Hyperelliptic sigma functions and the Kadomtsev-Petviashvili equation

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Dedicated to the memory of Vladimir Evgenievich Zakharov.

Abstract

In this paper, a theory of hyperelliptic functions based on multidimensional sigma functions is developed and explicit formulas for hyperelliptic solutions to the Kadomtsev-Petviashvili equations KP-I and KP-II are obtained. The long-standing problem of describing the dependence of these solutions on the variation of the coefficients of the defining equation of hyperelliptic curves, which are integrals of the equations, is solved.

1 Introduction

The Kadomtsev-Petviashvili equation (1970) is one of the most famous (2+1)-equations of the theory of nonlinear waves (cf. [21]). It is a natural generalization of the (1+1)-Korteweg-de Vries equation (1895). For a function $\mathcal{K}(t_1, t_2, t_3)$, the KP equation

$$\partial_{t_1}(\partial_{t_3}\mathcal{K} + \mathcal{K}\partial_{t_1}\mathcal{K} + \mathfrak{p}^2\partial_{t_1}^3\mathcal{K}) = \mathfrak{q}\partial_{t_2}^2\mathcal{K} \tag{1.1}$$

models the propagation of long waves on water with weakly nonlinear restoring forces and frequency dispersion $\mathfrak{p}$, where $\mathfrak{p}, \mathfrak{q} \in \mathbb{R}$ and $\partial_{t_i} = \partial/\partial t_i$. If the surface tension is weak compared to gravitational forces, then $\mathfrak{q} = 1$ is used. If the surface tension is strong, then $\mathfrak{q} = -1$ is used. For a function $\mathcal{L}(t_1, t_3)$, the KdV equation is the following equation:

$$\partial_{t_3}\mathcal{L} + c_0\partial_{t_1}\mathcal{L} + \mathfrak{a}\mathcal{L}\partial_{t_1}\mathcal{L} + \mathfrak{b}\partial_{t_1}^3\mathcal{L} = 0,$$

where $\mathcal{L}(t_1, t_3)$ is the free surface elevation, $\mathfrak{h}$ is the water depth, $\mathfrak{g}$ is the acceleration of gravity, $c_0 = \sqrt{\mathfrak{g}\mathfrak{h}}$ is the linear phase speed, that is, velocity of an infinitesimal linear

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sine wave, $\mathfrak{a} = 3c_0/(2\mathfrak{h})$, and $\mathfrak{b} = c_0\mathfrak{h}^2/6$ (cf. [31]). For a function $\Psi(t_1, t_2, t_3)$, the KP equation is also given by

$$\partial_{t_1}(\partial_{t_3}\Psi + c_0\partial_{t_1}\Psi + \mathfrak{a}\Psi\partial_{t_1}\Psi + \mathfrak{r}\partial_{t_1}^3\Psi) = -\frac{c_0}{2}\partial_{t_2}^2\Psi, \tag{1.2}$$

where $\Psi(t_1, t_2, t_3)$ is the surface elevation, T is the surface tension, ρ is the water density, and $\mathfrak{r} = \mathfrak{b} - c_0T/(18\rho\mathfrak{g})$. When the surface tension dominates, for water depths less than about a centimeter ($\mathfrak{r} < 0$), the equation (1.2) is referred to as KP-I equation. When the surface tension is negligible, for depths much larger than a centimeter ($\mathfrak{r} > 0$), the equation (1.2) is called KP-II equation (cf. [31]). We took the explanation of the physical meaning of the parameters of the KP equation in the form (1.1) from [21] and the KP equation in the form (1.2) from [31]. In the case $\mathfrak{r} < 0$, for a solution $\mathcal{K}(t_1, t_2, t_3)$ to (1.1) with $\mathfrak{q} = 1$, the function

$$\Psi(t_1, t_2, t_3) = -\mathfrak{a}^{-1}\mathfrak{u}^{-2}\mathcal{K}\Big(\mathfrak{u}(t_1 - c_0t_3), (2/c_0)^{1/2}t_2, -t_3/\mathfrak{u}\Big)$$

is a solution to (1.2), where $\mathfrak{u} = (\mathfrak{p}^2/|\mathfrak{r}|)^{1/4}$. In the case $\mathfrak{r} > 0$, for a solution $\mathcal{K}(t_1, t_2, t_3)$ to (1.1) with $\mathfrak{q} = -1$, the function

$$\Psi(t_1, t_2, t_3) = \mathfrak{a}^{-1}\mathfrak{u}^{-2}\mathcal{K}\Big(\mathfrak{u}(t_1 - c_0t_3), (2/c_0)^{1/2}t_2, t_3/\mathfrak{u}\Big)$$

is a solution to (1.2).

In [36] and [37], Zakharov and Shabat described solutions to the KP equation with the condition that they decrease rapidly at infinity. For a discussion of the integrability of the KP-II equation and non-integrability of the KP-I equation, see [35]. In [30], Sergei Novikov developed the theory of finite-zone integration of equations of mathematical physics. In [30], it was noted that "Our work is based on certain simple but fundamental algebraic properties of equations admitting the Lax representation which are strongly degenerate in the problem with rapidly decreasing functions (for $x \to \pm\infty$), and have therefore not been noted. Finally, it is essential to note the nonlinear "superposition law for waves" for the KdV equation which in the periodic case has an interesting algebraic-geometric interpretation." In [20], Its and Matveev constructed a solution to the KdV equation in terms of the Riemann theta function defined by the lattice of periods of holomorphic differentials on an algebraic curve of arbitrary genus and winding vectors defined by periods of abelian differentials of the second kind. Krichever added one direction vector in the theta functional solution of Its and Matveev to the KdV equation and constructed a solution to the KP equation (cf. [18], [26]). To be more precise, in [26], Krichever considered an algebraic curve of genus $g \geq 3$ with a marked point and proved that the function

$$\Psi(t_1, t_2, t_3) = q + 2\partial_{t_1}^2 \log\theta(U_1t_1 + U_2t_2 + U_3t_3 + W)$$

satisfies the KP equation in the form (1.2), where θ is the Riemann theta function defined by the lattice of periods of holomorphic differentials on the curve of genus g,

U_i with $1 \le i \le 3$ are g-dimensional vectors defined by periods of abelian differentials of the second kind, $q \in \mathbb{C}$, and $W \in \mathbb{C}^g$. Krichever's result on solutions to the KP equation led Novikov to the famous conjecture, an approach to solve the Riemann-Schottky problem in terms of the KP equation, [27], which was solved by T. Shiota [32]. In our work, explicit formulas for hyperelliptic solutions to the Kadomtsev-Petviashvili equations KP-I and KP-II are obtained in terms of the coefficients of the defining equation of hyperelliptic curves based on the sigma functions of these curves. Thus, we have not only obtained an explicit form of solutions but also related the coefficients of the defining equation of the curves to physical parameters. Our solutions require scaling, i.e., multiplication by scalars of variables and functions, and linear transformations of variables of the "traveling wave" type. To describe the behavior of the sigma function with shifts by periods, its expression through the theta function is used (see Proposition 2.1). The remarkable fact is that the sigma function of a hyperelliptic curve defined by a polynomial of odd degree is a solution to the system of multidimensional heat (multidimensional Schrödinger) equations in a nonholonomic frame, which is completely determined by the coefficients of the defining equation of the curve (cf. [8]). In this case, our solutions to the KP equation are determined only by the coefficients of the defining equation of the curves and the well-known problem of constructing real-valued solutions is solved.

Frobenius and Stickelberger described the Lie algebra of differentiations of elliptic functions with respect to the coefficients of the defining equation of the elliptic curve in the Weierstrass model (cf. [17]). In the works of Buchstaber, Leykin, and Bunkova, Lie algebras of differentiations of hyperelliptic functions with respect to the coefficients of the defining equation of hyperelliptic curves of genus $g > 1$ were described (cf. [14], [16]). Based on this result, Lie algebras of multidimensional heat (multidimensional Schrödinger) operators in a nonholonomic frame of annihilators of the hyperelliptic sigma function were described.

For a positive integer g, let C be the hyperelliptic curve of genus g defined by

$$Y^2 = X^{2g+1} + \lambda_2 X^{2g} + \lambda_4 X^{2g-1} + \cdots + \lambda_{4g} X + \lambda_{4g+2}, \qquad \lambda_i \in \mathbb{C}. \qquad (1.3)$$

We assign degrees for X, Y, and λ_i as $\deg X = 2$, $\deg Y = 2g+1$, and $\deg \lambda_i = i$. The equation (1.3) is homogeneous of degree $4g+2$ with respect to the coefficients λ_i and the variables X, Y. Let V be the hyperelliptic curve of genus g defined by

$$y^2 = \nu_0 x^{2g+2} + \nu_2 x^{2g+1} + \cdots + \nu_{4g+2} x + \nu_{4g+4}, \quad \nu_i \in \mathbb{C}, \quad \nu_0 \neq 0. \qquad (1.4)$$

We assign degrees for x, y, and ν_i as $\deg x = 2$, $\deg y = 2g+2$, and $\deg \nu_i = i$. The equation (1.4) is homogeneous of degree $4g+4$ with respect to the coefficients ν_i and the variables x, y. We consider the hyperelliptic functions $\wp_{i,j}(u_1, u_3, \ldots, u_{2g-1})$ with $i, j = 1, 3, \ldots, 2g-1$ defined by the sigma function of the curve C, which are meromorphic functions on $\mathbb{C}^g$, and assign degrees for u_i and $\wp_{i,j}$ as $\deg u_i = -i$ and $\deg \wp_{i,j} = i + j$. We consider the hyperelliptic functions $\mathcal{P}_{i,j}(v_{2g}, v_{2g-2}, \ldots, v_2)$ with $i, j = 2, 4, \ldots, 2g$ associated with the curve V, which are meromorphic functions on $\mathbb{C}^g$, and assign degrees for v_i and $\mathcal{P}_{i,j}$ as $\deg v_i = -i$ and $\deg \mathcal{P}_{i,j} = i + j$. Let us describe our solutions to the KP equation. We consider the case $g \geq 3$, assume $\lambda_{4g+2} \neq 0$, and for $g \geq 4$ take constants $b_i \in \mathbb{C}$ with $1 \leq i \leq g-3$. Let

$$\varphi(t_1, t_2, t_3) = -2\wp_{2g-1,2g-1}\left(b_1, \ldots, b_{g-3}, \mathfrak{c}t_3, \mathfrak{d}t_2, t_1 + \mathfrak{e}t_2\right) - \mathfrak{f},$$

where

$$\mathfrak{c} = -16\lambda_{4g+2}, \quad \mathfrak{d} = 2\sqrt{-3\lambda_{4g+2}}, \quad \mathfrak{e} = \frac{\lambda_{4g}}{\sqrt{-3\lambda_{4g+2}}}, \quad \mathfrak{f} = \frac{2}{3}\lambda_{4g-2} + \frac{\lambda_{4g}^2}{18\lambda_{4g+2}}.$$

