sum_{\beta=1}^{n} \frac{\partial \psi^*}{\partial w_\beta} \cdot c \cdot a_{\beta\alpha}; \quad \frac{\partial^2 \psi}{\partial \bar{t} \partial z_\alpha} = \sum_{\beta=1}^{n} \frac{\partial^2 \psi^*}{\partial \bar{t} \partial w_\beta} c\, a_{\beta\alpha}$$

$$\frac{\partial^2 \psi}{\partial \bar{z}_\alpha \partial z_\alpha} = \sum_{\beta,\gamma=1}^{n} \frac{\partial^2 \psi^*}{\partial \bar{w}_\gamma \partial w_\beta} |c|^2\, \overline{c_{\gamma\alpha}}\, c_{\beta\alpha}$$

and hence

$$\mathcal{L}\psi = \left|\frac{\partial \psi^*}{\partial t}\right|^2 \sum_{\alpha,\beta,\gamma=1}^{n} \frac{\partial^2 \psi^*}{\partial \bar{w}_\gamma \partial w_\beta} |c|^2 a_{\beta\alpha} \overline{a_{\gamma\alpha}}$$

$$-2\,\mathrm{Re}\left\{ \frac{\partial \psi^*}{\partial t} \sum_{\alpha,\beta,\gamma=1}^{n} \frac{\partial \psi^*}{\partial \bar{w}_\beta} \frac{\partial^2 \psi^*}{\partial \bar{t} \partial w_\gamma} |c|^2\, \overline{a_{\beta\alpha}}\, a_{\gamma\alpha} \right\}$$

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$$+ \frac{\partial^2 \psi^*}{\partial t \partial \bar{t}} \left| \sum_{\beta=1}^{n} \frac{\partial \psi^*}{\partial w_\beta} c\, a_{\beta\alpha} \right|^2 .$$

Since $\sum_{\alpha=1}^{n} a_{\beta\alpha} \bar{a}_{\gamma\alpha} = \delta_{\beta\gamma}$ (Kronecker's δ), we thus have

$$\mathcal{L}\psi = |c|^2 \mathcal{L}\psi^* .$$

On the other hand, we have

$$\| \operatorname{Grad}_{(z)} \psi \| = \sqrt{\sum_{\alpha=1}^{n} \sum_{\beta\gamma=1}^{n} \frac{\partial \psi^*}{\partial w_\beta} c\, a_{\beta\alpha} \frac{\partial \psi^*}{\partial w_\gamma} \bar{c}\, \bar{a}_{\gamma\alpha}}$$

$$= |c| \, \| \operatorname{Grad}_{(w)} \psi^* \|$$

It follows that

$$k_2(t,z) = \frac{\mathcal{L}\psi}{\| \operatorname{Grad}_{(z)} \psi \|^3} = \frac{1}{|c|} \mathcal{L}\psi^*$$

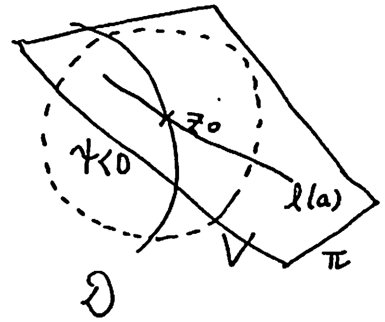
c.q.f.d.

§5 Function-theoretic variation.

Let $\mathcal{D}$ be a domain of $B \times \mathbb{C}^n$ ($n \geq 2$) such that
$$\mathcal{D} : t \to D(t) \ (t \in B)$$
is a smooth variation of smooth domains of $\mathbb{C}^n$ with parameter $t \in B$. In this §, we assume that
"$\mathcal{D}$ is a pseudoconvex domain in $B \times \mathbb{C}^n$".
In this lecture we simply say that, if $\mathcal{D}$ is pseudoconvex in $B \times \mathbb{C}^n$, then the variation $\mathcal{D}: t \to D(t)$ ($t \in B$) is a <u>function-theoretic</u> variation.

In the 1^{st} semester we studied the pseudoconvex domains over $\mathbb{C}^2$. All properties are valid in the case of $\mathbb{C}^n$ where $n \geq 3$.

Assume that $n \geq 3$. Let $\mathcal{D}$ be a domain in $\mathbb{C}^n$ and let $z_0 \in \partial\mathcal{D}$. Assume that $\partial\mathcal{D}$ is smooth near z_0, namely, there exists a real-valued function $\psi(z)$ of class C^2 in a neighborhood V of z_0 in $\mathbb{C}^n$ such that



$$\mathcal{D} \cap V = \{\psi < 0\};$$
$$\partial\mathcal{D} \cap V = \{\psi = 0\}$$
$$\mathrm{Grad}_{(z)}\, \psi \neq 0 \quad \text{on} \quad \psi = 0.$$

Consider a complex tangent plane π of dim $(n-1)$ which passes through z_0;
$$\pi : \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z_\alpha}(z_0)\,(z - z_{0\alpha}) = 0.$$

Draw any 1-dimensional line $l(a)$ $(\subset \pi)$ passing through z_0,

$$l(a): \quad z = z_0 + a\tau \quad (|\tau| \ll 1)$$

$$a \neq 0,\ a \in \mathbb{C}^n \text{ such that } \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z_\alpha}(z_0)\, a_\alpha = 0.$$

Consider the restriction of ψ to $l(a)$:

$$\begin{aligned}
\psi(z_0 + a\tau) &= \psi(z_{01} + a_1\tau, \cdots, z_{0n} + a_n\tau) \\
&= \psi(z_0) + 2\,\mathrm{Re}\left\{ \left(\sum_{\alpha=1}^{n} a_\alpha \frac{\partial \psi}{\partial z_\alpha}(z_0) \right) \tau \right\} + \left(\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta}(z_0)\, a_\alpha \bar{a}_\beta \right) |\tau|^2 \\
&\quad + \mathrm{Re}\left\{ \left(\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial z_\beta}(z_0)\, a_\alpha a_\beta \right) \tau^2 \right\} + O(|\tau|^3) \\
&= \left(\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta}(z_0)\, a_\alpha \bar{a}_\beta \right) |\tau|^2 + \mathrm{Re}\left\{ \left(\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial z_\beta}(z_0)\, a_\alpha a_\beta \right) \tau^2 \right\} \\
&\quad + O(|\tau|^3).
\end{aligned}$$

<u>Definition 5.1</u> If $\displaystyle\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta}(z_0)\, a_\alpha \bar{a}_\beta \geqq 0$ (resp. > 0) for all $a \neq 0$, $a \in \mathbb{C}^n$ such that $\displaystyle\sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z_\alpha}(z_0)\, a_\alpha = 0$, then z_0 is said to be a pseudoconvex (resp. strictly pseudoconvex) boundary point of $\mathcal{D}$.

<u>Definition 5.2</u> If each boundary point of $\mathcal{D}$ is pseudoconvex (resp. strictly pseudoconvex), then $\mathcal{D}$ is said to be a pseudoconvex (resp. strictly pseudoconvex) domain in $\mathbb{C}^n$.

These definitions do not depend on the choice of the defining function ψ of $\mathcal{D}$ at z_0. In fact, let

$$\varphi = f\psi \quad \text{in } V$$

where V is a neighborhood of z_0 in $\mathbb{C}^n$, f is >0 and of class C^2 on V. Then

$$\mathrm{Grad}_{(z)}\, \varphi(z_0) = f(z_0)\, \mathrm{Grad}_{(z)}\, \psi(z_0).$$

Hence $a \neq 0$, $a \in \mathbb{C}^n$ such that $\displaystyle\sum_{\alpha=1}^{n} \frac{\partial \varphi}{\partial z}(z_0)\, a_\alpha = 0$ satisfies

$\sum_{\alpha=1}^{n} \frac{\partial\psi}{\partial z_\alpha}(z_0)\, a_\alpha = 0$. Since

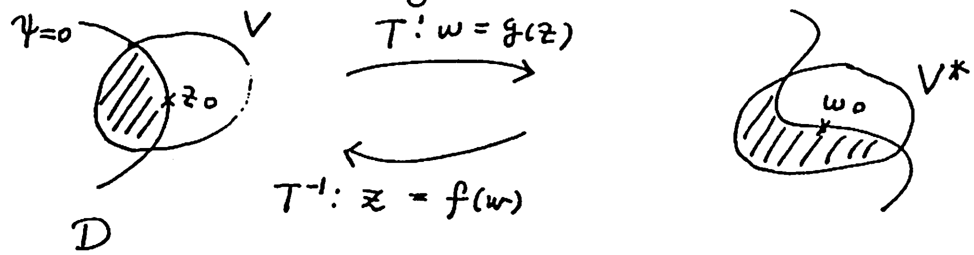
$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2\varphi}{\partial z_\alpha \partial \bar z_\beta}(z_0)\, a_\alpha \bar a_\beta = \sum_{\alpha,\beta=1}^{n}\left\{\frac{\partial f}{\partial \bar z_\beta}\frac{\partial \psi}{\partial z_\alpha} + \frac{\partial f}{\partial z_\alpha}\frac{\partial\psi}{\partial \bar z_\beta} + f\frac{\partial^2\psi}{\partial z_\alpha\partial\bar z_\beta}\right\}_{z_0} a_\alpha \bar a_\beta$$

$$= f(z_0)\Big(\sum_{\alpha,\beta=1}^{n}\frac{\partial^2\psi}{\partial z_\alpha\partial\bar z_\beta}(z_0)a_\alpha\bar a_\beta\Big) + 2\,\mathrm{Re}\Big\{\underbrace{\Big(\sum_{\alpha=1}^{n}\frac{\partial\psi}{\partial z_\alpha}a_\alpha\Big)}_{=0}\Big(\sum_{\beta=1}^{n}\frac{\partial f}{\partial \bar z_\beta}\bar a_\beta\Big)\Big\}$$

$$= f(z_0)\Big(\sum_{\alpha,\beta=1}^{n}\frac{\partial^2\psi}{\partial z_\alpha\partial\bar z_\beta}(z_0)a_\alpha\bar a_\beta\Big) \quad \text{where } f(z_0) > 0,$$

it follows that the definition of pseudoconvexity or strictly pseudoconvexity of z_0 does not depend on the choice of ψ.

One of most important property is the invariance of pseudoconvexity or strictly pseudoconvexity at $z_0 \in \partial D$ under the local holomorphic mappings:



Let $T: w = g(z)$ be a 1:1 holomorphic mapping from V to V^* where V is a nbd of z_0 in $\mathbb{C}^n$. We denote by $z = f(w)$ the inverse mapping of T. Now assume that z_0 is pseudoconvex (resp. strictly pseudoconvex). Then $w_0 = g(z_0)$ is pseudoconvex (resp. strictly pseudoconvex).

Proof. Let $\psi = 0$ defines $(\partial D)\cap V$ and let

$$\psi^*(w) = \psi(f(w)) = \psi(z)$$

on V^*. Then

$$\frac{\partial\psi^*(w)}{\partial w_\alpha} = \sum_{\gamma=1}^{n}\frac{\partial\psi}{\partial z_\gamma}\frac{\partial f_\gamma}{\partial w_\alpha};$$

$$\frac{\partial^2 \psi^*(w)}{\partial \overline{w}_\beta \, \partial w_\alpha} = \sum_{\gamma=1}^{n} \sum_{\delta=1}^{n} \left[\frac{\partial^2 \psi}{\partial \overline{z}_\delta \partial z_\gamma} \cdot \overline{\left(\frac{\partial f_\delta}{\partial w_\beta}\right)} \; \frac{\partial f_\gamma}{\partial w_\alpha} \right]$$

$$= \sum_{\gamma,\delta=1}^{n} \frac{\partial^2 \psi}{\partial z_\gamma \partial \overline{z}_\delta} \, \frac{\partial f_\gamma}{\partial w_\alpha} \overline{\left(\frac{\partial f_\delta}{\partial w_\beta}\right)} .$$

Let $a \neq 0$, $a \in \mathbb{C}^n$ such that $\sum_{\alpha=1}^{n} \frac{\partial \psi^*}{\partial w_\alpha}(w_0) a_\alpha = 0$. Hence if we set $A_\gamma = \sum_{\alpha=1}^{n} \frac{\partial f_\gamma}{\partial w_\alpha}(w_0) a_\alpha$ $(\gamma = 1, \cdots, n)$, then $A = (A_1, \cdots, A_n) \neq 0$ in $\mathbb{C}^n$ and $\sum_{\gamma=1}^{n} \frac{\partial \psi}{\partial z_\gamma}(z_0) A_\gamma = 0$.

Since $\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \psi^*}{\partial w_\alpha \partial \overline{w}_\beta}(w_0) a_\alpha \overline{a_\beta}$

$$= \sum_{\alpha,\beta=1}^{n} \left\{ \sum_{\gamma,\delta=1}^{n} \frac{\partial^2 \psi}{\partial z_\gamma \partial \overline{z}_\delta}(z_0) \, \frac{\partial f_\gamma}{\partial w_\alpha}(w_0) \, \overline{\frac{\partial f_\delta}{\partial w_\beta}(w_0)} \right\} a_\alpha \overline{a_\beta}$$

$$= \sum_{\gamma,\delta=1}^{n} \frac{\partial^2 \psi}{\partial z_\gamma \partial \overline{z}_\delta}(z_0) \left\{ \sum_{\alpha=1}^{n} \frac{\partial f_\gamma}{\partial w_\alpha} a_\alpha \right\} \overline{\left\{ \sum_{\beta=1}^{n} \frac{\partial f_\delta}{\partial w_\beta} a_\beta \right\}}$$

$$= \sum_{\gamma,\delta=1}^{n} \frac{\partial^2 \psi}{\partial z_\gamma \partial \overline{z}_\delta}(z_0) A_\gamma \overline{A_\delta} ,$$

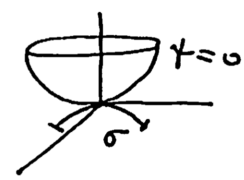
we see that w_0 is pseudoconvex (resp. strictly pseudoconvex) boundary point of $\varphi(V \cap D)$. c.q.f.d.

<u>Lemma 5.1</u> If z_0 is a strictly pseudoconvex boundary point of $\mathcal{D}$, then there exists a regular analytic set σ of dimension $n-1$ passing through z_0 such that

$$\sigma - \{z_0\} \subset (\mathcal{D} \cup \partial \mathcal{D})^c .$$

<u>Proof.</u> By the last fact, we may assume $z_0 = 0$ in $\mathbb{C}^n$ and the complex tangent plane π of $\mathcal{D}$ at $z_0 = 0$ is

$$\pi : \; z_n = 0 .$$



Let $\psi = 0$ define $\partial \mathcal{D}$ near 0. Moreover, $\mathrm{Grad}_{(z)} \psi(0) = (0, \cdots, 0, 1)$. Take any $(a_1, \cdots, a_{n-1}) \in \mathbb{C}^n - \{0\}$. Our condition is

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta}(0)\, a_\alpha \bar{a}_\beta > 0.$$

By Taylor development at 0, we have

$$\psi(z) = \psi(0) + 2\,\mathrm{Re}\left\{\sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z_\alpha}(0) z_\alpha\right\} + \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta}(0)\, z_\alpha \bar{z}_\beta$$
$$+ \mathrm{Re}\left\{\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial z_\beta}(0)\, z_\alpha z_\beta\right\} + O(\|z\|^3)$$

$$= \mathrm{Re}\left\{2 z_n + \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial z_\beta}(0)\, z_\alpha z_\beta\right\} + \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta}(0)\, z_\alpha \bar{z}_\beta + O(\|z\|^3).$$

We choose, as a regular analytic set σ, the following

$$\sigma : 2 z_n + \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial z_\beta}(0)\, z_\alpha z_\beta = 0 \quad \text{near } z = 0.$$

σ passes through 0 and is written easily

$$\sigma : \quad z_n = \sum_{\alpha,\beta=1}^{n-1} c_{\alpha\beta}\, z_\alpha z_\beta + O(\|\underset{(z_1,\cdots,z_{n-1})}{z}\|^3).$$

We thus have the restriction of ψ to σ:

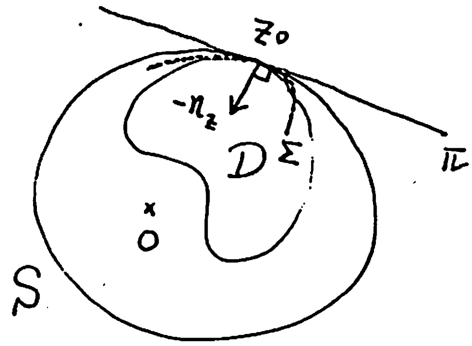
$$\psi\big|_\sigma = \sum_{\alpha,\beta=1}^{n-1} \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta}(0)\, z_\alpha \bar{z}_\beta + O(\|\underset{(z_1,\cdots,z_{n-1})}{z}\|^3)$$

It follows that, if $(z_1, \cdots, z_{n-1})$ is sufficiently close to 0, then $\psi|_\sigma > 0$ except 0, that is, $\sigma - \{0\} \subset (D \cup \partial D)^c$

c. q. f. d.

<u>Remark 5.1</u> Any bounded smooth domain D in $\mathbb{C}^n$ has at least one strictly pseudoconvex boundary point.

<u>Proof.</u> Consider the ball $[S] : \|z\| < R$ and the sphere $S : \|z\| = R$ such that S touches the surface ∂D from the outside. Namely, $[S] \supset D$ and $S \cap \partial D \neq \phi$. Take any point of contact in $S \cap \partial D$, say z_0. Then z_0 is one of strictly pseudoconvex boundary point of D.



In fact, let $\psi = 0$ define the boundary ∂D, and consider the function

$$u(z) = K\psi(z) - (\|z\|^2 - R^2)$$

where $K > 0$ is a constant.

If K is sufficiently large, then $\mathrm{Grad}_{(z)}\, u(z_0) \neq 0$ and $\frac{\partial u}{\partial n_z}(z_0) > 0$ where n_z denotes the outer normal with respect to ∂D, and hence to S. It follows that

$$u(z) < 0 \quad \text{on the inner normal } -n_z.$$

By definition of ψ, we have

$$u(z) \geq 0 \quad \text{on } S.$$

We consider the zero surface Σ of $u(z)$ i.e., $u(z) = 0$ near z_0. Since it is non-singular at z_0, it follows from the continuity of $u(z)$ that $\Sigma \subset [S]$.

Let π be the complex tangent plane:

$$\pi: \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z_\alpha}(z_0)(z_\alpha - z_{0\alpha}) = 0. \quad \text{Then } \pi \subset \{u(z) \geq 0\}.$$

Therefore the restriction of $u(z)$ to π attains its local minimum at z_0. For any $a \in \mathbb{C}^n$, $a \neq 0$ such that $\sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z_\alpha}(z_0) a_\alpha = 0$, we thus have

$$\frac{\partial^2}{\partial t \partial \bar{t}} u(z_0 + at) \geqq 0 \text{ at } t = 0$$

$$\|$$

$$\left\{ K \frac{\partial^2 \psi(z_0 + at)}{\partial t \partial \bar{t}} - \frac{\partial^2}{\partial t \partial \bar{t}} \|z_0 + at\|^2 \right\}_{t=0}$$

$$\|$$

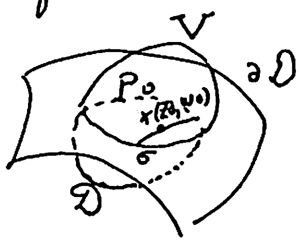
$$K \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta}(z_0) a_\alpha \bar{a}_\beta - \|a\|^2$$

$$\therefore \quad \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta}(z_0) a_\alpha \bar{a}_\beta \geqq \frac{\|a\|^2}{K} > 0.$$

c.q.f.d.

It was better that the following Levi's theorem was stated in Chapter IV in the 1st semester.

Let D be a domain in $\mathbb{C}^2$ of two complex variable z and w, and let $P_0 \in \partial D$ at which ∂D is smooth. Precisely, there exists a real valued function $\psi(z,w)$ of class C^2 in a neighborhood V of P_0 in $\mathbb{C}^2$ such that



$D \cap V = \{\psi(z,w) < 0\}$; $\partial D \cap V = \{\psi(z,w) = 0\}$
and $\mathrm{Grad}_{(z,w)}\, \psi \neq 0$ on $\partial D \cap V$.

Under this notation we have

<u>Levi's Theorem</u> Assume that the Levi's form

$$L\psi(z,w) = \frac{\partial^2\psi}{\partial z \partial \bar z}\left|\frac{\partial \psi}{\partial w}\right|^2 - 2\mathrm{Re}\left\{\frac{\partial^2\psi}{\partial z\partial \bar w}\,\frac{\partial\psi}{\partial \bar z}\frac{\partial \psi}{\partial w}\right\} + \frac{\partial^2\psi}{\partial w \partial \bar w}\left|\frac{\partial\psi}{\partial z}\right|^2 \equiv 0$$

on $(\partial D)\cap V$. Then, given $(z_0,w_0) \in \partial D \cap V$, there uniquely exists a regular analytic set σ (of one dimensional) passing through (z_0,w_0) such that

$$\sigma \subset (\partial D)\cap V.$$ ------"That is, $\partial D \cap V$ consists of real 1-parameter family of complex 1 dimensional analytic set."

The converse is true.

<u>Proof.</u> We write $z = x+iy$, $w = u+iv$. Since $L\psi \equiv 0$ does not depend on the choice of ψ and since $L\psi \equiv 0$ is invariant under the local holomorphic mapping, we may assume that

$$(z_0,w_0) = (0,0) \quad \text{and} \quad \psi(z,w) = v - \varphi(x,y,u)$$

where $\varphi(x,y,u)$ is a real-valued fn of class C^2 in a neighborhood of $(0,0,0)$ in $\mathbb{R}^3$ such that $\varphi(0,0,0) = 0$.

By use of complex notation, we have

$$\psi = \frac{w-\bar w}{2i} - \varphi\left(z, \frac{w+\bar w}{2}\right)$$

u: real
z: complex.

Hence
$$\frac{\partial \psi}{\partial z} = -\frac{\partial \varphi}{\partial z}$$
$$\frac{\partial^2 \psi}{\partial \bar{z}\partial z} = -\frac{\partial^2 \varphi}{\partial \bar{z}\partial z}$$
$$\frac{\partial \psi}{\partial w} = \frac{1}{2i} - \frac{\partial \varphi}{\partial u}\cdot\frac{1}{2}$$
$$\frac{\partial^2 \psi}{\partial \bar{w}\partial w} = -\frac{1}{4}\frac{\partial^2 \varphi}{\partial u^2}$$
$$\frac{\partial^2 \psi}{\partial \bar{w}\partial z} = -\frac{1}{2}\frac{\partial^2 \varphi}{\partial u\partial z}$$

$$\therefore L\psi = \left|\frac{\partial \varphi}{\partial z}\right|^2\left(-\frac{1}{4}\frac{\partial^2\varphi}{\partial u^2}\right) - 2\mathrm{Re}\left\{-\frac{\partial\varphi}{\partial z}\left(-\frac{1}{2i} - \frac{\partial\varphi}{\partial u}\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{\partial^2\varphi}{\partial u\partial \bar{z}}\right\}$$
$$+ \left|\frac{1}{2i} - \frac{\partial\varphi}{\partial u}\frac{1}{2}\right|^2\left(-\frac{\partial^2\varphi}{\partial z\partial \bar{z}}\right)$$
$$= -\frac{1}{4}\left\{\left|\frac{\partial\varphi}{\partial z}\right|^2\frac{\partial^2\varphi}{\partial u^2} - 2\mathrm{Re}\left\{\frac{\partial\varphi}{\partial z}\left(\frac{\partial\varphi}{\partial u} - i\right)\frac{\partial^2\varphi}{\partial u\partial\bar{z}}\right\} + \left(1 + \left(\frac{\partial\varphi}{\partial u}\right)^2\right)\frac{\partial^2\varphi}{\partial z\partial\bar{z}}\right\}$$
$$\equiv 0$$

We try to construct a regular analytic set σ on ∂D passing through $(0,0)$ of the form
$$\sigma : \begin{cases} u = u(z) \\ v = \varphi(z, u(z)) \end{cases}$$
where $u(z)$ is a real valued function of class C^1 near $z=0$. Our condition to $u(z)$ is thus that, if we put
$$f(z) = u(z) + i\,\varphi(z, u(z)) \qquad (|z| \ll 1),$$
then $f(0)=0$ and $f(z)$ is holomorphic for z near $z=0$, i.e., $u(0)=0$ and
$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\left(\frac{\partial\varphi}{\partial\bar{z}} + \frac{\partial\varphi}{\partial u}\frac{\partial u}{\partial\bar{z}}\right)_{(z,u(z))} \equiv 0,$$
or
$$\frac{\partial u(z)}{\partial\bar{z}} = \frac{-i\,\frac{\partial\varphi}{\partial\bar{z}}(z,u(z))}{1 + i\,\frac{\partial\varphi}{\partial u}(z,u(z))} \quad \text{in } |z| \ll 1.$$

If we set

$$K(z,u) = \frac{-i\,\frac{\partial\varphi}{\partial\bar z}(z,u)}{1+i\,\frac{\partial\varphi}{\partial u}(z,u)},$$

then $K(z,u)$ is a complex-valued function of class C^1 defined in a neighborhood $\mathcal{V}$ of $(0,0)$ in $\mathbb{C}\times\mathbb{R}$. Our claim is reduced to find a real-valued function $u(z)$ which satisfies the following system of differential equations near $z=0$

$$(1)\quad \frac{\partial u}{\partial\bar z} = K(z,u)$$

with initial value $u(0)=0$

By Lemma which will be shown right now, it is enough to show that

$$\mathrm{Im}\left\{\frac{\partial K}{\partial z} + \frac{\partial K}{\partial u}\,\overline{K}\right\} \equiv 0 \quad \text{in } \mathcal{V}.$$

Let us certify this equality: By direct calcul, we have

$$\mathrm{Im}\left\{\frac{\partial K}{\partial z} + \frac{\partial K}{\partial u}\,\overline{K}\right\}$$

$$= \mathrm{Im}\ i\left\{\frac{-\frac{\partial^2\varphi}{\partial z\partial\bar z}}{1+i\frac{\partial\varphi}{\partial u}} + \frac{\frac{\partial\varphi}{\partial\bar z}\left\{i\frac{\partial^2\varphi}{\partial z\partial u}\right\}}{\left(1+i\frac{\partial\varphi}{\partial u}\right)^2} + \left(\frac{-\frac{\partial^2\varphi}{\partial\bar z\partial u}}{1+i\frac{\partial\varphi}{\partial u}} + \frac{\frac{\partial\varphi}{\partial\bar z}\left(i\frac{\partial^2\varphi}{\partial u^2}\right)}{\left(1+i\frac{\partial\varphi}{\partial u}\right)^2}\right)\left(\frac{i\frac{\partial\varphi}{\partial z}}{1-i\frac{\partial\varphi}{\partial u}}\right)\right\}$$

$$= \frac{1}{\left(1+\left(\frac{\partial\varphi}{\partial u}\right)^2\right)^2}\,\mathrm{Re}\left\{-\left(\frac{\partial^2\varphi}{\partial z\partial\bar z}\right)\left(1-i\frac{\partial\varphi}{\partial u}\right)\left(1+\left(\frac{\partial\varphi}{\partial u}\right)^2\right) + i\frac{\partial\varphi}{\partial\bar z}\frac{\partial^2\varphi}{\partial z\partial u}\left(1-i\frac{\partial\varphi}{\partial u}\right)^2 - \frac{\partial^2\varphi}{\partial\bar z\partial u}\,i\,\frac{\partial\varphi}{\partial z}\left(1+\left(\frac{\partial\varphi}{\partial u}\right)^2\right) - \left|\frac{\partial\varphi}{\partial z}\right|^2\frac{\partial^2\varphi}{\partial u^2}\left(1-i\frac{\partial\varphi}{\partial u}\right)^2\right\}$$

$$= \frac{-1}{\left(1+\left(\frac{\partial\varphi}{\partial u}\right)^2\right)^2}\left\{\frac{\partial^2\varphi}{\partial z\partial\bar z}\frac{\partial\varphi}{\partial u} - \left|\frac{\partial\varphi}{\partial z}\right|^2\frac{\partial^2\varphi}{\partial u^2} + 2\,\mathrm{Re}\left\{\frac{\partial\varphi}{\partial z}\left(\frac{\partial\varphi}{\partial u}-i\right)\frac{\partial^2\varphi}{\partial u\partial\bar z}\right\}\right\}$$

$$= \frac{4}{\left(1+\left(\frac{\partial\varphi}{\partial u}\right)^2\right)^2}\cdot L\varphi \equiv 0.$$

Hence we have an analytic set $\sigma \ni (0,0)$ such that $\sigma\subset\partial D\cap V$

For the uniqueness, assume that a regular analytic set $\sigma \ni (0,0)$ on ∂D exists. Then σ has the form

$$\sigma : w = f(z) = u(z) + i\varphi(z, u(z))$$

where $f(z)$ is holomorphic near $z=0$ and $f(0)=0$.
It follows that the real-valued function $u(z)$ near $z=0$ satisfies the differential equation (1) with initial value $u(0)=0$. By use of Lemma which will be shown right now, such a function $u(z)$ must be unique and hence so does $f(z)$.

The converse is clear. For, take any $(z_0, w_0) \in \partial D \cap V$. Without loss of generality we assume $\frac{\partial \psi}{\partial w}(z_0, w_0) \neq 0$. Assume that there exists a regular analytic set σ passing through (z_0, w_0) on $\partial D \cap V$. Consequently, σ is of the form $\sigma : w = f(z)$ with $f(z_0) = w_0$ where $f(z)$ is holomorphic near z_0. Since $\sigma \subset \partial D \cap V$, we have

$$\psi(z, f(z)) \equiv 0 \quad \text{in } |z - z_0| \ll 1.$$

$$\therefore \frac{\partial \psi}{\partial z}(z, f(z)) + \frac{\partial \psi}{\partial w}(z, f(z)) f'(z) \equiv 0 \quad \text{in } |z - z_0| \ll 1$$

$$\therefore \frac{\partial^2 \psi}{\partial \bar{z} \partial z} + 2\,\mathrm{Re}\left\{ \frac{\partial^2 \psi}{\partial \bar{z} \partial w}(z, f(z)) f'(z) \right\} + \frac{\partial^2 \psi}{\partial w \partial \bar{w}}(z, f(z)) \overline{f'(z)} f'(z) \equiv 0$$

In particular, if we put $z = z_0$, we have

$$\frac{1}{\left|\frac{\partial \psi}{\partial w}\right|^2} \left\{ \frac{\partial^2 \psi}{\partial z \partial \bar{z}} \left|\frac{\partial \psi}{\partial z}\right|^2 - 2\,\mathrm{Re}\left\{ \frac{\partial^2 \psi}{\partial \bar{z} \partial w} \frac{\partial \psi}{\partial \bar{w}} \frac{\partial \psi}{\partial z} \right\} + \frac{\partial^2 \psi}{\partial w \partial \bar{w}} \left|\frac{\partial \psi}{\partial w}\right|^2 \right\}_{(z_0, w_0)} = 0$$

$$\therefore L\psi(z_0, w_0) = 0.$$

Since (z_0, w_0) is arbitrary point of $\partial D \cap V$, we have the converse.

c. q. f. d.

Lemma 1. Let $F(x,y,u)$, $G(x,y,u)$ be real-valued function of class C^2 defined in a domain D in $\mathbb{R}^3$ such that

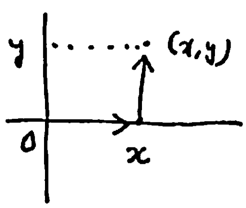
$$(2) \quad \frac{\partial F}{\partial y} + \frac{\partial F}{\partial u} G = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial u} F \quad \text{on } D.$$

Then, given $(a,b,c) \in D$, there exists a unique re solution $u = u(x,y)$ near (a,b) which satisfies the following system of differential equations

$$(3) \quad \begin{cases} \dfrac{\partial u(x,y)}{\partial x} = F(x,y,u(x,y)) \\[2ex] \dfrac{\partial u(x,y)}{\partial y} = G(x,y,u(x,y)) \end{cases}$$

with initial condition $u(a,b) = c$.

Proof. For the sake of convenience we may put $(a,b,c) = (0,0,0) \in D$.



First, we solve the following differential equation

$$\frac{d u_0(x)}{dx} = F(x, 0, u_0(x)) \quad \text{——— (a)}$$

$$\text{with } u_0(0) = 0 \quad \text{——— (a')}$$

Certainly $u_0(x)$ uniquely exists in a small interval $(-\rho, \rho)$. Next, given $x \in (-\rho, \rho)$, we solve the differential equation w. r. t. y; which is uniquely determined:

$$\frac{d u(x,y)}{dy} = G(x,y,u(x,y)) \quad \text{——— (b)}$$

$$\text{with } u(x,0) = u_0(x) \quad \text{——— (b')}$$

Hence $u(x,y)$ is defined near $(0,0)$. By (a'), (b'), we

have $u(0,0) = 0$. It suffices to prove that
$$\frac{\partial u(x,y)}{\partial x} = F(x,y,u(x,y)).$$

By integrating both sides of (b) w.r.t. y, we have
$$u(x,y) = \int_0^y G(x,y,u(x,y))\,dy + u(x,0)$$

By differentiating both sides w.r.t. x, we get
$$\frac{\partial u}{\partial x} = \int_0^y \left\{ \frac{\partial G}{\partial x} + \frac{\partial G}{\partial u}\frac{\partial u}{\partial x} \right\}_{(x,y,u(x,y))} dy + \frac{\partial u}{\partial x}(x,0)$$

By condition (2) and (a), (b), (b'), we have
$$\frac{\partial u}{\partial x} = \int_0^y \left\{ \frac{\partial F}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial G}{\partial u}\left(\frac{\partial u}{\partial x} - F\right)\right\}_{(x,y,u(x,y))} dy + F(x,0,u(x,0$$
$$= \int_0^y \frac{\partial F(x,y,u(x,y))}{\partial y}\,dy + F(x,0,u(x,0))$$
$$+ \int_0^y \left\{\frac{\partial G}{\partial u}\left(\frac{\partial u}{\partial x} - F\right)\right\}_{(x,y,u(x,y))} dy$$

$$\therefore \quad \frac{\partial u(x,y)}{\partial x} - F(x,y,u(x,y)) = \int_0^y \left\{\frac{\partial G}{\partial u}\left(\frac{\partial u}{\partial x} - F\right)\right\}_{(x,y,u(x,y))} dy$$

Namely, if we put
$$h(x,y) = \frac{\partial u(x,y)}{\partial x} - F(x,y,u(x,y));$$
$$\frac{\partial G}{\partial u}(x,y,u(x,y)) = c(x,y),$$
then
$$h(x,y) = \int_0^y c(x,y)\,h(x,y)\,dy. \qquad \text{—— (c)}$$

It follows that $h(x,0)=0$ for all x.
We regard (c) the integral equation w.r.t. y, or the differential equation with respect to y:

$$\frac{d h(x,y)}{dy} = c(x,y)\, h(x,y)$$

with initial value $h(x,0)=0$.
It is clear that $h(x,y) \equiv 0$ near $y=0$, and hence that

$$\frac{\partial u(x,y)}{\partial x} = F(x,y,u(x,y))$$

near $(x,y)=(0,0)$.

The existence of the solution (3) is proved. The uniqueness is clear because, if it exists, then it must be of the form which satisfies (a), (a') and (b), (b').

c.q.f.d.

Lemma 1 is written by use of complex notations into the following form:

<u>Lemma 1'</u> Let $K(z,u)$ be a <u>complex-valued</u> function of class C^2 defined in a domain D in $\mathbb{C}\times\mathbb{R}$ such that

$$\operatorname{Im}\left\{\frac{\partial K}{\partial z} + \frac{\partial K}{\partial u}\overline{K}\right\} \equiv 0 \quad \text{in } D.$$

Then, given $(z_0, u_0) \in D$, there uniquely exists a <u>real-valued</u> function $u(z)$ such that

$$\frac{\partial u(z)}{\partial \bar{z}} = K(z, u(z)) \quad \text{in } |z-z_0| \ll 1$$

with $u(z_0)=u_0$.

(1.1) Let $K = F + iG$
real valued function.

The condition is equivalent to

$$\operatorname{Im}\left\{ \frac{\partial K}{\partial \bar{z}} + \frac{\partial K}{\partial u}\overline{K} \right\}$$

$$= \frac{1}{2}\left\{ -\frac{\partial F}{\partial y} + \frac{\partial G}{\partial x} \quad -\frac{\partial F}{\partial u} G + \frac{\partial G}{\partial u} F \right\} = 0$$

$$\therefore \quad \frac{\partial F}{\partial y} + \frac{\partial F}{\partial u} G = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial u} F \quad \text{in } D.$$

The desired function satisfies

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} \right) = \left(F + iG \right)_{(x,y,u(x,y))}.$$

or equivalently,

$$\begin{cases} \dfrac{\partial u}{\partial x} = 2F(x,y,u) \\ \dfrac{\partial u}{\partial y} = 2G(x,y,u) \end{cases}$$

with $u(a,b) = c$.

It follows from Lemma 1 that Lemma 1' is proved.

c.q.f.d.

Let $\mathcal{D}: t \to D(t)$ $(t \in B)$ be a smooth variation of smooth domains such that $\mathcal{D} = \bigcup_{t \in B}(t, D(t))$ is a pseudoconvex domain of $B \times \mathbb{C}^n$ $(n \geqq 2)$. We say that $\mathcal{D}$ is function-theoretic variation.

Let $(\tilde{\mathcal{D}}, \psi)$ be a double which defines $\mathcal{D}$. Since $\mathcal{D}$ is pseudoconvex, we have, for $P \in \partial \mathcal{D}$,

$$(5.1) \quad \sum_{\alpha,\beta=0}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta}(P) a_\alpha \bar{a}_\beta \geqq 0$$

for any $a \neq 0$, $a \in \mathbb{C}^{n+1}$ such that

$$(5.2) \quad \sum_{\alpha=0}^{n} \frac{\partial \psi}{\partial z_\alpha}(P) a_\alpha = 0$$

where z_0 represents the variable t in B.

Let $1 \leqq \alpha \leqq n$. As $a \neq 0$, $a \in \mathbb{C}^n$, we take

$$a_0 = -\frac{\partial \psi}{\partial z_\alpha}(P), \quad a_1 = 0, \cdots, a_\alpha = \frac{\partial \psi}{\partial t}(P), \cdots, a_n = 0.$$

and have

$$\frac{\partial^2 \psi}{\partial t \partial \bar{t}} \left|\frac{\partial \psi}{\partial t}\right|^2 - 2\,\mathrm{Re}\left\{\frac{\partial^2 \psi}{\partial t \partial \bar{z}_\alpha} \frac{\partial \psi}{\partial t} \frac{\partial \psi}{\partial z_\alpha}\right\} + \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\alpha} \left|\frac{\partial \psi}{\partial t}\right|^2 \geqq 0$$

at $P \in \partial \mathcal{D}$. The left-hand side is $L_{(t,z_\alpha)}\psi$, and hence

$$\mathcal{L}\psi = \sum_{\alpha=1}^{n} L_{(t,z_\alpha)}\psi \geqq 0 \quad \text{on } \partial\mathcal{D},$$

or equivalently,

$$k_2(t,z) = \frac{\mathcal{L}\psi(t,z)}{\|\mathrm{Grad}_{(z)}\psi(t,z)\|^3} \geqq 0 \quad \text{on } \partial\mathcal{D}.$$

<u>Remark 5.2</u> If $k_2(t,z) = 0$ for $(t,z) \in \partial\mathcal{D}$, then $L_{(t,z_\alpha)}\psi(t,z) = 0$ $(\alpha = 1, \cdots, n)$ there.

Assume that $\mathcal{D}$ is pseudoconvex in $B \times \mathbb{C}^n$. We suppose that there exists a point ζ in $\mathbb{C}^n$ such that $\zeta \in D(t)$ for all $t \in B$. We thus ticure the green's function $g(t,z)$ and the Robin constant $\lambda(t)$ for $(D(t), \zeta)$. By Theorem 4.2, we see that
(*) "$-\lambda(t)$ is subharmonic on B."
But, in the present case that $\mathcal{D}$ is pseudoconvex, we can go further:
(i) Rigidity holds i.e., $\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(t_0) = 0$ induces $\frac{\partial g}{\partial t}(t_0, z) \equiv 0$ on $D(t_0)$
(ii) In stead of constant section ζ, we can a holomorphic section $\zeta : t \to \zeta(t)$ $(t \in B)$ of $\mathcal{D}$.

Theorem 5.1 (Rigidity)
Assume that $\mathcal{D} : t \to D(t)$ $(t \in B)$ is a function-theoreti variation such that $\mathcal{D} \supset B \times \{\zeta\}$. Then, if $\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(t_0) = 0$ for some $t_0 \in B$, then $\frac{\partial g}{\partial t}(t_0, z) \equiv 0$ on $D(t_0)$.

Proof. By Fundamental formula in Theorem 4.1, we have

$$\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(t_0) = -\frac{1}{(n-1)\omega_{2n}} \int_{\partial D(t_0)} k_2(t_0, z) \, \| \mathrm{Grad}_{(z)} \, g(t_0, z) \|^2 \, ds_z - \frac{4}{(n-1)\omega_{2n}} \iint_{D(t_0)} \left\{ \sum_{\alpha=1}^{n} \left| \frac{\partial^2 g}{\partial t \partial \bar{z}_\alpha}(t_0, z) \right|^2 \right\} dV.$$

By assumption that $\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(t_0) = 0$ and that $\mathcal{D}$ is pseudoconvex in $B \times \mathbb{C}^n$, we see that

$$\begin{cases} k_2(t_0, z) = 0 & \text{for } z \in \partial D(t_0) \\ \dfrac{\partial^2 g}{\partial t \partial \bar{z}_\alpha}(t_0, z) = 0 & \text{for } z \in D(t_0) \end{cases}$$

Hence $\frac{\partial g}{\partial t}(t_0, z)$ is holomorphic for z in $D(t_0)$. Since we can take $\psi = -g$, it follows that, for $z \in \partial D(t_0)$,

$$0 = k_2(t_0, z) = \frac{-\mathcal{L}\, g(t_0, z)}{\|\operatorname{Grad}_{(z)} g(t_0, z)\|^3}$$

$$= -\left\{ \frac{\frac{\partial^2 g}{\partial t \partial \bar{t}} \|\operatorname{Grad}_{(z)} g\|^2 - 2\operatorname{Re}\left\{ \sum_{\alpha=1}^{n} \frac{\partial^2 g}{\partial t \partial \bar{z}_\alpha} \frac{\partial g}{\partial \bar{t}} \frac{\partial g}{\partial z_\alpha} \right\} + (\Delta_{(z)} g) \left|\frac{\partial g}{\partial t}\right|^2}{\|\operatorname{Grad}_{(z)} g\|^3} \right\}_{(t_0,}$$

$$= - \frac{\frac{\partial^2 g}{\partial t \partial \bar{t}}(t_0, z)}{\|\operatorname{Grad}_{(z)} g(t_0, z)\|}$$

Hence $\frac{\partial^2 g}{\partial t \partial \bar{t}}(t_0, z) = 0$ for $z \in \partial D(t_0)$. By Remark 5.2, we have $(\mathcal{L}_{(t, z_\alpha)}(-g))(t_0, z) = 0$ for $z \in \partial D(t_0)$

$$(5.3) \qquad \frac{\partial^2 g}{\partial z_\alpha \partial \bar{z}_\alpha}(t_0, z) \left| \frac{\partial g}{\partial t}(t_0, z) \right|^2 = 0 \quad (\alpha = 1, \cdots, n)$$

for $z \in \partial D(t_0)$.

On the other hand, Remark 5.1 that there exists a strictly pseudoconvex boundary point z_0 of $D(t_0)$ in $\mathbb{C}^n$. Since our argument does not depend on the choice of the Euclidean coordinates system of $\mathbb{C}^n$, we may assume that

$$\operatorname{Grad}_{(z)} g(t_0, z) = \left(0, \cdots, 0, \frac{\partial g}{\partial z_n}(t_0, z_0)\right)$$
$$\neq 0.$$

It follows easily from (5.1) and (5.2) that

$$\sum_{\alpha,\beta=1}^{n-1} \left(-\frac{\partial^2 g}{\partial z_\alpha \partial \bar{z}_\beta}\right)(t_0, z_0)\, a_\alpha \bar{a}_\beta > 0$$

for <u>any</u> $a \in \mathbb{C}^{n-1}$ with $a \neq 0$. In particular, we take $a = (1, 0, \cdots, 0)$ and have $\frac{\partial^2 g}{\partial z_1 \partial \bar{z}_1}(t_0, z_0) < 0$.

There exists hence a neighborhood V of z_0 in $\mathbb{C}^n$ such that

$$\frac{\partial^2 g}{\partial z_1 \partial \bar{z}_1}(t_0, z) < 0 \quad \text{for } z \in V.$$

It follows from (5.3) that

$$\frac{\partial g}{\partial t}(t_0, z) = 0 \quad \text{for } \forall z \in V \cap \partial D(t_0).$$

Since $\frac{\partial g}{\partial t}(t_0, z)$ is holomorphic for z in $D(t_0)$, we have by a uniqueness theorem

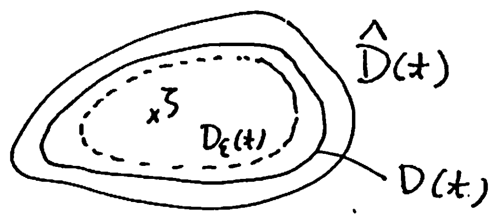
$$\frac{\partial g}{\partial t}(t_0, z) \equiv 0 \quad \text{on } D(t_0)$$

c.q.f.d.

Corollary 5.1 Under the same circumstances as in Theorem 5.1, assume that $\frac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} \equiv 0$ on B, i.e., $\lambda(t)$ is harmonic in B. Then we have $\mathcal{D} = B \times D(t_0)$ where t_0 is a fixed point of B.

Proof. By Theorem 5.1, $g(t, z)$ does not depend on t in B. Given $\varepsilon > 0$, we set $D_\varepsilon(t) = \{ z \in \hat{D}(t) \mid g(t, z) > \varepsilon \}$. Then $D_\varepsilon(t) = D_\varepsilon(t_0)$ where t_0 is a fixed point in B. Since $D(t) = \{ z \in \hat{D}(t_0) \mid g(t, z) > 0 \}$, it follows that $D(t) = D(t_0)$.

c.q.f.d



Here we consider a holomorphic section ζ of $\mathcal{D}$ defined on B

$$\zeta : t \to \zeta(t) \ (t \in B),$$

that is, ζ is a holomorphic mapping from B into $\mathcal{D}$ such that $\zeta(t) \in D(t)$ for all $t \in B$.

We thus have the green's function $g(t, z)$ and the Robin constant $\lambda(t)$ for $(D(t), \zeta(t))$. Consequently,

$$(5.4) \quad g(t,z) = \frac{1}{\|z-\zeta\|^{2n-2}} + \lambda(t) + h(t,z)$$

where $h(t,z)$ is harmonic for z near $z = \zeta(t)$ and

$$(5.5) \quad h(t, \zeta(t)) = 0$$

Under this notation, we get

Theorem 5.2 If $\mathcal{D}: t \to D(t) \ (t \in B)$ is function theoretic, then $\log(-\lambda(t))$, as well as $-\lambda(t)$, is subharmonic for t in B.

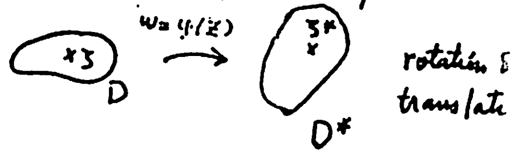
To prove this we need the following elementary and distinguished property of the Euclidean space $\mathbb{C}^n$:

Proposition 5.1. Consider an affine transformation of $\mathbb{C}^n$ of the form $w = \varphi(z) = a(Az) + b$, namely,

$$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = a \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

where $0 \neq a \in \mathbb{C}$, $b \in \mathbb{C}^n$ and A is an $n \times n$ unitary matrix i.e., ${}^t\bar{A}A = E_n$ (the $n \times n$ identity matrix).

Let D be a domain in $\mathbb{C}^n$ and let $\zeta \in D$. We set $D^* = \varphi(D)$ and $\zeta^* = \varphi(\zeta)$. Consider the Green's function $g(z)$ (resp. $g^*(z)$) and the Robin constant λ (resp. λ^*) for (D, ζ) (resp. (D^*, ζ^*)). Then



$$(5.6)\quad \begin{cases} g^*(w) = \dfrac{g(z)}{|a|^{2n-2}} & \text{where } w = \varphi(z) \\ \lambda^* = \dfrac{\lambda}{|a|^{2n-2}} \end{cases}$$

Proof. Since harmonicity is invariant under such transformation φ and φ^{-1}, if we set

$$G(w) = g(\varphi^{-1}(w)) = g(z),$$

then $G(w)$ is harmonic for w in $D^* - \{\zeta^*\}$. Moreover, $G(w) = 0$ for $w \in \partial D^*$. Since $\|w - \zeta^*\| = |a|\,\|z - \zeta\|$ where $w = \varphi(z)$, and since

$$g(z) = \frac{1}{\|z-\zeta\|^{2n-2}} + \lambda + h(z)$$

with $h(\zeta) = 0$, we see that

$$G(w) = \frac{|a|^{2n-2}}{\|w-\zeta^*\|^{2n-2}} + \lambda + H(w)$$

where $H(w) = h(\varphi^{-1}(w))$, so that $H(\zeta^*) = 0$. It follows that

$$g^*(w) = \frac{G(w)}{|a|^{2n-2}} = \frac{1}{\|w-\zeta^*\|^{2n-2}} + \frac{\lambda}{|a|^{2n-2}} + \frac{H(w)}{|a|^{2n-2}}$$

and hence that $\quad \lambda^* = \dfrac{\lambda}{|a|^{2n-2}}$.

c.q.f.d.

<u>Proof of Theorem 5.2</u> Take a disk $B_0 : |t - t_0| < \rho$ in B. Let $f(t)$ be any holomorphic function on B_0 such that $f(t) \neq 0$ at any $t \in B_0$. Let $f_1(t)$ be one of the branches of $\sqrt[2n+2]{f(t)}$ on B_0, and consider the <u>Hartogs transformation</u>

$$T: \begin{cases} t = t \\ w = f_1(t)\,(z - \zeta(t)) \end{cases}$$

where $t \in B_0$. We set

$$\mathcal{D}_{B_0} = \bigcup_{t \in B_0} (t, D(t)) \quad \text{and} \quad \mathcal{D}^* = T(\mathcal{D}_{B_0}).$$

Then $\mathcal{D}^*$, as well as $\mathcal{D}_{B_0}$, is pseudoconvex domain in $B \times \mathbb{C}^n$. Since the holomorphic section ζ of $\mathcal{D}$ is transformed to the constant zero section 0, we thus have the Green's function $g^*(t, w)$ of $\mathcal{D}^*$ and the Robin constant $\lambda^*(t)$ for $(D^*(t), 0)$. By Proposition 5.1, we have

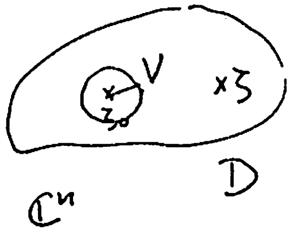
$$\lambda^*(t) = \frac{\lambda(t)}{|f(t)|} \quad \text{for } \forall t \in B.$$

By (*) in p.19, we see that $-\lambda^*(t)$ is subharmonic on B. It turns out that $-\lambda(t)/|f(t)|$ does not attain its local strict maximum in B_0. It follows from $\lambda(t) < 0$ that $\log(-\lambda(t))$ is subharmonic on B_0 and hence that $-\lambda(t)$ is subharmonic on B_0. c.q.f.d.

Chap. VII Strictly subharmonic functions

§1. The Robin constants $\Lambda(\zeta)$

Let D be a pseudoconvex domain in $\mathbb{C}^n$ $(n \geq 2)$ with smooth boundary. Let $\zeta \in D$. We have the Green's function $G(\zeta, z)$ and the Robin constant $\Lambda(\zeta)$ for (D, ζ), so that



$$G(\zeta, z) = \frac{1}{\|z-\zeta\|^{2n-2}} + \Lambda(\zeta) + H(\zeta, z) \tag{1.1}$$

where $H(\zeta, z)$ is harmonic for z in D with

$$H(\zeta, \zeta) = 0. \tag{1.2}$$

Hence $\Lambda(\zeta)$ defines a negative real-valued function on D.

<u>Lemma 1.1</u> $\Lambda(\zeta)$ is real-analytic in D.

<u>Proof.</u> Let $\zeta_0 \in D$ and take a ball $V: \|z - z_0\| < r$ such that $V \subset\subset D$. Consider the function $v(\zeta, z)$ in $\overline{V} \times \overline{V}$ such that

$$v(\zeta, z) = \begin{cases} G(\zeta, z) - \dfrac{1}{\|z-\zeta\|^{2n-2}} & (z \neq \zeta) \\ \Lambda(\zeta) & (z = \zeta). \end{cases}$$

By the well-known symmetry: $G(\zeta, z) = G(z, \zeta)$ in $D \times D$, we have

$$v(\zeta, z) = v(z, \zeta) \quad \text{in } \overline{V} \times \overline{V},$$

so that $v(\zeta, z)$ is harmonic for ζ as well as for z in $\overline{V}$.

By Poisson's formula, we have, for $\zeta \in V$,

$$\begin{aligned} \Lambda(\zeta) &= v(\zeta, \zeta) \\ &= \frac{1}{(r\omega_{2n})^2} \int_{\partial V} \left\{ \int_{\partial V} \frac{r^2 - \|\zeta - \zeta_0\|^2}{\|\zeta - \xi\|^{2n-2}} v(\eta, \xi)\, ds_\xi \right\} \frac{r^2 - \|\zeta - \zeta_0\|^2}{\|\eta - \zeta\|^{2n-2}} \end{aligned}$$

By (1.2) we see that $v(\zeta, z) \leqq 0$ on $\bar{V} \times \bar{V}$. It follows from Fubini's theorem that the right-hand repeated integral becomes the integral over $(\partial V)\times(\partial V)$, i.e.,

$$\Lambda(\zeta) = \frac{1}{(r\omega_{2n})^2} \iint_{(\partial V)\times(\partial V)} v(\zeta,\eta) \frac{r^2 - \|\zeta - \mathfrak{z}\|^2}{\|\zeta - \mathfrak{z}\|^{2n\cdot 2}} \cdot \frac{r^2 - \|\eta - \mathfrak{z}\|^2}{\|\eta - \mathfrak{z}\|^{2n\cdot 2}} (ds_\zeta \times ds_\eta)$$

for $\forall \mathfrak{z} \in V$. Hence $\Lambda(\mathfrak{z})$ is real analytic in V. c.q.f.d.

Definition 5.1 Let $s(z)$ be a real-valued function in a domain D in $\mathbb{C}^n$. If $s(z)$ satisfies the conditions:

(1) $-\infty \leqq s(z) < +\infty$ and $s(z)$ is uppersemi continuous in D;

(2) Let $z_0 \in D$ and $a \in \mathbb{C}^n$ with $a \neq 0$. Then $s(z_0 + a\tau)$ is subharmonic for τ in $|\tau| \ll 1$,

then $s(z)$ is called a plurisubharmonic function in D

Definition 5.2 Let $s(z)$ be a real-valued function of class C^2 in a domain D in $\mathbb{C}^n$. If $s(z)$ satisfies the condition:

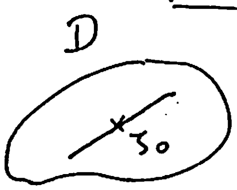
$$\left(\frac{\partial^2 \lambda(z)}{\partial z_\alpha \partial \bar{z}_\beta}\right)_{\alpha,\beta=1,\dots,n} > 0 \quad \text{for } z \in D,$$

Then $s(z)$ is called a strictly plurisubharmonic function in D.

It is clear that, under the assumption that $s(z)$ is of class C^2 in D, $s(z)$ is plurisubharmonic in D, if and only if

$$\left(\frac{\partial^2 \lambda(z)}{\partial z_\alpha \partial \bar{z}_\beta}\right)_{\alpha,\beta=1,\dots,n} \geqq 0 \quad \text{for } z \in D.$$

Lemma 1.2 $-\Lambda(\zeta)$ is strictly pluri subharmonic function in D.



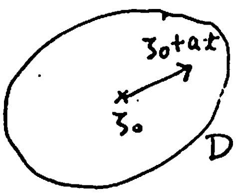
Proof. Take $\zeta_0 \in D$ and consider a complex line ζ through ζ_0 : $\zeta = \zeta(t) = \zeta_0 + at$ where $a \in \mathbb{C}^4$ with $a \neq 0$ and $t \in \mathbb{C}$. We choose a small disk $B : |t| < \rho$ such that

$$\zeta(t) = \zeta_0 + at \in D \text{ for all } t \in B.$$

Then $\mathcal{D} = B \times D$ is a pseudoconvex domains in $B \times \mathbb{C}^4$ with holomorphic section $\zeta : t \to \zeta(t)$ $(t \in B)$. It follows from Theorem 5.2 that $-\Lambda(\zeta(t))$ and $\log(-\Lambda(\zeta(t)))$ are subharmonic functions in B. Hence $-\Lambda(\zeta)$ and $\log(-\Lambda(\zeta))$ are plurisubharmonic in B.

We consider the translation

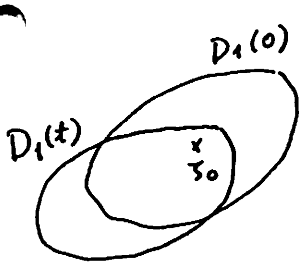
$$T_1 : (t, z) \to (t, w) = (t, z - at).$$



and set $\mathcal{D}_1 = T_1(B \times D)$. We thus have

(i) $\mathcal{D}_1$ is pseudoconvex in $B \times \mathbb{C}^4$

i.e., $\mathcal{D}_1 : t \to D_1(t)$ $(t \in B)$ is a function-theoretic variation.



(ii) $\mathcal{D}_1 \supset B \times \{\zeta_0\}$

We form the Green's function $g_1(t, w)$ and the Robin constant $\lambda_1(t)$ for $(D_1(t), \zeta_0)$. Since $\zeta(t) \in D$ corresponds to $\zeta_0 \in D_1(t)$, we have by Proposition 5.1

$$\begin{cases} g_1(t, w) = G(\zeta_0 + at, z) \\ \lambda_1(t) = \Lambda(\zeta_0 + at) \end{cases}$$

where $w = z - at$

Precisely, $g_1(t, w_1, \dots, w_n) = G(\zeta_{01} + a_1 t, \dots, \zeta_{0n} + a_n t\,;\, w_1 + a_1 t, \dots, w_n + a_n t)$.

$\lambda_1(t) \qquad = \Lambda(\zeta_{01} + a_1 t, \dots, \zeta_{0n} + a_n t)$

By direct calcul, we get

$$\frac{\partial^2 \lambda_1}{\partial t \partial \bar t}(0) = \sum_{\alpha, \beta = 1}^{n} \frac{\partial^2 \Lambda}{\partial \zeta_\alpha \partial \bar\zeta_\beta}(\zeta_0)\, a_\alpha \overline{a_\beta} \ ;$$

$$\frac{\partial g_1}{\partial t}(0, w) = \sum_{\beta=1}^{n} a_\beta \left(\frac{\partial G}{\partial \zeta_\beta} + \frac{\partial G}{\partial z_\beta} \right)(\zeta_0, z) \ ;$$

$$\frac{\partial^2 g_1}{\partial \overline{w}_\alpha \partial t}(0, w) = \sum_{\beta=1}^{n} a_\beta \frac{\partial}{\partial \bar z_\alpha} \left(\frac{\partial G}{\partial \zeta_\beta} + \frac{\partial G}{\partial z_\beta} \right)(\zeta_0, z)$$

Let $\psi(z)$ be a defining fn of the domain D. Then $\psi_1(t, w) = \psi(\quad w + a t)$ is the defining function of $\mathcal{D}_1$

It follows from Fundamental Formula that

$$\frac{\partial^2 \lambda_1}{\partial t \partial \bar t}(0) = -\frac{1}{(n-1)\omega_{2n}} \int_{\partial D_1(0)} k_2(0, w)\, \|\mathrm{Grad}_w\, g_1(0, w)\|^2 \, ds_w$$

$$- \frac{4}{(n-1)\omega_{2n}} \iint_D \sum_{\alpha=1}^{n} \left| \sum_{\beta=1}^{n} a_\beta \frac{\partial}{\partial \bar z_\alpha} \left(\frac{\partial G}{\partial \zeta_\beta} + \frac{\partial G}{\partial z_\beta} \right)(\zeta_0, z) \right|^2 dV$$

Since $\mathcal{D}_1$ is pseudoconvex in $B \times \mathbb{C}^n$, we see that $k_2(0, w) \geqq 0$ for $z \in \partial D_1(0)$. Hence

$$(1.3) \quad \sum_{\alpha, \beta = 1}^{n} \frac{\partial^2 (-\Lambda)}{\partial \zeta_\alpha \partial \bar\zeta_\beta}(\zeta_0)\, a_\alpha \overline{a_\beta} \geqq \frac{4}{(n-1)\omega_{2n}} \iint_D \sum_{\alpha=1}^{n} \left| \sum_{\beta=1}^{n} \frac{\partial}{\partial \bar z_\alpha} \left(\frac{\partial G}{\partial \zeta_\beta} + \frac{\partial G}{\partial z_\beta} \right)(\zeta_0, z) \right|^2 dV$$

$$\geqq 0$$

This inequality teaches us (that) $(-\Lambda)(\zeta)$ is plurisubharmonic in D and will (also) be useful later.

It remains to prove that $(-\Lambda)(\zeta)$ is strictly plurisubharmo.

in D. Let us prove it by contradiction.

Assume that there exist some $\zeta_0 \in D$ and some $a \in \mathbb{C}^n$ with $a \neq 0$ such that

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2(-\Lambda)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta_0)\, a_\alpha \overline{a_\beta} = 0 .$$

By the above argument, we have

$$\lambda_1(0) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2(-\Lambda)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta_0)\, a_\alpha \overline{a_\beta} = 0$$

where $\lambda_1(t)$ is the Robin constant for $(D_1(t), 0)$. Since $\mathcal{D}_1 : t \to D_1(t)$ $(t \in B)$ is function-theoretic variation, it follows from Rigidity (Theorem 5.1) that

$$\frac{\partial g_1}{\partial t}(0, w) \equiv 0 \quad \text{on } D_1(0),$$

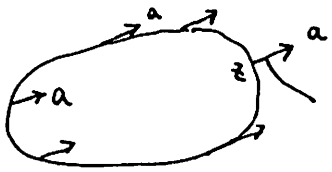
Or equivalently,

$$\sum_{\beta=1}^{n} a_\beta \left(\frac{\partial G}{\partial \zeta_\beta} + \frac{\partial G}{\partial z_\beta} \right)(\zeta_0, z) \equiv 0 \quad \text{on } D$$

Since $G(\zeta, z) \equiv 0$ for $(\zeta, z) \in D \times \partial D$, we have $\frac{\partial G}{\partial \zeta_\beta}(\zeta, z) \equiv 0$ on $D \times \partial D$. It follows that

$$(1.4) \qquad \sum_{\beta=1}^{n} a_\beta \frac{\partial G}{\partial z_\beta}(\zeta_0, z) = 0 \quad \text{on } \partial D$$

This means that the constant vector $a (\neq 0)$ is lying on the complex tangent plane Π_z of ∂D at <u>every</u> point z $(\in \partial D)$. This contradicts that D is a bounded smooth domain in $\mathbb{C}^n$.



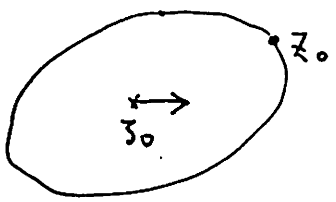
Here, a cannot be lie on the target plane Π_z !

c.q.f.d

We give another proof of "strictness".

Since $\Lambda(\zeta)$ does not depend on the choice of the Euclidean coordinates, it is sufficient to prove that

$$(1.5) \quad \frac{\partial^2(-\Lambda)}{\partial\zeta_1\partial\bar\zeta_1}(\zeta_0) > 0 .$$



To prove this, we take $a=(1,0,\dots,0)$ and consider the Hartogs Transformation

$$T: \begin{cases} t = t \\ w = z - (t,0,\dots,0). \end{cases} \quad \text{where } B = (|t| \ll 1)$$

Set $D_1 = T(B\times D)$ and consider the Green's function $g_1(t,w)$ and the Robin constant $\lambda_1(t)$ for $(D_1(t),\zeta_0)$, so that

$$\begin{cases} \lambda_1(t) = \Lambda(\zeta_{01}+t, \zeta_{02},\dots,\zeta_{0n}) \\ g_1(t,w) = G(\zeta_{01}+t,\zeta_{02},\dots,\zeta_{0n}; w_1+t, w_2,\dots,w_n) \end{cases}$$

We thus have $\quad \dfrac{\partial^2\lambda_1}{\partial t\partial\bar t}(0) = \dfrac{\partial^2\Lambda}{\partial\zeta_1\partial\bar\zeta_1}(\zeta_0)$

By Fundamental Formula, we get

$$\frac{\partial^2\lambda_1}{\partial t\partial\bar t}(0) = -\frac{1}{(n-1)\omega_n}\int_{\partial D_1(0)} k_2(0,w)\,\|\mathrm{Grad}_{(w)}\, g_1\|^2_{(0,w)}\, ds_w$$

$$-\frac{4}{(n-1)\omega_{2n}}\iint_{D_1(0)} \sum_{\alpha=1}^{n}\left|\frac{\partial^2 g_1}{\partial t\partial\bar w_\alpha}\right|^2 dV. \leqq -\frac{1}{(n-1)\omega_{2n}}\int_{\partial D_1(0)} k_2(0,w)\,\|\mathrm{Grad}\, g_1\|^2 ds$$

Here we calculate $k_2(0,w)$ exactly. Let $\psi(z)$ be a defining function of the domain D in $\mathbb{C}^n$. Then

$$\psi_1(t,w) = \psi(t+w_1, w_2,\dots,w_n)$$

becomes a defining function of the domain D_1 in $B\times\mathbb{C}^n$. It follows from simple calcul that

$L_{(t,w_1)}\psi_1 \equiv 0$;

$L_{(t,w_\alpha)}\psi_1 = \frac{\partial^2\psi}{\partial z_1\partial\bar z_1}\left|\frac{\partial\psi}{\partial z_\alpha}\right|^2 - 2\mathrm{Re}\left\{\frac{\partial^2\psi}{\partial z_1\partial\bar z_\alpha}\frac{\partial\psi}{\partial \bar z_1}\frac{\partial\psi}{\partial z_\alpha}\right\} + \frac{\partial^2\psi}{\partial z_\alpha\partial\bar z_\alpha}\left|\frac{\partial\psi}{\partial z_1}\right|^2 \left(\overline{\overline{\mathrm{def}}}\ L_{1\alpha}\psi\right)$

Since D is pseudoconvex in C^n, we see that $L_{1\alpha}\psi \geq 0$ on ∂D. Moreover, by Remark 5.1, there exists a strictly pseudoconvex boundary point z_0 of D. Hence

$$L_{1\alpha}\psi(z_0) > 0 \quad \text{for some } \alpha_1 \ (2 \leq \alpha \leq n).$$

It follows that there exists a point $w_0 = z_0$ such that

$$\mathcal{L}\psi_1(0,w_0) = \sum_{\alpha=1}^{n} L_{(t,w_\alpha)}\psi_1(0,w_0)$$

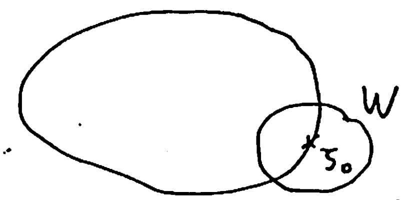
$$\geq L_{(t,w_{\alpha_1})}\psi_1(0,w_0) = L_{1\alpha_1}\psi(z_0) > 0.$$

so that $k_2(0,w_0) > 0$. Consequently,

$$\frac{\partial^2\lambda_1(0)}{\partial t\partial\bar t} < 0. \qquad \text{c.q.f.d.}$$

Lemma 1.3 $\lim_{\zeta\to\partial D} \Lambda(\zeta) = -\infty$.

Proof.



This is clear intuitively, because

$$\Lambda(\zeta) = H_{\frac{-1}{\|z-\zeta\|^{2n-2}}}(\zeta)$$

where H_u is the Dirichlet solution with boundary values u.

Precisely, take $\zeta_0 \in \partial D$. Given $M \gg 1$, we take a small ball W about ζ_0 such that

$$(*) \quad \frac{1}{\|z-\zeta\|^{2n-2}} > M \quad \text{for } \forall z, \forall \zeta \in W.$$

We consider the Dirichlet solution on D with boundary values $u_{-M}(z)$

$$u_{-M} = \begin{cases} -M & \text{on } \partial D \cap W \\ 0 & \text{on } \partial D - W. \end{cases}$$

Hence

$$-M < u_{-M} < 0 \quad \text{on } D.$$

Fix ζ in $D \cap W$ and form the harmonic function in D such that

$$s(z) = u_{-M}(z) - \underbrace{v(\zeta, z)}_{\parallel \; G(\zeta,z) - \frac{1}{\|z-\zeta\|^{2n-2}} \; (\leqq 0)}.$$

It is clear that

$$s(\zeta) = u_{-M}(\zeta) - \Lambda(\zeta) \quad \text{and} \quad -M \leqq s(z) \text{ on } D.$$

Moreover, (*) yields that

$$\varliminf_{z \to \partial D} s(z) \geqq 0.$$

It follows from the minimum principle that

$$s(z) \geqq 0 \quad \text{on } D.$$

In particular, $\quad s(\zeta) = u_{-M}(\zeta) - \Lambda(\zeta) \geqq 0$

It turns out

$$u_{-M}(\zeta) \geqq \Lambda(\zeta) \text{ for all } \zeta \in D \cap W$$

Therefore

$$\underbrace{\varlimsup_{\zeta \to \zeta_0} u_{-M}(\zeta)}_{= -M} \geqq \varlimsup_{\zeta \to \zeta_0} \Lambda(\zeta)$$

which means $\quad \lim_{\zeta \to \zeta_0} \Lambda(\zeta) = -\infty.$ c.q.f.d.

