Revised note on surface-link of trivial components

Akio Kawauchi

Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University

Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

kawauchi@omu.ac.jp

Abstract

In a previous note, it is claimed that every surface-link consisting of trivial components and having at most one non-sphere component is a ribbon surface-link, but it was false. In this revised note, this claim is replaced by the claim that a surface-link L with trivial components is a ribbon surface-link if and only if the surface-link obtained from L by every fusion is a ribbon surface-link if and only if the surface-link obtained from L by any one fusion is a ribbon surface-link. For any closed oriented disconnected surface $\mathbf{F}$ containing at least two non-sphere components, there is a pair of a ribbon $\mathbf{F}$-link L consisting of trivial components and a non-ribbon $\mathbf{F}$-link L' consisting of trivial components such that the fundamental groups of L and L' are the same group up to meridian-preserving isomorphisms and the pair of the $\mathbf{F}'$-links K and K' obtained from L and L' by every corresponding fusion is a pair of a ribbon surface-link and a non-ribbon surface-link such that the fundamental groups of K and K' are the same group up to meridian-preserving isomorphisms.

Keywords: Ribbon, Non-ribbon, Surface-link, Fusion.
Mathematics Subject Classification 2010: Primary 57Q45; Secondary 57N13

1. Introduction

Let $\mathbf{F}$ be a (possibly disconnected) closed surface. An $\mathbf{F}$-*link* in the 4-sphere S^4 is the image of a smooth embedding $\mathbf{F} \to S^4$. When $\mathbf{F}$ is connected, it is also called an $\mathbf{F}$-*knot*. An $\mathbf{F}$-link or $\mathbf{F}$-knot for an $\mathbf{F}$ is called a *surface-link* or *surface-knot* in S^4, respectively. If $\mathbf{F}$ consists of some copies of the 2-sphere S^2, then it is also called an S^2-*link* and an S^2-*knot* for $\mathbf{F} = S^2$. A *trivial surface-link* is a surface-link F which bounds disjoint handlebodies smoothly embedded in S^4. A *ribbon surface-link* is a surface-link F which is obtained from a trivial nS^2-link O for some n (where nS^2 denotes the disjoint union of n copies of S^2) by surgery along an embedded 1-handle system, [4], [7]. In a preprint version of a previous paper, it is claimed that every S^2-link L in S^4 consisting of trivial components is a ribbon surface-link without any restrictions, but it was false and revised, [8], [6]. This note is a revised version of a note based on the preprint's claim where it was erroneously claimed that every surface-link consisting of trivial components and having at most one non-sphere component is a ribbon surface-link, [5]. The error in the note as well as the preprint is the statement"The intersections $B(O) \cap D_i = B(O) \cap D'_i = \emptyset (i = 1, 2, \ldots, n)$ can be assumed by moving the 3-ball system $B(O) \cup B(O^H)$ in S^4."This move cannot be used there because this move generally changes the original surface-link.

A surface-link F' in S^4 is obtained from a surface-link F of $r(\geq 2)$ components in S^4 by *fusion* if F' is a surface-link of $r - s$ components obtained from F by surgery along an $s(\leq r-1)$ disjoint embedded 1-handle system on F in S^4. The following theorem giving a characterization of when a surface-link consisting of trivial components is a ribbon surface-link is a revised theorem of [5, Theorem 1].

Theorem 1. The following statements (1)-(3) on a surface-link F consisting of trivial components in S^4 are mutually equivalent.

(1) F is a ribbon surface-link.

(2) The surface-link obtained from F by every fusion is a ribbon surface-link.

(3) The surface-link obtained from F by any one fusion is a ribbon surface-link.

Proof of Theorem 1. Note that a trivial surface-link is a ribbon surface-link by definition. Thus, it is shown that (1) is equivalent to that the surface-knot obtained from F by any one fusion is a ribbon surface-knot, [6]. Since a surface-link obtained from F by a fusion is made a surface-knot by a fusion extending the fusion, the result is obtained. This completes the proof of Theorem 1.

