

APPLICATIONS OF EXTRAPOLATIONS TO WAVELET CHARACTERIZATION OF VARIOUS FUNCTION SPACES AND EXTENSION OPERATORS

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ABSTRACT. The aim of this paper is to apply an extrapolation result without relying on convexification. We characterize ball Banach function spaces in terms of wavelets, formulated in a way that takes into account the smoothness properties of the spaces under consideration. The same technique can also be applied to prove vector-valued inequalities, for example. Furthermore, the result presented here refines a recent extension operator result by Zhu, Yang, and Yuan.

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1. INTRODUCTION

Wavelets were originally defined as functions that form an orthonormal basis in the L^2 space, and their mathematical theory has seen significant development. Nowadays, various types of wavelets are known, possessing desirable properties such as smoothness, rapid decay, compact support, and band-limitedness. By leveraging these properties appropriately, it is possible to characterize function spaces via alternative norms and to construct bases with excellent properties across a wide range of spaces beyond just L^2 .

The wavelet characterization of function spaces is a fascinating subject in real analysis and applied mathematics. In particular, the interplay between weights and wavelets has been extensively developed. Such characterizations are especially useful when analyzing the boundedness of operators. Here and below, by a “weight”, we mean a non-negative measurable function that is positive almost everywhere.

In this paper, we explore the connection between wavelets and an extrapolation result for ball Banach function spaces. Our assumptions are stated entirely in terms of the ball Banach function space and its Köthe dual. As an application, we obtain wavelet characterizations of various function spaces.

As mentioned above, wavelet characterizations of function spaces can be effectively used to investigate the boundedness properties of operators. In recent years, the use of weights has become extremely useful in the analysis of operators arising in harmonic analysis.

The study of weighted function spaces has a rich history, grounded in Muckenhoupt’s theory in real analysis. For an overview, see [1, Introduction]. In particular, weighted Lebesgue spaces have been characterized via the boundedness of integral operators; see, for example, [13, 30]. This method extends to more general function spaces, including weighted Lebesgue spaces with variable exponent [21] and Herz-type spaces [20, 25].

Furthermore, the wavelet characterization of Herz-type spaces has been achieved via the φ -transform [16]. Another approach to the wavelet characterization of function spaces involves the use of extrapolation theorems. This method has been applied, for instance, in the setting of Lebesgue spaces with variable exponent [18, 27].

We recall the definition of ball Banach function spaces defined in [43]. The space $L^0(\mathbb{R}^n)$ denotes the space of all equivalence classes of measurable functions modulo null functions. Let ρ be a mapping which maps $L^0(\mathbb{R}^n)$ to $[0, \infty]$. Recall that it is called a *ball Banach function norm* (over $\mathbb{R}^n$) if, for all non-negative measurable functions f, g, f_j ($j = 1, 2, 3, \dots$), for all constants $a \geq 0$ and for all open balls B in $\mathbb{R}^n$, the following properties hold:

- (P1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e.; $\rho(af) = a\rho(f)$, $\rho(f + g) \leq \rho(f) + \rho(g)$,
- (P2) $\rho(g) \leq \rho(f)$ if $0 \leq g \leq f$ a.e.,
- (P3) the Fatou property; $\rho(f_j) \uparrow \rho(f)$ holds whenever $0 \leq f_j \uparrow f$ a.e.,
- (P4) $\rho(\chi_B) < \infty$,
- (P5) $\|\chi_B f\|_{L^1} \lesssim \rho(f)$ with the implicit constant depending on B and ρ but independent of f .

Accordingly, the space generated by such ρ , which is given by

$$X = X_\rho = \{f \in L^0(\mathbb{R}^n) : f \text{ is a measurable function satisfying } \|f\|_{X_\rho} := \rho(|f|) < \infty\},$$

is called a ball Banach function space. Furthermore its “associate norm” ρ' is defined for a non-negative measurable function g by $\rho'(g) := \sup \{\|f \cdot g\|_{L^1} : f \in X, \rho(f) \leq 1\}$. Likewise we can consider the Köthe dual $X' = (X_\rho)' = X_{\rho'}$ for Banach lattices.

We will use the Hardy–Littlewood maximal operator M to describe our assumption. For $f \in L^0(\mathbb{R}^n)$, one defines a function Mf by

$$Mf(x) := \sup_{r>0} \frac{1}{r^n} \int_{|y-x|<r} |f(y)| dy$$

for $x \in \mathbb{R}^n$. The mapping $f \mapsto Mf$ is called the Hardy–Littlewood maximal operator.

In [38] Rutsky characterized the condition for which the Riesz transforms, given by

$$\mathcal{R}_j f(x) := \lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) dy \quad (x \in \mathbb{R}^n),$$

for $j = 1, 2, \dots, n$, are bounded on X . In fact, he showed that M is bounded on X and on X' if and only if $\mathcal{R}_j$ is bounded on X for all $j = 1, 2, \dots, n$.

As we mentioned, weights are fundamental tools to investigate the boundedness of operators. Let w be a weight and $1 < p < \infty$. We also consider $L^p(w)$ which collects all $f \in L^0(\mathbb{R}^n)$ for which $\|f\|_{L^p(w)} := \|fw^{\frac{1}{p}}\|_{L^p}$ is finite. Using the Muckenhoupt class A_p , which will be recalled in (1.3) below, we begin by recalling an extrapolation theorem. We then extend this theorem to the case of the local Muckenhoupt class $A_{p,\text{loc}}$.

Our main focus in this paper is on applications of these extrapolation results. This approach clearly demonstrates the fundamental role that weights play in analyzing the boundedness properties of operators.

Theorem 1.1. *Let X be a ball Banach function space and $1 < p < \infty$. Assume that M is bounded on X and on X' . For a given increasing function $N : [1, \infty) \rightarrow (0, \infty)$, write*

$$(1.1) \quad \mathcal{F} := \{(f, g) \in L^0(\mathbb{R}^n)^2 : \|f\|_{L^p(w)} \leq N([w]_{A_p}) \|g\|_{L^p(w)} \text{ for all } w \in A_p\}.$$

Then there exists a constant $C > 0$ such that $\|f\|_X \leq C\|g\|_X$ for all $(f, g) \in \mathcal{F}$.

The proof of Theorem 1.1 is known; see [7, Theorem 10.1], [4, Theorem 3.1] and [36, Theorem A, Theorem 4.7 and Remark 4.8]. This theorem is a special case of [36, Theorem A] with $X = Y$, $r_1 = r_2 = 1$, and $s_1 = s_2 = \infty$. For clarity and convenience, we restate Theorem 1.1 in a more accessible form below.

Corollary 1.2. *Let X be a Banach function space and $1 < p < \infty$. Then the following are equivalent:*

- (A) *M is bounded on X and on X' .*
- (B) *The Riesz transform $\mathcal{R}_j$ is bounded on X for each $j = 1, 2, \dots, n$.*
- (C) *Define $\mathcal{F}$ by (1.1). Then there exists a constant $C > 0$ such that $\|f\|_X \leq C\|g\|_X$ for all $(f, g) \in \mathcal{F}$.*

Corollary 1.2 is also a known result. See [37] for more equivalent conditions. Nevertheless, we briefly recall its proof for completeness.

Corollary 1.2 follows directly from a combination of a known equivalence and Theorem 1.1. As previously mentioned, Rutsky established the equivalence between conditions (A) and (B). Theorem 1.1 asserts that (A) implies (C). Finally, it is known that $(\mathcal{R}_j f, f) \in \mathcal{F}$ for $N(t) = \alpha t^{\max(p, p')}$ with sufficiently large α ; see [42, Theorem 305], for example. This yields the implication (C) $\Rightarrow$ (B), completing the proof.