We summarize that

If D is a pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, then the Robin constant $-\Lambda(\zeta)$ in D defines "a positive valued, real analytic, strictly plurisubharmonic, exhaustion function" in D

We have proved

<u>Theorem 1.1</u> Let D be a pseudoconvex domain in $\mathbb{C}^n$ $(n \geqq 2)$ with smooth boundary. Let $\Lambda(\zeta)$ be the Robin constant for (D, ζ) where $\zeta \in D$. Then $-\Lambda(\zeta)$ is a positive real-valued, real analytic, strictly plurisubharmonic and exhaustion function on D.

We also prove

<u>Theorem 1.2</u> Under the same notation as in Theorem 1.1, we see that $\log(-\Lambda(\zeta))$ is strictly plurisubharmonic function on D.

<u>Proof.</u> It remains to prove "strictness", that is, for $\zeta_0 \in D$ and $a \in \mathbb{C}^n$ with $a \neq 0$, it holds

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta_0) a_\alpha \bar{a}_\beta > 0 .$$

Our aim is to construct the function-theoretic variation $\mathcal{D}_2 : t \to D_2(t)\ (t \in B)$ such that

$$\frac{\partial^2 \lambda_2}{\partial t}(0) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta_0) a_\alpha \bar{a}_\beta .$$

To realize it, take a small disk $B : |t| < \rho$ such that

$$\zeta(t) \underset{\text{df}}{=} \zeta_0 + at \in D \text{ for all } t \in B.$$

Translation with quantity $-at$.

We can consider the translation:

$$T_1 : (t, z) \longrightarrow (t, w) = (t, z - at)$$

$$B \times D \longrightarrow \mathcal{D}_1 = T_1(B \times D) \subset B \times \mathbb{C}^n$$

We then have a variation:

$$\mathcal{D}_1 : t \longrightarrow D_1(t) \quad (t \in B)$$

such that

(i) $\mathcal{D}_1 \supset B \times \{z_0\}$, and (ii) $\mathcal{D}_1$ is pseudoconvex in $B \times \mathbb{C}^n$ because D is pseudoconvex in $\mathbb{C}^n$. We denote by $g_1(t, w)$ and $\lambda_1(t)$ the Green's function and the Robin constant for $(D_1(t), z_0)$. By Proposition 5.1 we get, for each fixed $t \in B$,

$$(1.6) \begin{cases} g_1(t, w) = G(\zeta(t), z) ; \\ \lambda_1(t) = \Lambda(\zeta(t)) \end{cases}$$

where $z = w + at$. We note $D_1(0) = D$ and $z = w$ at $t = 0$.

Here we consider the Taylor development of $\log(-\lambda_1(t))$ at $t = 0$ as follows:

$$\log(-\lambda_1(t)) = \log(-\lambda_1(0)) + \mathrm{Re}\left\{ 2\left[\frac{\partial \log(-\lambda_1)}{\partial t}\right]_{t=0} t \right\} + k(t)$$

$$\underset{\text{def}}{=} \mathrm{Re}\{c_0 + c_1 t\} + k(t)$$

We set $\varphi(t) = c_0 + c_1 t$, and have

$$\log(-\lambda_1(t)) = \mathrm{Re}\{\varphi(t)\} + k(t)$$

so that $k(0)=0$; $\frac{\partial k}{\partial t}(0) = 0$;

$$\frac{\partial^2 k}{\partial \bar{t}\partial t}(0) = \frac{\partial^2 \log(-\lambda_1)}{\partial t \partial \bar{t}}(0) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta_0)\, a_\alpha \bar{a}_\beta .$$

We consider the following Hartogs' transformation:

$$T_2 : (t, w) \to (t, W) = \left(t, e^{\frac{\varphi(t)}{2n-2}}(w - \zeta_0)\right)$$

$$\mathcal{D}_1 \to \mathcal{D}_2 = T_2(\mathcal{D}_1) \subset B \times \mathbb{C}^n .$$

As usual we obtain the variation:

$$\mathcal{D}_2 : t \to D_2(t) \quad (t \in B)$$

which satisfies

(i) $\mathcal{D}_2 \supset B \times \{0\}$

(ii) $\mathcal{D}_2$, as well as $\mathcal{D}_1$, is pseudoconvex domain in $B \times \mathbb{C}^n$!

We thus have the Green's function $g_2(t, W)$ and the Robin constant $\lambda_2(t)$ for each $(D_2(t), \zeta)$. Proposition 5.1 implies that

$$(1.7)\quad \begin{cases} g_2(t, W) = e^{-\mathrm{Re}\,\varphi(t)}\, g_1(t, w) \\ \lambda_2(t) = e^{-\mathrm{Re}\,\varphi(t)}\, \lambda_1(t) \end{cases}$$

for all $t \in B$, where $w = \zeta_0 + e^{-\frac{\varphi(t)}{2n-2}} W$.

$$\begin{array}{ccccc}
B\times D & \underset{T_1^{-1}}{\overset{T_1}{\rightleftarrows}} & \mathcal{D}_1 & \underset{T_2^{-1}}{\overset{T_2}{\rightleftarrows}} & \mathcal{D}_2 \\
(t,\zeta(t)) & & (t,\zeta_0) & & (t,0) \cdots\cdots \text{pôle} \\
G(\zeta(t),z) & & g_1(t,w) & & g_2(t,W) \cdots\cdots \text{Green's function} \\
\Lambda(\zeta(t)) & & \lambda_1(t) & & \lambda_2(t) \cdots\cdots \text{Robin constant} \\
z & \longrightarrow & w = z - at & \longrightarrow & W = e^{\frac{\varphi(t)}{2n-2}}(w-\zeta_0) \\
We^{\frac{-\varphi(t)}{2n-2}} + at + \zeta_0 & \longleftarrow & w = \zeta_0 + e^{-\frac{\varphi(t)}{2n-2}} W & \longleftarrow & W
\end{array}$$

It follows that $\log(-\lambda_2(t)) = -\mathrm{Re}\,\varphi(t) + \log(-\lambda_1(t)) = k(t)$ and hence that

$$\lambda_2(t) = -e^{-k(t)} \quad ;$$

$$\frac{\partial \lambda_2}{\partial t}(t) = -e^{k(t)} \frac{\partial k(t)}{\partial t} \quad ;$$

$$\frac{\partial^2 \lambda_2}{\partial t \partial \bar t}(0) = -\left\{ e^{k(t)} \left|\frac{\partial k}{\partial t}\right|^2 + e^{k(t)} \frac{\partial^2 k}{\partial t \partial \bar t} \right\}_{t=0}$$

$$= -\frac{\partial^2 k}{\partial t \partial \bar t}(0)$$

$$= -\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial \zeta_\alpha \partial \bar\zeta_\beta}(\zeta_0)\, a_\alpha \bar a_\beta$$

By Fundamental Formula, we have

$$(1.8) \quad \frac{\partial^2 \lambda_2}{\partial t \partial \bar t}(0) = -\frac{1}{(n-1)\omega_{2n}} \int_{\partial D_2(0)} k_2(0,W)\, \|\mathrm{Grad}_{(W)}\, g_2(0,W)\|^2 \, dS_W$$
$$- \frac{4}{(n-1)\omega_{2n}} \iint_{D_2(0)} \left(\sum_{\alpha=1}^{n} \left| \frac{\partial^2 g_2}{\partial t \partial \bar W_\alpha} \right|^2_{(0,W)} \right) dV_W$$

We try to express $\frac{\partial^2 g_2}{\partial t \partial \bar W_\alpha}(0,W)$ and $k_2(0,W)$ by means of the original $G(\zeta,z)$, $\partial G/\partial\bar\zeta_\alpha \partial z_\beta, \cdots$ and $\psi(\zeta)$ (the defining function of D in $\mathbb{C}^n$).

From (1.6) and (1.7) we have

$$g_2(t,W) = e^{-\operatorname{Re}\varphi(t)}\, G\left(\zeta_0+at;\ We^{\frac{-\varphi(t)}{2n-2}} + \zeta_0 + at\right)$$

$$= e^{-c_0-\frac{c_1}{2}t-\frac{\bar c_1}{2}\bar t}\, G\left(\zeta_{01}+a_1t,\dots,\zeta_{0n}+a_nt;\ W_1 e^{\frac{-c_0-c_1t}{2n-2}}+\zeta_{01}+a_1t,\dots\dots, W_n e^{\frac{-c_0+c_1t}{2n-2}}+\zeta_{0n}+a_nt\right)$$

so that

$$\left[\frac{\partial g_2}{\partial t}(t,W)\right]_{t=0}$$

$$= e^{-c_0}\left(-\frac{c_1}{2}\right)G + e^{-c_0}\left\{\sum_{\alpha=1}^{n} a_\alpha \frac{\partial G}{\partial \zeta_\alpha} + \sum_{\alpha=1}^{n}\frac{\partial G}{\partial z_\alpha}\cdot\left[W_\alpha e^{\frac{-c_0}{2n-2}}\left(\frac{-c_1}{2n-2}\right)+a_\alpha\right]\right\} \quad \text{at } t=0$$

Since $We^{\frac{-c_0}{2n-2}} = z-\zeta_0$ at $t=0$ and $c_0=\log(-\Lambda(\zeta_0))$, we have

(1.9) $$\left[\frac{\partial g_2}{\partial t}(t,W)\right]_{t=0}$$

$$= \frac{1}{-\Lambda(\zeta_0)}\left[-\frac{c_1}{2}\left\{G+\frac{1}{n-1}\sum_{\alpha=1}^{n}(z_\alpha-\zeta_{0\alpha})\frac{\partial G}{\partial z_\alpha}\right\} + \sum_{\alpha=1}^{n} a_\alpha\left(\frac{\partial G}{\partial \zeta_\alpha}+\frac{\partial G}{\partial z_\alpha}\right)\right]_{(0,z)}$$

$$\underset{\text{def}}{=} \frac{1}{-\Lambda(\zeta_0)}\, H(a,\zeta_0,z)$$

Moreover, we have, from $W=(z-\zeta_0)(-\Lambda(\zeta_0))^{\frac{1}{2n-2}}$,

$$\frac{\partial^2 g_2}{\partial t\,\partial \bar W_\beta}(0,W) = \frac{1}{-\Lambda(\zeta_0)}\frac{\partial H}{\partial \bar z_\beta}\left(-\Lambda(\zeta_0)\right)^{-\frac{1}{2n-2}}$$

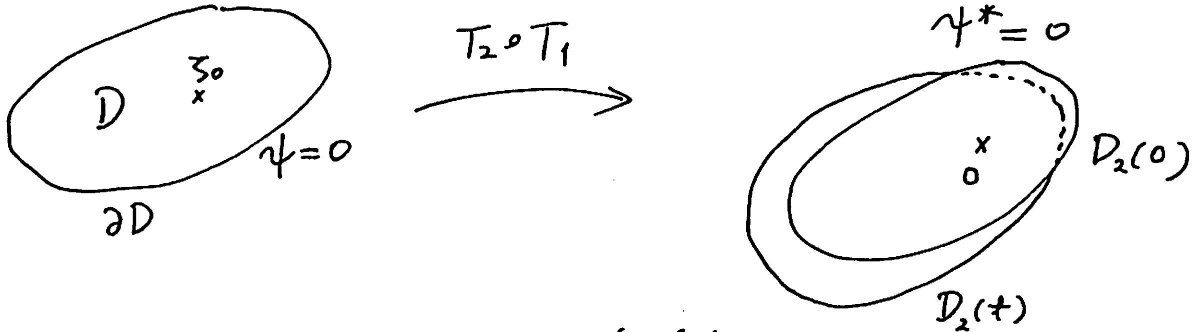
$$= \frac{1}{(-\Lambda(\zeta_0))^{\frac{2n-1}{2n-2}}}\frac{\partial H}{\partial \bar z_\beta}(a,\zeta_0,z)\ ;$$

$$dV_W = (dV_z)\cdot(-\Lambda(\zeta_0))^{\frac{2n}{2n-2}}$$

It follows that

(1.10) $$\iint_{D_1(0)}\sum_{\beta=1}^{n}\left|\frac{\partial^2 g_2}{\partial t\,\partial \bar W_\beta}\right|^2_{(0,W)} dV_W = \frac{1}{-\Lambda(\zeta_0)}\iint_{D}\sum_{\beta=1}^{n}\left|\frac{\partial}{\partial \bar z_\beta}H(a,\zeta_0,z)\right|^2 dV_z$$

Next we calculate $k_2(0, W)$ on $\partial D_2(0)$ by means of $\psi(z)$ (the defining function of D in $\mathbb{C}^n$) on ∂D.



$$\psi^*(t, W) = \psi\left(W e^{\frac{-c_0 - c_1 t}{2n-2}} + at + \zeta_0\right)$$

$$\|\mathrm{Grad}_{(W)} \psi^*\|_{t=0} = \|\mathrm{Grad}_{(z)} \psi\| \cdot (-\lambda)^{\frac{-1}{2n-2}}(\zeta_0)$$

By direct calcul we have

$$\frac{\partial \psi^*}{\partial t} = \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z_\alpha} \times \left\{ e^{\frac{-c_0 - c_1 t}{2n-2}} \frac{-c_1}{2n-2} W_\alpha + a_\alpha \right\}$$

$$\frac{\partial \psi^*}{\partial W_i} = \frac{\partial \psi}{\partial z_i} \times e^{\frac{-c_0 - c_1 t}{2n-2}} \quad ; \quad \frac{\partial^2 \psi^*}{\partial \overline{W}_i \partial W_i} = \frac{\partial^2 \psi}{\partial \overline{z}_i \partial z_i} \cdot \left| e^{\frac{-c_0 - c_1 t}{2n-2}} \right|^2$$

$$\frac{\partial^2 \psi^*}{\partial \overline{t} \partial W_i} = \sum_{\beta=1}^{n} \left[\frac{\partial}{\partial \overline{z}_\beta} \left(\frac{\partial \psi}{\partial z_i} \right) \right] \overline{\left\{ e^{\frac{-c_0 - c_1 t}{2n-2}} \frac{-c_1}{2n-2} W_\beta + a_\beta \right\}} \times e^{\frac{-c_0 - c_1 t}{2n-2}}$$

$$\frac{\partial^2 \psi^*}{\partial t \partial \overline{t}} = \sum_{\alpha=1}^{n} \left\{ e^{\frac{-c_0 - c_1 t}{2n-2}} \frac{-c_1}{2n-2} W_\alpha + a_\alpha \right\} \times \sum_{\beta=1}^{n} \frac{\partial^2 \psi}{\partial \overline{z}_\beta \partial z_\alpha} \overline{\left\{ e^{\frac{-c_0 - c_1 t}{2n-2}} \cdot \frac{-c_1}{2n-2} W_\beta + a_\beta \right\}}$$

$$\therefore L_{(t, W_i)} \psi^*(0, W)$$

$$= \left| \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z_\alpha} \left(e^{\frac{-c_0}{2n-2}} \cdot \frac{-c_1}{2n-2} W_\alpha + a_\alpha \right) \right|^2 \cdot \frac{\partial^2 \psi}{\partial z_i \partial \overline{z}_i} \cdot \left| e^{\frac{-c_0}{2n-2}} \right|^2$$

$$-2 \mathrm{Re} \left\{ \frac{\partial \psi}{\partial z_i} e^{\frac{-c_0}{2n-2}} \cdot \overline{\sum_{\alpha=1}^{n} \left(\frac{\partial \psi}{\partial z_\alpha} \right) \left(e^{\frac{-c_0}{2n-2}} \frac{-c_1}{2n-2} W_\alpha + a_\alpha \right)} \sum_{\beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\beta \partial \overline{z}_i} \left(e^{\frac{-c_0}{2n-2}} \frac{-c_1}{2n-2} W_\beta + a_\beta \right) \overline{\left(e^{\frac{-c_0}{2n-2}} \right)} \right.$$

$$+ \sum_{\alpha=1}^{n} \left\{ e^{\frac{-c_0}{2n-2}} \frac{-c_1}{2n-2} w_\alpha + a_\alpha \right\} \sum_{\beta=1}^{n} \frac{\partial^2 \psi}{\partial \bar{z}_\beta \partial z_\alpha} \overline{\left\{ e^{\frac{-c_0}{2n-2}} \frac{-c_1}{2n-2} w_\beta + a_\beta \right\}}$$

$$\times \left| \frac{\partial \psi}{\partial z_i} \right|^2 \cdot \left| e^{\frac{-c_0}{2n-2}} \right|^2$$

Since $z - \zeta_0 = e^{\frac{-c_0}{2n-2}} w$ at $t=0$, it follows that

$$L_{(t, w_i)} \psi^*(0, w)$$

$$= e^{\frac{-c_0}{n-1}} \left\{ \begin{array}{l} \left| \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z_\alpha} \left\{ \left(\frac{-c_1}{2n-2} \right) (z_\alpha - \zeta_{0\alpha}) + a_\alpha \right\} \right|^2 \cdot \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_i} \\ -2 \operatorname{Re} \left\{ \frac{\partial \psi}{\partial z_i} \sum_{\alpha, \beta=1}^{n} \overline{\left(\frac{\partial \psi}{\partial z_\alpha} \right)} \frac{\partial^2 \psi}{\partial z_\beta \partial \bar{z}_i} \overline{\left\{ \frac{-c_1}{2n-2} (z_\alpha - \zeta_{0\alpha}) + a_\alpha \right\}} \left\{ \frac{-c_1}{2n-2} (z_\beta - \zeta_{0\beta}) + a_\beta \right\} \right. \\ + \sum_{\alpha, \beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta} \left\{ \frac{-c_1}{2n-2} (z_\alpha - \zeta_{0\alpha}) + a_\alpha \right\} \overline{\left\{ \frac{-c_1}{2n-2} (z_\beta - \zeta_{0\beta}) + a_\beta \right\}} \cdot \left| \frac{\partial \psi}{\partial z_i} \right|^2 \end{array} \right\}$$

For the simplicity, we set

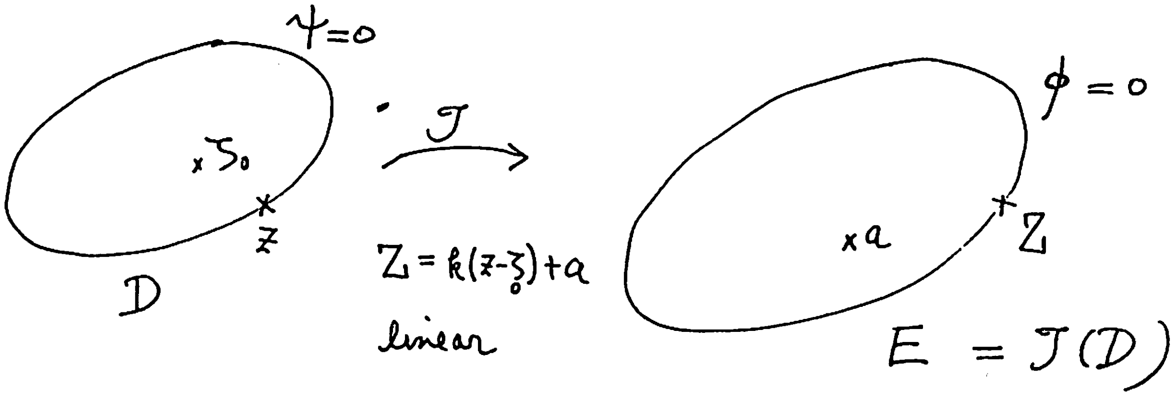
$$\left\{ \begin{array}{l} k = \frac{-c_1}{2n-2} ; \qquad \text{~~(We may assume } k \neq 0.)\text{~~} \\ Z = k(z - \zeta) + a . \end{array} \right.$$

Then

$$L_{(t, w_i)} \psi^*(0, w)$$

$$= e^{\frac{-c_0}{n-1}} \cdot \left\{ \begin{array}{l} \left| \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z_\alpha} Z_\alpha \right|^2 \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_i} \\ -2 \operatorname{Re} \left\{ \frac{\partial \psi}{\partial z_i} \sum_{\alpha, \beta=1}^{n} \overline{\left(\frac{\partial \psi}{\partial z_\alpha} \right)} \frac{\partial^2 \psi}{\partial z_\beta \partial \bar{z}_i} \bar{Z}_\alpha Z_\beta \right\} \\ + \left(\sum_{\alpha, \beta=1}^{n} \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta} Z_\alpha \bar{Z}_\beta \right) \cdot \left| \frac{\partial \psi}{\partial z_i} \right|^2 \end{array} \right\}$$

$$= e^{\frac{-c_0}{n-1}} \sum_{\alpha,\beta=1}^{n} \left\{ \frac{\partial^2 \psi}{\partial z_i \partial \bar z_i} \frac{\partial \psi}{\partial z_\alpha} \frac{\partial \psi}{\partial \bar z_\beta} - 2\operatorname{Re}\left[\frac{\partial \psi}{\partial \bar z_i} \frac{\partial \psi}{\partial z_\alpha} \frac{\partial^2 \psi}{\partial z_i \partial \bar z_\beta} \right] + \left| \frac{\partial \psi}{\partial z_i} \right|^2 \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar z_\beta} \right\} Z_\alpha \bar Z_\beta$$



$$\psi(z) \stackrel{\text{def}}{=} \phi\left(\tfrac{1}{k}(Z-a) + \zeta_0\right)$$

$$\frac{\partial \psi}{\partial z_i} = \frac{1}{k} \frac{\partial \phi}{\partial Z_i}$$

$$\frac{\partial^2 \psi}{\partial \bar z_i \partial z_i} = \left|\frac{1}{k}\right|^2 \frac{\partial^2 \phi}{\partial \bar Z_i \partial Z_i}$$

$$\frac{\partial^2 \psi}{\partial z_i \partial \bar z_\alpha} = \left|\frac{1}{k}\right|^2 \frac{\partial^2 \phi}{\partial Z_i \partial \bar Z_\alpha}$$

$$\therefore\ L_{(t,W_i)} \psi^*(0,W) = e^{\frac{-c_0}{n-1}} \cdot \left|\frac{1}{k}\right|^4 \cdot$$

$$\times \sum_{\alpha,\beta=1}^{n} \left\{ \frac{\partial^2 \phi}{\partial Z_i \partial \bar Z_i} \frac{\partial \phi}{\partial Z_\alpha} \frac{\partial \phi}{\partial \bar Z_\beta} - 2\operatorname{Re}\left[\frac{\partial \phi}{\partial \bar Z_i} \frac{\partial \phi}{\partial Z_\alpha} \frac{\partial^2 \phi}{\partial Z_i \partial \bar Z_\beta} \right] + \left| \frac{\partial \phi}{\partial Z_i} \right|^2 \frac{\partial^2 \phi}{\partial Z_\alpha \partial \bar Z_\beta} \right\} Z_\alpha \bar Z_\beta$$

$$\stackrel{\text{def}}{=} e^{\frac{-c_0 + 2\operatorname{Re} c_1}{n-1}} \cdot L_i \phi(Z) \qquad \text{where } Z \in \partial E$$

We want to show that $k_2(0,W) \gneq 0$ for some $W \in \partial D_2(0)$.
It suffices to prove that

(1.11) $\displaystyle \sum_{i=1}^{n} L_i \phi(Z) \gneq 0$ for some $Z \in \partial E$.

For example, assume that ∂E passes through the point $Z^* = (1, 0, \cdots, 0)$ and Z^* is a strictly pseudo-convex boundary point of E. Then

$$L_i\Phi(Z^*) = \left(\frac{\partial^2\Phi}{\partial Z_i \partial \bar Z_i} \left|\frac{\partial \Phi}{\partial Z_1}\right|^2 - 2\mathrm{Re}\left\{ \frac{\partial\Phi}{\partial \bar Z_i}\frac{\partial \Phi}{\partial Z_1}\frac{\partial^2\Phi}{\partial Z_i \partial \bar Z_1}\right\} + \left|\frac{\partial\Phi}{\partial Z_i}\right|^2 \frac{\partial^2\Phi}{\partial Z_1\partial \bar Z_1} \right)_{Z^*}$$

Since $\operatorname{grad}\Phi(Z^*) \neq 0$, there exists j $(1 \leqq j \leqq n)$ such that $\frac{\partial \Phi}{\partial Z_j}(Z^*) \neq 0$. If we take

$$a = \Big(\frac{\partial\Phi}{\partial Z_j}(Z^*), 0, \cdots 0, \underset{(j)}{-\frac{\partial\Phi}{\partial Z_1}(Z^*)}, \cdots, 0\Big) \neq 0,$$

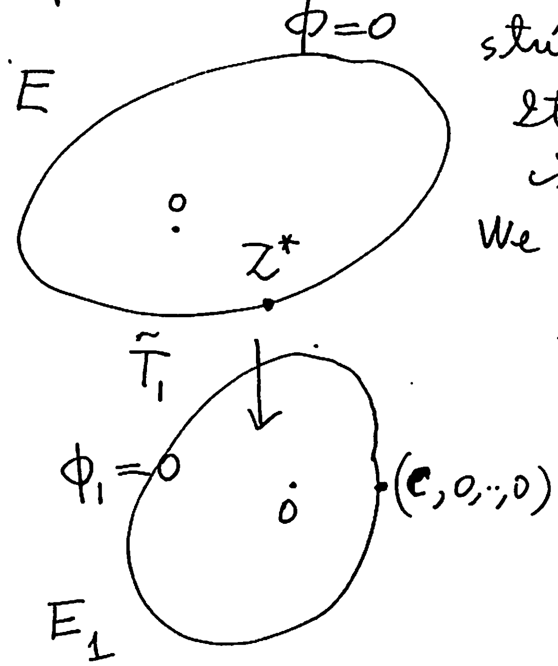
Then

$$\underset{\displaystyle \| \atop \displaystyle L_j\Phi(Z^*)}{\sum_{\alpha,\beta=1}^{n} \frac{\partial^2\Phi}{\partial Z_\alpha \partial \bar Z_\beta} a_\alpha \bar a_\beta} > 0 \text{ at } Z^* \qquad \therefore \sum_{i=1}^{n} L_i\Phi$$

We know $L_i\Phi(Z) \geqq 0$ for all i and all $Z \in \partial E$. Therefore

$$\sum_{i=1}^{n} L_i\Phi(Z^*) \geqq L_j\Phi(Z^*) > 0.$$

In the general case, we consider a unitary transformation as follows: Take $Z^* \in \partial E$ which is strictly pseudoconvex boundary point. It certainly exists, because E is bdd domain in $\mathbb{C}^n$. We may assume $Z^* \neq (0, \cdots, 0)$.



$$\tilde T_1 : \begin{pmatrix} \tilde W_1 \\ \vdots \\ \tilde W_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix}$$

such that ${}^t\bar A A = E_n$ & $T_1(Z^*) = (c, 0, \cdots, 0)$ $(c > 0)$

$$\phi(Z) = \Phi_1(\tilde W) = \Phi_1\left(a_{11}Z_1 + \cdots + a_{1n}Z_n, \cdots, a_{n1}Z_1 + \cdots + a_{nn}Z_n\right)$$

$$\frac{\partial\phi}{\partial Z_i} = \sum_{\gamma=1}^{n}\frac{\partial\Phi_1}{\partial\tilde W_\gamma}a_{\gamma i} \quad ; \quad \frac{\partial\phi}{\partial Z_\alpha} = \sum_{\varepsilon=1}^{n}\frac{\partial\Phi_1}{\partial\tilde W_\varepsilon}a_{\varepsilon\alpha}$$

$$\frac{\partial^2\phi}{\partial\bar Z_i\partial Z_i} = \sum_{\gamma,\delta=1}^{n}\frac{\partial^2\Phi_1}{\partial\bar{\tilde W}_\delta\partial\tilde W_\gamma}a_{\gamma i}\,\overline{a_{\delta i}} \quad ; \quad \frac{\partial^2\phi}{\partial\bar Z_\beta\partial Z_i} = \sum_{\gamma,\delta=1}^{n}\frac{\partial^2\Phi_1}{\partial\bar{\tilde W}_\delta\partial\tilde W_\gamma}a_{\gamma i}\,\overline{a_{\delta\beta}}$$

We thus have by definition

$$\hat L_i\phi(Z) = \sum_{\alpha,\beta=1}^{n}\Bigg\{\sum_{\gamma,\delta=1}^{n}\frac{\partial^2\Phi_1}{\partial\bar{\tilde W}_\delta\partial\tilde W_\gamma}a_{\gamma i}\overline{a_{\delta i}}\sum_{\varepsilon=1}\frac{\partial\Phi_1}{\partial\tilde W_\varepsilon}a_{\varepsilon\alpha}\sum_{\eta=1}\frac{\partial\Phi_1}{\partial\bar{\tilde W}_\eta}\overline{a_{\eta\beta}}$$
$$-2\,\mathrm{Re}\left[\sum_{\gamma=1}^{n}\frac{\partial\Phi_1}{\partial\bar{\tilde W}_\gamma}\overline{a_{\gamma i}}\sum_{\varepsilon=1}^{n}\frac{\partial\Phi_1}{\partial\tilde W_\varepsilon}a_{\varepsilon\alpha}\sum_{\eta,\delta=1}^{n}\frac{\partial^2\Phi_1}{\partial\bar{\tilde W}_\delta\partial\tilde W_\eta}a_{\eta i}\,\overline{a_{\delta\beta}}\right]$$
$$+\left|\sum_{\gamma=1}^{n}\frac{\partial\Phi_1}{\partial\tilde W_\gamma}a_{\gamma i}\right|^2\sum_{\varepsilon\,\eta=1}^{n}\frac{\partial^2\Phi_1}{\partial\tilde W_\varepsilon\partial\bar{\tilde W}_\eta}a_{\varepsilon\alpha}\overline{a_{\eta\beta}}\Bigg\}Z_\alpha\bar Z_\beta$$

$$=\sum_{\substack{\gamma,\delta\\ \varepsilon,\eta}}\Bigg[\left(\frac{\partial^2\Phi_1}{\partial\bar{\tilde W}_\delta\partial\tilde W_\gamma}\frac{\partial\Phi_1}{\partial\tilde W_\varepsilon}\frac{\partial\Phi_1}{\partial\bar{\tilde W}_\eta}a_{\gamma i}\overline{a_{\delta i}}\right)\left(\sum_{\alpha=1}^{n}a_{\varepsilon\alpha}Z_\alpha\right)\left(\sum_{\beta=1}^{n}\overline{a_{\eta\beta}}\bar Z_\beta\right)$$
$$-2\,\mathrm{Re}\left\{\left(\frac{\partial^2\Phi_1}{\partial\bar{\tilde W}_\delta\partial\tilde W_\eta}\frac{\partial\Phi_1}{\partial\bar{\tilde W}_\gamma}\frac{\partial\Phi_1}{\partial\tilde W_\varepsilon}\overline{a_{\gamma i}}\,a_{\eta i}\right)\left(\sum_{\alpha=1}^{n}a_{\varepsilon\alpha}Z_\alpha\right)\left(\sum_{\beta=1}^{n}\overline{a_{\delta\beta}}\bar Z_\beta\right)\right\}$$
$$+\left(\frac{\partial^2\Phi_1}{\partial\tilde W_\varepsilon\partial\bar{\tilde W}_\eta}\frac{\partial\Phi_1}{\partial\tilde W_\gamma}\frac{\partial\Phi_1}{\partial\bar{\tilde W}_\delta}a_{\gamma i}\overline{a_{\delta i}}\right)\left(\sum_{\alpha=1}^{n}a_{\varepsilon\alpha}Z_\alpha\right)\left(\sum_{\beta=1}^{n}\overline{a_{\eta\beta}}\bar Z_\beta\right)\Bigg]$$

$$\underset{\text{def}}{=} L_i\Phi_1(\tilde W)$$

We substitute $Z = Z^*$ i.e., $\tilde W = \tilde W^* = (c, 0, \cdots, 0)$.

We thus have

Kronecker's δ

$$L_i\Phi_1(Z^*) = \sum_{\substack{\gamma,\delta\\ \varepsilon,\eta}}^{n} \left\{ \begin{aligned} &\left(\frac{\partial^2\Phi_1}{\partial\bar{\tilde{W}}_\delta\partial\tilde{W}_\gamma}\frac{\partial\Phi_1}{\partial\tilde{W}_\varepsilon}\frac{\partial\Phi_1}{\partial\bar{\tilde{W}}_\eta}a_{\gamma i}\bar{a}_{\delta i}\right)\delta_{\varepsilon 1}\,\delta_{\eta 1}\,|c|^2 \\ &-2Re\left\{\frac{\partial^2\Phi_1}{\partial\bar{\tilde{W}}_\delta\partial\tilde{W}_\eta}\frac{\partial\Phi_1}{\partial\bar{\tilde{W}}_\gamma}\frac{\partial\Phi_1}{\partial\tilde{W}_\varepsilon}\bar{a}_{\gamma i}a_{\eta i}\right\}\delta_{\varepsilon 1}\,\delta_{\delta 1}\,|c|^2 \\ &+\left(\frac{\partial^2\Phi_1}{\partial\tilde{W}_\varepsilon\partial\bar{\tilde{W}}_\eta}\frac{\partial\Phi_1}{\partial\tilde{W}_\sigma}\frac{\partial\Phi_1}{\partial\bar{\tilde{W}}_\delta}a_{\gamma i}\bar{a}_{\delta i}\right)\delta_{\varepsilon 1}\,\delta_{\eta 1}\,|c|^2 \end{aligned}\right\}$$

$$= |c|^2\cdot\left\{ \begin{aligned} &\sum_{\gamma,\delta}^{n}\frac{\partial^2\Phi_1}{\partial\bar{\tilde{W}}_\delta\partial\tilde{W}_\gamma}\left|\frac{\partial\Phi_1}{\partial\tilde{W}_1}\right|^2 a_{\gamma i}\bar{a}_{\delta i} \\ &-2Re\left\{\sum_{\gamma,\eta}^{n}\frac{\partial^2\Phi_1}{\partial\bar{\tilde{W}}_1\partial\tilde{W}_\eta}\frac{\partial\Phi_1}{\partial\bar{\tilde{W}}_\gamma}\frac{\partial\Phi_1}{\partial\tilde{W}_1}\bar{a}_{\gamma i}a_{\eta i}\right\} \\ &+\sum_{\gamma,\delta=1}^{n}\frac{\partial^2\Phi_1}{\partial\tilde{W}_1\partial\bar{\tilde{W}}_1}\frac{\partial\Phi_1}{\partial\tilde{W}_\gamma}\frac{\partial\Phi_1}{\partial\bar{\tilde{W}}_\delta}a_{\gamma i}\bar{a}_{\delta i} \end{aligned}\right\}$$

It follows that

$$\sum_{i=1}^{n} L_i\Phi_1(Z^*)$$

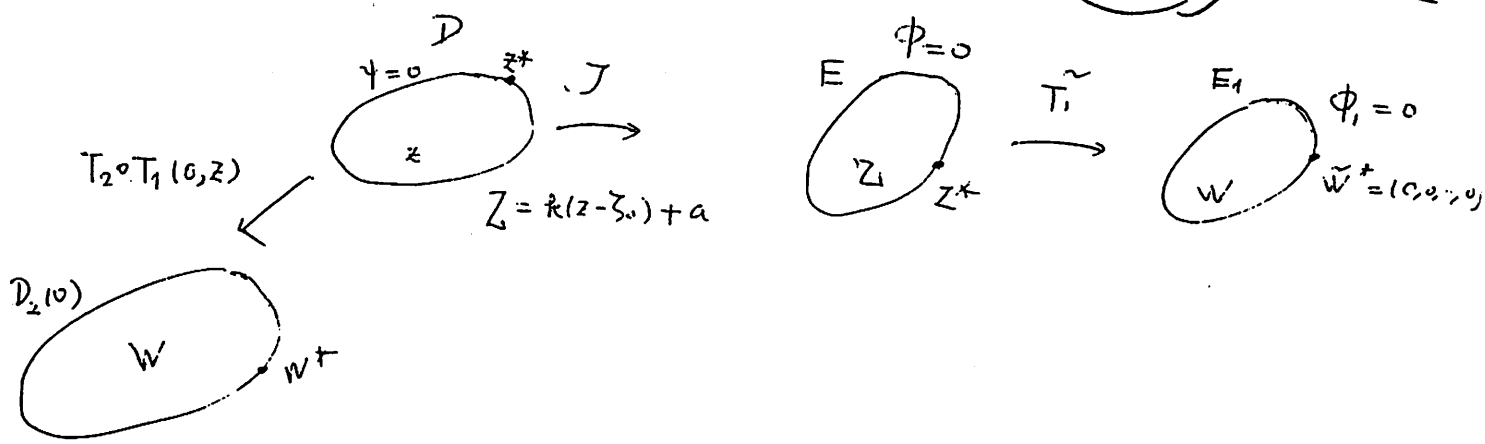
$$= |c|^2\cdot\left\{ \begin{aligned} &\sum_{\gamma,\delta}\frac{\partial^2\Phi_1}{\partial\bar{\tilde{W}}_\delta\partial\tilde{W}_\gamma}\left|\frac{\partial\Phi_1}{\partial\tilde{W}_1}\right|^2\left(\sum_{i=1}^{n}a_{\gamma i}\bar{a}_{\delta i}\right) \\ &-2Re\left\{\sum_{\gamma,\delta=1}^{n}\frac{\partial^2\Phi_1}{\partial\bar{\tilde{W}}_1\partial\tilde{W}_\eta}\frac{\partial\Phi_1}{\partial\bar{\tilde{W}}_\sigma}\frac{\partial\Phi_1}{\partial\tilde{W}_1}\left(\sum_{i=1}^{n}\bar{a}_{\gamma i}a_{\eta i}\right)\right\} \\ &+\sum_{\gamma,\delta=1}^{n}\frac{\partial^2\Phi_1}{\partial\tilde{W}_1\partial\bar{\tilde{W}}_1}\frac{\partial\Phi_1}{\partial\tilde{W}_\sigma}\frac{\partial\Phi_1}{\partial\bar{\tilde{W}}_\delta}\left(\sum_{i=1}^{n}a_{\gamma i}\bar{a}_{\delta i}\right) \end{aligned}\right\}$$

Since $\sum_{i=1}^{n}a_{\gamma i}\bar{a}_{\delta i} = \delta_{\gamma\delta}$, it becomes

$$= |c|^2\left\{\left|\frac{\partial\Phi_1}{\partial\tilde{W}_1}\right|^2\Delta_{(\tilde{W})}\Phi_1 - 2Re\left\{\sum_{\alpha=1}^{n}\frac{\partial^2\Phi_1}{\partial\bar{\tilde{W}}_1\partial\tilde{W}_\alpha}\frac{\partial\Phi_1}{\partial\bar{\tilde{W}}_\alpha}\frac{\partial\Phi_1}{\partial\tilde{W}_1}\right\} + \frac{\partial^2\Phi_1}{\partial\tilde{W}_1\partial\bar{\tilde{W}}_1}\left\|Grad_{(\tilde{W})}\Phi_1\right\|^2\right\}_{\tilde{W}^*}$$

$\underset{def}{=} |c|^2\cdot I$

Because Z^* is a strictly pseudoconvex boundary point of E, $\tilde{W}^*$ is a strictly pseudoconvex boundary point of E_1. It follows that the last $|c|^2\cdot I > 0$.



Therefore $\sum_{i=1}^{n} L_i\Phi(Z^*) = \sum_{i=1}^{n} L_i\Phi_1(W^*) = |c|^2 I > 0$, which proves (1.11).

We conclude that, if we put $W^* = T_2\circ T_1(0, z^*) \in \partial D_2(0)$, then

$$k_2(0, W^*) = e^{\frac{-c_0}{n-1}} \sum_{i=1}^{n} L_i\Phi(Z^*) \gneq 0$$

By (1.8), we have

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\lambda)}{\partial z\,\partial \bar z}(\zeta_0)\, a_\alpha \bar a_\beta \geq \frac{1}{(n-1)\omega_{2n}} \int_{\partial D_2(0)} k_2(0,W)\, \|\mathrm{Grad}_{(W)}\, g_2(0,W)\|^2\, dS_W$$

$$\gneq 0$$

Namely,

$\log(-\lambda)$ is strictly plurisubharmonic on D.

c.q.f.d

We have proved

Theorem 1.1 Let D be a pseudoconvex domain in $\mathbb{C}^n$ $(n \geq 2)$ with smooth boundary. Let $\Lambda(z)$ be the Robin constant for (D, z) where $z \in D$. Then $-\Lambda(z)$ is a positive real-valued, real analytic, strictly plurisubharmonic and exhaustion function on D.

We also prove

Theorem 1.2 Under the same notation as in Theorem 1.1, we see that $\log(-\Lambda(z))$ is strictly plurisubharmonic function on D.

Proof. It remains to prove "strictness", that is, for $z_0 \in D$ and $a \in \mathbb{C}^n$ with $a \neq 0$, it holds

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta}(z_0) a_\alpha \bar{a}_\beta > 0.$$

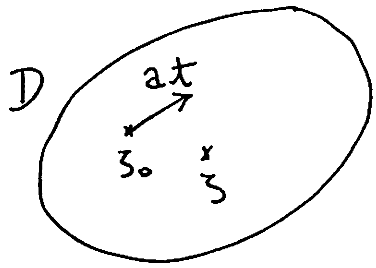
Our aim is to construct the function-theoretic variation $\mathcal{D}_2 : t \to D_2(t)\ (t \in B)$ such that

$$\frac{\partial^2 \lambda_2}{\partial t}(0) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta}(z_0) a_\alpha \bar{a}_\beta .$$

To realize it, take a small disk $B : |t| < \rho$ such that

$$z(t) \underset{\text{def}}{=} z_0 + at \in D \quad \text{for all } t \in B.$$

Let D be a pseudoconvex domain (with smooth boundary) in $\mathbb{C}^n$ ($n \geqq 2$) and let $\zeta \in D$. We denote by $G(\zeta, z)$ and $\Lambda(\zeta)$ the Green's function and the Robin constant for (D, ζ).



We give a qualitative proof of strictly plurisubharmonicity of $\log(-\Lambda(\zeta))$ in D. We prove it by contradiction.

Assume that $\log(-\Lambda(\zeta))$ is not strictly plurisubharmonic in D. Namely, there exist a point $\zeta_0 \in D$ and a vector $a \in \mathbb{C}^n$ with $a \neq 0$ such that

$$(1.12) \quad \left[\frac{\partial^2 \log(-\Lambda(\zeta_0 + at))}{\partial t \partial \bar{t}}\right]_{t=0} = 0 .$$

We already obtained a variation

$$\mathcal{D}_2 : t \longrightarrow D_2(t) \quad (t \in B)$$

where $B = (|t| < \rho)$ such that

(i) $\mathcal{D}_2$ is a function-theoretic variation;

(ii) $\mathcal{D}_2 \supset B \times \{0\}$;

(iii) $-\dfrac{\partial^2 \lambda_2}{\partial t \partial \bar{t}}(0) = \left[\dfrac{\partial^2 \log(-\Lambda(\zeta_0 + at))}{\partial t \partial \bar{t}}\right]_{t=0}$

where $\lambda_2(t)$ is the Robin constant for $(D_2(t), 0)$.

Our condition (1.12) gives us

$$\frac{\partial^2 \lambda_2}{\partial t \partial \bar{t}}(0) = 0 .$$

It follows from Rigidity (Theorem 5.1) that

$$\frac{\partial g_2}{\partial t}(0, W) \equiv 0 \quad \text{for } W \in D_2(0)$$

where $g_2(t, W)$ is the Green's function for $(D_2(t), 0)$

It follows from (1.9) and the continuity that

$$-\frac{c_1}{2}\left\{G(\zeta_0,z)+\frac{1}{n-1}\sum_{\alpha=1}^{n}(z_\alpha-\zeta_{0\alpha})\frac{\partial G}{\partial z_\alpha}(\zeta_0,z)\right\}+\sum_{\alpha=1}^{n}a_\alpha\left(\frac{\partial G}{\partial \zeta_\alpha}+\frac{\partial G}{\partial z_\alpha}\right)(\zeta_0,z)\equiv \quad \text{on } D\cup\partial D$$

where c_1 is a constant (mentioned in p.2 at March 1st)

Since $G(\zeta,z)=0$ for $(\zeta,z)\in D\times\partial D$, we have $(\partial G/\partial\zeta_\alpha)(\zeta,z)=0$ for $(\zeta,z)\in D\times\partial D$. Therefore, if we restrict z to ∂D, then

$$-\frac{c_1}{2}\left\{\frac{1}{n-1}\sum_{\alpha=1}^{n}(z_\alpha-\zeta_{0\alpha})\frac{\partial G}{\partial z_\alpha}(\zeta_0,z)\right\}+\sum_{\alpha=1}^{n}a_\alpha\frac{\partial G}{\partial z_\alpha}(\zeta_0,z)=0 \quad \text{on } \partial D$$

$$\therefore\quad \sum_{\alpha=1}^{n}\left(a_\alpha-\frac{c_1}{2n-2}(z_\alpha-\zeta_{0\alpha})\right)\frac{\partial G}{\partial z_\alpha}(\zeta_0,z)=0 \quad \text{for } z\in\partial D.$$

The function in the left-hand side is defined in a neighborhood V of ∂D and is of class C^2. It follows from Preliminary 1.1 (p.4 in Feb. 1st) that the function is divisible by $G(\zeta_0,z)$ in V. Precisely, there exists a function $Q(z)$ of class C^2 in V such that

$$(1.13)\quad \sum_{\alpha=1}^{n}\left(a_\alpha-\frac{c_1}{2n-2}(z_\alpha-\zeta_{0\alpha})\right)\frac{\partial G}{\partial z_\alpha}(\zeta_0,z)=Q(z)\,G(\zeta_0,z) \quad \text{in } V$$

Take any point $z_0=(z_{01},\dots,z_{0n})\in\partial D$ in $\mathbb{C}^n$. Consider the system of differential equations on $\mathbb{C}$:

$$\frac{dz_\alpha}{dt}=a_\alpha-\frac{c_1}{2n-2}(z_\alpha-\zeta_{0\alpha}) \qquad (1\leqq\alpha\leqq n)$$

with initial value $z(0)=z_0$

They are uniquely given in the whole $\mathbb{C}$ as follows:

In the case $c_1=0$: $z_\alpha(t)=z_{0\alpha}+a_\alpha t$

In the case $c_1\neq 0$: $z_\alpha(t)=A_\alpha+(z_{0\alpha}-A_\alpha)e^{kt}$

where $k = \frac{-c_1}{2n-2}$ and $A_\alpha = \zeta_{0\alpha} - \frac{a_\alpha}{k}$.
In both cases we put
$$z(t) = (z_\alpha(t))_{\alpha=1,\dots,n} \quad \text{for } t \in \mathbb{C}.$$

Let $K = \{t \in \mathbb{C} \mid z(t) \in \partial D\}$. Then we want to prove that
$$K = \mathbb{C}.$$

Indeed, we easily see that $K \neq \phi$ (because of $K \ni z_0$, and that K is closed in $\mathbb{C}$. Now let $z_1 \in K$. i.e., there exists $t_1 \in \mathbb{C}$ such that $z(t_1) = z_1 \in \partial D$.
We have a disk $B_0 : |t - t_1| < \rho$ such that
$$z(t) \in V \quad \text{for } \forall t \in B_0.$$
Consider the function
$$f(t) = G(\zeta_0, z(t)) \quad \text{for } t \in B_0.$$
Then $f(t_1) = 0$ and $f(t)$ satisfies the following differential equation:
$$\frac{\partial f}{\partial t} = Q(z(t))\, f \quad \text{for } t \in B_0$$
[For, by differentiating $f(t) = G(\zeta_0, z(t))$, we have
$$\frac{\partial f(t)}{\partial t} = \sum_{\alpha=1}^{n} \frac{\partial G}{\partial z_\alpha}(\zeta_0, z(t)) \frac{d z_\alpha(t)}{dt}$$
$$= \sum_{\alpha=1}^{n} \frac{\partial G}{\partial z_\alpha}(\zeta_0, z(t)) \left(a_\alpha - \frac{c_1}{2n-2}(z_\alpha(t) - \zeta_{0\alpha})\right)$$
It follows from (1.13) that
$$\frac{\partial f(t)}{\partial t} = Q(z(t))\, G(\zeta_0, z(t)) = Q(z(t))\, f(t) \quad \text{for } t \in B_0$$
Since $f(t)$ is <u>real-valued</u> on B_0, it follows from a uniqueness theorem that

$$f(t) \equiv 0 \quad \text{on } B_0.$$

Or equivalently,

$$G(\zeta_0, z(t)) = 0 \quad \text{for } t \in B_0$$

By Preliminary 1.1, we have

$$z(t) \subset \partial D \quad (t \in B_0)$$

Hence K is open in $\mathbb{C}$. We conclude that $K = \mathbb{C}$. and that

$$\partial D \supset \{ z(t) \mid t \in \mathbb{C} \}$$

From the beginning we may assume $z_0 \neq \zeta_0 - \frac{a}{k}$. It follows that $\{ z(t) \mid t \in \mathbb{C} \}$ is not bounded, which contradicts the boundedness of D.

c.q.f.d.

§2. Relation between Robin constants and Bergman kernels.

In the last semester we got the following theorem:

Let D be a domain in $\mathbb{C}$ with smooth boundary, and let $\zeta \in D$. We have the Robin constant $\Lambda(\zeta)$ for (D, ζ). Then

$$\frac{\partial^2 \Lambda(\zeta)}{\partial \zeta \partial \bar{\zeta}} = -\frac{\pi}{2} K(\zeta, \zeta) \qquad \text{for } \zeta \in D.$$

where $K(\zeta, z)$ is the Bergman kernel function such that, for any holomorphic fn $f(z)$ in D with $\|f\| < \infty$,

$$\iint_D f(z) \overline{K(\zeta, z)} \, dx dy = f(\zeta)$$

We try to generalize this theorem into the case where D is a pseudoconvex domain in $\mathbb{C}^n$ $(n \geq 2)$.

Theorem 2.1

$$-\sum_{\alpha=1}^{n} \frac{\partial^2 \Lambda(\zeta)}{\partial \zeta_\alpha \partial \bar{\zeta}_\alpha} \geq \frac{(n-1)\omega_{2n}}{n} K(\zeta, \zeta)$$

where $\Lambda(\zeta)$ is the Robin constant for (D, ζ) ;
$K(\zeta, z)$ is the Bergman kernel function such that

$$\iint_D f(z) \overline{K(\zeta, z)} \, dV = f(\zeta)$$

for any holomorphic fn $f(z)$ with $\|f\|_D^2 < +\infty$.

In order to prove Theorem 2.1 we need some preparations:
Let $G(\zeta, z)$ be the Green's function for (D, ζ) so that

(2.1) $$G(\zeta, z) = \frac{1}{\|z - \zeta\|^{2n-2}} + \Lambda(\zeta) + H(\zeta, z)$$

where $H(\zeta, z)$ is harmonic for z in D and

(2.2) $$H(\zeta, \zeta) = 0$$

We know that $G(\zeta, z)$ is real analytic function on $D \times D$ except for the diagonal and has a symmetric property:

(2.3) $$G(\zeta, z) = G(z, \zeta).$$

We set, for $(\zeta, z) \in D \times D$ and $1 \leq \alpha, \beta, \gamma \leq n$,

$$(2.4)\begin{cases} G_\alpha(\zeta,z) = \left(\frac{\partial G}{\partial \zeta_\alpha} + \frac{\partial G}{\partial z_\alpha}\right)(\zeta,z) \\ G_{\alpha\bar\beta}(\zeta,z) = \left(\frac{\partial G_\alpha}{\partial \bar\zeta_\beta} + \frac{\partial G_\alpha}{\partial \bar z_\beta}\right)(\zeta,z)\end{cases}$$

Under this notation we get

Proposition 2.1

G_α, $G_{\alpha\bar\beta}$ are real analytic symmetric functions on $D\times D$ such that

(1) $\frac{\partial\Lambda(\zeta)}{\partial\zeta} = G_\alpha(\zeta,\zeta)$;

(2) $\frac{\partial^2\Lambda(\zeta)}{\partial\zeta_\alpha\partial\bar\zeta_\beta} = G_{\alpha\bar\beta}(\zeta,\zeta) = 2\frac{\partial G_\alpha}{\partial \bar z_\beta}(\zeta,\zeta) = 2\frac{\partial G_\alpha}{\partial\bar\zeta_\beta}(\zeta,\zeta)$

Proof. We differentiate (2.1) with respect to ζ_α and z_α. Since $\left(\frac{\partial}{\partial\zeta_\alpha} + \frac{\partial}{\partial z_\alpha}\right)\frac{1}{\|z-\zeta\|^{2n-2}} \equiv 0$, it follows that $G_\alpha(\zeta,z)$ has no singularity at $z=\zeta$, and that $G_\alpha(\zeta,z)$ is real analytic on $D\times D$.

⌈Rigorously speaking, consider the function
$$v(\zeta,z) = G(\zeta,z) - \frac{1}{\|z-\zeta\|^{2n-2}}.$$
Then $v(\zeta,z)$ is symmetric and hence $v(\zeta,z)$ is harmonic with respect to ζ as well as z in D (except for $z=\zeta$). Since $v(\zeta,z)\leqq 0$ on $D\times D$, it follows from Fubini's Theorem and from Poisson Integral formula that $v(\zeta,z)$ is real analytic for $(\zeta,z)\in D\times D$. Consequently, $G_\alpha(\zeta,z)$ is real analytic in $D\times D$.⌋

Differentiate (2.3) with respect to ζ_α and z_α and we have
$$\frac{\partial G}{\partial\zeta_\alpha}(\zeta,z) = \frac{\partial G}{\partial z_\alpha}(z,\zeta)\ ;\ \frac{\partial G}{\partial z_\alpha}(\zeta,z) = \frac{\partial G}{\partial\zeta_\alpha}(z,\zeta)$$

It follows by summation that

$$G_\alpha(\zeta, \bar{z}) = G_\alpha(z, \zeta)$$

which means that $G_\alpha(\zeta, z)$ is symmetric for ζ and z.

By the same method, we see that $G_\alpha(\zeta, z)$ is real analytic symmetric function on $D\times D$.

By (2.1), we have

$$G_\alpha(\zeta,\zeta) = \left[\frac{\partial\Lambda(\zeta)}{\partial\zeta_\alpha} + \frac{\partial H(\zeta,z)}{\partial\zeta_\alpha} + \frac{\partial H(\zeta,z)}{\partial z_\alpha}\right]_{z=\zeta}.$$

If we differentiate (2.2) with respect to ζ_α, we have

$$\frac{\partial H}{\partial\zeta_\alpha}(\zeta,\zeta) + \frac{\partial H}{\partial z_\alpha}(\zeta,\zeta) = 0$$

It follows that

$$G_\alpha(\zeta,\zeta) = \frac{\partial\Lambda(\zeta)}{\partial\zeta_\alpha},$$

which proves (1).

If differentiate (1) with respect to $\bar{\zeta}_\beta$, then

$$\frac{\partial^2\Lambda(\zeta)}{\partial\bar{\zeta}_\beta\partial\zeta_\alpha} = \frac{\partial G_\alpha}{\partial\bar{\zeta}_\beta}(\zeta,\zeta) + \frac{\partial G_\alpha}{\partial\bar{z}_\beta}(\zeta,\zeta) = G_{\alpha\bar{\beta}}(\zeta,\zeta).$$

We differentiate

$$G_\alpha(\zeta, z) = G_\alpha(z,\zeta)$$

with respect to $\bar{\zeta}_\beta$, and have

$$\frac{\partial G_\alpha}{\partial\bar{\zeta}_\beta}(\zeta,z) = \frac{\partial G_\alpha}{\partial\bar{z}_\beta}(z,\zeta).$$

If we put $z=\zeta$, then $\frac{\partial G_\alpha}{\partial\bar{\zeta}_\beta}(\zeta,\zeta) = \frac{\partial G_\alpha}{\partial\bar{z}_\beta}(\zeta,\zeta)$. It follows that

$$\frac{\partial^2 \Lambda(\zeta)}{\partial \bar\zeta_\beta \partial \zeta_\alpha} = 2\frac{\partial G_\alpha}{\partial \bar z_\beta}(\zeta,\zeta),$$

which proves (2). c.q.f.d.

<u>Lemma 2.1</u> Let

$$H(\zeta, z) = \sum_{\alpha=1}^{n} \frac{\partial^2 G}{\partial \zeta_\alpha \partial z_\alpha}(\zeta, z)$$

for $(\zeta, z) \in D \times D$. Then

(1) $H(\zeta, z)$ is real analytic on $D \times D$ and harmonic with respect to ζ as well as z in D;

(2) $H(\zeta, z) = \bar H(z, \zeta)$ where $\bar H = \sum_{\alpha=1}^{n} \frac{\partial^2 G}{\partial \zeta_\alpha \partial \bar z_\alpha}$;

(3) $H(\zeta,\zeta) = \frac{1}{2}\sum_{\alpha=1}^{n}\frac{\partial^2 \Lambda(\zeta)}{\partial \zeta_\alpha \partial \bar\zeta_\alpha}$;

(4) For any holomorphic function $f(z)$ on $D \cup \partial D$, we have, for $\zeta \in D$,

$$\iint_D f(z)\,\overline{H(z,\zeta)}\,dV = -\frac{1}{2}(n-1)\omega_{2n} f(\zeta).$$

<u>Proof.</u> To prove (1), we observe the 1st formula in (2.4):

$$G_\alpha(\zeta, z) = \left(\frac{\partial G}{\partial \zeta_\alpha} + \frac{\partial G}{\partial z_\alpha}\right)(\zeta, z)$$

Differentiate this formula with respect to $\bar{\zeta}_\alpha$, and sum up for $\alpha = 1, \cdots, n$. Then we have

$$\sum_{\alpha=1}^{n} \frac{\partial G_\alpha}{\partial \bar\zeta_\alpha}(\zeta, z) = \sum_{\alpha=1}^{n}\left(\frac{\partial^2 G}{\partial \bar\zeta_\alpha \partial \zeta_\alpha} + \frac{\partial^2 G}{\partial \bar\zeta_\alpha \partial z_\alpha}\right)(\zeta, z)$$

Since $G(\zeta, z)$ is harmonic for ζ in D, we get

(2.5) $\displaystyle\sum_{\alpha=1}^{n} \frac{\partial G_\alpha}{\partial \bar{\zeta}_\alpha}(\zeta, z) = \sum_{\alpha=1}^{n} \frac{\partial^2 G}{\partial \bar{\zeta}_\alpha \partial z_\alpha}(\zeta, z) \underset{\text{def}}{=} H(\zeta, z)$

It follows from Proposition 2.1 that $H(\zeta, z)$ is real analytic on $D \times D$, and that $H(\zeta, z)$ is harmonic for ζ as well as z in D. Moreover, we have from (2) of Proposition 2.1

$$H(\zeta,\zeta) = \sum_{\alpha=1}^{n} \frac{\partial G_\alpha}{\partial \bar{\zeta}_\alpha}(\zeta,\zeta) = \sum_{\alpha=1}^{n} \frac{1}{2} \frac{\partial^2 \Lambda(\zeta)}{\partial \zeta_\alpha \partial \bar{\zeta}_\alpha},$$

which proves (3) of Lemma 2.1.

From the symmetry: $G_\alpha(\zeta, z) = G_\alpha(z, \zeta)$, we have

$$\frac{\partial G_\alpha}{\partial \bar{\zeta}_\alpha}(\zeta, z) = \frac{\partial G_\alpha}{\partial \bar{z}_\alpha}(z, \zeta) = \frac{\partial^2 G}{\partial \zeta_\alpha \partial \bar{z}_\alpha}(z,\zeta) + \frac{\partial^2 G}{\partial z_\alpha \partial \bar{z}_\alpha}(z,\zeta)$$

Sum up for $\alpha = 1, \cdots, n$ and have, from $\sum_{\alpha=1}^{n} \frac{\partial^2 G}{\partial z_\alpha \partial \bar{z}_\alpha}(z,\zeta) \equiv 0$,

$$\sum_{\alpha=1}^{n} \frac{\partial G_\alpha}{\partial \bar{\zeta}_\alpha}(\zeta, z) = \sum_{\alpha=1}^{n} \frac{\partial^2 G}{\partial \zeta_\alpha \partial \bar{z}_\alpha}(z, \zeta)$$

$$\therefore \quad H(\zeta, z) = \overline{H}(z, \zeta)$$

(2) of Lemma 2.1 is proved.

For the proof of (4) of Lemma 2.1 we recall the formula (1.6) at p. 8 (at Feb. 1st)

" For any harmonic function $u(z)$ on $D \cup \partial D$,

$$u(\zeta) = \frac{-1}{(2n-2)\,\omega_{2n}} \int_{\partial D} u(z) \frac{\partial G(\zeta, z)}{\partial n_z}\, ds_z \quad \text{for } \zeta \in D \text{ ''}$$

By complex notation we have, at every point $z \in \partial D$,

$$\frac{\partial}{\partial n_z} ds_z = \frac{-i^n}{2^{n-1}} \sum_{\alpha=1}^{n} \left(\frac{\partial}{\partial z_\alpha} dz_\alpha - \frac{\partial}{\partial \bar z_\alpha} d\bar z_\alpha\right) dz_1\wedge d\bar z_1 \wedge \cdots \wedge \widehat{dz_\alpha \wedge d\bar z_\alpha} \wedge \cdots \wedge dz_n \wedge d\bar z_n$$

where $\widehat{\square}$ represents the absence of $\square$.
Since $G(\zeta, z) = 0$ for $z \in \partial D$, we get

$$d_z G = \sum_{\alpha=1}^{n} \left(\frac{\partial G}{\partial z_\alpha} dz_\alpha + \frac{\partial G}{\partial \bar z_\alpha} d\bar z_\alpha\right) = 0 \quad \text{on } \partial D$$

It follows that

$$\frac{\partial G}{\partial n_z} ds_z = \frac{-i^n}{2^{n-2}} \sum_{\alpha=1}^{n} \frac{\partial G}{\partial z_\alpha} dz_\alpha \wedge dz_1\wedge d\bar z_1 \wedge \cdots \wedge \widehat{dz_\alpha \wedge d\bar z_\alpha} \wedge \cdots \wedge dz_n \wedge d\bar z_n$$

on ∂D. On the other hand, since $G(\zeta, z) = 0$ for $(\zeta, z) \in D \times \partial D$, we have $\frac{\partial G}{\partial \zeta_\alpha}(\zeta, z) = 0$ for $z \in \partial D$. It follows that

$$\frac{\partial G}{\partial n_z} ds_z = \frac{-i^n}{2^{n-2}} \sum_{\alpha=1}^{n} \left(\frac{\partial G}{\partial \zeta_\alpha} + \frac{\partial G}{\partial z_\alpha}\right) dz_\alpha \wedge dz_1\wedge d\bar z_1 \wedge \cdots \wedge \widehat{dz_\alpha \wedge d\bar z_\alpha} \wedge \cdots \wedge dz_n \wedge d\bar z_n .$$

$$= \frac{-i^n}{2^{n-2}} \sum_{\alpha=1}^{n} G_\alpha(\zeta, z)\, dz_\alpha \wedge dz_1\wedge d\bar z_1 \wedge \cdots \wedge \widehat{dz_\alpha \wedge d\bar z_\alpha} \wedge \cdots \wedge dz_n \wedge d\bar z_n$$

and hence that

$$u(\zeta) = \frac{i^n}{2^{n-1}(n-1)\omega_{2n}} \sum_{\alpha=1}^{n} \int_{\partial D} u(z)\, G_\alpha(\zeta, z)\, dz_\alpha \wedge dz_1\wedge d\bar z_1 \wedge \cdots \wedge \widehat{dz_\alpha \wedge d\bar z_\alpha} \wedge \cdots \wedge dz_n \wedge d\bar z_n .$$

Since $G_\alpha(\zeta, z)$ is regular for z in D, it follows from Stokes' formula that

$$u(\zeta) = \frac{i^n}{2^{n-1}(n-1)\omega_{2n}} \sum_{\alpha=1}^{n} \iint_D d\left(u\, G_\alpha\, dz_\alpha \wedge dz_1\wedge d\bar z_1 \wedge \cdots \wedge \widehat{dz_\alpha \wedge d\bar z_\alpha} \wedge \cdots \wedge dz_n \wedge d\bar z_n\right)$$

$$= \frac{i^n}{2^{n-1}(n-1)\omega_{2n}} \sum_{\alpha=1}^{n} \iint_D \left(\frac{\partial u}{\partial \bar z_\alpha} G_\alpha + u \frac{\partial G_\alpha}{\partial \bar z_\alpha}\right) d\bar z_\alpha \wedge dz_\alpha \wedge dz_1 \wedge d\bar z_1 \wedge \cdots \wedge \widehat{dz_\alpha \wedge d\bar z_\alpha} \wedge \cdots \wedge dz_n \wedge d\bar z_n$$

$$= \frac{-2}{(n-1)\omega_{2n}} \iint_D \left\{ \left(\sum_{\alpha=1}^{n} \frac{\partial u}{\partial \bar{z}_\alpha} G_\alpha \right) + \left(u \sum_{\alpha=1}^{n} \frac{\partial G_\alpha}{\partial \bar{z}_\alpha} \right) \right\} dV$$

$$= \frac{-2}{(n-1)\omega_{2n}} \iint_D \left\{ \sum_{\alpha=1}^{n} \frac{\partial u}{\partial \bar{z}_\alpha} G_\alpha + u \bar{H} \right\} dV.$$

Now assume that $u(z)$ is a holomorphic function $f(z)$ on $D \cup \partial D$. Then, because of $\partial f / \partial \bar{z}_\alpha \equiv 0$ in D, we get

$$(2.6) \quad f(\zeta) = \frac{-2}{(n-1)\omega_{2n}} \iint_D u \bar{H}(\zeta, z) \, dV.$$

(4) of Lemma 2.1 is thus proved. c.q.f.d.

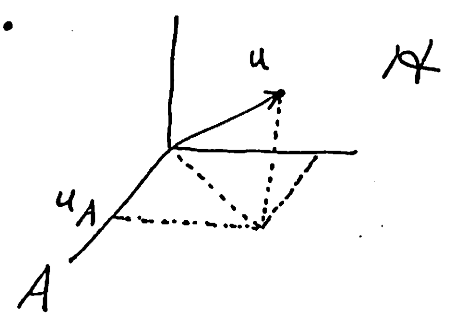
We consider, as usual, the space $\mathcal{A}(D)$ (resp. $\mathcal{H}(D)$) formed by all the square integrable holomorphic (resp. harmonic) functions in D. Both spaces $A(D)$ and $\mathcal{H}(D)$ with inner product

$$(f, g)_D = \iint_D f \bar{g} \, dV$$

are Hilbert spaces. Since $A(D)$ is closed in $\mathcal{H}(D)$, we have the orthogonal decomposition:

$$\mathcal{H}(D) = \mathcal{A}(D) + \mathcal{A}(D)^\perp.$$

Given $u \in \mathcal{H}(D)$, we denote by u_A the projection of u to $\mathcal{A}(D)$.



Given $\zeta \in D$, we consider the functional on $\mathcal{A}(D)$:

$$f \in \mathcal{A}(D) \longrightarrow f(\zeta) \in \mathbb{C}$$

Since it is continuous, there exists a unique function $K(\zeta, z)$ in $\mathcal{A}(D)$ such that

$$f(\zeta) = \iint_D f(z)\overline{K(\zeta,z)}\, dV,$$

$K(\zeta,z)$ is called the Bergman kernel function for (D, ζ).
By definition we have

$$(2.7) \qquad K(\zeta,\zeta) = \iint_D |K(\zeta,z)|^2 dV$$

and by (2.6) we have

$$(2.8) \qquad K(\zeta, z) = \frac{-2}{(n-1)\omega_{2n}} H(\zeta,z)_A$$

Till now we don't assume that D is pseudoconvex in $\mathbb{C}^n$. We only assume that D is a smooth domain in $\mathbb{C}^n$.

Here we assume that D is pseudoconvex in $\mathbb{C}^n$.
We got at p. 28 (at Feb. 24) the following inequality:

"$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2(-\lambda)}{\partial\zeta_\alpha \partial\bar\zeta_\beta}(\zeta) a_\alpha \bar a_\beta \geq \frac{4}{(n-1)\,\omega_{2n}} \iint_D \sum_{\alpha=1}^{n} \left| \sum_{\beta=1}^{n} a_\beta \frac{\partial G_\beta}{\partial \bar z_\alpha}(\zeta, z)\right|^2 dV \text{''}$$

for any $a \in \mathbb{C}^n$ with $a \neq 0$.

In particular, we put $a = (\underbrace{0,\cdots,0,1}_{\beta},0,\cdots 0)$. Then we have

$$\frac{\partial^2(-\lambda)}{\partial\zeta_\beta\partial\bar\zeta_\beta}(\zeta) \geq \frac{4}{(n-1)\omega_{2n}} \iint_D \sum_{\alpha=1}^{n} \left|\frac{\partial G_\beta}{\partial \bar z_\alpha}(\zeta,z)\right|^2 dV$$

If we sum up for $\beta = 1, \cdots, n$, then we have

$$\sum_{\beta=1}^{n} \frac{\partial^2(-\lambda)}{\partial\zeta_\beta\partial\bar\zeta_\beta}(\zeta) \geq \frac{4}{(n-1)\omega_{2n}} \iint_D \sum_{\alpha,\beta=1}^{n} \left|\frac{\partial G_\beta}{\partial \bar z_\alpha}(\zeta,z)\right|^2 dV$$

$$\geq \frac{4}{(n-1)\omega_{2n}} \iint_D \sum_{\alpha=1}^{n} \left| \frac{\partial G_\alpha}{\partial \bar{z}_\alpha}(\zeta, z) \right|^2 dV$$

$$\geq \frac{4}{n(n-1)\omega_{2n}} \iint_D \left| \sum_{\alpha=1}^{n} \frac{\partial G_\alpha}{\partial \bar{z}_\alpha}(\zeta, z) \right|^2 dV$$

$$= \frac{4}{n(n-1)\omega_{2n}} \iint_D \left| \overline{H}(\zeta, z) \right|^2 dV$$

$$= \frac{4}{n(n-1)\omega_{2n}} \iint_D \left| H(\zeta, z) \right|^2 dV$$

$$\geq \frac{4}{n(n-1)\omega_{2n}} \iint_D \left| H(\zeta, z)_A \right|^2 dV$$

$$= \frac{(n-1)\omega_{2n}}{n} K(\zeta, \zeta) \quad \text{by (2.7) and (2.8).}$$

Therefore Lemma 2.1 is proved. c.q.f.d.

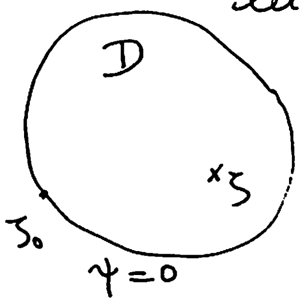
<u>Remark 2.1</u> Lemma 2.1 is rather curious. For, let $w = T(z)$ be an analytic transformation from D onto $\tilde{D}$ (in $\mathbb{C}^n$). Then we have

$$\tilde{K} \cdot |J_T|^2 = K$$

where J_T is Jacobian matrix of T. But, as we know, for Robin constants, we don't have any relation between $\tilde{\Lambda}$ and Λ.