For a surface-link F in S^4, let $\Delta(F; Z_2)$ be the subgroup of $H_1(F; Z_2)$ consisting of an element represented a loop ℓ in F which bounds an immersed disk d in S^4 with $d \cap F = \ell$. Let $\xi : \Delta(F; Z_2) \to Z_2$ be the function defined by $\xi([\ell])$ to be the Z_2-self-intersection number $\mathrm{Int}(d, d) \mod 2$ in S^4 with respect to the framing of the surface F, which defines a possibly singular Z_2-quadratic function

$$\xi(x + y) = \xi(x) + \xi(y) + x \cdot y \quad (x, y \in \Delta(F; Z_2),$$

where $x \cdot y$ denotes the z_2-intersection number of x and y in F and called the *null-homotopic quadratic function* of the surface-link F. The *null-homotopic Gauss sum* of F is the Gauss sum $nGS(F)$ of ξ defined by

$$GS(F) = \sum_{x \in \Delta(F; Z_2)} \exp(2\pi\sqrt{-1}\frac{\xi(x)}{2}).$$

This number $nGS(F)$ is an invariant of a surface-link F and calculable, [3]. In particular, it is known that if F is a ribbon surface-link of total genus g, then $GS(F) = 2^g$. The following result in the case that $\mathbf{F}$ has at least two non-sphere components is obtained by using this invariant $GS(F)$ which strengthens an earlier result, [5, Theorem 2].

Theorem 2. Let $\mathbf{F}$ be any closed oriented disconnected surface with at least two non-sphere components. Then there exist a pair of a ribbon $\mathbf{F}$-link L and a non-ribbon $\mathbf{F}$-link L' in S^4 that consist of trivial components and have the same fundamental group up to meridian-preserving isomorphisms. Further, for the ribbon $\mathbf{F}'$-link K obtained from L by every fusion, there is a non-ribbon $\mathbf{F}'$-link K' obtained from L' by the corresponding fusion such that the fundamental groups of K and K' are the same group up to meridian-preserving isomorphisms.