We have an analogy to local Hardy–Littlewood maximal operator M_{loc} . For $f \in L^0(\mathbb{R}^n)$, one defines a function $M_{\text{loc}}f$ by

$$M_{\text{loc}}f(x) := \sup_{0 < r < 1} \frac{1}{r^n} \int_{|y-x| < r} |f(y)| dy$$

for $x \in \mathbb{R}^n$. The mapping $f \mapsto M_{\text{loc}}f$ is called the local Hardy–Littlewood maximal operator.

Theorem 1.3. *Let X be a ball Banach function space and $1 < p < \infty$. Assume that M_{loc} is bounded on X and on X' . For a given increasing function $N : [1, \infty) \rightarrow (0, \infty)$, write*

$$(1.2) \quad \mathcal{F}_{\text{loc}} := \{(f, g) \in L^0(\mathbb{R}^n)^2 : \|f\|_{L^p(w)} \leq N([w]_{A_{p, \text{loc}}}) \|g\|_{L^p(w)} \text{ for all } w \in A_{p, \text{loc}}\}.$$

Then there exists a constant $C > 0$ such that $\|f\|_X \leq C\|g\|_X$ for all $(f, g) \in \mathcal{F}_{\text{loc}}$.

This result is also known: see [4, Theorem 3.1]. For readers' convenience, we provide a short proof of Theorem 1.3 as an appendix in Section 4. Here, we make a clarifying remark on the strength of the assumption in Theorems 1.1 and 1.3 regarding X .

Remark 1.4. If X satisfies the assumption in Theorem 1.1, meaning that M is bounded on both X and X' , then according to Lerner and Pérez [29], there exists a constant $\eta > 1$ such that the operator $f \mapsto (M[|f|^\eta])^{\frac{1}{\eta}}$ is bounded on both X and X' . In this scenario, we can establish $X \hookrightarrow L^\eta(w)$ for some $w \in A_1$. We refer to [34, Lemma 2.7]. See also [41, Lemma 3]. An analogy to M_{loc} is available to Theorem 1.3.

Curbera, García-Cuerva, Martell, and Pérez made a significant breakthrough by obtaining the extrapolation theorem for rearrangement invariant Banach function spaces [10]. The case where X is not rearrangement invariant has gained increasing attention. For instance, in [17, Theorem 3.3] and [11, Theorem 3.1], Ho employed p -convexification to achieve a vector-valued extension of the maximal inequality. For a ball Banach function

space X and $p > 0$, we consider its p -convexification given by $X^p := \{f \in L^0(\mathbb{R}^n) : |f|^{\frac{1}{p}} \in X\}$. The norm of X^p is given by $\|f\|_{X^p} := (\| |f|^{\frac{1}{p}} \|_X)^p$. Remarkably, Theorems 1.1 and 1.3 do not require p -convexification. Nor do we need the absolute continuity of the norm of X . Recall that X is said to have the absolutely continuous norm if $\|f\chi_{E_j}\|_X \rightarrow 0$ as $j \rightarrow \infty$ for any $f \in X$ and any sequence $\{E_j\}_{j=1}^\infty$ of sets decreasing to $\emptyset$. If the space X is separable, then X has the absolutely continuous norm. See [3].

Here and below, in addition to the above notation, we use the following convention in this paper.

- By a “cube” we mean a compact cube whose edges are parallel to the *coordinate axes*. The metric closed ball defined by ℓ^∞ is called a *cube*. If a cube has center x and radius r , we denote it by $Q(x, r)$. Namely, we write

$$Q(x, r) \equiv \left\{ y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n : \max_{j=1,2,\dots,n} |x_j - y_j| \leq r \right\}$$

when $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $r > 0$. The symbol $\mathcal{Q}$ stands for all cubes.

- Given $x \in \mathbb{R}^n$ and $r > 0$, we denote the open ball centered at x and having radius $r > 0$ by $B(x, r)$. We abbreviate $B(r) := B(0, r)$.
- Let $A, B \geq 0$. Then $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leq CB$, where C is usually independent of the functions we are considering. The symbol $A \sim B$ stands for the two-sided inequality $A \lesssim B \lesssim A$.
- Let $l \in \mathbb{N}$. We denote by M^l the l -fold composition of the Hardy-Littlewood maximal operator M . We write M^0 for the operator

$$f \in L^0(\mathbb{R}^n) \mapsto |f| \in L^0(\mathbb{R}^n).$$

The symbol M_{loc}^l corresponds to the one for the local Hardy-Littlewood maximal operator M_{loc} .

- Given a measurable set $E \subset \mathbb{R}^n$, the characteristic function of E is denoted by χ_E . We write $|E|$ for the Lebesgue measure of E .
- The space $L_c^\infty(\mathbb{R}^n)$ denotes the set of bounded and compactly supported measurable functions on $\mathbb{R}^n$.
- For a measurable set $E \subset \mathbb{R}^n$ and a weight w , the notation

$$w(E) := \int_E w(x) \, dx$$

stands for the integral of w over E .

- A weight is a measurable function $w : \mathbb{R}^n \rightarrow [0, \infty]$ that is positive and finite almost everywhere.

A locally integrable weight w belongs to the A_1 -class (or is an A_1 -weight) if $0 < w < \infty$ almost everywhere and

$$[w]_{A_1} := \left\| \frac{Mw}{w} \right\|_{L^\infty} < \infty.$$

The local A_1 constant is defined similarly, with M replaced by a localized maximal operator M_{loc} :

$$[w]_{A_{1,\text{loc}}} := \left\| \frac{M_{\text{loc}}w}{w} \right\|_{L^\infty} < \infty.$$

- Let $1 < p < \infty$. A locally integrable weight w belongs to the A_p -class (or is an A_p -weight) if $0 < w < \infty$ almost everywhere and

$$(1.3) \quad [w]_{A_p} := \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \int_Q w(x) \, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty,$$

where $\mathcal{Q}$ is the collection of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

The local A_p constant is defined by restricting the supremum in (1.3) to cubes with side length at most 1:

$$[w]_{A_{p,\text{loc}}} := \sup_{\substack{Q \in \mathcal{Q} \\ \ell(Q) \leq 1}} \left(\frac{1}{|Q|} \int_Q w(x) \, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1}.$$

- We denote by $\|M\|_{X \rightarrow X}$ the operator norm of the Hardy-Littlewood maximal operator M acting on a function space X . The operator norm $\|M_{\text{loc}}\|_{X \rightarrow X}$ is defined analogously for the localized maximal operator M_{loc} .
- Given two complex-valued functions f, g , we formally write the L^2 -inner product by

$$\langle f, g \rangle := \int_{\mathbb{R}^n} f(x) \overline{g(x)} \, dx.$$

Here is the organization of this paper. Section 2 is oriented to applications of Theorems 1.1 and 1.3. We obtain the boundedness properties of operators and the wavelet characterization in Section 2. We also take up the vector-valued inequality of M and the extension operator in Section 2. We will see that our method allows us to include more function spaces than are dealt with in [47]. We take up various function spaces in Section 3. We compare what we can say from Theorems 1.1 and 1.3 with known results in Sections 2 and 3. Finally, Section 4 gives a short proof of Theorem 1.3.

2. MAIN RESULTS: APPLICATIONS OF THE EXTRAPOLATION

We present applications. To this end, we introduce some terminology. We consider X -based Sobolev spaces over domains. First we recall the one over $\mathbb{R}^n$. Let $s > 0$. Recall that the Bessel potential $(1 - \Delta)^{\frac{s}{2}} := \mathcal{F}^{-1}[(1 + |\cdot|^2)^{\frac{s}{2}} \mathcal{F}f]$ is an isomorphism in $\mathcal{S}'(\mathbb{R}^n)$; see [40, p. 251].