§3 Boundary behavior of $\Lambda(\zeta)$

Let D be a domain with smooth boundary in $\mathbb{C}^n$ and let $\zeta \in D$. We have the Green's function $G(\zeta, z)$ and the Robin constant $\Lambda(\zeta)$ for (D, ζ). We already proved in Lemma 1.3



$$\lim_{\zeta \to \partial D} \Lambda(\zeta) = -\infty .$$

In this section we study the behavior of $\Lambda(\zeta)$ near ∂D more precisely.

Let $\psi(z)$ be a defining function of D, namely, $\psi(z)$ is of class C^∞ real-valued function in $\mathbb{C}^n$ such that

$$D = \{ z \in \mathbb{C}^n \mid \psi(z) < 0 \} ;$$
$$\partial D = \{ z \in \mathbb{C}^n \mid \psi(z) = 0 \} ;$$
$$\mathrm{Grad}\, \psi(z) \neq 0 \quad \text{for } z \in \partial D.$$

We may assume that

$$(3.1) \qquad \psi(z) \equiv \text{const.} > 0 \quad \text{in } \mathbb{B}_M = \{ \|z\| \geqq M \} .$$

Let us prove first

<u>Theorem 3.1</u> Let $\zeta_0 \in \partial D$. Then we have

$$\lim_{\zeta \to \zeta_0} \Lambda(\zeta)\, \psi(\zeta)^{2n-2} = - \| \mathrm{Grad}\, \psi(\zeta_0) \|^{2n-2}$$

<u>Proof.</u> For the sake of convenience, we put

$$E = \mathbb{C}^n - D \cup \partial D = \{ \psi(z) > 0 \}$$

so that $\quad \mathbb{C}^n = D \cup \partial D \cup E \quad$ (disjoint union)

Given $\zeta \in D \cup E$, we consider the transformation of $\mathbb{C}^n$:

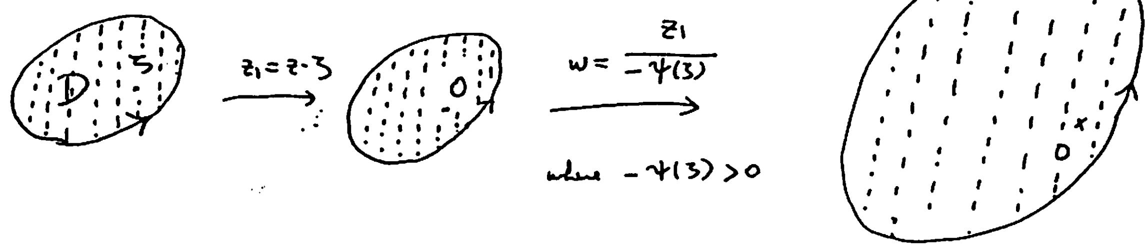
$$T_\zeta : z \longrightarrow w = \frac{z-\zeta}{-\psi(\zeta)}$$

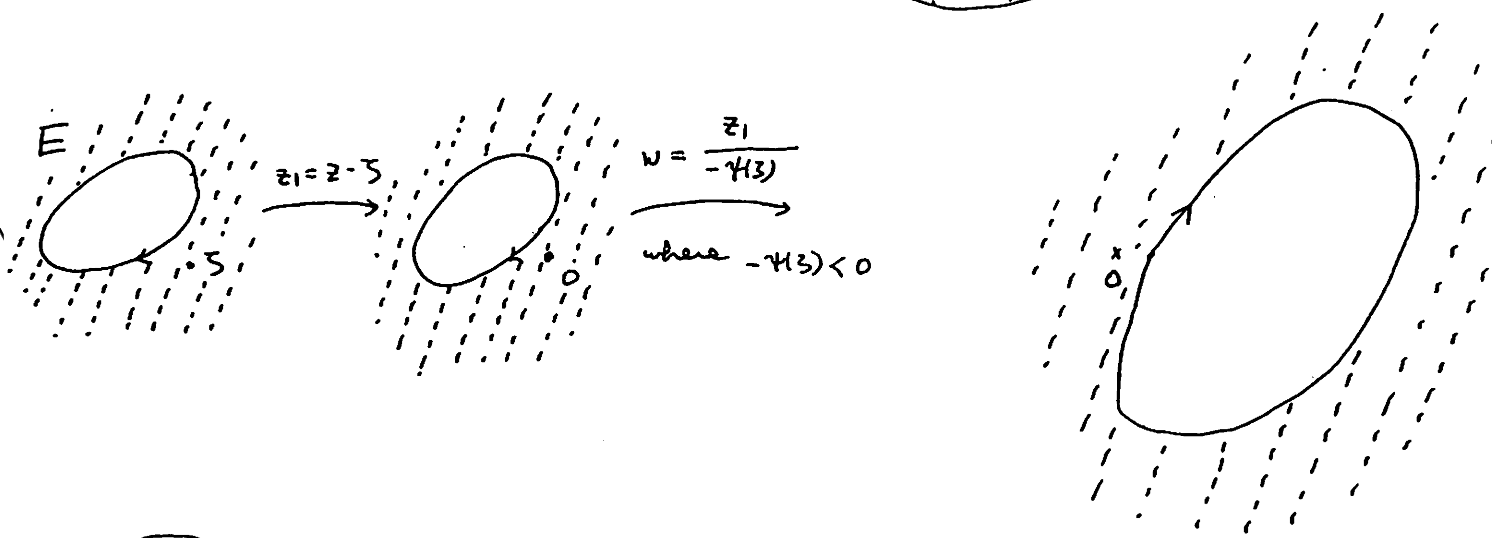
and put

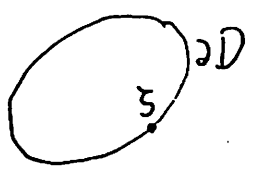
$$D(\zeta) \underset{\text{def}}{=} \begin{cases} T_\zeta(D) & (\zeta \in D) \\ T_\zeta(E) & (\zeta \in E) \end{cases}$$

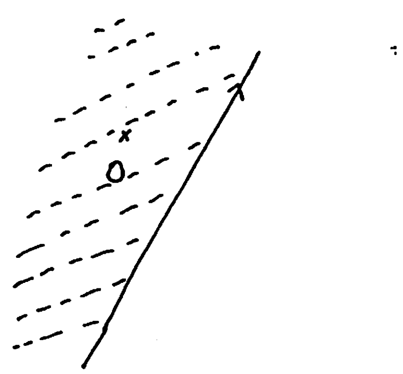
Given $\zeta \in \partial D$, we define

(3.2) $$D(\zeta) = \left\{ w \in \mathbb{C}^n \,\middle|\, 2\,\mathrm{Re}\left\{ \sum_{\alpha=1}^{n} w_\alpha \frac{\partial \psi}{\partial z_\alpha}(\zeta) \right\} - 1 < 0 \right\}$$









We set $\mathcal{D} = \bigcup_{\zeta \in \mathbb{C}^n} (\zeta, D(\zeta))$,
so that $\mathcal{D}$ defines a variation:

$$\mathcal{D} : \zeta \to D(\zeta) \quad (\zeta \in \mathbb{C}^n)$$

The followings are visible:

(i) $\mathcal{D}|_D : \zeta \to D(\zeta)\,(\zeta \in D)$ is diffeomorphically equivalent to the trivial one $D \times D : \zeta \to D\,(\zeta \in D)$;

(ii) Each $D(\zeta)$ $(\zeta \in D)$ is similar to D with similar ratio $1/-\psi(\zeta) > 0$;

(iii) $\mathcal{D}|_E : \zeta \to D(\zeta)\,(\zeta \in E)$ is diffeomorphically equivalent to the trivial one $E \times E : \zeta \to E\,(\zeta \in E)$;

(iv) Each $D(\zeta)$ $(\zeta \in E)$ is similar to E with similar ratio $1/-\psi(\zeta) < 0$;

(v) Each $D(\zeta)$ $(\zeta \in D$ (resp. E)) is defined by the function $\psi(\zeta - \psi(\zeta)w)$ (resp. $-\psi(\zeta - \psi(\zeta)w)$);

Each $D(\zeta)$ $(\zeta \in \partial D)$ is defined by the function

$$2\,\mathrm{Re}\left\{\sum_{\alpha=1}^{n} w_\alpha \frac{\partial \psi}{\partial \zeta_\alpha}(\zeta_0)\right\} - 1 \quad \text{in } \mathbb{C}^n.$$

From the beginning we define for $(\zeta, w) \in \mathbb{C}^n \times \mathbb{C}^n$

$$\tilde{\psi}(\zeta, w) \underset{\text{def}}{=} \psi(\zeta - \psi(\zeta)w).$$

Then $\tilde{\psi} \in C^\infty(\mathbb{C}^n \times \mathbb{C}^n)$ such that

$\tilde{\psi} \equiv 0$ on $\partial D \times \mathbb{C}^n$. ($4n-1$ dimensional smooth surfaces)

$\ulcorner \because$ For $\zeta \in \partial D$, we have $\psi(\zeta) = 0$.

Hence $\tilde{\psi}(\zeta, w) = \psi(\zeta - \psi(\zeta)w) = \psi(\zeta) = 0 \lrcorner$

On the other hand, the original function $\psi(\zeta)$ defined for $\zeta \in \mathbb{C}^n$ is regarded as a function

in $\mathbb{C}^n \times \mathbb{C}^n$ where $\psi(\zeta)$ is constant for $w \in \mathbb{C}^n$.
It follows that

$$\psi(\zeta) = 0 \quad \text{on} \quad \partial D \times \mathbb{C}^n \quad \text{(only there.)}$$

and that

$$\mathrm{Grad}_{(\zeta,w)}\, \psi(\zeta) = \left(\frac{\partial\psi}{\partial\zeta_1}(\zeta), \dots, \frac{\partial\psi}{\partial\zeta_n}(\zeta), 0, \dots, 0\right)$$
$$\neq 0 \text{ for any } \zeta \in \partial D \times \mathbb{C}^n.$$

It follows that $\tilde{\psi}(\zeta, w)$ is divisible by $\psi(\zeta)$, precisely, there exists a C^∞-$\underbrace{\quad}_{\text{real-valued function}}$ $f(\zeta, w)$ in $\mathbb{C}^n \times \mathbb{C}^n$ such that

(3.3) $\quad \tilde{\psi}(\zeta, w) = -f(\zeta, w)\,\psi(\zeta)$

We obtain the following properties:

(1) $f(\zeta, 0) = -1$ for $\forall \zeta \in \mathbb{C}^n$

($\because$ For $\zeta \in D \cup E$, we have

$$f(\zeta,0) = -\tilde{\psi}(\zeta,0)/\psi(\zeta)$$
$$= -\psi(\zeta - \psi(\zeta)\cdot 0)/\psi(\zeta) = -1$$

By continuity, $f(\zeta, 0) = -1$ for $\zeta \in \partial D$)

(2) $\dfrac{\partial f}{\partial w_\alpha}(\zeta, w) = \dfrac{\partial \psi}{\partial z_\alpha}(\zeta - \psi(\zeta) w)$ for $(\zeta, w) \in \mathbb{C}^n \times \mathbb{C}^n$

($\because$ $\psi(\zeta - \psi(\zeta) w) = -\psi(\zeta) f(\zeta, w)$

Differentiate both sides by w_α and we have

$$\frac{\partial\psi}{\partial z_\alpha}(\zeta - \psi(\zeta) w)\cdot\psi(\zeta) = -\psi(\zeta)\frac{\partial f}{\partial w_\alpha}(\zeta, w)$$

If $\zeta \in D \cup E$, then $\psi(\zeta) \neq 0$. It follows that

$$\left.\frac{\partial\psi}{\partial z_\alpha}\right|_{\zeta - \psi(\zeta) w} = \left.\frac{\partial f}{\partial w_\alpha}\right|_{(\zeta, w)}$$

By continuity, it is also true for $\zeta \in \partial D$.)

Let ζ be fixed in $\mathbb{C}^n$. Then , for any $w \in \mathbb{C}^n$,

$$f(\zeta,w) = \int_0^1 \left\{\frac{d}{dt} f(\zeta, tw)\right\} dt + f(\zeta,0)$$

$$= \int_0^1 \left\{\sum_{\alpha=1}^n w_\alpha \frac{\partial f}{\partial w_\alpha}(\zeta, tw) + \bar{w}_\alpha \frac{\partial f}{\partial \bar{w}_\alpha}(\zeta, tw)\right\} dt \qquad \text{by (1)}$$

$$= 2\,\mathrm{Re}\left\{\int_0^1 \sum_{\alpha=1}^n w_\alpha \left[\frac{\partial \psi}{\partial z_\alpha}\right]_{\zeta - \psi(\zeta) tw} dt\right\} - 1 \qquad \text{by (2)}$$

In particular, we put $\zeta \in \partial D$. Then

$$f(\zeta, w) = 2\,\mathrm{Re}\left\{\int_0^1 \sum_{\alpha=1}^n w_\alpha \left[\frac{\partial\psi}{\partial z_\alpha}\right](\zeta)\, dt\right\} - 1$$

$$= 2\,\mathrm{Re}\left\{\sum_{\alpha=1}^n w_\alpha \frac{\partial \psi}{\partial z_\alpha}(\zeta)\right\} - 1 .$$

It follows that, for $\zeta \in \partial D$,

$$D(\zeta) = \{ w \in \mathbb{C}^n \mid f(\zeta, w) < 0 \}.$$

On the other hand, (v) together with (3.3) easily yield that, for any $\zeta \in D \cup E$,

$$D(\zeta) = \{ w \in \mathbb{C}^n \mid f(\zeta, w) < 0 \}$$

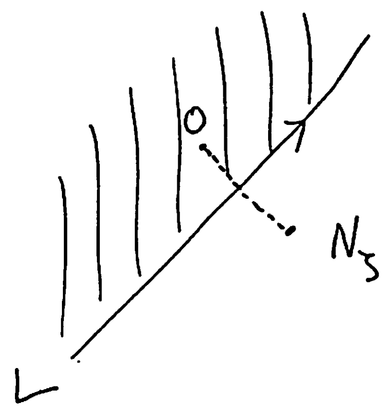
We conclude that our variation $\mathcal{D}: \zeta \to D(\zeta)$ ($\zeta \in \mathbb{C}^n$) is defined by the C^∞ function $f(\zeta, w)$ in $\mathbb{C}^n \times \mathbb{C}^n$!!

We form the Green's function $G(\zeta, z)$ and the Robin constant $\Lambda(\zeta)$ for (D, ζ) or for (E, ζ). Since $T_\zeta : w = (z - \zeta)/(-\psi(\zeta))$ is translation and enlargement, it follows from Proposition 5.1 that if we denote by $g(\zeta, w)$ and $\lambda(\zeta)$ the Green's functions and the Robin constant for $(D(\zeta), 0)$, then

$$(3.4) \quad \begin{cases} g(\zeta, w) = (\psi(\zeta))^{2n-2}\, G(\zeta, z)\ ; \\ \lambda(\zeta) \quad\ = (\psi(\zeta))^{2n-2}\, \Lambda(\zeta) \end{cases}$$

for $\zeta \in D \cup E$; $z = \zeta - \psi(\zeta) w$ and $w \in D(\zeta) \cup \partial D(\zeta)$.

Let $\zeta \in \partial D$. We then don't have $G(\zeta, z)$ or $\Lambda(\zeta)$, but we have the Green's function $g(\zeta, w)$ and the Robin constant $\lambda(\zeta)$.



$$f(\zeta, w) = 2\,\mathrm{Re}\Big\{\sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z_\alpha}(\zeta_0)\, w_\alpha\Big\} - 1 = 0$$

N_ζ = the symmetric point of 0 with respect to the hyperplane L

$$= \frac{\overline{\mathrm{Grad}\ \psi(\zeta)}}{\|\mathrm{Grad}\ \psi(\zeta)\|^2}$$

$$(3.5) \quad \left.\begin{aligned} \therefore\ g(\zeta, w) &= \frac{1}{\|w\|^{2n-2}} - \frac{1}{\|w - N_\zeta\|^{2n-2}} \\ \therefore\ \lambda(\zeta) &= -\|\mathrm{Grad}\ \psi(\zeta)\|^{2n-2} \end{aligned}\right\}$$

Let us prove Theorem 2.1. Let $\zeta_0 \in \partial D$. It suffices from (3.5) that

$$(3.6) \quad \lim_{\zeta \to \zeta_0} \lambda(\zeta) = \lambda(\zeta_0)$$

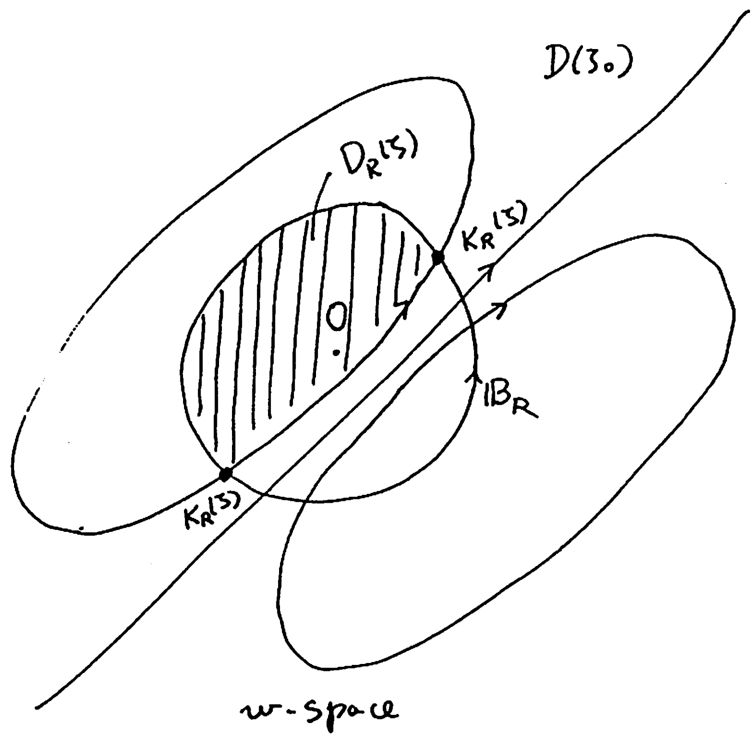
that is, $\lambda(\zeta)$ is continuous at ζ_0.

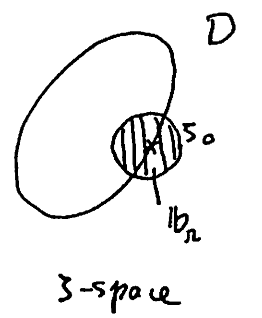
Given $R > 1$, let $\mathbb{B}_R = \{ \|w\| < R \}$ and consider

$$\mathcal{D}_R = \mathcal{D} \cap \mathbb{B}_R$$

or equivalently

$$\mathcal{D}_R = \{ (\zeta, w) \in \mathbb{C}^n \times \mathbb{C}^n \mid f(\zeta, w) < 0 \text{ and } \|w\| < R \}$$





Since $\zeta_0 \in \partial D$, $D_R(\zeta_0)$ is an almost half ball. We have a small ball b_r : $\|\zeta - \zeta_0\| < r$ such that the restriction of $\mathcal{D}_R$ to b_r

$$\mathcal{D}_{R, b_r} : \zeta \longrightarrow D_R(\zeta) \; (\zeta \in b_r)$$

is a smooth variation of domains of $\mathbb{C}^n$ with corners $K_R(\zeta) \underset{\mathrm{def}}{=} (\partial b_r) \cap (\partial D(\zeta))$, such that the variation $\mathcal{D}_{R, b_r}$ is diffeomorphically equivalent to the trivial one : $b_r \times D_R(\zeta_0) : \zeta \longrightarrow D_R(\zeta_0) \; (\zeta \in b_r)$. Each fiber $D_R(\zeta)$ $(\zeta \in b_r)$ carries the Green's function

$g_R(z,w)$ and the Robin constant $\lambda_R(z)$ with pole at 0. Then it is clear that $g_R(z,w)$ is of class C^3 with respect to (z,w) in $\mathcal{D}_{R,B_r} \cup \partial\mathcal{D}_{R,B_r}$ except for the pole $B_r \times \{0\}$ and the corners $K_R = \bigcup_{z\in B_r}(z, K_R(z))$.

We consider the function

$$u_R(z,w) = g_R(z,w) - g(z,w).$$

$u_R(z,w)$ is then harmonic for w in $D_R(z)$ with boundary values are

$$u_R(z,w) = \begin{cases} -g(z,w) & \text{on } (\|w\|=R)\cap D(z) \\ 0 & \text{on } (\|w\|<R)\cap \partial D(z). \end{cases}$$

On the other hand, By (1.2) (at Feb. 1) we see that

$$0 < g(z,w) < \frac{1}{\|w\|^{2n-2}} \quad \text{for } w\in D(z),$$

so that

$$0 > u_R(z,w) > \frac{1}{R^{2n-2}} \quad \text{on } \partial D_R(z).$$

It follows from the maximum principle that

$$0 > u_R(z,w) > \frac{1}{R^{2n-2}} \quad \text{on } D_R(z).$$

In particular, if we put $w=0$, then

$$0 > \lambda_R(z) - \lambda(z) > \frac{1}{R^{2n-2}} \quad \text{for all } z\in B_r$$

Now, let us prove (3.6.). Let $\varepsilon>0$ be given arbitrarily. Choose (and fix) $R>1$ so that $1/R^{2n-2} < \varepsilon/3$. Since

$$\mathcal{D}_{R,B_r} : z \to D_R(z) \quad (z\in B_r)$$

is diffeomorphically equivalent to the trivial one, we know that $\lambda_R(z)$ is of class C^2 in $B_r : \|z-z_0\|<r$.

We thus have a small $B_\delta : \|z-z_0\|<\delta$ $(\subset B_r)$ such that

$$|\lambda_R(z) - \lambda_R(z_0)| < \varepsilon/3 \quad \text{for } \forall z\in B_\delta$$

It follows that, for $\forall \zeta \in \mathbb{b}_\delta$,

$$
\begin{aligned}
&|\lambda(\zeta) - \lambda(\zeta_0)| \\
&\leqq |\lambda(\zeta) - \lambda_R(\zeta)| + |\lambda_R(\zeta) - \lambda_R(\zeta_0)| + |\lambda_R(\zeta_0) - \lambda(\zeta_0)| \\
&\leqq 1/R^{2u-2} + \varepsilon/3 + 1/R^{2u-2} \\
&< \varepsilon .
\end{aligned}
$$

and hence that $\lambda(\zeta)$ is continuous at $\zeta_0 \in \partial D$. c.q.f.d.

We repeat the last lecture briefly.

Let D be a smooth domain in $\mathbb{C}^n$ $(n \geqq 2)$. We put $E = \mathbb{C}^n - D \cup \partial D$, so that $\mathbb{C}^n = D \cup \partial D \cup E$ (disjoint union). Let $\zeta \in D$ (resp. E). We consider the Green's function $G(\zeta, z)$ and the Robin constant $\lambda(\zeta)$ for (D, ζ) (resp. (E, ζ)).

Let $\psi(z)$ be a defining function of D. That is, $\psi(z)$ is of class C^∞ in $\mathbb{C}^n$ such that

$$D = \{ z \in \mathbb{C}^n \mid \psi(z) < 0 \} ; \quad \partial D = \{ z \in \mathbb{C}^n \mid \psi(z) = 0 \}$$

$$\text{Grad}\, \psi(z) = \left(\frac{\partial \psi}{\partial z_1}, \dots, \frac{\partial \psi}{\partial z_n} \right)(z) \neq 0 \quad \text{for all } z \in \partial D$$

We put

$$(1) \qquad m_1 = \operatorname{Min} \{ \| \text{Grad}\, \psi(z) \| : z \in \partial D \} > 0 .$$

We have $E = \{ z \in \mathbb{C}^n \mid \psi(z) > 0 \}$. We may assume that $\psi(z) \equiv \text{const.} > 0$ near ∞, so that the derivatives $\frac{\partial \psi}{\partial z_\alpha}, \frac{\partial^2 \psi}{\partial z_\alpha \partial z_\beta}, \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta}$ $(1 \leqq \alpha, \beta \leqq n)$ are uniformly bounded in $\mathbb{C}^n$. Say

$$(2) \qquad \left| \frac{\partial \psi}{\partial z_\alpha} \right|, \left| \frac{\partial^2 \psi}{\partial z_\alpha \partial z_\beta} \right|, \left| \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta} \right| \leqq M_1 \quad \text{on } \mathbb{C}^n .$$

For arbitrarily fixed $\zeta \in D \cup E$, we form the linear transformation of $\mathbb{C}^n$:

$$T_\zeta : z \longrightarrow w = \frac{z - \zeta}{-\psi(\zeta)}$$

and put

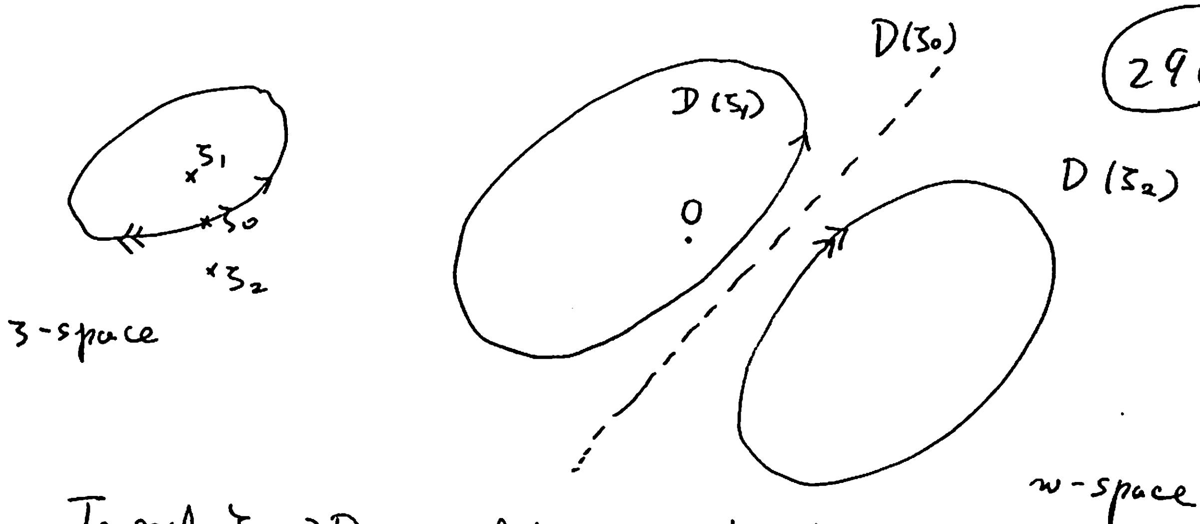
$$D(\zeta) = \begin{cases} T_\zeta(D) & \text{if } \zeta \in D, \\ T_\zeta(E) & \text{if } \zeta \in E . \end{cases}$$

It is clear that

① $T_\zeta(\zeta) = 0$;

② $D(\zeta) \backsim \begin{cases} D & \text{if } \zeta \in D \text{ with similar ratio } -\frac{1}{\psi(\zeta)} > 0, \\ E & \text{if } \zeta \in E \text{ with similar ratio } -\frac{1}{\psi(\zeta)} < 0 \end{cases}$

③ $\partial D(\zeta) = T_\zeta(\partial D) = \{ w \in \mathbb{C}^n \mid \psi(\zeta - \psi(\zeta) w) = 0 \}$



To each $\zeta_0 \in \partial D$, we let correspond the half-space

$$D(\zeta_0) = \left\{ w \in \mathbb{C}^n \,\middle|\, 2\,\mathrm{Re}\left\{ \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z_\alpha}(\zeta_0)\, w_\alpha \right\} - 1 < 0 \right\}$$

and define $\qquad \mathcal{D} = \bigcup_{\zeta \in \mathbb{C}^n} (\zeta, D(\zeta))$.

$\mathcal{D}$ is a domain of $\mathbb{C}^n \times \mathbb{C}^n$ of $2n$ complex variables ζ, w. We regard, as usual, $\mathcal{D}$ as a variation

$$\mathcal{D} : \zeta \longrightarrow D(\zeta) \quad (\zeta \in \mathbb{C}^n).$$

Clearly, $D(\zeta) \ni 0$ for all $\zeta \in \mathbb{C}^n$. We thus have the Green's function $g(\zeta, w)$ and the Robin constant $\lambda(\zeta)$ for $(D(\zeta), 0)$. Since $D(\zeta_0)$ $(\zeta_0 \in \partial D)$ is the concrete half-space, we easily have

$$(3) \quad \begin{cases} g(\zeta_0, w) = \dfrac{1}{\|w\|^{2n-2}} - \dfrac{1}{\|w - N_{\zeta_0}\|^{2n-2}} \, ; \\ \lambda(\zeta_0) = - \| \mathrm{Grad}\, \psi(\zeta_0) \|^{2n-2} \end{cases}$$

where N_{ζ_0} = the symmetric point of 0 with respect to the half-space $L : 2\,\mathrm{Re}\left\{ \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z_\alpha}(\zeta_0) \right\} - 1 = 0$

$$= - \overline{\mathrm{Grad}\, \psi(\zeta_0)} \,/\, \| \mathrm{Grad}\, \psi(\zeta_0) \|^2$$

The variation $\mathcal{D}: \zeta \to D(\zeta)$ ($\zeta \in \mathbb{C}^n$) is realized as follows:
Let

$$f(\zeta, w) = \frac{\psi(\zeta - \psi(\zeta)w)}{-\psi(\zeta)}$$

$$= 2\,\mathrm{Re}\left\{ \int_0^1 \sum_{\alpha=1}^{n} w_\alpha \left[\frac{\partial \psi}{\partial z_\alpha}\right]_{\zeta - \psi(\zeta) t w} dt \right\} - 1 .$$

It is clear that $f(\zeta, w)$ is real-valued C^∞ function in $\mathbb{C}^n \times \mathbb{C}^n$ and that

$$\mathcal{D} = \{ (\zeta, w) \in \mathbb{C}^n \times \mathbb{C}^n \;/\; f(\zeta, w) < 0 \};$$

$$\partial\mathcal{D} = \{ (\zeta, w) \in \mathbb{C}^n \times \mathbb{C}^n \;/\; f(\zeta, w) = 0 \};$$

so that for each $\zeta \in \mathbb{C}^n$

$$D(\zeta) = \{ w \in \mathbb{C}^n / \; f(\zeta, w) < 0 \};$$

$$\partial D(\zeta) = \{ w \in \mathbb{C}^n / \; f(\zeta, w) = 0 \}.$$

We had
(3')
$$\frac{\partial f}{\partial w_\alpha}(\zeta, w) = \left.\frac{\partial \psi}{\partial z_\alpha}\right|_{\zeta - \psi(\zeta) w} \quad \text{for all } (\zeta, w) \in \mathbb{C}^n \times \mathbb{C}^n$$

Consequently,

$$\min_{\substack{w \in \partial D(\zeta) \\ \zeta \in D \cup E}} \| \mathrm{Grad}_{(w)} f(\zeta, w) \| = \min_{z \in \partial D} \| \mathrm{Grad}\, \psi(z) \| ,$$

it follows from (1) that independent of ζ, w.

(4) $$\| \mathrm{Grad}_{(w)} f(\zeta, w) \| = m_1 > 0$$

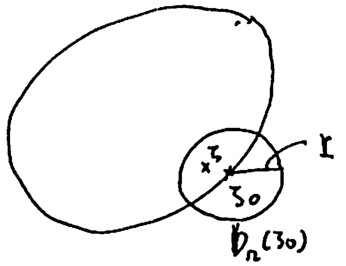
for all $\zeta \in D \cup E$ and all $w \in \partial D(\zeta)$.

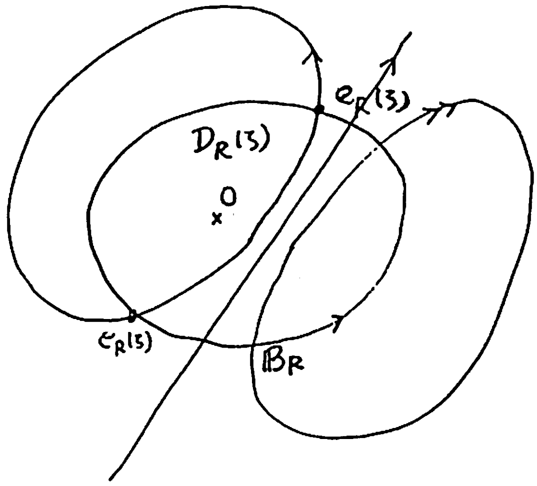
Since $f(\zeta, w)$ is of class C^∞ in $\mathbb{C}^n \times \mathbb{C}^n$, we have the following extension lemma:

Lemma 1 Let $\zeta_0 \in \partial D$ and let $R > 1$ be given. Then there exists a ball $B_r(\zeta_0)$: $\|\zeta - \zeta_0\| < r$ such that if we put $D_R(\zeta) = D(\zeta) \cap (\|w\| < R)$, then the variation

$$\mathcal{D}_{R, b_r(\zeta_0)} : \zeta \longrightarrow D_R(\zeta) \; (\zeta \in b_r(\zeta_0))$$

is a <u>smooth variation</u> of domains with corners $e_R(\zeta)$.





We consider the Green's function $g_R(\zeta, w)$ and the Robin constant $\lambda_R(\zeta)$ for $(D_R(\zeta), 0)$ where $\zeta \in b_r(\zeta_0)$. Then we can prove without difficulty that

(5) $\begin{cases} g_R(\zeta, w) \text{ is continuous with respect to } (\zeta, w) \\ \text{on } \mathcal{D}_{R, b_r(\zeta_0)} \cup \partial\mathcal{D}_{R, b_r(\zeta_0)} - b_r(\zeta_0) \times \{0\}; \\ \lambda_R(\zeta) \text{ is continuous with respect to } \zeta \text{ in } b_r(\zeta_0). \end{cases}$

In last lecture we have proved:

<u>Theorem 1</u> $\lambda(\zeta)$ is continuous on $\mathbb{C}^n$.

Today we will prove

<u>Theorem 2</u> $\lambda(\zeta)$ is of class C^1 on $\mathbb{C}^n$.

Proof: <u>1st step</u> $g(\zeta, w)$ is continuous with respect to (ζ, w) on $\mathcal{D} - \mathbb{C}^n \times \{0\}$.

In fact, Since

$\mathcal{D}|_D \cup \partial\mathcal{D}|_D : z \to \overline{D(z)}\ (z \in D)$, $\mathcal{D}|_E \cup \partial\mathcal{D}|_E : z \to \overline{D(z)}\ (z \in E)$

are equivalent diffeomorphically to the trivial ones

$D \times \overline{D} : z \to \overline{D}\ (z \in D)$, $E \times \overline{} : z \to E\ (z \in E)$

respectively, it follows that $g(z,w)$ is of class C^2 with respect to $\mathcal{D}|_{D\cup E} \cup \partial\mathcal{D}|_{D \cup E} - (D \cup E) \times \{0\}$. Hence $\lambda(z)$ is of class C^2 in $\mathbb{C}^n - \partial D$.

It rests to prove 1st step at a point (z_0, w_0) where $z_0 \in \partial D$ and $w_0 \in \overline{D(z_0)}$.

Let $\varepsilon > 0$ be given. Take $R > 1$ so large that

$$\|w_0\| < R \quad \text{and} \quad 1/R^{2n-2} < \varepsilon.$$

By Extension Lemma, we find a ball $\mathbb{b}_1(z_0) : \|z - z_0\| < r_1$ such that $\mathcal{D}_{R, \mathbb{b}_1(z_0)} : z \longrightarrow D_R(z) \quad (z \in \mathbb{b}_1(z_0))$

is a smooth variation of domains with corners $e_R(z)$.

We form the Green's function $g_R(z,w)$ and the Robin constant $\lambda_R(z)$ for $(D_R(z), z)$ where $z \in \mathbb{b}_1(z_0)$. By the maximum principle we have, for $z \in \mathbb{b}_1(z_0)$,

$$0 < g(z,w) - g_R(z,w) < 1/R^{2n-2} \qquad \text{for } \forall w \in \overline{D_R(z)}.$$

Since $g_R(z,w)$ is continuous for (z,w) on $\bigcup_{z \in \mathbb{b}_1(z_0)} (z, \overline{D_R(z)}) - \mathbb{b}_1(z_0) \times \{0\}$, it follows that there exists a neighborhood $\mathbb{b}_2(z_0) \times V_1(w_0)$ where $\mathbb{b}_2(z_0) : \|z - z_0\| < r_2\ (< r_1)$ and $V_1(w_0) : \|w - w_0\| < \rho_1\ (\subset B_R)$ such that $|g_R(z,w) - g_R(z_0,w_0)| < \varepsilon$ for all $w \in V_1(w_0) \cap \overline{D_R(z)}$ and $z \in \mathbb{b}_2(z_0)$.

It follows that, for $\forall w \in V_1(w_0) \cap \overline{D}(z)$ and $z \in \mathbb{b}_2(z_0)$,

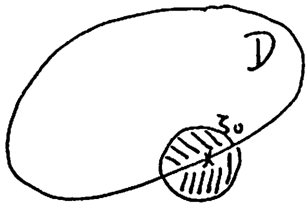
$$|g(z,w) - g(z_0,w_0)| < |g(z,w) - g_R(z,w)| + |g_R(z,w) - g_R(z_0,w_0)| + |g_R(z_0,w_0) - g(z_0,w_0)|$$
$$< \frac{1}{R^{2n-2}} + \varepsilon + \frac{1}{R^{2n-2}} < 3\varepsilon.$$

and hence that $g(z,w)$ is continuous at (z_0, w_0). c.q.f.d.

Since $\lambda(\zeta)$ is of class C^2 on $\mathbb{C}^n - \partial D$, we have $\frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta)$ for $\zeta \in \mathbb{C}^n - \partial D$ where $1 \leqq \alpha \leqq n$.

<u>2^{nd} step</u>. Let $\zeta_0 \in \partial D$ and let $\varepsilon > 0$. Then there exists an $r_0 > 0$ such that

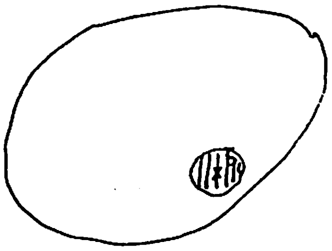
$$(6) \quad \left| \frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta) - \frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta') \right| < \varepsilon \quad \text{for any } \zeta, \zeta' \in B_{r_0}(\zeta_0) - \partial D.$$

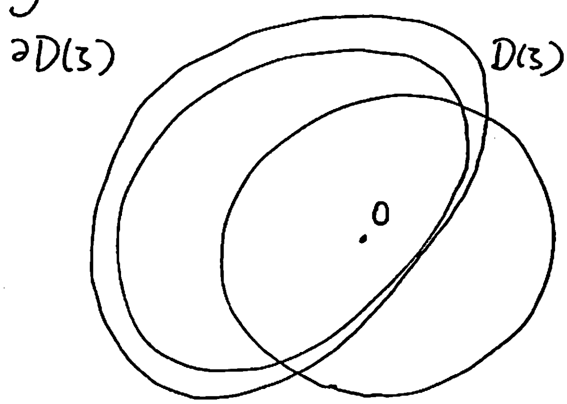


That is, $\frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta)$ $(\zeta \in \mathbb{C}^n - \partial D)$ forms a Cauchy sequence.

In fact, we get by Hadamard's formula

$$(7) \quad \frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta) = \frac{-1}{2(n-1)\,\omega_{2n}} \int_{\partial D(\zeta)} \frac{\partial g(\zeta,w)}{\partial \zeta_\alpha} \frac{\partial g(\zeta,w)}{\partial n_w}\, ds_w . \quad \text{for } \zeta \in D \cup E$$





Since $f(\zeta, w)$ and $-g(\zeta, w)$ defines the smooth variation $)_{D\cup E} \cup \partial \mathcal{D}_{D \cup E} : \zeta \to \overline{D(\zeta)}$ $(\zeta \in D \cup E)$, of smooth domains: it follows that, for $w \in \partial D(\zeta)$ and $\zeta \in D \cup E$,

$$\frac{-\frac{\partial g(\zeta,w)}{\partial \zeta_1}}{\frac{\partial f(\zeta,w)}{\partial \zeta_1}} = \cdots = \frac{-\frac{\partial g(\zeta,w)}{\partial \zeta_n}}{\frac{\partial f(\zeta,w)}{\partial \zeta_n}} = \frac{-\frac{\partial g(\zeta,w)}{\partial w_1}}{\frac{\partial f(\zeta,w)}{\partial w_1}} = \cdots = \frac{-\frac{\partial g(\zeta,w)}{\partial w_n}}{\frac{\partial f(\zeta,w)}{\partial w_n}} = \frac{\|\operatorname{Grad}_{(w)} g(\zeta,w)\|}{\|\operatorname{Grad}_{(w)} f(\zeta,w)\|}$$

$$\therefore \quad \frac{\partial g(\zeta,w)}{\partial \zeta_\alpha} = \frac{-\|\mathrm{Grad}_{(w)}\, g(\zeta,w)\|}{\|\mathrm{Grad}_{(w)}\, f(\zeta,w)\|}\, \frac{\partial f(\zeta,w)}{\partial \zeta_\alpha} \quad \text{for } w\in\partial D(\zeta) \text{ and } \zeta\in D\cup E$$

We substitute this into (7) and have

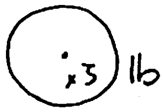
$$(8) \quad \frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta) = \frac{1}{2(n-1)\,\omega_{2n}} \int_{\partial D(\zeta)} \frac{\|\mathrm{Grad}_{(w)}\, g(\zeta,w)\|}{\|\mathrm{Grad}_{(w)}\, f(\zeta,w)\|}\, \frac{\partial f(\zeta,w)}{\partial \zeta_\alpha}\, \frac{\partial g(\zeta,w)}{\partial n_w}\, ds_w$$

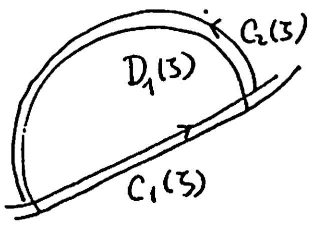
We need the following result (without proof):

<u>Preliminary 1</u> Let $\mathbb{B}$ a ball in $\mathbb{C}^n$. For each $\zeta\in\mathbb{B}$, let have a domain $D_1(\zeta)$ with corners $e_1(\zeta)$ such that the variation

$$\mathcal{D}_1 :\ \zeta \longrightarrow D_1(\zeta) \quad (\zeta\in\mathbb{B})$$

is a smooth variation of domains $D_1(\zeta)$.





$$\begin{cases} \partial D_1(\zeta) = C_2(\zeta)\cup C_1(\zeta) \\ C_2(\zeta)\cap C_1(\zeta) = e_1(\zeta) \end{cases}$$

Assume that $u(\zeta,w)$ is a continuous function with respect to (ζ,w) in $\mathcal{D}_1\cup\partial\mathcal{D}_1$ and that $u(\zeta,w)$ is harmonic for w in $D_1(\zeta)$ and vanishes on $C_1(\zeta)$.

Then, for $1\le\alpha\le n$,

$$\frac{\partial u}{\partial \zeta_\alpha}(\zeta,w) \text{ is continuous for } (\zeta,w) \text{ on } \mathcal{D}_1 \cup \Big[\bigcup_{\zeta\in\mathbb{B}} (\zeta, C_1(\zeta)-e_1(\zeta))\Big].$$

Now, let $R\gg 1$ be given and let $\mathbb{B}_R = (\|w\|<R)$. By 1st step and Preliminary 1, we see that

$$\|\mathrm{Grad}_{(w)}\, g(\zeta,w)\| \quad \Big(= -\frac{\partial g(\zeta,w)}{\partial n_w}\Big) \text{ is uniformly continuous}$$

for (ζ,w) on $\displaystyle\bigcup_{\zeta\in\mathbb{B}_1} (\zeta, \partial D.(\zeta)\cap\mathbb{B}_R)$, where

$$\mathbb{B}_1 : \|\zeta-\zeta_0\|<r_1 \quad\text{and}\quad D_R(\zeta) = D(\zeta)\cap\mathbb{B}_R.$$

It follows that

$$(9)\quad \int_{\partial D(\zeta)\cap \mathbb{B}_R} \frac{\|\operatorname{Grad}_{(w)} g(\zeta,w)\|}{\|\operatorname{Grad}_{(w)} f(\zeta,w)\|}\,\frac{\partial f(\zeta,w)}{\partial \zeta_\alpha}\,\frac{\partial g(\zeta,w)}{\partial n_w}\,d\delta_w$$

$$\longrightarrow -\int_{\partial D(\zeta_0)\cap \mathbb{B}_R} \frac{\|\operatorname{Grad}_{(w)} g(\zeta_0,w)\|^2}{\|\operatorname{Grad}_{(w)} f(\zeta_0,w)\|}\,\frac{\partial f}{\partial \zeta_\alpha}(\zeta_0,w)\,d\delta_w \quad \text{as } \zeta\in D\cup E\to\zeta_0\in\partial D.$$

— finite value —

We estimate the integral over $\partial D(\zeta)\cap(\mathbb{C}^n-\mathbb{B}_R)$.
First we do differentiation $\frac{\partial f}{\partial \zeta_\alpha}(\zeta,w)$:

$$f(\zeta,w) = \int_0^1\Big\{\sum_{\beta=1}^n w_\beta \frac{\partial\psi}{\partial z_\beta}\Big|_{\zeta-\psi(\zeta)tw} + \sum_{\beta=1}^n \bar w_\beta \frac{\partial\psi}{\partial \bar z_\beta}\Big|_{\zeta-\psi(\zeta)tw}\Big\}dt - 1 \quad \text{in } \mathbb{C}^n\times\mathbb{C}^n$$

$\therefore\quad \frac{\partial f}{\partial \zeta_\alpha}(\zeta,w)$

$$= \int_0^1\sum_{\beta=1}^n\Big\{ w_\beta\Big[\frac{\partial^2\psi}{\partial z_\beta\partial z_\alpha} + \sum_{\gamma=1}^n\Big(\Big[\frac{\partial^2\psi}{\partial z_\gamma\partial z_\beta}\Big]\cdot\Big(-\frac{\partial\psi}{\partial\zeta_\alpha}\Big)tw_\gamma + \Big[\frac{\partial^2\psi}{\partial\bar z_\gamma\partial z_\beta}\Big]\Big(-\frac{\partial\psi}{\partial\zeta_\alpha}\Big)t\bar w_\gamma\Big)\Big]$$

$$+\bar w_\beta\Big[\frac{\partial^2\psi}{\partial\bar z_\beta\partial z_\alpha} + \sum_{\gamma=1}^n\Big(\Big[\frac{\partial^2\psi}{\partial z_\gamma\partial\bar z_\beta}\Big]\Big(-\frac{\partial\psi}{\partial\zeta_\alpha}\Big)tw_\gamma + \Big[\frac{\partial^2\psi}{\partial\bar z_\gamma\partial\bar z_\beta}\Big]\Big(-\frac{\partial\psi}{\partial\zeta_\alpha}\Big)t\bar w_\gamma\Big)\Big]\Big\}.$$

$$\left(\begin{aligned}&\frac{\partial\psi}{\partial z_\beta}\Big|_{\zeta-\psi(\zeta)tw} \quad \text{precisely}\\ &=\frac{\partial\psi}{\partial z_\beta}(\zeta_1-\psi(\zeta_1,\dots,\zeta_n)w_1t,\ \dots,\ \zeta_n-\psi(\zeta_1,\dots,\zeta_n)w_nt)\end{aligned}\right)$$

at $z=\zeta-\psi(\zeta)tw$

$$\overset{\Vert}{\underset{(\zeta,w)}{=}} \int_0^1\sum_{\beta=1}^n\Big(w_\beta\frac{\partial^2\psi}{\partial z_\beta\partial z_\alpha} + \bar w_\beta\frac{\partial^2\psi}{\partial\bar z_\beta\partial z_\alpha}\Big)_{z=\zeta-\psi(\zeta)tw}dt$$

$$-\frac{\partial\psi}{\partial\zeta_\alpha}(\zeta)\sum_{\beta,\gamma=1}^n\int_0^1\Big\{\frac{\partial^2\psi}{\partial z_\beta\partial z_\gamma}w_\beta w_\gamma + \frac{\partial^2\psi}{\partial\bar z_\beta\partial w_\gamma}\bar w_\beta w_\gamma + \frac{\partial^2\psi}{\partial z_\beta\partial\bar w_\gamma}w_\beta\bar w_\gamma + \frac{\partial^2\psi}{\partial\bar w_\beta\partial\bar w_\gamma}\bar w_\beta\bar w_\gamma\Big\}_{z=\zeta-\psi(\zeta)tw}t\,dt$$

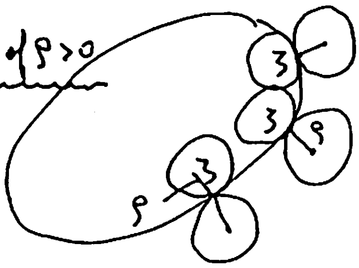
It follows from (2) that, for any $\|w\|>1$,

$$(10)\quad \Big|\frac{\partial f}{\partial\zeta_\alpha}(\zeta,w)\Big| \leqq M_2\|w\|^2 \quad \text{for all } \zeta\in\mathbb{C}^n.$$

where M_2 is independent of $(\zeta, w) \in \mathbb{C}^n \times (\|w\|>1)$.
Next, we evaluate

$$\|\mathrm{Grad}_{(w)}\, g(\zeta, w)\| \quad \text{for} \quad w \in \partial D(\zeta) \cap (\mathbb{C}^n - \mathbb{B}_R)$$

Def. of $\rho > 0$



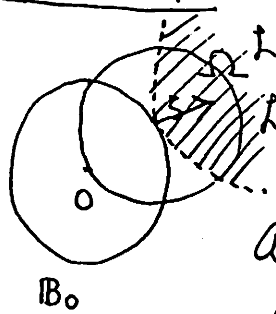
Since ∂D is smooth and is compact, we find a small $\rho > 0$ such that, at any $\zeta \in \partial D$, we can draw two sphere with radius ρ being tangent to the surface ∂D at ζ from the outside, and from the inside, respectively.

Def. of $A > 0$

$$A \underset{\text{def}}{=} 1 + \text{the diameter of } D \ (< +\infty).$$

We need the following easy Proposition:

Proposition 1



Let $\mathbb{B}_0 = (\|w\|<1)$ and $\mathbb{B}_\zeta = (\|w-\zeta\|<1)$ where $\zeta \in \partial\mathbb{B}_0$.
Let Ω is a domain which is lying outside of $\mathbb{B}_0$ and whose boundary $\partial\Omega$ is tangent to $\partial\mathbb{B}_0$ at ζ.
Assume that $u(z)$ is a harmonic fn on Ω such that

$$u(z) = \begin{cases} 0 & \text{on } z \in (\partial\Omega) \cap \mathbb{B}_\zeta \\ \in [0,1] & \text{on } z \in (\partial\mathbb{B}_\zeta) \cap \Omega \end{cases}$$

Then we find a $C > 0$ such that

$$(11) \qquad \|\mathrm{Grad}_{(z)}\, u(\zeta)\| \leq C$$

where C is independent of $\zeta \in \partial\mathbb{B}_0$, Ω and $u(z)$.

Proof. Consider a harmonic function $U(z)$ for z in $\mathbb{B}_\zeta - \mathbb{B}_0$ such that

$$U(z) = \begin{cases} 0 & \text{for } z \in (\partial\mathbb{B}_0) \cap \mathbb{B}_\zeta \\ 1 & \text{for } z \in (\partial\mathbb{B}_\zeta) \cap (\mathbb{B}_0^c). \end{cases}$$

Then the maximum principle implies that

$$0 \leq u(z) \leq U(z) \quad \text{on } \Omega \cap (\mathbb{B}_\zeta - \mathbb{B}_0)$$

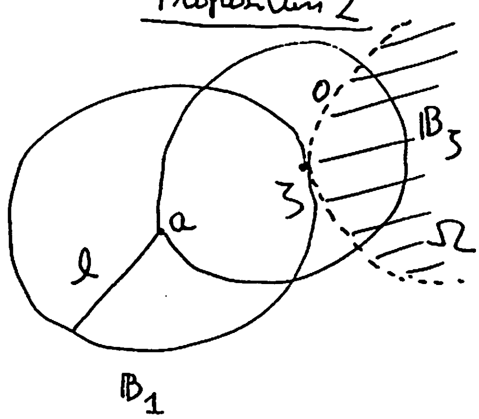
Since $\partial\Omega$ and ∂B_0 are tangent at the point ζ, it follows that

$$\|\mathrm{Grad}\, u(\zeta)\| = \frac{\partial u}{\partial n_z}(\zeta) \leqq \frac{\partial U}{\partial n_z}(\zeta) = \|\mathrm{Grad}\, U(\zeta)\|.$$

It is clear that $\|\mathrm{Grad}\, U(\zeta)\|$ is independent of $\zeta \in \partial B_0$ and Ω, $u(z)$. Hence $C = \|\mathrm{Grad}\, U(\zeta)\|$ satisfies (11). c.q.f.d.

Proposition 1 easily induce the following

<u>Proposition 2</u>



Let $\mathbb{B}_1$ be a ball of radius l, and $\mathbb{B}_\zeta = \{\|w-\zeta\| < l\}$ where $\zeta \in \partial\mathbb{B}_1$. Let Ω lie on the outside of $\mathbb{B}_1$ and $\partial\Omega$ be tangent to $\partial\mathbb{B}_1$ at the point ζ. Assume that $u(w)$ is a harmonic fn for w in Ω such that

$$u(w) = 0 \text{ on } \partial\Omega \cap \mathbb{B}_\zeta;$$
$$0 \leqq u(w) \leqq m \text{ on } (\partial\mathbb{B}_\zeta) \cap \Omega$$

(12) Then $\|\mathrm{Grad}_{(w)}\, u(\zeta)\| \leqq \dfrac{mC}{l}$

where C is the constant defined in (11).

(:) consider a harmonic function

$$u_1(z) \underset{\mathrm{def}}{=} \frac{u(a + l z)}{m} \quad \text{and} \quad \zeta_0 = \frac{\zeta - a}{l} \quad (\|z\| = 1)$$

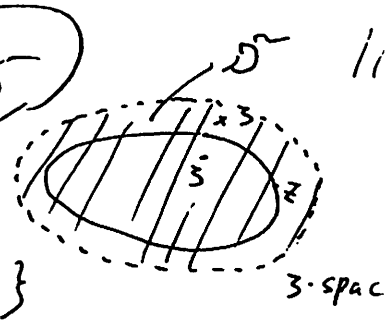
Then Proposition 1 implies that

$$\|\mathrm{Grad}_{(z)}\, u_1(\zeta_0)\| \leqq C$$
$$\|$$
$$\frac{l}{m}\|\mathrm{Grad}_{(w)}\, u(\zeta)\| \qquad \therefore \|\mathrm{Grad}_{(w)}\, u(\zeta)\| \leqq \frac{mC}{l}$$

<u>Proposition 3</u> For any $\zeta \in D \cup \left\{ \begin{array}{l} \text{the set of points } \zeta \text{ with} \\ \mathrm{dist}(\zeta, \partial D) < \frac{1}{2} \end{array} \right\}$ –

We have

(13) $$\mathrm{Max}\{\|w\| : w \in \partial D(\zeta)\} \leqq \frac{A}{|\psi(\zeta)|}$$



For the simplicity we put

$$\tilde{D} = D \cup \{\zeta \in \mathbb{C}^n \mid \operatorname{dist}(\zeta, \partial D) < \tfrac{1}{2}\}$$

<u>Proof.</u> Let $\zeta \in \tilde{D} - \partial D$. Take $w \in \partial D(\zeta)$. Then it exists uniquely $z \in \partial D$ such that $w = -\dfrac{z-\zeta}{\psi(\zeta)}$. Hence

$$\begin{aligned} &\operatorname{Max}\{\|w\| : w \in \partial D(\zeta)\} \\ &\leqq \max\{|z-\zeta|/|\psi(\zeta)| : z \in \partial D\} \\ &\leqq A/|\psi(\zeta)| \quad . \quad \text{c.q.f.d.} \end{aligned}$$

Consequently, given $\xi \in \partial D(\zeta)$, we have

$$\frac{\rho}{|\psi(\zeta)|} = \frac{A}{|\psi(\zeta)|}\cdot\frac{\rho}{A} \geqq \frac{\rho}{A}\cdot \operatorname{Max}\{\|w\| : w \in \partial D(\zeta)\} \geqq \frac{\rho}{A}\|\xi\|, \text{ i.e.,}$$

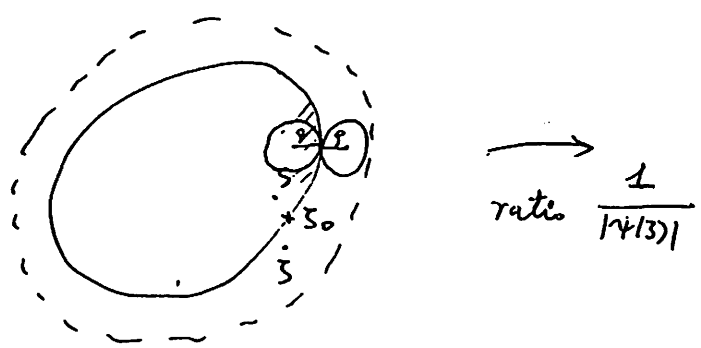
$$(14) \qquad \frac{\rho}{|\psi(\zeta)|} \geqq \frac{\rho}{A}\|\xi\| \quad \text{for all } \zeta \in \tilde{D} - \partial D \text{ and } \xi \in \partial D(\zeta)$$

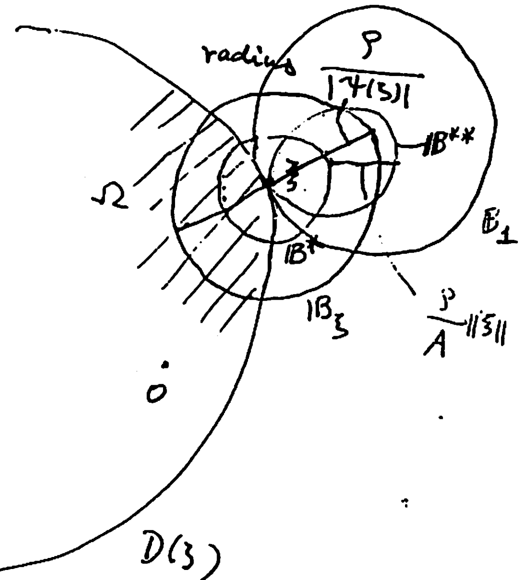
We may assume $\frac{\rho}{A} < 1$. If non, retake a smaller ρ.

We often used the following inequality:

$$0 < g(\zeta, w) \leqq \frac{1}{\|w\|^{2n-2}} \quad \text{for all } (\zeta, w) \in \mathcal{D}$$

Consider $D(\zeta)$ for $\zeta \in \tilde{D} - \partial D$





Take $\xi \in \partial D(\zeta)$. Draw a sphere $B^*: \|s - \xi\| < \frac{\rho}{A}\|\xi\|$. Then the sphere $\mathbb{B}^{**}$ with radius $\frac{\rho}{A}\|\xi\|$ is tangent to $\partial D(\zeta)$ at ξ from the outside, because of (14) and the definition of $\rho > 0$.

In $\mathbb{B}^* \cap D(\zeta)$, we have

$$0 < g(\zeta, w) < \frac{1}{(1-\frac{\rho}{A})^{2n-2}} \frac{1}{\|\xi\|^{2n-2}}$$

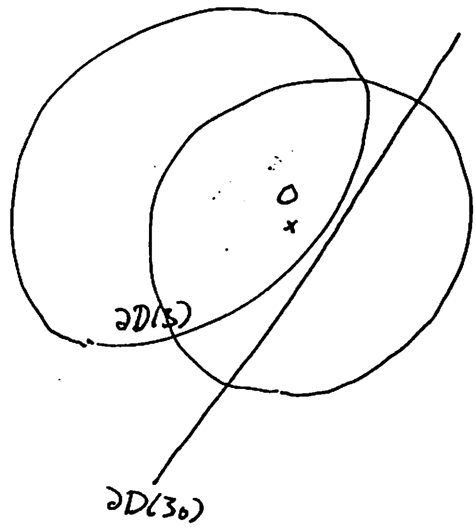
It follows from (12) that

$$(15) \quad \|\mathrm{Grad}_{(w)}\, g(\zeta, w)\|_{w=\xi} \leqq \frac{1}{(1-\frac{\rho}{A})^{2n-2}} \cdot \frac{1}{\|\xi\|^{2n-2}} \cdot \frac{1}{\frac{\rho}{A}\|\xi\|} = \frac{K_2}{\|\xi\|^{2n-1}}$$

where $K_2 = \frac{1}{(1-\frac{\rho}{A})^{2n-2}} \frac{A}{\rho}$, which does not depend on $\zeta \in \tilde{D} - \partial D$ and $\xi \in \partial D(\zeta)$.

Finally we observe

$$\int_{(\partial D(\zeta)) \cap \mathbb{B}_R} \frac{\partial g(\zeta, w)}{\partial n_w} ds_w .$$



We know that

$$-\frac{\partial g(\zeta, w)}{\partial n_w} ds_w > 0 \text{ on } \partial D(\zeta)$$

and that $\displaystyle -\int_{\partial D(\zeta)} \frac{\partial g(\zeta, w)}{\partial n_w} ds_w = 2(n-1)\omega_{2n}$ for all $\zeta \in \mathbb{C}^n$.

Moreover, as already noted,

$$-\int_{\partial D(\zeta) \cap \mathbb{B}_R} \frac{\partial g(\zeta, w)}{\partial n_w} ds_w \longrightarrow -\int_{\partial D(\zeta_0) \cap \mathbb{B}_R} \frac{\partial g(\zeta_0, w)}{\partial n_w} ds_w \quad \text{as } \zeta \in D \cup E \to \zeta_0 \in \partial D$$

Therefore if $R \gg 1$ is sufficiently large, then, given $0 < \varepsilon \ll 1$,

$$\frac{-1}{2(n-1)\omega_{2n}} \int_{\partial D(\zeta_0) \cap \mathbb{B}_R} \frac{\partial g(\zeta_0, w)}{\partial n_w} ds_w > 1 - \varepsilon$$

We fix such a R.

Hence, if $\zeta \in D \cup E$ is very close to ζ_0, then

$$\frac{-1}{(2n-2)\omega_{2n}} \int_{\partial D(\zeta) \cap \mathbb{B}_R} \frac{\partial g(\zeta,w)}{\partial n_w}\, ds_w > 1 - 2\varepsilon$$

i.e., there exists an $r_2 > 0 \ni$.

$$(16) \quad 0 < \frac{-1}{(2n-2)\omega_{2n}} \int_{\partial D(\zeta) \cap \mathbb{B}_R^c} \frac{\partial g(\zeta,w)}{\partial n_w}\, ds_w < 2\varepsilon$$

for all $\zeta \in D \cup E \ni (\|\zeta - \zeta_0\| < r_2) = \mathbb{B}_{r_2}(\zeta_0)$

Therefore

$$\left| \frac{1}{2(n-1)\omega_{2n}} \int_{\partial D(\zeta) \cap \mathbb{B}_R^c} \frac{\|\mathrm{Grad}_{(w)}\, g(\zeta,w)\|}{\|\mathrm{Grad}_{(w)}\, f(\zeta,w)\|} \frac{\partial f}{\partial \zeta_\alpha}(\zeta,w) \frac{\partial g(\zeta,w)}{\partial n_w}\, ds_w \right|$$

$$\leqq \frac{1}{2(n-1)\omega_{2n}} \int_{\partial D(\zeta) \cap \mathbb{B}_R^c} \frac{\left(\frac{K_2}{\|w\|^{2n-1}}\right)}{m_1} \cdot M_2 \|w\|^2 \left(-\frac{\partial g(\zeta,w)}{\partial n_w}\, ds_w\right)$$

$\because$ (from (4), (15) and (10))

$$= \frac{1}{2(n-1)\omega_{2n}} \cdot \frac{K_2 M_2}{m_1} \int_{\partial D(\zeta) \cap \mathbb{B}_R^c} \frac{1}{\|w\|^{2n-3}} \left(-\frac{\partial g(\zeta,w)}{\partial n_w}\, ds_w\right)$$

$$\leqq \frac{K_2 M_2}{m_1} \cdot \frac{1}{R^{2n-3}} \cdot \left\{ \int_{\partial D(\zeta) \cap \mathbb{B}_R^c} \frac{-1}{(2n-2)\omega_{2n}} \frac{\partial g(\zeta,w)}{\partial n_w}\, ds_w \right\}$$

$$\leqq \frac{K_2 M_2}{m_1 R^{2n-3}} \cdot 2\varepsilon \quad (\text{from } (16))$$

Since we may assume $\frac{1}{R^{2n-3}} < \varepsilon$, we conclude that for any $\zeta \in (D \cup E) \cap \mathbb{B}_{r_2}(\zeta_0)$,

$$(17) \quad \left| \frac{1}{(2n-2)\omega_{2n}} \int_{\partial D(\zeta) \cap \mathbb{B}_R^C} \frac{\|\mathrm{Grad}_{(w)} g(\zeta,w)\|}{\|\mathrm{Grad}_{(w)} f(\zeta,w)\|} \frac{\partial f(\zeta,w)}{\partial \zeta_\alpha} \frac{\partial g(\zeta,w)}{\partial n_w} ds_w \right| < K_3 \varepsilon^2$$

where $K_3 = 2K_2 M_2 / m_2$ which is independent of ζ, R.

On the other hand, we get from (9),

$\exists\, 0 < r_3 (\leq r_2)$ s.t. for any $\zeta, \zeta' \in (D \cup E) \cap \mathbb{B}_{r_3} : \|\zeta - \zeta_0\| < r_3$,

$$\frac{1}{(2n-2)\omega_{2n}} \left| \int_{\partial D(\zeta) \cap \mathbb{B}_R} \frac{\|\mathrm{Grad}_{(w)} g(\zeta,w)\|}{\|\mathrm{Grad}_{(w)} f(\zeta,w)\|} \frac{\partial f}{\partial \zeta_\alpha}(\zeta,w) \frac{\partial g(\zeta,w)}{\partial n_w} ds_w - \int_{\partial D(\zeta') \cap \mathbb{B}_R} \frac{\|\mathrm{Grad}_{(w)} g(\zeta',w)\|}{\|\mathrm{Grad}_{(w)} f(\zeta',w)\|} \frac{\partial f}{\partial \zeta_\alpha}(\zeta',w) \frac{\partial g(\zeta',w)}{\partial n_w} ds_w \right| < \varepsilon$$

It follows from (8) that for any $\zeta, \zeta' \in (D \cup E) \cap \mathbb{B}_{r_3}(\zeta_0)$

$$\left| \frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta) - \frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta') \right| < 2K_3 \varepsilon^2 + \varepsilon,$$

which proves 2nd step.

<u>3rd step.</u> $\frac{\partial \lambda}{\partial \zeta_\alpha}$ exists at $\zeta_0 \in \partial D$ and defines a continuous function on $\mathbb{C}^n$, where $1 \leqq \alpha \leqq n$.

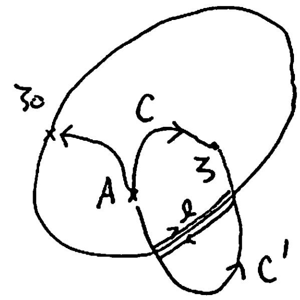
<u>Proof.</u> Let $\zeta_0 \in \partial D$. By 2nd step, we have a limit

$$\lambda_\alpha(\zeta_0) = \lim_{\substack{\zeta \to \zeta_0 \\ \zeta \in \mathbb{C}^n - \partial D}} \frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta).$$

It follows that, if we put

$$\lambda_\alpha(\zeta) = \begin{cases} \frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta) & \text{if } \zeta \in D \cup E \\ \lambda_\alpha(\zeta) & \text{if } \zeta \in \partial D, \end{cases}$$

then $\lambda_\alpha(z)$ is continuous fn on $\mathbb{C}^n$. Consider the integral:



$$F_C(z) \underset{df}{=} \sum_{\alpha=1}^{n} \int_C \lambda_\alpha(z)\, dz_\alpha + \bar{\lambda}_\alpha\, d\bar{z}_\alpha$$

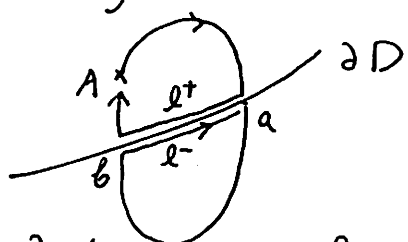
where C connects a fixed point A and the point z ($\in \mathbb{C}^n$).

By 1st step, we see that $F_C(z)$ does not depend on the choice of C and that $F_C(z_0) = \lambda(z_0) - \lambda(A)$ for $z_0 \in \partial D$.

In fact, let C' be another curve from A to z.

Then

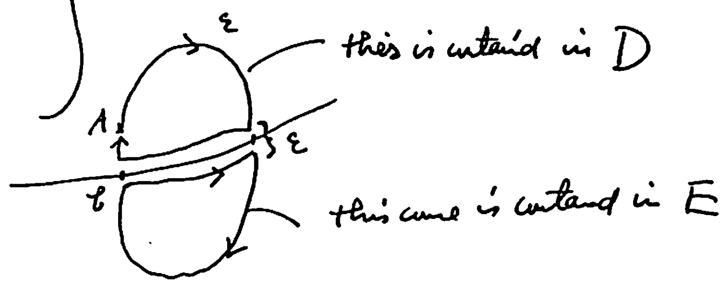
$$F_C(z) - F_{C'}(z) = \int \sum_{\alpha=1}^{n} \left(\lambda_\alpha(z)\, dz_\alpha + \bar{\lambda}_\alpha\, d\bar{z}_\alpha \right)$$



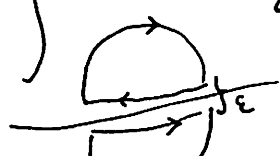
where $l \subset \partial D$

By continuity of $\lambda_\alpha(z)$, we have it is

$$= \lim_{\varepsilon \to 0} \int \sum_{\alpha=1}^{n} \left(\lambda_\alpha(z)\, dz_\alpha + \bar{\lambda}_\alpha\, d\bar{z}_\alpha \right)$$



$$= \lim_{\varepsilon \to 0} \int d\lambda(z) \quad \text{from definit. of } \lambda_\alpha(z)$$



$= 0$

$\therefore$ Integral does not depend on the curve C.