Proof of Theorem 2. Let $k \cup k'$ be a non-splittable link in the interior of a 3-ball B such that k and k' are trivial knots. For the boundary 2-sphere $S = \partial B$ and the disk D^2 with the boundary circle S^1, let L be the torus-link consisting of the torus-components $T = k \times S^1$ and $T' = k' \times S^1$ in the 4-sphere S^4 with $S^4 = B \times S^1 \cup S \times D^2$, which is a ribbon torus-link in S^4, [4]. In particular, $nGS(L) = 2^2$. Since k and k' are trivial knots in B, the torus-knots T and T' are trivial torus-knots in S^4 by construction. Since $k \cup k'$ is non-splittable in B, there is a simple loop $t(k)$ in T coming from the longitude of k in B such that $t(k)$ does not bound any disk not meeting T' in S^4, meaning that there is a simple loop c in T unique up to isotopies of T which bounds a disk d in S^4 not meeting T', where c and d are given by $c = p \times S^1$ and $d = a \times S^1 \cup q \times D^2$ for a simple arc a in B joining a point p of k to a point q in S with $a \cap (k \cup k') = \{p\}$ and $a \cap S = \{q\}$. Regard the 3-ball B as the product $B = B_1 \times [0, 1]$ for a disk B_1. Let τ_1 is a diffeomorphism of the solid torus $B_1 \times S^1$ given by one full-twist along the meridian disk B_1, and $\tau = \tau_1 \times 1$ the product diffeomorphism of $(B_1 \times S^1) \times [0, 1] = B \times S^1$. Let $\partial\tau$ be the diffeomorphism of the boundary $S \times S^1$ of $B \times S^1$ obtained from τ by restricting to the boundary, and the 4-manifold M obtained from $B \times S^1$ and $S \times D^2$ by pasting the boundaries $\partial(B \times S^1) = S \times S^1$ and $\partial(S \times D^2) = S \times S^1$ by the diffeomorphism $\partial\tau$. Since the diffeomorphism $\partial\tau$ of $S \times S^1$ extends to the diffeomorphism τ of $B \times S^1$, the 4-manifold M is diffeomorphic to S^4. Let $L_M = T_M \cup T'_M$ be the torus-link in the 4-sphere M arising from $L = T \cup T'$ in $B \times S^1$. There is a meridian-preserving isomorphism $\pi_1(S^4 \setminus L, x) \to \pi_1(M \setminus L_M, x)$ by van Kampen theorem. The loop $t(k)$ in T_M does not bound any disk not meeting T'_M in M, so that the loop c in T_M is a unique simple loop up to isotopies of T_M which bounds a disk $d_M = a \times S^1 \cup D^2_M$ in M not meeting T'_M, where D^2_M denotes a proper disk in $S \times D^2$ bounded by the loop $\partial\tau(q \times S^1)$. An important observation is that the self-intersection number $\mathrm{Int}(d_M, d_M)$ in M with respect to the surface-framing on L_M is ± 1. This means that the loop c in T_M is a non-spin loop. Similarly, there is a unique non-spin loop c' in T'_M which bounds a disk d'_M with the self-intersection number $\mathrm{Int}(d'_M, d'_M) = \pm 1$ with respect to the surface-framing on L_M. Then it is calculated that $nGS(L_M) = 0$ and the torus-link L_M in M is not any ribbon torus-link, [3]. Let $(S^4, L') = (M, L_M)$. If $\mathbf{F}$ consists of two tori, then the pair (L, L') forms a desired pair. If $\mathbf{F}$ is any surface consisting of two non-sphere components, then a desired $\mathbf{F}$-link pair is obtained from the pair (L, L') by taking connected sums of some trivial surface-knots, because every stabilization of a ribbon surface-link is a ribbon surface-link and every stable-ribbon surface-link is a ribbon surface-link, [5]. If $\mathbf{F}$ has some other surface $\mathbf{F}_1$ in addition to a surface $\mathbf{F}_0$ of two non-sphere components, then a desired $\mathbf{F}$-link pair is obtained from a desired $\mathbf{F}_0$-link pair by adding the trivial $\mathbf{F}_1$-link as a split sum. Thus, a desired $\mathbf{F}$-link pair (L, L') is obtained. In particular, $\mathbf{F}$ has total genus, $g \leq 2$, then $nGS(L) = 2^g$ and $nGS(L') = 2^{g-2}$. Let A be a 4-ball in S^4 such that $A \cap L = A \cap L'$ is a trivial disk system in A with one disk component from one component of L and of L'. The 1-handle system h used for every fusion of L in S^4 is deformed into A, so that h is also considered as a 1-handle system used for a fusion of L' in S^4. Thus, the surface-links K and K' obtained from L and L' by the fusion along h are $\mathbf{F}'$-links for the surface $\mathbf{F}'$ induced from $\mathbf{F}$ by the fusion. By van Kampen theorem, the fundamental groups of the $\mathbf{F}'$-links K and K' are the same group up to meridian-preserving isomorphisms. The total genus and the null-homotopic quadratic function of a surface-link are unchanged by any fusion. In particular, the null-homotopic Gauss sum is unchanged by any fusion. Thus, K is a ribbon $\mathbf{F}'$-link with $nGS(K) = nGS(L) = 2^g$ and K' is a non-ribbon $\mathbf{F}'$-link with $nGS(K') = nGS(L') = 2^{g-2}$. This completes the proof of Theorem 2.

In the proof of Theorem 2, the diffeomorphism $\partial\tau$ of $S \times S^1$ coincides with Gluck's non-spin diffeomorphism of $S^2 \times S^1$, [2]. The torus-link (M, T_M) called a *turned torus-link* of a link $k \cup k'$ in B is an analogy of a turned torus-knot of a knot in B, [1].

Acknowledgements. This work was partly supported by JSPS KAKENHI Grant Number JP21H00978 and MEXT Promotion of Distinctive Joint Research Center Program JPMXP0723833165.

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