Definition 2.1. Let $s > 0$. Define the X -based Sobolev space $W_X^s(\mathbb{R}^n)$ of order s to be the set of all $f \in X(\mathbb{R}^n)$ for which $(1 - \Delta)^{\frac{s}{2}} f$ is represented by an element in X . Equip $W_X^s(\mathbb{R}^n)$ with the norm given by $\|f\|_{W_X^s} := \|(1 - \Delta)^{\frac{s}{2}} f\|_X$ for all $f \in W_X^s(\mathbb{R}^n)$.

Assume that M is bounded on X and on X' . Recall that the inverse operator $(1 - \Delta)^{-\frac{s}{2}}$ of $(1 - \Delta)^{\frac{s}{2}}$ has the convolution kernel K satisfying $K \in L^1(\mathbb{R}^n)$ (see [42, Corollary 197]). Hence the Bessel potential $(1 - \Delta)^{-\frac{s}{2}}$ is bounded on X since we have a pointwise estimate $|(1 - \Delta)^{-\frac{s}{2}} f(x)| \lesssim Mf(x)$, $x \in \mathbb{R}^n$, for all $f \in X$; see [42, Proposition 159]. Thus, $W_X^s(\mathbb{R}^n)$ is a Banach space. Note that this is a natural extension of $L^{p,s}(w)$ considered in [19, 22] if $X = L^p(w)$ with $1 < p < \infty$ and $w \in A_p$. By definition $W_X^0(\mathbb{R}^n) = X$. Furthermore, if s is a positive integer, by virtue of the boundedness of singular integral operators, which is guaranteed by the boundedness of M on X and X' , the space W_X^s coincides with the space

of all $f \in X$ such that any weak partial derivative up to order s belong to X . Something similar applies if we merely assume that M_{loc} is bounded on X and on X' . In this case we need to replace $(1 - \Delta)^{-\frac{s}{2}}$ with $(1 - t_0^2 \Delta)^{-\frac{s}{2}}$ with some $0 < t_0 \ll 1$.

Our definitions can be extended to domains. Let D be a bounded Lipschitz domain. Let $k \in \mathbb{N}$, $1 < p < \infty$, and $w \in A_p$. Define $L^{p,k}(D, w)$ to be the set of all $f \in L^p(w)$ for which $\partial^\alpha f$, the weak partial derivative of order α , belongs to $L^p(w)$ for any multiindex α with $|\alpha| \leq k$.

For a ball Banach function space X , we write $X(D)$ to be the set of all $f \in L^1_{\text{loc}}(D)$ for which Zf belongs to X . Let Z be the zero extension operator. The norm of $f \in X(D)$ is given by $\|f\|_{X(D)} = \|Zf\|_X$.

We write $W_X^k(D)$ for the set of all $f \in L^1_{\text{loc}}(D)$ for which $\partial^\alpha f$ belongs to $X(D)$ for any multiindex α with $|\alpha| \leq k$.

Here is the organization of Section 2. Section 2.1 considers wavelet characterization. Section 2.2 deals with vector-valued inequalities of the Hardy–Littlewood maximal operator. We refine a recent result in [47] in Section 2.3.

2.1. Wavelet characterization. Based on the fundamental wavelet theory (see [15, 35] for example), we can choose compactly supported C^K -functions

$$(2.1) \quad \varphi \text{ and } \psi^l \quad (l = 1, 2, \dots, 2^n - 1)$$

so that the following conditions are satisfied:

- (1) For any $J \in \mathbb{Z}$, the system

$$\left\{ \varphi_{J,k}, \psi_{j,k}^l : k \in \mathbb{Z}^n, j \geq J, l = 1, 2, \dots, 2^n - 1 \right\}$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$. Here, given a function F defined on $\mathbb{R}^n$, we write

$$F_{j,k} := 2^{\frac{jn}{2}} F(2^j \cdot -k)$$

for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$.

- (2) Denote by $\mathcal{P}_0^\perp$ the set of all integrable functions having 0 integral. We have

$$(2.2) \quad \psi^l \in \mathcal{P}_0^\perp \quad (l = 1, 2, \dots, 2^n - 1).$$

In addition, they are real-valued and compactly supported with

$$(2.3) \quad \text{supp}(\varphi) = \text{supp}(\psi^l) = [0, 2N - 1]^n$$

for some $N \in \mathbb{N}$.

We also consider cubes $Q_{j,k}$ and $Q_{j,k}^*$ defined by

$$Q_{j,k} := \prod_{m=1}^n [2^{-j} k_m, 2^{-j}(k_m + 1)], \quad Q_{j,k}^* := \prod_{m=1}^n [2^{-j} k_m, 2^{-j}(k_m + 2N - 1)],$$

where $N \in \mathbb{N}$ satisfies (2.3), and

$$(2.4) \quad \chi_{j,k} := 2^{\frac{jn}{2}} \chi_{Q_{j,k}}, \quad \chi_{j,k}^* := 2^{\frac{jn}{2}} \chi_{Q_{j,k}^*}.$$

We note that the L^2 -inner product $\langle f, \phi \rangle$ exists for the complex-valued functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\phi \in C_c$. Then, using the L^2 -inner product, we define two square functions Vf , $W_s f$ by

$$Vf := V^\varphi f := \left(\sum_{k \in \mathbb{Z}^n} |\langle f, \varphi_{J,k} \rangle \chi_{J,k}|^2 \right)^{\frac{1}{2}},$$

$$W_s f := W_s^{\psi^l} f := \left(\sum_{l=1}^{2^n-1} \sum_{j=J}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| 2^{js} \langle f, \psi_{j,k}^l \rangle \chi_{j,k} \right|^2 \right)^{\frac{1}{2}}.$$

Here, $s \geq 0$ is a non-negative real number and J is a fixed integer. We abbreviate $Wf := W_0 f$.

The wavelet characterization of X can be obtained with ease.

Theorem 2.2. *Let $K > s > 0$. Suppose that we have $\varphi, \psi^l \in C^K(\mathbb{R}^n)$ as in (2.1). Assume that (2.2) holds, that M_{loc} is bounded on X and on X' . Then the following are equivalent for any $f \in X$:*

- (1) $Vf + W_s f \in X$.
- (2) $f \in W_X^s(\mathbb{R}^n)$.

Furthermore, in this case $\|f\|_{W_X^s} \sim \|Vf + W_s f\|_X$.

Proof. Let $0 < t_0 \ll 1$. Then we have

$$(2.5) \quad ((1 - t_0^2 \Delta)^{\frac{s}{2}} f, Vf + W_s f) \in \mathcal{F}_{\text{loc}}$$

and

$$(2.6) \quad (Vf + W_s f, (1 - t_0^2 \Delta)^{\frac{s}{2}} f) \in \mathcal{F}_{\text{loc}}$$

from [22, Theorem 4.6]. □

Despite the above observations, it is helpful to offer some words (2.5) and (2.6) when s is an integer.

Remark 2.3. Let $X = L^p(\mathbb{R}^n)$. Relations (2.5) and (2.6) are essentially proved in the books [15, 35].

Now let $X = L^p(w)$ with $w \in A_p$. In this weighted setting, Lemarié-Rieusset [30] has established the result for the case $s = 0$.

Further, consider $X = W^{s,p}(w)$ with $w \in A_p$ and $s \in \{0, 1, 2, \dots\}$. The first author [19] has extended the result to this broader context.

Therefore, we are in a position to apply Theorems 1.1 and 1.3 once again.

Letting $s = 0$, we derive the following corollary. It suffices to assume *a priori* that f is locally integrable.