For the curve C, we assign degrees for t_i with $i = 1, 2, 3$ as $\deg t_i = (1-2g)i$. We have $\deg \varphi = 4g-2$. In Corollary 2.13, we prove that the function φ satisfies the KP-I equation

$$\partial_{t_1}(\partial_{t_3}\varphi + 6\varphi\partial_{t_1}\varphi + \partial_{t_1}^3\varphi) = \partial_{t_2}^2\varphi.$$

In [11], it was pointed out that if $g \geq 3$, under certain restrictions on the coefficients of the defining equation of the curve, the function $\wp_{2g-1,2g-1}$ is a solution to the KP equation. In Corollary 2.13 of our paper, we give a simple explicit condition on the coefficients of the defining equation of the curve and under this condition we prove that the function $\wp_{2g-1,2g-1}$ is a solution to the KP equation. We consider the case $g \geq 3$ and for $g \geq 4$ take constants $c_i \in \mathbb{C}$ with $1 \leq i \leq g-3$. Let

$$\psi(t_1, t_2, t_3) = -2\mathcal{P}_{2,2}\left(c_1, \ldots, c_{g-3}, \alpha t_3, \beta t_2, t_1 + \gamma t_2\right) - \delta,$$

where

$$\alpha = -16\nu_0, \quad \beta = 2\sqrt{-3\nu_0}, \quad \gamma = \frac{\nu_2}{\sqrt{-3\nu_0}}, \quad \delta = \frac{2}{3}\nu_4 + \frac{\nu_2^2}{18\nu_0}.$$

For the curve V, we assign degrees for t_i with $i = 1, 2, 3$ as $\deg t_i = -2i$. We have $\deg \psi = 4$. We derive the differential relations between the hyperelliptic functions $\mathcal{P}_{i,j}$ explicitly for any g (see Theorem 4.6). In Corollary 4.8, by using these differential relations, we prove that the function ψ satisfies the KP-I equation

$$\partial_{t_1}(\partial_{t_3}\psi + 6\psi\partial_{t_1}\psi + \partial_{t_1}^3\psi) = \partial_{t_2}^2\psi.$$

Our solutions to the KP equation use not only differentiation operators but also argument shift operators. Our solutions to the KP equation are consistent with the grading. Grading is the fundamental difference between the sigma function and the theta function. The theta function does not allow grading since its arguments are normalized. We can obtain solutions to the KP-II equation (see Remark 2.14).

In [22] and [23], Klein generalized the Weierstrass elliptic sigma function to the multidimensional sigma functions associated with hyperelliptic curves. On this problem, Klein published 3 works (1886–1890). Pay attention to the papers [4] and [5] by Baker. In 1923, a 3-volume collection of Klein's scientific works was published. There is no doubt that Klein knew Baker's results, however in this collection in the preface to the works on the problem under discussion, Klein emphasized that the theory of hyperelliptic functions was still far from complete. Klein and Baker did not discuss the equations of mathematical physics. The development of the theory of multidimensional sigma functions in the direction of applications to problems of mathematical physics began with the works of Buchstaber, Enolski, and Leykin (cf. [9], [10], [12], [13]). Over the past 30 years, a number of authors have successfully joined in the development of the classical results of Klein and Baker with applications in mathematical physics.

In [5], Baker derived a fundamental formula on the hyperelliptic functions associated with the curve C. Baker did not set the problem of describing all relations in the field of meromorphic functions on the Jacobian variety of a hyperelliptic curve. In [9], this problem was set and the generators of the ideal of relations between the hyperelliptic functions associated with the curve C were described explicitly. The solution to this problem made it possible to specify the equations describing the Jacobian variety of a hyperelliptic curve explicitly. It is remarkable that these generators explicitly contain the KdV equation (cf. [9]). This ideal also contains the KP equation. In [15], Buchstaber and Mikhailov described the connection between hyperelliptic functions based on the sigma functions and the known constructions of the KdV hierarchy. In [34], in the case $g = 3$, a solution to the KP equation in terms of the hyperelliptic function associated with the curve C was considered.

In [5], Baker introduced basic hyperelliptic functions associated with the curve V (see Definition 3.5) and derived a fundamental formula on these functions (see Lemma 4.1). Further, in [5], the differential relations between the hyperelliptic functions associated with the curve V were described explicitly for $g = 1, 2, 3$. In [5], Baker used an explicit form of the inversion of the Abel-Jacobi map and did not actually need the sigma function. In [28], in the case $g = 3$, it was proved that the hyperelliptic function associated with the curve V satisfies the KP equation. Above we described the solution to the KP equation of degree $4g-2$ for the curve C and the solution to the KP equation of degree 4 for the curve V. We also note the following results of our paper:

- We construct a solution to the KP equation of degree 2 for the curve C (see Proposition 2.10).
- We describe the relations between the hyperelliptic functions associated with the curves C and V explicitly (see Proposition 6.3).

In the works of Buchstaber, Enolski, and Leykin, the focus was on a hyperelliptic curve defined by a polynomial of odd degree. In this paper, we also obtain results for a hyperelliptic curve defined by a polynomial of even degree. The problem to find hyperelliptic solutions to the KP equation is non-trivial. In the general approach of the theory of finite-zone integration, only hyperelliptic solutions to the KdV equation were discussed. This problem was posed in the works of Buchstaber, Enolski, and Leykin only on the basis of the analysis of differential relations in the field of hyperelliptic functions. We have obtained an interesting and new relationship of results in the odd and even cases. Functions admit two types of operations:

1. differentiation with respect to one of the variables;
2. linear transformation of variables.

For the abelian functions, in the first case the lattice of periods does not change and in the second case it is linearly transformed. In the odd case, we use a linear transformation of the arguments (see Remark 2.15). In the even case, this is not required.

An n-dimensional theta function is an entire function of n variables whose power series expansion is given in terms of a special lattice Γ of rank $2n$. In the case where an abelian variety $\mathbb{C}^n/\Gamma$ is the Jacobian variety of an algebraic curve, the lattice Γ is given by the periods of holomorphic differentials on this curve. In [3] and [29], for (n, s) curves and telescopic curves, the relations between the tau functions and the sigma functions were studied and solutions to the KP hierarchy in terms of the tau functions were described. In [24] and [25], Kodama described solutions to the KP problem based on Hirota's approach using the tau function and a description of solutions in terms of Grassmann manifolds. Our paper describes solutions to the KP equation in terms of multidimensional sigma functions. Unlike theta functions, the multidimensional sigma function is an entire function whose coefficients of the power series expansion are polynomials in the coefficients of the defining equation of the curve. The coefficients of the defining equation of the curve are integrals of the KP equation. By studying the deformations of the coefficients of the defining equation of the curve, we study the deformation of the integrals. Describing the dependence of the solution on the deformation of the integrals is an important task. This problem can be solved by studying the solutions to the KP equation in terms of multidimensional sigma functions.

In [1], [2], and [6], the identities for the hyperelliptic functions of genus 2 which are different from the hyperelliptic functions considered in our paper were studied.

The present paper is organized as follows. In Section 2, we review the definition and properties of the hyperelliptic sigma functions and the hyperelliptic functions defined by a polynomial of odd degree. We construct solutions to the KP equation in terms of the hyperelliptic functions defined by a polynomial of odd degree. In Section 3, we review the definition of the hyperelliptic functions defined by a polynomial of even degree, which was given in [5]. In Section 4, we derive the differential relations between the hyperelliptic functions defined by a polynomial of even degree and prove that the hyperelliptic function defined by a polynomial of even degree satisfies the KP equation for any genus. In Section 5, we consider the inversion problem of the Abel-Jacobi map for the hyperelliptic curve defined by a polynomial of even degree. In Section 6, we describe the relations between the hyperelliptic functions defined by a polynomial of odd degree and the hyperelliptic functions defined by a polynomial of even degree explicitly.

2 Hyperelliptic sigma functions

In this section, we review the definition of the hyperelliptic sigma functions and give facts about them which will be used later on. For details of the hyperelliptic sigma functions, see [9], [10], and [12].

For a positive integer g, let us consider the polynomial in X

$$M(X) = X^{2g+1} + \lambda_2 X^{2g} + \lambda_4 X^{2g-1} + \cdots + \lambda_{4g} X + \lambda_{4g+2}, \qquad \lambda_i \in \mathbb{C}.$$

We assume that $M(X)$ has no multiple roots and consider the non-singular hyperelliptic curve of genus g

$$C = \Big\{ (X,Y) \in \mathbb{C}^2 \ \Big| \ Y^2 = M(X) \Big\}.$$

We assign degrees for X, Y, and λ_i as $\deg X = 2$, $\deg Y = 2g+1$, and $\deg \lambda_i = i$. The equation $Y^2 = M(X)$ is homogeneous of degree $4g+2$ with respect to the coefficients λ_i and the variables X, Y. A basis of the vector space consisting of holomorphic 1-forms on C is given by

$$\omega_i = -\frac{X^{g-i}}{2Y} dX, \qquad 1 \le i \le g.$$

We set $\omega = {}^t(\omega_1, \ldots, \omega_g)$. Let us consider the following meromorphic 1-forms on C:

$$\eta_i = -\frac{1}{2Y} \sum_{k=g-i+1}^{g+i-1} (k+i-g)\lambda_{2g+2i-2k-2} X^k dX, \qquad 1 \le i \le g, \tag{2.1}$$

which are holomorphic at any point except ∞. In (2.1), we set $\lambda_0 = 1$. Let $\{A_i, B_i\}_{i=1}^g$ be a canonical basis in the one-dimensional homology group of the curve C. We define the period matrices by

$$2\omega' = \left(\int_{A_j} \omega_i \right), \quad 2\omega'' = \left(\int_{B_j} \omega_i \right), \quad -2\eta' = \left(\int_{A_j} \eta_i \right), \quad -2\eta'' = \left(\int_{B_j} \eta_i \right).$$

