RADON TRANSFORMATION AND FOURIER INTEGRAL OPERATORS

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Abstract. The purpose of this short note is to remark a relation between the *incidence relation* (named by S.S. Chern in [3] for a kind of submanifold in a product manifold $M \times N$) and the *composition of Fourier integral operators* defined by a double submersion, which we call Radon transformations. We remark that if the canonical relation is not a graph of homogeneous symplectic isomorphism, the ellipticity of Fourier integral operators does not necessarily imply the Fredholmness of the operator, that is, we show both of the dimensions of kernel and cokernel are infinite dimensional for an example.

Dedicated to Professor Wolfgang Schulze on the occasion of his 80th birthday.

Contents

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1. Introduction

A submersion naturally defines two Fourier integral operators between function spaces on the total space and the base space, one is the pull-back operation and the other is defined as the push-forward operation by fiber integration, although for the definition of the later operator we need a measure on the total space. If we have a double submersions

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & N \\ {\scriptstyle\pi}\big\downarrow & & \\ M & & \end{array}$$

from a manifold P to two base manifolds M and N satisfying a kind of dependence condition with which we may discuss an operator theory within the class of Fourier integral operators. Such a class of operators we call generalized Radon transformations, or simply a Radon transformation.

The study of Radon transformation started already more than 100 years ago as a topic in the integral geometry. Especially, inversion formulas guaranteed a basic principle for the CT scanner and S.S. Chern, in [3], *"On integral geometry in Klein spaces"*, Annals of Math., Vol. **43**, 178-189(1942), formulated a property named "incidence relation" which plays a central role in the theory of such type operators, although he was working in the group-theoretic setting, when there were no operator theories of pseudo-differential operators nor Fourier integral operators. This property highly relates to the composition of two Fourier integral operators whether the resulting operator is in the class of Fourier integral operators.

As the continuation of the earlier works by J. Radon [26], P. Funk [5] and S. S.Chern [3], I. M. Gelfand, S. Gindikin, M. I. Greav, S. Helgason and others established inversion formulas for many cases consisting of some class of symmetric spaces in the group theoretical framework (there are so many, here we may cite only a few [8], [9], [11], [13], [14], [15], [16] [10], [19], [20], [23], [12]).

The last one is including a short overview of the operator theory in relation with Fourier integral operators.

They worked on the double fibration consisting of quotient spaces of a Lie group,

$$\begin{array}{ccc} G & \longrightarrow & G/K \\ \big\downarrow & & \\ G/L & & \end{array} \qquad \text{or} \qquad \begin{array}{ccc} G/(K\cap L) & \longrightarrow & G/K \\ \big\downarrow & & \\ G/L & & \end{array} \tag{1.1}$$

S. Helgason [20] summed up several problems on such type operators in the setting (1.1). Of course, the inversion formula is central and the behavior of the support of the transformed functions (see [19] and there is a useful review [12] on the book by S. Helgason [20]), where he mentioned several problems

(A) How transformed the support of the functions ?

(B) How is the inversion formula?

(C) Are there any correspondence between the differential operator algebras commuting with the transformation?

(D) Special form of the inversion by orbital integrals.

After the paper *"Fourier integral operators I"* by L. Hörmander, Guillemin-Sternberg considered these operators in its framework and remarked a special condition called *Bolker* condition [18]. Under this condition the construction of the inverse reduces to the elliptic pseudo-differential operator theory, although all the cases do not necessarily satisfy such a condition.

Most of the contents of this note is an overview of the theory of Radon transformation defined by double submersion from the point of Fourier integral operator theory and remark the relation between the property called "*incidence relation*" and the *composition of Fourier integral operators* whose canonical relation are a normal bundle of a submanifold. We do not specify double submersions consisting of quotient spaces of Lie groups.

In §2 we resume a theory of Lagrangean distributions and Fourier integral operators following the papers [22], [2], [27] and a book [18].

In §3 as basic examples of Fourier integral operators we explain the pull-back operation and the push-forward operation and remark several L_2-operator theoretic properties.

In §4 we define a general Radon transformation and show it is a Fourier integral operator, if the double submersion satisfies a condition, which was named as "*incidence relation*" by S.S. Chern in [3]. The proof is just reduced to show this condition satisfies the clean product theorem ([17], [27]).

We discuss a Radon transformation from the Gromoll-Meyer exotic 7 sphere ([7], [4]) to the standard 7 sphere.

Also we note a question when such an operator is a Fredholm operator, what kinds of double submersions define Fredholm Radon transformations or never happens except trivial cases?

The manifolds in this note are all closed (= compact without boundary), connected and orientable.

2. Lagrangean distributions and Fourier integral operators

In this section we recall the definition of the Lagrangean distribution and related Fourier integral operator following the papers by L. Hörmander [22], J.J. Duistermaat [2], Guillemin-Sternberg [18] and A. Weinstein [27] within the necessity for our purpose.

2.1. **Lagrangean submanifold and Lagrangean distribution.** Let X be a manifold (put $\dim X = n$ and $\partial X = \phi$) and denote by θ^X and $\omega^X = d\theta^X$ the canonical one form and the canonical symplectic form on the cotangent bundle $T^*(X)$ respectively. Let Λ be a closed conic ($\equiv$ closed and closed under $\mathbb{R}_+$ action) Lagrangean submanifold Λ in $\dot{T}^*(X)$ (= the cotangent bundle minus zero section), which is equivalent to the vanishing of θ^X on Λ, that is $\theta^X = 0$ on Λ. In general, Lagrangean submanifolds are defined as a closed submanifold in the cotangent space $T^*(X)$ of the same dimension with the base manifold X on which the canonical symplectic form vanishes.

Let Λ be a conic Lagrangean submanifold, then for any point $\lambda \in \Lambda$ we can find a neighborhood U of $\pi_X(\lambda)$ ($\pi_X : T^*(X) \to X,$ the natural projection map), an open cone Γ in $\mathbb{R}^N\backslash\{0\}$ and a real valued smooth function $\phi(x,\theta)$ defined on $U\times\Gamma$, $(x,\theta) = (x_1,\dots,x_n,\theta_1,\dots,\theta_N) \in U\times\Gamma$, such that

(2.1) $\phi(x, t\cdot\theta) = t\phi(x,\theta),\ t>0,$ that is,

ϕ is homogeneous of degree one with respect to the oscillation variable θ.

$$\text{Put } C_\phi = \left\{(x,\theta)\in U\times\Gamma \;\middle|\; \frac{\partial\phi}{\partial\theta_i} = 0,\ i=1,\dots,N\right\},\ \text{then}$$

$$\text{the differentials } \left\{ d\left(\frac{\partial\phi}{\partial\theta_j}\right) = \sum_i \frac{\partial^2\phi}{\partial x_i\partial\theta_j}\,dx_i + \sum_k \frac{\partial^2\phi}{\partial\theta_k\partial\theta_j}\,d\theta_k \right\}_{j=1}^N \tag{2.2}$$

are linearly independent on C_ϕ, and the correspondence

$$\mathfrak{p}_\phi : C_\phi \ni (x,\theta) \longmapsto d\phi_x = \left(x, \frac{\partial\phi}{\partial x_1},\dots,\frac{\partial\phi}{\partial x_n}\right) = \sum_{i=1}^n \frac{\partial\phi}{\partial x_i}(x,\theta)dx_i \in \dot{T}^*(X) \tag{2.3}$$

is a diffeomorphism from C_ϕ to $\mathfrak{p}_\phi(C_\phi) := \Lambda_\phi \subset \Lambda$.

Any Lagrangean submanifold are covered by such subsets Λ_ϕ and we call the function ϕ a non-degenerate phase function and the correspondence $\mathfrak{p}_\phi$ above a local parametrization of the Lagrangean submanifold by the phase function ϕ. The dimension N is not unique, even for any point $\in\Lambda$ there are several choices of such phase functions ϕ and the open subsets $C_\phi \subset U\times\Gamma$ defined by ϕ.