Corollary 2.4. *Assume that M_{loc} is bounded on X and on X' , or equivalently, each $\mathcal{R}_j$ is bounded on X . Then the following are equivalent for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$:*

- (1) $Vf + Wf \in X$.
- (2) $f \in X$.

Furthermore, in this case $\|f\|_X \sim \|Vf + Wf\|_X$.

The proof is the same as Theorem 2.2, although we assume that f is merely locally integrable.

It is interesting to compare Corollary 2.4 with the following result:

Theorem 2.5. *Assume that M_{loc} is bounded on X and on X' and that X is separable. Let $f \in X$. Then*

$$(2.7) \quad f = \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{J,k} \rangle \varphi_{J,k} + \sum_{l=1}^{2^n-1} \sum_{j=J}^{\infty} \sum_{k \in \mathbb{Z}^n} \langle f, \psi_{j,k}^l \rangle \psi_{j,k}^l$$

holds unconditionally in the topology of X .

See [26, Theorem 8] for the case of $n = 1$. Once we use Corollary 2.4, Theorem 2.5 is easy to prove. In fact, since X is separable, it is straightforward to show that X has the absolutely continuous norm. Corollary 2.4 ensures that equation (2.7) converges in X .

Let $\mathcal{B}$ denote the set of all Borel measurable sets. We define the set

$$S_0 = \left\{ \sum_{j=1}^N a_j \chi_{E_j} : \{a_1, a_2, \dots, a_N\} \subset \mathbb{C}, \{E_1, E_2, \dots, E_N\} \subset \mathcal{B} \right\}.$$

Assume that M is bounded in both X and X' .

It is important to mention the approach taken by Karlovich, who refined the results in [21, Theorem 4.1], assuming that X is separable as we have mentioned. If either X is separable or if $L^2(\mathbb{R}^n) \cap X$ is dense in X , then Karlovich [26] and the authors in [21] proved unconditional convergence in (2.7), respectively. Among others, Karlovich used the fact that $S_0 \cap X$ is dense in X assuming that X is separable. It is noteworthy that X has absolutely continuous norm if X is separable.

Furthermore, in any of these cases the set

$$\mathcal{X} = \left\{ \psi_{j,k}^l : l = 1, 2, \dots, 2^n - 1, j \geq J, k \in \mathbb{Z}^n \right\} \cup \{ \varphi_{J,k} : k \in \mathbb{Z}^n \}$$

is dense in X . In particular, this implies that $\mathcal{X}$ is dense in X . It is clear that X is separable and that $L^2(\mathbb{R}^n) \cap X$ are contained in $\mathcal{X}$. This means that the assumptions in [26, Theorem 8] and [21, Theorem 4.1] are equivalent.

2.2. Vector-valued maximal inequality. A similar technique can be used to obtain the vector-valued inequality.

Example 2.6. Let $r \in (1, \infty]$. Assume that M_{loc} is bounded on X and on X' . Then there exists a constant $C > 0$ such that

$$(2.8) \quad \left\| \left(\sum_{j=1}^{\infty} M_{\text{loc}} f_j^r \right)^{\frac{1}{r}} \right\|_X \leq C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_X$$

for all $\{f_j\}_{j=1}^{\infty} \subset X$ since

$$\left(\left(\sum_{j=1}^{\infty} M_{\text{loc}} f_j^r \right)^{\frac{1}{r}}, \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right) \in \mathcal{F}_{\text{loc}}$$

for $N(t) = \alpha t^\beta$ where $t \gg 1$ and α, β are constants. This result recaptures [4, Theorem 3.1]. See [42, Theorem 303] for the case of M to find

$$\left\| \left(\sum_{j=1}^{\infty} M f_j^r \right)^{\frac{1}{r}} \right\|_{L^p(w)} \leq \alpha ([w]_{A_p})^\beta \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)}$$

for all $\{f_j\}_{j=1}^{\infty} \subset X$. Estimate (2.8) corresponding to the vector-valued maximal inequality for M obtained in [34, Theorem 2.1].

2.3. Extension operator. Let D be a bounded Lipschitz domain. Chua established that there exists a bounded linear operator $\Lambda : L^{p,k}(D, w) \rightarrow L^{p,k}(w)$ independent of w such that $\Lambda f|_D = f$ for all $f \in L^{p,k}(D, w)$. More precisely, letting Z the zero extension operator, Chua essentially proved that

$$\left(\Lambda f, \sum_{|\alpha| \leq k} |Z \partial^\alpha f| \right) \in \mathcal{F}$$

with some suitable choice of $N(\cdot)$. See [6, Theorem 1.1]. Based on this fact, we investigate the action of the extension operator Λ on X -based Sobolev spaces over D .

Example 2.7. Assume that M is bounded on X and on X' , or equivalently, each $\mathcal{R}_j$ is bounded on X . Let $f \in W_X^k(D)$. Then $f \in L^{p,k}(w, D)$ for some $w \in A_1$. Thus, Λf makes sense and $\|\Lambda f\|_X \sim \sum_{|\alpha| \leq k} \|Z \partial^\alpha f\|_X$.

We compare Example 2.7 with an existing result [47, Theorem 5.4].

Remark 2.8.

- (1) A recent work [47, Theorem 5.4] requires that $X^{\frac{1}{p}}$ is a ball Banach space for some $p > 1$ and that M is bounded on $(X^{\frac{1}{p}})'$. This assumption implies that M is bounded on both X and X' , since it implies that $\mathcal{R}_j$ is bounded on X .
- (2) A recent work [47, Theorem 5.4] requires that X has the absolutely continuous norm. According to Theorems 1.1 and 1.3, X need not have the absolutely continuous norm.

3. EXAMPLES OF BANACH LATTICES TO WHICH THEOREMS 1.1 AND 1.3 ARE APPLICABLE

We will exhibit examples of Banach lattices to which Theorems 1.1 and 1.3 are applicable. We concentrate on wavelet characterization. However, it is worth mentioning that we have a counterpart to vector-valued inequalities.

3.1. Weighted Lebesgue spaces. Let $1 < q < \infty$ and $w \in A_q$. It is well known that M and $\mathcal{R}_j$, $j = 1, 2, \dots, n$, are bounded on $L^q(w)$ [42, Theorems 290 and 306]. Thus, Theorems 1.1 and 1.3 are applicable to $L^q(w)$. If we merely assume that $w \in A_{q,\text{loc}}$, then only Theorem 1.3 is applicable to $L^q(w)$.

Consider a more concrete case. Let $1 < q < \infty$ and $v(x) = |x|^\alpha$ be a power weight of order $\alpha \in \mathbb{R}$. According to [42, Theorem 283], $v \in A_q$ if and only if $-n < \alpha < n(q-1)$.

This condition also applies to the weight W , defined by $W(x) = \max(1, |x|)^\alpha$ for $x \in \mathbb{R}^n$. Therefore, if $-n < \alpha < n(q-1)$, Theorems 1.1 and 1.3 applicable to both $L^q(v)$ and $L^q(W)$.

3.2. Lorentz spaces. Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Then the Lorentz space $L^{p,q}(\mathbb{R}^n)$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{L^{p,q}} := \left\{ \int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t} \right\}^{\frac{1}{q}}$$

is finite. Here f^* stands for the decreasing rearrangement of f . Remark that if $1 < p < \infty$, then there exists a norm $\|\cdot\|$ on $L^{p,q}(\mathbb{R}^n)$ such that

$$\|f\|_{L^{p,q}} \sim \|f\|$$

for all $f \in L^{p,q}(\mathbb{R}^n)$ [42, Corollary 39].