We define the lattice of periods $\Lambda = \left\{ 2\omega' m_1 + 2\omega'' m_2 \mid m_1, m_2 \in \mathbb{Z}^g \right\}$ and consider the Jacobian variety $\mathrm{Jac}(C) = \mathbb{C}^g/\Lambda$. The normalized period matrix is given by $\tau = (\omega')^{-1}\omega''$. Let $\tau\delta' + \delta''$ with $\delta', \delta'' \in \mathbb{R}^g$ be the Riemann constant with respect to $(\{A_i, B_i\}_{i=1}^g, \infty)$. We denote the imaginary unit by $\mathbf{i}$. The sigma function $\sigma(u)$ associated with the curve C, $u = {}^t(u_1, u_3, \ldots, u_{2g-1}) \in \mathbb{C}^g$, is defined by

$$\sigma(u) = \varepsilon \exp\left(\frac{1}{2} {}^t u \eta' (\omega')^{-1} u \right) \theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \big((2\omega')^{-1} u, \tau \big),$$

where $\theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} (u, \tau)$ is the Riemann theta function with the characteristics $\begin{bmatrix} \delta' \\ \delta'' \end{bmatrix}$ defined by

$$\theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} (u, \tau) = \sum_{n \in \mathbb{Z}^g} \exp\big\{ \pi \mathbf{i}\, {}^t(n+\delta')\tau(n+\delta') + 2\pi\mathbf{i}\, {}^t(n+\delta')(u+\delta'') \big\}$$

and ε is a non-zero constant.

Proposition 2.1 ([9, pp. 7–8]). *For $m_1, m_2 \in \mathbb{Z}^g$, let $\Omega = 2\omega' m_1 + 2\omega'' m_2$. Then, for $u \in \mathbb{C}^g$, we have*

$$\sigma(u+\Omega)/\sigma(u) = (-1)^{2({}^t\delta' m_1 - {}^t\delta'' m_2) + {}^t m_1 m_2} \exp\big\{ {}^t(2\eta' m_1 + 2\eta'' m_2)(u + \omega' m_1 + \omega'' m_2) \big\}.$$

For an integer $k \geq 2$, let $\wp_{i_1,\ldots,i_k} = -\partial_{u_{i_1}} \cdots \partial_{u_{i_k}} \log \sigma$, where $\partial_{u_l} = \partial/\partial u_l$. We assign degrees for $\wp_{i_1,\ldots,i_k}$ as $\deg \wp_{i_1,\ldots,i_k} = i_1 + \cdots + i_k$. For $1 \leq i \leq g$, we take points $S_i = (X_i, Y_i) \in C \setminus \{\infty\}$ such that $S_i \neq \tau_1(S_j)$ if $i \neq j$, where τ_1 is the hyperelliptic involution of C

$$\tau_1 : C \to C, \qquad (X, Y) \mapsto (X, -Y).$$

Let

$$u = \sum_{i=1}^{g} \int_{\infty}^{S_i} \omega.$$

For $1 \leq k \leq g$, let $h_k(z_1, \ldots, z_g)$ be the elementary symmetric polynomial of degree k in g variables $z_1, \ldots, z_g$. We set $h_0(z_1, \ldots, z_g) = 1$. For non-negative integers i and j, let $\delta_{i,j}$ be the Kronecker delta

$$\delta_{i,j} = \begin{cases} 1 & \text{if } \ i = j \\ 0 & \text{if } \ i \neq j \end{cases}.$$

Theorem 2.2 ([9, Theorem 2.2]). *We have*

$$h_k(X_1, \ldots, X_g) = (-1)^{k-1} \wp_{1,2k-1}(u), \qquad 1 \leq k \leq g.$$

Remark 2.3. In [5, pp. 155, 156], for $g = 1, 2, 3$, the functions $\wp_{i,j,k,l}$ with $i, j, k, l = 1, 3, \ldots, 2g-1$ were expressed in terms of $\wp_{m,n}$ explicitly.

Remark 2.4. In [10, pp. 57, 58, 88], for $g = 2, 3$, the functions $\wp_{i,j,k,l}$ with $i, j, k, l = 1, 3, \ldots, 2g-1$ were expressed in terms of $\wp_{m,n}$ explicitly.

Remark 2.5. In [19], for $g = 4$, the functions $\wp_{i,j,k,l}$ with $i, j, k, l = 1, 3, 5, 7$ were expressed in terms of $\wp_{m,n}$ explicitly.

If the number of i is d in the suffixes of $\wp_{i_1,\ldots,i_k}$, then we use the notation $i \cdot d$. For example, we denote $\wp_{1,1}$, $\wp_{1,1,3}$, and $\wp_{1,1,3,3}$ by $\wp_{1\cdot 2}$, $\wp_{1\cdot 2,3}$, and $\wp_{1\cdot 2,3\cdot 2}$, respectively. If $i_j \notin \{1, 3, \ldots, 2g-1\}$ for some $1 \leq j \leq k$, we set $\wp_{i_1,\ldots,i_k} = 0$.

Theorem 2.6 ([9, Corollaries 3.1.2, 3.1.3, Theorem 3.2]). (i) *For $1 \leq i \leq g$, the following expressions hold*:

$$\wp_{1\cdot 3,2i-1} = (6\wp_{1\cdot 2} + 4\lambda_2)\wp_{1,2i-1} + 6\wp_{1,2i+1} - 2\wp_{3,2i-1} + 2\delta_{1,i}\lambda_4.$$

(ii) *For $1 \leq i, j \leq g$, the following relations hold*:

$$\begin{aligned}
\wp_{1\cdot 2,2i-1}\wp_{1\cdot 2,2j-1} = {} & 4\wp_{1\cdot 2}\wp_{1,2i-1}\wp_{1,2j-1} - 2(\wp_{1,2i-1}\wp_{3,2j-1} + \wp_{1,2j-1}\wp_{3,2i-1}) \\
& + 4(\wp_{1,2j-1}\wp_{1,2i+1} + \wp_{1,2i-1}\wp_{1,2j+1}) + 4\wp_{2i+1,2j+1} \\
& - 2(\wp_{2j-1,2i+3} + \wp_{2i-1,2j+3}) + 4\lambda_2\wp_{1,2i-1}\wp_{1,2j-1} \\
& + 2\lambda_4(\delta_{i,1}\wp_{1,2j-1} + \delta_{j,1}\wp_{1,2i-1}) + 4\lambda_{4i+2}\delta_{i,j} + 2(\lambda_{4i}\delta_{i,j+1} + \lambda_{4j}\delta_{i+1,j}).
\end{aligned}$$

(iii) *For $1 \leq i, j \leq g$, the following relations hold*:

$$\wp_{1\cdot 2,2j-1}\wp_{1,2i-1} - \wp_{1\cdot 2,2i-1}\wp_{1,2j-1} + \wp_{1,2i+1,2j-1} - \wp_{1,2i-1,2j+1} = 0.$$

(iv) *For $1 \leq i, j \leq g$, the following relations hold*:

$$\wp_{1\cdot 3,2j-1}\wp_{1,2i-1} - \wp_{1\cdot 3,2i-1}\wp_{1,2j-1} + \wp_{1\cdot 2,2i+1,2j-1} - \wp_{1\cdot 2,2i-1,2j+1} = 0.$$

Remark 2.7. Theorem 2.6 was obtained in [9]. We changed the suffixes of $\wp_{i_1,\dots,i_k}$ in [9] to use the grading. The suffix g in [9] is replaced with 1.

Remark 2.8. The equations in Theorem 2.6 (i)–(iv) are homogeneous of degree $2i+2$, $2i+2j+2$, $2i+2j+1$, and $2i+2j+2$, respectively.

Remark 2.9. In [9, Theorem 4.12], by using the equation in Theorem 2.6 (i) with $i=1$, it was proved that $\mathcal{G}(t_1,t_3)=2\wp_{1\cdot 2}(t_1,t_3)+2\lambda_2/3$ satisfies the KdV equation

$$4\partial_{t_3}\mathcal{G}+6\mathcal{G}\partial_{t_1}\mathcal{G}-\partial_{t_1}^3\mathcal{G}=0.$$

We consider the case $g\ge 2$ and for $g\ge 3$ take constants $\varrho_i\in\mathbb{C}$ with $3\le i\le g$. Let us consider the function

$$\Upsilon(t_1,t_2,t_3)=-2\wp_{1\cdot 2}(t_1+2\sqrt{\lambda_2}t_2,-4t_3,\varrho_3,\dots,\varrho_g).$$

Proposition 2.10. *The function Υ satisfies the KP-*I *equation*

$$\partial_{t_1}(\partial_{t_3}\Upsilon+6\Upsilon\partial_{t_1}\Upsilon+\partial_{t_1}^3\Upsilon)=\partial_{t_2}^2\Upsilon.$$

Proof. From Theorem 2.6 (i) with $i=1$, we have

$$\wp_{1\cdot 4}=(6\wp_{1\cdot 2}+4\lambda_2)\wp_{1\cdot 2}+4\wp_{1,3}+2\lambda_4.$$

By differentiating this equation with respect to u_1 twice, we obtain

$$\wp_{1\cdot 6}=12\wp_{1\cdot 2}\wp_{1\cdot 4}+12\wp_{1\cdot 3}^2+4\lambda_2\wp_{1\cdot 4}+4\wp_{1\cdot 3,3}.$$

We have

$$\partial_{t_1}\partial_{t_3}\Upsilon=8\wp_{1\cdot 3,3},\quad (\partial_{t_1}\Upsilon)^2=4\wp_{1\cdot 3}^2,\quad \Upsilon\partial_{t_1}^2\Upsilon=4\wp_{1\cdot 2}\wp_{1\cdot 4},$$
$$\partial_{t_1}^4\Upsilon=-2\wp_{1\cdot 6},\quad \partial_{t_2}^2\Upsilon=-8\lambda_2\wp_{1\cdot 4}.$$

From the equations above, we obtain the statement of the proposition. □

Remark 2.11. Proposition 2.10 is an analog of the following well-known result: For $g=1$, the function $\mathcal{H}(t_1,t_3)=2\wp_{1\cdot 2}(t_1+\lambda_2 t_3)$ satisfies the KdV equation

$$4\partial_{t_3}\mathcal{H}+6\mathcal{H}\partial_{t_1}\mathcal{H}-\partial_{t_1}^3\mathcal{H}=0.$$

The statement of the following theorem was formulated in the conclusion in [5, p. 156]. Since it is used to derive the KP equation from Theorem 2.6, we give its proof.