Next, let $\zeta_0 \in \partial D$. We connect ζ_0 and the original point A by a curve C_0 included in D. Then

$$F(\zeta_0) = \int_{A,\,C_0}^{\zeta_0} \left(\sum_{\alpha=1}^{n} \lambda_\alpha d\zeta_\alpha + \bar{\lambda}_\alpha d\bar{\zeta}_\alpha \right)$$

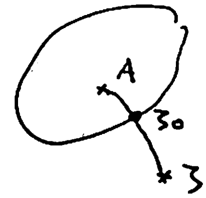
$$= \lim_{\substack{\zeta \to \zeta_0 \\ C_0 \cap D}} \int_A^{\zeta} \underbrace{\left(\sum_{\alpha=1}^{n} \lambda_\alpha d\zeta_\alpha + \lambda_{\bar{\alpha}} d\bar{\zeta}_\alpha \right)}_{= \, d\lambda \text{ in } D}$$

$$= \lim_{\zeta \to \zeta_0} \{\lambda(\zeta) - \lambda(A)\}$$

$$= \lambda(\zeta_0) - \lambda(A) \quad \text{by 1st step.}$$

We conclude that for any $\zeta \in \mathbb{C}^n$

$$\lambda(\zeta) = \lambda(a) + F(\zeta).$$



and hence that

$\lambda(\zeta)$ is differentiable even at $\zeta_0 \in \partial D$ and

$$\frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta) = \lambda_\alpha(\zeta) \quad \text{for all } \zeta \in \mathbb{C}^n.$$

c.q.f.d.

Remark 1 From the above proof of (9), we have for $\zeta_0 \in \partial D$,

$$(18) \quad \frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta_0) = \frac{-1}{2(n-1)\omega_{2n}} \int_{\partial D(\zeta_0)} \frac{\|\mathrm{Grad}_{(w)}\, g(\zeta_0, w)\|^2}{\|\mathrm{Grad}_{(w)}\, f(\zeta_0, w)\|} \frac{\partial f}{\partial \zeta_\alpha}(\zeta_0, w)\, ds_w .$$

We note that the right-hand side is written down explicitly by $\frac{\partial \psi}{\partial \zeta_\alpha}$, $\frac{\partial^2 \psi}{\partial \zeta_\alpha \partial \zeta_\beta}$, $\frac{\partial^2 \psi}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}$ at ζ_0, by use of (3), (3') and (9').

Differentiability

Until now we have dealt with smooth variations of smooth domains in (or over $\mathbb{C}^n$):

Let $\mathcal{D}: t \to D(t)$ $(t \in B)$ be a smooth variation of smooth domains in $\mathbb{C}^n$ with parameter $t \in B$, where B is a region of the complex t-plane.

We put $\mathcal{D} = \bigcup_{t \in B} (t, D(t))$ $(\subset B \times \mathbb{C}^n)$. Assume that $\exists\, \zeta \in \mathbb{C}^n$ such that $B \times \{\zeta\} \subset \mathcal{D}$. We thus have the Green's function $g(t,z)$ and the Robin constant $\lambda(t)$, for $(D(t), \zeta)$. Under this notation, we got the following Fundamental formula:

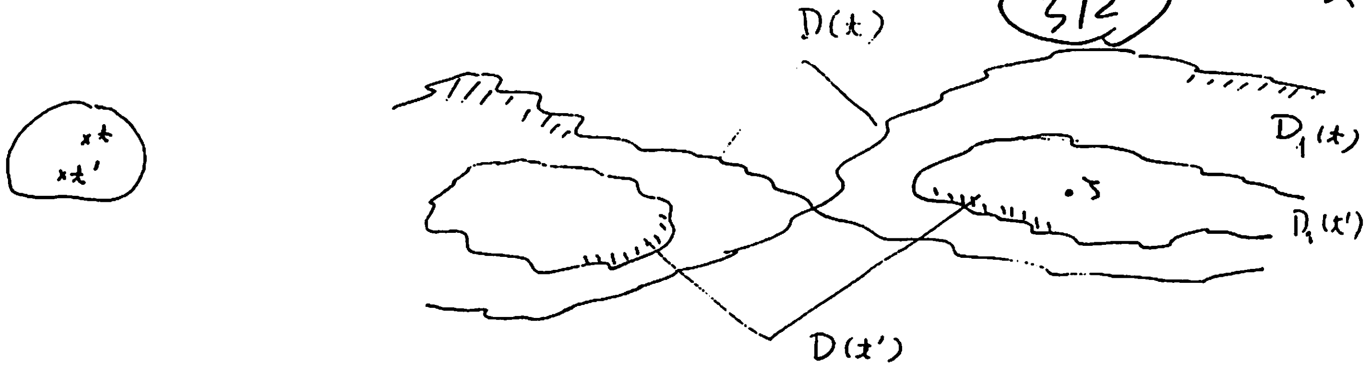
$$(1) \quad \frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(t) = \frac{-1}{(n-1)\omega_{2n}} \int_{\partial D(t)} k_2(t,z) \, \| \mathrm{Grad}_{(z)}\, g \|^2 ds_z$$

$$- \frac{4}{(n-1)\omega_{2n}} \iint_{D(t)} \Big\{ \sum_{\alpha=1}^{n} \Big| \frac{\partial^2 g}{\partial t \partial \bar{z}_\alpha} \Big|^2 \Big\} dV.$$

On the other hand, if $\mathcal{D}$ is pseudoconvex in $B \times \mathbb{C}^n$, then $k_2(t,z) \geq 0$ on $\partial D(t)$. It follows that

(2) If $\mathcal{D}$ is pseudoconvex in $B \times \mathbb{C}^n$, then $\lambda(t)$ is superharmonic for t in B.

Today, let $\mathcal{D}$ be any domain in $B \times \mathbb{C}^n$ $(n \geq 1)$. The fiber $D(t)$ $(t \in B)$ is no longer domain of $\mathbb{C}^n$ with smooth boundary. In general, $D(t)$ is an open set whose number of connected components will vary with $t \in B$.



We assume that $\exists\, \zeta \in \mathbb{C}^n$ such that $\mathcal{D} \supset B \times \{\zeta\}$. We define, as usual, the Green's function $g(t, z)$ and the Robin constant $\lambda(t)$ for $(D(t), \zeta)$ as follows:

Let $D_1(t)$ be the connected component of $D(t)$ which contains the point ζ. Then we consider the sequence of domains $D_{1n}(t)$ in $D(t)$, with smooth boundary $\partial D_{1n}(t)$ such that $D_{1n}(t) \ni \zeta$ and that $D_{11}(t) \Subset D_{12}(t) \Subset \cdots$ and $D_1(t) = \bigcup_{n=1}^{\infty} D_{1n}(t)$. Each $D_{1n}(t)$ $(n = 1, 2, \cdots)$ carries the Green's function $g_{1n}(t, z)$ and the Robin constant $\lambda_{1n}(t)$ for $(D_{1n}(t), \zeta)$. Since

$$g_{1n}(t, z) \leqq g_{1n+1}(t, z) \quad \text{for } z \in D_{1n}(t) \ ;$$
$$\lambda_{1n}(t) \leqq \lambda_{1n+1}(t)$$

it follows that the limits

$$g_1(t, z) = \lim_{n \to \infty} g_{1n}(t, z) \quad \text{for } z \in D_1(t) \ ;$$
$$\lambda_1(t) = \lim_{n \to \infty} \lambda_{1n}(t)$$

exist. We say that $g_1(t, z)$ and $\lambda_1(t)$ the Green's function and the Robin constant for $(D_1(t), \zeta)$. We put

$$g(t, z) = \begin{cases} g_1(t, z) & \text{for } z \in D_1(t) \\ 0 & \text{for } z \in D(t) - D_1(t) \end{cases} \ ;$$

$$\lambda(t) = \lambda_1(t).$$

$g(t, z)$ and $\lambda(t)$ are called the Green's function and the Robin constant for $(D(t), \zeta)$.

As already noted in the last semester, in the case $n=1$, it may happen that

$$g_1(t,z) \equiv +\infty \quad \text{on } D_1(t),$$

or equivalently,

$$\lambda_1(t) = +\infty.$$

We say that $D_1(t)$ with $\lambda_1(t) < +\infty$ (resp. $= +\infty$) is hyperbolic (resp. parabolic).

In the case $n \geq 2$, we always have

$$g_1(t,z) \leq \frac{1}{\|z-\zeta\|^{2n-2}} \quad \text{for } z \in D_1(t) \ ;$$

$$\lambda_1(t) \leq 0.$$

It is clear that, $g_1(t,z) \equiv \frac{1}{\|z-\zeta\|^{2n-2}}$ for $z \in D_1(t)$ if and only if $\lambda_1(t) = 0$.

<u>Definition 1</u> In the case $n \geq 2$, we say that $D_1(t)$ with $\lambda_1(t) < 0$ (resp. $=0$) is hyperbolic (resp. parabolic).

For general variations of domains we no longer hope a fundamental formula (1). But (2) is generalized as follows.

<u>Theorem 1</u> If $\mathcal{D} = \bigcup_{t\in B}(t, D(t))$ is a pseudo-convex domain in $B \times \mathbb{C}^n$, then $\lambda(t)$ is superharmonic function on B in any case $n \geq 1$. In the case $n \geq 2$, $\log(-\lambda(t))$ is subharmonic for t in B.

In order to pass from (2) to Theorem 1, the following differentiality lemma is essential:

Let $\mathcal{D}$ be a domain of $B \times \mathbb{C}^n$ $(n \geq 1)$ where B is a region in the complex t-plane. We impose the following conditions $1 \sim 4$ on $\mathcal{D}$:

Condition 1.

There exists a constant ζ in $\mathbb{C}^n$ such that $B \times \{\zeta\} \subset \mathcal{D}$.

Each fiber $D(t)$ (which is an open set in $\mathbb{C}^n$) then carries the Green's function $g(t, z)$ and the Robin constant $\lambda(t)$ for $(D(t), \zeta)$.

Condition 2.

There exists a double $(\widetilde{\mathcal{D}}, \psi(t, z))$ defining the domain $\mathcal{D}$ such that $\psi(t, z)$ is real analytic with respect to (t, z) in $\widetilde{\mathcal{D}}$.

i.e., $\widetilde{\mathcal{D}} \supset \mathcal{D}$; $\widetilde{D}(t) \gg D(t)$ for all $t \in B$.

$$\mathcal{D} = \{ (t, z) \in \widetilde{\mathcal{D}} \mid \psi(t, z) < 0 \} ;$$

$$\partial\mathcal{D} = \{ (t, z) \in \widetilde{\mathcal{D}} \mid \psi(t, z) = 0 \} ;$$

$$\mathrm{Grad}_{(t,z)}\, \psi(t, z) = \left(\frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial z_1}, \cdots, \frac{\partial \psi}{\partial z_n} \right)(t, z) \neq 0 \quad \text{on } \partial\mathcal{D}.$$

$\therefore \left\{ (t, z) \in \widetilde{\mathcal{D}} \mid \psi = \frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial z_1} = \cdots = \frac{\partial \psi}{\partial z_n} = 0 \text{ at } (t, z) \right\}$ is empty.

Here we consider the subset Γ of $\widetilde{\mathcal{D}}$ defined by

$$\Gamma = \left\{ (t, z) \in \widetilde{\mathcal{D}} \mid \psi = \frac{\partial \psi}{\partial z_1} = \cdots = \frac{\partial \psi}{\partial z_n} = 0 \text{ at } (t, z) \right\}$$

and set

$$\begin{aligned} \gamma &= \text{the projection of } \Gamma \text{ to } B \\ &= \{ t \in B \mid \exists\, w \in \mathbb{C}^n \text{ such that } (t, z) \in \Gamma \} \end{aligned}$$

The set Γ is thus determined by $(2n+1)$ real-analytic equations in the real $(2n+2)$ dimensional space $\widetilde{\mathcal{D}}$.

For the sake of convenience we introduce the notations:

$$\Gamma_{B_0} = \Gamma \cap (B_0 \times \mathbb{C}^n) \quad \text{and} \quad \Gamma(t) = \Gamma \cap (\{t\} \times \mathbb{C}^n)$$

for any open set $B_0 \subset B$ and $t \in B$.

Condition 3.

(a) For each open set $B_0 \subset\subset B$, the set Γ_{B_0} consists of a finite number of real 1-dimensional non-singular curves in $\mathcal{D}_{B_0}$ $(= \mathcal{D} \cap (B_0 \times \mathbb{C}^n))$ except perhaps for a finite number of singular points;

(b) The set $\Gamma(t)$ for $t \in \gamma$ consists of a finite number of points: $\Gamma(t) = \{ z^{(1)}(t), \dots, z^{(q)}(t) \}$ where q (≥ 1) may depend on $t \in \gamma$.

The set $\gamma \cap B_0$ then consists of a finite number of smooth real analytic curves in B_0 except perhaps for a finite number of singular points.

Condition 4.

The function $\psi(t, z)$ of condition 2 is plurisubharmonic with respect to (t, z) in $\tilde{\mathcal{D}}$.

To understand our situation better we give examples

Example 1 In $B \times \mathbb{C}$, let

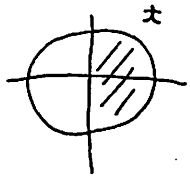
$$\psi(t, z) = 5(t + \bar{t}) + |(z - t^2) - 1|^2 |(z - t^2) + 1|^2 - 1$$

Given $|t| \, \rho$, we put

$$D(t) = \{ z \in \mathbb{C} \mid \psi(t, z) < 0 \}$$
$$= \{ z \in \mathbb{C} \mid |(z - t^2) - 1| \cdot |(z - t^2) + 1| < \sqrt{1 - 10 t_1} \}$$

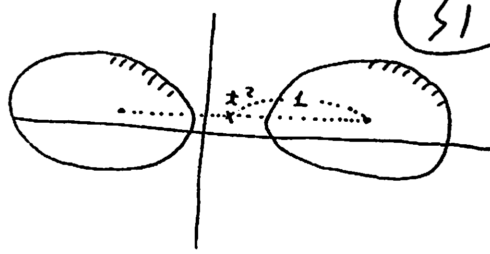
where $t = t_1 + i t_2$.

$\therefore$ $D(t)$ is a lemniscate of focuses $(-1 + t^2, 1 + t^2)$ with radius $\sqrt[4]{1 - 10 t_1}$.

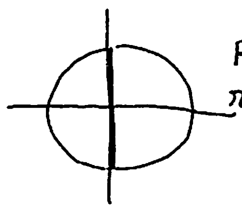


t i.e, $\mathrm{Re}\, t > 0$

radius < 1

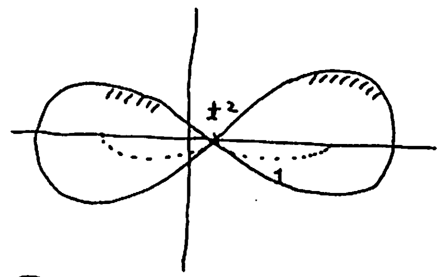


$D(t)$ consists of two components

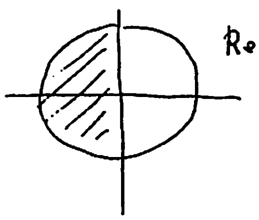


$\mathrm{Re}\, t = 0$

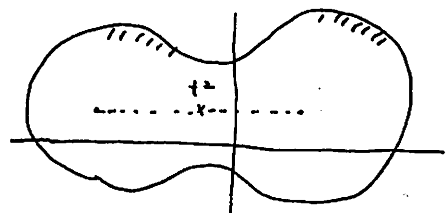
radius $= 1$



$D(t)$ consists of two components which meet together at $z = t^2$



$\mathrm{Re}\, t < 0$



$D(t)$ consists of only one component.

In this case

$$\gamma = \{ |t| < \rho \mid \mathrm{Re}\, t = 0 \} \; ; \quad \Gamma = \bigcup_{t \in \gamma} \{(t, t^2)\}$$

We note that

$$\frac{\partial \psi}{\partial t} = 5 - 2t \left\{ (\overline{z - t^2 - 1}) \, |z - t^2 + 1|^2 + |z - t^2 - 1|^2 \, \overline{z - t^2 + 1} \right\}$$

Hence for $|t| \ll 1$, $\frac{\partial \psi}{\partial t} \neq 0$ on $\mathcal{D} \cup \partial \mathcal{D}$.

Moreover, if t is very close to 0, then $\zeta = 1$ is always attained in each $D(t)$. It follows that Conditions $1 \sim 4$ are fullfilled.

Example 2 We would like to show, even if the dimension n of the fiber is ≥ 2, the singular set Γ is one real dimensional, by simple but typical example.

Assume $(z, w) \in \mathbb{C}^2$, and Consider

$$\psi(t, z, w) = 5(t + \bar{t}) + \left\{ |z - t^2 - 1|^2 + |z - t - 2|^2 \right\} \left\{ |z - t^2 + 1|^2 + |w - t + 2|^2 \right\} - 25$$

$\underset{\text{def}}{=} 5(t+\bar{t}) + AB - 25$

Given $|t| \ll 1$, we set

$$D(t) = \{(z,w) \in \mathbb{C}^2 \mid \psi(t,z) < 0\}.$$

$$\frac{\partial\psi}{\partial z} = \overline{z - t^2 - 1}\, B + \overline{z - t^2 + 1}\, A$$

$$\frac{\partial\psi}{\partial w} = \overline{w - t - 2}\, B + \overline{w - t + 2}\, A$$

Let $^\forall(t, z, w) \in \Gamma$. Then we have

$$\overline{(z - t^2 - 1)}\ \overline{(w - t + 2)}\, AB = \overline{(z - t^2 + 1)}\ \overline{(w - t - 2)}\, AB\ ;$$

$$5(t + \bar{t}) + AB - 25 = 0$$

Since $|t| \ll 1$, $AB \fallingdotseq 25 \overset{*}{\ne} 0$, so that

$$(z - t^2 - 1)(w - t + 2) = (z - t^2 + 1)(w - t - 2)$$

$$\therefore\quad w - t = 2(z - t^2).$$

Hence
$$A = 5\,|z - t^2 - 1|^2$$
$$B = 5\,|z - t^2 + 1|^2$$

$$\therefore \overline{\left(\frac{\partial\psi}{\partial z}\right)} = (z - t^2 - 1)\,5\,|z - t^2 + 1|^2 + (z - t^2 + 1)\,5\,|z - t^2 - 1|^2 = 0$$

$$(z - t^2 - 1)(z - t^2 + 1)\,\underbrace{\left[\overline{z - t^2 + 1} + \overline{z - t^2 - 1}\right]}_{=\,2\,\overline{(z - t^2)}} = 0$$

$$\therefore\quad z - t^2 = \pm 1 \text{ or } 0.$$

$$\therefore\ \psi(t, z; w) = 5\left\{2t_1 + 5\,|z - t^2 - 1|^2\,|z - t^2 + 1|^2 - 5\right\}$$

It follows from $|t| \ll 1$ that $z - t^2 = \pm 1$ is impossible.

In the case where $z - t^2 = 0$, we have

$$\psi(t, z, w) = 5\{2t_1\} = 0 \quad \therefore\quad \operatorname{Re} t = 0.$$

This means that, if $(t, z, w) \in \Gamma$, then $\operatorname{Re} t = 0$; $z = t^2$ and $w = t$. Conversely, consider the point (t, t^2, t) where $\operatorname{Re} t = 0$ and $|t| \ll 1$. Then it is clear that $\psi = \frac{\partial\psi}{\partial z} = \frac{\partial\psi}{\partial w} = 0$ at that point. We conclude that

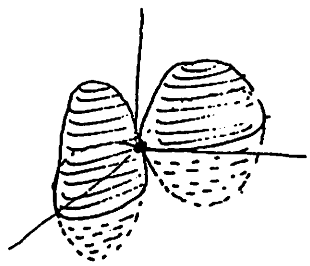
$$\gamma = \{|t| < \rho \mid \operatorname{Re} t = 0\};$$

$$\Gamma = \bigcup_{t \in \gamma} (t, t^2, t) \quad \therefore\quad \Gamma \text{ is one real dimensional in } B \times \mathbb{C}^2.$$

Homework : we draw the rough graph of $\partial D(0)$ determined by

$$\psi(0, z, w) = 0 \quad , \text{or}$$

$$\left(|z-1|^2 + |w-2|^2\right)\left(|z+1|^2 + |w+2|^2\right) = 25$$



<u>Lemma</u> Suppose that $\mathcal{D} = \bigcup_{t \in B} (t ; D(t))$ satisfies conditions $1 \sim 4$. Then $\lambda(t)$ is a function of class C^1 on B.

The proof will be divided into several short steps. Troughout these steps we set

$$B^* = B - \gamma \quad ; \quad B_0^* = B_0 - \gamma \quad ; \quad \mathcal{D}_{B_0} = \mathcal{D} \cap (B_0 \times \mathbb{C}^n)$$

for any open B_0 in B. The proof in the case $n = 1$ is quite same as that of the case $n \geqq 2$. So we may assume $n \geqq 2$.

<u>1st step</u>

(1) The function $g(t, z)$ is of class C^3 for (t, z) in $(\mathcal{D}_{B^*} - B^* \times \{\zeta\}) \cup \partial \mathcal{D}_{B^*}$;

(2) The function $\lambda(t)$ is superharmonic and of class C^3 on B^*.

Indeed, fix $t_0 \in B^*$ and take a disk B_0 of center t_0 such that $B_0 \subset\subset B^*$. Then condition 3 implies that the open set $\mathcal{D}_{B_0}$ in $B_0 \times \mathbb{C}^n$ consists of a finite number of domains $\mathcal{D}_j$ $(j = 1, \cdots, m)$ over $B_0 \times \mathbb{C}^n$ such that $(\mathcal{D}_j \cup \partial \mathcal{D}_j) \cap (\mathcal{D}_i \cup \partial \mathcal{D}_i) = \phi$ for $j \neq i$. One of them, say $\mathcal{D}_1$, has constant section ζ i.e., $B_0 \times \{\zeta\} \subset \mathcal{D}_1$, where ζ is the point in condition 1. By definition, for each $t \in B_0$, $g(t, z)$ is the green's function for $(D_1(t), \zeta)$ in $D_1(t)$ and is defined to be 0 in $D_j'(t)$ $(2 \leq j \leq m)$. Also, $\lambda(t)$ means the Robin constant for $(D_1(t), \zeta)$. So it is clear that $g(t,z)$ is of class C^3 for (t,z) on $\bigcup_{j=2}^{m} (\mathcal{D}_j \cup \partial \mathcal{D}_j)$. The domain $\mathcal{D}_1$ in $B_0 \times \mathbb{C}^n$ with $B_0 \times \{\zeta\}$ satisfies conditions for smooth variations of smooth domains. It follows that $g(t,z)$ is of class C^3 for (t,z) on $(\mathcal{D}_1 \cup \partial \mathcal{D}_1) - B_0 \times \{\zeta\}$, and that $\lambda(t)$ is of class C^3 and superharmonic on B_0. The first step is thus proved.

2nd step.

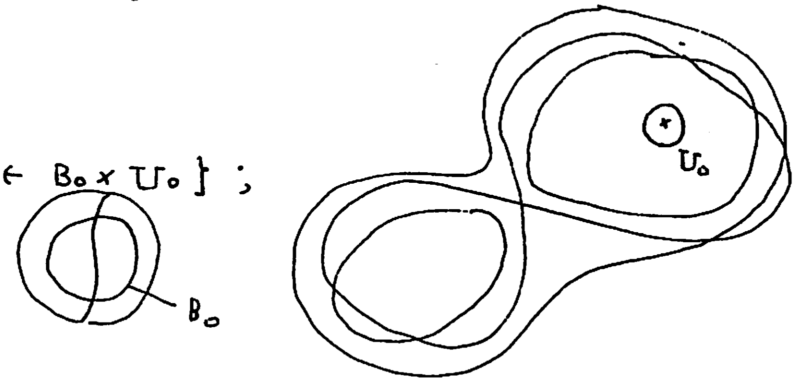
Let B_0 be a region of B such that $B_0 \Subset B$. Let $U_0 : \|z - \zeta\| < r_0$ be a ball with center at the pole ζ such that $B_0 \times U_0 \Subset$ Then there exists a constant $c > 0$ (depending on B_0 and U_0) such that the following inequalities hold:

$$(2.1) \qquad g(t,z) \leq -c\, \psi(t,z)$$

for all $z \in D(t) - U_0 \cup \partial U_0$ and all $t \in B_0$;

$$(2.2) \qquad \|\mathrm{Grad}_{(z)}\, g(t,z)\| \leq c\, \|\mathrm{Grad}_{(z)}\, \psi(t,z)\|$$

for all $z \in \partial D(t)$ and $t \in B_0$*



In fact, if we put

$$m = \mathrm{Inf}\,\{-\psi(t,z) \mid (t,z) \in B_0 \times U_0\};$$

$$C = \frac{1}{m} \cdot \frac{1}{r_0^{2n-2}},$$

then $m > 0$ and $c > 0$.

We shall verify that this c satisfies (2.1) and (2.2).

To this end, let t be an arbitrary point in B_0. Inequality:

$$g(t,z) \leq \frac{1}{\|z - \zeta\|^{2n-2}} \quad \text{for all } t \in B \text{ and } z \in D(t)$$

implies from the maximum principle that

$$(2.3) \qquad 0 < g(t,z) < \frac{1}{r_0^{2n-2}}$$

for all $z \in U(t) - U_0$. Let Ω be any open set in $D(t)$ with smooth boundary $\partial\Omega$ and such that

$$U_0 \subset\subset \Omega \subset\subset D(t).$$

We denote by $g_\Omega(t,z)$ the Green's function for $(\Omega, \mathfrak{z})$. Construct the function

$$v_\Omega(t,z) = c\,\psi(t,z) + g_\Omega(t,z)$$

on $\Omega - U_0 \cup \partial U_0$. Condition 4 implies that the restriction of ψ to $D(t)$ is plurisubharmonic for z in $D(t)$. Consequently, $v_\Omega(t,z)$ is subharmonic for z in Ω, i.e., $\Delta_{(z)} v_\Omega(t,z) \geqq 0$. Since $v_\Omega(t,z) < 0$ on $\partial(\Omega - U_0) = \partial\Omega \cup \partial U_0$, (by the definition of c) it follows that $v_\Omega(t,z) < 0$ on $\Omega - U_0 \cup \partial U_0$.
Because $g_\Omega(t,z) \nearrow g(t,z)$ as $\Omega \nearrow D(t)$, we have $c\,\psi(t,z) + g(t,z) \leqq 0$ on $D(t) - U_0 \cup \partial U_0$. Hence the constant c satisfies (2.1).

To prove (2.2), let $t \in B_0^*$. Then $g(t,z)$ can be extended of class C^3 beyond $\partial D(t)$ in $\tilde{D}(t)$. Since $g(t,z) = \psi(t,z) = 0$ on $\partial D(t)$, it follows from (2.1) that

$$\|\operatorname{Grad}_{(z)} g(t,z)\| = -\frac{1}{2}\frac{\partial g(t,z)}{\partial n_z} \leqq \frac{c}{2}\frac{\partial \psi(t,z)}{\partial n_z} = c\,\|\operatorname{Grad}_{(z)} \psi(t,z)\|$$

for all $z \in \partial D(t)$. Hence our c satisfies (2.2) and the 2nd step is proved.

Before proceeding to the following steps we shall state two preliminary results. Let $B_0 : |t - t_0| < \rho$ be a disk in the complex t-plane and let G be an open set in $\mathbb{C}^n$. Let $\psi_1(t,z)$ and $\psi_2(t,z)$ be real-valued, real-analytic functions with respect to (t,z) in $B_0 \times G$. We put

$$\mathcal{E}_j = \{(t,z) \in B_0 \times G \mid \psi_j(t,z) < 0\} \quad (j=1,2);$$
$$E_j(t) = \{z \in G \mid (t,z) \in \mathcal{E}_j\} \quad (t \in B_0);$$
$$\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2 \quad \text{and} \quad E(t) = E_1(t) \cap E_2(t).$$

We denote by $\partial\mathcal{E}$ and $\partial\mathcal{E}_j$ the boundary of $\mathcal{E}$ or $\mathcal{E}_j$ in $B_0 \times G$. Given $t \in B_0$, we denote by $\partial E(t)$ or $\partial E_j(t)$ the boundary of $E(t)$ or $E_j(t)$ in G. Assume that

(a) for each $t \in B_0$, $\mathrm{Grad}_{z}\, \psi_j(t,z) \neq 0$ for all $z \in \partial E_j(t)$ $(j=1,2)$;

(b) $\partial E_j(t) \cap \partial E(t) \neq \phi$ $(j=1,2)$ for all $t \in B_0$;

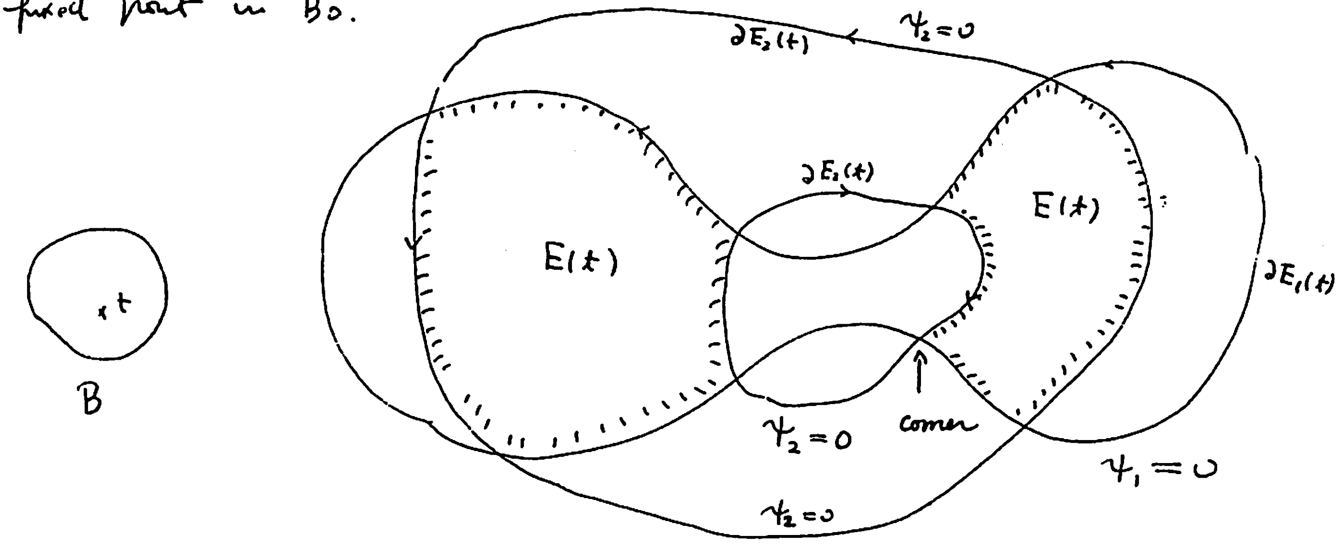
(c) for each $t \in B_0$, $\partial E_1(t)$ and $\partial E_2(t)$ intersect transversally in G;

(d) $E(t) \subset\subset G$ for each $t \in B_0$

For the sake of simplicity we say that such a set $\mathcal{E}$ is an open set with corners in $B_0 \times G$. Also we say that the double $(B_0 \times G, \{\psi_1, \psi_2\})$ defines the open set $\mathcal{E}$. In this case, each $E(t)$ $(t \in B_0)$ is bounded by a finite number of smooth surfaces. Moreover, the variation

$$\mathcal{E} \cup \partial\mathcal{E} : t \longrightarrow E(t) \cup \partial E(t) \quad (t \in B_1)$$

where B_1 is a disk $\subset\subset B_0$, is diffeomorphically equivalent to the trivial one: $t \longrightarrow E(t_0) \cup \partial E(t_0)$ $(t \in B_0)$ where t_0 is a fixed point in B_0.



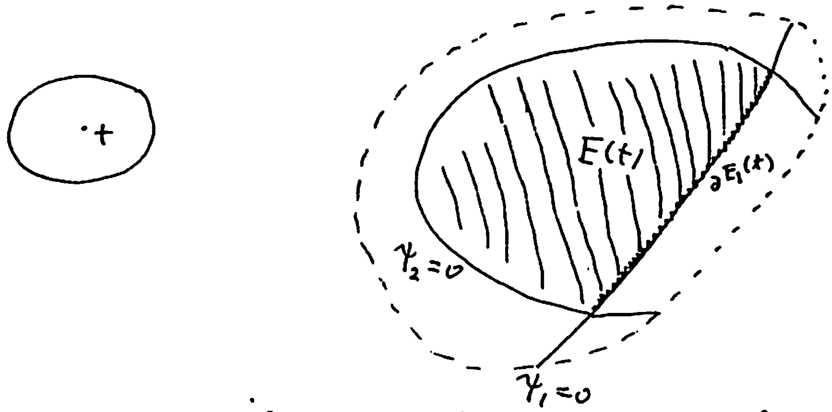
Using this notation, we have

Preliminary 1

Let $\mathcal{E}$ be an open set with corners in $B_0 \times G$. Assume that there exists a point $\zeta \in G$ such that $B_0 \times \zeta \subset \mathcal{E}$. For $t \in B_0$, we denote by $g(t, z)$ the Green's function for $(E(t), \zeta)$. Then $g(t, z)$ is continuous with respect to (t, z) in $\mathcal{E}$ except for the pole $B_0 \times \{\zeta\}$.

Preliminary 2

Let $\mathcal{E}$ be an open set with corners defined by the double $(B_0 \times G, \{\psi_1, \psi_2\})$. Assume that $u(t, z)$ is a continuous function with respect to (t, z) in $\mathcal{E} \cup \partial\mathcal{E}$ such that for any fixed $t \in B_0$, $u(t, z)$ is harmonic for z in $E(t)$ and vanishes on $\partial E(t) \cap \partial E_1(t)$. Then $\frac{\partial u(t,z)}{\partial z_\alpha}$ $(1 \leq \alpha \leq n)$ is continuous with respect to $(t, z) \in \mathcal{E} \cup [\partial\mathcal{E} \cap (\partial\mathcal{E}_1 - \partial\mathcal{E}_2)]$



These can be proved without difficult by following the concrete construction of Green's function by means of the theory of Fredholm's integral equations.

Let us return to the proof of Lemma 1.

3^{rd} step.

(1) The function $g(t, z)$ is continuous with respect to

(t,z) on $(\mathcal{D} - B\times\{\zeta\}) \cup \partial\mathcal{D}$, and vanishes on $\partial\mathcal{D}$;

(2) The function λ is continuous for $t \in B$.

In fact, by the 1st step, it remains to prove the 3rd step for t on γ. We assume $t_0 \in \gamma$. By condition 3 (b), the boundary surface $\partial D(t_0)$ have the singular points:

$$\Gamma(t_0) = \{ z^{(1)}(t_0), \dots, z^{(\ell)}(t_0) \}.$$

First, let $(t_0, z_0) \in \partial\mathcal{D}$. Take a disk B_0 of center t_0 and a ball U_0 with center at the pole ζ such that $B_0 \times U_0 \subset\subset \mathcal{D}$. By the 2nd step we can find a constant $c > 0$ satisfying (1). Therefore, if $(t,z) \in \mathcal{D}$ tends to (t_0, z_0), then

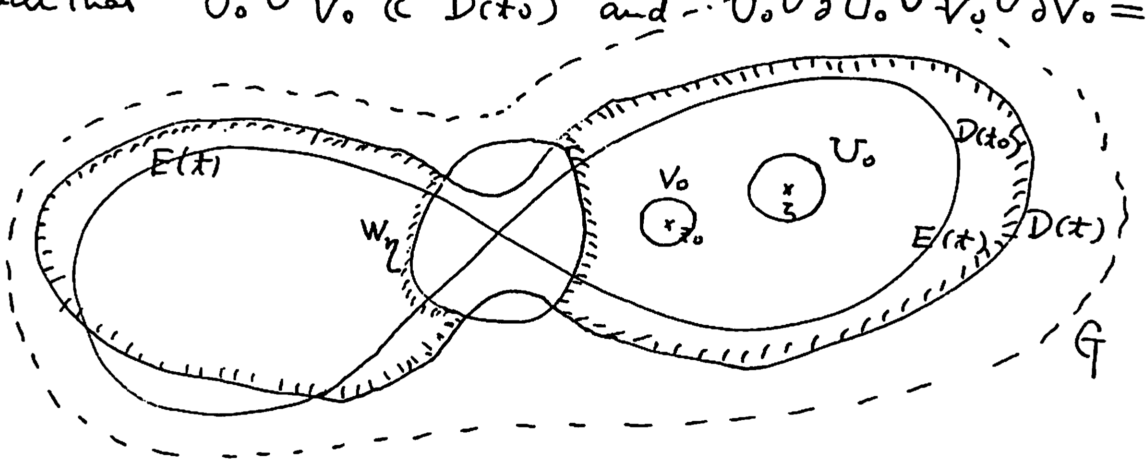
$$0 \leq \overline{\lim_{(t,z)\to(t_0,z_0)}} g(t,z) \leq -c \overline{\lim_{(t,z)\to(t_0,z_0)}} \psi(t,z) = -c\,\psi(t_0,z_0) = 0.$$

This means that $g(t_0, z_0) = 0$ and $g(t,z)$ is continuous at (t_0, z_0).

Next, let $(t_0, z_0) \in \mathcal{D} - B\times\{\zeta\}$. Fix balls

$$U_0 : \|z - \zeta\| < r_0 \quad \text{and} \quad V_0 : \|z - z_0\| < \delta_0$$

such that $U_0 \cup V_0 \subset\subset D(t_0)$ and $(U_0 \cup \partial U_0) \cap (V_0 \cup \partial V_0) = \phi$.



We also take an open set G of $\tilde{D}(t_0)$ such that $D(t_0) \subset\subset G \subset\subset \tilde{D}(t_0)$ and such that the boundary ∂G is smooth in $\tilde{D}(t_0)$.

We use the following notation: Given $\rho > 0$ and $\eta > 0$, we put

$$B_\rho = \{ t \in B \mid |t - t_0| < \rho \} \text{ and } W_\eta = \bigcup_{k=1}^{q} W_\eta^{(k)}$$

where $W_\eta^{(k)} = \{ z \in \widetilde{D(t_0)} \mid \| z - z^{(k)}(t_0) \| < \eta \} \quad (1 \le k \le q)$.

By condition 3, we can choose small numbers $\eta_1 > 0$ and $\rho_1 > 0$ such that

(i) $W_{\eta_1} \subset\subset G$ and $(W_{\eta_1} \cup \partial W_{\eta_1}) \cap (\bar{U}_0 \cup \bar{V}_0) = \phi$;

(ii) $U_0 \cup V_0 \subset\subset D(t) \subset\subset G$ for each $t \in B_{\rho_1}$;

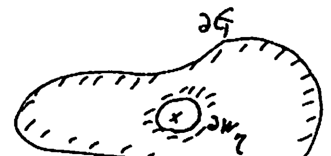
(iii) given $\eta > 0$ such that $\eta < \eta_1$, we can find a number $\rho(\eta)$ such that $0 < \rho(\eta) < \rho_1$ and such that the set $\mathcal{E} = \mathcal{D}_{B_{\rho(\eta)}} - (B_{\rho(\eta)} \times \bar{W}_\eta)$ is an open set with corners in $B_{\rho(\eta)} \times G$.

To achieve (iii), it is sufficient to take as a double defining $\mathcal{E}$, $(B_{\rho(\eta)} \times G, \{\psi_1, \psi_2\})$ where $\psi_1 = \psi$ and $\psi_2 = \prod_{k=1}^{q} \psi_{2k}$ with

$$\psi_{2,k}(t, z) = \eta^2 - \| z - z^{(k)}(t_0) \|^2.$$

Now fix η $(0 < \eta < \eta_1)$. For $t \in B_{\rho_1}$, we denote by $g_\eta(t, z)$ the Green's function for $(D(t) - \bar{W}_\eta, z)$. We construct the harmonic function $u_\eta(z)$ defined on $G - \bar{W}_\eta$ whose boundary values are

$$u_\eta(z) = \begin{cases} \dfrac{1}{r_0^{2n-2}} & \text{on } \partial W_\eta \\ 0 & \text{on } \partial G \end{cases}$$



It is clear that, as $\eta \searrow 0$, $u_\eta(z) \searrow 0$ uniformly on any compact set in $G - \{ z^{(k)}(t_0) \}_{k=1}^{q}$ and, in particular, on $\bar{V}_0$. On the other hand, from the maximum principle and (2.3), for each $t \in B_{\rho_1}$,

$$0 < g(t, z) - g_\eta(t, z) < u_\eta(z)$$

for all $z \in D(t) - \overline{W_\eta}$. It follows from (i) and (ii) that

$$\left| g(t,z) - g(t_0,z_0) \right| \leqq u_\eta(z) + u_\eta(z_0) + \left| g_\eta(t,z) - g_\eta(t_0,z) \right|$$

for $(t,z) \in B_{\rho_1} \times V_0$. Given $\varepsilon > 0$, we take, first, a number η_0 such that $0 < \eta_0 < \eta_1$ and such that $0 < u_{\eta_0}(z) < \varepsilon/3$ for $z \in \overline{V_0}$. Preliminary 1 together with (iii) imply that $g_{\eta_0}(t,z)$ is continuous with respect to (t,z) in $\mathcal{E} = \mathcal{D}_{B_{\rho(\eta_0)}} - (B_{\rho(\eta_0)} \times \overline{W_{\eta_0}})$. Because $\mathcal{E} \ni (t_0, z_0)$, we can find a neighborhood $\mathcal{V}$ of (t_0, z_0) in $\mathcal{E}$ such that $|g_{\eta_0}(t,z) - g_{\eta_0}(t_0,z_0)| < \varepsilon/3$ for $(t,z) \in \mathcal{V}$. It follows that

$$|g(t,z) - g(t_0,z_0)| < \varepsilon \text{ for } (t,z) \in \mathcal{V}.$$

Consequently, (1) of the 3rd step is proved. Since the Robin constant is determined by the values of the Green function on the sphere about the pole:

$$\lambda(t) = -\frac{1}{r_0^{2n-2}} + \frac{1}{r_0^{2n-1}\omega_{2n}} \int_{\partial U_0} g(t,z)\, ds_z$$

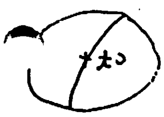
it follows that $\lambda(t)$ is continuous for t in B. (2) is proved.

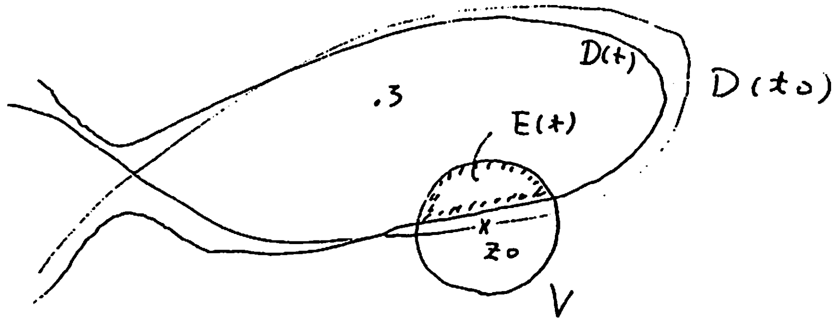
4th step

The derivative $\frac{\partial g}{\partial z_\alpha}(t,z)$ $(1 \leqq \alpha \leqq n)$ is continuous with respect to (t,z) in $(\mathcal{D} - B \times \{\zeta\}) \cup (\partial\mathcal{D} - \Gamma)$. Precisely, $\frac{\partial g}{\partial z_\alpha}(t,z)$, which is certainly defined in $\mathcal{D} - B \times \{\zeta\}$, can be continuously extended to $\partial\mathcal{D} - \Gamma$.

Indeed, first suppose $(t_0, z_0) \in \mathcal{D} - B \times \{\zeta\}$. We take $B_0 : |t - t_0| < \rho$ and $V : \|z - z_0\| < r$ such that $B_0 \times V \subset\subset \mathcal{D} - B \times \{\zeta\}$. Then Poisson's integral formula combined with (1) of the 3rd step imply that $\frac{\partial g}{\partial z_\alpha}(t,z)$ is continuous for (t,z) in

$B_0 \times V$. Next, assume that $(t_0, z_0) \in \partial\mathcal{D} - \Gamma$. Then we find $B_0 : |t - t_0| < \rho$ and $V : \|z - z_0\| < r$ such that $B_0 \times V \Subset \tilde{\mathcal{D}} - B \times \{3\}$ and $\mathrm{Grad}_{(z)}\, \psi(t,z) \neq 0$ for all $(t,z) \in \partial\mathcal{D} \cap (B_0 \times V)$. Therefore $\mathcal{E} = \mathcal{D} \cap (B_0 \times V)$ is an open set with corners. It follows from Preliminary 2 and (1) of the 3^{rd} step that $\frac{\partial g}{\partial z_\alpha}(t,z)$ is continuous on $(\mathcal{D} \cup \partial\mathcal{D}) \cap (B_0 \times V)$. Consequently, the 4^{th} step is proved.





From (1) of the 1^{st} step, the derivative $\frac{\partial g}{\partial t}(t,z)$ exists for any $(t,z) \in \mathcal{D}_{B^*} \cup \partial\mathcal{D}_{B^*}$. Thus we consider its restriction to the boundary $\partial\mathcal{D}_{B^*}$ and put

$$u(t,z) = \frac{\partial g(t,z)}{\partial t} \quad \text{for } (t,z) \in \partial\mathcal{D}_{B^*}.$$

With this terminology we shall state

<u>5^{th} step</u>

(1) The function $u(t,z)$ defined on $\partial\mathcal{D}_{B^*}$ can be uniquely extended to a continuous function $\hat{u}(t,z)$ on $\partial\mathcal{D} - \Gamma$;

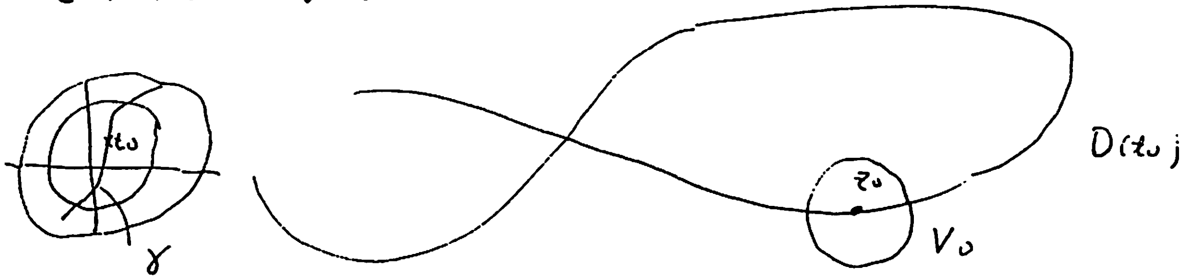
(2) Let B_0 be a disk $\Subset B$. Then there exists a constant $K > 0$ (depending on B_0) such that

$$(5.1) \qquad |\hat{u}(t,z)| \leq K \quad \text{and} \quad \left|\hat{u}(t,z)\,\frac{\partial g}{\partial n_z}(t,z)\right| \leq K$$

for all $(t,z) \in \partial\mathcal{D}_{B_0} - \Gamma$, where n_z denotes the unit outer normal vector to the $(2n-1)$-dimensional surface $\partial D(t)$ at the point z.

In fact, by (1) of the 1st step, $u(t,z)$ is continuous for $(t,z) \in \partial \mathcal{D}_{B^*}$. By condition (3), $\partial\mathcal{D}_{B^*}$ is dense in $\partial\mathcal{D} - \Gamma$. Thus to prove (1) of the 5th step, it suffices to verify the following fact: Let $(t_0, z_0) \in \partial\mathcal{D} - \Gamma$ with $t_0 \in \gamma$, and let $(t,z) \in \partial\mathcal{D}_{B^*}$ tend to (t_0, z_0), then the limit of $u(t,z)$ exists.

Indeed, because $\partial D(t_0)$ is non-singular at z_0, we can find a neighborhood $B_0 \times V_0$ of (t_0, z_0) in $\tilde{\mathcal{D}}$ where $B_0 : |t - t_0| < \rho$ and $V_0 : \|z - z_0\| < r$ such that $\operatorname{Grad}_{(z)} \psi(t,z) \neq 0$ for all $(t,z) \in B_0 \times V_0$.



On the other hand, $(\tilde{\mathcal{D}}_{B^*}, \psi)$ and $(\hat{\mathcal{D}}_{B^*}, -g)$ define $\mathcal{D}_{B^*}$. It follows that

$$\frac{-\frac{\partial g}{\partial t}}{\frac{\partial \psi}{\partial t}} = \frac{-\frac{\partial g}{\partial z_1}}{\frac{\partial \psi}{\partial z_1}} = \cdots = \frac{-\frac{\partial g}{\partial z_n}}{\frac{\partial \psi}{\partial z_n}} = \frac{-\|\operatorname{Grad}_{(z)} g(t,z)\|}{\|\operatorname{Grad}_{(z)} \psi(t,z)\|}$$

for $\forall z \in \partial D(t)$ and $t \in B^*$. In particular, we get

$$(5.2) \qquad u(t,z) = \frac{\partial g}{\partial t}(t,z) = \frac{-\left(\frac{\partial \psi}{\partial t}\right) \cdot \|\operatorname{Grad}_{(z)} g(t,z)\|}{\|\operatorname{Grad}_{(z)} \psi(t,z)\|}$$

for all $(t,z) \in \partial\mathcal{D}_{B^*}$.

By 4th step, $\frac{\partial g}{\partial z_\alpha}(t,z)$ $(1 \leq \alpha \leq n)$ is a continuous function for (t,z) on $(\mathcal{D} - B \times \{\zeta\}) \cup (\partial\mathcal{D} - \Gamma)$. By condition 2, $\frac{\partial \psi}{\partial t}(t,z) \Big/ \|\operatorname{Grad}_{(z)} \psi(t,z)\|$ is continuous for (t,z) in $B_0 \times V_0$.

It follows that

$$\lim_{(t,z)\to(t_0,z_0)} u(t,z) = -\left(\frac{\partial\psi}{\partial t}(t_0,z_0)\right)\frac{\|\operatorname{Grad}_{(z)} g(t_0,z_0)\|}{\|\operatorname{Grad}_{(z)}\psi(t_0,z_0)\|}$$

where $(t,z)\in(\partial\mathcal{D}_{B^*})\cap(B_0\times V_0)$. This proves (1) of the 5^{th} step.

For the proof of (2), let B_0 be a disk such that $B_0 \subset\subset B$. By definition of $u'(t,z)$ for $t\in\gamma$ and by the 4^{th} step, it suffices to prove the existence of a constant $K>0$ such that

$$(5.1')\quad \left|\frac{\partial g}{\partial t}(t,z)\right| \leq K \quad\text{and}\quad \left|\frac{\partial g}{\partial t}(t,z)\cdot\frac{\partial g}{\partial n_z}(t,z)\right|\leq K$$

for all $(t,z)\in\partial\mathcal{D}_{B_0^*}$. Since $B_0\subset\subset B$, we can find a ball $U_0: \|z-3\|<r_0$ such that $B_0\times U_0\subset\subset\mathcal{D}$. By the 2^{nd} step, we can find a constant $c>0$ (depending on B_0 and U_0) which satisfies

$$\|\operatorname{Grad}_{(z)} g(t,z)\| \leq c\,\|\operatorname{Grad}_{(z)}\psi(t,z)\|$$

for all $(t,z)\in\partial\mathcal{D}_{B_0^*}$. It follows from (5.2) that

$$\left|\frac{\partial g}{\partial t}\right|\leq c\left|\frac{\partial\psi}{\partial t}\right| \quad\text{and}\quad \left|\frac{\partial g}{\partial t}\cdot\frac{\partial g}{\partial n_z}\right|\leq c^2\left|\frac{\partial\psi}{\partial t}\right|\|\operatorname{Grad}_{(z)}\psi\|$$

for all $(t,z)\in\partial\mathcal{D}_{B_0^*}$. Since $\psi(t,z)$ is real analytic for (t,z) in $\tilde{\mathcal{D}}$ and since $\mathcal{D}_{B_0}\subset\subset\tilde{\mathcal{D}}$, we can find a number $M>0$ such that

$$\left|\frac{\partial\psi}{\partial t}\right|,\ \left|\frac{\partial\psi}{\partial z_\alpha}\right|\leq M \quad (1\leq\alpha\leq n) \quad\text{in } \mathcal{D}_{B_0}\cup\partial\mathcal{D}_{B_0}.$$

Consequently, if we put $K=\max\{cM, \sqrt{n}\,c^2M^2\}$, then K satisfies inequality (5.1'). Thus (2) is proved.

It must be noted that (1) implies neither the existence nor the continuity of $\frac{\partial g}{\partial t}(t,z)$ as a function with respect to (t,z) in $\mathcal{D}\cup(\partial\mathcal{D}-\Gamma)$ at $(t_0,z_0)\in\partial\mathcal{D}-\Gamma$ with $t_0\in\gamma$.

6th step

The function $\lambda(t)$ is of class C^1 on B.

In fact, by (1) of the 1st step, $\lambda(t)$ is of class C^2 on B^*. By (2) of the 3rd step, $\lambda(t)$ is continuous on all of B. Since B^* is dense in B, it suffices to prove the following

Property (A) Let t_0 be any fixed point of γ. Then, given $\varepsilon > 0$ there exists a disk $B_0 \Subset B$ of center t_0 such that

$$\left| \frac{\partial \lambda}{\partial t}(t) - \frac{\partial \lambda}{\partial t}(t') \right| < \varepsilon \quad \text{for all } t, t' \in B_0^* = B_0 - \gamma.$$

Now, given $a \in \mathbb{C}^n$ and $\eta > 0$, we consider the ball $V(a,\eta)$: $\|z - a\| < \eta$ in $\mathbb{C}^n$. Since $\psi(t,z)$ is real analytic with respect to (t,z) in $\tilde{\mathcal{D}}$ and since $\mathcal{D}_{B_0} \Subset \tilde{\mathcal{D}}$ for any $B_0 \Subset B$, the following fact is clear:

Let B_0 be a disk $\Subset B$. Then, given $\delta > 0$, there exists a number $\eta > 0$ such that

$$(6.1) \qquad \int_{(\partial D(t) - P(t)) \cap V(a,\eta)} dS_z < \delta$$

for all $(t,a) \in B_0 \times \mathbb{C}^n$.

To show property (A), let $t_0 \in \gamma$. The surface $\partial D(t_0)$ then has the singular points $P(t_0) = \{ z^{(k)}(t_0) \}_{k=1,\dots,q}$. We choose a disk $B_0 \Subset B$ center at t_0 and a ball $U_0 \Subset D(t_0)$ with center at the pole ζ such that $B_0 \times U_0 \Subset \mathcal{D}$. From (2) of the 5th step, we can find a constant $K > 0$ (depending on B_0) which satisfies (5.1) for $(t,z) \in \partial \mathcal{D}_{B_0} - P$.

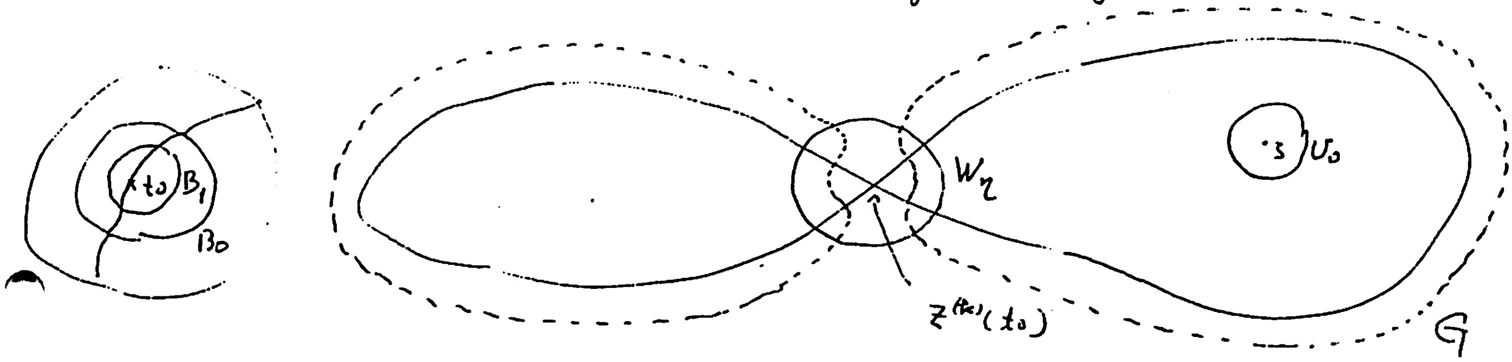
Let $\varepsilon > 0$ be given. We put

$$\delta = 2(n-1)\omega_{2n} / (3K) > 0.$$

With this $\delta > 0$, we can find a number $\eta > 0$ for which (6.1) holds for all $(t, z) \in B_0 \times \mathbb{C}^n$. If we put

$$W_\eta = \bigcup_{k=1}^{q} W_\eta^{(k)} \quad \text{where} \quad W_\eta^{(k)} = V(z^{(k)}(t_0), \eta) \quad (1 \leq k \leq q)$$

then $\partial D(t_0) - \overline{W_\eta}$ consists of only non-singular points.



Take an open set G with $D(t_0) - \overline{W_\eta} \subset\subset G \subset\subset \hat{D}(t_0)$. As already noted in the 3rd step, we can find a disk $B_1 : |t - t_0| < \rho(\eta)$ in B_0 such that $\mathcal{E} = \mathcal{D}_{B_1} - B_1 \times \overline{W_\eta}$ is an open set with corners in $B_1 \times G$. Therefore, the surfaces, together with their unit normal vectors, approach those of $\partial D(t_0) - \overline{W_\eta}$ in continuous way as $t \in B_1$ tend to t_0. Also, by (1) of the 5th step, $\hat{u}(t, z)$ is uniformly continuous on relatively compact set $\bigcup_{t \in B_1} (t, \partial D(t) - \overline{W_\eta})$ of $\partial\mathcal{D} - \Gamma$. It follows from the 4th step that

$$\lim_{\substack{t \to t_0 \\ t \in B_1}} \int_{\partial D(t) - \overline{W_\eta}} \hat{u}(t, z) \frac{\partial g}{\partial n}(t, z)\, dS_z = \int_{\partial D(t_0) - \overline{W_\eta}} \hat{u}(t_0, z) \frac{\partial g}{\partial n_z}(t_0, z)\, dS_z .$$

Hence there exists a small disk $B_2 : |t - t_0| < \rho_2$ in B_1 such that

$$(6.2) \quad \left| \int_{\partial D(t) - \overline{W_\eta}} \hat{u}(t, z) \frac{\partial g}{\partial n_z}(t, z)\, dS_z - \int_{\partial D(t') - \overline{W_\eta}} \hat{u}(t', z) \frac{\partial g}{\partial n_z}(t', z)\, dS_z \right| < \frac{2(n-1)\omega_{2n}}{3} \varepsilon$$

for $t, t' \in B_2$

Since $B_2 \subset B_1 \subset B_0 \Subset B$, the inequalities (5.1) for $(t,z) \in \partial\mathcal{D}_{B_2}$, (6.1) for $(t,a) \in B_2 \times \mathbb{C}^n$ and (6.2) for $t, t' \in B_2$ remain valid. Therefore, in view of Hadamard's variation formula, we see that for every $t, t' \in B_2^*$,

$$\left|\frac{\partial \lambda}{\partial t}(t) - \frac{\partial \lambda}{\partial t}(t')\right|$$

$$= \left| \frac{-1}{2(n-1)\omega_{2n}} \left\{ \int_{\partial D(t)} \left(\frac{\partial g}{\partial t} \frac{\partial g}{\partial n_z}\right)(t,z)\, ds_z - \int_{\partial D(t')} \left(\frac{\partial g}{\partial t}\frac{\partial g}{\partial n_z}\right)(t',z)\, ds_z \right\} \right|$$

$$\leq \frac{1}{2(n-1)\omega_{2n}} \left\{ \left| \int_{\partial D(t) - \overline{W_\eta}} \left(u \cdot \frac{\partial g}{\partial n_z}\right)(t,z)\, dS_z - \int_{\partial D(t') - \overline{W_\eta}} \left(u \frac{\partial g}{\partial n_z}\right)(t',z)\, dS_z \right| \right.$$
$$\left. + \sum_{k=1}^{q} \left(\int_{\partial D(t) \cap \overline{W_\eta^k}} K\, dS_z + \int_{\partial D(t') \cap W_\eta^{(k)}} K\, ds_z \right) \right\}$$

$$\leq \frac{1}{2(n-1)\omega_{2n}} \left(\frac{2(n-1)\omega_{2n}\varepsilon}{3} + 2Kq\delta \right)$$
$$= \varepsilon .$$

We thus Property (A). Lemma 1 is completely proved.

<u>Remark 1</u> By means of the proof we conclude that at each $t_0 \in \gamma$, the improper integral

$$\int_{\partial D(t_0) - P} \hat{u}(t_0, z) \frac{\partial g}{\partial n_z}(t_0, z)$$

exists and is equal to $2(n-1)\omega_{2n} \frac{\partial \lambda}{\partial t}(t_0)$.

As already noted in the last semester, we have the following

<u>Corallary 1</u> Under the same conditions as in Lemma 1, the function $\lambda(t)$ is superharmonic and of class C^1 in B.

Variations of Harmonic modulus

Let D be a finite Riemann surface with at least two boundary components. We devide the boundary components into two disjoint group α, β i.e.,

$$\partial D = \alpha \cup \beta$$

Consider the harmonic function $u(z)$ whose boundary values are

$$u = \begin{cases} 1 & \text{on } \alpha \\ 0 & \text{on } \beta \end{cases}$$

$u(z)$ is called the harmonic measure for (D, α).
We consider its Dirichlet integral:

$$I = \iint_D \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] dx\,dy$$

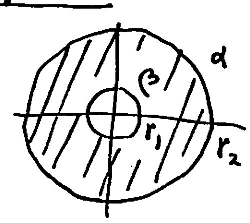
where $z = x + iy$ local parameter

Define its reciprocal:

$$\mu = \frac{1}{I}$$

μ is called the harmonic module for (D, α).

Example 1



Let $D = (r_1 < |z| < r_2)$ and $\alpha = (|z| = r_2)$.
Then $u(z) = \log \frac{|z|}{r_1} \Big/ \log \frac{r_2}{r_1}$;

$$I = 2\pi \left(\log \frac{r_2}{r_1} \right)^{-1} ;$$

and $\mu = \frac{1}{2\pi} \log \frac{r_2}{r_1}$

<u>Example 2</u> Let $B = (|t| < \rho)$ and $D(t) = \{ r_1(t) < |z| < r_2(t) \}$ for $t \in B$. Assume that

$$\mathcal{D} = \bigcup_{t \in B} (t, D(t)) \text{ is a pseudoconvex domain in } B \times \mathbb{C}.$$

By Hartogs' Theorem, it is equivalent to the fact that $\log r_2(t)$ and $-\log r_1(t)$ are superharmonic for $t \in B$.

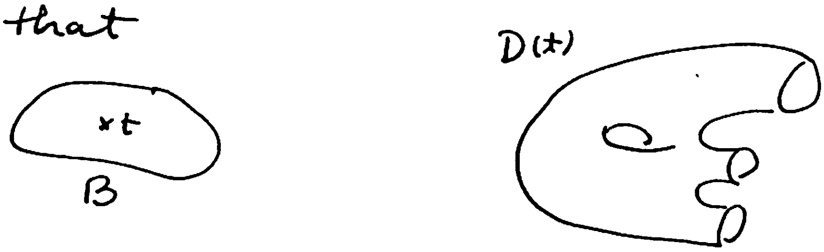
By example 1, we have

$$\mu(t) = \text{the harmonic module for } (D(t), |z| = r_2(t))$$
$$= \frac{1}{2\pi} \log \frac{r_2(t)}{r_1(t)}$$

It follows that $\mu(t)$ is superharmonic for $t \in B$.

Today's main purpose is to generalize this result to general pseudoconvex domain $\mathcal{D}$.

Let B be a region in the complex t-plane. To each $t \in B$ we let correspond a Riemann surface $D(t)$ such that



(1) $D(t)$ spread over the complex z-plane without branch points;

(2) $\partial D(t)$ is smooth and consists of at least

two boundary components ;

(3) The variation $\mathcal{D} : t \to D(t)$ $(t \in B)$ is smooth.

From (1) and (3), $\mathcal{D} : t \to D(t)$ $(t \in B)$ is diffeomorphically equivalent to the trivial one : $t \to D(t_0)$ $(t \in B)$ where t_0 is a fixed point in B. It follows by (2) that we can devide the boundary component of $D(t)$ into two groups $\alpha(t)$, $\beta(t)$:

$$\partial D(t) = \alpha(t) \cup \beta(t)$$

such that $\alpha(t)$ and $\beta(t)$ move continuously with $t \in B$. We put

$$\alpha = \bigcup_{t \in B} (t, \alpha(t)) \, , \quad \beta = \bigcup_{t \in B} (t, \beta(t))$$

$$\therefore \quad \partial \mathcal{D} = \alpha \cup \beta$$

For any fixed $t \in B$, we construct

the harmonic measure : $u(t, z)$ for $(D(t), \alpha(t))$,

the Dirichlet integral : $I(t) = \iint_{D(t)} \left[\left(\frac{\partial u(t,z)}{\partial x}\right)^2 + \left(\frac{\partial u(t,z)}{\partial y}\right)^2 \right] dx\,dy$,

the harmonic module for $(D(t), \alpha(t))$: $\mu(t) = \dfrac{1}{I(t)}$.

Since $\mathcal{D} : t \to D(t)$ $(t \in B)$ is smooth, we easily see that $u(t, z)$ can be extended to be a function of class C^∞ in a neighborhood $\widehat{\mathcal{D}}$ of $\mathcal{D} \cup \partial \mathcal{D}$ over $B \times \mathbb{C}$.

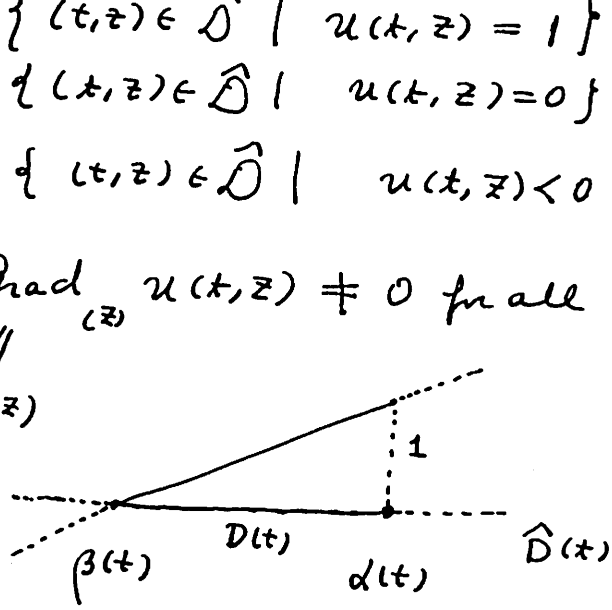
Moreover, we get by the maximum principle

<u>Proposition 1</u>

$$\mathcal{D} = \{ (t, z) \in \widehat{\mathcal{D}} \mid 0 < u(t, z) < 1 \}$$

$$\alpha = \{(t,z) \in \hat{D} \mid u(t,z) = 1\}$$
$$\beta = \{(t,z) \in \hat{D} \mid u(t,z) = 0\}$$
$$\hat{D} - D = \{(t,z) \in \hat{D} \mid u(t,z) < 0 \text{ or } u(t,z) > 1\}$$

and $\underbrace{\mathrm{Grad}_{(z)}}_{\frac{\partial u}{\partial z}(t,z)} u(t,z) \neq 0$ for all $(t,z) \in \alpha \cup \beta$.



It follows that

$$(1)\quad \begin{cases} \dfrac{\partial u}{\partial n_z}(t,z) = \underbrace{2\|\mathrm{Grad}_{(z)}\, u(t,z)\|}_{2\left|\frac{\partial u}{\partial z}\right|} & \text{for } z \in \alpha(t) \\[2ex] \dfrac{\partial u}{\partial n_z}(t,z) = -2\|\mathrm{Grad}_{(z)}\, u(t,z)\| & \text{for } z \in \beta(t). \end{cases}$$

Under these circumstances we have the following fundamental formulas

Lemma 1

$$(2)\quad \frac{\partial I(t)}{\partial t} = 4 \iint_{D(t)} \frac{\partial^2 u(t,z)}{\partial t\, \partial \bar{z}} \, \frac{\partial u(t,z)}{\partial z} \, dx\,dy \;;$$

$$(3)\quad \frac{\partial^2 I(t)}{\partial \bar{t}\, \partial t} = 2 \int_{\partial D(t)} k_2(t,z) \left|\frac{\partial u(t,z)}{\partial z}\right|^2 ds_z + 8 \iint_{D(t)} \left|\frac{\partial^2 u(t,z)}{\partial \bar{t}\, \partial z}\right|^2 dx\,dy.$$

Here $k_2(t,z) = L\varphi(t,z) \Big/ \left|\frac{\partial \varphi}{\partial z}\right|^3$

$$= \frac{\frac{\partial^2\varphi}{\partial t\partial\bar{t}}\left|\frac{\partial\varphi}{\partial z}\right|^2 - 2\,\mathrm{Re}\left\{\frac{\partial^2\varphi}{\partial\bar{t}\partial z}\frac{\partial\varphi}{\partial\bar{z}}\frac{\partial\varphi}{\partial t}\right\} + \frac{\partial^2\varphi}{\partial z\partial\bar{z}}\left|\frac{\partial\varphi}{\partial t}\right|^2}{\left|\frac{\partial\varphi}{\partial z}\right|^3}$$

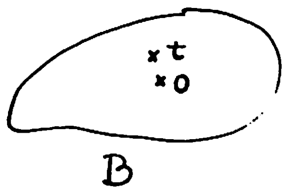
where φ is a defining function of the domain $\mathcal{D}$ in $\tilde{\mathcal{D}}$

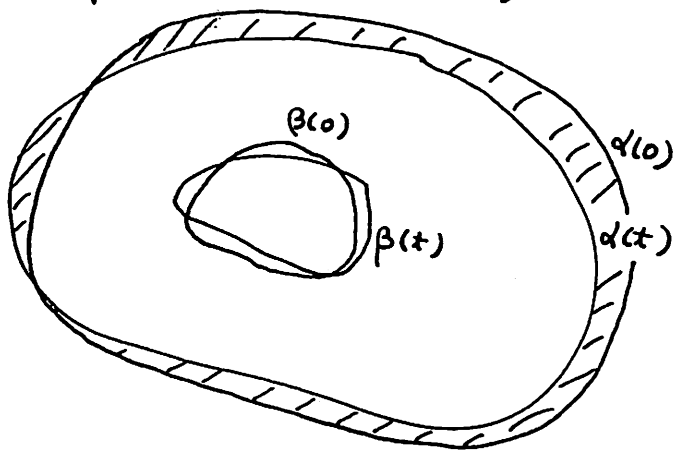
i.e., $\mathcal{D} = \{(t,z)\in\tilde{\mathcal{D}} \mid \varphi(t,z)<0\}$

$\mathrm{Grad}_{(z)}\,\varphi(t,z) \neq 0$ for $\forall z \in \partial D(t)$.

