Kervaire conjecture on weight of group via fundamental group of ribbon sphere-link

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ABSTRACT

Kervaire conjecture that the weight of the free product of every non-trivial group and the infinite cyclic group is not one is affirmatively confirmed by confirming affirmatively Conjecture **Z** on the knot exterior introduced by González Acuña and Ramírez as a conjecture equivalent to Kervaire conjecture.

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1. Introduction

A *weight system* of a group G is a system of elements w_i, $(i = 1, 2, \dots, n)$ of G such that the normal closure $N(w_1, w_2, \dots, w_n)$ of w_i, $(i = 1, 2, \dots, n)$ in G (=: the smallest normal subgroup generated by w_i, $(i = 1, 2, \dots, n)$ in G) is equal to G. The *weight* of a group G is the least cardinal number $w(G)$ of a weight system of G. By convention, $w(G) = 0$ if and only if G is the trivial group. The *rank* of G is the least cardinal number $r(G)$ of generators of G. The difference $r(G) - w(G)$ is non-negative and in general taken sufficiently large. For example, let $G = \pi_1(S^3 \setminus k, x)$ be the fundamental group of a polygonal knot k in S^3. Then $G \cong \mathbf{Z}$ and $r(G) = 1$ for the trivial knot k, $r(G) = 2$ for the trefoil knot $k = 3_1$, and $r(G) = n$ for the $n-1(\geq 2)$-fold connected sum $k = \#_{n-1} 3_1$ of the trefoil knot 3_1. On the other hand,

$w(G) = 1$ for every knot k, because $G/N(m(k)) = \{1\}$ for a meridian element $m(k)$ of k. Let $G * \mathbf{Z}$ denote the free product of a group G and the infinite cyclic group $\mathbf{Z}$. *Kervaire's conjecture* on the weight of a group is the following conjecture (see Kervaire [6], Magnus-Karrass-Solitar [8, p. 403]):

Kervaire Conjecture. $w(G * \mathbf{Z}) > 1$ for every non-trivial group G.

Some partial affirmative confirmations of this conjecture are known. For example, the following result of Klyachko [7] is used in this paper:

Theorem (Klyachko). $w(G * \mathbf{Z}) > 1$ for every non-trivial torsion-free group G.

A *knot exterior* is a compact 3-manifold $E = \mathrm{cl}(S^3 \setminus N(k))$ for a tubular neighborhood $N(k)$ of a polygonal knot k in the 3-sphere S^3. Let F be a compact connected orientable non-separating proper surface of E where the boundary ∂F of F may be disconnected. Let $E(F) = \mathrm{cl}(E \setminus F \times I)$ be the compact piecewise-linear 3-manifold for a normal line bundle $F \times I$ of F in $E(F)$ where $I = [-1, 1]$. Let $E(F)^+$ be the 3-complex obtained from $E(F)$ by adding the cone $\mathrm{Cone}(v, F \times \partial I)$ over the base $F \times \partial I$ with a vertex v disjoint from E, where $\partial I = \{1, -1\}$. The 3-complex $E(F)^+$ is also considered to be obtained from E by shrinking the normal line bundle $F \times I$ into the vertex v. The result of Conjecture $\mathbf{Z}$ due to González Acuña and Ramírez in [2] is stated as follows:

Theorem (González Acuña-Ramírez). Kervaire's conjecture is equivalent to the following conjecture:

Conjecture Z. The fundamental group $\pi_1(E(F)^+, v)$ is isomorphic to $\mathbf{Z}$ for every knot exterior E and every compact connected orientable non-separating proper surface F in E.

See [1, 2, 9] for some knot theoretical investigations of this surface F and some partial confirmations. In this paper, Kervaire conjecture is confirmed affirmatively by confirming Conjecture $\mathbf{Z}$ affirmatively.

Theorem 1. Conjecture $\mathbf{Z}$ is true.

González Acuña-Ramírez theorem and Theorem 1 imply:

Corollary 2. Kervaire conjectureis true.

An outline of the proof of Theorem 1 is explained as follows.

Outline of the proof of Theorem 1. Let

$$E(F)^{++} = E(F) \cup \mathrm{Cone}(v_+, F \times 1) \cup \mathrm{Cone}(v_-, F \times (-1))$$

be a 3-complex for distinct vertexes v_+ and v_- disjoint from E. Then the 3-complex $E(F)^+$ is homotopy equivalent to the bouquet $E(F)^{++} \vee S^1$. Hence the fundamental group $\pi_1(E(F)^+, v)$ is isomorphic to the free product $\pi_1(E(F)^{++}, v) * \mathbf{Z}$. Thus, $\pi_1(E(F)^+, v) \cong \mathbf{Z}$ if and only if $\pi_1(E(F)^{++}, v) = \{1\}$ and Conjecture **Z** is equivalent to the claim that $\pi_1(E(F)^{++}, v) = \{1\}$. The following observation is used.