We denote by $S^m(U\times\Gamma)$ (Γ is an open cone in $\mathbb{R}^N\backslash\{0\}$) a class of functions $a = a(x,\theta) \in C^\infty(U\times\Gamma)$, called amplitude functions or Hörmander class functions, satisfying the condition : for any compact subset $K\subset U$, any multi-indeces $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\beta = (\beta_1,\dots,\beta_N)$, there exists a constant $C = C_{K,\alpha,\beta} > 0$ such that

$$\sup_{x\in K,\ \theta\in\Gamma} \left|\frac{\partial^{|\alpha+\beta|}a(x,\theta)}{\partial x^\alpha\,\partial\theta^\beta}\right| \le C_{K,\alpha,\beta}\big(1+|\theta|\big)^{m-|\beta|}, \tag{2.4}$$

where $|\alpha| = \sum\alpha_i$, $|\beta| = \sum\beta_j$ and $|\theta| = \sqrt{\sum{\theta_j}^2}$. Also we assume the limit

$$\lim_{t\to+\infty}\frac{a(x,t\cdot\theta)}{t^m} =: a_m(x,\theta) \tag{2.5}$$

exits. This is a homogeneous function of the degree m.

The integral

$$C_0^\infty(U) \ni u \longmapsto \int_{U\times\Gamma} e^{\sqrt{-1}\phi(x,\theta)}\, a(x,\theta)\, u(x) dx d\theta \tag{2.6}$$

in the sense of *oscillatory integral* defines a distribution on U (see [22] for the definition of the oscillatory integral, Proposition 1.2.2, p. 90).

To treat such distributions in the global sense, we need to work on the distributions defined on half densities instead of on function spaces and on the space of amplitude functions taking values in the "*Maslov line bundle*" valued half densities (of course we may consider by fixing a volume form in advance).

So let denote by $\Omega^{1/2}(X)$, or simply by $\Omega^{1/2}$, the half density line bundle on a manifold X.

Let Λ be a closed conic Lagrangean submanifold in $\dot{T}^*(X)$. We consider two non-degenerate phase functions ϕ and ψ defined on $U \times \Gamma$ and $V \times \tilde{\Gamma}$, where U and V are local coordinates neighborhoods of a point $x_0 = \pi_X(\lambda_0) \in \pi_X(\Lambda)$ $(\lambda_0 \in \Lambda)$, Γ and $\tilde{\Gamma}$ are open cones in $\mathbb{R}^N\backslash\{0\}$ and $\mathbb{R}^{\tilde{N}}\backslash\{0\}$, respectively. Here for the simplicity, we may consider the same local coordinates $x = (x_1, \dots, x_n)$ for points in $U \cap V$ around the point x_0. Each phase function ϕ and ψ gives a local parametrization $\mathfrak{p}_\phi : C_\phi \to \Lambda_\phi$ and $\mathfrak{p}_\psi : C_\psi \to \Lambda_\psi$ around a point $\lambda_0 \in \Lambda, \pi_X(\lambda_0) = x_0$, respectively.

Now let $a = a(x,\theta) \in S^{\mu+(n-2N)/4}(U \times \Gamma)$ and $\tilde{a} = \tilde{a}(x,\tilde{\theta}) \in S^{\mu+(n-2\tilde{N})/4}(V \times \tilde{\Gamma})$ and assume for any $u \in C_0^\infty(U \cap V)$,

$$\begin{aligned}(2\pi)^{-(n+2N)/4} &\int_{U\times\Gamma} e^{\sqrt{-1}\phi(x,\theta)}\, a(x,\theta)\, u(x) dx d\theta \\ &= (2\pi)^{-(n+2\tilde{N})/4} \int_{V\times\tilde{\Gamma}} e^{\sqrt{-1}\psi(x,\tilde{\theta})}\, \tilde{a}(x,\tilde{\theta})\, u(x) dx d\tilde{\theta}.\end{aligned}$$

That is, they define the same distribution on $U \cap V$. Then

$$e^{\sqrt{-1}\pi\sigma/4} a(x,\theta)\sqrt{d_{C_\phi}} - \tilde{a}(x,\tilde{\theta})\sqrt{d_{C_\psi}} \in S^{\mu+n/4-1}(\Lambda, \Omega^{1/2}), \tag{2.7}$$

where d_{C_ϕ} and d_{C_ψ} are volume forms on C_ϕ (and also C_ψ) defined as follows: Let $\lambda_1, \dots, \lambda_n$ be a local coordinates of the submanifold C_ϕ extended to an open neighborhood (existence is guarnteed by the non-degeneracy of the phase function, that is the map

$$\mathfrak{q}_\phi : U \times \Gamma \ni (x,\theta) \mapsto (\phi_{\theta_1}, \dots, \phi_{\theta_N}) \in \mathbb{R}^N$$

is a submersion around the point ${\mathfrak{p}_\phi}^{-1}(0)$), then

$$d_{C_\phi} = \left\{\det\left(\frac{\lambda_1, \dots, \lambda_n, \phi_1, \dots, \phi_N}{x_1, \dots, x_n, \theta_1, \dots, \theta_N}\right)\right\}^{-1} d\lambda_1 \wedge \cdots \wedge d\lambda_n.$$

Note that

$$d_{C_\phi} \wedge d\phi_{\theta_1} \wedge \cdots \wedge \phi_{\theta_N} = dx_1 \wedge \cdots \wedge dx_n \wedge d\theta_1 \wedge \cdots \wedge d\theta_N. \tag{2.8}$$

The difference (2.7) is rewritten as

$$\sqrt{-1}^{\frac{\sigma-N+\tilde{N}}{2}} e^{\frac{\sqrt{-1}N\pi}{4}} a(x,\theta)\sqrt{d_{C_\phi}} - e^{\frac{\sqrt{-1}\tilde{N}\pi}{4}} \tilde{a}(x,\tilde{\theta})\sqrt{d_{C_\psi}} \in S^{\mu+n/4-1}(\Lambda, \Omega^{1/2}). \tag{2.9}$$

The quantity

$$(\sigma - N + \tilde{N})/2 = \Big(\mathrm{sign}(\phi_{\theta\,\theta}) - N - \big(\mathrm{sign}(\psi_{\tilde{\theta}\,\tilde{\theta}}) - \tilde{N}\big)\Big)/2, \tag{2.10}$$

where the signature of the matrices $\{\phi_{\theta\,\theta}\}$ and $\{\psi_{\tilde{\theta}\,\tilde{\theta}}\}$ are of their values at the points $(x,\theta) \in C_\phi$ and $(x,\tilde{\theta}) \in C_\psi$ satisfying $\mathfrak{p}_\phi(x,\theta) = \mathfrak{p}_\psi(x,\tilde{\theta}) \in \Lambda_\phi \cap \Lambda_\psi$, is a locally constant integer valued function on $\Lambda_\phi \cap \Lambda_\psi$. Hence, by considering an open covering by open neighborhoods of the form $\mathfrak{p}_\phi(C_\phi) = \Lambda_\phi$ these functions define an integral cochain defining a class $\in H^1(\Lambda, \mathbb{Z})$. The transition functions $\left\{e^{\sqrt{-1}\pi(\sigma-N+\tilde{N})/4}\right\}$ define a flat complex line bundle, which is called a *Maslov line bundle*. We denote it by L_Λ.

By the definition $\overset{4}{\otimes} L_\Lambda$ is trivial and L_Λ itself is trivial in the smooth sense, however in general there are no way to trivialize it in a canonical way. Our case below when Λ is a conormal bundle, we can see easily it is trivial naturally, since we can take non-degenerate phase functions by linear functions with respect to the oscillatory variables of the same dimension, the quantity $\sigma - N + \tilde{N}$ is alway zero.

Now we see that the amplitude functions

$$\left\{e^{\sqrt{-1}\pi N/4} a(x,\theta)\sqrt{d_{C_\phi}} \;\middle|\; \phi \text{ are non-degenerate phase functions}\right\}$$

mod $S^{\mu+n/4-1}$ define a well-defined global section of the line bundle $L_\Lambda \otimes \Omega^{1/2}$ on Λ.

We denote by $S^{\mu+n/4}(\Lambda, L_\Lambda \otimes \Omega^{1/2})$ the space of the Maslov line bundle valued half densities of the form

$$a(x,\theta)\sqrt{d_{C_\phi}}$$

with the coefficient function $a \in S^\mu(U \times \Gamma)$ being an amplitude function satisfying the condition (2.4).

Also we denote by $I^\mu(X,\Lambda)$ the space of distributions micro-locally defined on the space of half densities of the form (2.6) and call them Lagrangean distributions whose phase function being defined by the conic Lagrangean submanifold Λ, or simply Lagrangian distributions.

Then it holds the isomorphism:

Theorem 2.1 ([22])**.**

$$I^\mu(X,\Lambda)/I^{\mu-1}(X,\Lambda) \cong S^{\mu+n/4}(\Lambda, L_\Lambda \otimes \Omega^{1/2})/S^{\mu-1+n/4}(\Lambda, L_\Lambda \otimes \Omega^{1/2}). \tag{2.11}$$

For a distribution $I \in I^{\mu}(X,\Lambda)$, the corresponding amplitude function by (2.11) is not unique, however the limit function (2.5) is uniquely determined. We call this function the principal symbol of the distribution I.