Theorem 2.12. *For any $g\ge 1$, the following expression holds*:

$$\wp_{(2g-1)\cdot 4}=6\wp_{(2g-1)\cdot 2}^2+4\lambda_{4g}\wp_{2g-1,2g-3}+4\lambda_{4g+2}(4\wp_{2g-1,2g-5}-3\wp_{(2g-3)\cdot 2})$$
$$+4\lambda_{4g-2}\wp_{(2g-1)\cdot 2}-8\lambda_{4g+2}\lambda_{4g-6}+2\lambda_{4g}\lambda_{4g-4},$$

where we set $\lambda_0=1$ and $\lambda_i=0$ for $i<0$.

Proof. For $g=1$, this theorem is well known. For $g=2,3$, this theorem was given in [10, pp. 58, 88]. For $g \geq 4$, we prove this theorem from Theorem 2.6 directly. For a positive integer d, we set

$$p_d = \wp_{1\cdot d,2g-1}, \qquad q_d = \wp_{1\cdot d,2g-3}, \qquad r_d = \wp_{1\cdot d,2g-5}, \qquad s_d = \wp_{1\cdot d,2g-7}.$$

For a meromorphic function $H(u_1,u_3,\dots,u_{2g-1})$ on $\mathbb{C}^g$, we use the notation $\dot{H} = \partial_{u_{2g-1}}H$. From Theorem 2.6 (ii) with $i=j=g-1$, we have

$$\wp_{(2g-1)\cdot 2} = \frac{1}{4}q_2^2 - q_1^2(\wp_{1\cdot 2}+\lambda_2) + q_1(\wp_{3,2g-3}-2p_1) - \lambda_{4g-2}. \tag{2.2}$$

From Theorem 2.6 (i) with $i=g-1$, we have

$$\wp_{3,2g-3} = q_1(3\wp_{1\cdot 2}+2\lambda_2) + 3p_1 - \frac{1}{2}q_3. \tag{2.3}$$

By substituting (2.3) into (2.2), we obtain

$$\wp_{(2g-1)\cdot 2} = \frac{1}{4}q_2^2 + q_1^2(2\wp_{1\cdot 2}+\lambda_2) + q_1\left(p_1 - \frac{1}{2}q_3\right) - \lambda_{4g-2}.$$

From Theorem 2.6 (iii) and (iv), we have

$$\begin{aligned}
\dot{p}_1 &= q_2p_1 - p_2q_1, & \dot{q}_1 &= r_2p_1 - p_2r_1, & \dot{r}_1 &= s_2p_1 - p_2s_1,\\
\dot{p}_2 &= q_3p_1 - p_3q_1, & \dot{q}_2 &= r_3p_1 - p_3r_1, & \dot{r}_2 &= s_3p_1 - p_3s_1.
\end{aligned}$$

We have

$$\dot{q}_3 = \partial_{u_1}\dot{q}_2 = \partial_{u_1}(r_3p_1 - p_3r_1) = r_4p_1 + r_3p_2 - p_4r_1 - p_3r_2.$$

From the equations above, we have

$$\begin{aligned}
\wp_{(2g-1)\cdot 3} = {}& \frac{1}{2}q_2(r_3p_1-p_3r_1) + (r_2p_1-p_2r_1)\left(4q_1\wp_{1\cdot 2} + 2\lambda_2q_1 + p_1 - \frac{1}{2}q_3\right)\\
&- \frac{1}{2}q_1(r_4p_1+r_3p_2-p_4r_1-p_3r_2) + q_1(p_2q_1+p_1q_2).
\end{aligned}$$

From Theorem 2.6 (i) and (iii), we have

$$\begin{aligned}
\dot{p}_2 &= q_3p_1 - p_3q_1 = 6p_1^2 - 2p_1\wp_{3,2g-3} + 2q_1\wp_{3,2g-1},\\
\dot{q}_2 &= r_3p_1 - p_3r_1 = 6p_1q_1 - 2p_1\wp_{3,2g-5} + 2r_1\wp_{3,2g-1},\\
\dot{r}_2 &= s_3p_1 - p_3s_1 = 6p_1r_1 - 2p_1\wp_{3,2g-7} + 2s_1\wp_{3,2g-1} + 2\delta_{g,4}\lambda_4p_1,\\
\dot{p}_3 &= \partial_{u_1}\dot{p}_2 = 10p_1p_2 + 2\wp_{1\cdot 2}(p_1q_2 - p_2q_1) + 2q_2\wp_{3,2g-1} - 2p_2\wp_{3,2g-3},\\
\dot{q}_3 &= \partial_{u_1}\dot{q}_2 = 4p_1q_2 + 6p_2q_1 + 2\wp_{1\cdot 2}(p_1r_2 - p_2r_1) + 2r_2\wp_{3,2g-1} - 2p_2\wp_{3,2g-5},\\
\dot{r}_3 &= \partial_{u_1}\dot{r}_2 = 4p_1r_2 + 6p_2r_1 + 2\wp_{1\cdot 2}(p_1s_2 - p_2s_1) + 2s_2\wp_{3,2g-1} - 2p_2\wp_{3,2g-7} + 2\delta_{g,4}\lambda_4p_2,\\
p_4 &= 4p_1\wp_{1\cdot 3} + 8p_2\wp_{1\cdot 2} + 4\lambda_2p_2,\\
r_4 &= 4r_1\wp_{1\cdot 3} + 8r_2\wp_{1\cdot 2} + 4\lambda_2r_2 + 4q_2,\\
\dot{p}_4 &= \partial_{u_1}\dot{p}_3 = 8p_2^2 + 10p_1p_3 + 4\wp_{1\cdot 3}(p_1q_2 - p_2q_1) + 2\wp_{1\cdot 2}(p_1q_3 - p_3q_1) + 2q_3\wp_{3,2g-1}\\
&\qquad - 2p_3\wp_{3,2g-3},\\
\dot{r}_4 &= \partial_{u_1}\dot{r}_3 = 8p_2r_2 + 4p_1r_3 + 6p_3r_1 + 4\wp_{1\cdot 3}(p_1s_2 - p_2s_1) + 2\wp_{1\cdot 2}(p_1s_3 - p_3s_1)\\
&\qquad + 2s_3\wp_{3,2g-1} - 2p_3\wp_{3,2g-7} + 2\delta_{g,4}\lambda_4p_3.
\end{aligned}$$

From the equations above, Theorem 2.6 (i), and (ii), we have $\wp_{(2g-1)\cdot 4} = J\big(\{\wp_{i,j}\}\big)$, where $J\big(\{\wp_{i,j}\}\big)$ is a polynomial in $\{\wp_{i,j}\}$. Let

$$\begin{aligned}
Z_1 &= (\wp_{1\cdot 2,2g-5}\wp_{1\cdot 2,2g-3})(\wp_{1\cdot 2,2g-3}\wp_{1\cdot 2,2g-1}) - (\wp_{1\cdot 2,2g-5}\wp_{1\cdot 2,2g-1})(\wp_{1\cdot 2,2g-3}\wp_{1\cdot 2,2g-3}),\\
Z_2 &= (\wp_{1\cdot 2,2g-5}\wp_{1\cdot 2,2g-1})(\wp_{1\cdot 2,2g-5}\wp_{1\cdot 2,2g-1}) - (\wp_{1\cdot 2,2g-5}\wp_{1\cdot 2,2g-5})(\wp_{1\cdot 2,2g-1}\wp_{1\cdot 2,2g-1}),\\
Z_3 &= (\wp_{1\cdot 2,2g-7}\wp_{1\cdot 2,2g-3})(\wp_{1\cdot 2,2g-1}\wp_{1\cdot 2,2g-1}) - (\wp_{1\cdot 2,2g-7}\wp_{1\cdot 2,2g-1})(\wp_{1\cdot 2,2g-3}\wp_{1\cdot 2,2g-1}),
\end{aligned}$$

where the parentheses mean that the substitutions from Theorem 2.6 (ii) are made before expanding (cf. [9, Corollary 3.2.2]). We have $Z_i = 0$ for $i = 1, 2, 3$. From the direct calculations, we can check that $J\big(\{\wp_{i,j}\}\big) + (Z_1 + Z_2 + Z_3)/2$ is equal to the right hand side of the equation in Theorem 2.12. □

We consider the case $g \geq 3$, assume $\lambda_{4g+2} \neq 0$, and for $g \geq 4$ take constants $b_i \in \mathbb{C}$ with $1 \leq i \leq g-3$. Let

$$\varphi(t_1, t_2, t_3) = -2\wp_{(2g-1)\cdot 2}\,(b_1, \dots, b_{g-3}, \mathfrak{c}t_3, \mathfrak{d}t_2, t_1 + \mathfrak{e}t_2) - \mathfrak{f},$$

where

$$\mathfrak{c} = -16\lambda_{4g+2}, \quad \mathfrak{d} = 2\sqrt{-3\lambda_{4g+2}}, \quad \mathfrak{e} = \frac{\lambda_{4g}}{\sqrt{-3\lambda_{4g+2}}}, \quad \mathfrak{f} = \frac{2}{3}\lambda_{4g-2} + \frac{\lambda_{4g}^2}{18\lambda_{4g+2}}.$$

Corollary 2.13. *If $\lambda_{4g+2} \neq 0$, the function φ satisfies the KP-I equation*

$$\partial_{t_1}(\partial_{t_3}\varphi + 6\varphi\partial_{t_1}\varphi + \partial_{t_1}^3\varphi) = \partial_{t_2}^2\varphi.$$