Furthermore, since the space $L^{p,q}(\mathbb{R}^n)$ can be obtained through the real interpolation of the spaces $L^{p_0}(\mathbb{R}^n)$ and $L^{p_1}(\mathbb{R}^n)$ as shown in [2], the operators M and $\mathcal{R}_j$ (for $j = 1, 2, \dots, n$) are bounded on $L^{p,q}(\mathbb{R}^n)$ for $1 < p < \infty$ and $1 \leq q \leq \infty$. Consequently, Theorems 1.1 and 1.3 can be applied to $L^{p,q}(\mathbb{R}^n)$.

To our knowledge, the wavelet characterization of the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$ for $1 < p < \infty$ and $1 \leq q \leq \infty$ has been established by Soardi [45].

It is important to note that Lorentz spaces $L^{p,\infty}(\mathbb{R}^n)$, also known as the weak Lebesgue spaces $WL^p(\mathbb{R}^n)$, are not separable. Therefore, in this case, we cannot apply Theorem 2.5 to obtain the unconditional convergence. However, according to Remark 1.4, for any $\eta \in (1, p)$ and any $f \in WL^p(\mathbb{R}^n)$, there exists an A_1 -weight w (which is independent of f) such that the convergence of the expansion in Theorem 2.5 occurs in $L^\eta(w)$.

3.3. Herz spaces. For each $k \in \mathbb{Z}$, we set $C_k := B(2^k) \setminus B(2^{k-1})$. Let $p, q \in [1, \infty]$ and $\alpha \in \mathbb{R}$. We write $\chi_k = \chi_{C_k}$ for each $k \in \mathbb{Z}$. The non-homogeneous Herz space $K_p^{\alpha,q}(\mathbb{R}^n)$ consists of all $f \in L^0(\mathbb{R}^n)$ for which

$$\|f\|_{K_p^{\alpha,q}} := \|f\|_{L^p(B(1))} + \left(\sum_{k=1}^{\infty} (2^{k\alpha} \|f\chi_k\|_{L^p})^q \right)^{\frac{1}{q}} < \infty$$

and the homogeneous Herz space $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ consists of all $f \in L^0(\mathbb{R}^n)$ for which

$$\|f\|_{\dot{K}_p^{\alpha,q}} := \left(\sum_{k=-\infty}^{\infty} (2^{k\alpha} \|f\chi_k\|_{L^p})^q \right)^{\frac{1}{q}} < \infty.$$

Let $\alpha \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq q \leq \infty$ satisfy

$$(3.1) \quad -\frac{n}{p} < \alpha < n - \frac{n}{p}.$$

Then M and $\mathcal{R}_j$ are bounded on $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ and $K_p^{\alpha,q}(\mathbb{R}^n)$ for all $j = 1, 2, \dots, n$. See [31, 32, 33]. Thus, Theorems 1.1 and 1.3 are applicable to $K_p^{\alpha,q}(\mathbb{R}^n)$ and $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$. As far as we know, there are some results [16, 25] on wavelet characterization of Herz spaces. Since M is bounded on $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$, $K_p^{\alpha,q}(\mathbb{R}^n)$ and their preduals, it follows that many of these results fall within the scope of Corollary 2.4. Remark that $\dot{K}_p^{\alpha,\infty}(\mathbb{R}^n)$ and $K_p^{\alpha,\infty}(\mathbb{R}^n)$ are not separable. However, in view of Remark 1.4, we see that for any $\eta \in (1, p)$ and

$f \in \dot{K}_p^{\alpha,\infty}(\mathbb{R}^n) \cup K_p^{\alpha,\infty}(\mathbb{R}^n)$ there exists an A_1 -weight w independent of f such that the convergence of the expansion in Theorem 2.5 takes place in $L^7(w)$.

3.4. Weighted Lebesgue spaces with variable exponents. Let w be a weight. The Lebesgue space $L^{p(\cdot)}(w)$ with variable exponent consists of all $f \in L^0(\mathbb{R}^n)$ for which

$$\|f\|_{L^{p(\cdot)}(w)} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} w(x) dx \leq 1 \right\} < \infty.$$

We postulate some conditions on variable exponents. We use the following terminology: we use the following standard notation on exponents:

Definition 3.1.

- (1) Let $r(\cdot)$ be a variable exponent. We write

$$r_- := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} r(x), \quad r_+ := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} r(x).$$

- (2) The class $\mathcal{P}_0 = \mathcal{P}_0(\mathbb{R}^n)$ collects all the variable exponents $r(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ that satisfy $0 < r_- \leq r_+ < \infty$, while $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ collects all the variable exponents $r(\cdot)$ in $\mathcal{P}_0$ that satisfy $r_- > 1$.

Definition 3.2. Let $r(\cdot) \in L^0(\mathbb{R}^n)$.

- (1) The variable exponent $r(\cdot)$ satisfies the local log-Hölder continuity condition if

$$(3.2) \quad |r(x) - r(y)| \leq \frac{c_*}{\log(|x - y|^{-1})} \quad \text{for } x, y \in \mathbb{R}^n \quad \text{with } |x - y| \leq \frac{1}{2}.$$

- (2) If (3.2) is satisfied only for $y = 0$, then $r(\cdot)$ is log-Hölder continuous at the origin. The class $\text{LH}_0 = \text{LH}_0(\mathbb{R}^n)$ stands for the class of exponents which are log-Hölder continuous at the origin.

- (3) The exponent $r(\cdot)$ satisfies the log-Hölder-type decay condition at infinity if

$$(3.3) \quad |r(x) - r_\infty| \leq \frac{c^*}{\log(e + |x|)} \quad \text{for } x \in \mathbb{R}^n.$$

The class $\text{LH}_\infty = \text{LH}_\infty(\mathbb{R}^n)$ represents the class of exponents that satisfy (3.3).

- (4) The class $\text{LH} = \text{LH}(\mathbb{R}^n)$ contains all exponents that satisfy (3.2) and (3.3).

Here c_* , c^* and r_∞ are constants independent of x and y .

For a variable exponent $r(\cdot)$ satisfying $1 < r_- \leq r_+ < \infty$, write $r'(\cdot) = \frac{r(\cdot)}{r(\cdot)-1}$. We employ the definition in [8].

Definition 3.3. Suppose $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A weight w is said to be an $A_{p(\cdot)}$ -weight if w satisfies

$$\sup_{Q \in \mathcal{Q}} \frac{1}{|Q|} \left\| w^{\frac{1}{p(\cdot)}} \chi_Q \right\|_{L^{p(\cdot)}} \left\| w^{-\frac{1}{p'(\cdot)}} \chi_Q \right\|_{L^{p'(\cdot)}} < \infty.$$

The Muckenhoupt class $A_{p(\cdot)}$ adapted to $p(\cdot)$ consists of all $A_{p(\cdot)}$ weights.

Note that this definition extends the classical class A_p . In fact, when $p(\cdot)$ is a constant function p , the two definitions coincide. It is known that M is bounded on $L^{p(\cdot)}(w)$ if $p(\cdot) \in \text{LH}$ satisfies $1 < p_- \leq p_+ < \infty$ and $w \in A_{p(\cdot)}$. Since this condition is symmetric; in this case $p'(\cdot) \in \text{LH}$, $1 < p'_- \leq p'_+ < \infty$ and $w^{-\frac{1}{p'(\cdot)-1}} \in A_{p'(\cdot)}$, Theorems 1.1 and 1.3 are applicable to $L^{p(\cdot)}(w)$. This recaptures a result by [8, 9, 21].

3.5. Herz spaces with variable exponents. Modelled on the previous section, it seems useful to study Herz spaces with variable exponents. The first author [20] has established a method of wavelet characterization applying the boundeness of sublinear operators and norm estimates for characteristic functions with variable exponent. Recently more and more papers have appeared in this direction. In this subsection we establish a new method of wavelet characterization as an application of the extrapolation theorem.