Especially, if $\Lambda = \overset{\bullet}{\mathcal{N}}(Y)$, the co-normal bundle minus zero section of a submanifold Y in X, then the Maslov line bundle is canonically trivial and the isomorphism (2.11) is expressed as

$$I^{\mu}(X,\overset{\bullet}{\mathcal{N}}(Y))/I^{\mu-1}(X,\overset{\bullet}{\mathcal{N}}(Y)) \cong S^{\mu+n/4}(\overset{\bullet}{\mathcal{N}}(Y),\Omega^{1/2})/S^{\mu-1+n/4}(\overset{\bullet}{\mathcal{N}}(Y),\Omega^{1/2}).$$

2.2. **Homogeneous canonical relation and Fourier integral operator.** The symplectic form $\omega^{X\times Y}$ of the cotangent bundle of the product manifold $X\times Y$ is expressed as $\omega^{X\times Y} = \hat{\omega}^X + \hat{\omega}^Y$, where we denote by $\hat{\omega}^X$ and $\hat{\omega}^Y$ the pull-back forms on $T^*(X)\times T^*(Y) \cong T^*(X\times Y)$ by the natural projection maps $\rho_X : T^*(X\times Y) \cong T^*(X)\times T^*(Y) \to T^*(X)$ and $\rho_Y : T^*(X\times Y) \to T^*(Y)$, respectively. The projection ρ_X is the dual of the differential of the projection $X\times Y \to X$ and so on.

Also the product $T^*(X)\times T^*(Y) = T^*(X\times Y)$ has a symplectic form $\hat{\omega}^X - \hat{\omega}^Y$. Let $C : \overset{\bullet}{T^*}(X) \to \overset{\bullet}{T^*}(Y)$ be a homogeneous symplectic isomorphism (= homogeneous symplectomorphism), that is it satisfies the properties that $C^*(\omega^Y) = \omega^X$ and if $C(x,\xi) = (y,\eta)$, then $C(x, t\cdot\xi) = (y, t\cdot\eta), t > 0$. In this case its graph $G_C = \{(x,\xi,C(x,\xi)) \mid (x,\xi) \in \overset{\bullet}{T^*}(X)\} \subset \overset{\bullet}{T^*}(X)\times\overset{\bullet}{T^*}(Y)$ is a conic Lagrangean submanifold with respect to the symplectic form $\hat{\omega}^X - \hat{\omega}^Y$. If we denote by ${G_C}' = \{(x,\xi,y,-\eta) \mid C(x,\xi) = (y,\eta)\}$, then ${G_C}'$ is a Lagrangean submanifold with respect to the symplectic form $\hat{\omega}^X + \hat{\omega}^Y = \omega^{X\times Y}$.

Let $\Lambda \subset \overset{\bullet}{T^*}(X\times Y)$ be a conic Lagrangean submanifold (with respect to the symplectic form $\omega^{X\times Y}$) which need not be included in the subset $\overset{\bullet}{T^*}(X)\times\overset{\bullet}{T^*}(Y)$, then the linear operators corresponding to distributions $\in I^m(\Lambda,\Omega^{1/2})$ do not necessarily map smooth functions to smooth functions. However, if the Lagrangean submanifold $\Lambda \subset \overset{\bullet}{T^*}(X)\times\overset{\bullet}{T^*}(Y) \subset \overset{\bullet}{T^*}(X\times Y)$, then the corresponding operator to a Lagrangean distribution $\in I^m(\Lambda,\Omega^{1/2})$ maps always smooth functions to smooth functions. In this case the phase function $\phi(x,y,\theta)$ defined on a cone $U\times V\times\Gamma$ ($U \subset M, V \subset N$, and $\Gamma \subset \mathbb{R}^N\backslash\{0\}$ is a open cone), satisfies the condition (2.1), (**??**) and (2.3) with respect to (x,θ) and (y,θ). We call such a phase function a *functional* phase function and the operator corresponding to a Lagrangean distribution $\in I^m(\Lambda,\Omega^{1/2})$ defined by a Lagrangean submanifold $\Lambda \subset \overset{\bullet}{T^*}(X)\times\overset{\bullet}{T^*}(Y)$ a *Fourier integral operator*. Of course if $\Lambda = \Delta_{\overset{\bullet}{T^*}(X)}$, the graph of the identity map $Id : \overset{\bullet}{T^*}(X) \to \overset{\bullet}{T^*}(X)$, then the operators are pseudo-differential operators.

In general, let $C \subset \overset{\bullet}{T^*}(X)\times\overset{\bullet}{T^*}(Y)$ be a conic Lagrangean submanifold with respect to the symplectic form $\hat{\omega}^X - \hat{\omega}^Y$, we call it a *homogeneous canonical relation*, that is

C' is a conic Lagrangean submanifold with respect to the natural symplectic form $\omega^{X\times Y} = \hat{\omega}^X + \hat{\omega}^Y$ on the product manifold $X \times Y$.

2.3. **Composition of Fourier integral operators.** First, we prepare a notation for describing the composition of canonical relations.

Let $A \subset X \times Y$ and $B \subset Y \times Z$, then we denote by $A \circ B$, the composition of A and B,

$$A \circ B = \left\{(a,b) \in X \times Z \mid {}^{\exists}y \in Y \text{ such that } (a,y) \in A \text{ and } (y,b) \in B\right\}.$$

This coincides with the image of the natural projection of the intersection

$$(A \times B) \bigcap (X \times \Delta_Y \times Z) \subset X \times Y \times Y \times Z$$

to the product $X \times Z$.

In particular, if $A = G_f = \{(a, f(a) \mid a \in A\}$, the graph of a map $f : X \to Y$ and $B = G_g$, the graph of a map $g : Y \to Z$, then $A \circ B = G_{g\circ f}$, the graph of the composition $g \circ f : X \to Z$.

We consider the composition of two Fourier integral operators F_1 and F_2, each corresponds to Lagrangean distribution $K_i \in I^{m_i}(C_i', \Omega^{1/2}), i = 1, 2$, where $C_2 \subset \dot{T}^*(Z) \times \dot{T}^*(X)$ and $C_1 \subset \dot{T}^*(X) \times \dot{T}^*(Y)$ are homogeneous canonical relations.

Theorem 2.2. [22], [17]. *Assume $C_2 \times C_1$ and $\dot{T}^*(Z) \times \Delta_{\dot{T}^*(X)} \times \dot{T}^*(Y)$ intersects transversally (or more generally “”cleanly“, see [17] and [27] and the Definition 2.3 below), and assume $C_2 \circ C_1$ is a submanifold in $\dot{T}^*(Z) \times \dot{T}^*(X)$, then $C_2 \circ C_1$ is a canonical relation in $\dot{T}^*(Z) \times \dot{T}^*(Y)$ and the composition $F_2 \circ F_1$ is a Fourier integral operator corresponding to a Lagrangean distribution $\in I^{m_1+m_2}\big((C_2 \circ C_1)', \Omega^{1/2}\big)$.*

We omit the description of the principal symbol how is given by two principal symbols following the isomorphism (2.1). However for our case it will be seen that it is a constant function micro-locally.

We call this Theorem *“transversal product Theorem” or “clean product Theorem”* according the the conditions satisfied by the canonical relations C_1 and C_2.

Definition 2.3. *Let $f : M \to X$ and $g : N \to X$ be smooth maps. We call the maps f and g intersect cleanly, if the subset (= the fiber product of f and g)*

$$M \times_X N = \{(x, y) \mid f(x) = g(y)\}$$

is a submanifold in $M \times N$ and the tangent bundle of the fiber product $M \times_X N$ is the fiber product $T(M) \times_{T(X)} T(N)$ of the maps

$$\begin{array}{ccc} & & T(N) \\ & & \Big\downarrow{\scriptstyle dg} \\ T(M) & \xrightarrow{df} & T(X). \end{array}$$

In a special case, let M and N be submanifolds in X. In this case M and N intersect cleanly, if and only if, the intersection $M \cap N$ is a submanifold in X and $T(M \cap N) = T(M) \bigcap T(N)$. That is a tangent vector $V \in T_z(M \cap N)$ is not only tangent to both of M and N, it is tangent to a small submanifold $M \cap N$.

So if M and N intersect transversely, then they intersect cleanly.