Proof. In [11, p. 170], it was pointed out that if $g \geq 3$, under certain restrictions on the coefficients of the defining equation of the curve, $\wp_{(2g-1)\cdot 2}$ is a solution to the KP equation. We give a proof of this corollary. By differentiating the formula in Theorem 2.12 with respect to u_{2g-1} twice, we obtain

$$\begin{aligned}
\wp_{(2g-1)\cdot 6} = {}& 12\wp_{(2g-1)\cdot 3}^2 + 12\wp_{(2g-1)\cdot 2}\wp_{(2g-1)\cdot 4} + 4\lambda_{4g}\wp_{(2g-1)\cdot 3,2g-3}\\
&+ 16\lambda_{4g+2}\wp_{(2g-1)\cdot 3,2g-5} - 12\lambda_{4g+2}\wp_{(2g-1)\cdot 2,(2g-3)\cdot 2} + 4\lambda_{4g-2}\wp_{(2g-1)\cdot 4}.
\end{aligned}$$

We have

$$\begin{aligned}
\partial_{t_1}\partial_{t_3}\varphi &= -2\mathfrak{c}\wp_{(2g-1)\cdot 3,2g-5}, \qquad (\partial_{t_1}\varphi)^2 = 4\wp_{(2g-1)\cdot 3}^2,\\
\varphi\partial_{t_1}^2\varphi &= 2(2\wp_{(2g-1)\cdot 2} + \mathfrak{f})\wp_{(2g-1)\cdot 4}, \qquad \partial_{t_1}^4\varphi = -2\wp_{(2g-1)\cdot 6},\\
\partial_{t_2}^2\varphi &= -2\mathfrak{e}^2\wp_{(2g-1)\cdot 4} - 4\mathfrak{d}\mathfrak{e}\wp_{(2g-1)\cdot 3,2g-3} - 2\mathfrak{d}^2\wp_{(2g-1)\cdot 2,(2g-3)\cdot 2}.
\end{aligned}$$

From the equations above, we obtain the statement of the corollary. □

Remark 2.14. In general, let $\Phi(t_1, t_2, t_3)$ be a solution to the KP-I equation

$$\partial_{t_1}(\partial_{t_3}\Phi + 6\Phi\partial_{t_1}\Phi + \partial_{t_1}^3\Phi) = \partial_{t_2}^2\Phi.$$

Then the function $\widetilde{\Phi}(t_1, t_2, t_3) = \Phi(t_1, \sqrt{-1}t_2, t_3)$ is a solution to the KP-II equation

$$\partial_{t_1}(\partial_{t_3}\widetilde{\Phi} + 6\widetilde{\Phi}\partial_{t_1}\widetilde{\Phi} + \partial_{t_1}^3\widetilde{\Phi}) = -\partial_{t_2}^2\widetilde{\Phi}. \tag{2.4}$$

Let ξ_8 be a primitive 8th root of unity. The function $\overline{\Phi}(t_1, t_2, t_3) = \xi_8^2\Phi(\xi_8 t_1, t_2, \xi_8^3 t_3)$ is also a solution to the KP-II equation in the form (2.4).

Remark 2.15. By the shift $X' = X + X_0$ for some $X_0 \in \mathbb{C}$, it is always possible to achieve the condition $\lambda_{4g+2} \neq 0$. Thus, we have obtained a solution to the KP equation for any hyperelliptic curve up to normalization.

3 $(2, 2g+2)$ functions

In this section, we define the fundamental meromorphic functions on the Jacobian variety of the hyperelliptic curve defined by a polynomial of even degree in accordance with [5, p. 145].

For a positive integer g, let us consider the polynomial in x

$$N(x) = \nu_0 x^{2g+2} + \nu_2 x^{2g+1} + \cdots + \nu_{4g+2} x + \nu_{4g+4}, \quad \nu_i \in \mathbb{C}, \quad \nu_0 \neq 0.$$

We assume that $N(x)$ has no multiple roots and consider the non-singular hyperelliptic curve of genus g

$$V = \Big\{ (x, y) \in \mathbb{C}^2 \;\Big|\; y^2 = N(x) \Big\}.$$

We assign degrees for x, y, and ν_i as $\deg x = 2$, $\deg y = 2g+2$, and $\deg \nu_i = i$. The equation $y^2 = N(x)$ is homogeneous of degree $4g+4$ with respect to the coefficients ν_i and the variables x, y. A basis of the vector space consisting of holomorphic 1-forms on V is given by

$$\mu_i = \frac{x^{i-1}}{2y} dx, \qquad 1 \leq i \leq g.$$

We set $\mu = {}^t(\mu_1, \ldots, \mu_g)$. Let $\{\mathfrak{a}_i, \mathfrak{b}_i\}_{i=1}^g$ be a canonical basis in the one-dimensional homology group of the curve V. We define the period matrices by

$$2\mu' = \left(\int_{\mathfrak{a}_j} \mu_i \right), \qquad 2\mu'' = \left(\int_{\mathfrak{b}_j} \mu_i \right).$$

We define the lattice of periods $L = \big\{ 2\mu' m_1 + 2\mu'' m_2 \mid m_1, m_2 \in \mathbb{Z}^g \big\}$ and consider the Jacobian variety $\mathrm{Jac}(V) = \mathbb{C}^g / L$. We call a meromorphic function on $\mathrm{Jac}(V)$ a $(2, 2g+2)$ *function*. We take $a \in \mathbb{C}$ such that $N(a) = 0$. Let $\mathrm{Sym}^g(V)$ be the g-th symmetric product of V. Let $\mathcal{F}\big(\mathrm{Sym}^g(V)\big)$ and $\mathcal{F}\big(\mathrm{Jac}(V)\big)$ be the fields of meromorphic functions on $\mathrm{Sym}^g(V)$ and $\mathrm{Jac}(V)$, respectively. Let us consider the Abel-Jacobi map

$$I: \quad \mathrm{Sym}^g(V) \to \mathrm{Jac}(V), \qquad \sum_{i=1}^g Q_i \mapsto \sum_{i=1}^g \int_{(a,0)}^{Q_i} \mu.$$

The map I induces the isomorphim of fields

$$I^*: \quad \mathcal{F}\big(\mathrm{Jac}(V)\big) \to \mathcal{F}\big(\mathrm{Sym}^g(V)\big), \qquad \phi \mapsto \phi \circ I.$$

For $(x_i, y_i) \in V$ with $1 \leq i \leq g$, let

$$R(x) = (x-a)(x-x_1) \cdots (x-x_g), \qquad R'(x) = \frac{d}{dx} R(x).$$

For variables e_1, e_2, we set

$$\nabla = \sum_{i=1}^{g} \frac{y_i}{(e_1 - x_i)(e_2 - x_i)R'(x_i)}, \quad f(e_1, e_2) = \sum_{i=0}^{g+1} e_1^i e_2^i \{2\nu_{4g+4-4i} + \nu_{4g+2-4i}(e_1 + e_2)\},$$

where we set $\nu_{-2} = 0$.

Lemma 3.1 ([5, p. 146])**.** *We have*

$$f(e_1, e_1) = 2N(e_1), \qquad \left.\frac{\partial f}{\partial e_2}\right|_{e_2=e_1} = \left.\frac{dN}{dx}\right|_{x=e_1}.$$

Lemma 3.2 ([7, p. 315])**.** *For a symmetric polynomial $\widehat{f}(e_1, e_2) \in \mathbb{C}[e_1, e_2]$, we assume that the degree of $\widehat{f}(e_1, e_2)$ is $g+1$ in each variable and*

$$\widehat{f}(e_1, e_1) = 2N(e_1), \qquad \left.\frac{\partial \widehat{f}}{\partial e_2}\right|_{e_2=e_1} = \left.\frac{dN}{dx}\right|_{x=e_1}.$$

Then there exist complex numbers $\{\mathfrak{m}_{i,j}\}_{i,j=1}^{g}$ such that $\mathfrak{m}_{i,j} = \mathfrak{m}_{j,i}$ and

$$\widehat{f}(e_1, e_2) = f(e_1, e_2) + (e_1 - e_2)^2 \sum_{i,j=1}^{g} \mathfrak{m}_{i,j} e_1^{i-1} e_2^{j-1}.$$

We set

$$F(e_1, e_2) = f(e_1, e_2)R(e_1)R(e_2) + (e_1 - e_2)^2 R(e_1)^2 R(e_2)^2 \nabla^2 - N(e_1)R(e_2)^2 - N(e_2)R(e_1)^2.$$

Note that $F(e_1, e_2)$ is a symmetric polynomial in e_1 and e_2.

Lemma 3.3. *The polynomial $F(e_1, e_2)$ can be divided by $R(e_1)R(e_2)$.*

Proof. First, we will prove $F(x_1, e_2) = 0$. We have $f(x_1, e_2)R(x_1)R(e_2) = N(e_2)R(x_1)^2 = 0$ and $\nabla^2 = F_1(e_1, e_2) + F_2(e_1, e_2)$, where

$$\begin{aligned} F_1(e_1, e_2) &= \frac{N(x_1)}{(e_1 - x_1)^2 (e_2 - x_1)^2 R'(x_1)^2}, \\ F_2(e_1, e_2) &= \sum_{i=2}^{g} \frac{N(x_i)}{(e_1 - x_i)^2 (e_2 - x_i)^2 R'(x_i)^2} \\ &\quad + \sum_{1 \le i < j \le g} \frac{2y_i y_j}{(e_1 - x_i)(e_1 - x_j)(e_2 - x_i)(e_2 - x_j)R'(x_i)R'(x_j)}. \end{aligned}$$

We have $R(x_1)^2 R(e_2)^2 F_2(x_1, e_2) = 0$ and

$$R(e_1)^2 R(e_2)^2 F_1(e_1, e_2) = \frac{N(x_1)(e_1 - a)^2 (e_2 - a)^2 \prod_{i=2}^{g} (e_1 - x_i)^2 (e_2 - x_i)^2}{R'(x_1)^2}.$$

Thus, we have

$$(x_1 - e_2)^2 R(x_1)^2 R(e_2)^2 F_1(x_1, e_2) = N(x_1)(e_2 - a)^2 \prod_{i=1}^{g} (e_2 - x_i)^2 = N(x_1) R(e_2)^2.$$

Therefore, we have $F(x_1, e_2) = 0$. Similarly, we have $F(x_i, e_2) = 0$ for any $1 \leq i \leq g$. From $N(a) = R(a) = 0$, we can check $F(a, e_2) = 0$. Therefore, the polynomial $F(e_1, e_2)$ can be divided by $R(e_1)$. Since $F(e_1, e_2)$ is a symmetric polynomial in e_1 and e_2, it can be divided by $R(e_2)$. Thus, we obtain the statement of the lemma. □

Lemma 3.4. *The polynomial* $F(e_1, e_2)$ *can be divided by* $(e_1 - e_2)^2$.