We noted that the quantity (which seems a kind of mean curvature) $k_2(t,z)$ does not depend on the choice of the defining function φ of D.

<u>Proof.</u> It suffices to prove (2) and (3) at $t=0$ in B.





over z-plane.

As usual, we have the following expression by Stokes' formula:

$$I(t) = \iint_{D(t)} \left\{\left(\frac{\partial u(t,z)}{\partial x}\right)^2 + \left(\frac{\partial u(t,z)}{\partial y}\right)^2\right\} dx\,dy$$

$$= \int_{\partial D(t) = \alpha(t)\cup\beta(t)} u(t,z)\,\frac{\partial u(t,z)}{\partial n_z}\,ds_z$$

$$= \int_{\alpha(t)} \frac{\partial u(t,z)}{\partial n_z} ds_z$$

We denote by $[\alpha(0), \alpha(t)]$ the domain bounded by the contours $\alpha(0)$ and $\alpha(t)$. By Stokes' formula in $[\alpha(0), \alpha(t)]$ we have

$$I(t) = \int_{\alpha(0)} \frac{\partial u(t,z)}{\partial n_z} ds_z + \iint_{[\alpha(0),\alpha(t)]} \Delta_{(z)} u(t,z)\,dx\,dy$$

$$= \int_{\underset{\underset{\partial D(0)}{\parallel}}{\alpha(0)\cup\beta(0)}} u(0,z) \frac{\partial u(t,z)}{\partial n_z} ds_z + \iint_{[\alpha(0),\alpha(t)]} \Delta_{(z)} u(t,z)\,dx\,dy$$

$$\left(\because \begin{array}{l} u(0,z) = 1 \text{ on } \alpha(0) \\ u(0,z) = 0 \text{ on } \beta(0) \end{array} \right)$$

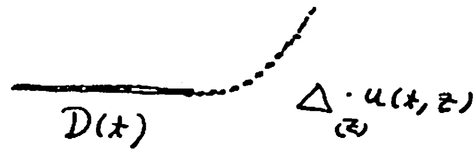
We apply the Stokes' formula to the first integral in $D(0)$. Since $u(0,z)$ is harmonic in $D(0)$ and since $u(t,z)$ is harmonic in $D(t) - D(0)$, it follows that

$$I(t) = \int_{\partial D(0)} u(t,z) \frac{\partial u(0,z)}{\partial n_z} ds_z + \iint_{D(0)-D(t)} u(0,z)\,\Delta_z u(t,z)\,dx\,dy$$

$$+ \iint_{[\alpha(0),\alpha(t)]} \Delta_{(z)} u(t,z)\,dx\,dy.$$

Since $u(t,z)$ is extended of class C^4 beyond $\partial D(t)$ and since $\Delta_{(z)} u(t,z) \equiv 0$ on $D(t)$, it follows easily that

$$\left|\Delta_{(z)} u(t,z)\right| \leqq O(|t|^2)$$



for all z near $\partial D(0)$.

Since the variation $\mathcal{D}: t \to D(t)$ $(t \in B)$ is smooth, we have

$$\left|[d(t), d(0)]\right| = \iint_{[d(t), d(0)]} dx\,dy \leqq O(|t|) \ ;$$

$$\left|D(0) - D(t)\right| = \iint_{D(0)-D(t)} dx\,dy \leqq O(|t|) \ .$$

Therefore we conclude that

$$I(t) = \int_{\partial D(0)} u(t,z) \frac{\partial u(0,z)}{\partial n_z} ds_z + O(|t|^3) \qquad \text{for } |t| \ll 1$$

It follows that

$$\boxed{\begin{aligned} &(4) \quad \left[\frac{\partial I(t)}{\partial t}\right]_{t=0} = \int_{\partial D(0)} \left[\frac{\partial u(t,z)}{\partial t}\right]_{t=0} \frac{\partial u(0,z)}{\partial n_z} ds_z \ ; \\ &(5) \quad \left[\frac{\partial^2 I(t)}{\partial t \partial \bar{t}}\right]_{t=0} = \int_{\partial D(0)} \left[\frac{\partial^2 u(t,z)}{\partial t \partial \bar{t}}\right]_{t=0} \frac{\partial u(0,z)}{\partial n_z} ds_z \ . \end{aligned}}$$

Since $\quad (\partial/\partial n_z)\, ds_z = \frac{1}{i}\left[(\partial/\partial z)\, dz - (\partial/\partial \bar{z})\, d\bar{z}\right]$

on $\partial D(0)$, and since $\quad \dfrac{\partial u(0,z)}{\partial z} dz + \dfrac{\partial u(0,z)}{\partial \bar{z}} d\bar{z} = 0$

along $\partial D(0)$ $\quad (\because\ u(0,z) \equiv \text{const}, 0 \text{ or } 1 \text{ on } \partial D(0))$,

we thus have

$$\left[\frac{\partial I(t)}{\partial t}\right]_{t=0} = \frac{2}{i}\int_{\partial D(0)} \frac{\partial u}{\partial t}(0,z)\,\frac{\partial u}{\partial z}(0,z)\,dz$$

By Stokes' formula, it becomes

$$= \frac{2}{i}\iint_{D(0)} d_{z}\left(\frac{\partial u}{\partial t}(0,z)\,\frac{\partial u}{\partial z}(0,z)\,dz\right)$$

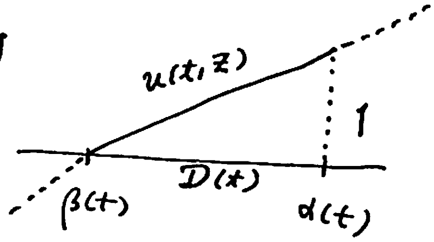
$$= \frac{2}{i}\iint_{D(0)} \left\{\frac{\partial^2 u}{\partial t\,\partial \bar z}(0,z)\,\frac{\partial u}{\partial z}(0,z) + \frac{\partial u}{\partial t}(0,z)\,\frac{\partial^2 u}{\partial \bar z\,\partial z}(0,z)\right\} d\bar z \wedge d$$

Since $u(0,z)$ is harmonic in $D(0)$ and $d\bar z \wedge dz = 2i\,dx\,dy$, it becomes that

$$\left[\frac{\partial I(t)}{\partial t}\right]_{t=0} = 4\iint_{D(0)} \frac{\partial^2 u}{\partial t\,\partial \bar z}(0,z)\,\frac{\partial u}{\partial z}(0,z)\,dx\,dy$$

Hence (2) of Lemma 1 is proved.

For the proof of (2) of Lemma 1, we remark that
Near α in $\hat{\mathcal{D}}$,
$u(t,z) - 1$ $(= \varphi(t,z))$ is a defining function of $\mathcal{D}$;
Near β in $\hat{\mathcal{D}}$
$-u(t,z)$ $(= \varphi(t,z))$ is a defining function of $\mathcal{D}$.



It follows that

$$k_2(t,z) = \begin{cases} \dfrac{L\,(u(t,z)-1)}{\left|\frac{\partial}{\partial z}(u(t,z)-1)\right|^3} & \text{on } \alpha \\[2ex] \dfrac{L\,(-u(t,z))}{\left|\frac{\partial}{\partial z}(-u(t,z))\right|^3} & \text{on } \beta \end{cases}$$

By observing that $\partial^2 u(t,z)/\partial z\partial\bar z = 0$, we thus have

$$k_2(t,z) = \begin{cases} \dfrac{\frac{\partial^2 u}{\partial t\partial \bar t}\left|\frac{\partial u}{\partial z}\right|^2 - 2\,\mathrm{Re}\left\{\frac{\partial^2 u}{\partial t\partial z}\,\frac{\partial u}{\partial t}\,\frac{\partial u}{\partial \bar z}\right\}}{\left|\frac{\partial u}{\partial z}\right|^3} & \text{on } \alpha \\[2ex] \dfrac{-\frac{\partial^2 u}{\partial t\partial \bar t}\left|\frac{\partial u}{\partial z}\right|^2 + 2\,\mathrm{Re}\left\{\frac{\partial^2 u}{\partial t\partial z}\,\frac{\partial u}{\partial t}\,\frac{\partial u}{\partial \bar z}\right\}}{\left|\frac{\partial u}{\partial z}\right|^3} & \text{on } \beta \end{cases}$$

It follows that

$$\frac{\partial^2 u}{\partial t\partial \bar t} = \begin{cases} k_2(t,z)\left|\frac{\partial u}{\partial z}\right| + 2\,\mathrm{Re}\left\{\dfrac{\frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial t\partial z}}{\frac{\partial u}{\partial z}}\right\} & \text{on } \alpha \\[2ex] -k_2(t,z)\left|\frac{\partial u}{\partial z}\right| + 2\,\mathrm{Re}\left\{\dfrac{\frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial t\partial z}}{\frac{\partial u}{\partial z}}\right\} & \text{on } \beta \end{cases}$$

This combined with the formula (1) yields that

$$\int_{\alpha(0)} \frac{\partial^2 u}{\partial t\partial \bar t}(0,z)\,\frac{\partial u}{\partial n_z}(0,z)\,ds_z = \int_{\alpha(0)} k_2(0,z)\left|\frac{\partial u}{\partial z}\right| 2\left|\frac{\partial u}{\partial z}\right| ds_z + 2\,\mathrm{Re}\left\{\int_{\alpha(0)} \frac{\frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial t\partial z}}{\frac{\partial u}{\partial z}}\,\frac{\partial u}{\partial n_z}\,ds_z\right\}$$

$$\int_{\beta(0)} \frac{\partial^2 u}{\partial t\partial \bar t}(0,z)\,\frac{\partial u}{\partial n_z}(0,z)\,ds_z = \int_{\beta(0)} -k_2(0,z)\left|\frac{\partial u}{\partial z}\right|(-2)\left|\frac{\partial u}{\partial z}\right| ds_z + 2\,\mathrm{Re}\left\{\int_{\alpha(0)} \frac{\frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial t\partial z}}{\frac{\partial u}{\partial z}}\,\frac{\partial u}{\partial n_z}\,ds_z\right\}$$

By summing up for $\alpha(0)$ and $\beta(0)$, we get

$$\int_{\partial D(0)} \frac{\partial^2 u}{\partial t\partial \bar t}(0,z)\,\frac{\partial u}{\partial n_z}(0,z)\,ds_z = 2\int_{\partial D(0)} k_2(0,z)\left|\frac{\partial u}{\partial z}(0,z)\right|^2 ds_z + 2\,\mathrm{Re}\left\{\int_{\partial D(0)} \frac{\frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial t\partial z}}{\frac{\partial u}{\partial z}}\,\frac{\partial u}{\partial n_z}\,ds_z\right\}$$

Since $\frac{\partial u}{\partial n_z}(0,z)\,ds_z = \frac{2}{i}\,\frac{\partial u}{\partial z}(0,z)\,dz$ on $\partial D(0)$, it follows that

$$\underset{\substack{\partial D(0)\\ \text{at } t=0}}{\int} \frac{\partial^2 u}{\partial t\partial \bar t}\frac{\partial u}{\partial n_z}\, ds_z = 2\underset{\substack{\partial D(0)\\ \text{at } t=0}}{\int} k_2 \left|\frac{\partial u}{\partial z}\right|^2 ds_z + 2\,\mathrm{Re}\Big\{2\underset{\partial D(0)}{\int} \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial \bar t\partial z}\,\frac{1}{i}\,dz\Big\}_{t=0}$$

$$= 2\underset{\partial D(0)}{\int} k_2 \left|\frac{\partial u}{\partial z}\right|^2 ds_z + 4\,\mathrm{Im}\underset{D(0)}{\iint} \underbrace{d\left[\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial \bar t\partial z}\,dz\right]}_{\left[\frac{\partial^2 u}{\partial t\partial \bar z}\frac{\partial^2 u}{\partial \bar t\partial z} + \frac{\partial u}{\partial t}\underbrace{\frac{\partial^3 u}{\partial \bar t\partial z\partial \bar z}}_{=0}\right]\underbrace{d\bar z\wedge dz}_{2i\,dx\,dy}}$$

$$= 2\underset{\partial D(0)}{\int} k_2 \left|\frac{\partial u}{\partial z}\right|^2 ds_z + 8\underset{D(0)}{\iint}\left|\frac{\partial^2 u}{\partial t\partial \bar z}\right|^2 dx\,dy$$

By Formula (5), we have thus

$$\frac{\partial^2 I}{\partial t\partial \bar t}(0) = 2\underset{\partial D(0)}{\int} k_2(0,z)\left|\frac{\partial u}{\partial z}(0,z)\right|^2 ds_z + 8\underset{D(0)}{\iint}\left|\frac{\partial^2 u}{\partial t\partial \bar z}(0,z)\right|^2 dx\,dy$$

(2) of Lemma 1 is proved. c. q. f. d.

<u>Theorem 1</u> If $\mathcal{D} = \bigcup_{t\in B}(t, D(t))$ is pseudoconvex over $B\times\mathbb{C}$, then

(i) $I(t)$ is subharmonic function for t in B:

(ii) $\mu(t)$ is $\geqq 0$ and superharmonic for t in B.

<u>Proof.</u>

Assume that $\mathcal{D}$ is pseudoconvex over $B\times\mathbb{C}$. Then it is equivalent that $k_2(t,z)\geqq 0$ on $\partial\mathcal{D}$. (This is a result of E.E. Levi). By (3) of Lemma 1 (i) is proved. For the proof of (ii), we remark that

$$\frac{\partial^2 \mu(t)}{\partial t \partial \bar{t}} = - \frac{I(t) \frac{\partial^2 I(t)}{\partial t \partial \bar{t}} - 2\left|\frac{\partial I(t)}{\partial t}\right|^2}{I(t)^3}$$

because of $\mu(t) = 1/I(t)$ $(t \in B)$.

By (2) of Lemma 1, we have

$$\left|\frac{\partial I(t)}{\partial t}\right|^2 \leqq 16\left(\iint_{D(t)} \left|\frac{\partial^2 u}{\partial t \partial \bar{z}}\right|^2 dxdy\right) \cdot \left(\iint_{D(t)} \left|\frac{\partial u}{\partial z}\right|^2 dxdy\right)$$

$$= \frac{1}{2}\left(8 \iint_{D(t)} \left|\frac{\partial^2 u}{\partial t \partial \bar{z}}\right|^2 dxdy\right)\left(4 \iint_{D(t)} \left|\frac{\partial u}{\partial z}\right|^2 dxdy\right).$$

Since $\mathcal{D}$ is pseudoconvex, we have from (3) of Lemma 1

$$\frac{\partial^2 I(t)}{\partial t \partial \bar{t}} \geqq 8 \iint_{D(t)} \left|\frac{\partial^2 u}{\partial t \partial \bar{z}}\right|^2 dxdy.$$

It follows from $I(t) = 4 \iint_{D(t)} \left|\frac{\partial u}{\partial z}\right|^2 dxdy$ that

$$\left|\frac{\partial I(t)}{\partial t}\right|^2 \leqq \frac{1}{2} \frac{\partial^2 I(t)}{\partial t \partial \bar{t}} \cdot I(t)$$

$$\therefore \quad \frac{\partial^2 \mu(t)}{\partial t \partial \bar{t}} \leqq 0 \quad \text{on } B$$

$\mu(t)$ is thus superharmonic for t in B c.q.f.d.

We have treated variations of a domain D with smooth boundary in $\mathbb{C}^n$ $(n \geqq 1)$. In this section we study variations of an open set D without smooth boundary.

Let $\mathcal{D}$ be a domain of $B \times \mathbb{C}^n$, where B is a region in the complex t-plane. For each $t \in B$, we denote by $D(t)$ the fiber of $\mathcal{D}$ at t, that is,

$$D(t) = \{ z \in \mathbb{C}^n \mid (t, z) \in \mathcal{D} \}.$$

Hence $D(t)$ is an open set, not necessarily a region with smooth boundary.

We briefly recall the Green's function for an open set D. First, let D be a region in $\mathbb{C}^n$ and let $\zeta \in D$. We choose a sequence of domains $\{\Omega_n\}_{n=1,2,\cdots}$ with smooth boundary $\partial\Omega_n$ such that

$$\zeta \in \Omega_1 \Subset \Omega_2 \Subset \cdots \quad \text{and that} \quad \bigcup_{n=1}^{\infty} \Omega_n = D.$$

We thus have the Green's function $g_n(z)$ for (Ω_n, ζ) and the Robin constant λ_n. By the maximum principle we have

$$g_n(z) < g_{n+1}(z) \quad \text{for } z \in \Omega_n ;$$
$$\lambda_n < \lambda_{n+1}.$$

Therefore the limits

$$g(z) = \lim_{n\to\infty} g_n(z) \quad \text{for } z \in D ;$$
$$\lambda = \lim_{n\to\infty} \lambda_n$$

exist. In the case of $n = 1$, it may happen to be

$$g(z) \equiv +\infty \quad \text{on } D$$

or equivalently,

$$\lambda = +\infty$$

In the case of $n \geqq 2$,

$$0 < g(z) \leqq \frac{1}{\|z-\zeta\|^{2n-2}} \quad \text{for } z \in D;$$

$$-\infty < \lambda \leqq 0$$

Moreover, it is clear that

$$g(z) \equiv \frac{1}{\|z-\zeta\|^{2n-2}} \text{ on } D \text{ iff } \lambda = 0.$$

The function $g(z)$ and the constant term λ are said to the Green's function and the Robin constant for (D, ζ).

Definition 1 A domain D with $\lambda = +\infty$ (resp. $\lambda = 0$) in the case $n = 1$ (resp. $n \geqq 2$) is said to be parabolic.

Next, let D be an open set and let $\zeta \in D$. We denote by D_1 the connected component of D which contains ζ, and consider the Green's function $g_1(z)$ and the Robin constant λ_1 for (D_1, ζ). We put

$$g(z) = \begin{cases} g_1(z) & \text{on } D_1 \\ 0 & \text{on } D - D_1 \end{cases};$$

$$\lambda = \lambda_1$$

and say that $g(z)$ and λ are the Green's function and the Robin constant for (D, ζ).

Now, let $\mathfrak{D}$ be a domain of $B \times \mathbb{C}^n$ $(n \geq 1)$. We assume that there exists a point ζ in $\mathbb{C}^n$ such that

$B \times \{3\} \subset \mathcal{D}$.

We thus have the green's function $g(t, z)$ and the Robin constant $\lambda(t)$ for $(D(t), 3)$. Of cource, $\lambda(t)$ is no longer of class C^2, in general.

We prove the following

<u>Theorem 1</u>

If $\mathcal{D}$ is a pseudoconvex domain in $B \times \mathbb{C}^n$ then $\lambda(t)$ is a superharmonic function on B. Moreover, in the case of $n \geq 2$, $\log(-\lambda(t))$ is subharmonic on B.

<u>Proof.</u> Since $\mathcal{D}$ is pseudoconvex, we construct a sequence of subdomains $\{\mathcal{D}_p\}$ of $\mathcal{D}$ such that $\mathcal{D}_1 \subset\subset \mathcal{D}_2 \subset\subset \cdots$; $\bigcup_{p=1}^{\infty} \mathcal{D}_p = \mathcal{D}$ and such that $\mathcal{D}_p$ is pseudoconvex of $\mathbb{C}^{n+1}$ with smooth boundary $\partial \mathcal{D}_p$. (In the last semester we got such a sequence $\{\mathcal{D}_p\}$ in the case $n = 1$, following K. Oka). For example, by considering the Robin constant $\Lambda_p(t, z)$ for $(\mathcal{D}_p, (t, z))$, we have a real analytic strictly plurisubharmonic function $\psi_p(t, z)$ such that $\psi_p(t, z) \nearrow +\infty$ as $(t, z) \to \partial \mathcal{D}_p$.

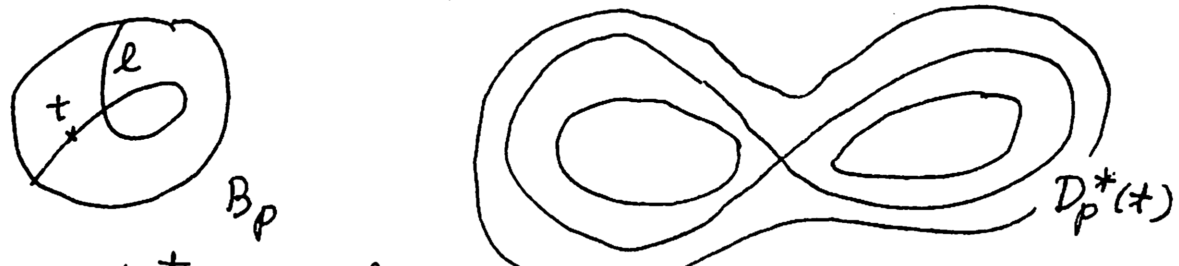
Now, let $\{B_p\}$ be a sequence of subregions of B such that $B_1 \subset\subset B_2 \subset\subset \cdots$ and $\bigcup_{p=1}^{\infty} B_p = B$. We choose a subsequence $\mathcal{D}_{j(p)}$ of $\mathcal{D}_p$ such that $\mathcal{D}_{j(p)} \supset\supset 3(B_{p+1})$. We relabel $j(p) = p$ $(p = 1, 2, \ldots)$. For each $t \in B_{p+1}$, we consider the Robin constant $\lambda_p(t)$

for $(D_p(t), \zeta)$. Since $\alpha_p > 0$ with the following property: If we put $\mathcal{D}_p^* = \{(t,z) \in \mathcal{D}_p : \psi_p(t,z) < \alpha_p\}$, then $\mathcal{D}_p \gg \mathcal{D}_p^* \gg \mathcal{D}_{p-1}$, the domain

$$\mathcal{D}_{p, B_p}^* = \bigcup_{t \in B_p} (t, \mathcal{D}_p^*(t))$$

has smooth boundary in $B_p \times \mathbb{C}^n$ and

(*)



there exists a real analytic curve l in B_p such that each $D_p^*(t)$ has smooth boundary $\partial D_p^*(t)$ for $t \in B_p - l$ and such that $\partial D_p^*(t)$ has a finite number of singular points for $t \in l$.

For each $t \in B_p$, let $\lambda_p^*(t)$ denote the Robin constant for $(D_p^*(t), \zeta)$. Under this notation, we showed that $\lambda_p^*(t)$ is a superharmonic function of class C^1 on B_p. Moreover, in the case of $n \geq 2$, $\log(-\lambda_p^*(t))$ is a subharmonic function of class C^1 on B_p. On the other hand, because $D_{p-1}(t) \Subset D_p^*(t) \Subset D_p(t) \Subset D(t)$ for $t \in B_p$, we have $\lambda_{p-1}(t) < \lambda_p^*(t) < \lambda_p(t) < \lambda(t)$ for $t \in B_p$. Since $\lambda_p(t) \nearrow \lambda(t)$ as $p \nearrow +\infty$ for $t \in B$, it follows that $\lambda(t)$ is a superharmonic function on B and that $\log(-\lambda(t))$ is subharmonic on B in the case $n \geq 2$. Q.E.D.

We study (*) in the proof more precisely:
Here we treat the case $n = 1$, but our argument will be available for $n \geq 2$.

Let $\psi(t, z)$ be any real analytic and strictly plurisubharmonic and exhaustion function in $\tilde{D}$. By Sard's Theorem:

⌈Let $f: U \to \mathbb{R}^p$ be a smooth map, defined on an open set $\subset \mathbb{R}^q$, and let

$$C = \{ x \in U \mid \text{rank } df_x < q \}$$

then the image $f(C) \subset \mathbb{R}^p$ has Lebesgue measure zero.
there exists $\{\alpha_n\}_{n=1,2,\dots}$ such that $\alpha_n \nearrow +\infty$ and, if we put

$$\mathcal{D}_n = \{ \psi(t, z) < \alpha_n \},$$

then

$$\mathcal{D}_n \subset\subset \tilde{\mathcal{D}} \text{ and } \partial \mathcal{D}_n \text{ is smooth.}$$

In particular, $\partial \mathcal{D}_n = \{ \psi(t, z) = \alpha_n \}$.

For the sake of convenience, we put $\mathcal{D}_n = \mathcal{D}$ and $\alpha_n = 0$. We denote by $\mathcal{D}(t)$ the fiber of $\mathcal{D}$ at t in B. Take $t \in B$ such that $D(t) \neq \phi$. It is clear that

$D(t)$ is an open set such that $D(t) \subset\subset \tilde{D}(t)$;

$\partial D(t) \subset \{ z \in \tilde{D}(t) \mid \psi(t, z) = 0 \}$,

(but the converse inclusion is not always true)

or equivalently,

$$D(t) \subset \{ \psi(t, z) < 0 \}$$

<u>Theorem 1</u> Under this notation, $\partial D(t)$ has at most finite number of singularities.

In order to prove this, we need the following

<u>Proposition 1</u>

Let $u(z)$ $(= u(x,y))$ be a (non-zero) real-valued real analytic function near $(x,y) = (0,0)$. Assume that there exists a sequence of points (x_n, y_n) $(= z_n)$ $\neq (0,0)$ such that $u(z_n) = 0$ $(n=1,2,\cdots)$ and such that $z_n \to 0$ as $n \to \infty$. Then there exists a real analytic curve C through $(0,0)$ such that

(i) C is regular except $(0,0)$

(ii) C passes through infinitely many z_{n_j} $(j=1,2,\cdots)$ of $\{z_n\}$.

<u>Proof.</u> We may assume $x_n \geqq 0$ $(n=1,2,\cdots)$.

We put $z=(x,y)$, $Z = (X, Y)$, $X = x+ix'$, $Y = y+iy'$ where x', y' are real. Since $u(x,y)$ is real analytic, we get

$$u(x,y) = \sum_{n,m=0}^{\infty} a_{mn} x^m y^n$$

near $(x,y) = (0,0)$. We define

$$\hat{u}(X,Y) = \sum_{m,n=0}^{\infty} a_{mn} X^m Y^n$$

so that $\hat{u}(x,y) = u(x,y)$ and $\hat{u}(X,Y)$ is holomorphic with respect to (X,Y) near $X=Y=0$.

Since $\hat{u}(0,0) = u(0,0) = \lim_{n\to\infty} u(x_n, y_n) = 0$, it follows from Weierstrass preparation theorem that

The set (which is ignored the order of zero) of $\hat{u}(X,Y) = 0$ near $(0,0)$ consists of finite number of the graphes of the form

$$\Sigma : Y = g(X) = a_k X^{\frac{k}{p}} + a_{k+1} X^{\frac{k+1}{p}} + \cdots \quad \begin{pmatrix} p \geq 1 \\ k \geq 1 \end{pmatrix}$$

where $a_k \neq 0$.

or equivalently

$$\begin{cases} X = t^p \\ Y = \sum_{j=k}^{\infty} a_j t^j \end{cases} \qquad (|t| < \rho) \text{ in } \mathbb{C}$$

Since $\{(x_n, y_n)\}_{n=1,2,\dots}$ are infinitely different points, one of Σ contains infinitely many $\{(x_{n_j}, y_{n_j})\}_{j=1,2,\dots}$. For the simplicity, we relabel $(x_{n_j}, y_{n_j}) = (x_n, y_n)$. Since $x_n \geq 0$, we put $t_n = x_n^{1/p} > 0$. Therefore

$$y_n = \sum_{j=k}^{\infty} a_j (t_n)^j.$$

Since y_n is real and since $(x_n, y_n) \to (0,0)$, it follows that all a_j $(j \geq k)$ are real.

Therefore, if we put

$$C : \begin{cases} x = t^p \\ y = \sum_{j=k}^{\infty} a_j t^j \end{cases} \qquad -\rho < t < \rho \text{ in } \mathbb{R}$$

in the (x,y)-plane, then C is a real analytic curve such that $C \ni (x_n, y_n)$ $(n = 1, 2, \dots)$ and such that C is non-singular except $(0,0)$.

q.e.d.

Proof of Theorem 1

Let $(x_0, y_0) \in \partial D(t)$. For the sake of convenience, we put $t = 0$ & $(x_0, y_0) = (0,0)$, and set $\psi(0, z) = \varphi(z)$. Hence $\varphi(z)$ is a real-valued, real analytic function with

$$\varphi(0) = 0 \quad \& \quad \frac{\partial^2 \varphi}{\partial z \partial \bar{z}}(0) > 0.$$

It follows that

$$\varphi(z) = A|z|^2 + \frac{B z^2 + \bar{B}\bar{z}^2}{2} + O(|z|^3)$$

where $A > 0$, $B \in \mathbb{C}$.

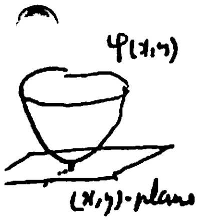
If we put $B = a + ib$ (a, b: real), then

$$\varphi(x,y) = (A+a)x^2 + (A-a)y^2 - 2bxy + O(|z|^3)$$

Case 1 $\quad a^2 + b^2 < A^2$.

Since $(A+a) + (A-a) = 2A > 0$, we have

$$\varphi(x,y) = (\alpha x + \beta y)^2 + \gamma y^2 + O(|z|^3)$$



where $(\alpha, \beta) \neq (0,0) \in \mathbb{R}^2$, $\gamma \neq 0 \in R$. $\gamma > 0$

Hence $\varphi(z) > 0$ except at $z = 0$ in a neighborhood of 0. $z = 0$ is not thus a boundary point of $\partial D(0)$. Case 1 never occurs.

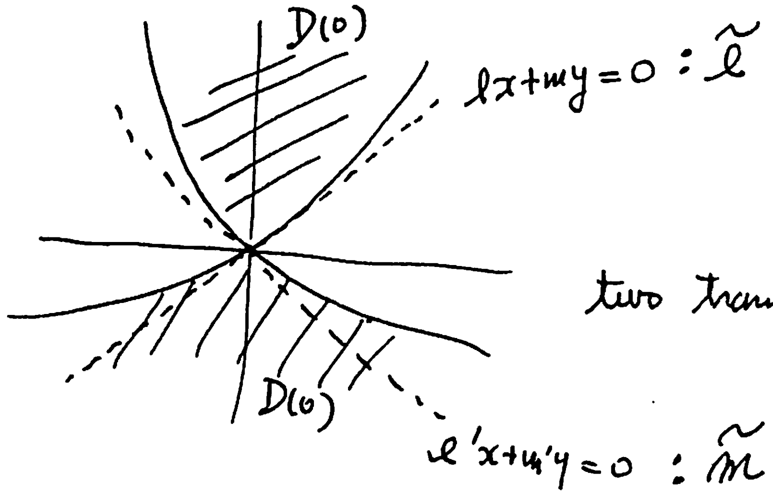
Case 2 $\quad a^2 + b^2 > A^2$.

We then get

$$\varphi(z) = (lx + my)(l'x + m'y) + O(|z|^3)$$

where $\begin{vmatrix} l & m \\ l' & m' \end{vmatrix} \neq 0$.

It follows that, in a neighborhood of 0, we have



saddle point.

two transversal lines: $\tilde{l}, \tilde{m}$

Therefore $z=0$ is isolated point of $\partial D(0)$, certainly.

<u>Case 3</u> $a^2+b^2 = A^2$ i.e., $\dfrac{\partial^2\varphi}{\partial z \partial \bar{z}} = \left|\dfrac{\partial^2 \varphi}{\partial z^2}\right|^2$ at $(0,0)$

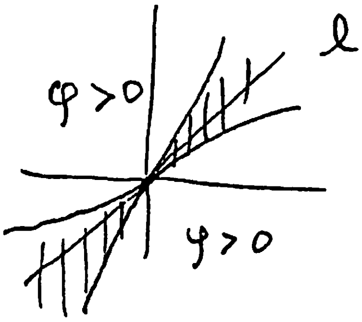
Then we have

$$((*)) \quad \varphi(x,y) = \left(\sqrt{\sqrt{a^2+b^2}+a}\; x - \sqrt{\sqrt{a^2+b^2}-a}\; y\right)^2 + O(|z|$$

We note that

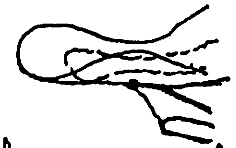
$$\left(\sqrt{\sqrt{a^2+b^2}+a}, \sqrt{\sqrt{a^2+b^2}-a}\right) \neq (0,0)$$

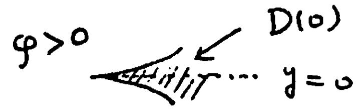
because of $A = \sqrt{a^2+b^2} > 0$.



$l : \sqrt{\sqrt{a^2+b^2}+a}\; x - \sqrt{\sqrt{a^2+b^2}-a}\; y =$

at present, we dont know $\varphi > 0$ or < 0 in the shaded part, tangential domain along l:





Of course, the case of isolated occurs ; $\varphi(z) = y^2 - x^3$ for example.
Assume that $z = 0$ is not isolated singular point of $\partial D(0)$. Namely, there exists

$$z_n = (x_n, y_n) \longrightarrow (0,0)$$

such that $z_n \neq 0$ and z_n is a singular point of $\partial D(0)$.
We may assume that each $x_n > 0$.
Since (x_n, y_n) is singular boundary point of $D(0)$, it follows at least that

$$\varphi(x_n, y_n) = \frac{\partial \varphi}{\partial x}(x_n, y_n) = \frac{\partial \varphi}{\partial y}(x_n, y_n) = 0 \quad (n = 1, 2, \dots)$$

By Proposition 1, there exists a real analytic curve

$$C : \begin{cases} x = t^p & (\infty > p \geq 1) \text{ integer}. \\ y = \sum_{j=k}^{\infty} a_k t^k & (-\rho < t < \exists \rho) \end{cases}$$

where a_k are real, such that

$$C \ni (x_n, y_n) \quad (n = 1, 2, \cdots) \qquad \text{(relabeled)}$$

Hence at each (x_n, y_n) we may suppose C is non-singular. Consequently, (x_n, y_n) is neither of case 1 nor of case 2, so that each (x_n, y_n) is of case 3. We have thus

$$0 < \frac{\partial^2 \varphi}{\partial z \partial \bar{z}}(x_n, y_n) = \left| \frac{\partial^2 \varphi}{\partial z^2}(x_n, y_n) \right|^2$$

$$\therefore \quad \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \equiv \left| \frac{\partial^2 \varphi}{\partial z^2} \right|^2 \quad \text{on } C$$

because of real analyticity.
By the same reasoning,

$$\varphi = \frac{\partial \varphi}{\partial z} = 0 \quad \text{on } C.$$

It follows that for each $t \in (-\rho, \rho)$, $(x(t), y(t))$ is a singular point of $\partial D(t)$ of case 3.

Put $A(x,y) = \frac{\partial^2\varphi}{\partial z \partial \bar{z}}(x,y) > 0$

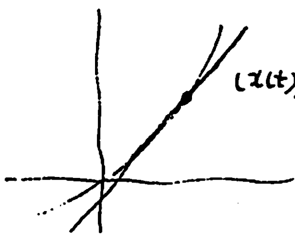
$B(x,y) = a(x,y) + ib(x,y) = \frac{\partial^2\varphi}{\partial z^2}(x,y)$ ((x,y) close to $(0,0)$

and put

$$C : z = (x(t), y(t)) = z(t) \quad (-\rho < t < \rho)$$

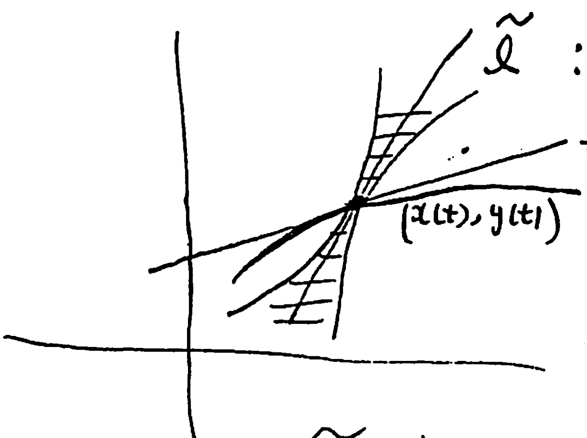
Then $a(z)^2 + b(z)^2 = A(z) > 0$ for $z \in C$.

For each fixed $t \in (-\rho, \rho)$, we have, by the case 3,



$$(x'(t), y'(t)) = \left(\sqrt{\sqrt{a(z(t))^2 + b(z(t))^2} + a(z(t))}, \sqrt{\sqrt{a(z(t))^2 + b(z(t))^2} - a(z(t))}\right)$$

⌜ (∵) If not, we have the following circumstance:



Assume $\tilde{l}$ intersect $\tilde{\tilde{l}}$ transversally. Then $\tilde{\tilde{l}}$ is not contained in the shaded part. i.e, $\varphi|_{\tilde{\tilde{l}}} > 0$ except $(x(t), y(t))$. But $\varphi = 0$ on C. C and $\tilde{\tilde{l}}$ are tangent at $(x(t), y(t))$. This is a contradiction. ⌟

Namely, C coincide with the integral curve passing through $(0,0)$ of the real analytic vector field:

$$X(z) = \left(\sqrt{a(z)^2 + b(z)^2} + b(z),\ \sqrt{a(z)^2 + b(z)^2} - b(z) \right)$$

Since $C \ni (0,0)$ and $A(0) = a^2(0) + b^2(0) > 0$, the curve C must be non-singular at $(0,0)$. So we may assume

$$C: \quad y = s(x) \qquad (-\rho' < x < \rho')$$

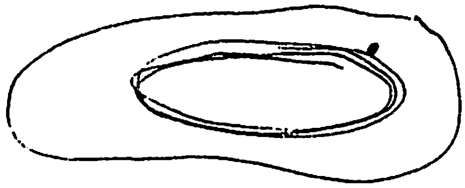
where $s(x)$ is a real analytic function in $(-\rho', \rho')$.
At each fixed x, as an equation of y, $\varphi(x,y) = 0$ has a solution $y = s(x)$ of order 1 just (see ((*)) in case 3). It follows from Weierstrass theorem that

$$\varphi(x,y) = \left(y - s(x)\right)^2 H(x,y)$$

where $H(x,y)$ is real analytic & $H(0,0) > 0$.

$\therefore \quad \varphi(x,y) > 0$ near $(0,0)$ except C.
Hence $(0,0)$ is not boundary point of $D(0)$.
Consequently, case 3 never occurs.

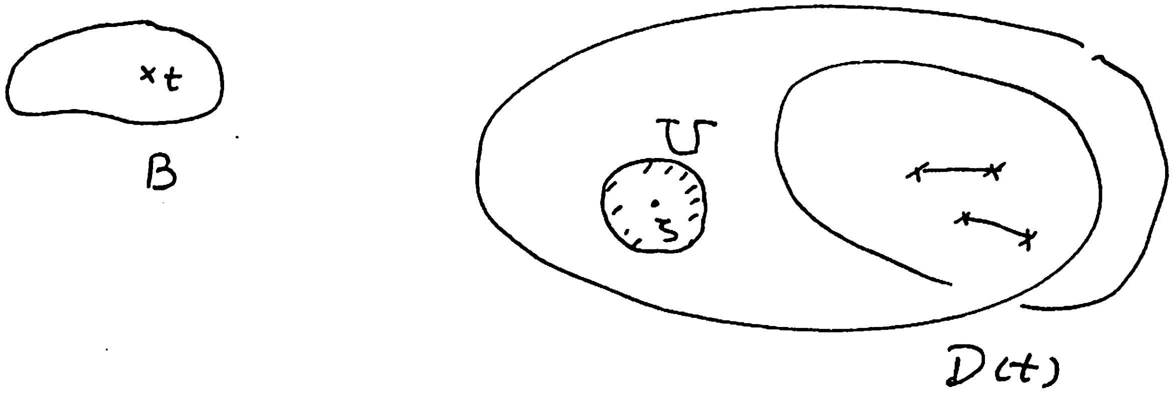
Remark. We have proved in case 3 that the curve C is a closed Jordan curve in $\tilde{D}(0)$. (because C is an integral curve of the field X and C is non-singular ($\because A > 0$). Moreover $\varphi = 0$ is compact in $\tilde{D}(0)$. If C is accumulated, then



$\tilde{D}(0)$

We easily have a contradiction at any accumulation point. For, of case 1, 2 or 3 never occurs at that point.

Let $\mathcal{D}$ be a ramified domain over $B\times\mathbb{C}$ and denote by $D(t)$ the fiber of $\mathcal{D}$ at $t\in B$. Assume that there exists a univalent domain $B\times U$ where $U=\{|z-\zeta|<r\}\subset$ of $\mathcal{D}$.



We denote by $g(t,z)$ and $\lambda(t)$ the Green's function and the Robin constant for $(D(t),\zeta)$.

<u>Corollary 1</u>. Let $\mathcal{S}$ be a branch surface of $\mathcal{D}$, and set $\mathcal{D}^*=\mathcal{D}-\mathcal{S}$, so that $\mathcal{D}^*$ is an unramified covering domain over $B\times\mathbb{C}$. Then, if $\mathcal{D}^*$ is pseudoconvex over $B\times\mathbb{C}$, then $\lambda(t)$ is superharmonic on B.

(:) Let $\lambda^*(t)$ be the Robin constant for $(D^*(t),\zeta)$ where $D^*(t)$ is the fiber of $\mathcal{D}^*$ at $t\in B$. Since $D(t)-D^*(t)$ consists of isolated points, we have
$$\lambda^*(t)=\lambda(t).$$
Because $\mathcal{D}^*$ is an unramified pseudoconvex domain over $B\times\mathbb{C}$, we know that $\lambda^*(t)$ is

superharmonic on B, and so is $\lambda(t)$. Q.E.D.

In order to show a "uniformization theorem", we prepare some elementary facts which are known in the theory of Riemann surface.

Definition 1 Let R be a non-compact Riemann surface. If R is topologically equivalent to a domain of $\mathbb{C}$, then R is said to be planar (schlichtartig).

Definition 2 Let R be a non-compact Riemann surface. Let $\zeta \in R$. If the Robin constant for (R, ζ) is $+\infty$, then R is said to be parabolic.

As already noted, $\lambda = +\infty$ does not depend on the choice of $\zeta \in R$.

Theorem 1 (Koebe)

Let R be a non-compact planar and parabolic Riemann surface. Then R is analytically isomorphic to a domain D of $\mathbb{C}$ such that $K = \mathbb{C} - D$ is of logarithmic capacity zero.

This Theorem is standard, so we assume it from now on.

Let R be a Riemann surface which is parabolic and planar. Take a point $P_0 \in R$ and fix a local parameter, say $|z| < 2$.

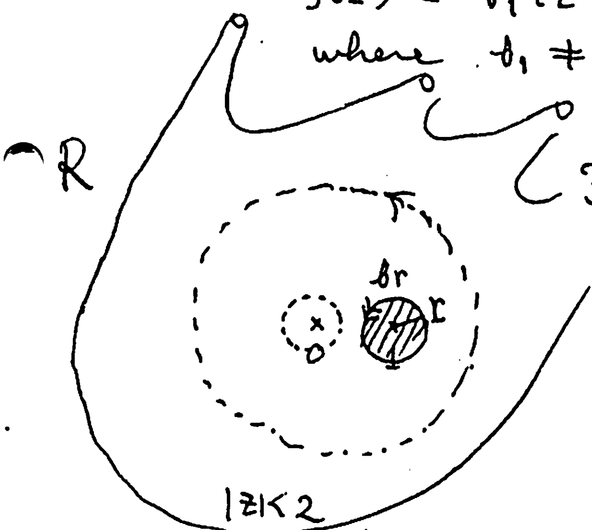
By Koebe's Theorem, R is uniquely and conformally mapped onto a domain D of $\mathbb{C}\cup\{\infty\}$ by a mapping $f(z)$ such that

$$(*)\quad \begin{cases} f(z) = \frac{1}{z} + c_0 + c_1 z + \cdots \\ f(1) = 0 \end{cases}$$

So $w=\infty$ is an interior point of D, and at $z=1$ we have

$$f(z) = b_1(z-1) + b_2(z-1)^2 + \cdots$$

where $b_1 \neq 0$.



Fix $0<r<1$ and let $b_r = \{|z-1|<r\}$.

We set

$$R_r = R - b_r \;;\; D_r = f(R_r)$$

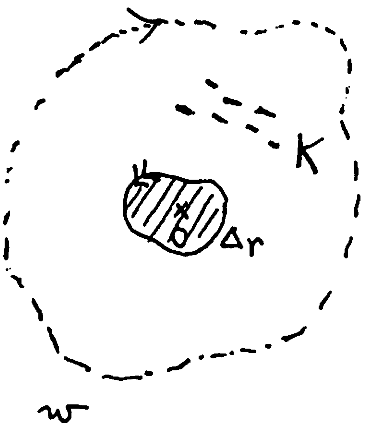
$$\Delta_r = f(b_r) \;;\; \Delta_r^c = \mathbb{C} - \Delta_r$$

We note that

$$\Delta_r^c - D_r = K = \mathbb{C} - D \text{ is of logarithmic}$$

capacity zero.

$w=f(z)$



Here we form the Green's function

$$g_r(z) \text{ for } (R_r, 0),$$

so that

$$(**)\quad g_r(z) = \log\frac{1}{|z|} + \lambda_r + h_r(z)$$

where $h_r(0)=0$.

We put

$$G_r(w) = g(f^{-1}(w)) \quad \text{on } D_r$$

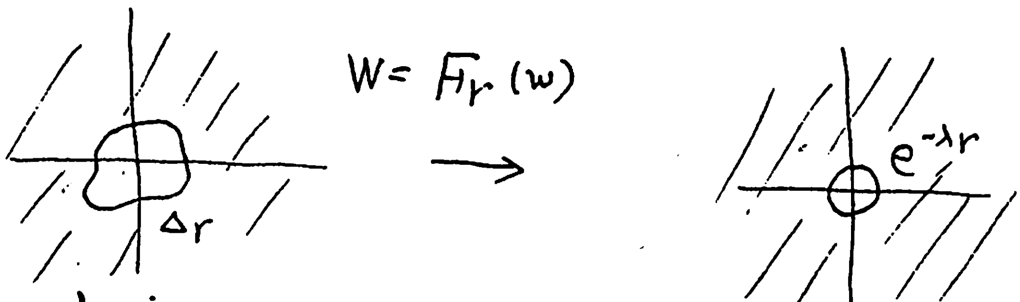
Then $G_r(w)$ is harmonic on D_r, and is $=0$ on $\partial\Delta_r$. Moreover, at $w=\infty$, ≥ 0 we have from $(*)$, $(**)$

$$G_r(w) = \log|w| + \lambda_r + H_r(w)$$

where $H_r(w)$ is harmonic near $w=\infty$ and $H_r(\infty)=0$.
Since $\Delta_r^c (\supset D_r)$ is different from D_r by a compact set K of logarithmic capacity zero, it follows that $G_r(w)$ is the Green's function of (Δ_r^c, ∞). Since Δ_r^c is simply connected the function

$$W = F_r(w) = e^{G_r(w) + iG_r^*(w) - \lambda_r}$$

where $G_r^*(w)$ is a harmonic conjugate of $G_r(w)$ such that $H_r^*(\infty)=0$, conformally mapps Δ_r^c onto $(|W| > e^{-\lambda_r})$



Since $\lambda_r \nearrow +\infty$ as $r \downarrow 0$, it follows that

$$F_r(w) \longrightarrow W \quad \underbrace{\text{on any compact set on } \mathbb{C}-\{0\}}_{\text{uniformly}}.$$

Let us prove the following

<u>Lemma 1</u> $\displaystyle \lim_{r\to 0}\left(-\lambda_r + \log\frac{1}{r}\right) = \log|b_1|$

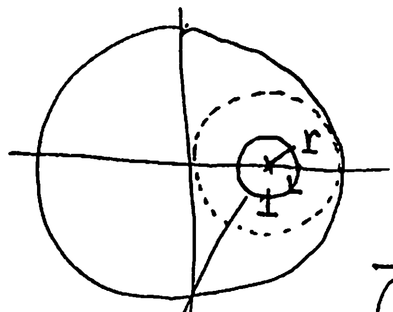
In order to prove this, we need Koebe's Distortion Theorem:

Let $f(z)$ be any univalent function on $|z|<1$ such that $f(0)=0$, $f'(0)=1$. Then

(A) $f(|z|<1) \supset (|w| < \frac{1}{4})$

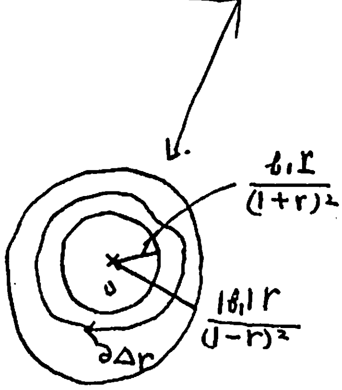
(B) $$\frac{|z|}{(1+|z|)^2} < |f(z)| < \frac{|z|}{(1-|z|)^2}$$

In the proof, see for example "Conformal mapping" by NEHARI.



We apply (B) to the function $f(z)/b_1$ on $|z-1|<1$. Then

$$\frac{|z-1|}{(1+|z-1|)^2} \leqq \left|\frac{f(z)}{b_1}\right| \leqq \frac{|z-1|}{(1-|z-1|)^2}$$



$$\frac{|b_1|\,r}{(1+r)^2} \leqq |\partial\Delta_r| \leqq \frac{|b_1|\,r}{(1-r)^2}.$$

$$\therefore \left\{|w| > \frac{|b_1| r}{(1+r)^2}\right\} \supset \Delta_r^c \supset \left\{|w| > \frac{|b_1| r}{(1-r)^2}\right\}$$

On the other hand, the Robin constant λ_c for $(\{|w|>c\}, \infty)$ is explicitly written into the form

$$\lambda_c = \log\frac{1}{c}$$

$$\therefore \quad \log\frac{(1+r)^2}{|b_1| r} > \lambda_r > \log\frac{(1-r)^2}{|b_1| r}$$

or equivalently

$$\log|b_1| - \log(1+r)^2 < -\lambda_r + \log\frac{1}{r} < \log|b_1| - \log(1-r)^2$$

$$\therefore \quad \lim_{r\to 0}\left(-\lambda_r + \log\frac{1}{r}\right) = \log|b_1|. \qquad \text{Q.E.D.}$$

<u>Remark 1</u> (1) $\frac{1}{2} < |f(z)| < \frac{16}{3}$ in $\left\{\frac{1}{3} < |z| < \frac{1}{2}\right\}$

(2) $|f(z)| < 4$ in $R - \{|z|<1\}$

Proof.

We apply Koebe's distortion theorem (A) to the function $F(z) = 1/f(z)$ in $|z|<1$, and have

$$|F(z)| < \frac{1}{4} \quad \text{in } |z|<1$$

Or equivalently, $|f(z)| > 4$ in $|z|<1$.
Since $f(z)$ is univalent in R, we have

$$f(R - (|z|<1)) \subset \{|w|<4\}.$$

which proves (2).
We apply Koebe's distortion theorem (B) to the function $F(z)$ in $|z|<1$, and have

$$\frac{|z|}{(1+|z|)^2} < |F(z)| < \frac{|z|}{(1-|z|)^2} \quad \text{for } |z|<1$$

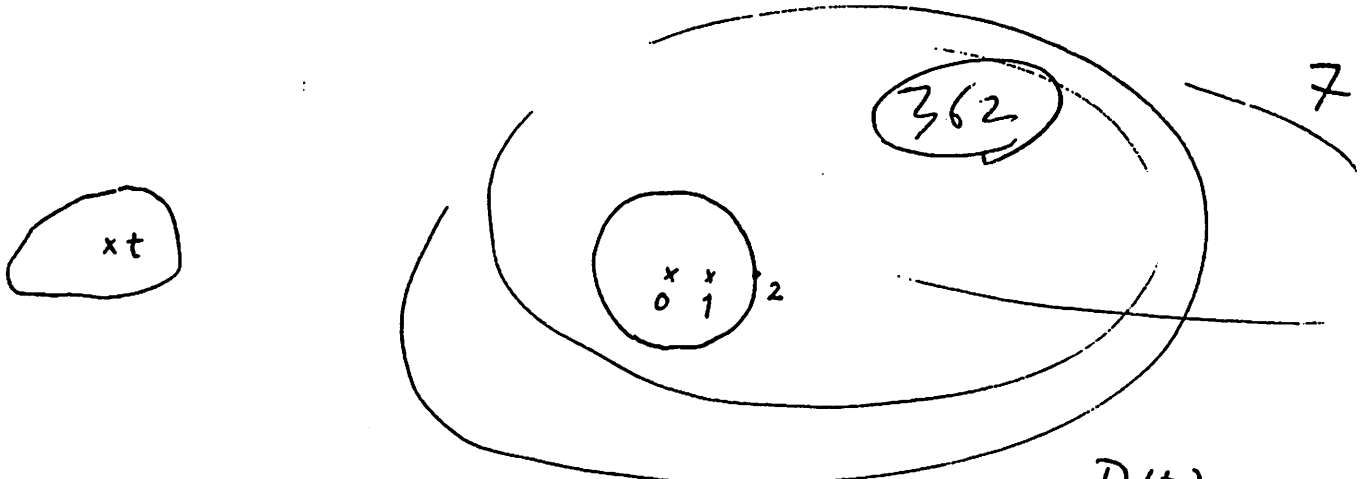
Therefore

$$\frac{16}{3} = \frac{(1+\frac{1}{3})^2}{\frac{1}{3}} > |f(z)| > \frac{(1-\frac{1}{2})^2}{\frac{1}{2}} = \frac{1}{2} \quad \text{for } \frac{1}{3}<|z|<\frac{1}{2}$$

which proves (1). Q.E.D.

Under these notations, we return to the variation of Riemann surfaces.

Let $\mathcal{D}$ be a ramified domain over $B\times\mathbb{C}$. Assume that there exists a univalent part $B\times U_0$ of $\mathcal{D}$ where $U_0 = \{|z|<2\}$. We denote by $D(t)$ the fiber of $\mathcal{D}$ at t.



$D(t)$

Assume that each fiber $D(t)$ is planar & parabolic. By Koebe's uniformization theorem, there uniquely exists a holomorphic function $f(t,z)$ on $D(t)$ such that

$$f(t,z) = \frac{1}{z} + c_0(t) + c_1(t)z + c_2(t)z^2 + \cdots \qquad \text{near } z=0$$

$$f(t,z) = b_1(t)(z-1) + b_2(t)(z-1)^2 + \cdots \qquad \text{near } z=1$$

$f(t, D(t)) = E(t)$ is a domain of $\mathbb{C}\cup\{\infty\}$.
If we put $K(t) = \mathbb{C}\cup\{\infty\} - D(t)$, then $K(t)$ is a compact set of $\mathbb{C}$ of logarithmic capacity zero.

We put $\mathcal{S}$ the branch surface of $\mathcal{D}$ and write
$$\mathcal{D}^* = \mathcal{D} - \mathcal{S}$$
so that $\mathcal{D}^*$ is unramified over $B\times\mathbb{C}$.

<u>Lemma 2</u> $f(t,z)$ is continuous for $(t,z)\in\mathcal{D}^* - B\times\{0\}$

<u>Proof.</u> Let $(t_0, z_0)\in\mathcal{D}^* - B\times\{0\}$ and take a neighborhood $B_0\times V_0$ of (t_0,z_0) of the form
$$B_0 = |t-t_0| < \rho_0, \qquad V_0 = |z-z_0| < \eta_0$$
By Remark 1 (1), (2), we easily see that

the family $\{f(t,z)\}_{t\in B_0}$ forms a normal family in V_0, and any limiting function is univalent in V_0. Since V_0 is arbitrary, we can choose, given family $\{f(t_n,z)\}_{n=1,2,\cdots}$ where $t_n \to t_0$, a subsequence $\{f(t_{n_j},z)\}$ which uniformly converges on any compact set on $D^*(t_0)$. If we denote by $F(z)$ the limiting function, then

$$F(z) = \frac{1}{z} + A_0 z + \cdots \quad \text{near } z=0$$

$$F(1) = 0$$

$F(z)$ is univalent on $D^*(t_0)$.

Since $D(t_0) - D^*(t_0)$ consists of isolated points, we thus have $F(z) = f(t_0, z)$ which is independent of the choice of subsequence. It follows that

$$f(t,z) \to f(t_0,z) \quad (t \to t_0)$$

uniformly on any compact set on $D^*(t_0)$. Because $f(t,z)$ is holomorphic for z, we see that

$f(t,z)$ is continuous for (t,z) in $\mathcal{D}^* - B\times\{0\}$.

<u>Remark 2</u> This continuity proof by use of Koebe's distortion theorem is due to T. Nishino.

Lemma 1 together with Lemma 2 yields the following <u>uniformization theorem</u>

If $\mathcal{D}^*$ is pseudoconvex over $B\times\mathbb{C}$, then $f(t,z)$ is holomorphic for (t,z) in $\mathcal{D} - B\times\{0\}$.

We divide the proof into several short steps :

1st step

$\log|b_1(t)|$ is continuous subharmonic function on B

(∵) Since $b_1(t) = \frac{\partial f}{\partial z}(t,1)$, it follows from Lemma 2 that $b_1(t)$ is continuous on B. Moreover $b_1(t) \neq 0$. Hence $\log|b_1(t)|$ is continuous on B. By Lemma 1,

$$\log|b_1(t)| = \lim_{r\to 0}\left(-\lambda_r(t) + \log\frac{1}{r}\right) \quad \text{for } \forall t \in B,$$

and the convegence is uniform on B. On the other hand,

$$\mathcal{D} - B\times\{|z-1|<r\}$$

satisfies condition of Corollary 1. Hence $\lambda_r(t)$ is superharmonic for $t\in B$. It turns out $\log|b_1(t)|$ is subharmonic on B. Q.E.D.

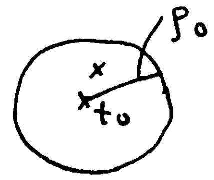
2nd step

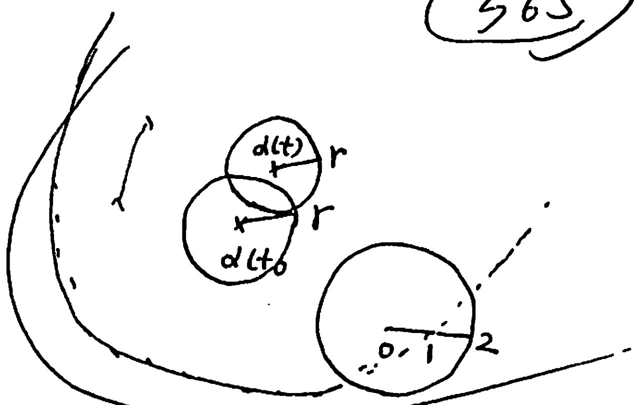
$\log\left|\frac{\partial f(t,z)}{\partial z}\right|$ is continuous plurisubharmonic function on $\mathcal{D}^* - B\times\{0\}$

(∵) Lemma 2 implies that $\frac{\partial f}{\partial z}(t,z)$ is continuous for $(t,z)\in\mathcal{D}^* - B\times\{0\}$. Since $f(t,z)$ is univalent $\log|\partial f/\partial z|(t,z)$ is continuous for $\mathcal{D}^* - B\times\{0\}$. For the proof of plurisubharmonicity, it suffices to prove that given $w = \alpha(t)$ $(t\in B_0 (\subset B))$: analytic set in $\mathcal{D}^* - B\times\{0\}$

$$\log\left|\frac{\partial f}{\partial z}\right|(t,\alpha(t)) \text{ is subharmonic on } B_0.$$

In fact





Consider a domain

$$\mathcal{D}_{B_0,r} \underset{\text{def}}{=} \mathcal{D}_{B_0} - \bigcup_{t\in B_0} (t, |z-\alpha(t)|<r) \quad \text{for } 0<r\ll 1.$$

Then $\mathcal{D}^*_{B_0,r}$ is pseudoconvex over $B_0\times\mathbb{C}$. If we denote by $g^\alpha_r(t,z)$ the Green's function and by $\lambda^\alpha_r(t)$ the Robin constant for $\left(D(t)-(|z-\alpha(t)|<r),\ 0\right)$, then

$$e^{[g^\alpha_r(t,z)+i\,g^\alpha_r(t,z)]-\lambda^\alpha_r} \longrightarrow f_\alpha(t,z)$$

where $f_\alpha(t,z)$ is the holomorphic fn (univalent and) on $D(t)$ such that

$$f_\alpha(t,z) = \frac{1}{z} + C_{\alpha 0}(t) + C_{\alpha 1}(t)\,z + \cdots$$

$$f_\alpha(t,\alpha(t)) = b_{\alpha 1}(t)(z-\alpha(t)) + b_{\alpha 2}(t)(z-\alpha(t))^2 + \cdots.$$

Since $\lambda^\alpha_r(t)$ is superharmonic in B_0, it follows by the same argument as 1^{st} that

$$\log|b_{\alpha 1}(t)| \text{ is continuous subharmonic on } B_0.$$

On the other hand, by the uniqueness theorem, we have

$$f_\alpha(t,z) = f(t,z) - f(t,\alpha(t)) \quad \text{for } t\in B_0$$

It follows that
$$\log|b_{\alpha 1}(t)| = \log\left|\frac{\partial f_\alpha}{\partial z}(t,\alpha(t))\right| = \log\left|\frac{\partial f}{\partial z}(t,\alpha(t))\right|.$$

3rd step

$\left(\frac{\partial^2 f}{\partial z^2} \Big/ \frac{\partial f}{\partial z}\right)(t,z)$ is holomorphic for $(t,z) \in \mathcal{D}^* - B\times\{0\}$

Proof. Since $\log\left|\frac{\partial f}{\partial z}\right|$ is continuous plurisubharmonic for (t,z) in $\mathcal{D}^* - B\times\{0\}$, it follows by using the molifier that

$$(*) \quad {}^{\exists} s_n(t,z) \longrightarrow \log\left|\frac{\partial f}{\partial z}\right| \text{ uniformly}$$

on any $B_0 \times V_0 \; (\subset \mathcal{D}^* - B\times\{0\}$ where $s_n(t,z)$ is plurisubharmonic of class C^2. Moreover, the construction of $s_n(t,z)$ teaches us that $s_n(t,z)$ is harmonic for z (as well as $\log\left|\frac{\partial f}{\partial z}\right|$) Hence
$$\begin{pmatrix} \frac{\partial^2 s_n}{\partial t\partial \bar t} & \frac{\partial^2 s_n}{\partial t\partial \bar z} \\ \frac{\partial^2 s_n}{\partial \bar t\partial z} & \underset{\overset{\parallel}{0}}{\frac{\partial^2 s_n}{\partial z\partial \bar z}} \end{pmatrix} \geq 0$$

It follows that $\frac{\partial^2 s_n}{\partial \bar t\partial z} \equiv 0$ on $B_0 \times V_0$, and that $\frac{\partial s_n}{\partial z}(t,z)$ is holomorphic for (t,z) in $B_0\times V_0$. Because $s_n(t,z)$ is $\underbrace{\text{harmonic}}_{\text{for } z}$, and because $(*)$ is uniform, we see that
$$\frac{\partial s_n}{\partial z}(t,z) \to \left(\frac{\partial^2 f}{\partial z^2}\Big/\frac{\partial f}{\partial z}\right)(t,z)$$
uniformly on $B_0\times V_0$. We thus conclude that $\left(\frac{\partial^2 f}{\partial z^2}\Big/\frac{\partial f}{\partial z}\right)(t,z)$ is holomorphic for (t,z) in $B_0\times V_0$.

3rd step is proved.

<u>4th step</u> $f(t,z)$ is holomorphic for $(t,z)\in \mathcal{D}-B\times\{0\}$.

<u>Proof.</u> By 3rd step, we put $\varphi(t,z)= \frac{\partial^2 f}{\partial z^2}\Big/\frac{\partial f}{\partial z}$ on $\mathcal{D}^*-B\times\{0\}$, which is holomorphic for (t,z). In other words, $f(t,z)$ satisfies the differential equation:

$$\frac{\partial^2 f}{\partial z^2}(t,z) = \varphi(t,z)\,\frac{\partial f}{\partial z}(t,z)$$

such that $\quad [z\ f(t,z)]_{z=0} = 1 \quad ; \quad f(t,1)=0$

Since

$$f(t,z) = \frac{1}{z} + c_0(t)z + \cdots\cdots ,$$

we easily have

$$\frac{\frac{\partial^2 f}{\partial z^2}}{\frac{\partial f}{\partial z}} = -\frac{2}{z} + d_0(t) + d_1(t)z + \cdots\cdots . \left(=\varphi(t,z)\right)$$

Therefore, if we put

$$F(t,z) = -z^2\,\frac{\partial f}{\partial z}(t,z)\ ; \text{ and } \psi(t,z) = \varphi(t,z) + \frac{2}{z}$$

then $\psi(t,z)$ is holomorphic for (t,z) in $B\times U$ where $U=(|z|<2)$ and $F(t,z)$ satisfies the differential equation:

$$\frac{\partial F}{\partial z}(t,z) = \psi(t,z)$$

such that $\quad F(t,0) = 1.$

$$\therefore\quad F(t,z) = e^{\int_0^z \psi(t,z)dz} \qquad \text{for } B\times U$$

In particular, for $|z-1|<1$

$$\frac{\partial f}{\partial z} = -\frac{1}{z^2}\,e^{\int_0^z \psi(t,z)dz}$$

Since $f(t,1)=0$, we have

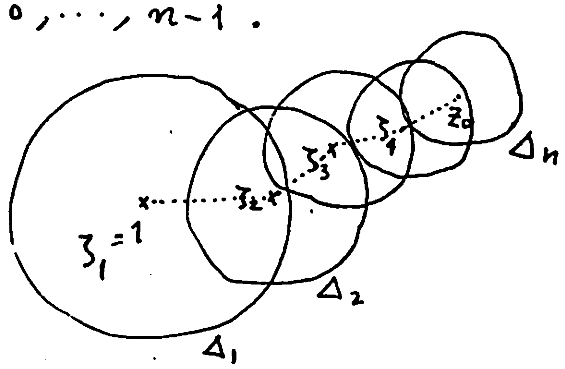
$$f(t,z) = -\int_1^z \frac{1}{\zeta^2}\,e^{\int_0^\zeta \psi(t,\zeta)d\zeta}\,d\zeta .$$

It follows that $f(t,z)$ is holomorphic for $(t,z)\in B\times(|z-1|<$

Take $t_0 \in B$ and $z_0 \in D^*(t_0) - \{0\}$. Then there exists a finite number of points $1 = \zeta_1, \cdots, \zeta_{n-1}, \zeta_n = z_0$ in $D^*(t_0) - \{0\}$ such that

$$\Delta_k : \{z : |z - \zeta_k| < |\zeta_{k+1} - \zeta_k|\} \subset\subset D^*(t_0) - \{0\}$$

for $k = 0, \cdots, n-1$.



Since $D^*(t_0)$ has no branch points, it follows that $\exists B_0 : |t - t_0| \subset\subset B$ such that $\Delta_k \Subset D^*(t) - \{0\}$ $(k = 1, \ldots, n)$ for all $t \in B_0$. We consider the Taylor development of $f(t, z)$ at z_1 in Δ_2 and have

$$f(t, z) = \sum_{n=0}^{\infty} a_n(t)(z - z_1)^n$$

So $a_n(t)$ $(n = 0, 1, 2, \cdots)$ are holomorphic for $t \in B_0$. Hence $f(t, z)$ is holomorphic for $(t, z) \in B_0 \times \Delta_2$. By the same procedure it turns out that $f(t, z)$ is holomorphic for (t, z) in $B_0 \times \Delta_n$, so that $f(t, z)$ is holomorphic near (t_0, z_0). $f(t, z)$ is holomorphic in $\mathcal{D}^* - B \times \{0\}$.

Since $f(t, z)$ is bounded in $\mathcal{D} - B \times \{|z| < 1\}$ and since the branch surface $\mathcal{S}$ is 1-dimensional analytic set in $\mathcal{D} - B \times \{|z| < 1\}$, it follows that $f(t, z)$ is holomorphic in $\mathcal{D} - B \times \{$

Q.E.D.

Corollary 2 Under the same circumstances as in Uniformization Theorem, we assume that each $D(t)$ $(t \in B)$ is conformally equivalent to the plane $\mathbb{C}$. Then $\mathcal{D}$ is biholomorphic to the product $B \times \mathbb{C}$.

Proof. We set $\alpha(t) = \mathbb{C} \cup \{\infty\} - f(t, D(t))$ for $t \in B$, and $A = \bigcup_{t \in B} (t, \alpha(t))$. Since $D(t) \sim \mathbb{C}$, $\alpha(t)$ is a point in $\mathbb{C}$. Moreover we set

$$\mathcal{E}^* = \bigcup_{t \in B} (t, E^*(t)) \subset B \times (\mathbb{C} \cup \{\infty\})$$

where $E^*(t) = f(t, D^*(t))$. Since $\mathcal{D}^*$ is pseudoconvex over $B \times \mathbb{C}$ and $f(t,z)$ is holomorphic in $\mathcal{D}^*$, it follows that $\mathcal{E}^*$ is pseudoconvex in $B \times (\mathbb{C} \cup \{\infty\})$. We put

$$K = \mathbb{C} \cup \{\infty\} - \mathcal{E}^*.$$

Then K is thus bounded and pseudoconcave set in $B \times \mathbb{C}$ and each fiber $K(t)$ is isolated set except $\alpha(t)$. It follows from Hartogs' theorem that $K - \bigcup_{t \in B} (t, \alpha(t))$ is locally analytic sets in $B \times \mathbb{C}$. If we take the derived set K' of K in $B \times \mathbb{C}$, then

$$K' = \bigcup_{t \in B} (t, \alpha(t)) \text{ and } K' \text{ is pseudoconcave in } B \times \mathbb{C}.$$

Again Hartogs theorem yields that $\alpha(t)$ is holomorphic for $t \in B$. Therefore, if we consider

$$T : \begin{cases} t = t \\ w = 1 \big/ (f(t,z) - \alpha(t)), \end{cases}$$

then $T(\mathcal{D}) = B \times \{|w| < \infty\}$. Q.E.D.