3. Pull-back and push forward operations

Let $\varphi : M \to N$ be a smooth map between smooth manifolds M and N. We identify always volume forms and nowhere vanishing highest degree differential forms. We denote by $\varphi^* : C^\infty(N) \to C^\infty(M)$, the map defining the pull-back of smooth functions on N.

Let σ be a highest degree differential form on M (if it is not cited particularly, it is smooth always), then we define its push-forward $\varphi_*(\sigma)$ as the highest degree "distributional form" (= the dual of the function space $C^\infty(N)$) on N by the relation

$$< h, \varphi_*(\sigma) >:= \int_M \varphi^*(h) \cdot \sigma, \tag{3.1}$$

where $h \in C^\infty(N)$, that is, the right hand side defines a liner functional on the space $C^\infty(N)$, which we denote by $\varphi_*(\sigma)$.

Remark 1. *The map φ^* can be defined for measurable functions on M. The push-forward map φ_* is extended to the differential forms with measurable coefficients.*

If a map $\varphi : M \to N$ is a submersion, that is, its differentials $d\varphi_x : T_x(M) \to T_{\varphi(x)}(N)$ are always surjective for any $x \in M$, then the map φ is open and surjective, and $\varphi_*(\sigma)$ coincides with a highest degree differential form on N. In this case we may express the left hand side of (3.1) by the integral $\int_N h \cdot \varphi_*(\sigma)$.

Now we assume that the map φ is a *submersion.* Then both maps φ^* and φ_* are Fourier integral operators whose canonical relations are essentially the normal bundle $\mathcal{N}(G_\varphi)$ of the graph $G_\varphi = \{(x, \varphi(x)) \,|\, x \in M\}$ of the map φ. In fact, for a function $f \in C_0^\infty(V)$, V is a small coordinate neighborhood around a point $\varphi(x)$ with coordinates $(y_1, \ldots, y_n)$, then the Fourier inversion formula just says that, for $x \in \varphi^{-1}(V)$,

$$\varphi^*(f)(x) = (2\pi)^{-n} \int e^{\sqrt{-1}<\varphi(x)-y,\theta>} f(y)\, dy\, d\theta \ (n = \dim N), \tag{3.2}$$

so that the function

$$\phi(x,y,\theta) :=< \varphi(x) - y, \theta >= \sum \big(\varphi_i(x) - y_i\big)\,\theta_i$$

satisfies the conditions for the functional phase function and

$$\mathfrak{p}_\phi : C_\phi = \left\{\frac{\partial\phi}{\partial\theta} = 0\right\}$$

$$\ni (x,y,\theta) \mapsto \left(\sum_j \theta_j \frac{\partial\varphi_j}{\partial x_1} dx_1, \ldots, \sum_j \theta_j \frac{\partial\varphi_j}{\partial x_m} dx_m, -\theta_1 dy_1, \ldots, -\theta_n dy_n\right) \in \dot{\mathcal{N}}(G_\varphi)$$

$$\subset \dot{T}^*(M) \times \dot{T}^*(N),\ (m = \dim M,\ (x_1, \ldots, x_m)\text{:local coordinates around a point } x).$$

Hence

(1) the canonial relation C_{φ^*} of the operator φ^* is

$$C_{\varphi^*} = \dot{\mathcal{N}}(G_\varphi)' = \{(x,\, {}^t d\varphi_x(\theta), \varphi(x), \theta\,) \,|\, x \in M,\, T^*_{\varphi(x)}(N) \ni \theta \neq 0\}. \tag{3.3}$$

Here ${}^t d\varphi_x$ is the dual map of $d\varphi_x : T_x(M) \to T_{\varphi(x)}(N)$, ${}^t d\varphi_x : T^*_{\varphi(x)}(N) \to T^*_x(M)$. Note that ${}^t d\varphi_x(\theta) = 0$, then $\theta = 0$, so that

$$C_{\varphi^*} = \dot{\mathcal{N}}(G_\varphi)' \subset \dot{T}^*(M) \times \dot{T}^*(N).$$

From the expression (3.2) we see that the principal symbol of the corresponding Lagrangean distribution $K_{\varphi^*} \in I^0\big(M \times N, \dot{\mathcal{N}}(G_\varphi)\big)$

$$< K_\varphi, g \otimes f >= \int_{U\times V} g(x)\varphi^*(f)(x) = (2\pi)^{-n} \int e^{\sqrt{-1}<\varphi(x)-y,\theta>} g(x) f(y)\, dy\, d\theta dx$$

micro-loccally is a constant function *mod* half density $\in S^{(\dim M + \dim N)/4}(\dot{\mathcal{N}}(G_\varphi), \Omega^{1/2})$ (see (2.8)). (2) For defining the push-forward operator from the function space $C^\infty(M)$ to the function space $C^\infty(N)$, we need to fix a volume form dv_M on M and put $dv_N := \varphi_*(dv_M)$ which is also a volume form on N. Then we define a map $\Phi_\varphi : C^\infty(M) \to C^\infty(N)$ by

$$C^\infty(M) \ni f \longmapsto \Phi_\varphi(f) \in C^\infty(N),\ \Phi_\varphi(f)\cdot dv_N := \varphi_*(f \cdot dv_M).$$

Then the canonical relation C_Φ of the operator Φ_φ is

$$C_\Phi = \{(\varphi(x), \eta, x, {}^t d\varphi_x(\eta) \,\big|\, x \in M,\, T^*_{\varphi(x)}(N) \ni \eta \neq 0\}. \tag{3.4}$$

Let $L_2(M, dv_M)$ ($L_2(N, dv_N)$) be the Hilbert space on M (N) with respect to the inner product $(\cdot\,,\,\cdot)_M$ ($(\cdot\,,\,\cdot)_N$) defined as

$$(\cdot\,,\,\cdot)_M = \int_M f \cdot \overline{g} \cdot dv_M \quad \text{and } (\cdot\,,\,\cdot)_N = \int_N h \cdot \overline{k} \cdot dv_N = \int_N h \cdot \overline{k}\, \varphi_*(dv_M)$$

for functions f, g on M (h, k on N), respectively.

Proposition 3.1. *Let M and N be manifolds as assumed aboved and $\varphi : M \to N$ a submersion. Then, by the definitions of the maps φ^* and Φ_φ the following properties hold:*

(1) *The operator $\varphi^* : L_2(N, dv_N) \to L_2(M, dv_M)$ is isometric.*
(2) $(\varphi^*(h)\,,\, f)_M = (h\,,\, \Phi_\varphi(f))_N$*, for $h \in L_2(N, dv_N), f \in L_2(M, dv_M)$.*
(3) $\Phi_\varphi \circ \varphi^* = Id$*,*
(4) *The operator $\varphi^* \circ \Phi_\varphi := P_\varphi$ is an orthogonal projection, i.e., ${}^tP_\varphi = P_\varphi$ and ${P_\varphi}^2 = P_\varphi$.*
(5) $||\Phi_\varphi(f)|| = \sqrt{(\Phi_\varphi(f), \Phi_\varphi(f))_N} \;\leq\; ||f||,\;\; f \in L_2(M, dv_M)$.

Remark 2. *We use the notation $\bullet_*$ and $\bullet^*$ only for the push-forward operator and the pull-back operator induced from a manifold map and denote the adjoint (or dual) operators defined with respect to a specific inner product by ${}^t\bullet$ (or the operators induced in the dual spaces). So (2) says that ${}^t(\varphi^*) = \Phi_\varphi$.*

Proof. All these are proved by the similar way based on their definitions.

(1) $$(\varphi^*(h),\, \varphi^*(k))_M = \int_M \varphi^*(h)\cdot \overline{\varphi^*(k)}\cdot dv_M = \int_M \varphi^*(h\cdot \overline{k})\cdot dv_M$$
$$= \int_N h\cdot \overline{k}\cdot \varphi_*(dv_M) = \int_N h\cdot \overline{k}\cdot dv_N = (h,\, k)_N.$$

(2) $$(\varphi^*(h),\, f)_M = \int_M \varphi^*(h)\cdot \overline{f}\cdot dv_M$$
$$= \int_N h\cdot \varphi_*(\overline{f}\cdot dv_M) = \int_N h\cdot \overline{\Phi_\varphi(f)}\cdot dv_N = (h,\, \Phi_\varphi 1(f))_N.$$

(3) $$(\Phi_\varphi(\varphi^*(h),\, k)_N = \int_N \Phi_\varphi(\varphi^*(h))\cdot \overline{k}\cdot dv_N = \int_N \overline{k}\cdot \varphi_*(\varphi^*(h)\cdot dv_M)$$
$$= \int_M \varphi^*(\overline{k})\cdot \varphi^*(h)\cdot dv_M = \int_M \varphi^*(\overline{k}\cdot h)\cdot dv_M = \int_M \overline{k}\cdot h\cdot \varphi_*(dv_M) = (h,\, k)_N.$$

(4) From (3) $\varphi^*\circ\Phi_\varphi\circ\varphi^*\circ\Phi_\varphi = \varphi^*\circ\Phi$ and from (2) ${}^t(\varphi^*\circ\Phi_\varphi) = {}^t\Phi_\varphi\circ{}^t\varphi^* = \varphi^*\circ\Phi_\varphi$.