Proof. From Lemma 3.1 and the direct calculations, we can check

$$F(e_1, e_1) = 0, \qquad \left.\frac{\partial F}{\partial e_2}\right|_{e_2 = e_1} = 0.$$

Thus, we obtain the statement of the lemma. □

Let $G(e_1, e_2) = F(e_1, e_2)/\{(e_1 - e_2)^2 R(e_1) R(e_2)\}$. Then $G(e_1, e_2)$ is a symmetric polynomial in e_1 and e_2 of degree at most $g - 1$ in each variable. We assign degrees for a, x_i, y_i, and e_i as $\deg a = \deg x_i = \deg e_i = 2$ and $\deg y_i = 2g + 2$. Then $G(e_1, e_2)$ is homogeneous of degree $4g$.

Definition 3.5 ([5, p. 145])**.** (i) For $1 \leq i, j \leq g$, we define $P_{2g+2-2i, 2g+2-2j} \in \mathcal{F}\big(\mathrm{Sym}^g(V)\big)$ by

$$\sum_{i,j=1}^{g} P_{2g+2-2i, 2g+2-2j} e_1^{i-1} e_2^{j-1} = G(e_1, e_2). \tag{3.1}$$

(ii) For $i, j = 2, 4, \ldots, 2g$, we define the $(2, 2g+2)$ functions $\mathcal{P}_{i,j}$ by $\mathcal{P}_{i,j} = (I^*)^{-1}(P_{i,j})$.

Remark 3.6. Since $G(e_1, e_2)$ is a symmetric polynomial in e_1 and e_2, we have $\mathcal{P}_{i,j} = \mathcal{P}_{j,i}$ for any i, j.

Example 3.7. For $g = 1$, we have

$$P_{2,2} = \frac{a(\nu_2 + 2a\nu_0)x_1 + \nu_6 + 2a\nu_4 + 2a^2\nu_2 + 2a^3\nu_0}{x_1 - a}.$$

For $v = (v_{2g}, v_{2g-2}, \ldots, v_2) \in \mathbb{C}^g$, let $\mathcal{P}_{i,j,k_1,\ldots,k_l} = \partial_{v_{k_1}} \cdots \partial_{v_{k_l}} \mathcal{P}_{i,j}$. From (3.1), we have $\deg P_{i,j} = i + j$ with respect to x_i, y_i, ν_i, and a. We assign degrees for $\mathcal{P}_{i,j}$ as $\deg \mathcal{P}_{i,j} = i + j$. For $1 \leq i \leq g$, let

$$\chi_i(x) = x^i - h_1(x_1, \ldots, x_g)x^{i-1} + h_2(x_1, \ldots, x_g)x^{i-2} - \cdots + (-1)^i h_i(x_1, \ldots, x_g).$$

We have $\chi_g(x) = (x - x_1) \cdots (x - x_g)$. We set $\chi_0(x) = 1$. Note that $\chi_i(x)$ is homogeneous of degree $2i$ with respect to x and x_i. For $1 \leq j \leq g$, we have

$$\frac{\chi_g(x)}{(x - x_j)} = x^{g-1} + \chi_1(x_j)x^{g-2} + \chi_2(x_j)x^{g-3} + \cdots + \chi_{g-1}(x_j)$$

(cf. [5, p. 136]).

Lemma 3.8 ([5, p. 137]). *By the isomorphism I^*, for $1 \le i \le g$, the derivation $\partial_{v_{2g+2-2i}}$ of $\mathcal{F}\big(\mathrm{Jac}(V)\big)$ corresponds to the following derivation of $\mathcal{F}\big(\mathrm{Sym}^g(V)\big)$:*

$$\sum_{j=1}^{g} \frac{2y_j}{\chi_g'(x_j)} \chi_{g-i}(x_j) \partial_{x_j}.$$

From Lemma 3.8, we can assign degrees for $\mathcal{P}_{i_1,\dots,i_k}$ as $\deg \mathcal{P}_{i_1,\dots,i_k} = i_1 + \cdots + i_k$.

4 Solution to the KP equation in terms of a $(2, 2g+2)$ function

Let

$$E(e_1, e_2) = (e_1 - e_2) \left\{ f(e_1, e_2) - (e_1 - e_2)^2 \sum_{i,j=1}^{g} \mathcal{P}_{2g+2-2i, 2g+2-2j} e_1^{i-1} e_2^{j-1} \right\}.$$

Lemma 4.1 ([5, p. 144]). *For variables e_1, e_2, e_3, e_4, the following relation holds*:

$$\begin{aligned} &\frac{1}{2}(e_2 - e_1)(e_3 - e_2)(e_3 - e_1)(e_4 - e_3)(e_4 - e_2)(e_4 - e_1)\times \\ &\sum_{i,j,k,l=1}^{g} \mathcal{P}_{2g+2-2i, 2g+2-2j, 2g+2-2k, 2g+2-2l} e_1^{i-1} e_2^{j-1} e_3^{k-1} e_4^{l-1} \\ &= E(e_2, e_3)E(e_4, e_1) + E(e_3, e_1)E(e_4, e_2) + E(e_1, e_2)E(e_4, e_3). \end{aligned}$$

Remark 4.2. The formula in Lemma 4.1 is homogeneous of degree $8g + 12$.

Lemma 4.3 ([5, p. 144]). *The functions $\mathcal{P}_{i,j,k,l}$ have values independent of the order of the suffixes i, j, k, l.*

Proof. For the sake to be complete and self-contained, we give a proof of this lemma. Let

$$\widetilde{E}(e_1, e_2, e_3, e_4) = E(e_2, e_3)E(e_4, e_1) + E(e_3, e_1)E(e_4, e_2) + E(e_1, e_2)E(e_4, e_3).$$

From $\widetilde{E}(e_1, e_1, e_3, e_4) = 0$, the polynomial $\widetilde{E}(e_1, e_2, e_3, e_4)$ can be divided by $(e_2 - e_1)$. Similarly, we find that the polynomial $\widetilde{E}(e_1, e_2, e_3, e_4)$ can be divided by $(e_2 - e_1)(e_3 - e_2)(e_3 - e_1)(e_4 - e_3)(e_4 - e_2)(e_4 - e_1)$. We can check that $\widetilde{E}(e_1, e_2, e_3, e_4)/\big\{(e_2 - e_1)(e_3 - e_2)(e_3 - e_1)(e_4 - e_3)(e_4 - e_2)(e_4 - e_1)\big\}$ is a symmetric polynomial in e_1, e_2, e_3, e_4. Therefore, we obtain the statement of the lemma. □

Remark 4.4. In [5, pp. 155, 156], for $g = 1, 2, 3$, the functions $\mathcal{P}_{i,j,k,l}$ with $i, j, k, l = 2, 4, \dots, 2g$ were expressed in terms of $\mathcal{P}_{m,n}$ explicitly.

Remark 4.5. In [5, p. 156], for any g, the functions $\mathcal{P}_{i,j,k,l}$ with $i, j, k, l = 2g - 2, 2g$ were expressed in terms of $\mathcal{P}_{m,n}$ explicitly.

If the number of i is d in the suffixes of $\mathcal{P}_{i_1,\dots,i_k}$, then we use the notation $i\cdot d$. For example, we denote $\mathcal{P}_{2,2}$, $\mathcal{P}_{2,2,4}$, and $\mathcal{P}_{2,2,4,4}$ by $\mathcal{P}_{2\cdot 2}$, $\mathcal{P}_{2\cdot 2,4}$, and $\mathcal{P}_{2\cdot 2,4\cdot 2}$, respectively. If $i_j \notin \{2,4,\dots,2g\}$ for some $1\le j\le k$, we set $\mathcal{P}_{i_1,\dots,i_k}=0$.

Theorem 4.6. *For $1\le k\le g$, we have the following relations*:

$$\begin{aligned}\mathcal{P}_{2\cdot 3,2k} &= 2\mathcal{P}_{2,2k}(3\mathcal{P}_{2\cdot 2}+2\nu_4)+2\nu_2(\delta_{1,k}\nu_6-\mathcal{P}_{4,2k}+3\mathcal{P}_{2,2k+2})\\ &\quad +4\nu_0(3\mathcal{P}_{2,2k+4}-3\mathcal{P}_{4,2k+2}+\mathcal{P}_{6,2k}-2\delta_{1,k}\nu_8-\delta_{2,k}\nu_{10}).\end{aligned}$$

Proof. For $1\le i\le g$, the coefficient of $e_1^{i-1}e_2^g e_3^{g+1}e_4^{g+2}$ in the left hand side of the formula in Lemma 4.1 is $\mathcal{P}_{2\cdot 3,2g+2-2i}/2$. For $1\le i\le g$, the coefficients of $e_1^{i-1}e_2^g e_3^{g+1}e_4^{g+2}$ in $E(e_2,e_3)E(e_4,e_1)$, $E(e_3,e_1)E(e_4,e_2)$, and $E(e_1,e_2)E(e_4,e_3)$ are $(3\mathcal{P}_{2\cdot 2}+2\nu_4)\mathcal{P}_{2,2g+2-2i}$, $\nu_2(\delta_{g,i}\nu_6-\mathcal{P}_{4,2g+2-2i}+3\mathcal{P}_{2,2g+4-2i})$, and $2\nu_0(3\mathcal{P}_{2,2g+6-2i}-3\mathcal{P}_{4,2g+4-2i}+\mathcal{P}_{6,2g+2-2i}-2\delta_{g,i}\nu_8-\delta_{g-1,i}\nu_{10})$, respectively. From Lemma 4.1, we obtain the statement of the theorem. □

Remark 4.7. The formula in Theorem 4.6 is homogeneous of degree $2k+6$.