Note T. Nishino made a study on value distribution of entire functions of two complex variables:

Nouvelles recherches sur les fonctions entières de plusieurs variables complexes [I] ~ [V], J. Math. Kyoto Univ. 8 (1968) 49-100, 9 (1969) 221-274, 10 (1970) 245-271, 13 (1973) 217-272, 15 (1975) 527-553.

His main result is the following:

Let $f(x,y)$ be an entire function in $\mathbb{C}^2$. Assume that each irreducible component S_c of every constant surface $f(x,y)=c$ when $c \in \mathbb{C}$ is conformally equivalent to a punctured Riemann surface, that is

$S_c \sim$  a Compact Riemann surface − finite points

(Of course, a punctured Riemann surface does depend on S_c.)

Then there exists an analytic automorphism of $\mathbb{C}^2$;

$$\begin{cases} u = \xi(x,y) \\ v = \eta(x,y) \end{cases}$$

a polynomial $P(u,v)$ in $\mathbb{C}^2$ and an entire function $F(z)$ of one complex variable z in $\mathbb{C}$ such that

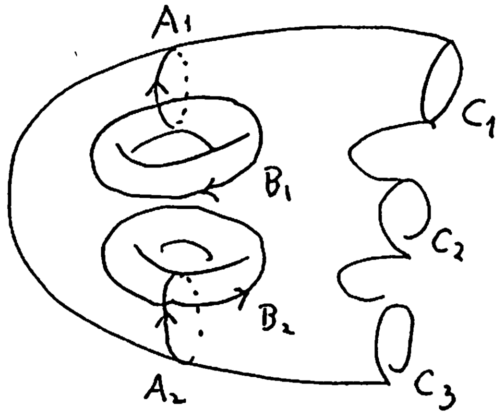
$$f(x,y) = F\left(P(\xi(x,y), \eta(x,y))\right).$$

Corollary 2 was proved in [II] as "Fundamental Lemma" under the condition that "$\mathfrak{D}$ is a Stein manifold".

Equivalencies of holomorphic families of non-compact Riemann surfaces

§1 Reproducing differentials

Let R be a finite Riemann surface, that is, there exists a Riemann surface $\tilde{R}$ such that $\tilde{R} \supset\supset R$ and the boundary ∂R of R in $\tilde{R}$ consists of finite number of closed curves.



We let R be of topological type (g, n) where g is of genus and n is the number of boundary components.
We set

$$\chi = 2g + n - 1. \ (\geqq 0)$$

If $\chi = 0$, then R is topologically equivalent to a disk.
If $\chi = 1$, then 〃 to an annulus.

In general, we can take, as a canonical homology basis, χ cycles

$$A_1, B_1, \cdots, A_g, B_g, C_1, \cdots, C_{n-1}$$

such that

$$\begin{cases} A_i \times B_j \text{ (intersection number)} = \delta_{ij} \\ A_i \times A_j = B_i \times B_j = A_i \times C_j = B_i \times C_j = C_i \times C_j = 0 \end{cases}$$

A_i B_i

Following L. V. Ahlfors - L. Sario "Riemann surfaces" Ch. V we set Γ_h the set of all the harmonic 1-form (differenti

$\omega = a(z)dx + b(z)dy$ where $z = x+iy$ on R such that its norm is finite:

$$\|\omega\|^2 = \iint_R \omega \wedge \overline{\omega^*} = \iint_R \left(|a|^2 + |b|^2\right) dx\,dy < \infty$$

where $\omega^* = -b(z)dx + a(z)dy$. With inner product,

$$(\omega, \sigma) = \iint_R \omega \wedge \bar{\sigma}^* = \iint_R (a\bar{c} + b\bar{d})\, dx\,dy$$

where $\omega = a\,dx + b\,dy$, $\sigma = c\,dx + d\,dy$, the space Γ_h becomes Hilbert space over $\mathbb{C}$.

Now, let γ be a smooth closed curve on R. Then there exists a unique $\sigma_\gamma \in \Gamma_h$ such that

$$(1.1) \qquad \int_\gamma \omega = (\omega, \sigma_\gamma^*)_R \qquad \text{(by Riesz's Theorem)}$$

for any $\omega \in \Gamma_h$. σ_γ is called the <u>reproducing differential</u> for (R, γ). The construction of σ_γ is as follows:

<u>Case 1</u> $\gamma = C_i$ (dividing cycle)

We consider the harmonic function $u(z)$ on R such that

$$u(z) = \begin{cases} 1 & \text{on } C_i \\ 0 & \text{on } \partial R - C_i \end{cases}$$

Define $\sigma_\gamma = du$.

Then σ_γ satisfies (1.1)

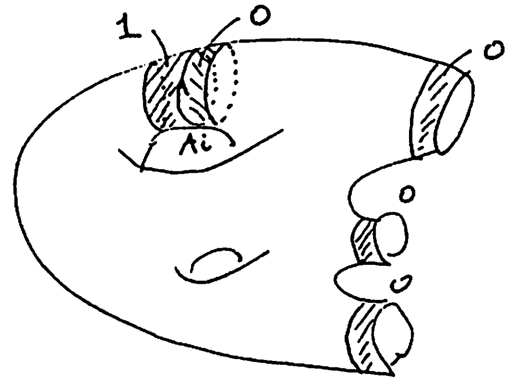
$\ulcorner$ ($\because$) First, it is clear that $\sigma_\gamma \in \Gamma_h$. Next, given $\omega \in \Gamma_h$, we form

$$(\omega, du^*) = \iint_R \omega \wedge (du^*)^* = \iint_R du \wedge \omega$$

$$= \iint_R d(u \cdot \omega) = \int_{\partial R} u\,\omega = \int_{C_i} \omega \quad \lrcorner$$

Consequently, we see that
σ_γ has no period, i.e., exact on R;
$\sigma_\gamma = 0$ along ∂R, i.e., $u = $ const. on each C_j $(\subset \partial R)$

<u>Case 2</u> $\gamma = A_i$



We form a real-valued function $f(z)$ of class C^∞ on $R - A_i$ such that

$$f(z) \equiv \begin{cases} 1 & \text{on the left-hand side of } A_i \\ 0 & \text{on the right-hand side of } A_i \\ & \text{and a neighborhood of } \partial R \end{cases}$$

It follows that, although $f(z)$ is not uniform on R, df is a closed differential on R. It is well-known that the space Γ_c of the <u>closed differential</u> on R with finite norm has the following orthogonal decomposition:

$$\Gamma_c = \Gamma_h \dotplus \Gamma_{eo}$$

where Γ_{eo} is the completion of the set of all the differentials of support compact C^∞-function on R.

$$\therefore\ df = \underset{\Gamma_h}{\underset{\Uparrow}{\sigma_{A_i}}} + \underset{\Gamma_{eo}}{\underset{\Uparrow}{dF_{eo}}} \quad \text{(unique representation)}$$

Then we get

$$\int_{A_i} \omega = (\omega, \sigma_{A_i}^*)$$

$\ulcorner$ ∵ $(\omega, \sigma_{A_i}^*) = \left(\omega, (df)^* - (dF_{eo})^*\right)$

$= (\omega, (df)^*) - (\omega, (dF_{eo})^*)$

On the other hand, $\Gamma_h \perp \Gamma_{eo}^*$

i.e., $(\omega, dF_{eo}^*) = -\iint_R \omega \wedge dF = \iint_R d(F\omega) = \int_{\partial R} F\omega = 0.$

It follows that

$$(\omega, \sigma_{A_i}^*)_R = (\omega, (df)^*)_R$$

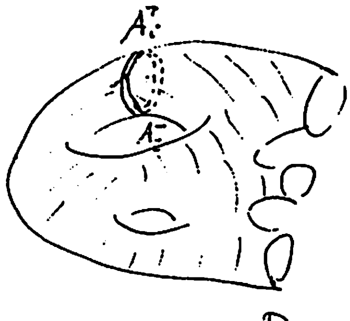
$$= (\omega, (df)^*)_{R-A}$$

$$= \iint_{R-A_i} f\omega$$

$$= \int_{\partial R - A_i} f\omega$$

$$= \int_{A_i^+ - A_i^-} f\omega \qquad (\because f = 0 \text{ on } \partial R)$$

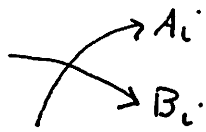
$$= \int_{A_i^+} \omega = \int_A \omega \qquad \lrcorner$$



Consequently, we see that

- σ_{A_i} has no periods except B_i such that

$$\int_{B_i} \sigma_{A_i} = A_i \times B_i = 1 \;;$$



- $\sigma_{A_i} = 0$ along ∂R. Precisely, there exists a harmonic function $u_i(z) = f - F_{eo}$ on $R - A_i$ such that $u_i(z) = 0$ on ∂R and such that $\sigma_{A_i} = du_i$ on R.

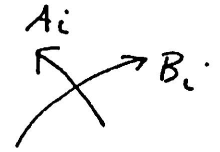
<u>Case 3</u> $\gamma = B_i$

By the same construction of σ_{A_i}, we get σ_{B_i} such that

$\int_{B_i} \omega = (\omega, \sigma_{B_i}{}^{*})$ for all $\omega \in \Gamma_h$;

σ_{B_i} has no periods except A_i such that

$$\int_{A_i} \sigma_{B_i} = -1 = B_i \times A_i \; ;$$



$\sigma_{B_i} = 0$ along ∂R. Precisely, there exists a harmonic function $v_i(z)$ on $R - B_i$ such that $dv_i = \sigma_{B_i}$ on R and such that $v_i(z) = 0$ on ∂R.

Now we consider the following subset Γ_1 of Γ_h such that

$$\Gamma_1 = \left\{ \sigma \in \operatorname{Re} \Gamma_h \;\middle|\; \sigma = \sum_{i=1}^{g} x_i \sigma_{A_i} + y_i \sigma_{B_i} + \sum_{j=1}^{n-1} z_j \sigma_{C_j} \quad \text{where } x_i, y_i, z_j \in \mathbb{R} \right\}$$

So Γ_1 is of real dimension $\chi = 2g + n - 1$. It is convenient to write

$$\Gamma_1 = \left\{ \sigma_\gamma \;\middle|\; \gamma = \sum_{i=1}^{g} x_i A_i + y_i B_i + \sum_{j=1}^{n-1} z_j C_j \quad \text{where } x_i, y_i, z_j \in \mathbb{R} \right\}$$

Let $\gamma = \sum_{i=1}^{g} x_i A_i + y_i B_i + \sum_{j=1}^{n-1} z_j C_j$ and construct σ_γ. Then we see that

(P1) $\int_\gamma \omega = (\omega, \sigma_\gamma{}^{*})$ for all $\omega \in \Gamma_h$;

(P2) σ_γ has periods as follows :

$$\int_{A_i} \sigma_\gamma = y_i \, , \quad \int_{B_i} \sigma_\gamma = -x_i \, , \quad \int_{C_j} \sigma_\gamma = 0$$

(P3) there exists a harmonic function $u_\gamma(z)$ on $R - [A_i \cup B_i]$ such that $\sigma_\gamma = du_\gamma$ on R and such that $u_\gamma(z) = z_j$ on C_j $(1 \leq j \leq n-1)$ and $= 0$ on C_n

We introduce the following notations:

Γ_a = the space of the analytic differentials on R with finite norm.

$$\Omega_\gamma = \sigma_\gamma + i\sigma_\gamma^* \in \Gamma_a$$

We note that

$$\|\Omega_\gamma\|^2 = 2\|\sigma_\gamma\|^2 \; ; \quad \Omega_\gamma = 2\frac{\partial u_\gamma}{\partial z}dz$$

On ∂R, we have

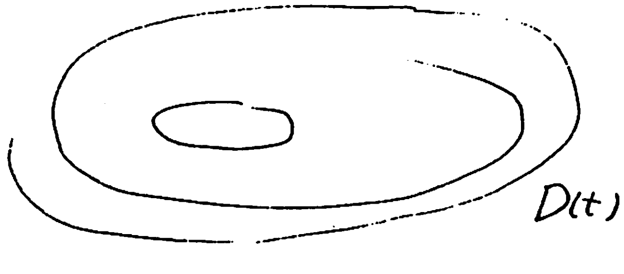
$$(1.2) \qquad \Omega_\gamma = i\sigma_\gamma^* = 2\frac{\partial u_\gamma}{\partial z}dz$$

The reciprocal μ_γ of $\|\sigma_\gamma\|^2$ is called the harmonic measure for (R, γ) :

$$\mu_\gamma = \frac{1}{\|\sigma_\gamma\|^2}$$

§2 Variation formula for $\|\sigma_\gamma(t,\cdot)\|^2_{D(t)}$.

Let $B = (|t| < \rho)$ and let $\mathcal{D}$ be a domain spreading over $B \times \mathbb{C}$ without branch surfaces, so each fiber $D(t)$ $(t \in B)$ is a Riemann surface sheeted on $\mathbb{C}$ without branch points.



Abstractly,

We assume that the variation

$$\mathcal{D} : t \to D(t) \quad (t \in B)$$

is smooth variation of smooth domain $D(t)$. Precisely there exists a double $(\tilde{\mathcal{D}}, \psi)$ where $\tilde{\mathcal{D}} \supset \mathcal{D}$ and ψ is a real valued C^∞-fn on $\tilde{\mathcal{D}}$ such that

$$\mathcal{D} = \{ (t,z) \in \tilde{\mathcal{D}} \mid \psi(t,z) < 0 \} ;$$

$$\partial\mathcal{D} = \{ (t,z) \in \tilde{\mathcal{D}} \mid \psi(t,z) = 0 \} ;$$

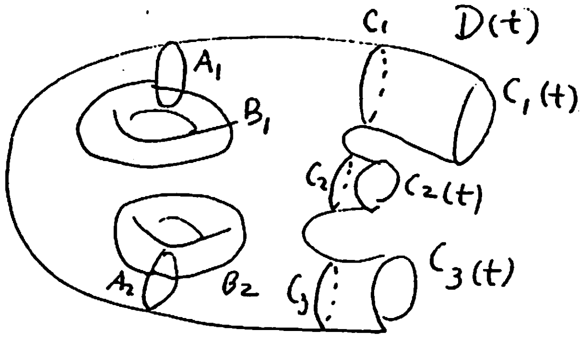
$$\mathrm{Grad}_{(t,z)} \psi(t,z) \neq 0 \quad \text{for all } (t,z) \in \partial\mathcal{D} ;$$

moreover, for each $t \in B$, the double $(\tilde{D}(t), \psi(t,\cdot))$ defines the domain $D(t)$, i.e.,

$$\frac{\partial \psi}{\partial z}(t,z) \neq 0 \quad \text{for } z \in \partial D(t).$$

$$D(t) = \{ z \in \tilde{D}(t) \mid \psi(t,z) < 0 \} .$$

Under this condition, $\mathcal{D}$ is diffeomorphically equivalent to the trivial $B \times D(0)$. Hence the topological type (g, n) of $D(t)$ is independent of $t \in B$. We can take a canonical base : A_i, B_i, C_j $(i = 1, \dots, g,\ j = 1, \dots, n-1)$ of the cycles on $D(0)$ which realizes also a canonical base of the cycles on $D(t)$. If necessary, we choose $B_0 (= |t| < \rho_0) \subset\subset B$.



Let $x_i, y_i, z_j \in \mathbb{R}$. and Construct $\sigma_\gamma(t,\cdot) \in \Gamma_1(D(t))$

where $\gamma = \sum x_i A_i + y_i B_i + \sum z_j C_j$. From (P1)~(P3), we choose a harmonic function $u_\gamma(t,z)$ on $D(t) - [A_i, B_i]_{i=1,\dots,g}$ such that

$$\sigma_\gamma(t,z) = du_\gamma(t,z) \text{ on } D(t);$$

$$\int_{A_i} du_\gamma(t,z) = y_i, \quad \int_{B_i} du_\gamma(t,z) = -x_i, \quad \int_{C_j} du_\gamma(t,z) = 0;$$

$$u_\gamma(t,z) = z_j \text{ on } C_j(t) \quad (j=1,\dots,n-1)$$
$$= 0 \text{ on } C_n(t)$$

We remark that x_i, y_i, z_j does not depend on $t \in B$.

For the sake of simplicity, we assume that $\partial\mathcal{D}$ and each $\partial D(t)$ are real analytic.

Hence, (if necessary, take a small enough $B_0 (\subset B)$) $\sigma_\gamma(t,z)$, or equivalently $u_\gamma(t,z)$ is harmonic on $\hat{D}(t)$ such that $\hat{D}(t) \supset\supset D(t) \cup D(0)$ for all $t \in B$.

We want to represent $\|\sigma_\gamma(t,z)\|^2_{D(t)}$ by integral along $\partial D(0)$

$$\|\sigma_\gamma(t,z)\|^2_{D(t)} = (\sigma^*_\gamma(t,z), \sigma^*_\gamma(t,z))_{D(t)}$$

$$= \int_\gamma \sigma^*_\gamma(t,z) \qquad \text{by (P1)}$$

$$= (\sigma^*_\gamma(t,z), \sigma^*_\gamma(0,z))_{D(0)}$$

$$= (\sigma_\gamma(t,z), \sigma_\gamma(0,z))_{D(0)}$$

$$= (du_\gamma(t,z) - du_\gamma(0,z), \sigma_\gamma(0,z))_{D(0)} + \|\sigma_\gamma(0,z)\|^2_{D(0)}$$

$\therefore \quad \frac{\partial u_\gamma}{\partial t}(t,z)$ is harmonic fn on $D(t)$.

$$\therefore \quad \|\sigma_\gamma(t,z)\|^2_{D(t)} = \iint_{D(0)} d(u_\gamma(t,z) - u_\gamma(0,z)) \wedge \sigma_\gamma^*(0,z) \quad - \|\sigma_\gamma(0,z)\|^2_{D(0)}$$

Since $u_\gamma(t,z) - u_\gamma(0,z)$ is a harmonic function on the whole $D(0)$ and since $\sigma_\gamma(0,z) = du_\gamma(0,z)$, it becomes

$$= \int_{\partial D(0)} \left(u_\gamma(t,z) - u_\gamma(0,z)\right) du_\gamma^*(0,z) \quad - \|\sigma_\gamma(0,z)\|^2_{D(0)}$$

where $u_\gamma^*(0,z)$ is a conjugate function (locally) of $u_\gamma(0,z)$.
It follows that

$$(2.1) \quad \left[\frac{\partial}{\partial t} \|\sigma_\gamma(t,z)\|^2_{D(t)}\right]_{t=0} = \int_{\partial D(0)} \left[\frac{\partial u_\gamma(t,z)}{\partial t}\right]_{t=0} du_\gamma^*(0,z)$$

$$(2.2) \quad \left[\frac{\partial^2}{\partial t \partial \bar t} \|\sigma_\gamma(t,z)\|^2_{D(t)}\right]_{t=0} = \int_{\partial D(0)} \left[\frac{\partial^2 u_\gamma(t,z)}{\partial t \partial \bar t}\right]_{t=0} du_\gamma^*(0,z).$$

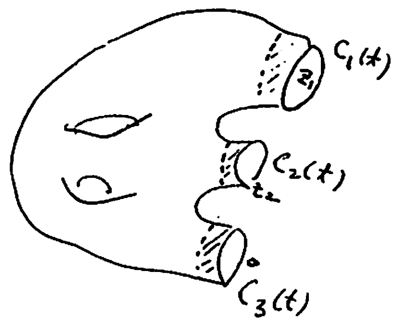
We note by (P3) that $u_\gamma(t,z)$ is defined near $\partial\mathcal{D}$ and

$$u_\gamma(t,z) = z_j \quad \text{on } C_j(t) \quad (1 \leqq j \leqq n)$$

where $z_n = 0$.

z_j $(1 \leqq j \leqq n)$ does not depend on $t \in B$.

It is clear that $\frac{\partial u_\gamma}{\partial z}(0,z) = 0$ has only finite number of zero on $\partial D(0)$.



Take $\zeta \in \partial D(0)$ such that $\frac{\partial u_\gamma}{\partial z}(0,\zeta) \neq 0$.

Assume that $\zeta \in C_j(0)$.

<u>Case 1</u> $u_\gamma(0,z) < z_j$ for $z \in D(0)$ close to ζ.

$u_\gamma(t,z) - z_j$ (then) defines the boundary $\partial\mathcal{D}$ at $(0,\zeta)$, and hence

$$k_2(t,z) = \frac{L\psi}{\left|\frac{\partial \psi}{\partial z}\right|^3} = \frac{L(u_\gamma - z_j)}{\left|\frac{\partial (u - z_j)}{\partial z}\right|^3} \quad \text{at } (0,\zeta)$$

where $L = \partial^2/\partial t\partial\bar t\, |\partial/\partial z|^2 - 2\,\mathrm{Re}\{\partial^2/\partial \bar t\partial z\ \partial/\partial t\ \partial/\partial \bar z\} + \partial^2/\partial z\partial\bar z\, |\partial/\partial t|^2$

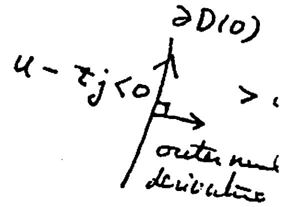
$$\therefore\ \frac{\partial^2 u_\gamma}{\partial t\partial\bar t} = k_2\left|\frac{\partial u_\gamma}{\partial z}\right| + 2\,\mathrm{Re}\left\{\frac{\frac{\partial^2 u_\gamma}{\partial\bar t\partial z}\frac{\partial u_\gamma}{\partial t}}{\frac{\partial u}{\partial z}}\right\} \quad \text{at } (0,\zeta)$$

Case 2 $\quad u(0,z) > z_j$ for $z \in D(0)$ close to ζ.

Then $z_j - u(t,z)$ is one of Levi's functions of $\partial\mathcal{D}$ at $(0,\zeta)$
By the same reasoning as above we get

$$\frac{\partial^2 u_\gamma}{\partial t\partial\bar t} = -k_2\left|\frac{\partial u_\gamma}{\partial z}\right| + 2\,\mathrm{Re}\left\{\frac{\frac{\partial^2 u_\gamma}{\partial\bar t\partial z}\frac{\partial u_\gamma}{\partial t}}{\frac{\partial u_\gamma}{\partial z}}\right\} \quad \text{at } (0,\zeta)$$

On the other hand,



In the case 1, we have

$$du_\gamma^* = \frac{\partial u_\gamma}{\partial n_z}\,ds > 0 \quad \text{at } (0,\zeta).$$

In the case 2, we have

$$du_\gamma^* = \frac{\partial u_\gamma}{\partial n_z}\,ds < 0 \quad \text{at } (0,\zeta)$$

Since $\frac{\partial u_\gamma}{\partial z}dz = \frac{1}{2}(du_\gamma + i\,du_\gamma^*) = \frac{i}{2}\,du_\gamma^*$ on $\partial D(t)$,
it follows that

$$\left|\frac{\partial u_\gamma}{\partial z}\right| = \begin{cases} \frac{1}{2}\frac{\partial u_\gamma}{\partial n_z} & \text{in case 1} \\ -\frac{1}{2}\frac{\partial u_\gamma}{\partial n_z} & \text{in case 2.} \end{cases}$$

In both cases we get

$$\frac{\partial^2 u}{\partial t\partial\bar t}\,du_\gamma^* = \frac{1}{2}k_2\left|\frac{\partial u_\gamma}{\partial n_z}\right|^2 ds_z + 2\,\mathrm{Re}\left\{\frac{\frac{\partial^2 u_\gamma}{\partial\bar t\partial z}\frac{\partial u}{\partial t}}{\frac{\partial u}{\partial z}}\cdot du_\gamma^*\right\} \quad \text{at } (0,\zeta)$$

$$= 2k_2\left|\frac{\partial u_\gamma}{\partial z}\right|^2 ds + 4\,\mathrm{Im}\left\{\frac{\partial^2 u_\gamma}{\partial\bar t\partial z}\frac{\partial u_\gamma}{\partial t}\,dz\right\} \quad \text{at } (0,\zeta)$$

By (2.2) we have

$$\left[\frac{\partial^2}{\partial t\partial\bar{t}}\|\sigma_\gamma(t,z)\|^2_{D(t)}\right]_{t=0}$$

$$= 2\int_{\partial D(0)} k_2 \left|\frac{\partial u_\gamma}{\partial z}\right|^2 ds_z + 4\,\mathrm{Im}\int_{\partial D(0)}\left\{\frac{\partial^2 u_\gamma}{\partial\bar{t}\partial z}\frac{\partial u_\gamma}{\partial t}dz\right\}$$

Since $\frac{\partial u_\gamma}{\partial t}(t,z)$ is harmonic function in $D(0)$, we have by Green's formula

$$\int_{\partial D(0)}\frac{\partial^2 u_\gamma}{\partial\bar{t}\partial z}\frac{\partial u_\gamma}{\partial t}dz = \iint_{D(0)} d\left(\frac{\partial^2 u_\gamma}{\partial\bar{t}\partial z}\frac{\partial u_\gamma}{\partial t}dz\right)$$

$$= \iint_{D(0)}\left|\frac{\partial^2 u_\gamma}{\partial\bar{t}\partial z}\right|^2 \underbrace{d\bar{z}\wedge dz}_{2i\,dxdy}$$

Consequently,

$$(2.3)\quad \left[\frac{\partial^2}{\partial t\partial\bar{t}}\|\sigma_\gamma(t,z)\|^2_{D(t)}\right]_{t=0} = 2\int_{\partial D(0)} k_2\left|\frac{\partial u_\gamma}{\partial z}\right|^2 ds_z + 8\iint_{D(0)}\left|\frac{\partial^2 u_\gamma}{\partial\bar{t}\partial z}\right|^2 dxdy.$$

By (2.1) we have

$$(2.4)\quad \left[\frac{\partial}{\partial t}\|\sigma_\gamma(t,z)\|^2_{D(t)}\right]_{t=0} = \frac{2}{i}\int_{\partial D(0)}\frac{\partial u_\gamma}{\partial t}\frac{\partial u_\gamma}{\partial z}dz$$

$$= \frac{2}{i}\iint_{D(0)} d\left(\frac{\partial u_\gamma}{\partial t}\frac{\partial u_\gamma}{\partial z}dz\right)$$

$$= \frac{2}{i}\iint_{D(0)}\frac{\partial^2 u_\gamma}{\partial t\partial\bar{z}}\frac{\partial u_\gamma}{\partial z}d\bar{z}\wedge dz$$

$$= 4\iint_{D(0)}\frac{\partial^2 u_\gamma}{\partial t\partial\bar{z}}\frac{\partial u_\gamma}{\partial z}dxdy$$

By definition, $\|\sigma_\gamma(0,z)\|^2_{D(0)} = 4 \iint_{D(0)} \left|\frac{\partial u_\gamma(0,z)}{\partial z}\right|^2 dxdy$.
It follows that

$$\left[\frac{\partial}{\partial t}\|\sigma_\gamma(t,z)\|^2_{D(t)}\right]^2_{t=0} \leqq 16 \left|\iint_{D(0)} \frac{\partial^2 u_\gamma}{\partial t\,\partial \bar{z}} \, \frac{\partial u_\gamma}{\partial z}\, dxdy\right|^2$$

$$= \frac{1}{2}\left\{8\iint_{D(0)} \left|\frac{\partial^2 u}{\partial t \partial \bar z}\right|^2_{t=0} dxdy\right\}\left\{4\iint_{D(0)}\left|\frac{\partial u_\gamma}{\partial z}\right|^2_{t=0} dxdy\right\}$$

$$(2.5)\quad 2\left[\frac{\partial}{\partial t}\|\sigma_\gamma(t,z)\|^2_{D(t)}\right]^2_{t=0} \leq \left\{8 \iint_{D(0)} \left|\frac{\partial^2 u}{\partial \bar t \partial \bar z}\right|^2_{t=0} dxdy\right\}\cdot \|\sigma_\gamma(0,z)\|^2_{D(0)}.$$

§3. Function-theoretic variation.

Under the same circumstance as in §2, we assume that $\mathcal{D}$ is a pseudoconvex domain over $B\times\mathbb{C}$, i.e., the variation

$$\mathcal{D} : t \longrightarrow D(t) \quad (t\in B)$$

is a function-theoretic variation.

Then we have $k_2(t,z) \geqq 0$ on $\partial\mathcal{D}$. It follows by (2.3) the following

<u>Lemma 1</u>

$$(3.1)\quad \frac{\partial^2}{\partial t \partial \bar t}\|\sigma_\gamma(t,z)\|^2_{D(t)} \geqq 8 \iint_{D(t)} \left|\frac{\partial^2 u_\gamma}{\partial \bar t \partial \bar z}\right|^2 dxdy \quad \text{for } \forall t\in B$$

and that the equality holds for all $t\in B$ if and only if $k_2(t,z)=0$ on $\partial\mathcal{D}$, i.e., $\mathcal{D}$ is Levi-flat

By (2.5) we also have

(3.2) $$\frac{\partial^2}{\partial t \partial \bar{t}} \frac{1}{\|\sigma_\gamma(t,z)\|^2_{D(t)}} \geq 0 \quad \text{for } t \in B.$$

Namely, the harmonic measure $\mu_\gamma(t) = \dfrac{1}{\|\sigma_\gamma(t,z)\|^2_{D(t)}}$ is positive superharmonic function on B.

We will give two applications of (3.1) and (3.2):

<u>Theorem 1</u> Let (g,n) be the topological type of $D(t)$. Assume that $\chi = 2g+n-1 \geq 2$. Assume that $\mathcal{D}$ is pseudoconvex domain over $B \times \mathbb{C}$. Then, if each $D(t)$ is conformally equivalent to $D(0)$, then $\mathcal{D}$ is biholomorphically equivalent to the product $B \times D(0)$.

<u>Proof.</u> Since $\chi \geq 2$, we have two independent reproducing differentials $\sigma_1(t,z)$ and $\sigma_2(t,z)$ for $(D(t),\gamma_1)$ and $(D(t),\gamma_2)$. By (3.1), we have, for each $t \in B$,

$$8 \iint_{D(t)} \left| \frac{\partial^2 u_i}{\partial \bar{t} \partial z} \right|^2 dx\,dy \leqq \frac{\partial^2}{\partial t \partial \bar{t}} \|\sigma_i(t,z)\|^2_{D(t)} \qquad (i=1,2)$$

By assumption that $D(t) \sim D(0)$ as Riemann surface, we have

$$\|\sigma_i(t,z)\|^2_{D(t)} = \|\sigma_i(0,z)\|^2_{D(0)} \quad \text{for all } t \in B.$$

$$\therefore \quad \frac{\partial^2}{\partial t \partial \bar{t}} \|\sigma_i(t,z)\|^2_{D(t)} \equiv 0 \quad \text{for } t \in B.$$

It follows from the above inequality that

$$\frac{\partial^2 u_i}{\partial \bar{t}\, \partial z} \equiv 0 \quad \text{on } D(t).$$

that is, $\frac{\partial u_i'}{\partial z}(t,z)$ is holomorphic for (t,z).

Let $\Omega_i(t,z) = \sigma_i(t,z) + i\sigma_i(t,z)^* = 2\,\frac{\partial u_i'(t,z)}{\partial z}dz$, which is holomorphic differential on $D(t)$. Hence,

$$w = f(t,z) = \frac{\Omega_2(t,z)}{\Omega_1(t,z)} = \frac{\partial u_2}{\partial z}(t,z) \Big/ \frac{\partial u_1}{\partial z}(t,z)$$

defines a meromorphic function on $D(t)$ such that $f(t,z)$ holomorphically depend on $t \in B$.

On the other hand, $D(t) \sim D(0)$ as Riemann surface teaches us

$$\Omega_i(t,z) = \Omega_i(0,z) \quad (i=1,2)$$

as holomorphic differentials

It follows that $f(t, D(t))$, which is a ramified Riemann surface over $\mathbb{P}^1 = \{|w| \leqq +\infty\}$, is independent of $t \in B$, say $D^*(0)$. Consequently, $\mathcal{D}$ is mapped onto $B \times D^*(0)$ by the meromorphic mapping for (t,z)

$$\begin{cases} t = t \\ w = f(t,z). \end{cases}$$

Since $D^*(0) \sim D(0)$ as Riemann surfaces, we prove Theorem 1.

q.e.d.

<u>Theorem 2</u> Assume that $B = \mathbb{C}$ and $\chi \geqq 2$. Assume that $\mathcal{D}$ is a pseudoconvex domain over $\mathbb{C} \times \mathbb{C}$. Then $\mathcal{D}$ is biholomorphic to the trivial $\mathbb{C} \times D(0)$

<u>Proof.</u> By (3.2), we see that the harmonic module $\mu_i(t) = 1/\|\sigma_{\gamma_i}(t,z)\|^2_{D(t)}$ is positive superharmonic function for t in $\mathbb{C}$. Hence it must be constant, so that $\|\sigma_{\gamma_i}(t,z)\|^2_{D(t)}$ is constant for $t \in \mathbb{C}$.

$$\therefore \quad \frac{\partial^2}{\partial t \partial \bar{t}} \| \sigma_{\gamma_i}(t, z) \|^2_{D(t)} \equiv 0 \text{ for } t \in \mathbb{C}$$

It follows by the same reasoning as Theorem 1 that $\mathcal{D}$ is biholomorphically equivalent to the product $\mathbb{C} \times D(0)$.

q.e.d.

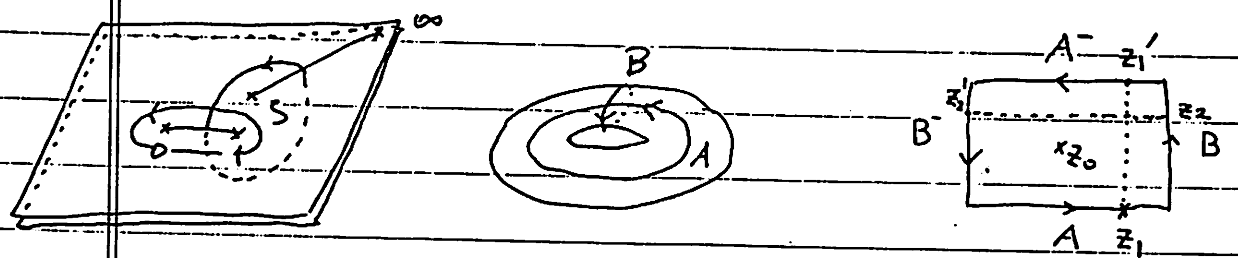
An application of variations of harmonic modules

First we recall Picard's Theorem and its proof:

Picard's Theorem

Let $f(t)$ be a holomorphic function for t in $\mathbb{C}$. If $f(t) \neq 0, 1, \infty$ for all $t \in \mathbb{C}$, then $f(t)$ must be a constant.

Proof. Let $\zeta \neq 0, 1, \infty$. Consider the two-sheeted covering surface $S (= S(\zeta))$ whose branch points are $0, 1, \zeta, \infty$ and is of order 1



We draw 2 cycles A, B as above, and cut S along A and B. We thus have a simply connected domain $R = S - \{A, B\}$ such that $\partial R = A + B + A^- + B^-$. Consider the following holomorphic differential

$$\omega(z) = \omega(\zeta, z) = \frac{1}{\sqrt{z(z-1)(z-\zeta)}}$$

and set

$$F(z) = \int_{z_0}^{z} \omega(z) \quad \text{for } z \in R$$

where z_0 is a fixed point in R. $F(z)$ is holomorphic function in R. By Cauchy's Theorem, we have for $z_1 \in A$ and $z_2 \in B$

$$F(z_1') - F(z_1) = \int_B \omega(z) \quad ;$$

$$F(z_2') - F(z_2) = - \int_A \omega(z)$$

Here we consider the following integral:

$$\|\omega(z)\|^2_S = \|\omega(\zeta,z)\|^2_{S(\zeta)}$$

$$= i\iint_S \left|\frac{1}{\sqrt{z(z-1)(z-3)}}\right|^2 dz\wedge d\bar{z} = -i\iint_{S-\{A,B\}} \frac{dz}{\sqrt{z(z-1)(z-3)}}\cdot\overline{\omega(z)}$$

$$= i\iint_R d\left(F(z)\,\overline{\omega(z)}\right)$$

$$= -i\int_{A+B+A^-+B^-} F\,\overline{\omega(z)}$$

$$= i\left\{-\int_A \left(F(z')-F(z)\right)\overline{\omega(z)} + \int_B \left(F(z')-F(z)\right)\overline{\omega(z)}\right\}$$

$$= i\left\{\int_A \omega(z)\int_B \overline{\omega(z)} - \int_A \overline{\omega(z)}\int_B \omega(z)\right\}.$$

Or equivalently,

$$\int_A \omega(z)\int_B \overline{\omega(z)} - \int_A \overline{\omega(z)}\int_B \omega(z) = -i\|\omega(z)\|^2_S \neq 0$$

Hence $\int_A \omega(z) \neq 0$. Both sides being divided by $\left|\int_A \omega(z)\right|^2 > 0$ we have

$$\frac{\int_B \omega(z)}{\int_A \omega(z)} - \overline{\left(\frac{\int_B \omega(z)}{\int_A \omega(z)}\right)} = i\left\|\frac{\omega(z)}{\int_A \omega(z)}\right\|^2_S.$$

Consequently, if we set

$$\varphi(\zeta) = \frac{\int_B \omega(\zeta,z)}{\int_A \omega(\zeta,z)},$$

then $\operatorname{Im}\varphi(\zeta) > 0$.

By concrete construction:

$$\varphi(\zeta) = \frac{\int_B \frac{1}{\sqrt{z(z-1)(z-\zeta)}}\,dz}{\int_A \frac{1}{\sqrt{z(z-1)(z-\zeta)}}\,dz}$$

$\varphi(\zeta)$ is (locally) holomorphic function for ζ. That means $\varphi(\zeta)$ is a holomorphic function of the universal covering space of $\mathbb{P}^1 - \{0, 1, \infty\}$. It is clear that $\varphi(\zeta)$ is not constant. For, because of

$$\varphi(\zeta) = -\int_1^\zeta \frac{dz}{\sqrt{z(z-1)(z-\zeta)}} \Big/ \int_0^1 \frac{dx}{\sqrt{x(x-1)(x-\zeta)}},$$

we have $\varphi(\zeta) \to +\infty$ as $\zeta \to +\infty$ along x-axis.

Let us prove Picard's Theorem. Let $f(t)$ be holomorphic in $\mathbb{C}$ such that $f(t) \neq 0, 1, \infty$. Putting $\zeta = f(t)$, we form the Riemann surface $S(\zeta(t)) (\equiv S(t))$ and the function $w = \varphi(f(t))$. By monodromy theorem, $\varphi(f(t))$ is holomorphic for t in $\mathbb{C}$, whose imaginary part is > 0. Hence $\varphi(f(t)) \equiv$ const. for $t \in \mathbb{C}$, so that $f(t) \equiv$ const. for $t \in \mathbb{C}$.

Q.E.D.

By use of the same idea of making a double sheeted Riemann surface over $\mathbb{C}$, we can prove the following

Theorem 1

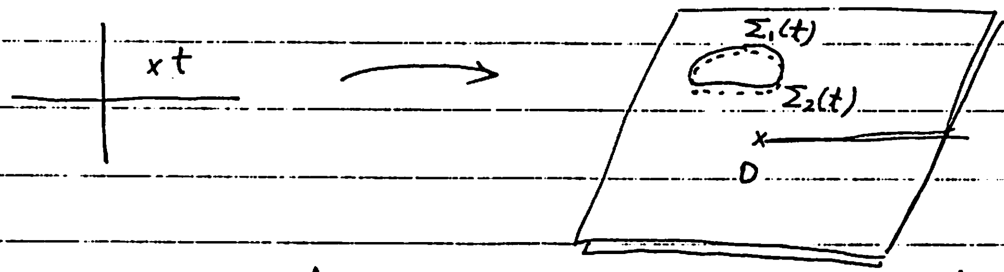
To each $t \in \mathbb{C}$ we correspond a closed Jordan domain $\Sigma(t)$ in the z-plane such that

(i) $\Sigma(t) \not\ni 0$;

(ii) If we put $\Sigma = \bigcup_{t \in \mathbb{C}} (t, \Sigma(t))$, then

Σ is smooth pseudoconcave set in $\mathbb{C}\times\mathbb{C}$.
Then Σ is biholomorphically equivalent to $\mathbb{C}\times\Sigma(0)$.

Proof.



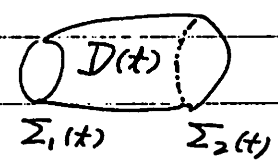
We consider a doubly sheeted Riemann surface R whose branch points are 0 and ∞ of order 1. Condition (i) implies that there exist two copies $\Sigma_1(t)$, $\Sigma_2(t)$ of $\Sigma(t)$ in R. Put

$$D(t) = R - \Sigma_1(t)\cup\Sigma_2(t);$$

$$\mathcal{D} = \bigcup_{t\in\mathbb{C}} (t, D(t))$$

By condition (ii), $\mathcal{D}$ is a smooth pseudoconvex domain in $\mathbb{C}\times$. Since each $D(t)$ $(t\in\mathbb{C})$ is topologically annulus, we have the harmonic function $u(t,z)$ on $D(t)$ such that

$$u(t,z) = \begin{cases} 0 & \text{on } \Sigma_1(t) \\ 1 & \text{on } \Sigma_2(t) \end{cases}$$



We form the integral

$$\|du(t,z)\|^2_{D(t)} = \iint_{D(t)} \left[\left(\frac{\partial u(t,z)}{\partial x}\right)^2 + \left(\frac{\partial u(t,z)}{\partial y}\right)^2\right] dxdy$$

and put

$$\mu(t) = 1\Big/\|du(t,z)\|^2_{D(t)}.$$

This is known as the harmonic module for $D(t)$.
We have shown that

$\mu(t)\geq 0$ is superharmonic on $\mathbb{C}$.

Consequently, $\mu(t) \equiv \text{const.}$ for $t \in \mathbb{C}$.

$$\therefore \|du(t,\cdot)\|^2_{D(t)} = \text{const.} \text{ for } t \in \mathbb{C}.$$

Moreover, we got

$$0 = \frac{\partial^2}{\partial t \partial \bar{t}} \|du(t,\cdot)\|^2_{D(t)}$$

$$= 2\int_{\partial D(t)} k_2(t,z) \left|\frac{\partial u(t,z)}{\partial z}\right|^2 ds_z + 8 \iint_{D(t)} \left|\frac{\partial^2 u(t,z)}{\partial \bar{t} \partial z}\right|^2 dx\,dy \geq$$

where $k_2(t,z)$ is a kind of mean curvature of $\partial\mathcal{D}$ such that $k_2(t,z) \geq 0$ if and only if $\mathcal{D}$ is pseudoconvex and such that $k_2(t,z) = 0$ if and only if $\mathcal{D}$ is Levi flat. Condition (ii) implies that

(a) $k_2(t,z) = 0$ on $\partial\mathcal{D} = \bigcup_{t \in \mathbb{C}} (t, \partial D(t))$

(b) $\dfrac{\partial^2 u(t,z)}{\partial \bar{t} \partial z} \equiv 0$ on $\mathcal{D}$

By Levi's Theorem and Monodromy Theorem we have the following function $f(t,\zeta)$ where $t \in \mathbb{C}$ and $\zeta \in \partial\Sigma(0)$ such that

(1) $f(t,\zeta)$ is holomorphic for $t \in \mathbb{C}$ with $f(0,\zeta) = \zeta$;

(2) $f(t,\zeta) \subset \partial\Sigma(t)$; $\quad f(t, \partial\Sigma(0)) = \partial\Sigma(t)$.

(3) $f(t,\zeta_1) \neq f(t,\zeta_2)$ for $\zeta_1 \neq \zeta_2$.

Fix $\zeta_0 \in \partial\Sigma(0)$. Then, if we set, for $\zeta \in \partial\Sigma(0)$,

$$\psi(t,\zeta) = f(t,\zeta) / f(t,\zeta_0),$$

then $\psi(t,\zeta) \neq 0, 1, \infty$ and $\psi(t,\zeta)$ is holomorphic for $t \in \mathbb{C}$. Hence, By Picard's Theorem we have

$$\Psi(t,\zeta) = \text{const.} \quad \text{for } t \in \mathbb{C}$$

$$\therefore \quad \Psi(t,\zeta) = \Psi(0,\zeta) = \zeta/\zeta_0 \quad \text{by} \quad (1).$$

Hence (※) $f(t,\zeta) = \dfrac{\zeta}{\zeta_0} \cdot f(t,\zeta_0)$

We consider the automorphism of $\mathbb{C} \times \mathbb{C}$:

$$T : (t, z) \longrightarrow (t, w) = \left(t, \frac{z}{\zeta_0} f(t,\zeta_0)\right)$$

T is biholomorphic and

$$T(t, \partial\Sigma(0)) = \partial\Sigma(t) \quad \text{by (2) and (※)}.$$

It follows that

$$T(t, \Sigma(0)) = \Sigma(t)$$

or equivalently

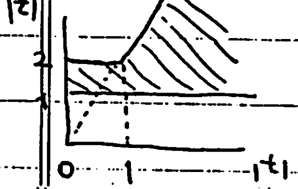
$$T : \mathbb{C} \times \Sigma(0) \longrightarrow \Sigma,$$

$\mathbb{C} \times \Sigma(0)$ is biholomorphically equivalent to Σ.

Q.E.D.

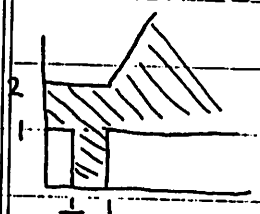
Remark 1 In the proof of Theorem 1, the conditions :
(1) $\Sigma(t) \not\ni 0$ for all $t \in \mathbb{C}$; (2) $\mathbb{P}^1 - \Sigma(t)$ is connected
are indispensable.

In fact, consider the following examples:

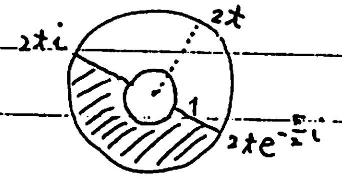


$$\Sigma = \begin{cases} 1 \leq |z| \leq 2|t| & \text{for } |t| \geq 1 \\ 1 \leq |z| \leq 2 & \text{for } |t| \leq 1 \end{cases}$$

Then $\Sigma(t) \not\ni 0$, Σ is pseudoconcave in $\mathbb{C}\times\mathbb{C}$, but not biholomorphically equivalent to $\mathbb{C}\times\mathbb{C}$. Clearly, (2) is not satisfied. Next, consider

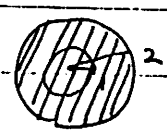


For $|t| \geq 1$, let $\Sigma(t) =$



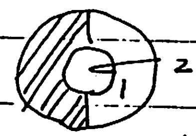
for $\frac{1}{2} \leq |t| \leq 1$, let

$\Sigma(t) =$



and for $|t| \leq \frac{1}{2}$,

$\Sigma(t) =$



Then $\Sigma = \bigcup_{t\in\mathbb{C}} \Sigma(t)$ is pseudoconcave in $\mathbb{C}\times\mathbb{C}$ and each $\Sigma(t)$ is simply connected, but there is no analytic set $z = f(t)$ $(t \in \mathbb{C})$ such that $\Sigma \supset (t, f(t))$ for all $t \in \mathbb{C}$. Clearly, $\Sigma(t) \not\ni 0$ is not satisfied when $\frac{1}{2} \leq |t| \leq 1$.

Our examples Σ are not smooth, but we easily modify Σ to be smooth.

Theorem 2 Let $B = \{|t| < \rho\}$ in the t-plane and $\mathbb{P}$ the z-Riemann sphere. Let $\mathcal{D}$ be a smooth pseudoconvex domain over $B\times\mathbb{P}$, precisely, there exists another $\widetilde{\mathcal{D}}$ (ramified) over $B\times\mathbb{P}$ such that

(i) $\widetilde{\mathcal{D}}$ is a Stein manifold;

(ii) $\widetilde{\mathcal{D}} \supset \mathcal{D}$; $\widetilde{D}(t) \supset\supset D(t)$ for all $t \in B$;

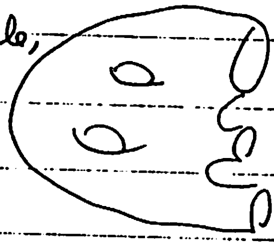
(iii) $\mathcal{D}$ is a smooth pseudoconvex domain in $\widetilde{\mathcal{D}}$.

We assume that each fiber $D(t)$ of $\mathcal{D}$ at $t \in B$ is a finite Riemann surface which is not topologically equivalent to a disk.

Suppose that $D(t)$ $(t \in B)$ are conformally equivalent to each other, i.e., $D(t) \sim D(0)$. Then $\mathcal{D}$ is biholomorphically equivalent to the trivial one : $B \times D(0)$.

Proof. We let (g, n) denote the topological type of $D(t)$ and $\chi = 2g+n-1$, so that $\chi \geq 1$.

For example,



$g=2$, $n=3$ $\therefore \chi = 6$.

Case 1 $\chi \geq 2$.

We then have homologously $\overbrace{\text{independent}}^{\text{two}}$ curves $C_1(t), C_2(t)$ on each $D(t)$ such that $C_i(t)$ $(i=1,2)$ moves in continuous way for $t \in B$. We form the harmonic reproducing differential $\sigma_i(t,\cdot)$ for $(D(t), C_i(t))$. Locally we can write

$$\sigma_i(t,z) = d\,u(t,z)$$

where $u(t,z)$ is harmonic function for z. If we put

$$\Omega_i(t,z) = \sigma_i(t,z) + \sqrt{-1}\,\sigma_i^*(t,z)$$

where $\sigma_i^*(t,z)$ is harmonic conjugate differential of $\sigma_i(t,z)$ then

$$\Omega_i(t,z) = 2\frac{\partial u}{\partial z}(t,z)\,dz.$$

$\Omega_i(t,z)$ is a holomorphic differential on $D(t)$.

Since $D(t) \sim D(0)$, we have

$$\|\sigma_i(t,\cdot)\|^2_{D(t)} = \|\sigma_i(0,\cdot)\|^2_{D(0)} \quad \text{for } t \in B.$$

$$\therefore \frac{\partial^2}{\partial t \partial \bar{t}}\left(\|\sigma_i(t,\cdot)\|^2_{D(t)}\right) = 0 \quad \text{for } t \in B.$$

On the other hand, by Fundamental equality, we have

$$\frac{\partial^2}{\partial t \partial \bar{t}}\left(\|\sigma_i(t,\cdot)\|^2_{D(t)}\right) = 2\int_{\partial D(t)} k_2(t,z)\left|\frac{\partial u_i(t,z)}{\partial z}\right|^2 ds_z + 8\iint_{D(t)}\left|\frac{\partial^2 u_i(t,z)}{\partial \bar{t}\,\partial z}\right|^2 dx\,dy.$$

Since $\mathcal{D}$ is pseudoconvex, it follows that $k_2(t, z) \geq 0$ on $\partial D(t)$ for all $t \in B$, i.e., on $\partial \mathcal{D}$. Consequently,

(a) $k_2(t, z) = 0$ on $\partial \mathcal{D}$;

(b) $\Omega_i(t, z) = \dfrac{\partial u_i(t,z)}{\partial z} dz$ is holomorphic for $t \in B$.

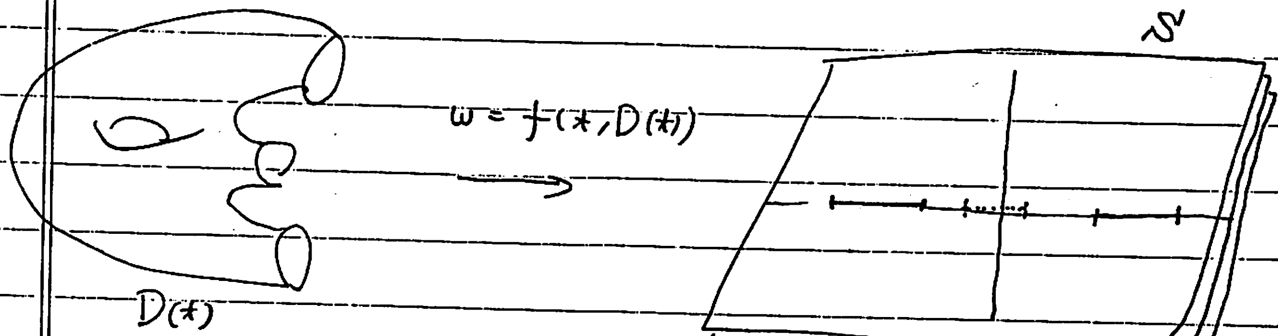
By (b), we see that

$$w = f(t, z) = \frac{\Omega_2(t,z)}{\Omega_1(t,z)} \quad \text{for } (t,z) \in \mathcal{D}$$

is a meromorphic non-constant function for $(t, z) \in \mathcal{D}$. Since $D(t) \sim D(0)$, it follows that

$$f(t, D(t)) = f(0, D(0)) \quad (= S) \quad (\underset{\text{over}}{\subset} \mathbb{P}_w)$$

as Riemann surface. S is clearly conformally equivalent to $D(0)$. Because $\sigma_i(t, z) = 0$ along $\partial D(t)$, $f(t, \partial D(t))$ = real values on $\partial D(t)$, so that S is a slit domain along some arcs on the real axis.



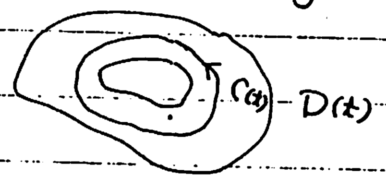
Hence $\mathcal{D} \approx B \times S$ by the analytic transformation
$T : (t, z) \longrightarrow (t, w) = (t, f(t,z))$.

Case 2 $\chi = 1$, i.e., $D(0) \sim$ annulus.

We then have only one cycle $c(t)$ in $D(t)$ which is not homologous zero. We consider the harmonic reproducing differential $\sigma(t, z) = du(t,z)$ for $(D(t), c(t))$.

In this case, $u(t,z)$ is the harmonic function on $D(t)$ whose boundary values are 0 and 1 on each boundary component. We remark that

$$\int_{C(t)} \sigma^*(t,z) = \|\sigma(t,z)\|^2_{D(t)}$$



$$= \text{const.}\ \alpha \quad (\text{by condition}: D(t) \sim D(0))$$

By the same reasoning as in the case 1 we have

(a) $k_2(t,z) = 0$ on $\partial\mathcal{D}$;

(b) $\Omega(t,z) = 2\frac{\partial u}{\partial z}(t,z)\,dz$ is holomorphic for $(t,z) \in \mathcal{D}$.

By (a), $\mathcal{D}$ is Levi flat. By Levi's Theorem and Monodromy Theorem, we find a holomorphic function $\varphi(t)$ for $t \in B$ such that

$$(t, \varphi(t)) \subset \partial\mathcal{D} \quad \text{for } t \in B.$$

On each $D(t)$ we form the integral

$$F(t,z) = \int_{\varphi(t)}^{z} \Omega(t,z)\,dz.$$

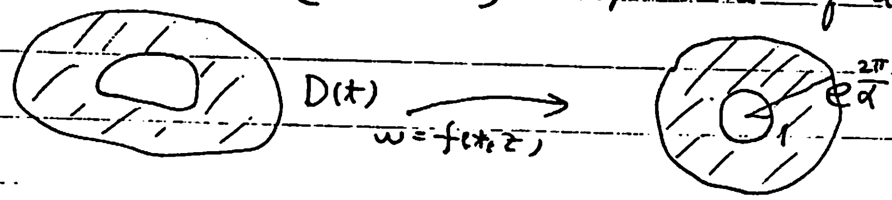
Then $F(t,z)$ is (multivalued) holomorphic function for $(t,z) \in \mathcal{D}$ such that

$$F(t, \varphi(t)) = 0 \quad \text{for } t \in B$$

and such that $F(t,z)$ has a period $\sqrt{-1}\,\alpha$ along $C_1(t)$. It follows that

$$w = f(t,z) = e^{\frac{2\pi}{\alpha} F(t,z)}$$

is a uniform holomorphic function for $(t,z) \in \mathcal{D}$. If we put $\varphi(t, D(t)) = S$, then S is an annulus with radii 1 and $e^{2\pi/\alpha}$, independent of $t \in B$.



Hence $\mathcal{D} \approx B \times S$ by the transformation T:
$(t, z) \longrightarrow (t, w) = (t, f(t, z))$ Q.E.D.

We study the equivalence of variations of finite Riemann surfaces.

Let $(\mathcal{D}, B, \pi)$ be a triple such that

$\mathcal{D}$: complex 2-dimensional manifold,

B: a disk of the complex t-plane; $B = \{|t| < \rho\}$

π: $\mathcal{D} \to B$ analytic mapping (function)

with the following properties

(i) π is surjective and submersion; that is,

$$\pi(\mathcal{D}) = B \quad \text{and} \quad \left(\frac{\partial \pi(x,y)}{\partial x}, \frac{\partial \pi(x,y)}{\partial y} \right) \neq (0, 0)$$

for all $(x, y) \in \mathcal{D}$

(ii) each $\pi^{-1}(t)$ $(t \in B)$ is irreducible and its topological type (g, n) is finite

(iii) There exists another triple $(\tilde{\mathcal{D}}, B, \tilde{\pi})$ such that

(a) $\tilde{\mathcal{D}}$ is Stein manifold;

(b) $\tilde{\mathcal{D}} \supset \mathcal{D}$ and $\tilde{\pi} = \pi$ on $\mathcal{D}$

$\tilde{\pi}^{-1}(t) \supset\supset \pi^{-1}(t)$ for all $t \in B$

(c) $\mathcal{D}$ is a smooth domain in $\tilde{\mathcal{D}}$, and $\pi^{-1}(t)$ is also smooth domain in $\tilde{\pi}^{-1}(t)$.

(iv) $\mathcal{D}$ is pseudoconvex in $\tilde{\mathcal{D}}$

We set $D(t) = \pi^{-1}(t)$ $(t \in B)$ and write

$$\mathcal{D} : t \to D(t) \quad (t \in B)$$

so that $\mathcal{D}$ is a smooth variation of smooth finite Riemann surfaces from a nbd V of y_0 to a nbd $B_0 = |t-t_0|<\rho_0$ $(\subset B)$. Given $t \in B_0$, we have, by Weierstrass preparation theorem,

$$\pi(x,y)=t \longleftrightarrow y=\xi(t,x)$$

where $x\in U$: a nbd of x_0

such that $\xi(t,x_0)=y$ with $\pi(x_0,y)=t$.

Since $\pi(x,\xi(t,x))=t$, we have

$$\frac{\partial \pi}{\partial y}\cdot\frac{\partial \xi}{\partial t}=1, \text{ so that } \frac{\partial \xi}{\partial t}(t,x)\neq 0.$$

Moreover, by construction $\xi(t,x)$ is holomorphic for (t,x).

Consider a mapping $T : (t,x)\to(x,y)=(x,\xi(t,x))$ from $B_0\times U$ into $U\times V$. Then

$$|J_T| = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \\ \frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\partial \xi}{\partial t} \\ 1 & \frac{\partial y}{\partial x} \end{vmatrix} = -\frac{\partial \xi}{\partial t}\neq 0.$$

Hence T is one to one from a Nbd of (t_0,x_0) onto a Nbd of (x_0,y_0). This means, there exists local coordinate (t,x) $\in B_0\times U_0$ where $B_0=|t-t_0|<\rho_0$ and $U_0=(|z|<\rho_1)$ in the z-plane such that $\pi(t,x)=t$ for all $t\in B_0$.

We say such local coordinates <u>π-coordinates</u>. The transition function between two π-coordinate $B_0\times U_0$ and $B_0\times V_0$ where $V_0=(|w|<\rho_2)$ is of the form

$$\begin{cases} t=t \\ w=f(t,z) \end{cases}$$

where $f(t,z)$ is holomorphic for t,z

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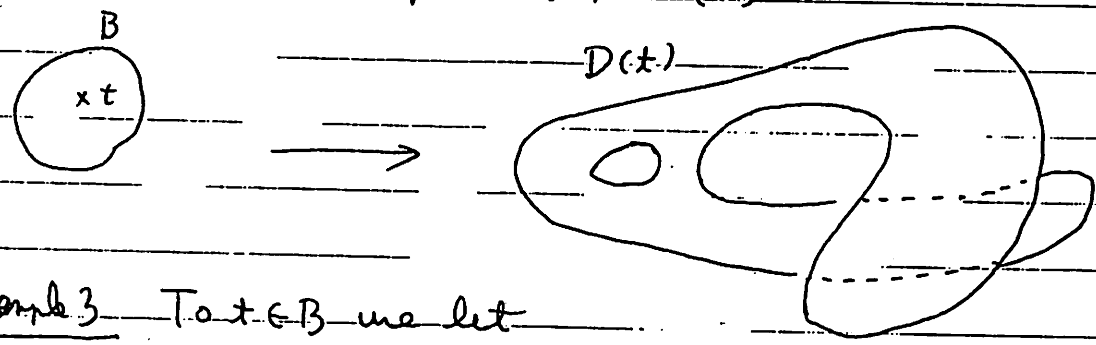
Let us give some examples of triples with (i) ~ (iv):

Example 1. Trivial one: $(B \times R, B, \pi)$ where R is a finite Riemann surface and π is the first projection.

Example 2. To each $t \in B = (|t| < \rho)$, we let correspond a Riemann surface spread over $\mathbb{C}$ with smooth boundary and without branch points. Let $\mathcal{D} = \bigcup_{t \in B} (t, D(t))$ and assume that $\mathcal{D}$ is smooth and pseudoconvex domain over $B \times \mathbb{C}$. Then the triple $(\mathcal{D}, B, \pi)$ where $\pi^{-1}(t) = D(t)$ for $t \in B$ satisfies (i) ~ (iv)



Example 3. To $t \in B$ we let correspond a Riemann surface $D(t)$ with smooth boundary $\partial D(t)$ and with finite number of branch points $(f_1(t), \cdots, f_m(t))$ such that

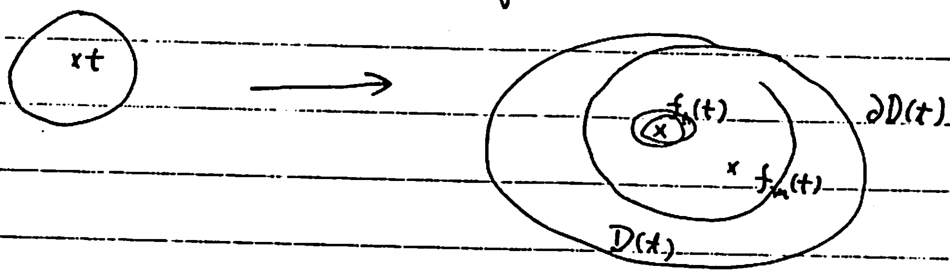
(a) $\partial D(t)$ smoothly varies with $t \in B$;

(b) each $f_i(t)$ is holomorphic for $t \in B$ such that $f_i(t) \neq f_j(t)$ $(i \neq j)$ and such that $f_i(t)$ does not meet $\partial D(t)$;

(c) $f_i(t)$ is of order of ramification $l_i - 1$ (≥ 1) independent of $t \in B$;

(d) Let $\mathcal{D} = \bigcup_{t \in B} (t, D(t))$. Take $z_0 \in \partial D(t_0)$ where $t_0 \in B$. Then we find $B_0 \times U_0$ where $B_0 = (|t - t_0| < r_0)$, $U_0 = (|z - z_0| < \rho_0)$ such that $(B_0 \times U_0) \cap \mathcal{D}$ is pseudoconvex domain in $B \times \mathbb{C}$.

The triple $(\mathcal{D}, B, \pi)$ then satisfies conditions (i) ~ (iv).



In example 3, let $t_0 \in B$, and $f_i(t_0) \in D(t_0)$ be a branch point of order $l_i - 1$ (≥ 1). Put

$$B_0 = \{|t - t_0| < r\}, \quad U_0 = \{|w| < \rho_0 \mid w = \sqrt[l_i]{z - f_i(t)}$$

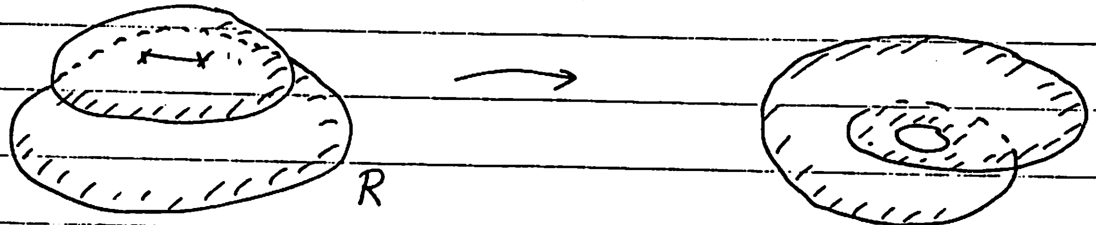
where z runs near $f_i(t)$ in $D(t)\}$. Then $B_0 \times U_0$ is a typical π-neighborhood of $(t_0, f_i(t_0))$ in $\mathcal{D}$.

We use the following Theorems without proofs:

(1) Gunning-Narasimhan (Math. Ann. 1967) "Immersion of open Riemann surface"

Any open Riemann surface R is realized as a Riemann surface over $\mathbb{C}$ without branch point, i.e., there exists a holomorphic function f on R such that $f'(z) \neq 0$ anywhere.

In particular, let R be a finite Riemann surface with smooth boundary and with some branch points. Then R is realized as a Riemann surface over $\mathbb{C}$ with smooth boundary but not with branch points.



(2) Y. Nishimura (Kyoto Univ. 1978) "Immersion analytique d'une famille de surfaces de Riemann ouvertes"

Let $(\mathcal{D}, B, \pi)$ be a triple with (i), (ii) and condition that $\mathcal{D}$ is Stein manifold. Then

Given $t_0 \in B$, there exists a disk B_0 such that $t_0 \in B_0 \subset\subset B$ such that $\pi^{-1}(B_0)$ is realized as a covering domain over $B_0 \times \mathbb{C}$ without branch surfaces.

In particular, let $(\mathcal{D}, B, \pi)$ be a triple with (i) ~ (iv). Let $t_0 \in B$. Then there exists a disk $t_0 \in B_0 \subset\subset B$ such that the triple $(\pi^{-1}(B_0), B_0, \pi)$ is one of Example 2.

Therefore, as long as we study the local property with respect to $t \in B$, Example 2 is general.

To study Problem in p. 12 we prepare some results:

<u>Proposition 1</u>

Let $(\mathcal{D}_0, B, \pi_0)$ and $(\mathcal{D}_1, B, \pi_1)$ two triples with (i). Assume that they are B-diffeomorphism by $T = (t, \varphi(t,z)) : \mathcal{D}_0 \to \mathcal{D}_1$ such that $\varphi(t,z)$ is holomorphic for $z \in D_0(t)$. Hence each fiber $D_0(t)$ $(t \in B)$ is conformally equivalent to $D_1(t)$.

Let $\Omega_0(t,z)$ and $\Omega_1(t,z)$ be holomorphic differentials on $D_0(t)$ and $D_1(t)$ such that they are of class C^1 for $(t,z) \in \mathcal{D}_1$ and for $(t,w) \in \mathcal{D}_1$ respectively and such that

(1.1) $\Omega_0(t,z) = \varphi(t,z) \# \Omega_1(t,z)$ for $z \in D_0(t)$.

Then

(1.2) $\dfrac{\partial \Omega_0(t,z)}{\partial \bar{t}} = \varphi(t,z) \# \dfrac{\partial \Omega_1}{\partial \bar{t}} + d_z F(t,z)$ on $D_0(t)$

where $F(t,z) = \dfrac{\partial \varphi}{\partial \bar{t}} \cdot \Omega_0 \Big/ \dfrac{\partial \varphi}{\partial z} dz$ is a holomorphic function on $D(t)$.

<u>Remark</u> (1) $\dfrac{\partial \Omega_0(t,z)}{\partial \bar{t}}$ is well-defined to be a holomorphic differential on $D(t)$;

(2) $\dfrac{\partial \varphi}{\partial \bar{t}} \Omega \Big/ \dfrac{\partial \varphi}{\partial z} dz$ is well-defined to be a holomorphic function on $D(t)$

In fact, let $(t,z) \in B_0 \times U$ and $(t,\tilde{z}) \in B_0 \times \tilde{U}$ be π-neighborhoods at the same point in $\mathcal{D}_0$. Hence $\exists$ $f(t,z)$: holomorphic for $(t,z) \in B_0 \times U$ such that $(t,z) \Longrightarrow (t,\tilde{z}) = (t, f(t,z))$ is a transition function. We denote by $g_0(t,z)$ and $\tilde{g}_0(t,\tilde{z})$ the representation of $\Omega_0(t,z)$ with respect to $z \in U$ and $\tilde{z} \in \tilde{U}$, i.e., $\Omega_0 = g_0(t,z)dz = \tilde{g}_0(t,\tilde{z})d\tilde{z}$

$$g_0(t,z) = \tilde{g}_0(t,\tilde{z}) \frac{df(t,z)}{dz} \quad \text{for } z \in U.$$

We differentiate both sides by $\bar{t}$ and have

$$\frac{\partial g_0}{\partial \bar{t}} = \left(\frac{\partial \tilde{g}_0}{\partial \bar{t}} + \frac{\partial f}{\partial \bar{t}} \right) \frac{\partial f}{\partial z} + \tilde{g}_0(t,\tilde{z}) \frac{\partial^2 f}{\partial \bar{t} \partial z}$$

$$= \frac{\partial \tilde{g}_0}{\partial \bar{t}} \frac{df}{dz} \qquad \text{by } \frac{\partial f}{\partial \bar{t}} \equiv 0,$$

Hence, $\frac{\partial \Omega_0}{\partial \bar{t}} = \frac{\partial g_0}{\partial \bar{t}} dz$ defines a holomorphic differential on $D_0(t)$.