(5) $$\int_N |\Phi_\varphi(f)|^2\cdot dv_N = \int_N \Phi_\varphi(f)\cdot \overline{\Phi_\varphi(f)}\cdot dv_N = \int_N \overline{\Phi_\varphi(f)}\cdot \varphi_*(f\cdot dv_M)$$
$$= \int_M \varphi^*(\overline{\Phi_\varphi(f)})\cdot f\cdot dv_M = (f,\, P_\varphi(f))_M.$$

Hence from (4), $||\Phi_\varphi(f)|| \;\leq\; ||f||$.

$\square$

The canonical relation of the operator $\Phi_\varphi \circ \varphi^* = Id$ is, of course, the diagonal $\Delta_{\dot{T}^*(N)}$ of $\dot{T}^*(N) \times \dot{T}^*(N)$, which can be seen as follows:

$$\begin{aligned}
&(C_\Phi \times C_{\varphi^*}) \bigcap \left(\dot{T}^*(N) \times \Delta_{\dot{T}^*(M)} \times \dot{T}^*(N)\right)\\
&= \left\{(\varphi(y), \theta, y, {}^t d\varphi_y(\theta))\right\} \times \left\{(x, {}^t d\varphi_x(\eta), \varphi(x), \eta)\right\} \bigcap \left(\dot{T}^*(N) \times \Delta_{\dot{T}^*(M)} \times \dot{T}^*(N)\right)\\
&= \left\{ \left(\varphi(y), \theta, y, {}^t d\varphi_y(\theta); x, {}^t d\varphi_x(\eta), \varphi_x, \eta\right) \mid x = y \text{ and } {}^t d\varphi_x(\theta) = {}^t d\varphi_x(\eta) \ \leftrightarrow\ \theta = \eta\right\}
\end{aligned}$$

where $x, y \in M$, $\theta \in T^*_y(N)$ and $\eta \in T^*_x(N)$, and since ${}^t d\varphi_x$ is injective $\theta = \eta$. Hence

$$\begin{aligned}
C_\Phi \circ C_{\varphi^*} &= \left\{(\varphi(x), \theta, \varphi(x), \theta) \mid x \in M, \theta \in \dot{T}^*{}_{\varphi(x)}(N)\right\}\\
&= \Delta_{\dot{T}^*(N)}{}'.
\end{aligned}$$

Like wise we have

Proposition 3.2. *The canonical relation C_P of the composition operator $P = \varphi^* \circ \Phi$ is*

$$C_{\varphi^*} \circ C_\Phi = \left\{(y, {}^t d\varphi_y(\theta), x, {}^t d\varphi_x(\theta) \big| x, y \in M,\ \varphi(x) = \varphi(y), \theta \in \dot{T}^*{}_{\varphi(x)}(M) = \dot{T}^*{}_{\varphi(y)}(M)\right\},$$

that is, if we denote the fiber product of the maps

$$\begin{array}{ccc} & & M \\ & & \big\downarrow \varphi \\ M & \xrightarrow{\ \varphi\ } & N \end{array}$$

by F, then it coincides with $\dot{\mathcal{N}}\ (F)'$.

This is a special case of the following

Proposition 3.3. *Let $\alpha : X \to Z$ and $\beta : Y \to Z$ be two submersions and we fix a volume form dv_Y on Y. The operator $\Phi_\beta : C^\infty(Y) \to C^\infty(Z)$ is defined as before, where we use the volume form $dv_Z := \beta_*(dv_Y)$. Then the composition $\alpha^* \circ \Phi_\beta : C^\infty(Y) \to C^\infty(X)$ is a Fourier integral operator whose canonical relation $C_{\alpha^* \circ \Phi_\beta}$ is given by*

$$C_{\alpha^* \circ \Phi_\beta} = \dot{\mathcal{N}}\ (X \times_Z Y)' \subset \dot{T}^*(X) \times \dot{T}^*(Y), \tag{3.5}$$

where we denote by $X \times_Z Y := \{(x, y) \in X \times Y \mid \alpha(x) = \beta(y)\}$, the fiber product of the two submersions

$$\begin{array}{ccc} & & Y \\ & & \big\downarrow \beta \\ X & \xrightarrow[\alpha]{} & Z. \end{array}$$

Proof. The canonical relation C_{Φ_β} of the operator Φ_β is

$$C_{\Phi_\beta} = \{\big(\beta(y), \eta, y, {}^t d\beta_y(\eta)\big) \mid y \in Y, \eta \in \overset{\bullet}{T^*}_{\beta(y)}(Z)\} \subset \overset{\bullet}{T^*}(Z)\} \times \overset{\bullet}{T^*}(Y)\}$$

and the canonical relation C_{α^*} of the operator α^* is

$$C_{\alpha^*} = \{\big(x, {}^t d\alpha_x(\theta), \alpha(x), \theta\big) \mid x \in X, \theta \in \overset{\bullet}{T^*}_{\alpha(x)}(Z)\} \subset \overset{\bullet}{T^*}(X) \times \overset{\bullet}{T^*}(Z).$$

Based on the submersion property we see in terms of local coordinates that the product space $C_{\alpha^*} \times C_{\Phi_\beta}$ intersects with the space $\overset{\bullet}{T^*}(X) \times \Delta_{\overset{\bullet}{T^*(Z)}} \times \overset{\bullet}{T^*}(Y)$ transversely. This is seen by the local expression around the point $(\lambda, \lambda^*) \in C_{\alpha^*} \times C_{\Phi_\beta} \bigcap T^*X \times \Delta_{T^*Z} \times T^*Y$. In fact, let's denote

$$(\lambda, \lambda^*) = \big(x, {}^t d\alpha(\eta), \alpha(x), \eta\big) \times \big(\beta(y), \theta, y, {}^t d\beta(\theta)\big) \in C_{\alpha^*} \times C_{\Phi_\beta}$$

and

$$(x', \eta', z, \tau, z', \tau', y', \theta') \in T^*(X) \times T^*(Z) \times T^*(Z) \times T^*(Y),$$

then at the point $(\lambda, \lambda^*) \in \Big(C_{\alpha^*} \times C_{\Phi_\beta}\Big) \bigcap \Big(T^*X \times \Delta_{T^*Z} \times T^*Y\Big)$, the coordinates must satisfy the conditions

$$\alpha(x) = \beta(y), \eta = \theta, z = z', \tau = \tau'.$$

Then the variables $x', \eta', y', \theta', z, \tau, x, \eta$ can take any value of the coordinates around the point (λ, λ^*), since the map β is a submersion, so that the equality $\alpha(x) = \beta(y)$ has solutions for any x. Hence we have

$$T(C_{\alpha^*} \times C_{\Phi_\beta}) + T\big(T^*(X) \times \Delta_{T^*(Z)} \times T^*(Y)\big) = T\big(T^*(X) \times T^*(Z) \times T^*(Z) \times T^*(Y)\big)$$

on the intersection (λ, λ^*).

The projection of the intersection to the space $\overset{\bullet}{T^*}(X) \times \overset{\bullet}{T^*}(Y)$ is the space (3.5), the normal bundle of the fiber product *minus zero section*. Hence by Theorem 2.2 gives the desired conclusion. □

If Z is a point, then the operators are smooth kernel operators so that we have non-trivial Fourier integral operators when $\dim Z \geq 1$.

4. Double submersion and Radon transformation

Let $\pi : \mathbb{P} \to M$ and $\varphi : \mathbb{P} \to N$ be two submersions from a total space $\mathbb{P}$. We assume that the manifolds $\mathbb{P}$, M and N satisfy the conditions as in the last section. Also we fix a nowhere vanishing highest degree differential form $dv_{\mathbb{P}}$ on the total space $\mathbb{P}$, by which the manifold $\mathbb{P}$ is oriented and put $dv_M = \pi_*(dv_{\mathbb{P}})$ and $dv_N = \varphi_*(dv_{\mathbb{P}})$. Each is a nowhere vanishing highest degree differential form on M and N respectively.