We consider the case $g\ge 3$ and for $g\ge 4$ take constants $c_i\in\mathbb{C}$ with $1\le i\le g-3$. Let

$$\psi(t_1,t_2,t_3) = -2\mathcal{P}_{2\cdot 2}\left(c_1,\dots,c_{g-3},\alpha t_3,\beta t_2,t_1+\gamma t_2\right)-\delta,$$

where

$$\alpha=-16\nu_0,\quad \beta=2\sqrt{-3\nu_0},\quad \gamma=\frac{\nu_2}{\sqrt{-3\nu_0}},\quad \delta=\frac{2}{3}\nu_4+\frac{\nu_2^2}{18\nu_0}.$$

Corollary 4.8. *The function ψ satisfies the KP-*I *equation*

$$\partial_{t_1}(\partial_{t_3}\psi+6\psi\partial_{t_1}\psi+\partial_{t_1}^3\psi)=\partial_{t_2}^2\psi.$$

Proof. By differentiating the formula in Theorem 4.6 with $k=1$ with respect to v_2 twice, we obtain

$$\mathcal{P}_{2\cdot 6}=12\mathcal{P}_{2\cdot 3}^2+12\mathcal{P}_{2\cdot 2}\mathcal{P}_{2\cdot 4}+4\nu_4\mathcal{P}_{2\cdot 4}+4\nu_2\mathcal{P}_{2\cdot 3,4}+16\nu_0\mathcal{P}_{2\cdot 3,6}-12\nu_0\mathcal{P}_{2\cdot 2,4\cdot 2}.$$

We have

$$\begin{aligned}&\partial_{t_1}\partial_{t_3}\psi=-2\alpha\mathcal{P}_{2\cdot 3,6},\qquad (\partial_{t_1}\psi)^2=4\mathcal{P}_{2\cdot 3}^2,\qquad \psi\partial_{t_1}^2\psi=2(2\mathcal{P}_{2\cdot 2}+\delta)\mathcal{P}_{2\cdot 4},\\ &\partial_{t_1}^4\psi=-2\mathcal{P}_{2\cdot 6},\qquad \partial_{t_2}^2\psi=-2\beta^2\mathcal{P}_{2\cdot 2,4\cdot 2}-4\beta\gamma\mathcal{P}_{2\cdot 3,4}-2\gamma^2\mathcal{P}_{2\cdot 4}.\end{aligned}$$

From the equations above, we obtain the statement of the corollary. □

Example 4.9. Let us consider the case $N(x)=x^{2g+2}+\nu_{4g+4}$. We have $f(e_1,e_2)=2(e_1^{g+1}e_2^{g+1}+\nu_{4g+4})$. From Theorem 4.6, for $g=1$, we have

$$\mathcal{P}_{2\cdot 4}=6\mathcal{P}_{2\cdot 2}^2-8\nu_8.$$

For $g\ge 2$ and $1\le k\le g$, we have

$$\mathcal{P}_{2\cdot 3,2k}=6\mathcal{P}_{2\cdot 2}\mathcal{P}_{2,2k}+4(3\mathcal{P}_{2,2k+4}-3\mathcal{P}_{4,2k+2}+\mathcal{P}_{6,2k}).$$

From Corollary 4.8, the function

$$\psi(t_1,t_2,t_3)=-2\mathcal{P}_{2\cdot 2}\left(c_1,\dots,c_{g-3},-16t_3,2\sqrt{-3}t_2,t_1\right)$$

satisfies the KP-I equation

$$\partial_{t_1}(\partial_{t_3}\psi+6\psi\partial_{t_1}\psi+\partial_{t_1}^3\psi)=\partial_{t_2}^2\psi.$$

5 Inversion problem of the Abel-Jacobi map for the curve V

We consider the curve V defined in Section 3 and set

$$N(x)=\nu_0(x-a)\prod_{i=1}^{2g+1}(x-a_i), \qquad a_i\in\mathbb{C}.$$

Let $\mathfrak{s}=N'(a)^{2/(2g+1)}$ and $\mathfrak{t}=N'(a)^{1/2}$. We have $\deg\mathfrak{s}=4$ and $\deg\mathfrak{t}=2g+1$. Let us consider the polynomial

$$\widetilde{M}(X)=\prod_{i=1}^{2g+1}\left(X-\frac{\mathfrak{s}}{a_i-a}\right)$$

and the hyperelliptic curve $\widetilde{C}$ of genus g defined by

$$\widetilde{C}=\left\{(X,Y)\in\mathbb{C}^2 \;\middle|\; Y^2=\widetilde{M}(X)\right\}.$$

Proposition 5.1. *We have the following isomorphism from V to $\widetilde{C}$:*

$$\zeta:\quad V\to\widetilde{C}, \qquad (x,y)\mapsto(X,Y)=\left(\frac{\mathfrak{s}}{x-a},\frac{\mathfrak{t}\,y}{(x-a)^{g+1}}\right). \tag{5.1}$$

Proof. By the direct calculations, we obtain the statement of the proposition. □

Let D be the $g\times g$ regular matrix defined by ${}^t\big(\zeta^*(\omega_1),\ldots,\zeta^*(\omega_g)\big)=D\mu$, where $\zeta^*(\omega_i)$ is the pullback of the holomorphic 1-form ω_i on $\widetilde{C}$ with respect to the map ζ. For $1\le i\le g$, we take points $T_i=(x_i,y_i)\in V\backslash\{\pm\infty,(a,0)\}$ such that $T_i\neq\tau_2(T_j)$ if $i\neq j$, where τ_2 is the hyperelliptic involution of V

$$\tau_2: V\to V, \qquad (x,y)\mapsto(x,-y).$$

Let

$$v=\sum_{i=1}^{g}\int_{(a,0)}^{T_i}\mu.$$

Proposition 5.2. *For $1\le k\le g$, we have*

$$h_k(x_1-a,\ldots,x_g-a)=(-\mathfrak{s})^k\frac{\wp_{1,2g-2k-1}(Dv)}{\wp_{1,2g-1}(Dv)},$$

where $\wp_{1,2i-1}$ with $1\le i\le g$ are the hyperelliptic functions associated with the curve $\widetilde{C}$ and we set $\wp_{1,-1}(Dv)=-1$.

Proof. For $1 \le i \le g$, let $(X_i, Y_i) = \zeta\big((x_i, y_i)\big)$. We have

$$v = \sum_{i=1}^{g} \int_{\infty}^{(X_i,Y_i)} D^{-1}\omega = D^{-1} \sum_{i=1}^{g} \int_{\infty}^{(X_i,Y_i)} \omega.$$

Thus, we have

$$\sum_{i=1}^{g} \int_{\infty}^{(X_i,Y_i)} \omega = Dv.$$

From Theorem 2.2, we have

$$h_k(X_1, \ldots, X_g) = (-1)^{k-1} \wp_{1,2k-1}(Dv), \qquad 0 \le k \le g.$$

From Proposition 5.1, for $1 \le k \le g$, we have

$$\begin{aligned} h_k(x_1 - a, \ldots, x_g - a) &= \mathfrak{s}^k \frac{h_{g-k}(X_1, \ldots, X_g)}{h_g(X_1, \ldots, X_g)} \\ &= (-\mathfrak{s})^k \frac{\wp_{1,2g-2k-1}(Dv)}{\wp_{1,2g-1}(Dv)}. \end{aligned}$$

□

Remark 5.3. Proposition 5.2 is a generalization of [33, Sections 20.6 and 20.7] for $g = 1$ to any g.

6 Relationships between the hyperelliptic functions associated with the curves V and $\widetilde{C}$

We consider the hyperelliptic functions $\wp_{i,j}$ associated with the curve $\widetilde{C}$. For $0 \le i \le 2g+1$, we define $\widetilde{\lambda}_{2i} \in \mathbb{C}$ such that the following relation holds:

$$\widetilde{M}(X) = \widetilde{\lambda}_0 X^{2g+1} + \widetilde{\lambda}_2 X^{2g} + \widetilde{\lambda}_4 X^{2g-1} + \cdots + \widetilde{\lambda}_{4g} X + \widetilde{\lambda}_{4g+2}.$$

Let

$$\widetilde{R}(X) = (X - X_1) \cdots (X - X_g),$$

where $X_1, \ldots, X_g$ are defined in Section 5. For variables $\widetilde{e}_1, \widetilde{e}_2$, we set

$$\widetilde{\nabla} = \sum_{i=1}^{g} \frac{Y_i}{(\widetilde{e}_1 - X_i)(\widetilde{e}_2 - X_i)\widetilde{R}'(X_i)},$$

$$\widetilde{f}(\widetilde{e}_1, \widetilde{e}_2) = \sum_{i=0}^{g} (\widetilde{e}_1)^i (\widetilde{e}_2)^i \left\{ 2\widetilde{\lambda}_{4g+2-4i} + \widetilde{\lambda}_{4g-4i}(\widetilde{e}_1 + \widetilde{e}_2) \right\}.$$

Lemma 6.1 ([5, p. 146])**.** *We have*

$$\widetilde{f}(\widetilde{e}_1, \widetilde{e}_1) = 2\widetilde{M}(\widetilde{e}_1), \qquad \left.\frac{\partial \widetilde{f}}{\partial \widetilde{e}_2}\right|_{\widetilde{e}_2 = \widetilde{e}_1} = \left.\frac{d\widetilde{M}}{dX}\right|_{X = \widetilde{e}_1}.$$

Theorem 6.2 ([5, p. 138], [7, pp. 328, 329], [28, p. 4729])**.** *The following relation holds*:

$$\sum_{i,j=1}^{g} \wp_{2g+1-2i,2g+1-2j}(Dv)(\widetilde{e}_1)^{i-1}(\widetilde{e}_2)^{j-1}$$
$$= \frac{\widetilde{f}(\widetilde{e}_1,\widetilde{e}_2)}{(\widetilde{e}_1-\widetilde{e}_2)^2} + \widetilde{R}(\widetilde{e}_1)\widetilde{R}(\widetilde{e}_2)\widetilde{\nabla}^2 - \frac{1}{(\widetilde{e}_1-\widetilde{e}_2)^2}\left(\frac{\widetilde{M}(\widetilde{e}_1)\widetilde{R}(\widetilde{e}_2)}{\widetilde{R}(\widetilde{e}_1)} + \frac{\widetilde{M}(\widetilde{e}_2)\widetilde{R}(\widetilde{e}_1)}{\widetilde{R}(\widetilde{e}_2)}\right).$$

Proposition 6.3. (i) *For g^2 variables $z=\{z_{2k-1,2l-1}\}_{k,l=1}^{g}$, there exist g^2 polynomials $K_{i,j}(z)\in\mathbb{C}[z]$ with $1\le i,j\le g$ such that the degree of $K_{i,j}(z)$ is 1 and*

$$\mathcal{P}_{2g+2-2i,2g+2-2j}(v) = K_{i,j}\Big(\big\{\wp_{2k-1,2l-1}(Dv)\big\}_{k,l=1}^{g}\Big), \qquad 1\le i,j\le g,$$

where in this equation we substitute $\wp_{2k-1,2l-1}(Dv)$ into $z_{2k-1,2l-1}$ for any k and l.