We now set up some notation to recall the definition of Herz spaces with three variable exponents established by the authors' previous work in [24]. Let $p(\cdot)$, $q(\cdot)$ be variable exponents. The space $\ell^{q(\cdot)}(L^{p(\cdot)}(w))$ is the set of all sequences $\{f_j\}_{j \in \mathbb{Z}}$ in $L^0(\mathbb{R}^n)$ such that

$$\|\{f_j\}_{j \in \mathbb{Z}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} := \inf \left\{ \mu > 0 : \rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}(\{\mu^{-1} f_j\}_{j \in \mathbb{Z}}) \leq 1 \right\} < \infty,$$

where

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}(\{\mu^{-1} f_j\}_{j \in \mathbb{Z}}) := \sum_{j=-\infty}^{\infty} \|\mu^{-1} f_j\|_{L^{p(\cdot)/q(\cdot)}(w)}^{q(\cdot)}.$$

We sometimes consider the case where $\mathbb{Z}$ is replaced by $\mathbb{N}$.

Now, we recall the definition of two-weighted Herz spaces with three variable exponents.

Definition 3.4. Let v, w be weights on $\mathbb{R}^n$, $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$.

- (1) The non-homogeneous two-weighted Herz space $K_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(v, w)$ consists of all $f \in L^0(\mathbb{R}^n)$ for which

$$\|f\|_{K_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(v, w)} := \|f \chi_{B_0}\|_{L^{p(\cdot)}(w)} + \left\| \left\{ [v(B_k)]^{\alpha(\cdot)/n} f \chi_{C_k} \right\}_{k=1}^{\infty} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}$$

is finite.

- (2) The homogeneous two-weighted Herz space $\dot{K}_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(v, w)$ consists of all $f \in L^0(\mathbb{R}^n)$ for which

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(v, w)} := \left\| \left\{ [v(B_k)]^{\alpha(\cdot)/n} f \chi_k \right\}_{k=-\infty}^{\infty} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} < \infty.$$

- (3) In the above, if $v = w = 1$, then we write $\dot{K}_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(\mathbb{R}^n)$ instead of $\dot{K}_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(v, w)$ and omit (v, w) in the norm $\|\cdot\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(v, w)}$. Analogously, we define $K_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(\mathbb{R}^n)$.

The space $\dot{K}_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(\mathbb{R}^n)$ is called homogeneous (non-weighted) Herz spaces with three variable exponents, while the space $K_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(\mathbb{R}^n)$ is called non-homogeneous (non-weighted) Herz spaces with three variable exponents.

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap \text{LH}(\mathbb{R}^n)$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap \text{LH}_0(\mathbb{R}^n) \cap \text{LH}_\infty(\mathbb{R}^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \text{LH}_0(\mathbb{R}^n) \cap \text{LH}_\infty(\mathbb{R}^n)$, $v \in A_{p_v}$ for some $p_v \in [1, \infty)$, $w \in A_{p(\cdot)}$. We know that there exist constants $C > 0$ and $\delta \in (0, 1)$ such that

$$(3.4) \quad \frac{v(B)}{v(E)} \leq C \left(\frac{|B|}{|E|} \right)^{p_v},$$

$$(3.5) \quad \frac{v(E)}{v(B)} \leq C \left(\frac{|E|}{|B|} \right)^\delta$$

for all open balls B and all measurable sets $E \subset B$. See [12] for fundamental properties of the Muckenhoupt weights. Among others, we refer to [12, (7.3)] and [12, Corollary 7.6] for (3.4) and (3.5), respectively. Define

$$(3.6) \quad v^- := \begin{cases} \delta & \text{if } \alpha_- \geq 0, \\ p_v & \text{if } \alpha_- < 0, \end{cases} \quad v^+ := \begin{cases} p_v & \text{if } \alpha_+ \geq 0, \\ \delta & \text{if } \alpha_+ < 0. \end{cases}$$

In our earlier paper [23], the first and second authors established that there exist positive constants $\delta_1 = \delta_1(w, p(\cdot))$, $\delta_2 = \delta_2(w, p(\cdot))$, $\delta_3 = \delta_3(w, p(\cdot))$, $\delta_4 = \delta_4(w, p(\cdot)) \in (0, 1)$ and $C > 0$ such that

$$(3.7) \quad \frac{\|\chi_k\|_{L^{p(\cdot)}(w)}}{\|\chi_l\|_{L^{p(\cdot)}(w)}} \sim \frac{\|\chi_k\|_{(L^{p'(\cdot)}(w^{-p(\cdot)/p'(\cdot)})')'}}{\|\chi_l\|_{(L^{p'(\cdot)}(w^{-p(\cdot)/p'(\cdot)})')'}} \leq C \left(\frac{|C_k|}{|C_l|} \right)^{\delta_1},$$

that

$$(3.8) \quad \frac{\|\chi_k\|_{(L^{p(\cdot)}(w))'}}{\|\chi_l\|_{(L^{p(\cdot)}(w))'}} \leq C \left(\frac{|C_k|}{|C_l|} \right)^{\delta_2}$$

that

$$(3.9) \quad \chi_{\mathbb{N} \times \mathbb{N}}(k, l) \frac{\|\chi_k\|_{L^{p(\cdot)}(w)}}{\|\chi_l\|_{L^{p(\cdot)}(w)}} \leq C \left(\frac{|C_k|}{|C_l|} \right)^{\delta_3},$$

and that

$$(3.10) \quad \chi_{\mathbb{N} \times \mathbb{N}}(k, l) \frac{\|\chi_k\|_{(L^{p(\cdot)}(w))'}}{\|\chi_l\|_{(L^{p(\cdot)}(w))'}} \leq C \left(\frac{|C_k|}{|C_l|} \right)^{\delta_4}$$

for all $k, l \in \mathbb{Z}$ with $k \leq l$. Finally, assume that $\alpha(\cdot)$ satisfies

$$(3.11) \quad -n\delta_1 < v^- \alpha_-$$

and

$$(3.12) \quad v^+ \alpha_+ < n\delta_2.$$

Then in [24] the authors showed that there exists a constant $C > 0$ such that

$$\|\mathcal{R}_m f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(v, w)} \leq C \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(v, w)}$$

for all $f \in L_c^\infty(\mathbb{R}^n)$ and $m = 1, 2, \dots, n$. We have an analogy to the non-homogeneous space $\dot{K}_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(v, w)$ if we replace (3.11) and (3.12) by

$$(3.13) \quad -n\delta_3 < v^- \alpha_-$$

and

$$(3.14) \quad v^+ \alpha_+ < n\delta_4.$$

Hence, Theorems 1.1 and 1.3 are applicable to $\dot{K}_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(v, w)$ and $K_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(v, w)$ in this case.

3.6. Orlicz spaces. First, let us recall the notion of Young functions. A function $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a *Young function*, if it satisfies the following conditions:

- (1) $\Phi(0) = 0$.
- (2) Φ is continuous.
- (3) Φ is convex. That is, $\Phi((1-\theta)t_1 + \theta t_2) \leq (1-\theta)\Phi(t_1) + \theta\Phi(t_2)$ for all $t_1, t_2 \in [0, \infty)$ and $0 < \theta < 1$.

Next, we recall the definition of Orlicz spaces which a Young function $\Phi : [0, \infty) \rightarrow [0, \infty)$ generates. We define the *Luxemburg–Nakano norm* $\|\cdot\|_{L^\Phi}$ by

$$\|f\|_{L^\Phi} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\} \cup \{\infty\}$$

for $f \in L^0(\mathbb{R}^n)$. The *Orlicz space* $L^\Phi(\mathbb{R}^n)$ over $\mathbb{R}^n$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which $\|f\|_{L^\Phi}$ is finite.