4.1. **Incidence relation and Radon transformation.**

Definition 4.1. *We call a transformation*

$$C^\infty(N) \ni f \longmapsto \mathcal{R}(\mathbb{P}; N, M)(f) := \Phi_\pi(\varphi^*(f)) \in C^\infty(M)$$

a Radon transformation.

Remark 3. *Let D be a pseudo-differential operator on the total space $\mathbb{P}$, then we can define an operator of a form*

$$\Phi_\pi(D(\varphi^*(f))),$$

which may be called a generalized Radon transformation, but in this note we do not treat this type operator, although the conclusion in Theorem 4.5 is still valid for such type operators.

Proposition 4.2.

$$\int_M h \cdot \mathcal{R}(\mathbb{P}; N, M)(f) \cdot dv_M = \int_N \mathcal{R}(\mathbb{P}; M, N)(h) \cdot f \cdot dv_N. \tag{4.1}$$

Proof. The left hand side is, by definition of the volume forms on N and M,

$$\int_M h \cdot \mathcal{R}(\mathbb{P}; N, M)(f) \cdot dv_M = \int_N h \cdot \pi_*(\varphi^*(f) \cdot dv_P) = \int_{\mathbb{P}} \pi^*(h) \cdot \varphi^*(f) \cdot dv_P.$$

The same calculation gives the same expression for the right hand side, which shows the equality (4.1). □

Let $\overline{\varphi} : \overline{P} \to N$ and $\overline{\pi} : \overline{P} \to M$ be two submersions and assume there exisits a submersion $\rho : P \to \overline{P}$ such that

$$\overline{\pi} \circ \rho = \pi \text{ and } \overline{\varphi} \circ \rho = \varphi, \tag{4.2}$$

and we define all the volumes forms on $\overline{P}, M$ and N defined as the push forwards of dv_P.

Then

Proposition 4.3.

$$\mathcal{R}(\mathbb{P}; N, M) = \mathcal{R}(\overline{\mathbb{P}}; N, M) \tag{4.3}$$

Proof. We define the map $\Phi_{\overline{\rho}}$ by making use of the measure $dv_{\overline{P}} = \rho_*(dv_P)$. Then by the assumption (4.2) we have

$$\Phi_{\overline{\pi}} \circ \Phi_\rho = \Phi_{\overline{\pi}\circ\rho}.$$

Hence

$$\Phi_{\overline{\pi}} \circ \overline{\varphi}^* = \Phi_{\overline{\pi}} \circ \Phi_\rho \circ \rho^* \circ \overline{\varphi}^* = \Phi_\pi \circ \varphi^*,$$

since

$$\Phi_\rho \circ \rho^* = Id.$$

□

By this property, it is enough to study the transformation $\mathcal{R}(\mathbb{P}; N, M)$, in a sense, with respect to a minimal dimensional total space $\mathbb{P}$.

If the fibers $\pi^{-1}(\pi(p))$ and $\varphi^{-1}(\varphi(p))$ at any point $p \in \mathbb{P}$ intersect transversely, that is

$$T_p(\pi^{-1}(\pi(p)))) + T_p(\varphi^{-1}(\varphi(p))) = T_p(\mathbb{P}),$$

then

Proposition 4.4. *The operator $\mathcal{R}(\mathbb{P}; N, M)$ is a smooth kernel operator.*

Proof. By the condition, since the map

$$T_p(\pi^{-1}(\pi(p)))) \hookrightarrow T_p(\pi^{-1}(\pi(p)))) + T_p(\varphi^{-1}(\varphi(p))) = T_p(\mathbb{P}) \longrightarrow T_{\pi(p)}(N)$$

is surjective, the restriction of the map φ to the each fiber of the map π, $\varphi_{|\pi^{-1}(\pi(p))}$, is already submersion. So the map $\rho := \pi \times \varphi : \mathbb{P} \to M \times N$ is surjective. If we put $\overline{\mathbb{P}} = M \times N$ in the proceeding Proposition 4.3, then the map $\mathcal{R}(\mathbb{P}; N, M) = \mathcal{R}(\overline{\mathbb{P}}, N, M)$. Hence

$$\int_M h(x) \cdot \mathcal{R}(\overline{\mathbb{P}}; N, M)(f)(x)\pi_*(dv_{\mathbb{P}}) = \int_{M \times N} h(x) \cdot f(y) dv_{\mathbb{P}}.$$

The smooth measure $dv_{\mathbb{P}}$ is the kernel of this distribution. □

Remark 4. *By the proof above it will be seen that the same conclusion holds only by the assumtion that at any point $p \in \mathbb{P}$ one of the restiction map $d\pi : T_p(\varphi^{-1}(\varphi(p))) \to T_{\pi(p)}(N)$ or $d\varphi : T_p(\pi^{-1}(\pi(p))) \to T_{\pi(p)}(M)$ is surjective, since then the map $\rho = \pi \times \varphi : \mathbb{P} \to M \times N$ is a submersiom.*

Also we remark that if at a point $p \in \mathbb{P}$ the restriction $\pi : \varphi^{-1}(\varphi(p))$ is surjective, then the restrictions of the map π to any fibers $\varphi^{-1}(y)$ ($\forall y \in N$) are subjective.

By the remark above and Proposition 4.4 we consider only such a pair of submersions $\varphi : \mathbb{P} \to N$ and $\pi : \mathbb{P} \to M$ that any restrictions of the maps π to $\varphi^{-1}(y)$ ($\forall y \in N$) and φ to $\pi^{-1}(x)$ ($\forall x \in M$) are not surjective.

We are interested in the case that among the (generalized) Radon transformations $\mathcal{R}(\mathbb{P}; N, M) = \Phi_\pi \circ \varphi^*$, when it is a Fourier integral operator other than smooth kernel operators.

So by Theorem 2.2, we consider the case that the canonical relations C_{φ^*} and C_{Φ_π} intersect cleanly, that is,

(4.4) $C_{\varphi^*} \times C_{\Phi_\pi}$ intersects cleanly with $\dot{T}^*(M) \times \Delta_{\dot{T}^*(\mathbb{P})} \times \dot{T}^*(N)$,

(4.5) the projection of the intersection $\left(C_{\varphi^*} \circ C_{\Phi_\pi}\right) \bigcap \left(\dot{T}^*(M) \times \Delta_{\dot{T}^*(\mathbb{P})} \times \dot{T}^*(N)\right)$

to $\dot{T}^*(M) \times \dot{T}^*(N)$ is a submanifold.

Theorem 4.5. *If the image*

$$\tau = \pi \times \varphi : \mathbb{P} \to M \times N \tag{4.6}$$

is a submanifold in $M \times N$ (π and φ are submersions), then two conditions (4.4) *and* (4.5) *are satisfied.*

Hence the composition $\Phi_\pi \circ \varphi^*$ is a Fourier integral operator.

Proof. By Proposition 4.3, we may consider from the begining that the total space $\mathbb{P}$ is a submanifold in $M \times N$, $\dim \mathbb{P} < \dim M + \dim N$ and the natural projection maps coincide with $\pi_{M|\mathbb{P}} = \pi$ and $\pi_{N|\mathbb{P}} = \varphi$ respectively, and they are submersions. Then by making use of the surjectivity of the maps $d\pi$ and $d\varphi$ we see that

$$C_{\Phi_\pi} \circ C_{\varphi^*} = \dot{\mathcal{N}}(\mathbb{P})' \subset \dot{T}^*(M) \times \dot{T}^*(N) \tag{4.7}$$

is a submanifold.