(ii) *If $a=0$, then the (i,j) element of D is $(\mathfrak{s}^{g+1-i}/\mathfrak{t})\delta_{i,j}$ for $1\le i,j\le g$ and we have*

$$\mathcal{P}_{2g+2-2i,2g+2-2j}(v) = \nu_{4g+2}^{-1}\mathfrak{s}^{2g-i-j+2}\wp_{2i-1,2j-1}(Dv), \qquad 1\le i,j\le g.$$

(iii) *If $a=0$, $\nu_{4g+2}=1$, and $\mathfrak{s}=1$, then we have*

$$\mathcal{P}_{2g+2-2i,2g+2-2j}(v) = \wp_{2i-1,2j-1}(v), \qquad 1\le i,j\le g.$$

Proof. We substitute $\widetilde{e}_i=\mathfrak{s}/(e_i-a)$ for $i=1,2$ into the both sides of the equation in Theorem 6.2 and multiply this equation by $(e_1-a)^{g-1}(e_2-a)^{g-1}$. We have

$$(e_1-a)^{g-1}(e_2-a)^{g-1}\frac{\widetilde{f}(\widetilde{e}_1,\widetilde{e}_2)}{(\widetilde{e}_1-\widetilde{e}_2)^2} = N'(a)\mathfrak{s}^{-2}\frac{\overline{f}(e_1,e_2)}{(e_1-e_2)^2}, \tag{6.1}$$

where

$$\overline{f}(e_1,e_2) = \frac{1}{N'(a)}(e_1-a)^{g+1}(e_2-a)^{g+1}\widetilde{f}\left(\frac{\mathfrak{s}}{e_1-a},\frac{\mathfrak{s}}{e_2-a}\right).$$

The polynomial $\overline{f}(e_1,e_2)$ is a symmetric polynomial. From $\widetilde{\lambda}_{4g+2}\neq 0$, the degree of $\overline{f}(e_1,e_2)$ is $g+1$ in each variable. From (5.1), we have

$$\frac{1}{N'(a)}(e_1-a)^{2g+2}\widetilde{M}\left(\frac{\mathfrak{s}}{e_1-a}\right) = N(e_1). \tag{6.2}$$

From Lemma 6.1 and (6.2), we have $\overline{f}(e_1,e_1)=2N(e_1)$. By differentiating the both sides of (6.2) with respect to e_1, we have

$$-\frac{\mathfrak{s}}{N'(a)}(e_1-a)^{2g}\widetilde{M}'\left(\frac{\mathfrak{s}}{e_1-a}\right) = N'(e_1) - \frac{(2g+2)}{N'(a)}(e_1-a)^{2g+1}\widetilde{M}\left(\frac{\mathfrak{s}}{e_1-a}\right). \tag{6.3}$$

We have

$$\frac{\partial\overline{f}(e_1,e_2)}{\partial e_2} = \frac{g+1}{N'(a)}(e_1-a)^{g+1}(e_2-a)^{g}\widetilde{f}\left(\frac{\mathfrak{s}}{e_1-a},\frac{\mathfrak{s}}{e_2-a}\right)$$
$$-\frac{\mathfrak{s}}{N'(a)}(e_1-a)^{g+1}(e_2-a)^{g-1}\frac{\partial\widetilde{f}}{\partial\widetilde{e}_2}\left(\frac{\mathfrak{s}}{e_1-a},\frac{\mathfrak{s}}{e_2-a}\right).$$

From Lemma 6.1 and (6.3), we have

$$\left.\frac{\partial \overline{f}}{\partial e_2}\right|_{e_2=e_1} = N'(e_1).$$

From Lemma 3.2, there exist complex numbers $\{\mathfrak{n}_{i,j}\}_{i,j=1}^{g}$ such that $\mathfrak{n}_{i,j}=\mathfrak{n}_{j,i}$ and

$$\overline{f}(e_1,e_2) = f(e_1,e_2) + (e_1-e_2)^2 \sum_{i,j=1}^{g} \mathfrak{n}_{i,j} e_1^{i-1} e_2^{j-1}.$$

From (5.1), we can check

$$(e_1-a)^{g-1}(e_2-a)^{g-1}\widetilde{R}(\widetilde{e}_1)\widetilde{R}(\widetilde{e}_2)\widetilde{\nabla}^2 = N'(a)\mathfrak{s}^{-2}R(e_1)R(e_2)\nabla^2, \tag{6.4}$$

$$(e_1-a)^{g-1}(e_2-a)^{g-1}\frac{\widetilde{M}(\widetilde{e}_1)\widetilde{R}(\widetilde{e}_2)}{(\widetilde{e}_1-\widetilde{e}_2)^2\widetilde{R}(\widetilde{e}_1)} = N'(a)\mathfrak{s}^{-2}\frac{N(e_1)R(e_2)}{(e_1-e_2)^2R(e_1)}, \tag{6.5}$$

$$(e_1-a)^{g-1}(e_2-a)^{g-1}\frac{\widetilde{M}(\widetilde{e}_2)\widetilde{R}(\widetilde{e}_1)}{(\widetilde{e}_1-\widetilde{e}_2)^2\widetilde{R}(\widetilde{e}_2)} = N'(a)\mathfrak{s}^{-2}\frac{N(e_2)R(e_1)}{(e_1-e_2)^2R(e_2)}. \tag{6.6}$$

From Theorem 6.2, for g^2 variables $z=\{z_{2k-1,2l-1}\}_{k,l=1}^{g}$, there exist g^2 polynomials $K_{i,j}(z)\in\mathbb{C}[z]$ with $1\le i,j\le g$ such that the degree of $K_{i,j}(z)$ is 1 and

$$\sum_{i,j=1}^{g} K_{i,j}\Big(\big\{\wp_{2k-1,2l-1}(Dv)\big\}_{k,l=1}^{g}\Big)e_1^{i-1}e_2^{j-1} = G(e_1,e_2).$$

From Definition 3.5, we obtain the statement of (i). Let us consider the case $a=0$. We have $N'(0)=\nu_{4g+2}$. The (i,j) element of D is $(\mathfrak{s}^{g+1-i}/\mathfrak{t})\delta_{i,j}$ for $1\le i,j\le g$. We have

$$\begin{aligned} & e_1^{g-1}e_2^{g-1}\sum_{i,j=1}^{g}\wp_{2g+1-2i,2g+1-2j}(Dv)(\widetilde{e}_1)^{i-1}(\widetilde{e}_2)^{j-1} \\ &= \sum_{i,j=1}^{g}\mathfrak{s}^{2g-i-j}\wp_{2i-1,2j-1}(Dv)e_1^{i-1}e_2^{j-1}. \end{aligned} \tag{6.7}$$

The polynomial $\overline{f}(e_1,e_2)$ has the form

$$\overline{f}(e_1,e_2) = \sum_{i=0}^{g+1} e_1^i e_2^i\big\{2\rho_{4g+4-4i}+\rho_{4g+2-4i}(e_1+e_2)\big\}, \qquad \rho_j\in\mathbb{C}, \quad \rho_{-2}=0.$$

From $\overline{f}(e_1,e_1)=2N(e_1)$, we have $\rho_j=\nu_j$ for any j. Thus, we have $\overline{f}(e_1,e_1)=f(e_1,e_2)$. From Theorem 6.2, (6.1), and (6.4)–(6.7), we have

$$\sum_{i,j=1}^{g}\mathfrak{s}^{2g-i-j}\wp_{2i-1,2j-1}(Dv)e_1^{i-1}e_2^{j-1} = \nu_{4g+2}\mathfrak{s}^{-2}G(e_1,e_2).$$

From Definition 3.5, we obtain the statement of (ii). For $1\le i,j\le g$, the function $\wp_{2i-1,2j-1}$ is an even function. From (ii), we obtain the statement of (iii). □

Corollary 6.4. *We have*

$$\mathcal{P}_{2\cdot 2}(v) = N'(a)^{-1}\mathfrak{s}^2\wp_{(2g-1)\cdot 2}(Dv) - \kappa,$$

where $\kappa = N'(a)^{-1}\{g(g+1)a^2\widetilde{\lambda}_{4g+2} - ga\mathfrak{s}\widetilde{\lambda}_{4g}\}$.

Proof. Since the coefficients of $e_1^{g+1}e_2^{g-1}$ in $\overline{f}(e_1, e_2)$ and $f(e_1, e_2)$ are κ and 0, respectively, we have $\mathfrak{n}_{g,g} = \kappa$. From Proposition 6.3 (i), we obtain the statement of the corollary. □

Example 6.5. Let us consider the case $g = 2$. We have

$$D = \frac{\mathfrak{s}}{\mathfrak{t}}\begin{pmatrix} \mathfrak{s} & 0 \\ -a & 1 \end{pmatrix}.$$

From Proposition 6.3 (i), we have

$$\begin{aligned}
\mathcal{P}_{2,4}(v) &= N'(a)^{-1}\{\mathfrak{s}^3\wp_{1,3}(Dv) - a\mathfrak{s}^2\wp_{3\cdot 2}(Dv) - a^2\mathfrak{s}\widetilde{\lambda}_8 + 2a^3\widetilde{\lambda}_{10}\}, \\
\mathcal{P}_{4\cdot 2}(v) &= N'(a)^{-1}\{\mathfrak{s}^4\wp_{1\cdot 2}(Dv) - 2a\mathfrak{s}^3\wp_{1,3}(Dv) + a^2\mathfrak{s}^2\wp_{3\cdot 2}(Dv) \\
&\quad + a\mathfrak{s}^3\widetilde{\lambda}_4 - 2a^2\mathfrak{s}^2\widetilde{\lambda}_6 + 4a^3\mathfrak{s}\widetilde{\lambda}_8 - 6a^4\widetilde{\lambda}_{10}\}.
\end{aligned}$$

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