Lemma 3. $w(\pi_1(E(F)^+, v)) = w(\pi_1(E(F)^{++}, v) * \mathbf{Z}) = 1$.

Proof of Lemma 3. Because the fundamental group $\pi_1(E(F)^+, v)$ is a non-trivial quotient group of $\pi_1(E, v)$ and $w(\pi_1(E, v)) = 1$, the desired result is obtained. This completes the proof of Lemma 3. □.

The following lemma is proved in Section 2.

Lemma 4. The fundamental group $\pi_1(E(F)^+, v)$ is a torsion-free group.

By assuming Lemma 4, the proof of Theorem 1 is completed as follows:

Proof of Theorem 1. Klyachko Theorem says that if G is a torsion-free group and $w(G * \mathbf{Z}) = 1$, then $G = \{1\}$. Hence by this theorem and Lemmas 3, 4, $\pi_1(E(F)^{++}, v) \cong \{1\}$ and $\pi_1(E(F)^+, v) \cong \mathbf{Z}$. This completes the proof of Theorem 1. □.

In the first draft of this research, the author tried to show that every finitely presented group G with $w(G * \mathbf{Z}) = 1$ is torsion-free. This trial succeeds for a group G of deficiency 0, but failed for a group G of negative deficiency. The main point of this failure is the attempt to construct a finitely presented group of deficiency 0 from the group of negative deficiency, which forced the author to show that G is a torsion-free group while the deficiency remains negative. Fortunately, the fundamental group $\pi_1(E(F)^+, v)$ of the 3-complex $E(F)^+$ was an excellent object to this consideration, so it could be done.

2. Proof of Lemma 4

The proof of Lemma 4 is done as follows by using the consept of collapse in [3].

Proof of Lemma 4. Collapse F into a triangulated graph γ by using that F is a bounded surface. Enlarge the fiber I of a normal line bundle $F \times I$ of F in E into a fiber J of a normal line bundle $F \times J$ of F in E so that $I \subset J \setminus \partial J$. Let $J^c = \mathrm{cl}(J \setminus I)$. Let $E(F)^- = \mathrm{cl}(E \setminus F \times J)$. Collapse $F \times J^c$ into $\gamma \times J^c$. Triangulate $\gamma \times J^c$ without introducing new vertexes. The 3-complex $E(F)^+$ is collapsed into a finite 3-complex

$$E(F)^- \cup \gamma \times J^c \cup \mathrm{Cone}(v, \gamma \times \partial I)$$

and thus collapsed into a finite 2-complex

$$P = P^- \cup \gamma \times J^c \cup \mathrm{Cone}(v, \gamma \times \partial I)$$

obtained by taking any 2-complex P^- collapsed from $E(F)^-$. This 2-complex P is a subcomplex of a 3-complex

$$Q = \mathrm{Cone}(v, P^- \cup \gamma \times J^c).$$

Since every 2-complex of $\gamma \times J^c$ contains at most one 1-simplex of $\gamma \times \partial I$, every 3-simplex of $\mathrm{Cone}(v, \gamma \times J^c)$ contains at most one 2-simplex of $\mathrm{Cone}(v, \gamma \times \partial I)$. Collapse every 3-simplex of $\mathrm{Cone}(v, \gamma \times J^c)$ from a 2-face containing v and not belonging to $\mathrm{Cone}(v, \gamma \times \partial I)$. Then collapse every 3-simplex of $\mathrm{Cone}(v, P^-)$ from any 2-face containing the vertex v. Thus, the 3-complex Q is collapsed to a finite 2-complex C containing the 2-complex P as a subcomplex. Since Q is collapsed to the vertex v, C is a finite contractible 2-complex. It is shown in [4] that every connected subcomplex of a finite contractible 2-complex is aspherical (see also [5]). Since the fundamental group of a connected aspherical complex is a torsion-free group, the group $\pi_1(P, v)$ is a torsion-free group. Note that this torsion-freeness comes from the torsion-freeness of the fundamental group of a ribbon S^2-link in the 4-sphere S^4 which corresponds bijectively to the fundamental group of a connected subcomplex of a finite contractible 2-complex, as it is discussed in [4]. Actually, the group $\pi_1(P, v)$ is a ribbon S^2-knot group, for $H_1(P; \mathbf{Z}) \cong \mathbf{Z}$. Since $\pi_1(E(F)^+, v)$ is isomorphic to $\pi_1(P, v)$, the group $\pi_1(E(F)^+, v)$ is a torsion-free group. This completes the proof of Lemma 4. $\square$

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