It will be apparent that the inclusion holds

$$\begin{aligned} &T\Big(\{C_{\Phi_\pi} \times C_{\varphi^*}\} \bigcap \{\dot{T}^*(M) \times \Delta_{\dot{T}^*(\mathbb{P})} \times \dot{T}^*(N)\}\Big) \\ &\subset T\big(C_{\Phi_\pi} \times C_{\varphi^*}\big) \bigcap T\Big(\dot{T}^*(M) \times \Delta_{\dot{T}^*(\mathbb{P})} \times \dot{T}^*(N)\Big). \end{aligned}$$

Conversely let

$$A \in \Big\{C_{\Phi_\pi} \times C_{\varphi^*}\Big\} \bigcap \Big\{\dot{T}^*(M) \times \Delta_{\dot{T}^*(\mathbb{P})} \times \dot{T}^*(N)\Big\}$$

and assume

$$T \in T_A\big(C_{\Phi_\pi} \times C_{\varphi^*}\big) \bigcap T_A\Big(\dot{T}^*(M) \times \Delta_{\dot{T}^*(\mathbb{P})} \times \dot{T}^*(N)\Big).$$

The tangent vector T is realized as a differentiation along a curve

$$\begin{aligned} &\{c(s)\}_{|s|\ll 1} \subset C_{\Phi_\pi} \times C_{\varphi^*} \subset \dot{T}^*(M) \times \dot{T}^*(\mathbb{P}) \times \dot{T}^*(\mathbb{P}) \times \dot{T}^*(N),\ c(0) = A, \\ &\{e(s)\}_{|s|\ll 1} \subset \dot{T}^*(M) \times \Delta_{\dot{T}^*(\mathbb{P})} \times \dot{T}^*(N),\ e(0) = A. \end{aligned}$$

We may put the curves as the forms that

$$\begin{aligned} c(s) &= \big(\pi(p(s)), \eta(s), p(s), {}^t d\pi_{\pi(p(s))}(\eta(s)\big) \times \big(q(s), {}^t d\varphi_{\varphi(q(s))}(\theta(s)), \varphi(q(s)), \theta(s)\big), \\ e(s) &= \big(a(s), \alpha(s)\big) \times \big(r(s), \gamma(s)\big) \times \big(r(s), \gamma(s)\big) \times \big(b(s), \beta(s)\big). \end{aligned}$$

By these expressions, especially at the point A

$$p(0) = r(0) = q(0), \tag{4.8}$$

$$\frac{d\,p}{ds}(0) = \frac{d\,r}{ds}(0) = \frac{d\,q}{ds}(0), \tag{4.9}$$

$$\gamma(0) = {}^t d\pi_{\pi(p(0))}(\eta(0)) = {}^t d\varphi_{\varphi(p(0))}(\theta(0)), \tag{4.10}$$

$$\frac{d\,\gamma(s)}{ds}(0) = \frac{d\,{}^t d\pi_{\pi(p(s))}(\eta(s))}{ds}(0) = \frac{d\,{}^t d\varphi_{\varphi(p(s))}(\theta(s))}{ds}(0). \tag{4.11}$$

We note here that since we assume $\dim \mathbb{P} < \dim M + \dim N$,

$$\dim \Big({}^t d\pi_{\pi(p(s))}\big(T^*_{\pi(p(s))}(M)\big) \bigcap {}^t d\pi_{\varphi(p(s))}\big(T^*_{\varphi(p(s))}(N)\big) \Big) = \dim M + \dim N - \dim \mathbb{P} \,\geq\, 1.$$

The proof is to find a curve $\{u(s)\}$ in

$$\Big\{ C_{\Phi_\pi} \times C_{\varphi^*} \Big\} \bigcap \Big\{ \dot{T}^*(M) \times \Delta_{\dot{T}^*(\mathbb{P})} \times \dot{T}^*(N) \Big\}$$

expressing the tangent vector $T = \dfrac{d\,u(s)}{ds}(0)$ at $A = u(0)$.

By the properties (4.8) and (4.9), first we replace $\{q(s)\}$ by $\{p(s)\}$.

Let $U \times V \ni (x, x')$ be a local coordinates system at the point $p(0)$ where the projection map π is given by $\pi : (x, x') \mapsto x$, so that U is a coordinate neighborhood of the point $\pi(x, x') = x$. Then the space C_{Φ_π} is parametrized by the coordinates (x, x', η), that is

$$\rho : U \times V \times \mathbb{R}^{\dim M} \ni (x, x', \eta) \mapsto \big(x, \eta, (x, x'), (\eta, 0)\big) \in C_{\Phi_\pi} \cong \pi^*(T^*M). \tag{4.12}$$

We consider the line segment $\ell(s)$ in the space $T^*_{\pi(p(0))}M \cong \{(x, \eta)\} \times \mathbb{R}^{\dim M}$, which starts from the point $\eta(0)$ to the direction $\dfrac{d\eta(s)}{ds}(0)$, that is

$$\ell(s) = \eta(0) + s \cdot \frac{d\eta(s)}{ds}(0), \ |s| \ll 1.$$

We denote by $\{\tilde{\ell}(s)\}$ the curve in C_{Φ_π}, which is the image of the line segment $\{\ell(s)\}$ under the trivialization (4.12). We express this curve as

$$\rho(\ell(s)) = \big(\pi(p(s)), \ell(s), p(s), {}^t d\pi_{p(s)}(\ell(s))\big) := \tilde{\ell}(s)$$

Then by the properties (4.10) and (4.11) we can define a curve $\theta^*(s)$ by making use of the identity

$${}^t d\pi_{\pi(p(s))}(\ell(s)) = {}^t d\varphi_{\varphi(p(s))}(\theta^*(s)).$$

Then the new curve
(4.13)

$$\begin{aligned}\tilde{c}(s) &= \big(\pi(p(s)), \ell(s), p(s), {}^t d\pi_{\pi(p(s))}(\ell(s))\big) \times \big(p(s), {}^t d\varphi_{\varphi(p(s))}(\theta^*(s)), \varphi(p(s)), \theta^*(s)\big)\\ &\in \Big\{C_{\Phi_\pi} \times C_{\varphi^*}\Big\} \bigcap \Big\{\overset{\bullet}{T}{}^*(M) \times \Delta_{\overset{\bullet}{T}{}^*(\mathbb{P})} \times \overset{\bullet}{T}{}^*(N)\Big\} \text{ for any } s, |s| \ll 1,\end{aligned}$$

and $\tilde{c}(0) = A$ and $\dfrac{d\,\tilde{c}(s)}{ds}(0) = T$. Hence we may replace the curve $\{c(s)\}$ by the new curve (4.13) and showed the both conditions (4.4) and (4.5) are satisfied.

□

If the image of the product map $\tau = \varphi \times \pi : \mathbb{P} \to M \times N$ of two submersions $\varphi : \mathbb{P} \to N$ and $\pi : \mathbb{P} \to M$ is a submanifold $\tau(\mathbb{P})$ in the product manifold $M \times N$, then the conormal bundle minus zero section $\overset{\bullet}{\mathcal{N}}(\tau(\mathbb{P}))$ is included in $\overset{\bullet}{T}{}^*(M) \times \overset{\bullet}{T}{}^*(N)$ and by Theorem 4.5, from now on we only consider such a submanifold $\mathbb{P} \subset M \times N$ that the normal bundle

$$\mathcal{N}(\mathbb{P}) = \{\theta \in T^*(M \times N) \cong T^*(M) \times T^*(N) \mid \theta(X) = 0 \text{ for}^\forall X \in T(\mathbb{P})\}$$

has a property that

$$\overset{\bullet}{\mathcal{N}}(\mathbb{P}) = \mathcal{N}(\mathbb{P})\backslash\{0\} \subset \overset{\bullet}{T}{}^*(M) \times \overset{\bullet}{T}{}^*(N). \tag{4.14}$$

Then the map $\mathbb{P} \to \tau(\mathbb{P})$ is necesarily a submersion and of course we have

Proposition 4.6. *The projection $\pi_M : \mathbb{P} \to M$ is a submersion (hence, the projection $\pi_N : \mathbb{P} \to N$ too).*

The condition (4.14) was already introduced in [3] and the submanifold $\mathbb{P}$ in $M \times N$ satisfying such condition is called an *incidence relation*, when there were no theory of pseudo-differential nor Fourier integral operators. This property implies the clean product condition for the composition of two Fourier integral operators φ^* and Φ_φ. The special case discussed in Proposition 3.3 corresponds to the transversal product condition.

Remark 5. *A typical example of such a submanifold is given in Proposition 3.3, that is a fiber product of two submersions.*

4.2. **Elliptic, but non-Fredholm case.** Our operator $\mathcal{F}(\mathbb{P}; N, M)$ is a Fourier integral operator whose canonical relation is the co-normal bundle $\overset{\bullet}{\mathcal{N}}(\mathbb{P})$ of a submanifold $\mathbb{P} \subset M \times N$ satisfying the condition (4.14). However this type operator $\mathcal{R}(\mathbb{P}; N, M)$ need not be a Fredholm operator, even if it is elliptic in the sense that the principal symbol does not vanish at any point on the canonical relation.

In fact, let $\alpha : X \to Z$ and $\beta : Y \to Z$ be two submersions as in Proposition 3.3. Then the fiber product $X \times_Z Y$ satisfies the property (4.14).