In order that M is bounded on both $L^\Phi(\mathbb{R}^n)$ and on its Köthe dual, we recall the following classes.

- (1) A function $\varphi : (0, \infty) \rightarrow [0, \infty)$ or $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a *doubling function*, if there exists a constant $C > 0$, called a *doubling constant*, such that $C^{-1}\varphi(t) \leq \varphi(s) \leq C\varphi(t)$ for all t and s in the domain of φ satisfying $s \leq t \leq 2s$.
- (2) Denote by Δ_2 the set of all convex bijections $\Phi : [0, \infty) \rightarrow [0, \infty)$ satisfying the doubling condition; $\Phi(2r) \lesssim \Phi(r)$ for $r > 0$. The implicit constant is again called a *doubling constant*. In this case Φ also satisfies the Δ_2 -condition.
- (3) The set ∇_2 is the set of all convex bijections $\Phi : [0, \infty) \rightarrow [0, \infty)$ such that there exists $C > 1$ such that $\Phi(2t) \geq 2C\Phi(t)$ for all $t \geq 0$. In this case one says that Φ satisfies the ∇_2 -condition.

If $\Phi \in \Delta_2 \cap \nabla_2$, then M is bounded on both $L^\Phi(\mathbb{R}^n)$ and on its Köthe dual. See [42, Theorem 53] and [42, Example 71]. Thus, Theorems 1.1 and 1.3 are applicable to $L^\Phi(\mathbb{R}^n)$. See [45] for wavelet characterization of Orlicz spaces.

3.7. Morrey spaces. Let $1 \leq r \leq r_0 < \infty$. For an $L^r_{\text{loc}}(\mathbb{R}^n)$ -function f , its (classical) Morrey norm is defined by

$$(3.15) \quad \|f\|_{\mathcal{M}_r^{r_0}} := \sup_{x \in \mathbb{R}^n, R > 0} |B(x, R)|^{\frac{1}{r_0} - \frac{1}{r}} \left(\int_{B(x, R)} |f(y)|^r dy \right)^{\frac{1}{r}}.$$

The Morrey space $\mathcal{M}_r^{r_0}(\mathbb{R}^n)$ is the set of all $f \in L^r_{\text{loc}}(\mathbb{R}^n)$ for which the norm $\|f\|_{\mathcal{M}_r^{r_0}}$ is finite. This is a natural extension of $L^{r_0}(\mathbb{R}^n)$ since $\mathcal{M}_{r_0}^{r_0}(\mathbb{R}^n)$ and $L^{r_0}(\mathbb{R}^n)$ are the same with coincidence of norms.

Let $1 < r \leq r_0 < \infty$. Then M and $\mathcal{R}_j$, $j = 1, 2, \dots, n$, are bounded on $\mathcal{M}_r^{r_0}(\mathbb{R}^n)$. See [5]. Thus, Theorems 1.1 and 1.3 are applicable to $\mathcal{M}_r^{r_0}(\mathbb{R}^n)$.

Let $1 < r < r_0 < \infty$. Remark that the Morrey space $\mathcal{M}_r^{r_0}(\mathbb{R}^n)$ is not separable but that M is bounded on $\mathcal{M}_r^{r_0}(\mathbb{R}^n)$ [42, Theorem 382] and $\mathcal{M}_r^{r_0}(\mathbb{R}^n)'$ [44]. Thus, in general Morrey spaces do not fall within the scope of Theorem 2.5. Remark that $\mathcal{M}_r^{r_0}(\mathbb{R}^n)$ fails to be reflexive [42, Example 148]. Although $L^{r_0}(\mathbb{R}^n)$ is a subset of $\mathcal{M}_r^{r_0}(\mathbb{R}^n)$, it fails to be dense [42, Theorem 20], it is still possible to obtain the characterization of Morrey spaces in terms of wavelets as in [39].

3.8. Besov–Bourgain–Morrey spaces. Let $1 \leq q \leq p < \infty$ and $r, \tau \in [1, \infty]$. The Besov–Bourgain–Morrey space $B\mathcal{M}_{q,r}^{p,\tau}(\mathbb{R}^n)$ is defined to be the set of all the $f \in L_{\text{loc}}^q(\mathbb{R}^n)$ satisfying that

$$\|f\|_{B\mathcal{M}_{q,r}^{p,\tau}(\mathbb{R}^n)} := \left\| \left\{ \left\| \left\{ |Q_{\nu,m}|^{\frac{1}{p}-\frac{1}{q}} \left[\int_{Q_{\nu,m}} |f(y)|^q dy \right]^{\frac{1}{q}} \right\}_{m \in \mathbb{Z}^n} \right\|_{\ell^r(\mathbb{Z}_m^n)} \right\}_{\nu \in \mathbb{Z}} \right\|_{\ell^\tau(\mathbb{Z}_\nu)} < \infty.$$

Here the inner ℓ^r norm is taken with respect to m and the outer ℓ^τ norm is taken with respect to ν . See [46] for more. The Bourgain–Morrey space $\mathcal{M}_{q,r}^p(\mathbb{R}^n)$ is the Besov–Bourgain–Morrey space $B\mathcal{M}_{q,r}^{p,r}(\mathbb{R}^n)$ [14]. If $1 < q < p < r < \infty$ and $1 \leq \tau < \infty$ or if $1 < q < p \leq r < \tau = \infty$, then M and $\mathcal{R}_j$, $j = 1, 2, \dots, n$, are bounded on $B\mathcal{M}_{q,r}^{p,\tau}(\mathbb{R}^n)$. Thus, Theorems 1.1 and 1.3 are applicable to $B\mathcal{M}_{q,r}^{p,\tau}(\mathbb{R}^n)$.

Remark that Besov–Bourgain–Morrey spaces fall within the scope of Theorem 2.5 unlike Morrey spaces.

4. APPENDIX–PROOF OF THEOREM 1.3

Let $\varepsilon > 0$ and $(f, g) \in \mathcal{F}_{\text{loc}}$. It suffices to show that

$$(4.1) \quad \|f\|_X \leq C_1 \|g\|_X + C_2 \varepsilon \|f\|_X,$$

where $C_1 > 0$ depends on ε , while $C_2 > 0$ is independent of ε and the pair (f, g) .

In fact, for any $r > 0$, since $(\chi_{B(r)} \chi_{[0,r]}(|f|)f, g) \in \mathcal{F}_{\text{loc}}$, we have

$$\|\chi_{B(r)} \chi_{[0,r]}(|f|)f\|_X \leq C_1 \|g\|_X + C_2 \varepsilon \|\chi_{B(r)} \chi_{[0,r]}(|f|)f\|_X$$

once we establish (4.1). Note that $\chi_{B(r)} \chi_{[0,r]}(|f|)f \in X$, since $L_c^\infty(\mathbb{R}^n) \subset X$. By taking $\varepsilon = \frac{1}{2C_2}$, we obtain

$$\|\chi_{B(r)} \chi_{[0,r]}(|f|)f\|_X \leq 2C_1 \|g\|_X.$$

Letting $r \rightarrow \infty$, we obtain the desired result. Thus, we may assume that $f \in L_c^\infty(\mathbb{R}^n)$.

We now prove (4.1). Since $(f, g) \in \mathcal{F}_{\text{loc}}$ if and only if $(|f|, |g|) \in \mathcal{F}_{\text{loc}}$ for any $f, g \in L^0(\mathbb{R}^n)$, we may assume that f, g are non-negative from now on.

Furthermore, by adding a small constant multiple of

$$L := \sum_{l=0}^{\infty} \frac{1}{(2\|M_{\text{loc}}\|_{X' \rightarrow X'})^l} M_{\text{loc}}^l \chi_{B(1)}$$

to g , we may assume that g is positive everywhere. Finally, we may assume $g \in X$; otherwise, the conclusion is trivial.