To certify the second assertion, we consider the following diagram:

$$\begin{tikzcd}
z \in U \arrow[r, "f(t,z)", "\text{transition fn.}"'] \arrow[d, "\varphi(t,z) \text{ diffeomorphism}"] & \tilde{z} \in \tilde{U} \arrow[d, "\tilde{\varphi}(t,\tilde{z})"] \arrow[r, dotted, dash] & \mathcal{D}_0 \arrow[d, "\varphi"] \\
w \in V \arrow[r, "f_1(t,w)", "\text{transition fn}"'] & \tilde{w} \in \tilde{V} \arrow[r, dotted, dash] & \mathcal{D}_1
\end{tikzcd}$$

By difinition,

$$f_1(t, \varphi(t,z)) = \tilde{\varphi}(t, f(t,z))$$

We differentiate both sides with respect to $\bar{t}$ and to $\bar{z}$, and have

$$\frac{\partial f_1}{\partial w} \frac{\partial \varphi}{\partial \bar{t}} = \frac{\partial \tilde{\varphi}}{\partial \bar{t}} \quad ;$$

$$\frac{\partial f_1}{\partial w} \frac{\partial \varphi}{\partial \bar{z}} = \frac{\partial \tilde{\varphi}}{\partial \bar{\tilde{z}}} \frac{\partial f}{\partial z}$$

Since $\Omega_0 = g_0 dz = \tilde{g_0} \frac{\partial f}{\partial z} dz$, it follows that

$$\frac{\partial\varphi}{\partial\bar{t}} g_0 \Big/ \frac{\partial\varphi}{\partial z} = \frac{\frac{\partial\tilde\varphi}{\partial\bar t}}{\frac{\partial f_1}{\partial w}} \tilde{g_0} \frac{\partial f}{\partial z} \Bigg/ \frac{\frac{\partial\tilde\varphi}{\partial\tilde z}\frac{\partial f}{\partial z}}{\frac{\partial f_1}{\partial w}} = \frac{\partial\tilde\varphi}{\partial\bar t}\tilde{g_0} \Big/ \frac{\partial\tilde\varphi}{\partial\tilde z}$$

which means that $\frac{\partial\varphi}{\partial\bar t}\Omega \Big/ \frac{\partial\varphi}{\partial z}dz$ defines a uniform function in $D(t)$. It is clear that this is holomorphic for $z \in D(t)$.

Proof of Proposition 1

By (1.1) we have

$$\Omega_0(t,z) = \varphi(t,z) \# \Omega_1(t,w) \quad \text{for } z \in D_0(t)$$

Precisely, let $B_0 \times U$ be π-neighborhood of $\mathcal{D}_0$, and $B_0 \times V$ the correspondant π-neighborhood of $\mathcal{D}_1$ by φ. Then

$$(1.3) \quad g_0(t,z) = g_1(t, \varphi(t,z)) \frac{\partial\varphi}{\partial z}(t,z)$$

where $\Omega_0 = g_0 dz$ and $\Omega_1 = g_1 dw$. By differentiation with respect to $\bar t$, we have

$$\frac{\partial g_0}{\partial\bar t} = \left(\frac{\partial g_1}{\partial\bar t} + \frac{\partial g_1}{\partial w}\frac{\partial\varphi}{\partial\bar t}\right)\frac{\partial\varphi}{\partial z} + g_1 \frac{\partial^2\varphi}{\partial\bar t\partial z}$$

$$= \frac{\partial g_1}{\partial\bar t}\frac{\partial\varphi}{\partial z} + \frac{\partial g_1}{\partial w}\frac{\partial\varphi}{\partial\bar t}\frac{\partial\varphi}{\partial z} + g_1\frac{\partial^2\varphi}{\partial\bar t\partial z}.$$

On the other hand, by (1.3) we have

$$\frac{\partial g_0}{\partial z} = \frac{\partial g_1}{\partial w}\left(\frac{\partial\varphi}{\partial z}\right)^2 + g_1\frac{\partial^2\varphi}{\partial z^2}$$

$$= \frac{\partial g_1}{\partial w}\left(\frac{\partial\varphi}{\partial z}\right)^2 + \left(g_0 \Big/ \frac{\partial\varphi}{\partial z}\right)\cdot\frac{\partial^2\varphi}{\partial z^2}$$

$$\therefore \quad \frac{\partial g_1}{\partial w} = \frac{\frac{\partial g_0}{\partial z}\cdot\frac{\partial\varphi}{\partial z} - g_0\frac{\partial^2\varphi}{\partial z^2}}{\left(\frac{\partial\varphi}{\partial z}\right)^3}$$

Consequently,

$$\frac{\partial g_0}{\partial\bar t} = \frac{\partial g_1}{\partial\bar t}\frac{\partial\varphi}{\partial z} + \frac{\left(\frac{\partial g_0}{\partial z}\frac{\partial\varphi}{\partial z} - g_0\frac{\partial^2\varphi}{\partial z^2}\right)\cdot\frac{\partial\varphi}{\partial\bar t}}{\left(\frac{\partial\varphi}{\partial z}\right)^2} + \frac{g_0\frac{\partial^2\varphi}{\partial\bar t\partial z}}{\frac{\partial\varphi}{\partial z}}$$

$$= \frac{\partial g_1}{\partial\bar t}\frac{\partial\varphi}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial\bar t}g_0 \Big/ \frac{\partial\varphi}{\partial z}\right)$$

which means

$$\frac{\partial \Omega_0}{\partial \bar{t}} = \varphi \# \frac{\partial \Omega_1}{\partial \bar{t}} + dF \quad \text{on } D(t)$$

where $F = \frac{\partial \varphi}{\partial \bar{t}} \Omega_0 \Big/ \frac{\partial \varphi}{\partial z} dz$ on $D(t)$. Q.E.D.

Corollary 1 Under the same conditions as in Proposition 1, we assume $\varphi(t,z)$ is holomorphic for $t \in B$. Then

$$\frac{\partial \Omega_0}{\partial \bar{t}} = \frac{\partial \Omega_1}{\partial \bar{t}} \quad \text{on } D(t).$$

Conversely, assume that there exist two independent holomorphic differentials Ω_0, $\tilde{\Omega}_0$ and Ω_1, $\tilde{\Omega}_1$ on $D_0(t)$ and on $D_1(t)$ respectively, such that

$$\Omega_0 = \varphi \# \Omega_1, \quad \tilde{\Omega}_0 = \varphi \# \tilde{\Omega}_1$$

and such that

$$\frac{\partial \Omega_0}{\partial \bar{t}} = \varphi \# \frac{\partial \Omega_1}{\partial \bar{t}}, \quad \frac{\partial \tilde{\Omega}_0}{\partial \bar{t}} = \varphi \# \frac{\partial \tilde{\Omega}_1}{\partial \bar{t}}.$$

Then $\varphi(t,z)$ is holomorphic for $t \in B$.

Proof. The first assertion is clear by (1.2). For the second assertion, we have by (1.2)

$$\frac{\partial \varphi}{\partial \bar{t}} \Omega_0 \Big/ \frac{\partial \varphi}{\partial z} dz \quad \text{and} \quad \frac{\partial \varphi}{\partial \bar{t}} \tilde{\Omega}_0 \Big/ \frac{\partial \varphi}{\partial z} dz$$

are constant functions on $D(t)$. If $\frac{\partial \varphi}{\partial \bar{t}} \neq 0$ on B, then we have $\exists t \in B$ such that Ω_0 and $\tilde{\Omega}_0$ are not linearly independent on $D(t)$. Hence φ is holo. for $t \in B$. Q.E.D.

Lemma 1 Let $(\mathcal{D}_0, B, \pi_0)$, $(\mathcal{D}_1, B, \pi_1)$ two triples with (i) ~ (iv). Assume that they are B-diffeomorphism by $\varphi(t,z)$ such that $\varphi(t,z)$ is holomorphic for z. Take a closed cycle $C_0(t)$

on $D_0(t)$ such that $C_0(t)$ continuously varies for $t \in B$. Let $C_1(t) = \varphi(t, C_0(t)) \subset D_1(t)$. Let $\omega_0(t, z)$ the harmonic reproducing differential for $(D_0(t), C_0(t))$ and let $\omega_1(t, w)$ the corresponding one for $(D_1(t), C_1(t))$, so that, if we put $\Omega_0 = \omega_0(t, z) + \sqrt{-1}\,\omega_0^*(t, z)$ and $\Omega_1 = \omega_1(t, w) + \sqrt{-1}\,\omega_1^*(t, w)$, then

$$\Omega_0(t, z) = \varphi(t, z) \# \Omega_1(t, w).$$

Assume that $\mathcal{D}_0$ is Leviflat. Then

$$(1.5) \qquad 2\,\mathrm{Re}\left(\frac{\partial \Omega_0}{\partial \bar t}, d_z F\right)_{D_0(t)} \geqq \| d_z F \|^2_{D(t)}$$

where $F = \frac{\partial \varphi}{\partial \bar t} \Omega_0 / \frac{\partial \varphi}{\partial z} dz$ on $\mathcal{D}_0$.

Proof. From (2.3) in the note on April 25, p. 11 we have

$$\frac{\partial^2}{\partial t \partial \bar t} \| \omega_0 \|^2_{D_0(t)} = 8 \iint_{D_0(t)} \left| \frac{\partial^2 u_0}{\partial \bar t \partial z} \right|^2 dx\,dy$$

Because of $\mathcal{D}_0$ is Leviflat, we have $k_2(t, z) = 0$ on $\partial \mathcal{D}_0$.
By using complex notation, it turns out

$$\frac{\partial^2}{\partial t \partial \bar t} \| \Omega_0 \|^2_{D_0(t)} = \left\| \frac{\partial \Omega_0}{\partial \bar t} \right\|^2_{D_0(t)}.$$

Since $\mathcal{D}_1$ is pseudoconvex, we analogously have

$$\frac{\partial^2}{\partial t \partial \bar t} \| \Omega_1 \|^2_{D_1(t)} \geqq \left\| \frac{\partial \Omega_1}{\partial \bar t} \right\|^2_{D_1(t)}.$$

On the other hand, $\Omega_0 = \varphi \# \Omega_1$ on $D(t)$ $(t \in B)$ implies

$$\| \Omega_0 \|^2_{D_0(t)} = \| \Omega_1 \|^2_{D_1(t)}.$$

Consequently, $\left\| \frac{\partial \Omega_0}{\partial \bar t} \right\|^2_{D_0(t)} \geqq \left\| \frac{\partial \Omega_1}{\partial \bar t} \right\|^2_{D_1(t)}$.

By Proposition 1, we have

$$\varphi \# \frac{\partial \Omega_1}{\partial \bar t} = \frac{\partial \Omega_0}{\partial \bar t} - dF \quad \text{on } D_0(t),$$

so that $\left\| \frac{\partial \Omega_1}{\partial \bar t} \right\|^2_{D_1(t)} = \left\| \frac{\partial \Omega_0}{\partial \bar t} - dF \right\|^2_{D_0(t)}$

Hence $\left\| \frac{\partial \Omega_0}{\partial \bar{t}} \right\|^2_{D_0(t)} \geqq \left\| \frac{\partial \Omega_0}{\partial \bar{t}} - d_z F \right\|^2_{D_0(t)}$

$\therefore \; 2 \,\mathrm{Re} \left(\frac{\partial \Omega_0}{\partial \bar{t}}, d_z F \right)_{D_0(t)} \geqq \| d_z F \|^2_{D_0(t)}$

Q.E.D.

Corollary 2 Under the same conditions as in Lemma 1, assume that, given $t_0 \in B$, there exist two independent cycles $C_0(t)$, $\tilde{C}_0(t)$ (with real coefficients) on $D_0(t)$ such that

$$\frac{\partial \Omega_0}{\partial \bar{t}}(t_0, z), \; \frac{\partial \tilde{\Omega}_0}{\partial \bar{t}}(t_0, z) = 0 \quad \text{on } D_0(t_0)$$

where Ω_0, $\tilde{\Omega}_0$ are the harmonic reproducing differentials for $(D_0(t), C_0(t))$ and $(D_0(t), \tilde{C}_0(t))$. (Note that $C_0(t)$, $C_1(t)$ may depend on t_0). Then φ is holomorphic for $t \in B$

(::) Such conditions imply $d_z F = d_z \tilde{F} = 0$ on $D_0(t_0)$ by means of (1.5). Hence $\frac{\partial \varphi}{\partial \bar{t}} = 0$ on $D_0(t_0)$. Since $t_0 \in B$ is arbitrary, we see that φ is holomorphic for $t \in B$.

Example 1 Let $(\mathcal{D}_0, B, \pi_0)$ be trivial, i.e., $\mathcal{D}_0 = B \times R$ where R is a finite Riemann surface of topological type (g, n). Assume that $\chi = 2g + n - 1 \geqq 2$. Then our problem in p. 12 is true

(::) There exist two independent cycles C_0, $\tilde{C}_0$ on R. Since $D_0(t)$ is independent of $t \in B$, it follows that $\frac{\partial \Omega_0}{\partial \bar{t}} \equiv 0$ and $\frac{\partial \tilde{\Omega}_0}{\partial \bar{t}} \equiv 0$. Corollary 2 implies φ is holo. for $t \in B$.

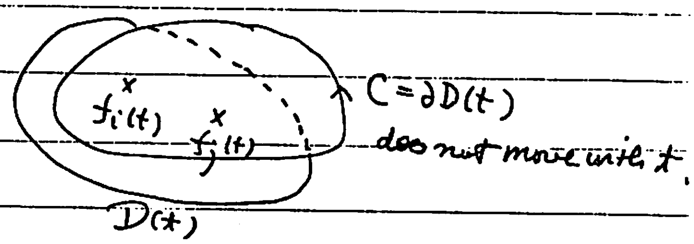
Example 2 Let $(\mathcal{D}_0, B, \pi)$ be a type of Example 3 in p. 12 such that $\partial D(t)$ does not vary with $t \in B$. Let $\chi = 2g + n - 1$ $(\geqq 1)$ and let $\mathcal{M} = (l_1 + \cdots + l_m) - m$ where (g, n) is of topological type of $D_0(t)$; $l_i - 1$ $(1 \leqq i \leqq m)$ is of order of branch point $f_i(t)$ of $D_0(t)$. Assume that

$$\chi \geq 2(\mu+1)$$

Then $\varphi(t,z)$ in our problem in p.12 is holomorphic for $t \in B$.

Proof:

Let $C(t)$ be a cycle on $D_0(t)$, and let $\omega(t,z)$ be the reproducing differential for $(D_0(t), C(t))$. Set $\Omega(t,z) = \omega(t,z) + \sqrt{-1}\,\omega(t,z)^*$ on $D_0(t)$.



By the construction of $u(t,z)$ with $\omega(t,z) = du(t,z)$, $\frac{\partial u}{\partial t}(t,z)$ is uniform harmonic function on $D(t)$ except at branch points $\{f_i(t)\}_{i=1}^m$. Moreover, since $C = \partial D(t)$ does not vary with $t \in B$, we have $\frac{\partial u(t,z)}{\partial t} = 0$ on $\partial D(t)$.

We observe the singularity of $\frac{\partial u}{\partial t}$ at $f_i(t)$ of order $l_i - 1$. For the sake of convenience, we write $f_i(t) = f(t)$ and $l_i = l$. Since $u(t,z)$ is harmonic (bounded) near $f(t)$, we get

$$u(t,z) = \mathrm{Re}\left\{ \sum_{j=0}^{\infty} A_j(t)\,(z - f(t))^{j/l} \right\}.$$

$$\therefore \quad \frac{\partial u}{\partial t} = \frac{1}{2} \sum_{j=1}^{\infty} \frac{j}{l} A_j(t)\,(z - f(t))^{\frac{j}{l}-1} \cdot f'(t) \quad + \text{ bdd part.}$$

$$= \frac{f'(t)}{2 \cdot l} \left\{ A_1(t)\,(z - f(t))^{\frac{1-l}{l}} + \cdots + A_{l-1}(t)\,(z - f(t))^{-\frac{1}{l}} \right\} + \text{bdd part.}$$

It follows that $\frac{\partial u}{\partial t}$ is regular at $f(t)$ if and only if $f'(t) = 0$ or $A_1(t) = \cdots = A_{l-1}(t) = 0$, that is, $\Omega(t,z)$ is 0 at least of order l at $f(t)$. Consequently, if $\Omega(t,z)$ is zero at least of order l_i at each branch point $f_i(t)$, then $\frac{\partial u}{\partial t}(t,z) \equiv 0$. For, $\frac{\partial u}{\partial t}(t,z)$ is regular harmonic on $D_0(t)$ with boundary values 0. In particular, $\frac{\partial \Omega}{\partial t} \equiv 0$.

Assume that $\chi \geqq 2(\mu+1)$. Then we have χ linearly independent (over $\mathbb{R}$) reproducing differentials $\{\omega_1(t,z), \cdots, \omega_\chi(t,z)\}$. Put
$\Omega_i(t,z) = \omega_i(t,z) + \sqrt{-1}\,\omega_i(t,z)^*$ on $D_0(t)$, and

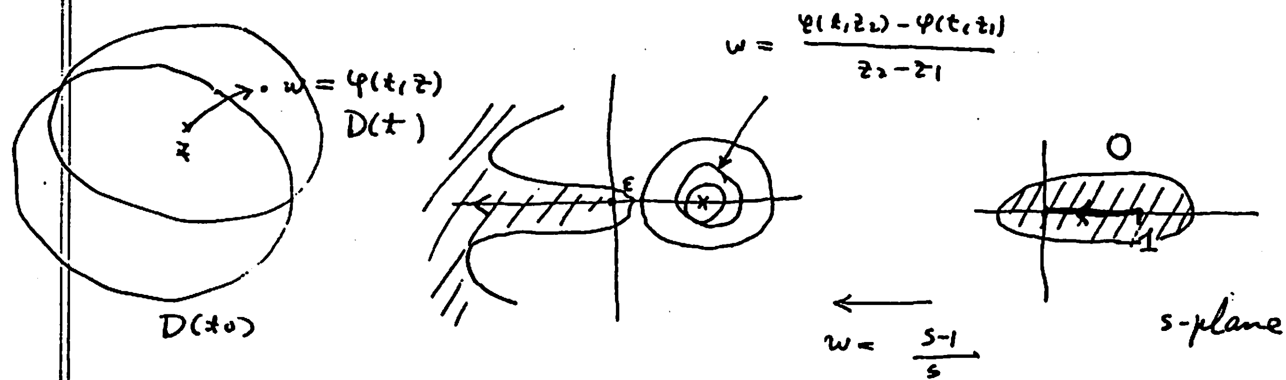
$$\Omega_C = \sum_{i=1}^{\chi} c_i \Omega_i(t,z)$$

where $C = (c_1, \cdots, c_\chi)$ are real constants. Let $t_0 \in B$ be given. Since $\chi \geqq 2(\mu+1)$, it follows that there exist at least two $C' = (c_1', \cdots, c_\chi')$ and $C'' = (c_1'', \cdots, c_\chi'')$ such that $\Omega_{C'}(t_0,z)$ and $\Omega_{C''}(t_0,z)$ are zero at least of order $l_i - 1$ at each branch point $f_i(t_0)$ on $D_0(t_0)$. Therefore, $\frac{\partial \Omega_{C'}}{\partial \bar{t}}(t_0,z) = \frac{\partial \Omega_{C''}}{\partial \bar{t}}(t_0,z) = 0$ on $D_0(t_0)$. By Corollary 2, $\varphi(t,z)$ is holomorphic for $t \in B$. Q.E.D.

We consider the general problem in P.12. Now, assume that $\mathcal{D}_0$ and $\mathcal{D}_1$ are B-analytically isomorphic by $T: (t,z) \to (t,w) = (t, \varphi(t,z))$ such that T is close to identity mapping in the sense that

$$(*) \qquad \left| \frac{\varphi(t,z_2) - \varphi(t,z_1)}{z_2 - z_1} - 1 \right| < 1 - \varepsilon, \quad \exists \varepsilon > 0$$

for all $t \in B$ and z_1, z_2, $z_1 \neq z_2$ in $D(t)$.



We consider the mapping, for $s \in {}^\exists O\ (\supset [0,1])$ in the complex s-plane,

$$\varphi(s,t,z) = (1-s)z + s\varphi(t,z)$$

and define

$$D(s,t) = \varphi(s,t,D_0(t)).$$

Then $D(0,t) = D_0(t)$, $D(1,t) = D_1(t)$. By assumption ($\ast$)
$D(s,t)$ defines a domain in the complex plane:
($(\because)$ assume $z_2 \neq z_1$ in $D_0(t)$. then $\varphi(s,t,z_1)$
$= (1-s)z_1 + s\varphi(t,z_1) \neq (1-s)z_2 + s\varphi(t,z_1) = \varphi(s,t,z_2)$.)
We thus have, for each $s \in O$, the variation

$$\mathcal{D}_s : t \longrightarrow D_s(t) = D(s,t) \quad (t \in B)$$

such that $\mathcal{D}_0$ and $\mathcal{D}_1$ are original ones. This means, we can find an analytic homotopy variation

$$\mathcal{D}_0 \rightsquigarrow \mathcal{D}_s \rightsquigarrow \mathcal{D}_1 \qquad (s \in O)$$

such that each $\mathcal{D}_s$ is B-analytically isomorphic to $\mathcal{D}_0$.

We show the converse, which may be called ANALYTIC HOMOTOPY THEOREM. Let O be a disk $|s|<r$; B a disk $|t|<\rho$. Let $(\mathcal{D}, O\times B, \pi)$ be a triple with (i) ~ (iv). Precisely, instead of (iii), $\mathcal{D}$ is a smooth pseudoconvex domain of dimension 3 in a Stein manifold. We thus have the variation

$$\mathcal{D} : (s,t) \longrightarrow D(s,t) \quad ((s,t) \in O\times B)$$

where $D(s,t)$ is a finite Riemann surface with smooth boundary.

Given $t \in B$, we denote by $\mathcal{D}^t = \bigcup_{s\in O} (s, D(s,t))$, i.e.,

$$\mathcal{D}^t : s \longrightarrow D^t(s) = D(s,t) \quad (s \in O).$$

Given $s \in O$, we denote by $\mathcal{D}_s = \bigcup_{t \in B} (t, D(s,t))$,

i.e., $\mathcal{D}_s : t \to D_s(t) = D(s,t) \quad (t \in B)$.

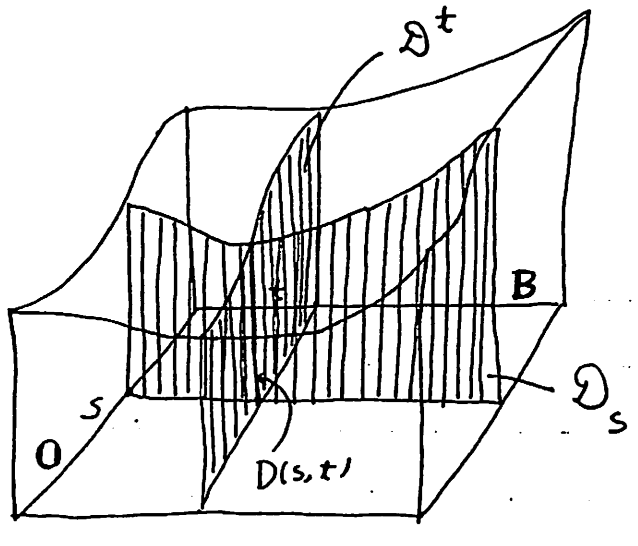
Theorem Under this notation, assume that

(1) the topological type (g, n) of $D(s,t)$ $((s,t) \in O \times B)$ is such that $\chi = 2g + n - 1 \geq 2$;

(2) $\mathcal{D}_0$ is Levi flat;

(3) Given $s \in O$, $\mathcal{D}_0$ is B-diffeomorphic to $\mathcal{D}_s$ by $\varphi_s(t,z)$ where $\varphi_s(t,z)$ is of class C^1 for (s,t,z) and holomorphic for $z \in D_0(t)$, so that, for $t \in B$, $D(0,t) \sim D(s,t)$ as Riemann surface by $\varphi_s(t,z)$.

Then $\varphi_s(t,z)$ is holomorphic for $(s,t) \in O \times B$.



Proof. 1^{st} step. $\varphi_s(t,z)$ is holomorphic for $s \in O$.
In fact, let $t \in B$. Condition (3) implies that the variation $\mathcal{D}^t : s \to D^t(s)$ $(s \in O)$ is O-diffeomorphic to the trivial $O \times D(0,t)$ by $\varphi_s^{-1}(t,z)$ such that $\varphi_s^{-1}(t,z)$ is holomorphic for $z \in D^t(s)$. It follows that $\varphi_s^{-1}(t,z)$, or equivalently $\varphi_s(t,z)$ is holomorphic for $s \in O$.

<u>2nd step</u> $\mathcal{D}$ is Levi flat.

In fact, since $\mathcal{D}$ is a smooth pseudoconvex domain, for each $(0, t_0, \zeta)$ where $\zeta \in \partial D(0, t_0)$, we draw a 2-dim. analytic set σ : $z = \zeta(s, t, \zeta)$ where $s \in O_0 = \{|s| < r_0\}$ and $t \in B_0 = \{|t - t_0| < \rho_0\}$ through $(0, t_0, \zeta)$ such that $\sigma \subset \mathcal{D}^c$. Since $\mathcal{D}_0$ is Levi flat, we have

$$z = \zeta(0, t, \zeta) \subset \partial D_0(t) \text{ for all } t \in B_0.$$

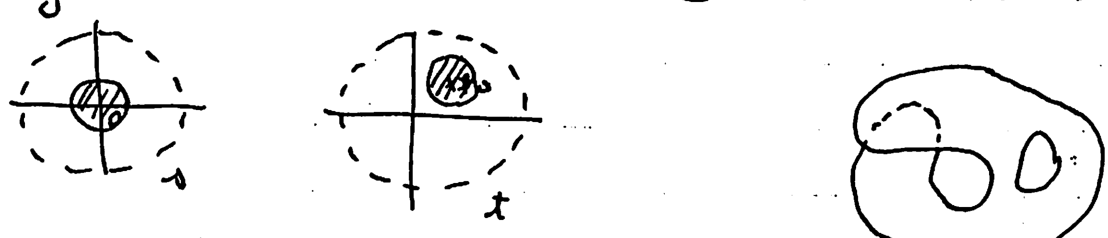
Hence since $\mathcal{D}^{t_1}$ $(t_1 \in B_1)$ is, by 1st step, Levi flat, we have

$$z = \zeta(s, t_1, \zeta) \subset \partial D^{t_1}(s) \text{ for all } s \in O_0$$

$$\therefore \ \sigma \subset \partial \mathcal{D}$$

Consequently, $\exists\, O^* = \{|s| < r^*\} \subset O$ such that the restriction of $\mathcal{D}$ to $O^* \times B$ is Levi flat. By repeating the same argument at $s_0 \in \partial O^*$ instead of $s = 0$, we see that $\mathcal{D}$ is Levi flat.

Let $t_0 \in B$. By Nishimura's Theorem, there exist $O_0 = \{|s| < r_0\}$ and $B_0 = \{|t - t_0| < \rho_0\}$ such that the restriction $\mathcal{D}$ to $O_0 \times B_0$ is realized as an unramified covering domain over $O_0 \times B_0 \times \mathbb{C}$ with smooth boundary.

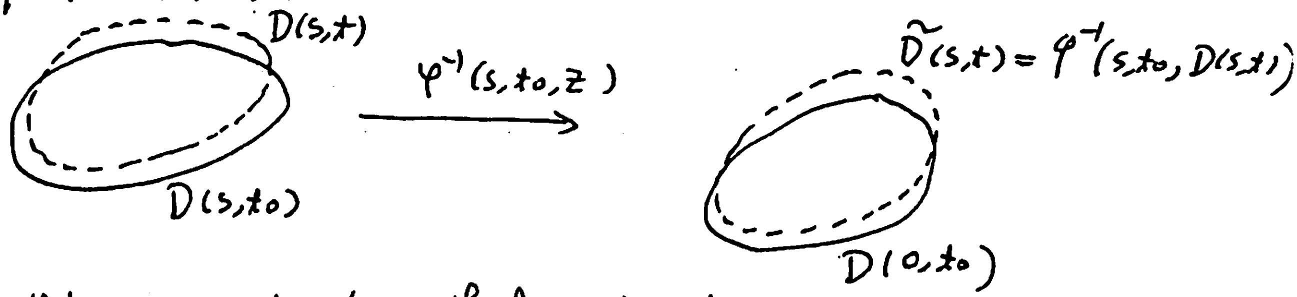


$D(s,t)$

$$\mathcal{D}|_{O_0 \times B_0} : (s, t) \longrightarrow D(s,t) \quad ((s,t) \in O_0 \times B_0)$$

<u>3rd step</u> We may assume $\varphi(s, t_0, z) \equiv z$, i.e.,
$D(s, t_0) = D(0, t_0)$.

In fact, by 1st step, $\varphi(s, t_0, z)$ is holomorphic for $s \in O_0$, and so is $\varphi^{-1}(s, t_0, z)$.



Since $\varphi^{-1}(s,t_0,z)$ is extended holomorphically beyond $\partial D(s,t_0)$, we may assume that $\varphi^{-1}(s,t_0,z)$ is defined on $D(s,t)$ for all $t \in B_0$ and $s \in O_0$. We set thus

$$\tilde{D}(s,t) = \varphi^{-1}(s, t_0, D(s,t))$$

The variation $\tilde{\mathcal{D}} : (s,t) \to \tilde{D}(s,t)$ $((s,t) \in O_0 \times B_0)$ is $O_0 \times B_0$-analytically isomorphic to the original $\mathcal{D} : (s,t) \to D(s,t)$ $((s,t) \in O_0 \times B_0)$ by the transformation: $(s,t,z) \to (s,t,\varphi(s,t_0,z))$

It follows that, if we put

$$\psi(s,t,z) = \varphi^{-1}(s, t_0, \varphi(s,t,z))$$

for $(s,t) \in O_0 \times B_0$ and $z \in D(0,t)$, then $\psi(s,t,z)$ maps $D(0,t)$ onto $\tilde{D}(s,t)$. $\psi(s,t,z)$ is holomorphic for s, z and $\underline{\psi(s,t_0,z) \equiv z}$ for $z \in D(0,t_0)$, that is, $\underline{\tilde{D}(s,t_0) = D(0,t_0)}$. Since $\frac{\partial\psi}{\partial\bar{t}}(s,t_0,z) = \frac{\partial\varphi}{\partial\bar{t}}(s,t_0,z) \div \frac{\partial\varphi}{\partial z}(s,t_0,z)$, it suffices for our purpose: $\frac{\partial\varphi}{\partial\bar{t}}(s,t_0,z) = 0$ to prove

<u>4th step</u> $\frac{\partial\psi}{\partial\bar{t}}(s,t_0,z) = 0$ for $z \in D(0,t_0)$

Under the notation of 3rd step, let $C(t)$ be a cycle on $D(0,t)$ $(t \in B_0)$ which varies continuously with $t \in B_0$, and form the harmonic reproducing differential $\omega(t,z)$ for $(D(0,t), C(t))$. We put $\Omega(t,z) = \omega(t,z) + i\,\omega(t,z)^*$ and $\omega(t,z) = du(t,z)$. By construction, $u(t,z)$ is harmonic function on $D(0,t) - C(t)$ such that $u(t,z) \equiv 0$ on $\partial D(0,t)$ and $\frac{\partial u}{\partial t}(t,z)$ is regular harmonic function on $D(0,t)$.

Now, fix $s \in O_0$. Since $\mathcal{D}_0|_{B_0}$ is B_0-diffeomorphic to $\mathcal{D}_s|_{B_0}$ by $(t,z) \to (t, w = \psi(s,t,z))$ where ψ is holomorphic for z, it follows from Lemma 1 that

$$2\,\mathrm{Re}\left(\frac{\partial \Omega}{\partial \bar t}, \frac{\partial F}{\partial \bar z} dz\right)_{D(0,t)} \geq \left\| \frac{\partial F}{\partial \bar z} dz \right\|^2_{D(0,t)}$$

where $F = \frac{\partial \psi}{\partial \bar t}(s,t,z)\, \Omega(t,z) \Big/ \frac{\partial \psi}{\partial z}(s,t,z)\,dz$.

Since $\psi(s,t_0,z) \equiv z$, we have, in particular,

$$2\,\mathrm{Re} \iint_{D(0,t_0)} \frac{\partial^2 u}{\partial \bar t \partial z}(t_0,z)\, \overline{\frac{\partial \psi}{\partial \bar t}(s,t_0,z) \frac{\partial u}{\partial z}(t_0,z)}\; dz \wedge d\bar z$$

$$\geqq i \iint_{D(0,t_0)} \left| \frac{\partial}{\partial \bar z}\left(\frac{\partial \psi}{\partial \bar t}(s,t_0,z) \frac{\partial u}{\partial z}(t_0,z)\right)\right|^2 dz \wedge d\bar z$$

By Stokes' formula, we get

$$(1.7)\quad \mathrm{Im} \int_{\partial D(0,t_0)} \frac{\partial^2 u}{\partial \bar t \partial z}(t_0,z)\, \overline{\frac{\partial \psi}{\partial \bar t}(s,t_0,z) \frac{\partial u}{\partial z}(t_0,z)}\; dz$$

$$\geqq \iint_{D(0,t_0)} \left| \frac{\partial}{\partial \bar z}\left(\frac{\partial \psi}{\partial \bar t}(s,t_0,z) \frac{\partial u}{\partial z}(t_0,z)\right)\right|^2 dx\,dy.$$

Let $\zeta \in \partial D(0,t_0) = \partial D(s,t_0)$ be fixed, and draw the analytic set σ : $z = \xi(s,t,\zeta)$ where $(s,t) \in O_0 \times B_0$

(415)

It follows that, for all $\zeta \in \partial D(0, t_0)$,

$$\frac{\partial \psi}{\partial t}(s, t_0, \zeta)\, \frac{\partial u}{\partial z}(t_0, \zeta)$$

$$= -\frac{\partial u}{\partial \bar z}(t_0, \zeta)\, \frac{\partial \bar\psi}{\partial t}(s, t_0, \zeta) + \frac{\partial u}{\partial \bar z}(t_0, \zeta) \overline{\left\{ \frac{\partial \zeta}{\partial t}(s, t_0, \zeta) - \frac{\partial \zeta}{\partial t}(0, t_0, \zeta) \right\}}$$

Substituting this into (1.7) we have

$$\operatorname{Im} \int_{\partial D(0,t_0)} \frac{\partial^2 u}{\partial t \partial z}(t_0, z) \left\{ -\frac{\partial u}{\partial \bar z}(t_0, z) \frac{\partial \psi}{\partial t}(s, t_0, z) + \frac{\partial u}{\partial \bar z}(t_0, z) \left[\frac{\partial \zeta}{\partial t}(s, t_0, z) - \frac{\partial \zeta}{\partial t}(0, t_0, z) \right] \right\} d\bar z$$

$$\geq \iint_{D(0,t_0)} \left| \frac{\partial}{\partial \bar z} \left(\frac{\partial \psi}{\partial t}(s, t_0, z)\, \frac{\partial u}{\partial z}(t_0, z) \right) \right|^2 dx\,dy$$

Since $\frac{\partial^2 u}{\partial t \partial z}(t_0, z)$, $\frac{\partial \psi}{\partial t}(s, t_0, z)$, $\frac{\partial u}{\partial z}(t_0, z)$ are holomorphic for z in $D(0, t_0)$, we have by Cauchy's Theorem

$$\int_{\partial D(0,t_0)} \frac{\partial^2 u}{\partial t \partial z}(t_0, z)\, \frac{\partial u}{\partial z}(t_0, z)\, \frac{\partial \psi}{\partial t}(s, t_0, z)\, dz = 0.$$

It turns out that, for any $s \in O_0$,

$$I(s) = \operatorname{Im} \int_{\partial D(0,t_0)} \frac{\partial^2 u}{\partial t \partial z}(t_0, z)\, \frac{\partial u}{\partial z}(t_0, z) \left\{ \frac{\partial \zeta}{\partial t}(s, t_0, z) - \frac{\partial \zeta}{\partial t}(0, t_0, z) \right\} d\bar z$$

$$\geq \iint_{D(0,t_0)} \left| \frac{\partial}{\partial \bar z} \left(\frac{\partial \psi}{\partial t}(s, t_0, z)\, \frac{\partial u}{\partial z}(t_0, z) \right) \right|^2 dx\,dy \geq 0.$$

Since $\zeta(s, t, z)$ is holomorphic for $(s, t) \in O_0 \times B_0$, it follows that the left-hand side $I(s)$ is a harmonic function for s on O_0 !! Moreover, $I(0) = 0$, $I(s) \geq 0$ by the above inequality. By the maximum principle, we get

$$I(s) \equiv 0 \quad \text{on} \quad O_0 = (|s| < r_0),$$

and hence

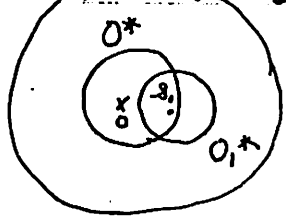
$\frac{\partial\psi}{\partial\bar z}(s,t_0,z)\ \frac{\partial u}{\partial z}(t_0,z) =$ const. $c(s)$ for $z\in D(0,t_0)$ where $s\in O_0$. By condition of $\chi = 2g+n-1 \geqq 2$, we have another reproducing differential $\omega_1 = du_1$ which is independent of $\omega = du$ (which is defined above). We thus conclude that given $t_0\in B$, $\forall s\in O_0$,

$$\frac{\partial\psi}{\partial\bar z}(s,t_0,z) = 0 \quad \text{for all } z\in D(0,t_0).$$

It follows that our original $\varphi(s,t,z)$ is holomorphic for $(s,t)\in O^*\times B$ and $z\in D(0,t)$ where $O^* = \{|s|<r^*\}$.

5th step $\varphi(s,t,z)$ is holo for $(s,t)\in O\times B$ and ~~$z\in D(0,t)$~~.

Fix $s_1\in O^*$ close to ∂O^*.



Let $s\in O$ and form

$$\psi(s,t,z) = \varphi(s,t,\varphi^{-1}(s_1,t,z))$$

which maps $D(s_1,t)$ onto $D(s,t)$. By repeating the same process at s_1 instead of $s=0$, we have $O_1^* = (|s-s_1|<r_1)$ as in the above figure such that $\psi(s,t,z)$ and hence $\varphi(s,t,z)$ is holomorphic for $s\in O_1^*$ and $t\in B$. We eventually conclude that $\varphi(s,t,z)$ is holomorphic for $(s,t)\in O\times B$ and $z\in D(0,t)$.

Q.E.D

Metric induced by Robin constants

§1. Definition of metric induced by Robin constants.

Let D be a domain in $\mathbb{C}^n$ $(n \geq 2)$ with smooth boundary ∂D. Given $\zeta \in D$, we consider the Green's function $G(\zeta, z)$ and the Robin constant $\Lambda(\zeta)$ for (D, ζ). We thus have

$$G(\zeta, z) = \frac{1}{\| z - \zeta \|^{2n-2}} + \Lambda(\zeta) + H(\zeta, z)$$

where $H(\zeta, z)$ is harmonic for z in D and $H(\zeta, \zeta) = 0$.
We know that

$\Lambda(\zeta) < 0$;
$\Lambda(\zeta)$ is real analytic for ζ in D ;
$\lim_{\zeta \to \partial D} \Lambda(\zeta) = -\infty$.

Moreover, if D is pseudoconvex in $\mathbb{C}^n$, then
$\log(-\Lambda(\zeta))$ and $-\Lambda(\zeta)$ are strictly plurisubharmonic function for ζ in D.

Therefore, the quadratic form

$$ds^2 = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda(\zeta))}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta} d\zeta_\alpha \otimes d\bar{\zeta}_\beta$$

defines a Kähler metric on D. In this note, we say that ds^2 is the metric induced by Robin constants.

At present, I don't know ds^2 is complete or not.
But, in some cases, for exemple,

D is strictly pseudoconvex in $\mathbb{C}^n$;
D is convex in $\mathbb{R}^m$

ds^2 is complete in D.

I guess that ds^2 is complete in any bounded pseudoconvex domain with smooth boundary in $\mathbb{C}^n$.

§2. Boundary behavior of $\Lambda(\zeta)$ for any bounded domain D with smooth boundary.

Let D be a domain with smooth boundary ∂D. D may not be pseudoconvex. Let $(\mathbb{C}^n, \psi)$ be a double which defines D, that is,

$\psi(z)$ is of class C^∞ in $\mathbb{C}^n$ such that

$$D = \{ z \in \mathbb{C}^n \mid \psi(z) < 0 \};$$

$$\partial D = \{ z \in \mathbb{C}^n \mid \psi(z) = 0 \}$$

& $\mathrm{Grad}_{(z)} \psi = \left(\frac{\partial \psi}{\partial z_1}, \dots, \frac{\partial \psi}{\partial z_n} \right) \neq 0$ at all $\zeta \in \partial D$.

We can take $\psi(z)$ such that $\psi(z) \equiv$ const. >0 near the infinity. First, let us show the following

<u>Lemma 2.1</u> Let $\zeta_0 \in \partial D$. Then

(1) $$\lim_{D \ni \zeta \to \zeta_0} \Lambda(\zeta) \psi(\zeta)^{2n-2} = - \| \mathrm{Grad}\, \psi(\zeta_0) \|^{2n-2} \quad ;$$

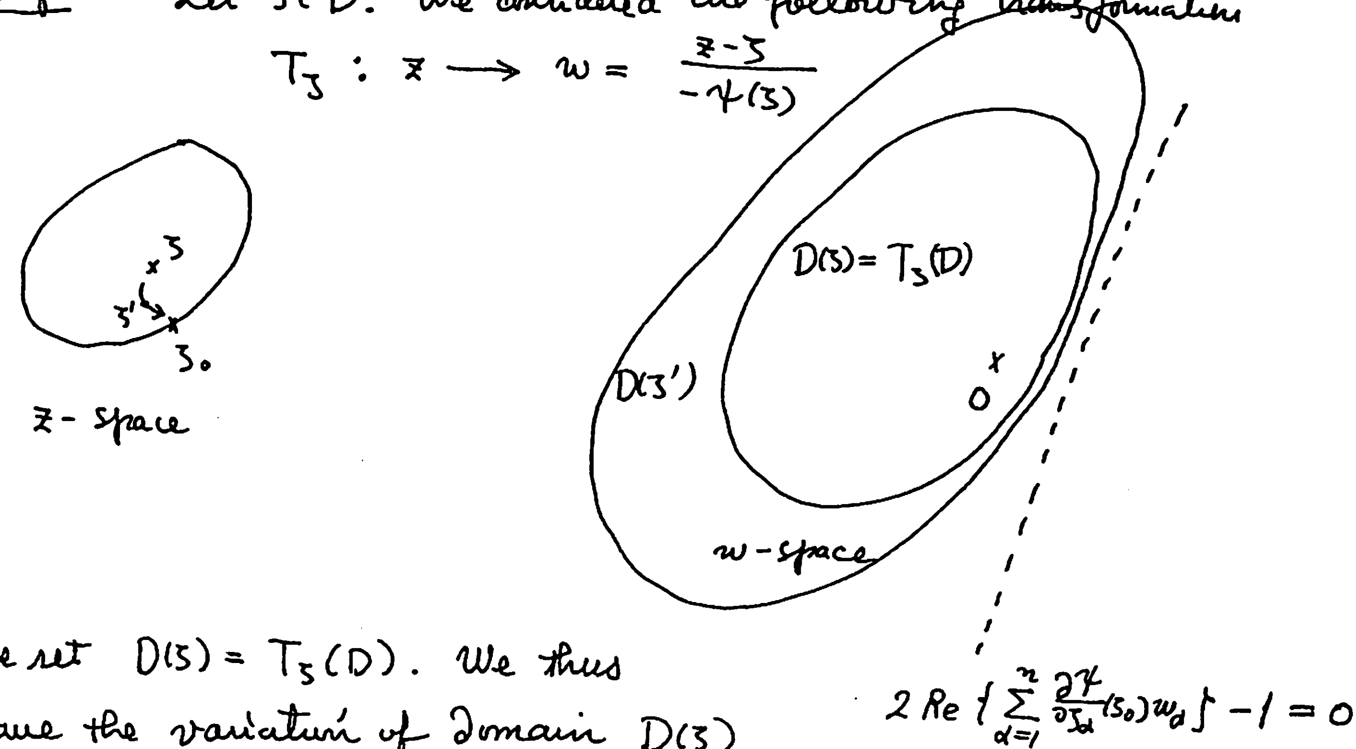
(2) $$\lim_{\zeta \to \zeta_0} \frac{\partial \Lambda}{\partial \zeta_\alpha}(\zeta) \psi(\zeta)^{2n-1} = (2n-2) \| \mathrm{Grad}\, \psi(\zeta_0) \|^{2n-2} \frac{\partial \psi}{\partial \zeta_\alpha}(\zeta_0) \quad ;$$

(3) $$\lim_{\zeta \to \zeta_0} \frac{\partial^2 \Lambda}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta) \cdot \psi(\zeta)^{2n} = -(2n-2)(2n-1) \| \mathrm{Grad}\, \psi(\zeta_0) \|^{2n-2} \frac{\partial \psi}{\partial \zeta_\alpha}(\zeta_0) \frac{\partial \psi}{\partial \bar{\zeta}_\beta}(\zeta_0) \quad ;$$

(4) $$\lim_{\zeta \to \zeta_0} \frac{\partial^2 \log(-\Lambda)(\zeta)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta} \cdot \psi(\zeta)^2 = (2n-2) \frac{\partial \psi}{\partial \zeta_\alpha}(\zeta_0) \frac{\partial \psi}{\partial \bar{\zeta}_\beta}(\zeta_0)$$

<u>Proof.</u> Let $\zeta \in D$. We considered the following transformation

$$T_\zeta : z \longrightarrow w = \frac{z-\zeta}{-\psi(\zeta)}$$



We set $D(\zeta) = T_\zeta(D)$. We thus have the variation of domain $D(\zeta)$ in $\mathbb{C}^n$,

$$\mathcal{D} : \zeta \longrightarrow D(\zeta) \quad (\zeta \in D)$$

As usual, we consider $\mathcal{D}$ a domain in $D \times \mathbb{C}^n$

$$\mathcal{D} = \bigcup_{\zeta \in D} (\zeta, D(\zeta)).$$

Each $D(\zeta)$ is similar to D with similar ratio $-\frac{1}{\psi(\zeta)} > 0$.
As already noted, if $\zeta \to \zeta_0 \in \partial D$, then $D(\zeta)$ approaches the half-space defined by

$$D(\zeta_0) = \left\{ w \in \mathbb{C}^n \,/\, 2\,\mathrm{Re}\left\{\sum_{\alpha=1}^{n} \frac{\partial\psi}{\partial\zeta_\alpha}(\zeta_0) w_\alpha\right\} - 1 < 0 \right\}$$

Since $\zeta \in D$ is mapped by T_ζ to $0 \in D(\zeta)$, we have the Green's function $g(\zeta, w)$ and the Robin constant $\lambda(\zeta)$ for $(D(\zeta), 0)$. Moreover, we got

(2.1) $g(\zeta, w) = \psi(\zeta)^{2n-2}\, G(\zeta, z)$;

(2.2) $\lambda(\zeta) = \psi(\zeta)^{2n-2}\, \Lambda(\zeta)$

where $w = \dfrac{z-\zeta}{-\psi(\zeta)}$

The variation $\mathcal{D} : \zeta \to D(\zeta)$ $(\zeta \in D)$ is smooth, $\lambda(\zeta)$ is of class C^2 on D and $g(\zeta, w)$ is of class C^2 with respect to (ζ, w) in $\mathcal{D}$ except the pole $D \times \{0\}$.

We have proved in the note of March 14 that

(2.3) $\lambda(\zeta)$ is of class C^1 up to ∂D.

By the same proof, we see that

(2.4) $g(\zeta, w)$ is of class C^1 for (ζ, w) in $\mathcal{D} \cup \bigcup_{\zeta_0 \in \partial D} (\zeta_0, D(\zeta_0))$

where $D(\zeta_0) = \{ w \in \mathbb{C}^n \mid 2\,\mathrm{Re}\{\sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial \zeta_\alpha}(\zeta_0) w_\alpha\} - 1 < 0 \}$.

We note that

$$\frac{\partial g}{\partial \zeta_\alpha}(\zeta, 0) = \frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta) \quad \text{for } \zeta \in D \cup \partial D ;$$

In the note of March 7 we have shown

(2.5) $$\frac{\partial \Lambda(\zeta)}{\partial \zeta_\alpha} = G_\alpha(\zeta, \zeta) ;$$

(2.6) $$\frac{\partial^2 \Lambda(\zeta)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta} = 2\, \frac{\partial G_\alpha}{\partial \bar{z}_\beta}(\zeta, \zeta)$$

where $G_\alpha(\zeta, z) = \dfrac{\partial G}{\partial \zeta_\alpha}(\zeta, z) + \dfrac{\partial G}{\partial z_\alpha}(\zeta, z)$ in $D \times D$.
$G_\alpha(\zeta, z)$ is a real analytic function for $(\zeta, z) \in D \times D$.

We want to represent $G_\alpha(\zeta, z)$ by means of $g(\zeta, w)$ and its derivatives :

By differentiation of (2.1) by $\bar{z}_\alpha$, we have

$$\text{(i)} \quad \frac{\partial g}{\partial \bar{w}_\alpha} \cdot \frac{1}{-\psi} = \psi^{2n-2} \cdot \frac{\partial G}{\partial \bar{z}_\alpha}$$

Formula (2.1) is precisely written as follows:

$$g\left(\zeta_1, \cdots, \zeta_n, \frac{z_1-\zeta_1}{-\psi(\zeta_1,\cdots,\zeta_n)}, \cdots, \frac{z_n-\zeta_n}{-\psi(\zeta_1,\cdots,\zeta_n)}\right) = \psi(\zeta)^{2n-2} G(\zeta, z).$$

By differentiation of both sides by ζ_α, we have

$$\text{(ii)} \quad \frac{\partial g}{\partial \zeta_\alpha} + \left(\frac{1}{-\psi}\right)\left(-\frac{\partial g}{\partial w_\alpha}\right) + \sum_{i=1}^n (z_i-\zeta_i)\frac{\frac{\partial \psi}{\partial \zeta_\alpha}}{(-\psi)^2}\frac{\partial g}{\partial w_i} + \sum_{i=1}^n (\overline{z_i-\zeta_i})\frac{\frac{\partial \psi}{\partial \zeta_\alpha}}{(-\psi)^2}\frac{\partial g}{\partial \bar{w}_i}$$

$$= (2n-2)\psi^{2n-3}\frac{\partial \psi}{\partial \zeta_1} G + \psi^{2n-2}\frac{\partial G}{\partial \zeta_\alpha}.$$

By (i)+(ii), we get

$$\frac{\partial g}{\partial \zeta_\alpha} + \sum_{i=1}^n \left(\frac{z_i-\zeta_i}{-\psi}\right)\frac{\frac{\partial \psi}{\partial \zeta_\alpha}}{-\psi}\cdot\frac{\partial g}{\partial w_i} + \sum_{i=1}^n \left(\overline{\frac{z_i-\zeta_i}{-\psi}}\right)\frac{\frac{\partial \psi}{\partial \zeta_\alpha}}{-\psi}\cdot\frac{\partial g}{\partial \bar{w}_i}$$

$$= (2n-2)\cdot\frac{\frac{\partial \psi}{\partial \zeta_\alpha}}{\psi}\cdot\psi^{2n-2}G + \psi^{2n-2}\left(\frac{\partial G}{\partial \bar{z}_\alpha} + \frac{\partial G}{\partial \zeta_\alpha}\right)$$

$$\therefore \quad \frac{\partial g}{\partial \zeta_\alpha} - \frac{\frac{\partial \psi}{\partial \zeta_\alpha}}{\psi}\left\{\sum_{i=1}^n\left(w_i\frac{\partial g}{\partial w_i} + \bar{w}_i\frac{\partial g}{\partial \bar{w}_i}\right)\right\} \qquad \text{by 2.1}$$

$$= (2n-2)\frac{\frac{\partial \psi}{\partial \zeta_\alpha}}{\psi}g + \psi^{2n-2}G_\alpha$$

It follows that

$$G_\alpha(\zeta, z) = \frac{1}{\psi^{2n-1}}\left\{\psi\frac{\partial g}{\partial \zeta_\alpha} - (n-1)\frac{\partial \psi}{\partial \zeta_\alpha}\left[\left(g + \frac{1}{n-1}\sum_{i=1}^n w_i\frac{\partial g}{\partial w_i}\right) + \overline{\left(g + \frac{1}{n-1}\sum_{i=1}^n w_i\frac{\partial g}{\partial w_i}\right)}\right]\right\}$$

Note that $\frac{\partial g}{\partial \zeta_\alpha}$ has no singularity in $D(\zeta)$ and assumes $\frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta)$ at $w=0$.
and that $g + \frac{1}{n-1}\sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i}$ has no singularity in $D(\zeta)$
and assumes λ at $w=0$.

For the sake of convenience, we set

(2.7) $H_0(\zeta, w) = g + \frac{1}{n-1}\sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i}$;

(2.8) $H_\alpha(\zeta, w) = \psi \frac{\partial g}{\partial \zeta_\alpha} - (n-1)\cdot\frac{\partial \psi}{\partial \zeta_\alpha}\cdot(H_0 + \overline{H_0})$

for $\zeta \in D \cup \partial D$ and $w \in D(\zeta) \cup \partial D(\zeta)$.

We thus have

$$G_\alpha(\zeta, z) = \frac{1}{\psi^{2n-1}} H_\alpha(\zeta, w)$$

for $(\zeta, z) \in D \times D$ and $w = \frac{z-\zeta}{-\psi(\zeta)}$

Although $G_\alpha(\zeta, z)$ is not defined for $\zeta \in \partial D$, $z \in D$, the product

$$G_\alpha(\zeta, z)\cdot \psi^{2n-1}(\zeta) = H_\alpha(\zeta, w)$$

is defined continuously in $\zeta \in D \cup \partial D$ and $w \in D(\zeta)$.

By (2.6) we have

$$\frac{\partial^2 \Lambda}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta) = 2\,\frac{\partial G_\alpha}{\partial \bar{z}_\beta}(\zeta, \zeta)$$

$$= 2\,\frac{1}{\psi^{2n-1}}\cdot\frac{\partial H_\alpha}{\partial \bar{w}_\beta}(\zeta, 0)\cdot\left(\frac{1}{-\psi(\zeta)}\right)$$

$$= \frac{-2}{\psi^{2n}}\left\{\psi \frac{\partial^2 g}{\partial \bar{w}_\beta \partial \zeta_\alpha}(\zeta, 0) - (n-1)\frac{\partial \psi}{\partial \zeta_\alpha}\left[\frac{\partial}{\partial \bar{w}_\beta}(H_0 + \overline{H_0})\right]_{(\zeta,0)}\right\}$$

Now, let $\zeta_0 \in \partial D$. Since $g(\zeta, w)$ is of class C^1 for (ζ, w) where $\zeta \in D \cup \partial D$ and $w \in D(\zeta)$, and since $g(\zeta, w)$ is harmonic in w, it follows that $\frac{\partial g}{\partial \bar{w}_\beta}$ is also of class C^1 there. Consequently,

$$\lim_{\zeta \to \zeta_0} \frac{\partial^2 g}{\partial \zeta_\alpha \partial \bar{w}_\beta}(\zeta, 0) = \frac{\partial^2 g}{\partial \zeta_\alpha \partial \bar{w}_\beta}(\zeta_0, 0),$$

which is finite. Hence

$$\lim_{\zeta \to \zeta_0} \psi(\zeta) \frac{\partial^2 g}{\partial \zeta_\alpha \partial \bar{w}_\beta}(\zeta, 0) = 0 .$$

On the other hand, as $\zeta \longrightarrow \zeta_0$, we have

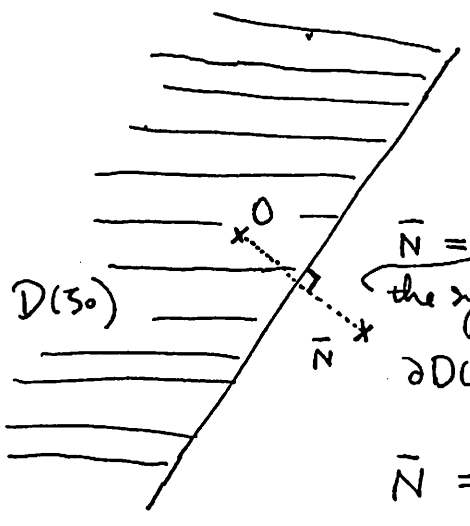
$$H_0(\zeta, w) = g(\zeta, w) + \frac{1}{n-1} \sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i}(\zeta, w)$$

$$\longrightarrow H_0(\zeta_0, w) = g(\zeta_0, w) + \frac{1}{n-1} \sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i}(\zeta_0, w)$$

on any compact set in $D(\zeta_0)$ where

$$g(\zeta_0, w) = \text{the Green's function for } (D(\zeta_0), \zeta)$$

$$= \frac{1}{\|w\|^{2n-2}} - \frac{1}{\|w - \bar{N}\|^{2n-2}}$$



$\bar{N}$ = the symmetric point of O with respect to the half-plane

$$\partial D(\zeta_0) : \quad 2 \operatorname{Re}\left\{ \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial \zeta_\alpha}(\zeta_0) w_\alpha \right\} - 1 = 0$$

$$\bar{N} = \frac{\overline{\operatorname{Grad} \psi(\zeta_0)}}{\|\operatorname{Grad} \psi(\zeta_0)\|^2}$$

$$\bar{N}_i = \frac{(\partial \psi / \partial \bar{\zeta}_i)(\zeta_0)}{\|\operatorname{Grad} \psi(\zeta_0)\|^2}$$

$$\therefore \quad H_0(\zeta_0, w) = g(\zeta_0, w) + \frac{1}{n-1} \sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i}(\zeta_0, w)$$

$$= \frac{\sum_{i=1}^{n} \bar{w}_i \bar{N}_i - \|N\|^2}{\|w - \bar{N}\|^{2n}}$$

In particular, we get

$$\lim_{\zeta \to \zeta_0} \left[\frac{\partial}{\partial \bar{w}_\beta} (H_0 + \bar{H}_0) \right]_{(\zeta, 0)}$$

$$= \left[\frac{\bar{N}_\beta}{\|w - \bar{N}\|^{2n}} - n \frac{(w_\beta - \bar{N}_\beta)(\sum \bar{w}_i \bar{N}_i - \|N\|^2)}{\|w - \bar{N}\|^{2n+2}} - n \frac{(w_\beta - \bar{N}_\beta)(\sum w_i N_i - \|N\|^2)}{\|w - \bar{N}\|^{2n+2}} \right]_{w=0}$$

$$= \frac{\bar{N}_\beta}{\|N\|^{2n}} - 2n \frac{\bar{N}_\beta \|N\|^2}{\|N\|^{2n+2}}$$

$$\left(\text{where } \|N\|^2 = \frac{1}{\|\operatorname{Grad} \psi\|^2} \right)$$

$$= -(2n-1) \|\operatorname{Grad} \psi\|^{2n-2} \frac{\partial \psi}{\partial \bar{z}_\beta}(\zeta_0)$$

$$\therefore \quad \lim_{\zeta \to \zeta_0} \frac{\partial^2 \Lambda}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta) \cdot \psi^{2n}(\zeta) = -(2n-2)(2n-1) \|\operatorname{Grad} \psi(\zeta_0)\|^{2n-2} \frac{\partial \psi}{\partial z_\alpha}(\zeta_0) \frac{\partial \psi}{\partial \bar{z}_\beta}(\zeta_0)$$

which proves (3) of Lemma 2.1.

(1) and (2) are more easily to prove:

$$\lim_{\zeta \to \zeta_0} \Lambda(\zeta) \psi(\zeta)^{2n-2} = \lim_{\zeta \to \zeta_0} \lambda(\zeta)$$

$$= \lambda(\zeta_0) = \lim_{w \to 0} \left(g(\zeta_0, w) - \frac{1}{\|w\|^{2n-2}} \right)$$

$$= \frac{-1}{\|0 - \bar{N}\|^{2n-2}} = -\|\operatorname{Grad} \psi(\zeta_0)\|^{2n-2}$$, which proves (1).

Since $\lambda(\zeta) = \psi(\zeta)^{2n-2} \Lambda(\zeta)$, we have

$$\frac{\partial \lambda}{\partial \zeta_\alpha} = (2n-2) \psi^{2n-3} \frac{\partial \psi}{\partial \zeta_\alpha} \Lambda + \psi^{2n-2} \frac{\partial \Lambda}{\partial \zeta_\alpha}.$$

$$(2,9) \quad \therefore \quad \psi^{2n-1} \frac{\partial \Lambda}{\partial \zeta_\alpha} = \psi \frac{\partial \lambda}{\partial \zeta_\alpha} - (2n-2) \frac{\partial \psi}{\partial \zeta_\alpha} \lambda$$

Because $\frac{\partial\lambda}{\partial\zeta_\alpha}(\zeta)$ is continuous up to ∂D, we see that

$$\lim_{\zeta\to\zeta_0}\psi^{2n-1}\frac{\partial\Lambda}{\partial\zeta} = -(2n-2)\frac{\partial\psi}{\partial\zeta_\alpha}(\zeta_0)\,\lambda(\zeta_0)$$
$$= (2n-2)\,\|\mathrm{Grad}\,\psi(\zeta_0)\|^{2n-2}\,\frac{\partial\psi}{\partial\zeta_\alpha}(\zeta_0),$$

which proves (2).

Also, (4) is proved by (1), (2) and (3) as follows:

$$\lim_{\zeta\to\zeta_0}\frac{\partial^2\log(-\Lambda)}{\partial\zeta_\alpha\partial\bar\zeta_\beta}\psi(\zeta)^2$$
$$= \lim_{\zeta\to\zeta_0}\frac{\Lambda\frac{\partial^2\Lambda}{\partial\zeta_\alpha\partial\bar\zeta_\beta}-\frac{\partial\Lambda}{\partial\zeta_\alpha}\frac{\partial\Lambda}{\partial\bar\zeta_\beta}}{\Lambda^2}\cdot\psi(\zeta)^2$$
$$= \lim_{\zeta\to\zeta_0}\frac{(\Lambda\psi^{2n-2})\left(\frac{\partial^2\Lambda}{\partial\zeta_\alpha\partial\bar\zeta_\beta}\psi^{2n}\right)-\left(\frac{\partial\Lambda}{\partial\zeta_\alpha}\psi^{2n-1}\right)\left(\frac{\partial\Lambda}{\partial\bar\zeta_\beta}\psi^{2n-1}\right)}{(\Lambda\psi^{2n-2})^2}$$
$$= \frac{\left(-\|\mathrm{Grad}\,\psi(\zeta_0)\|^{2n-2}\right)\left(-(2n-1)(2n-2)\|\mathrm{Grad}\,\psi(\zeta_0)\|^{2n-2}\frac{\partial\psi}{\partial\zeta_\alpha}(\zeta_0)\frac{\partial\psi}{\partial\bar\zeta_\beta}(\zeta_0)\right)-(2n-2)^2\|\mathrm{Grad}\,\psi(\zeta_0)\|^{4n-4}\frac{\partial\psi}{\partial\zeta_\alpha}(\zeta_0)\frac{\partial\psi}{\partial\bar\zeta_\beta}(\zeta_0)}{\|\mathrm{Grad}\,\psi(\zeta_0)\|^{4n-4}}$$
$$= (2n-2)\frac{\partial\psi}{\partial\zeta_\alpha}(\zeta_0)\frac{\partial\psi}{\partial\bar\zeta_\beta}(\zeta_0).$$

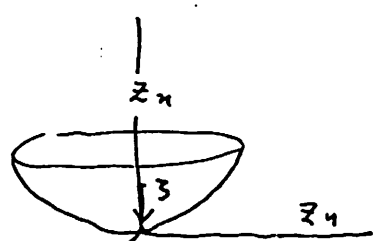
C.Q.F.D.

By Lemma 2.1, we have

$$\psi(\zeta)^2\cdot\left(\frac{\partial^2\log(-\Lambda)}{\partial\zeta_\alpha\partial\bar\zeta_\beta}\right)_{\alpha,\beta=1,\cdots,n}\xrightarrow{\zeta\to\zeta_0\in\partial D}(2n-2)\begin{pmatrix}\frac{\partial\psi}{\partial\zeta_1}\frac{\partial\psi}{\partial\bar\zeta_1} & \cdots & \frac{\partial\psi}{\partial\zeta_1}\frac{\partial\psi}{\partial\bar\zeta_n}\\ \vdots & & \vdots\\ \frac{\partial\psi}{\partial\zeta_n}\frac{\partial\psi}{\partial\bar\zeta_1} & \cdots & \frac{\partial\psi}{\partial\zeta_n}\frac{\partial\psi}{\partial\bar\zeta_n}\end{pmatrix}(\zeta_0)$$

for $\zeta\in D$, it is positive definite

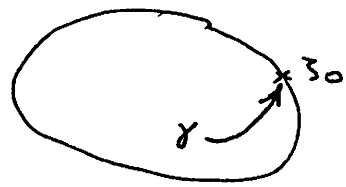
for $\zeta_0\in D$, it is non-negative definite with rank 1.



For example, let $\mathrm{Grad}\,\psi(\zeta_0)=(0,\cdots,0,1)$. Then the limit matrix is $\begin{pmatrix}0&\cdots&0\\ \vdots&0&\vdots\\ 0&\cdots 0&1\end{pmatrix}$

It follows that if $\zeta\to\zeta_0$ along z_n-axis, then $\left(\frac{\partial^2\log(-\Lambda)}{\partial\zeta_\alpha\partial\bar\zeta_\beta}\right)_{\alpha,\beta=1,\cdots,n}$ behaves like as $(2n-2)\frac{1}{\psi(\zeta)^2}$. So we easily have the following

Theorem 2.1 Let D be any smooth domain in $\mathbb{C}^n$, and let $\zeta_0 \in \partial D$. Let $\gamma : t \to \zeta(t)$ $(0 \leqq t \leqq 1)$ be a differentiable curve such that



$\zeta(t) \in D$ $(0 \leqq t < 1)$; $\zeta(1) = \zeta_0$;

(i) $\displaystyle\sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial \zeta_\alpha}(\zeta_0) \frac{d\zeta_\alpha}{dt}(1) \neq 0$.

Then, if we put

$$ds_\gamma^2 = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\psi)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta(t)) \frac{d\zeta_\alpha(t)}{dt} \overline{\left(\frac{d\zeta_\beta(t)}{dt}\right)} \qquad (0 \leqq t < 1),$$

then there exists $0 < t_0 < 1$ such that

$$ds_\gamma^2(t) > 0 \text{ for all } 0 < t_0 < {}^\forall t < 1 \ ;$$

(ii) $\displaystyle\int_{t_0}^{1} \sqrt{ds_\gamma^2(t)} = +\infty$

<u>Proof</u> By Lemma 2 1, we have

$$\lim_{t \to 1} ds_\gamma^2 \cdot \left(\psi(\zeta(t))\right)^2 = \lim_{t \to 1} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\psi)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta(t)) \cdot \psi(\zeta(t))^2 \frac{d\zeta_\alpha(t)}{dt} \overline{\left(\frac{d\zeta_\beta(t)}{dt}\right)}$$

$$= (2n-2) \sum_{\alpha,\beta=1}^{n} \frac{\partial \psi}{\partial \zeta_\alpha}(\zeta_0) \frac{\partial \psi}{\partial \bar{\zeta}_\beta}(\zeta_0) \frac{d\zeta_\alpha}{dt}(1) \overline{\frac{d\zeta_\beta}{dt}(1)}$$

$$= (2n-2) \left| \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial \zeta_\alpha}(\zeta_0) \frac{d\zeta_\alpha}{dt}(1) \right|^2 > 0 \quad \text{by (i)}.$$

By continuity of $\zeta(t)$ at $t = 1$ and $\zeta(1) = \zeta_0$, there exists $1 > t_0 > 0$ such that for ${}^\forall t \geqq t_0$

$$ds_\gamma^2 \ \psi(\zeta(t))^2 > (n-1) \left| \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial \zeta_\alpha}(\zeta(t)) \frac{d\zeta_\alpha}{dt}(t) \right|^2$$

$$\therefore \ ds_\gamma^2(t) > (n-1) \left| \sum_{\alpha=1}^{n} \frac{\frac{\partial \psi}{\partial \zeta_\alpha}(\zeta(t)) \frac{d\zeta_\alpha}{dt}(t)}{\psi(\zeta(t))} \right|^2$$

$$= (n-1) \left| \frac{\partial \log(-\psi(\zeta(t)))}{\partial t} \right|^2$$

$$\geqq \frac{n-1}{4} \left| \frac{d \log(-\psi(t))}{dt} \right|^2 \quad \text{for } t \geqq t_0$$

$$\therefore \int_{t_0}^{1} \sqrt{ds_\gamma^2} \geq \frac{\sqrt{n-1}}{2} \int_{t_0}^{1} \left| \frac{d}{dt} \log(-\psi(\zeta(t))) \right| dt$$

$$\geq \frac{\sqrt{n-1}}{2} \left| \int_{t_0}^{1} \frac{d}{dt} \log(-\psi(\zeta(t))) dt \right|$$

$$= \frac{\sqrt{n-1}}{2} \left| \log(-\psi(\zeta_0)) - \log(-\psi(\zeta(t_0))) \right|$$

$$= +\infty . \qquad \text{C.Q.F.D.}$$

Remark 2.1 If the condition (i) is not satisfied, then the statement (ii) is not true, in general.