We denote the projection maps $\pi : X \times_Z Y \to X$ and $\varphi : X \times_Z Y \to Y$. Then we have two Fourier integral operators

$$\mathcal{R}(X \times_Z Y; Y, X) = \Phi_\pi \circ \varphi^* : C^\infty(Y) \to C^\infty(X) \tag{4.15}$$

$$\alpha^* \circ \Phi_\beta : C^\infty(Y) \to C^\infty(X). \tag{4.16}$$

Their canonical relations are same (= co-normal bundle of the fiber product $X \times_Z Y$ in $X \times Y$, see Proposition 3.3) and through the isomorphism explained in Theorem 2.11 their symbol classes considered in

$$S^{(\dim X + \dim Y)/4}(\overset{\bullet}{\mathcal{N}}\ (X \times_Z Y), \Omega^{1/2}) / S^{(\dim X + \dim Y)/4-1}(\overset{\bullet}{\mathcal{N}}\ (X \times_Z Y), \Omega^{1/2})$$

coincide and in the sense of ellipticity that the symbol does not vanish on the canonical relation. However

Theorem 4.7. *The second operator will have infinite dimensional kernel if* $\dim Y > \dim Z$*, so that it is never a Fredholm operator.*

Also if $\beta : Y \to Z$ *has a local trivial part, that is, there is an open set* $U \subset Z$ *such that* $\beta^{-1}(U) \cong U \times F$ *with a manifold* F*, then* $\mathcal{R}(X \times_Z Y; Y, X) = \Phi_\pi \circ \varphi^*$ *is not a Fredholm operator, if we choose a suitable measure of the form* $dv_{X \times_Z Y} = v(x)w(y)dx \wedge dy$ *on* $(x, y) \in \alpha^{-1}(U) \times F \cong \alpha^{-1}(U) \times_U \beta^{-1}(U) \subset X \times_Z Y$*.*

Proof. Consider the functions $f \in C^\infty(F)$ such that

$$\int_F f(y)w(y)dy = 0.$$

Then for any $g \in C_0^\infty(U)$ and $h \in C^\infty\big(\alpha^{-1}(U)\big)$

$$\begin{aligned}
&\int_{\alpha^{-1}(U)} h(x) \cdot \Phi_\pi(\varphi^*(g \otimes f))(x) \pi_*(dv_{X \times_Z Y}) \\
&= \int \pi^*(h)(x, y) \cdot \varphi^*(g \otimes f)(x, y) dv_{X \times_Z Y} \\
&= \int h(x) \cdot g(\varphi(x)) f(y)\, v(x) w(y) dx dy \\
&= \int h(x) \cdot g(\varphi(x)) v(x) dx \cdot \int f(y)\, w(y) dy = 0.
\end{aligned}$$

Hence we showed that the Radon transform $\mathcal{R}(X \times_Z Y; Y, X)$ has the infinite dimensional kernel.

So if $\beta : Y \to Z$ is a Riemann submersion (in general, if Y is a complete Riemannian manifold and β is a Riemannian submersion, see [21]) or from the beginning assume it is a locally trivial fiber bundle, then the generalized Radon transformation need not be a Fredholm operator. □

By noting the equality $\Phi_\alpha \circ \Phi_\pi \circ \varphi^* = \Phi_\alpha \circ \mathcal{R}(X \times_Z Y; Y, X) = \Phi_\beta$

Proposition 4.8. *The Radon transformation* $\mathcal{R}(X\times_Z Y; Y, X)$ *is injective on* $\mathrm{Ker}\,(\Phi_\beta)^\perp$. *Moreover* $\mathcal{R}(X\times_Z Y, Y, X)(C^\infty(N)) \bigcap \mathrm{Ker}\,(\Phi_\alpha) = \{0\}$.

4.3. **An example.** ([7], [4]). We show a concrete example of the case in the above Theorem 4.7.

Let $Sp(2)$ be the group of quaternion 2×2 matrices :

$$Sp(2) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\Big|\; a, b, c, d \in \mathbb{H},\ gg^* = Id \right\},$$

where $\mathbb{H}$ denote the quaternion number field and $g^* = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$.

We consider the action of the group $Sp(1)\times Sp(1) \ni (q, r)$ by

$$G_0 : Sp(1)\times Sp(1)\times Sp(2) \longrightarrow Sp(2), (q, r, g) \longmapsto \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}\cdot g\cdot \begin{pmatrix} \overline{r} & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.17}$$

and two more actions

$$G_1 : Sp(1)\times Sp(2) \longrightarrow Sp(2), (q, g) \longmapsto \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}\cdot g\cdot \begin{pmatrix} \overline{q} & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.18}$$

$$G_2 : Sp(1)\times Sp(2) \longrightarrow Sp(2), (r, g) \longmapsto g\cdot \begin{pmatrix} \overline{r} & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.19}$$

All these action G_i are free. The orbit space by the action G_0 is the standard sphere S^4, which is known through the correspondence

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (2\overline{b}d, |b|^2 - |d|^2) \in \mathbb{R}^5,$$

the orbit space by the action G_1 is known as the Gromoll-Meyer exotic seven sphere Σ^7_{GM}, and the last one is the standard 7-sphere S^7, which is seen by the correspondence

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (b, d) \in \mathbb{H}\oplus\mathbb{H} \cong \mathbb{R}^8.$$

The group of the action G_0 is the diagonal $\Delta_{Sp(1)}$ of $Sp(1)\times Sp(1)$ and that of G_1 is the subgroup $\{Id\}\times Sp(1)$ of the action G_0. Also

$$\Delta_{Sp(1)} \bigcap \big(\{Id\}\times Sp(1)\big) = \{Id\}.$$

Hence we have the following diagram:

$$\begin{array}{ccc} Sp(2) & \xrightarrow{\pi} & \Sigma^7_{GM} \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ S^7 & \xrightarrow[\alpha]{} & S^4, \end{array}$$

and the fiber product of the maps (α, β) is $\alpha^*(S^4) = S^7 \times_{S^4} \Sigma^7_{GM} \cong Sp(2)$.

These imply that the Radon transformation $\mathcal{R}(Sp(2); \Sigma^7_{GM}, S^7)$ from Σ^7_{GM} to S^7 and its adjoint $\mathcal{R}(Sp(2); S^7, \Sigma^7_{GM})$ have infinite dimensional kernels. Hence the image of $\mathcal{R}(Sp(2); \Sigma^7_{GM}, S^7)$ also has the infinite dimensional orthogonal complement.

4.4. **Final remark.** In some cases (for examples [23], [13], [14] and there are many cases) of the double fibrations consisting of quotient spaces of a compact Lie group G

$$\begin{array}{ccc} G & \longrightarrow & G/K \\ \big\downarrow & & \\ G/L, & & \end{array} \tag{4.20}$$

it holds an inversion formula for the Radon transformation in the form that

$$P \circ \mathcal{R}(G/(K \cap L); G/L, G/K) \circ \mathcal{R}(G/(K \cap L); G/K, G/L) = Id,$$

where P is a pseudo-differential operator of a positive order.

Of course, the inversion formula of the original Radon transformation is of this type, although the spaces are not compact. In [1] the author discussed this case from the point of Fourier integral operator theory, but did not mention the incidence relation of the canonical relations there.

Not always P is elliptic ([16], [13], [23]). If it is elliptic, then with a parametrics Q we have

$$\mathcal{R}\big(G/(K \cap L)\,;\, G/L,\, G/K\big) \circ \mathcal{R}\big(G/(K \cap L)\,;\, G/K,\, G/L\big) = Q + S,$$

where S is a smooth kernel operator. Hence the composition

$$\mathcal{R}\big(G/(K \cap L); G/L,\, G/K\big) \circ \mathcal{R}\big(G/(K \cap L)\,;\, G/K, G/L\big)$$

is an elliptic pseudo-differential operator of negative order. Although the canonical relations of $\mathcal{R}\big(G/(K \cap L)\,;\, G/L, G/K\big)$ and $\mathcal{R}\big(G/(K \cap L)\,;\, G/K, G/L\big)$ do not satisfy the transversal (nor clean) intersection condition, never the less their composition is a pseudo-differential operator.

Here we ask are there any case of double fibration for which the Radon transformation is a Fredholm operator?

From our Theorem 4.7 in the double fibration (4.20), if there is a subgroup $H \subset G$ including both of K and L and if the fiber product of the maps $G/K \to G/H$ and $G/L \to G/H$ coincides with $G/(K \cap L)$, then such Radon transformation do not have inversion formulas.

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