Define, for any measurable function k ,

$$R_k(x) := \sum_{l=0}^{\infty} \frac{M_{\text{loc}}^l k(x)}{(\alpha \|M_{\text{loc}}\|_{X \rightarrow X})^l}, \quad R'_k(x) := \sum_{l=0}^{\infty} \frac{M_{\text{loc}}^l k(x)}{(\alpha \|M_{\text{loc}}\|_{X' \rightarrow X'})^l}, \quad x \in \mathbb{R}^n,$$

where $\alpha \geq 2$ is a constant to be fixed later (see (4.13)).

We will use the inequality

$$(4.2) \quad ab^{p-1} \leq C_\varepsilon a^p + \varepsilon b^p \quad (a, b, \varepsilon > 0),$$

where $C_\varepsilon > 0$ depends only on ε and p .

To estimate $\|f\|_X$, fix a non-negative function $h \in X'$ with $\|h\|_{X'} < 1$ to dualize. It suffices to show that there exist constants C_0 (independent of ε , f , g , and h) and C_ε (dependent on ε but not on f , g , or h) such that

$$(4.3) \quad \int_{\mathbb{R}^n} f(x)h(x)dx \leq C_\varepsilon \|g\|_X + C_0 \varepsilon \|f\|_X.$$

Indeed, since $X = (X')'$ as in [42] and $f \in L_c^\infty(\mathbb{R}^n) \subset X$, this implies (4.1).

We may also assume that h is positive by adding a small constant multiple of L if necessary. So from here on, assume $f \in L_c^\infty(\mathbb{R}^n)$ and $g, h > 0$.

Assuming $f \geq 0$, we observe that $R_{g+f} > 0$ and $R'_h > 0$. Moreover, we have

$$[R_{g+f}]_{A_{1,\text{loc}}} \leq \alpha \|M_{\text{loc}}\|_{X \rightarrow X}, \quad [R'_h]_{A_{1,\text{loc}}} \leq \alpha \|M_{\text{loc}}\|_{X' \rightarrow X'}.$$

Hence,

$$(4.4) \quad [R_{g+f}^{1-p} R'_h]_{A_{p,\text{loc}}} \leq [R_{g+f}]_{A_{1,\text{loc}}}^{p-1} [R_{g+f}^{1-p} R'_h]_{A_{1,\text{loc}}} \lesssim 1,$$

with the implicit constant independent of f and g .

Since $h \leq R'_h$, we get

$$(4.5) \quad \int_{\mathbb{R}^n} f(x)h(x)dx \leq \int_{\mathbb{R}^n} f(x)R'_h(x)dx = \int_{\mathbb{R}^n} f(x)R_{g+f}(x)^{p-1} \cdot R_{g+f}(x)^{1-p} R'_h(x)dx.$$

Applying (4.2), we obtain

$$(4.6) \quad \begin{aligned} & \int_{\mathbb{R}^n} f(x)R_{g+f}(x)^{p-1} R_{g+f}(x)^{1-p} R'_h(x)dx \\ & \leq C_\varepsilon \int_{\mathbb{R}^n} f(x)^p R_{g+f}(x)^{1-p} R'_h(x)dx + \varepsilon \int_{\mathbb{R}^n} R_{g+f}(x)^p R_{g+f}(x)^{1-p} R'_h(x)dx. \end{aligned}$$

We estimate the first term. Since $(f, g) \in \mathcal{F}_{\text{loc}}$, we have

$$(4.7) \quad \begin{aligned} \int_{\mathbb{R}^n} f(x)^p R_{g+f}(x)^{1-p} R'_h(x)dx & \leq N \left([R_{g+f}^{1-p} R'_h]_{A_{p,\text{loc}}} \right)^p \int_{\mathbb{R}^n} g(x)^p R_{g+f}(x)^{1-p} R'_h(x)dx \\ & \lesssim \int_{\mathbb{R}^n} g(x)^p R_{g+f}(x)^{1-p} R'_h(x)dx, \end{aligned}$$

by (4.4). Since $R_{g+f} \geq g$, it follows that

$$(4.8) \quad \int_{\mathbb{R}^n} g(x)^p R_{g+f}(x)^{1-p} R'_h(x)dx \leq \int_{\mathbb{R}^n} g(x)R'_h(x)dx.$$

Applying Hölder's inequality and the boundedness of R'_h on X' , we obtain

$$(4.9) \quad \int_{\mathbb{R}^n} g(x)R'_h(x)dx \lesssim \|g\|_X \|R'_h\|_{X'} \lesssim \|g\|_X \|h\|_{X'} \lesssim \|g\|_X.$$

Putting together (4.7)–(4.9), we find

$$(4.10) \quad \int_{\mathbb{R}^n} f(x)^p R_{g+f}(x)^{1-p} R'_h(x)dx \lesssim \|g\|_X.$$

We handle the second term using the inequality

$$\left(\sum_{l=1}^{\infty} a_l \right)^p \leq \sum_{l=1}^{\infty} 2^{pl} a_l^p$$

for non-negative sequences $\{a_l\}_{l=1}^\infty$. Applying this to R_{g+f} , we get

$$(4.11) \quad \begin{aligned} \int_{\mathbb{R}^n} R_{g+f}(x) R'_h(x) dx &= \int_{\mathbb{R}^n} R_{g+f}(x)^p R_{g+f}(x)^{1-p} R'_h(x) dx \\ &\leq 2^p \sum_{l=0}^{\infty} \frac{2^{pl}}{(\alpha \|M_{\text{loc}}\|_{X \rightarrow X})^{pl}} \int_{\mathbb{R}^n} M_{\text{loc}}^l[f+g](x)^p R_{g+f}(x)^{1-p} R'_h(x) dx. \end{aligned}$$

We abbreviate

$$W := R_{g+f}^{1-p} R'_h \in A_{p,\text{loc}}.$$

If we argue using the dyadic grids and using the idea of Lerner [28], we find a constant $\beta > 1$ such that

$$\int_{\mathbb{R}^n} M_{\text{loc}} F(x)^p W(x) dx \leq \beta [W]_{A_{p,\text{loc}}}^{p'} \int_{\mathbb{R}^n} |F(x)|^p W(x) dx$$

for all measurable functions F . Applying this estimate iteratively, we obtain

$$(4.12) \quad \int_{\mathbb{R}^n} M_{\text{loc}}^l[f+g](x)^p W(x) dx \leq (\beta [W]_{A_{p,\text{loc}}}^{p'})^l \int_{\mathbb{R}^n} (f(x) + g(x))^p W(x) dx.$$

Assume

$$\alpha > \frac{2 \left(\beta [W]_{A_{p,\text{loc}}}^{p'} \right)^{1/p}}{\|M_{\text{loc}}\|_{X \rightarrow X}}.$$

Then we have

$$(4.13) \quad \sum_{l=0}^{\infty} \frac{2^{pl} \left(\beta [W]_{A_{p,\text{loc}}}^{p'} \right)^l}{(\alpha \|M_{\text{loc}}\|_{X \rightarrow X})^{pl}} < \infty.$$

Since $R_{g+f} \geq f + g$, it follows from (4.12) and related estimates that

$$(4.14) \quad \int_{\mathbb{R}^n} R_{g+f}(x)^p R_{g+f}(x)^{1-p} R'_h(x) dx \lesssim \int_{\mathbb{R}^n} (f(x) + g(x)) R'_h(x) dx \lesssim \|f\|_X + \|g\|_X.$$

Combining (4.6), (4.10), and (4.14), we obtain (4.3), and hence the desired estimate (4.1) is proved. This completes the proof of Theorem 1.3.

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