For, we can find a simple example:

Let $\hat{D} = (\|z\| > 1)$ in $\mathbb{C}^2$ and let $\zeta \in D$. Then the Green's function $\hat{g}(\zeta, z)$ and the Robin constant $\hat{\lambda}(\zeta)$ for $(\hat{D}, \zeta)$ is given by

$$\begin{cases} \hat{g}(\zeta, z) = \dfrac{1}{\|z-\zeta\|^2} - \dfrac{1}{\|z-\zeta^*\|^2} & \text{where } \zeta^* = \dfrac{\zeta}{\|\zeta\|^2} \\ \hat{\lambda}(\zeta) = -\dfrac{1}{(\|\zeta\|^2-1)^2} \end{cases}$$

Consider the curve $\gamma : t \to (1, 1-t) = \zeta(t) \quad (0 \leq t \leq 1)$

Then $\gamma \subset \hat{D}$ for $0 \leq t \leq 1$; $\zeta(1) = (1, 0) \in \partial\hat{D}$ and

$\left(\frac{d\zeta_1}{dt}, \frac{d\zeta_2}{dt}\right) = (0, -1)$ for all t.

$$\therefore d\hat{s}_\gamma(t) = \sum_{\alpha,\beta=1}^{n} \left(\frac{\partial^2 \left(\log \frac{1}{\|\zeta\|^2-1}\right)}{\partial\zeta_\alpha \partial\bar{\zeta}_\beta} \right)_{\zeta(t)} \frac{d\zeta_\alpha}{dt} \overline{\left(\frac{d\zeta_\beta}{dt}\right)}$$

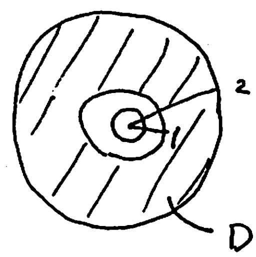
$$= - \left[\frac{\partial^2 \log(|\zeta_1|^2 + |\zeta_2|^2 - 1)}{\partial\zeta_2 \partial\bar{\zeta}_2} \right]_{\zeta=(1,1-t)}$$

$$= \left[\frac{|\zeta_1|^2 - 1}{|\zeta_1|^2 + |\zeta_2|^2 - 1} \right]_{(1,1-t)}$$

$$= 0 \quad \text{for } 0 \leq {}^\forall t < 1$$

Since $\hat{D}$ is not bdd, we have to make some modifications:

Let $D = \{1 < \|z\| < 2\}$ in $\mathbb{C}^2$ and let $\zeta \in D$. We consider



the Green's function $g(\zeta, z)$ and the Robin constant $\lambda(\zeta)$ for $(D; \zeta)$. Then $\hat{g} - g = u(\zeta, z)$ is harmonic for z in D and $= 0$ on $\|z\| = 1$. It follows that $u(\zeta, z)$ is harmonically extended to $1/2 < \|z\| < 2$. Moreover, $u(\zeta, z) > 0$; $u(\zeta, z) = u(z, \zeta)$ and $u(\zeta, \zeta) = \hat{\lambda} - \lambda$. Hence $u(\zeta, z)$ is harmonic for ζ as well as z, so that $u(\zeta, z)$ is real analytic for (ζ, z) in $(1/2 < \|z\| < 2) \times (1/2 < \|z\| < 2)$. In particular, $u(\zeta, \zeta)$ is real analytic in $(1/2 < \|z\| < 2)$.

$$\therefore \quad \lambda(\zeta) = -\frac{1}{(\|\zeta\|^2 - 1)^2} + u(\zeta, \zeta)$$

$$= -\frac{1 - u(\zeta,\zeta)(\|\zeta\|^2 - 1)}{(\|\zeta\|^2 - 1)^2}.$$

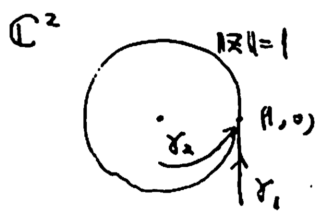
If we set $v(\zeta) = u(\zeta,\zeta)(\|\zeta\|^2 - 1)$, then $v(\zeta)$ is real analytic in $(1/2 < \|\zeta\| < 2)$ and $|v(\zeta)| \ll 1$ near $\|\zeta\| = 1$. It follows that

$$ds_\gamma^2 = \left[\frac{\partial^2 \log\left(\frac{1 - v(\zeta)}{(\|\zeta\|^2 - 1)^2}\right)}{\partial \zeta_2 \partial \bar{\zeta}_2}\right]_{\zeta = (1, 1-t)},$$

$$= \left[\frac{\partial^2 \log(1 - v(\zeta))}{\partial \zeta_2 \partial \bar{\zeta}_2}\right]_{\zeta = (1, 1-t)}$$

$$< {}^{\exists} M \text{ independent of } t \text{ near } t = 1.$$

Thus, (ii) does not occur for the pair (D, γ). $\lrcorner$



Both inside and outside of $\|z\| = 1$, we have the same formula:

$$\lambda(\zeta) = \frac{-1}{(\|\zeta\|^2 - 1)^2}$$

But $\int_{\gamma_1} ds \neq +\infty$, $\int_{\gamma_2} ds = +\infty$ (tangential γ_i)

§3. *Inequalités.*

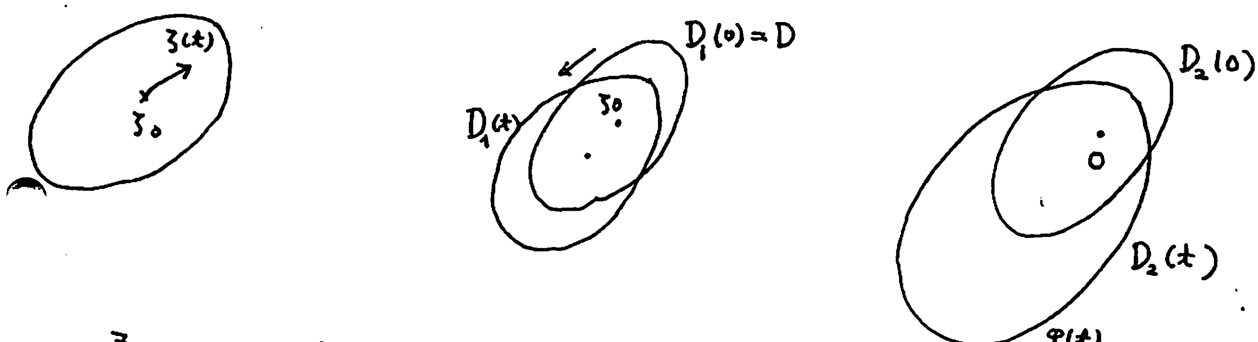
We recall a formula in the note of March 1st :
Let D be a domain in $\mathbb{C}^n$ with smooth boundary. We denote by $G(\zeta, z)$ and $\Lambda(\zeta)$ the Green's fn and the Robin const for (D, ζ). Let $\zeta_0 \in D$ and let $a \in \mathbb{C}^n$ with $\|a\| = 1$. We want to construct the variation which realize the value

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(\zeta_0)\, a_\alpha \overline{a_\beta} .$$

We set $\zeta = \zeta(t) = \zeta_0 + a t$ where $t \in B = (|t| < \rho)$ and $\varphi(t) = c_0 + c_1 t$ where

$$\begin{cases} c_0 = \log(-\Lambda)(\zeta_0) \\ c_1 = 2 \displaystyle\sum_{\alpha=1}^{n} a_\alpha \frac{\partial \log(-\Lambda)}{\partial \zeta_\alpha}(\zeta_0). \end{cases}$$

We consider the following transformations :



$$z \longrightarrow w = z - at \longrightarrow W = e^{\frac{\varphi(t)}{2n-2}}(w - \zeta_0)$$

$$z = W e^{\frac{-\varphi(t)}{2n-2}} + at + \zeta_0 \longleftarrow w = \zeta_0 + e^{\frac{-\varphi(t)}{2n-2}} W \longleftarrow$$

: $(t, \zeta(t))$	(t, ζ_0)	$(t, 0)$
: $G(\zeta(t), z)$	$g_1(t, w)$	$g_2(t, W)$
: $\Lambda(\zeta(t))$	$\lambda_1(t)$	$\lambda_2(t)$

We have the relations:

$$\begin{cases} g_1(t,w) = G(\zeta(t), z) \\ \lambda_1(t) = \Lambda(\zeta(t)) \end{cases}$$

$$\begin{cases} g_2(t, W) = e^{-\mathrm{Re}\,\varphi(t)} \cdot g_1(t,w) \\ \lambda_2(t) = e^{-\mathrm{Re}\,\varphi(t)} \cdot \lambda_1(t) \end{cases}$$

$$\therefore \quad g_2(t,W) = e^{-\mathrm{Re}\,\varphi(t)} G(\zeta_0 + at;\, W e^{\frac{-\varphi(t)}{2n-2}} + \zeta_0 + at)$$

$$= e^{-c_0 - \frac{c_1}{2}t - \frac{\bar c_1}{2}\bar t} G\left(\zeta_{01} + a_1 t, \dots, \zeta_{0n} + a_n t;\, W_1 e^{\frac{-c_0 - c_1 t}{2n-2}} + \zeta_{01} + a_1 t, \dots, W_n e^{\frac{-c_0-c_1 t}{2n-2}} + \zeta_{0n} + a_n t\right)$$

and $$\frac{\partial^2 \lambda_2}{\partial t \partial \bar t}(0) = -\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial \zeta_\alpha \partial \bar\zeta_\beta}(\zeta_0)\, a_\alpha \bar a_\beta .$$

Since the variation $\mathcal{D}_2 : t \to D_2(t)$ $(t \in B)$ is smooth, we get by fundamental formula (Theorem 4.1 in the note of Feb 15)

$$\frac{\partial^2 \lambda_2}{\partial t \partial \bar t}(0) = -\frac{1}{(n-1)\omega_{2n}} \int_{\partial D_2(0)} k_2(0,W)\, \|\mathrm{Grad}_W\, g_2(0,W)\|^2 dS_W$$

$$- \frac{4}{(n-1)\omega_{2n}} \iint_{D_2(0)} \left\{ \sum_{\alpha=1}^{n} \left| \frac{\partial^2 g_2}{\partial t \partial \bar W_\alpha} \right|^2_{(0,W)} \right\} dV_W .$$

We write the right-hand side by means of $G(\zeta, z)$ and its derivatives as follows:

$$\left[\frac{\partial g_2}{\partial t}\right]_{t=0} = \frac{1}{-\Lambda(\zeta_0)} \cdot H(a, \zeta_0, z) ;$$

$$H(a, \zeta_0, z) = -\frac{c_1}{2}\left\{G + \frac{1}{n-1}\sum_{\alpha=1}^{n}(z_\alpha - \zeta_{0\alpha})\frac{\partial G}{\partial z_\alpha}\right\} + \sum a_\alpha\left(\frac{\partial G}{\partial \zeta_\alpha} + \frac{\partial G}{\partial z_\alpha}\right)$$

Since $W = (z - \zeta_0)(-\Lambda(\zeta_0))^{1/(2n-2)}$, we have

$$\frac{\partial^2 g_2}{\partial t\,\partial \overline{W}_\alpha}(0,W) = \frac{1}{(-\lambda(\zeta_0))^{\frac{2n-1}{2n-2}}}\;\frac{\partial H}{\partial \overline{z}_\beta}(a,\zeta_0,z)$$

$$dV_W = (dV_z)\cdot\left(-\lambda(\zeta_0)\right)^{\frac{2}{2n-2}}$$

$$\therefore \iint_{D_2(0)}\left\{\sum_{\alpha=1}^{n}\left|\frac{\partial^2 g_2}{\partial t\,\partial \overline{W}_\alpha}\right|^2(0,W)\right\}dV_W = \frac{1}{-\lambda(\zeta_0)}\iint_{D}\left\{\sum_{\alpha=1}^{n}\left|\frac{\partial}{\partial \overline{z}_\alpha}H(a,\zeta_0,z)\right|^2\right\}dV_z$$

Assume that $\mathcal{D}$ is pseudoconvex in $\mathbb{C}^n$. Then the variation $\mathcal{D}_2 : t \to D_2(t)\ (t\in B)$ is function-theoretic, so that

$$k_2(t,W) \geqq 0 \quad \text{on } \partial\mathcal{D}$$

$$\therefore \frac{\partial^2\lambda}{\partial t\,\partial \overline{t}}(0) \leqq -\frac{4}{(n-1)\omega_{2n}}\iint_{D_2(0)}\left\{\sum_{\alpha=1}^{n}\left|\frac{\partial^2 g_2}{\partial t\,\partial \overline{W}_\alpha}(0,W)\right|^2\right\}dV_W$$

We thus obtain

$$\sum_{\alpha,\beta=1}^{n}\frac{\partial^2 \log(-\lambda)}{\partial \zeta_\alpha\,\partial \overline{\zeta}_\beta}(\zeta_0)\,a_\alpha\overline{a}_\beta \geqq \frac{4}{(n-1)\omega_{2n}}\,\frac{1}{-\lambda(\zeta_0)}\iint_{D}\left\{\sum_{\alpha=1}^{n}\left|\frac{\partial}{\partial \overline{z}_\alpha}H(a,\zeta_0,z)\right|^2\right\}dV \geqq 0.$$

Since $\zeta_0 \in D$ is arbitrary, it turns out, for $\forall \zeta \in D$,

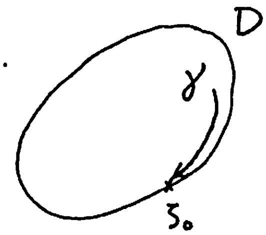
$$\boxed{\begin{gathered}\sum_{\alpha,\beta=1}^{n}\frac{\partial^2 \log(-\lambda)}{\partial \zeta_\alpha\,\partial \overline{\zeta}_\beta}\,a_\alpha\overline{a}_\beta \geqq \frac{4}{(n-1)\omega_{2n}}\,\frac{1}{-\lambda(\zeta)}\iint_{D}\left\{\sum_{\alpha=1}^{n}\left|\frac{\partial}{\partial \overline{z}_\alpha}H(a,\zeta,z)\right|^2\right\}dV_z \\ \text{where}\quad a\in\mathbb{C}^n \text{ with } \|a\|=1 \\ H(a,\zeta,z) = -\left(\sum_{\alpha=1}^{n}a_\alpha\frac{\partial \log(-\lambda)}{\partial \zeta_\alpha}\right)\left(G+\frac{1}{n-1}\sum_{\alpha=1}^{n}(z_\alpha-\zeta_\alpha)\frac{\partial G}{\partial z_\alpha}\right)+\sum_{\alpha=1}^{n}a_\alpha\left(\frac{\partial G}{\partial \zeta_\alpha}+\frac{\partial G}{\partial z_\alpha}\right)\end{gathered}}$$

Let $\zeta \to \zeta_0 \in \partial D$. Then $\frac{1}{-\lambda(\zeta)} \to 0$ and the integral of the right-hand side will become probably $+\infty$. We analyze the right-hand side and prove the following two Theorems:

Theorem 3.1 Assume that D is pseudoconvex in $\mathbb{C}^n$. Let $\gamma : t \to \zeta(t)$ $(t \in [0,1))$ be a curve in D such that

$$\lim_{t \to 1} \zeta(t) = \zeta(1) = \zeta_0 \in \partial D \, ;$$

$$\int_\gamma |dz| < +\infty \quad \text{i.e., Euclidean length of } \gamma \text{ is finite.}$$



Then $\displaystyle\int_\gamma ds = +\infty.$

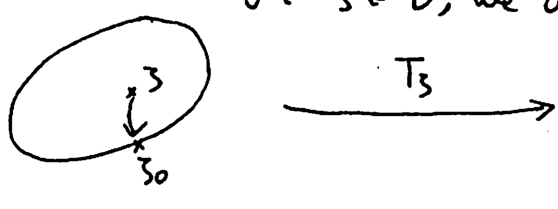
Theorem 3.2 Assume that D is strictly pseudoconvex in $\mathbb{C}^n$. Then there exists a number $c > 0$ (depending on D) such that

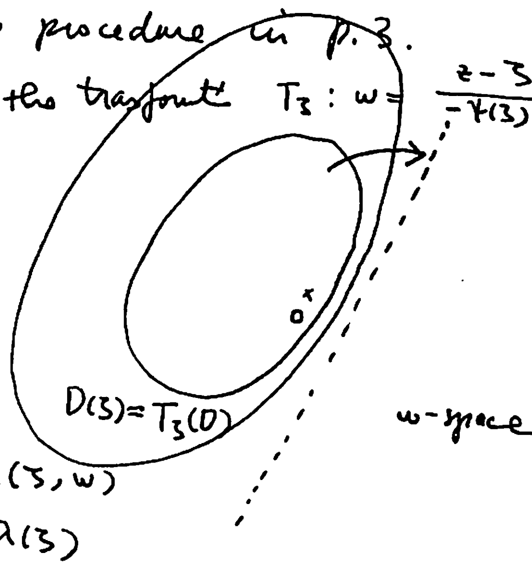
$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta} a_\alpha \bar{a}_\beta \geqq c \left| \sum_{\alpha=1}^{n} \frac{\partial \log(-\Lambda)}{\partial \zeta_\alpha} a_\alpha \right|^2 \quad \text{in } D.$$

Consequently, ds^2 is complete in D.

Proof. We also follow the procedure in p. 3.

For $\zeta \in D$, we consider the transformation $T_\zeta : w = \dfrac{z - \zeta}{-\psi(\zeta)}$





fn : $G(\zeta, z)$ $\qquad$ $g(\zeta, w)$

cnt : $\Lambda(\zeta)$ $\qquad$ $\lambda(\zeta)$

$$g(\zeta, w) = G(\zeta, z) \cdot \psi(\zeta)^{2n-2}$$
$$\lambda(\zeta) = \Lambda(\zeta) \cdot \psi(\zeta)^{2n-2} \qquad \text{where } w = \frac{z-\zeta}{-\psi(\zeta)}$$

We see that $g(\zeta, w)$ and $\lambda(\zeta)$ are of class C^1 up to ∂D.

We write the integral of the right-hand side of (3.1) by means of $g(z,w)$ and its derivatives:

In p.6 we got the formula

$$G_\alpha(\zeta,z) = \left(\frac{\partial G}{\partial \zeta_\alpha} + \frac{\partial G}{\partial z_\alpha}\right)(\zeta,z) = \frac{1}{\psi^{2n-1}} H_\alpha(\zeta,w)$$

where
$$H_\alpha(\zeta,w) = \psi \frac{\partial g}{\partial \zeta_\alpha} - (n-1)\frac{\partial \psi}{\partial \zeta_\alpha}(H_0 + \overline{H}_0)$$
$$H_0(\zeta,w) = g + \frac{1}{n-1}\sum_{i=1}^n w_i \frac{\partial g}{\partial w_i}$$

so that H_0 is independent of $\alpha = 1, \cdots, n$. By (2.1) we have

$$\frac{\partial g}{\partial w_\alpha}\frac{1}{-\psi} = \psi^{2n-2}\frac{\partial G}{\partial z_\alpha}$$

$$\therefore\ G + \frac{1}{n-1}\sum_{\alpha=1}^n (z_\alpha - \zeta_\alpha)\frac{\partial G}{\partial z_\alpha} = \frac{g}{\psi^{2n-2}} + \frac{1}{n-1}\sum_{\alpha=1}^n(-\psi)w_\alpha \frac{1}{-\psi^{2n-1}}\frac{\partial g}{\partial w_\alpha}$$
$$= \frac{1}{\psi^{2n-2}}\left(g + \frac{1}{n-1}\sum_{\alpha=1}^n w_\alpha \frac{\partial g}{\partial w_\alpha}\right)$$
$$= \frac{1}{\psi^{2n-2}} H_0(\zeta,w)$$

We simply set

$$L_2(a,\zeta) = \sum_{\alpha,\beta=1}^n \frac{\partial^2 \log(-\lambda)}{\partial \zeta_\alpha \partial \overline{\zeta}_\beta} a_\alpha \overline{a}_\beta \ ; \quad L_1(a,\zeta) = \sum_{\alpha=1}^n \frac{\partial \log(-\lambda)}{\partial \zeta_\alpha} a_\alpha .$$

By (3.1) we have

$$L_2(a,\zeta)$$
$$\geqq \frac{4}{(n-1)\omega_{2n}} \frac{\psi^{2n-2}}{-\lambda(\zeta)} \iint_{D(\zeta)} \left\{ \sum_{\alpha=1}^n \left| \frac{\partial}{\partial \overline{w}_\alpha}\left[-L_1 \frac{H_0}{\psi^{2n-2}} + \sum_{i=1}^n \frac{a_i}{\psi^{2n-1}}\left(\psi \frac{\partial g}{\partial \zeta_i} - (n-1)\frac{\partial \psi}{\partial \zeta_i}(H_0 + \overline{H}_0)\right)\right]\frac{1}{-\psi}\right|^2 \right\}$$
$$\times (-\psi)^{2n}\, dV_w$$

$$= \frac{4}{(n-1)\omega_{2n}} \cdot \frac{1}{-\lambda(\zeta)} \iint_{D(\zeta)} \left\{ \sum_{\alpha=1}^n \left| \frac{\partial}{\partial \overline{w}_\alpha}\left[-L_1 \cdot H_0 + \sum_{i=1}^n a_i \frac{\partial g}{\partial \zeta_i} - (n-1)\left(\sum_{i=1}^n \frac{\frac{\partial \psi}{\partial \zeta_i}}{\psi}\right)(H_0 + \overline{H}_0)\right]\right|^2 \right\} dV_w$$

We observe (2.9) : $\psi^{2n-1}\frac{\partial \Lambda}{\partial \zeta_\alpha} = \psi\frac{\partial \lambda}{\partial \zeta_\alpha} - (2n-2)\frac{\partial \psi}{\partial \zeta_\alpha}\lambda$

$$\frac{\frac{\partial \psi}{\partial \zeta_i}}{\psi} = \frac{1}{(2n-2)\lambda}\left(\frac{\partial \lambda}{\partial \zeta_i} - \psi^{2n-2}\frac{\partial \Lambda}{\partial \zeta_i}\right)$$

(3.1') $$\therefore \sum_{i=1}^{n} a_i \frac{\frac{\partial \psi}{\partial \zeta_i}}{\psi} = \frac{1}{(2n-2)\lambda}\left(\sum_{i=1}^{n} a_i\frac{\partial \lambda}{\partial \zeta_i} - \lambda\sum_{i=1}^{n} a_i\frac{\frac{\partial \Lambda}{\partial \zeta_i}}{\Lambda}\right)$$

$$= \frac{1}{2n-2}\left(\sum_{i=1}^{n} a_i\frac{\frac{\partial \lambda}{\partial \zeta_i}}{\lambda} - L_1\right)$$

Therefore, we have

$$L_2(a,\zeta)$$

$$\geq \frac{4}{(n-1)\omega_{2n}}\cdot\frac{1}{-\lambda(\zeta)}\iint_{D(\zeta)}\left\{\sum_{\alpha=1}^{n}\left|\frac{\partial}{\partial \bar{w}_\alpha}\left[L_1\cdot\left(\frac{\bar{H}_0 - H_0}{2}\right) + \sum_{i=1}^{n} a_i\frac{\partial g}{\partial \zeta_i} - \left(\sum_{i=1}^{n} a_i\frac{\frac{\partial \lambda}{\partial \zeta_i}}{\lambda}\right)\cdot\frac{H_0 + \bar{H}_0}{2}\right|^2\right\} dV_w$$

We know that, if $\zeta \to \zeta_0 \in \partial D$, then

$$H_0(\zeta, w) \to H_0(\zeta_0, w) = \frac{\sum \bar{w}_i \bar{N}_i - \|N\|^2}{\|w - \bar{N}\|^{2n}}$$ uniformly on any compact set in $D(\zeta_0)$

$$\lambda(\zeta) \longrightarrow -\|\mathrm{Grad}_{(z)}\psi(\zeta_0)\|^{2n-2}$$

$$\frac{\partial g}{\partial \zeta_i}(\zeta, w) \longrightarrow \frac{1}{\|\mathrm{Grad}\,\psi(\zeta_0)\|}\cdot\int_{\partial D(\zeta_0)} \|\mathrm{Grad}\, g(\zeta_0, \tilde{w})\|\,\frac{\partial f}{\partial \zeta_i}(\zeta_0, \tilde{w})\, d\Omega_w(\tilde{w}) \underset{\mathrm{def}}{=} g_i(\zeta_0, w)$$

uniformly on any compact in $D(\zeta_0)$

where $$g(\zeta_0, \tilde{w}) = \frac{1}{\|\tilde{w}\|^{2n-2}} - \frac{1}{\|\tilde{w} - \bar{N}\|^{2n-2}}$$

$$d\Omega_w(\tilde{w}) = \frac{-1}{2(n-2)\,\omega_{2n}}\frac{\partial g_w(\zeta_0, \tilde{w})}{\partial n_{\tilde{w}}} ds_{\tilde{w}}$$; $g_w(\zeta_0, \tilde{w})$ is the Green's function for $(D(\zeta_0), w)$

$$\frac{\partial f}{\partial \zeta_i}(\zeta_0, w) = \sum_{\beta=1}^{n}\left(\frac{\partial^2 \psi}{\partial z_i \partial z_\beta}(\zeta_0) w_\beta + \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_\beta}(\zeta_0)\bar{w}_\beta\right)$$
$$-\frac{1}{2}\frac{\partial \psi}{\partial \zeta_i}(\zeta_0)\sum_{\beta,\gamma=1}^{n}\left\{\frac{\partial^2 \psi}{\partial z_\beta \partial z_\gamma}(\zeta_0) w_\beta w_\gamma + \frac{\partial^2 \psi}{\partial z_\beta \partial \bar{z}_\gamma}(\zeta_0) w_\beta \bar{w}_\gamma + \frac{\partial^2 \psi}{\partial \bar{z}_\beta \partial z_\gamma}(\zeta_0)\bar{w}_\beta w_\gamma + \frac{\partial^2 \psi}{\partial \bar{z}_\beta \partial \bar{z}_\gamma}(\zeta_0)\bar{w}_\beta \bar{w}_\gamma\right\}$$

$$\frac{\partial \lambda}{\partial \zeta_i}(\zeta) = \frac{\partial g}{\partial \zeta_i}(\zeta, 0) \longrightarrow g_i(\zeta_0, 0)$$

We can take a small ball $V : \|w\| < \delta$ such that

$$V \subset\subset D(\zeta) \text{ for all } \zeta \in D \cup \partial D$$

For the sake of convenience, we set

$$H_0(a,\zeta,w) = \underset{\text{Real part}}{A(a,\zeta,w)} + i \underset{\text{imaginary part}}{B(a,\zeta,w)}$$

Then

$$A(a,\zeta,w) \to \frac{1}{2} \frac{\sum_{\alpha=1}^{n} \left(\bar{w}_\alpha \frac{\partial \psi}{\partial z_\alpha}(\zeta_0) + w_\alpha \frac{\partial \psi}{\partial \bar{z}_\alpha}(\zeta_0)\right) - 2}{\|w - \bar{N}\|^{2n} \|\mathrm{Grad}\, \psi(\zeta_0)\|^2} \underset{\text{def}}{=} A(a,\zeta_0,w)$$

$$B(a,\zeta,w) \to \frac{1}{2i} \frac{\sum_{\alpha=1}^{n} \left(\bar{w}_\alpha \frac{\partial \psi}{\partial z_\alpha}(\zeta_0) - w_\alpha \frac{\partial \psi}{\partial \bar{z}_\alpha}(\zeta_0)\right)}{\|w - \bar{N}\|^{2n} \|\mathrm{Grad}\, \psi(\zeta_0)\|^2} \underset{\text{def}}{=} B(a,\zeta_0,w)$$

Because $\mathrm{Grad}\, \psi(\zeta_0) \neq 0$, we see that $A(a,\zeta_0,w)$, $B(a,\zeta_0,w)$ are non-constant real valued functions on $D(\zeta_0)$.

We take the following constants:

$$0 < {}^\exists m_1 < -\lambda(\zeta) < {}^\exists M_1 < +\infty \quad \text{for all } \zeta \in D \cup \partial D$$

$$\iint_V \left\{ \sum_{\alpha=1}^{n} \left| \frac{\partial}{\partial \bar{w}_\alpha} \left(\sum_{i=1}^{n} a_i \frac{\partial g}{\partial \zeta_i}(\zeta,w) - \left(\sum_{i=1}^{n} a_i \frac{\frac{\partial \lambda}{\partial \zeta_i}}{\lambda} \right) \cdot A(a,\zeta,w) \right) \right|^2 \right\} dV < {}^\exists M_2$$

for all $a \in \mathbb{C}^n$ with $\|a\| = 1$
& all $\zeta \in D \cup \partial D$

$$\iint_V \left\{ \sum_{\alpha=1}^{n} \left| \frac{\partial}{\partial \bar{w}_\alpha} B(a,\zeta,w) \right|^2 \right\} dV \geq {}^\exists m_2 \quad \text{for all } a \in \mathbb{C}^n \text{ with } \|a\| = 1 \ \& \ \zeta \in D \cup \partial D.$$

Then (3.2) leads us

$L_2(a,\zeta)$

$$\geq \frac{4}{(n-1)\omega_{2n}} \frac{1}{-\lambda(\zeta)} \iint_V \left\{ \sum_{\alpha=1}^{n} \left| \frac{\partial}{\partial \bar{w}_\alpha} \left[(-iL_1(a,\zeta)) \cdot B(a,\zeta,w) + \sum_{i=1}^{n} a_i \frac{\partial g}{\partial \zeta_i}(\zeta,w) - \left(\sum_{i=1}^{n} a_i \frac{\frac{\partial \lambda}{\partial \zeta_i}}{\lambda} \right) A(a,\zeta,w) \right] \right|^2 \right\} dV_w$$

$$\geq \frac{1}{(n-1)\omega_{2n} M_1} \left\{ \sqrt{\iint_V \left\{ \sum_{\alpha=1}^{n} \left| \frac{\partial}{\partial \bar{w}_\alpha} L_1(a,\zeta) \cdot B(a,\zeta,w) \right|^2 \right\} dV_w} - \sqrt{\iint_V \sum_{\alpha=1}^{n} \left| \frac{\partial}{\partial \bar{w}_\alpha} \left(\sum_{i=1}^{n} a_i \frac{\partial g}{\partial \zeta_i} - \left(\sum_{i=1}^{n} a_i \frac{\frac{\partial \lambda}{\partial \zeta_i}}{\lambda} \right) A \right) \right|^2 dV_w} \right\}^2$$

$$\geq \frac{1}{(n-1)\omega_{2n} M_1} \left\{ |L_1(a,\zeta)| \sqrt{m_2} - \sqrt{M_2} \right\}^2 \quad \text{for all } a \in \mathbb{C}^n \text{ with } \|a\| = 1,\ \zeta \in D$$

It follows that, if $|L_1(a,\zeta)| > \frac{2\sqrt{M_2}}{\sqrt{m_2}}$, then

$$L_2(a,\zeta) \geqq \frac{1}{(n-1)\omega_{2n}M_1}\left\{|L_1(a,\zeta)|\cdot\sqrt{m_2} - \frac{\sqrt{m_2}}{2}|L_1(a,\zeta)|\right\}^2$$

$$= \frac{m_2}{4(n-1)\omega_{2n}M_1}|L_1(a,\zeta)|^2.$$

We set $p = \frac{2\sqrt{M_2}}{\sqrt{m_2}} > 0$, $q = \frac{\omega_2}{4(n-1)\omega_{2n}M_1} > 0$ which are determined by the domain D. We conclude that

$$(3.3)\quad \boxed{\begin{array}{l}\displaystyle\sum_{\alpha,\beta=1}^{n}\frac{\partial^2\log(-\lambda)}{\partial\zeta_\alpha\partial\bar\zeta_\beta}a_\alpha\bar a_\beta \geqq q\left|\sum_{\alpha=1}^{n}\frac{\partial\log(-\lambda)}{\partial\zeta_\alpha}a_\alpha\right|^2 \\ \text{for all } (a,\zeta) \text{ where } \begin{pmatrix}a\in\mathbb{C}^n, \text{ with } \|a\|=1\\ \zeta\in D\end{pmatrix} \text{ such that} \\ \displaystyle\left|\sum_{\alpha=1}^{n}\frac{\partial\log(-\lambda)}{\partial\zeta_\alpha}a_\alpha\right| > p.\end{array}}$$

This fact easily implies Theorem 3.1 as follows:

Let $\gamma : t \to \zeta(t)$ $(0 \leqq t < l)$ be any differentiable curve in D such that $\zeta_0 = \zeta(l) \in \partial D$. We may assume the parameter t is arc length. i.e., $\|\frac{d\zeta}{dt}\| = 1$ for $\forall t \in [0,$

We assume $l < +\infty$. Let

$$I_0 = \{t \in [0,l) \mid |L_1(\tfrac{d\zeta}{dt}(t), \zeta(t))| > p\}\ ;\quad I_1 = [0,l) - I_0.$$

Then
$$\int_\gamma ds = \int_0^l \sqrt{L_2(\tfrac{d\zeta}{dt}, \zeta(t))}\,dt \geqq \int_{I_0}\sqrt{L_2(\tfrac{d\zeta}{dt}, \zeta(t))}\,dt$$

$$\geqq \sqrt{q}\int_{I_0}|L_1(\tfrac{d\zeta}{dt}, \zeta(t))|\,dt \quad (\text{by } (3.3))$$

$$= \sqrt{q}\left\{\int_0^l |L_1(\tfrac{d\zeta}{dt}, \zeta(t))|\,dt - p\int_{I_1}dt\right\}$$

$$= \sqrt{q}\left|\int_0^l \sum_{\alpha=1}^{n}\frac{\partial\log(-\lambda)}{\partial\zeta_\alpha}(\zeta(t))\frac{d\zeta_\alpha}{dt}\,dt\right| - \sqrt{q}\,p$$

$$\geqq (\sqrt{q}/2)\left|\int_0^l \frac{d\log(-\lambda)(\zeta(t))}{dt}\right| - \sqrt{q}\,p$$

$$= +\infty \quad (\text{by } -\lambda(\zeta_0) = +\infty)$$

which proves Theorem 3.1.

§4 Strictly pseudoconvex domain

To prove Theorem 3.2 (that, in the case of strictly pseudoconvex domain, the metric induced by Robin constants is complete.), we recall some results already shown:

Let $\mathcal{D}: t \to D(t)$ $(t \in B)$ be a smooth variation of smooth domain $D(t)$ $(\subset \mathbb{C}^n)$. Let $\zeta \in D(t)$ for all $t \in B$. We have the green's function $g(t, z)$ and the Robin constant $\lambda(t)$ for $(D(t), \zeta)$

We take a defining function $\psi(t, z)$ of the domain $\mathcal{D} = \bigcup_{t \in B} (t, D(t))$ $(\subset B \times \mathbb{C}^n)$ and consider the quantity:

$$k_1(t, z) = \frac{\frac{\partial \psi}{\partial t}(t, z)}{\|\mathrm{Grad}_{(z)} \psi(t, z)\|} \qquad \text{for } z \in \partial D(t),\ t \in B.$$

Then it is clear that $k_1(t, z)$ does not depend on the choice of the defining fun $\psi(t, z)$. Since $-g(t, z)$ is regarded as a defining function of $\mathcal{D}$, it follows that

$$k_1(t, z) = \frac{-\frac{\partial g}{\partial t}(t, z)}{\|\mathrm{Grad}_{(z)}\, g(t, z)\|} \qquad \text{for } z \in \partial D(t)$$

Consequently,

$$(4.1) \quad \frac{\partial g}{\partial t}(t, z) = -k_1(t, z)\, \|\mathrm{Grad}_{(z)}\, g(t, z)\| \qquad \text{for } z \in \partial D(t)$$

Under this notation, we have

<u>Proposition 4.1</u> For $z_0 \in D(t)$, we have

$$\frac{\partial g}{\partial t}(t, z_0) = \frac{1}{2(n-1)\omega_{2n}} \int_{\partial D(t)} k_1(t,z) \left\| G_{\mathrm{rad}\,(z)}\, g(t,z) \right\| \frac{\partial g_{z_0}(t,z)}{\partial n_z} \, dS_t$$

where $g_{z_0}(t,z)$ is the Green's function for $(D(t), z_0)$.

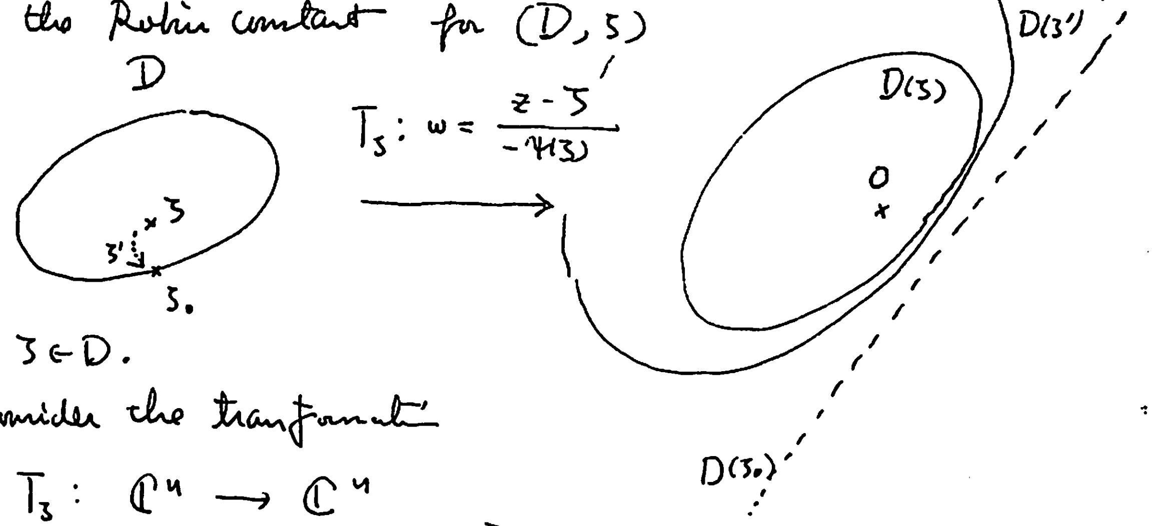
⌜(:) Since $\frac{\partial g}{\partial t}(t,z)$ is harmonic for z on $D(t)$ and is continuous up to $\partial D(t)$, we have by (4.1) proposition 4.1.⌟

For the sake of convenience, we write

$$d\Omega_{z_0}(t,z) = -\frac{1}{2(n-1)\omega_{2n}} \frac{\partial g_{z_0}(t,z)}{\partial n_z} \, dS_t > 0$$

for $z \in \partial D(t)$.

We apply Proposition 4.1 to our situation: Let D be a domain in $\mathbb{C}^n$ with smooth boundary ∂D. Let $\psi(z)$ be a defining function of D in $\mathbb{C}^n$. We denote by $G(\zeta, z)$ and $\Lambda(\zeta)$ the Green's function and the Robin constant for (D, ζ)



Let $\zeta \in D$.

We consider the transformation

$$T_\zeta : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$
$$z \longrightarrow w = \frac{z-\zeta}{-\psi(\zeta)}$$

and put

$$T_\zeta(D) = D(\zeta).$$

Since $D(\zeta) \ni 0$, we have the Green's function $g(\zeta, z)$ and the Robin

Robin constant $\lambda(\zeta)$ for $(D(\zeta), 0)$. We thus have the smooth variation $\mathcal{D}$ of smooth domain $D(\zeta)$ where $\zeta \in D$

$$\mathcal{D} : \zeta \longrightarrow D(\zeta) \quad (\zeta \in D).$$

One of the defining functions of $\mathcal{D} = \bigcup_{\zeta \in D} (\zeta, D(\zeta)) \quad (\subset D \times \mathbb{C}^n)$ was

$$(4.2) \quad f(\zeta, w) = \frac{\psi(-\psi(\zeta) w + \zeta)}{-\psi(\zeta)}$$

$$= 2\mathrm{Re}\left\{ \int_0^1 \sum_{\beta=1}^n w_\beta \left[\frac{\partial \psi}{\partial z_\beta}\right]_{z = \zeta - \psi(\zeta) t w} dt \right\} - 1 \qquad (t : \text{real})$$

This $f(\zeta, w)$ was of class C^∞ on $\mathbb{C}^n \times \mathbb{C}^n$.
It follows that

$$\frac{\partial f}{\partial w_\alpha}(\zeta, w) = \frac{\partial \psi}{\partial z_\alpha}\Big|_{z = \zeta - \psi(\zeta) w} \quad ;$$

$$\frac{\partial f}{\partial \zeta_\alpha}(\zeta, w) = \int_0^1 \sum_{\beta=1}^n \left(w_\beta \left[\frac{\partial^2 \psi}{\partial z_\beta \partial z_\alpha}\right] + \bar{w}_\beta \left[\frac{\partial^2 \psi}{\partial \bar{z}_\beta \partial z_\alpha}\right] \right)_{z = \zeta - \psi(\zeta) t w} dt$$

$$- \left[\frac{\partial \psi}{\partial z_\alpha}\right] \sum_{\beta, \gamma = 1}^n \int_0^1 \left\{ \left[\frac{\partial^2 \psi}{\partial z_\beta \partial z_\gamma}\right] w_\beta w_\gamma + \left[\frac{\partial^2 \psi}{\partial \bar{z}_\beta \partial z_\gamma}\right] \bar{w}_\beta w_\gamma + \left[\frac{\partial^2 \psi}{\partial z_\beta \partial \bar{z}_\gamma}\right] w_\beta \bar{w}_\gamma + \left[\frac{\partial^2 \psi}{\partial \bar{z}_\beta \partial \bar{z}_\gamma}\right] \bar{w}_\beta \bar{w}_\gamma \right\}_{z = \zeta - \psi(\zeta)} t\, dt$$

Let $w_0 \in D(t)$. Proposition 4.1 implies

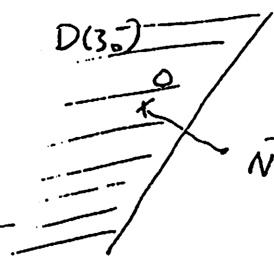
$$\frac{\partial g}{\partial \zeta_\alpha}(\zeta, w_0) = - \int_{\partial D(\zeta)} \frac{\frac{\partial f}{\partial \zeta_\alpha}(\zeta, w)}{\| \mathrm{Grad}_{(w)} f(\zeta, w) \|} \| \mathrm{Grad}_{(w)} g(t, w) \| \, d\Omega_{w_0}(\zeta, w)$$

where

$$d\Omega_{w_0}(\zeta, w) = - \frac{1}{(2n-2) w_{2n}} \cdot \frac{\partial g_{w_0}(\zeta, w)}{\partial n_w} ds_w \quad \text{for } w \in \partial D(\zeta).$$

Let $\zeta_0 \in \partial D$ and let $\zeta \longrightarrow \zeta_0$. Then

Then $D(\zeta) \longrightarrow D(\zeta_0) =$ the half space defined by $2\,\mathrm{Re} \sum_{\alpha=1}^{n} w_\alpha \frac{\partial \psi}{\partial \zeta_\alpha}(\zeta_0) - 1 < 0$. Moreover, if we denote by $g(\zeta_0, w)$ the Green's function for $(D(\zeta_0), 0)$ i.e.,



$$g(\zeta_0, w) = \frac{1}{\|w\|^{2n-2}} - \frac{1}{\|w - \bar{N}\|^{2n-2}},$$

then $g(\zeta, w) \longrightarrow g(\zeta_0, w)$ uniformly on any compact set in $D(\zeta_0)$. We have also

$$(4.3)\quad \frac{\partial g}{\partial \zeta_\alpha}(\zeta, w_0) \longrightarrow - \int_{\partial D(\zeta_0)} \frac{\frac{\partial f}{\partial \zeta_\alpha}(\zeta_0, w)}{\|\mathrm{Grad}_{(w)} f(\zeta_0, w)\|} \|\mathrm{Grad}_{(w)} g(\zeta_0, w)\| \, d\Omega_{w_0}(\zeta_0, w)$$

$$= - \int_{\partial D(\zeta_0)} \left\{ \frac{\sum_{\beta=1}^{n} \left(w_\beta \frac{\partial^2 \psi}{\partial z_\beta \partial z_\alpha}(\zeta_0) + \bar{w}_\beta \frac{\partial^2 \psi}{\partial \bar{z}_\beta \partial z_\alpha}(\zeta_0) \right) - \frac{1}{2} \frac{\partial \psi}{\partial z_\alpha}(\zeta_0) \sum_{\beta,\gamma=1}^{n} \left(\frac{\partial^2 \psi}{\partial z_\beta \partial z_\gamma}(\zeta_0) w_\beta w_\gamma + \cdots + \frac{\partial^2 \psi}{\partial \bar{z}_\beta \partial \bar{z}_\gamma}(\zeta_0) \bar{w}_\beta \bar{w}_\gamma \right)}{\|\mathrm{Grad}_{(z)} \psi(\zeta_0)\|} \right\} \|\mathrm{Grad}_w g(\zeta_0, w)\| \, d\Omega_{w_0}$$

We gave a rigorous proof of the limit formula in the note of March 14

$$(4.4)\quad \frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta) \to \lambda_\alpha(\zeta_0) = - \int_{\partial D(\zeta_0)} \left\{ \qquad \text{''} \qquad \right\} \|\mathrm{Grad}_w g(\zeta_0, w)\| \, d\Omega_0(\zeta_0, w)$$

where $d\Omega_0(\zeta_0, w) = \frac{-1}{(2n-2)\omega_{2n}} \frac{\partial g(\zeta_0, w)}{\partial n_w} d\sigma_w$.

That argument is quite available for the present case and we see that the convergence is uniform on any compact set in $D(\zeta_0)$. Since $\frac{\partial g}{\partial \zeta_\alpha}(\zeta, 0) = \frac{\partial \lambda}{\partial \zeta_\alpha}(\zeta)$, (4.4) is a special case of (4.3). In shorts, we write

$$\frac{\partial g}{\partial \zeta_\alpha}(\zeta, w) \longrightarrow g_\alpha(\zeta_0, w) \quad \text{for } w \in D(\zeta_0),$$

<u>Notation 4.1</u> (1) Given a complex-valued function $\varphi(\zeta)$ on $\partial D(\zeta_0)$, we set

$$S_\varphi(w) = \text{the Dirichlet solution on } D(\zeta_0) \text{ with boundary values } \varphi$$
$$= \int_{\partial D(\zeta_0)} \varphi(\cdot) \, d\Omega_w(\zeta_0, \cdot)$$

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(2) $\|\mathrm{Grad}_{(w)}\, g(\zeta_0, w)\| = \Gamma(w)$ for $w \in \partial D(\zeta_0)$

For exemple, $\mathcal{S}_{\Gamma w_\alpha}(w_0) =$ the Dirichlet solution on $D(\zeta_0)$ with boundary values $\Gamma(w) w_\alpha$

$$= \int_{\partial D(\zeta_0)} w_\alpha \, \|\mathrm{Grad}_{(w)}\, g(\zeta_0, w)\| \, d\Omega_{w_0}(\zeta_0, w).$$

It follows that

$$(4.5)\quad g_\alpha(\zeta_0, w) = -\frac{1}{\|\mathrm{Grad}_{(z)} \psi(\zeta_0)\|} \left\{ \sum_{\beta=1}^{n} \left(\frac{\partial^2 \psi}{\partial z_\beta \partial \bar z_\alpha}(\zeta_0) \mathcal{S}_{\Gamma w_\beta} + \frac{\partial^2 \psi}{\partial \bar z_\beta \partial \bar z_\alpha}(\zeta_0) \mathcal{S}_{\Gamma \bar w_\beta} \right) \right.$$

$$\left. - \frac{1}{2}\left(\frac{\partial \psi}{\partial \zeta_\alpha}\right)(\zeta_0) \underbrace{\sum_{\beta,\gamma=1}^{n} \left(\frac{\partial^2 \psi}{\partial z_\beta \partial z_\gamma}(\zeta_0) \mathcal{S}_{\Gamma w_\beta w_\gamma} + \frac{\partial^2 \psi}{\partial z_\beta \partial \bar z_\gamma}(\zeta_0) \mathcal{S}_{\Gamma w_\beta \bar w_\gamma} + \frac{\partial^2 \psi}{\partial \bar z_\beta \partial z_\gamma}(\zeta_0) \mathcal{S}_{\Gamma \bar w_\beta w_\gamma} + \frac{\partial^2 \psi}{\partial \bar z_\beta \partial \bar z_\gamma}(\zeta_0) \mathcal{S}_{\Gamma \bar w_\beta \bar w_\gamma} \right)}_{\text{this is independent of } \alpha = 1, \dots, n.} \right\}$$

Last week we had the following inequality:
Given $\zeta \in D$ and $a \in \mathbb{C}^n$ with $\|a\| = 1$, let

$$L_2(a, \zeta) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\lambda)}{\partial \zeta_\alpha \partial \bar\zeta_\beta} a_\alpha \bar a_\beta \;;$$

$$L_1(a, \zeta) = \sum_{\alpha=1}^{n} \frac{\partial \log(-\lambda)}{\partial \zeta_\alpha} a_\alpha \;;$$

$$H_0(\zeta, w) = g(\zeta, w) + \frac{1}{n-1} \sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i}(\zeta, w) \quad (\text{independent of } a \in \mathbb{C}^n),$$

Then, under the condition that D is pseudoconvex, we have

$L_2(a, \zeta)$

$$\geq \frac{4}{(n-1)\omega_{2n}} \frac{1}{-\lambda(\zeta)} \iint_{D(\zeta)} \sum_{\alpha=1}^{n} \left| \frac{\partial}{\partial \bar w_\alpha} \left[L_1(a,\zeta) \left(\frac{\overline{H_0} - H_0}{2} \right)_{(\zeta,w)} + \sum_{i=1}^{n} a_i \frac{\partial g}{\partial \zeta_i}(\zeta, w) - \left(\sum_{i=1}^{n} a_i \frac{\frac{\partial \lambda}{\partial \zeta_i}}{\lambda} \right)_{\zeta} \left(\frac{H_0 + \overline{H_0}}{2} \right)_{(\zeta,w)} \right] \right|^2 dV_w$$

$$\geq \frac{4}{(n-1)\omega_{2n}} \frac{1}{-\lambda(\zeta)} \iint_{V} \sum_{\alpha=1}^{n} \left| \frac{\partial}{\partial \bar w_\alpha} \left[\quad '' \quad \right] \right|^2 dV_w$$

where $V = \{ w \in \mathbb{C}^n \mid \|w\| < r \} \subset\subset D(\zeta)$ for all $\zeta \in D \cup \partial D$.

So $\lambda(\zeta) \geq -1/r^{2n-2}$ for $\forall \zeta \in D \cup \partial D$. In the sake of convenience we put

$$\mathcal{L}(a,\zeta,w) = L_1 \cdot \left(\frac{\overline{H_0} - H_0}{2}\right) + \sum_{i=1}^{n} a_i \frac{\partial g}{\partial \zeta_i} - \left(\sum_{i=1}^{n} a_i \frac{\frac{\partial \lambda}{\partial \zeta_i}}{\lambda}\right) \cdot \left(\frac{H_0 + \overline{H_0}}{2}\right);$$

$$C = \frac{4}{(n-1)\omega_{2n}} \cdot r^{2n-2} > 0 \quad (\text{which depends only on } D),$$

so that

(4.6) $$L_2(a,\zeta) \geq C \cdot \iint_V \left(\sum_{\alpha=1}^{n} \left|\frac{\partial}{\partial \overline{w}_\alpha} \mathcal{L}(a,\zeta,w)\right|^2\right) dV \quad \text{for } \forall a \in \mathbb{C}^n \text{ with } \|a\|=1, \ \forall \zeta \in D.$$

It suffices for Theorem 3.2 to prove the following

Lemma 4.1 If D is strictly pseudoconvex, then there exists a positive number k (depending only on D) such that

(4.7) $$L_2(a,\zeta) \geq k\, |L_1(a,\zeta)|^2 \quad \text{for } \forall a \in \mathbb{C}^n \text{ with } \|a\|=1, \ \forall \zeta \in D$$

Proof. We prove Lemma 4.1 by contradiction. We assume that (4.7) is not true. Then we find sequences:

$k_\nu > 0$, $a_\nu \in \mathbb{C}^n$ with $\|a_\nu\| = 1$, $\zeta_\nu \in D$ such that $k_\nu \searrow 0$ and $L_2(a_\nu, \zeta_\nu) \leq k_\nu\, |L_1(a_\nu, \zeta_\nu)|^2$.

By (3.3), we have $|L_1(a_\nu, \zeta_\nu)| \leq p$. We may suppose that, as $\nu \to \infty$, we have

$$a_\nu \longrightarrow a \in \mathbb{C}^n \quad \text{with } \|a\|=1$$
$$\zeta_\nu \longrightarrow \zeta_0 \in \partial D$$
$$L_1(a_\nu, \zeta_\nu) \longrightarrow p_0 \quad \text{where } |p_0| \leq p.$$

It follows that

$$\lim_{\nu \to \infty} L_2(a_\nu, \zeta_\nu) = 0$$

By (4.6) we have

$$\iint_V \sum_{\alpha=1}^{n} \left| \frac{\partial}{\partial \bar w_\alpha} L(z_\nu, \zeta_\nu, w) \right|^2 dV_w \longrightarrow 0. \quad (\nu \to \infty)$$

We may assume that

$$\mathrm{Grad}_{(z)}\, \psi(\zeta_0) = (0, \cdots, 0, i)$$

Hence

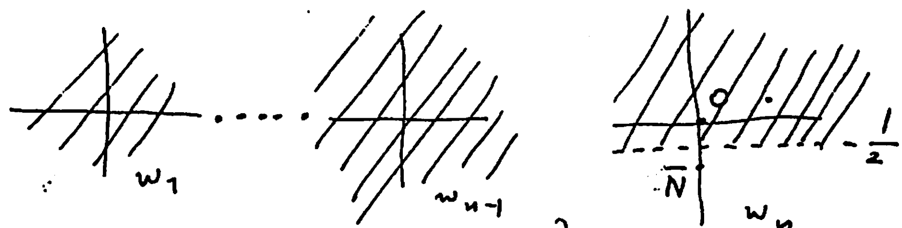
$$D(\zeta_\nu) \to D(\zeta_0) = \left\{ w \in \mathbb{C}^n \;\middle|\; 2\,\mathrm{Re} \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z_\alpha}(\zeta_0)\, w_\alpha - 1 < 0 \right\}$$

$$= \left\{ w \in \mathbb{C}^n \;\middle|\; y_{2n} > -\tfrac{1}{2} \right\}$$

where $w_\alpha = y_{2\alpha-1} + \sqrt{-1}\, y_{2\alpha}$ $(1 \leq \alpha \leq n)$.

$y_{2\alpha-1}$, $y_{2\alpha}$ are real.

$\partial D(\zeta_0)$ is



$$\partial D(\zeta_0) = \{ w = (\underbrace{w_1, \cdots, w_{n-1}}_{\text{free complex}},\ \underbrace{y_{2n-1}}_{\text{free real}} - \tfrac{1}{2}\sqrt{-1}) \}$$

Consequently,

$$(4.8) \quad \begin{aligned} w_n &= y_{2n-1} - \tfrac{1}{2}\sqrt{-1} \\ \bar w_n &= y_{2n-1} + \tfrac{1}{2}\sqrt{-1} \end{aligned} \qquad \therefore\ w_n = \overline{w_n + i} \quad \text{on } \partial D(\zeta_0)$$

We need the following

<u>Proposition 4.2</u> Let $D(\zeta_0) = \{ y_{2n} > -\frac{1}{2} \}$. Then

$$g(\zeta_0, w) = \frac{1}{\|w\|^{2n-2}} - \frac{1}{\|w - (0, \cdots, 0, -i)\|^{2n-2}}$$

$$H(\zeta_0, w) = g(\zeta_0, w) + \frac{1}{n-1} \sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i}(\zeta_0, w)$$

$$= \frac{-i\, (\overline{w_n + i})}{\|w - (0, \cdots, 0, -i)\|^{2n-2}}$$

$$\mathrm{Grad}_{(w)}\, g(\zeta_0, w) = -(n-1) \left(\frac{\bar w_\alpha}{\|w\|^{2n}} - \frac{\bar w_\alpha - \delta_{\alpha n}\, i}{\|w - (0, \cdots, 0, -i)\|^{2n}} \right)$$

$\therefore$ $\quad \mathrm{Grad}_{(w)}\, g(z_0,w) = -(n-1)\left(0,\cdots,0,\dfrac{i}{\|w\|^{2n}}\right)$ for $w \in \partial D(z_0)$

$\Gamma = \|\mathrm{Grad}_{(w)}\, g(z_0,w)\| = (n-1)/\|w\|^{2n}$ for $w \in \partial D(z_0)$

Moreover,
$$\mathcal{S}_{\Gamma w_\alpha} = \frac{(n-1)\, w_\alpha}{\|w-(0,\cdots,0,-i)\|^{2n}} \quad (1 \leqq \alpha \leqq n-1)$$

$$\mathcal{S}_{\Gamma w_n} = \frac{(n-1)\,\overline{w_n+i}}{\|w-(0,\cdots,0,-i)\|^{2n}}$$

$$H_0(z_0,w) = \frac{-i}{(n-1)}\,\mathcal{S}_{\Gamma w_n}$$

Γ $\because$ $\dfrac{1}{\|w-(0,\cdots,0,-i)\|^{2n-2}}$ is harmonic in $D(z_0)$. If we differentiate with respect to $\bar w_\alpha$ $(1 \leqq \alpha \leqq n-1)$, then

$$\frac{\partial}{\partial \bar w_\alpha}\left(\frac{1}{\|w-(0,\cdots,0,-i)\|^{2n-2}}\right) = \frac{-(n-1)\, w_\alpha}{\|w-(0,\cdots,0,-i)\|^{2n}} \quad \text{in } D(z_0).$$

If we restrict w to $\partial D(z_0)$, then we have $\|w-(0,\cdots,0,-i)\|$ $= \|w\|$, and hence $\dfrac{(n-1)\,w_\alpha}{\|w-(0,\cdots,0,-i)\|^{2n}} = \dfrac{(n-1)}{\|w\|^{2n}}\cdot w_\alpha$ $= \Gamma w_\alpha$ on $\partial D(z_0)$. Consequently,

$$\mathcal{S}_{\Gamma w_\alpha} = (n-1)w_\alpha / \|w-(0,\cdots,0,-i)\|^{2n}.$$

By the same method,

$$\frac{\partial}{\partial w_n}\frac{1}{\|w-(0,\cdots,0,-i)\|^{2n-2}} = \frac{-(n-1)\,\overline{w_n+i}}{\|w-(0,\cdots,0,-i)\|^{2n}}$$

is harmonic for $w \in D(z_0)$. On the boundary $\partial D(z_0)$, we see by (4.8) that

$$\frac{(n-1)\,\overline{w_n+i}}{\|w-(0,\cdots,0,-i)\|^{2n}} = \frac{(n-1)}{\|w\|^{2n}}\cdot w_n = \Gamma\cdot w_n$$

and hence that $\mathcal{S}_{\Gamma w_n} = \dfrac{(n-1)\,\overline{w_n+i}}{\|w-(0,\cdots,0,-i)\|^{2n}}$

$$\therefore \quad = (n-1)\, i\, H(z_0,w)$$

We use the notation : $\psi_\alpha = \partial\psi/\partial z_\alpha$, $\psi_{\alpha\beta} = \partial^2\psi/\partial z_\alpha \partial z_\beta$, ...

We remark the condition $\mathrm{Grad}_{(z)}\, \psi(\zeta_0) = (0, \cdots, 0, i)$ and the fact that

$$\lambda(\zeta_0) = -\|\mathrm{Grad}_{(z)}\, \psi(\zeta_0)\|^{2n-2} = -1 \; ;$$

$$\frac{\partial\lambda}{\partial\zeta_\alpha}(\zeta) \to \lambda_\alpha(\zeta_0) = \frac{\partial g}{\partial\zeta_\alpha}(\zeta_0, 0)$$

$$= \begin{cases} (n-1)i\,(\psi_{\alpha n}(\zeta_0) - \psi_{\alpha\bar n}(\zeta_0)) & \cdots\cdots (1 \leq \alpha \leq n-1) \\ (n-1)i\,(\psi_{\alpha n}(\zeta_0) - \psi_{\alpha\bar n}(\zeta_0)) + \frac{i}{2}\Big\{\sum_{\beta,\gamma=1}^{n} \psi_{\beta\gamma}(\zeta_0)\,\delta_{p w_\beta w_\gamma} + \cdots + \psi_{\bar\beta\bar\gamma}(\zeta_0)\,\delta_{p\bar w_\beta \bar w_\gamma}\Big\}_{w=0} & \cdots\cdots (\alpha = n) \end{cases}$$

It follows from (4.5) and Proposition 4.1 that

$$\mathcal{L}(a\nu, \zeta_0, w) \to p_0\left(\frac{\overline{H_0} - H_0}{2}\right)_{(\zeta_0,w)} + \sum_{\alpha=1}^{n} a_\alpha \frac{\partial g}{\partial\zeta_\alpha}(\zeta_0, w) - \left(\sum_{\alpha=1}^{n} a_\alpha \frac{\frac{\partial\lambda}{\partial\zeta_\alpha}}{\lambda}\right)\left(\frac{H_0 + \overline{H_0}}{2}\right)_{(\zeta_0,w)}$$

$$= \frac{i p_0}{2(n-1)}\left[\delta_{p\bar w_n} + \delta_{p w_n}\right]$$

$$-\sum_{\alpha=1}^{n} a_\alpha \Big(\sum_{\beta=1}^{n}\big(\psi_{\alpha\beta}(\zeta_0)\,\delta_{p w_\beta} + \psi_{\alpha\bar\beta}(\zeta_0)\,\delta_{p\bar w_\beta}\big)\Big)$$

$$+\frac{i}{2} a_n \sum_{\beta,\gamma=1}^{n}\Big(\psi_{\beta\gamma}(\zeta_0)\,\delta_{p w_\beta w_\gamma} + \cdots + \psi_{\bar\beta\bar\gamma}(\zeta_0)\,\delta_{p\bar w_\beta \bar w_\gamma}\Big)$$

$$+\left(\begin{array}{l} (n-1)i \sum_{\alpha=1}^{n} a_\alpha(\psi_{\alpha n}(\zeta_0) - \psi_{\alpha\bar n}(\zeta_0)) \\ \quad + \frac{i}{2}a_n\Big[\sum_{\beta,\gamma=1}^{n}\Big(\psi_{\beta\gamma}(\zeta_0)\,\delta_{p w_\beta w_\gamma} + \cdots + \psi_{\bar\beta\bar\gamma}(\zeta_0)\,\delta_{p\bar w_\beta \bar w_\gamma}\Big)\Big]_{w=0} \end{array}\right)$$

$$\times \frac{-i}{2(n-1)}\left[\delta_{p w_n} - \delta_{p\bar w_n}\right]$$

$$\overset{\mathrm{def}}{=} \mathcal{L}(a, \zeta_0, w)$$

We observe that $\mathcal{L}(a_\nu, \zeta_\nu, 0) = 0$ so that $\mathcal{L}(a, \zeta_0, 0) = 0$.

We see that the limiting function $\mathcal{L}(a, \zeta_0, w)$ is a linear combination of functions on $D(\zeta_0)$:

$$\left\{\delta_{p w_\alpha}, \delta_{p\bar w_\alpha}, \delta_{p w_\beta w_\gamma}, \delta_{p w_\beta \bar w_\gamma}, \delta_{p\bar w_\beta w_\gamma}, \delta_{p\bar w_\beta \bar w_\gamma}\right\}$$

where $\alpha, \beta, \gamma = 1, \ldots, n$.

They are linearly independent over the complex number.

Case 1 $a_n \neq 0$.

Since D is strictly pseudoconvex, we have $\psi_{1\bar{1}}(z_0) \neq 0$, for example. Hence $\mathcal{L}(a, z_0, w)$ has a non-zero term $\delta_{p w_1 \bar{w}_1}$; $\therefore \mathcal{L}(a, z_0, w) \not\equiv$ const. Then it is clear that $\mathcal{L}(a, z_0, w)$ is not holomorphic for $w \in D(z_0)$. Hence we have the contradiction that

$$0 = \lim_{\nu \to \infty} L_2(a_\nu, z_\nu) \geq \iint_V \left(\sum_{\alpha=1}^n \left| \frac{\partial}{\partial \bar{w}_\alpha} \mathcal{L}(a, z_0, w) \right|^2 \right) dV > 0.$$

Case 2 $a_n = 0$. $\quad \therefore (a_1, \cdots, a_{n-1}) \neq (0, \cdots, 0)$.

Then we have

$$\begin{aligned}
\mathcal{L}(a, z_0, w) = {} & \frac{i p_0}{2(n-1)} (\delta_{p\bar{w}_n} + \delta_{p w_n}) \\
& - \sum_{\alpha=1}^{n-1} a_\alpha \left(\sum_{\beta=1}^{n} (\psi_{\alpha\beta} \delta_{p w_\beta} + \psi_{\alpha\bar{\beta}} \delta_{p \bar{w}_\beta}) \right) \\
& - \sum_{\alpha=1}^{n-1} a_\alpha (\psi_{\alpha n} \delta_{p w_n} + \psi_{\alpha \bar{n}} \delta_{p \bar{w}_n}) \\
& + \frac{1}{2} \sum_{\alpha=1}^{n-1} a_\alpha (\psi_{\alpha n} - \psi_{\alpha \bar{n}})(\delta_{p w_n} - \delta_{p \bar{w}_n}) \\
= {} & \delta_{p w_n} \left\{ \frac{i p_0}{2(n-1)} - \frac{1}{2} \sum_{\alpha=1}^{n-1} a_\alpha (\psi_{\alpha n} + \psi_{\alpha \bar{n}}) \right\} \\
& + \delta_{p \bar{w}_n} \left\{ \frac{i p_0}{2(n-1)} - \frac{1}{2} \sum_{\alpha=1}^{n-1} a_\alpha (\psi_{\alpha n} + \psi_{\alpha \bar{n}}) \right\} \\
& - \sum_{\beta=1}^{n-1} \delta_{p w_\beta} \left(\sum_{\alpha=1}^{n-1} a_\alpha \psi_{\alpha\beta} \right) - \sum_{\beta=1}^{n-1} \delta_{p \bar{w}_\beta} \left(\sum_{\alpha=1}^{n-1} a_\alpha \psi_{\alpha\bar{\beta}} \right)
\end{aligned}$$

Since D is strictly pseudoconvex, we have

$$\det (\psi_{\alpha\bar{\beta}})_{\alpha,\beta=1,\cdots,n-1} \neq 0 \text{ at } z_0.$$

It follows from $(a_1, \cdots, a_{n-1}) \neq 0$ that some $\sum_{\alpha=1}^{n} a_\alpha' \psi_{\alpha\bar{\beta}}(z_0) \neq 0$. Consequently, $\mathcal{L}(a, z_0, w)$ has a non-zero term of $S_{\beta}\bar{w}_{\beta}$. Then it is clear that $\mathcal{L}(a, z_0, w)$ is not holomorphic for $w \in D(z_0)$. We have the same contradiction as in the case 1.

Lemma 4.1 is proved, and Theorem 3.2 is proved.

Example.

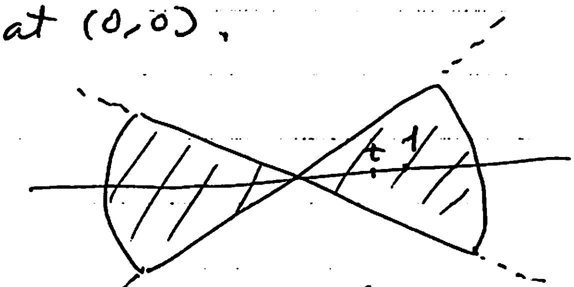
Let $D = \{(x,y) \in \mathbb{C}^2 \mid \psi(x,y) = |y|^2 - |x|^2 < 0\}$

$D_R = D \cap (|x|^2 + |y|^2 < R)$ where $R > 1$.

Hence ∂D_R has a singular point at $(0,0)$.

Put $\vec{t} = (t, 0)$, $\vec{1} = (1, 0)$

$z = (x, y)$



Denote by

$G_R(\vec{t}, z)$, $\Lambda_R(\vec{t})$ the green's function for $(D_R, \vec{t})$ and the Robin constant.

$G(\vec{t}, z)$, $\Lambda(\vec{t})$ the ones for $(D, \vec{t})$.

Since ∂D_R has strictly pseudoconvex boundary points, we see that

$ds^2 = \partial\bar{\partial} \log \Lambda_R(\zeta)$ defines a Kähler metric on D_R.

(But not for $\partial\bar{\partial} \log \Lambda(\zeta)$).

We notice that

D is invariant under the transformation

$T_c : z \to w = cz$

where $c \neq 0$ in $\mathbb{C}$.

It follows that

(1) $G(\vec{t}, z) = \dfrac{1}{|t|^2} G(1, w)$

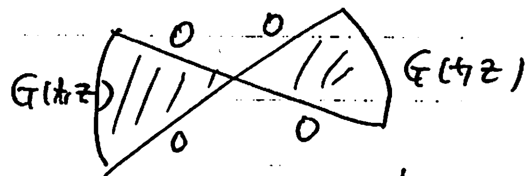
where $w = \dfrac{1}{t} z$

and that

(2) $\Lambda(\vec{t}) = \dfrac{\Lambda(\vec{1})}{|t|^2}$

<u>Lemma</u> $\Lambda_R(\vec{t}) = \dfrac{\Lambda(\vec{1})}{|t|^2} - \varepsilon(t)$ where $\varepsilon(t) \to 0$ as $|t| \to 0$.

<u>Proof</u> Let $u_R(t,z) = G(t,z) - G_R(t,z)$ on D_R. Then $u_R(t,z)$ is harmonic on D_R whose boundary values are



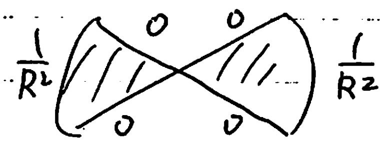
and $u_R(t,t) = \Lambda(\vec{t}) - \Lambda_R(\vec{t})$

Since $G(t,z) = \dfrac{1}{|t|^2} G(1,w)$ for $w = \dfrac{z}{t}$, we have, for $\|z\| = R$,

$$G(1,w) \leqq \frac{1}{\|w\|^2} = \frac{1}{\left(\frac{R}{|t|}\right)^2} = \frac{|t|^2}{R^2}$$

$$\therefore\ u_R(t,z) = G(t,z) \leqq \frac{1}{|t|^2}\,\frac{|t|^2}{R^2} = \frac{1}{R^2} \quad \text{on } |z| = R.$$

We consider the harmonic function $w(z)$ on D_R whose boundary values are



, so that

$$0 < u_R(t,z) \leqq w(z) \quad \text{for all } t \text{ and } z \in D_R.$$

Since the origin $(0,0)$ is regular for Dirichlet problem, it follows that $\lim\limits_{z \to 0} w(z) = 0$, and hence that

$$0 \leqq \lim_{t \to 0} u_R(t,t) \leqq \lim_{t \to 0} w(\vec{t}) = 0.$$

That is

$$\lim_{t \to 0} \left(\Lambda(\vec{t}) - \Lambda_R(\vec{t})\right) = \lim_{t \to 0} u_R(t,t) = 0$$

It follows from (2) that Lemma holds. ∎

We thus have

$$\log(-\Lambda_R(\vec{t})) = \log\left(\frac{-\Lambda(1)}{|t|^2} - \varepsilon(t)\right)$$

$$= -2\log|t| + \log\left(-\Lambda(1) - \varepsilon(t)|t|^2\right),$$

so that, observing $-\Lambda(1) \gneqq 0$, we see that

$$\frac{\partial^2 \log(-\Lambda_R(\vec{t}))}{\partial t\,\partial \bar{t}} = o(t)$$

where $o(t) \to 0$ as $|t| \to 0$.

Now consider a curve $C: (t, 0)$ $(0 < t \leqq 1)$ in D_R.

Then

$$\int_C ds = \int_0^1 \sqrt{o(t)}\,dt < +\infty.$$

Consequently, <u>ds^2 is not complete in D_R</u>.