CONSTRUCTION AND CLASSIFICATION OF INTEGRAL STRUCTURES ON PSEUDO H-TYPE LIE ALGEBRAS

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Abstract. Pseudo H-type Lie algebras are a special class of 2-step nilpotent metric Lie algebras, intimately related to Clifford algebras $\mathrm{Cl}_{r,s}$. In this work, we propose the classification method for integral orthonormal structures of pseudo H-type Lie algebras for full range of positive parameters $(r,s) \in \mathbb{Z}^2$. The existence of integral orthonormal structures gives rise to the integral discrete uniform subgroups or lattices of the pseudo H-type Lie groups. We apply the developed method for the full classification of the integral orthonormal structures for $0 < r+s \leq 16$, and minimal admissible Clifford modules. The cases $0 < r+s \leq 16$ form a core for further extensions by making use of the Atiyah-Bott periodicity and the reducibility of admissible Clifford modules.

Contents

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1. Introduction

Two-step nilpotent Lie algebras attracted the attention of G. Métivier [M8́0] in an attempt to describe hypoelliptic operators in a non-Euclidean setting. The condition of hypo-ellipticity required the adjoint map on the Lie algebra with the value on the centre to be surjective. This type of Lie algebras was studied under different names and for different purposes, for instance, in [Ebe94, LT99, MS04, OW10, GMKMV18]. A. Kaplan [Kap80] showed that if the adjoint map is an isometry, then the sub-Laplacian on two-step nilpotent Lie groups, admits a fundamental solution, reminiscent of that in Euclidean space. His result extended a theorem obtained by G. Folland on the Heisenberg group [Fol73]. Therefore, the class of these Lie algebras received the name H(eisenberg)-type Lie algebras. The H-type Lie algebras are in a bijective relation to Clifford algebras $\mathrm{Cl}_{r,0}$, generated by the Euclidean space $\mathbb{R}^r$ [Rei01a]. The definition of H-type Lie algebras related to Clifford algebras $\mathrm{Cl}_{r,s}$, $s > 0$, generated by pseudo Euclidean spaces $\mathbb{R}^{r,s}$ was extended by P. Ciatti [Cia00] and received the name pseudo H-type Lie algebras, see also [GMKM13]. The pseudo H-type Lie algebras, which will be denoted by $\mathfrak{n}_{r,s}$ is a fruitful source for studies of Damek-Ricci spaces [BTV95], Iwasawa decomposition of symmetric spaces [CDKR98], Riemannian nilmanifolds [Kap81], rigidity problems [Rei01b], properties of PDE on Lie groups [CS12, MR92, BFM20] and many others topics in geometry, analysis, and geometric measure theory. The classification of the pseudo H-type Lie algebras was completed in [FM17, FM19].

Our work is motivated by the study of uniform discrete subgroups on nilpotent Lie groups, which are crucial for the study of homogeneous spaces, compact nilmanifolds, and spectral problems. The existence of a uniform subgroup is guaranteed by a presence of a rational structure on the associated Lie algebra by seminal work of A. I. Malčev [Mc49]. The existence of rational structures on pseudo H-type Lie algebras was proved in [CD02, Ebe03, FM14]. A complete classification of rational structures in the class of pseudo H-type Lie algebras exists only on the Heisenberg algebra (related to the Clifford algebra $\mathrm{Cl}_{1,0}$) [GW86]. Some progress in the study of lattices can be found in [CP08].

In the present work, we describe the set of invariant integral structures, which are at the core of rational structures of the Lie algebras. An invariant integral structure is a span over $\mathbb{Z}$ of an orthonormal basis, constructed as an action of a subgroup $G(B_{r,s})$ of the invertible elements $\mathrm{Pin}(r,s)$ in the Clifford algebra $\mathrm{Cl}_{r,s}$ on a suitably chosen normal vector $v \in V$ in the Clifford module V, see Section 3. As a result, the basis of the Clifford module V is invariant under the action of $G(B_{r,s})$ and the non-vanishing structure constants of the H-type Lie algebra are equal to ± 1. We emphasize that invariant integral structures are particular cases of integral structures (having structure constants $0, \pm 1$) that are included in a general class of rational structures on a Lie algebra (having rational structure constants). Two invariant integral structures are isomorphic, if and only if the isotropy subgroups $\mathcal{S}_v^{(1)} \subset \mathrm{Cl}_{r,s}$ and $\mathcal{S}_v^{(2)} \subset \mathrm{Cl}_{r,s}$ of $v \in V$ belongs to the same equivalence class, see Definition 3.21 and Section 5. The isomorphism of invariant integral structures of the Lie algebras leads to the isomorphism of uniform discrete subgroups on the corresponding Lie groups, which is always extended to an automorphism of ambient pseudo H-type Lie groups, see [Rag72].

We apply the classification algorithm to isotropy groups $\mathcal{S}_v$ for parameters $0 < r + s \leq 16$ in Section 4. We note that the range $0 < r + s \leq 16$ corresponds to the first and the second period in r of pseudo H-type Lie groups originated from the Atiyah-Bott periodicity of Clifford algebras. The reader can notice that the second period $r \in \{9, \ldots, 16\}$ contains more non-equivalent subgroups with phenomena, such as disconnectedness, that can not appear in the first period $r \in \{3, \ldots, 8\}$ due to the lack of dimension of the center of the Lie algebra. The forthcoming paper will be dedicated to the study of new features in the increasing of the parameters r, s and the study of the periodicity. Despite this, the theorems and the characterizations proved in Sections 3 and 5 have general character and are valid for arbitrary parameters (r, s).

2. Preliminaries

In this section we remind some classical objects and introduce the main ones of our interest.

2.1. **Clifford algebras.** We denote by $\mathbb{R}^{r,s}$ the pseudo Euclidean space, that is the vector space $\mathbb{R}^{r+s}$ endowed with the non-degenerate symmetric bilinear form

$$\langle x, y\rangle_{r,s} = \sum_{k=1}^{r} x_k y_k - \sum_{k=r+1}^{r+s} x_k y_k.$$

Let $\mathrm{Cl}_{r,s}$ be a Clifford algebra over $\mathbb{R}$ generated by $\mathbb{R}^{r,s}$. Remind that $\mathrm{Cl}_{r,s}$ is a quotient of the tensor algebra

$$\mathcal{T}(U) := \mathbb{R} \oplus \mathbb{R}^{r,s} \oplus \Big(\overset{2}{\otimes}\, \mathbb{R}^{r,s} \Big) \oplus \Big(\overset{3}{\otimes}\, \mathbb{R}^{r,s} \Big) \oplus \Big(\overset{4}{\otimes}\, \mathbb{R}^{r,s} \Big) \oplus \ldots$$

by a two sided ideal $I_{r,s}$ generated by elements of the form

$$x \otimes x + \langle x, x\rangle_{r,s} \mathbf{1}, \quad x \in \mathbb{R}^{r+s},$$

and $\mathbf{1}$ is the identity element of the Clifford algebra $\mathrm{Cl}_{r,s}$. Consider a representation of $\mathrm{Cl}_{r,s}$ on a real vector space V

$$J \colon \mathrm{Cl}_{r,s} \to \mathrm{End}(V).$$

We call V the $\mathrm{Cl}_{r,s}$-module, or simply module if we do not want to specify the signature (r, s), and will denote by $J_z v$ the action of $z \in \mathbb{R}^{r,s}$ on $v \in V$. Assume also that the module V is equipped with a non-degenerate symmetric bilinear form $\langle . \, , . \rangle_V$ satisfying the condition

$$\langle J_z u, v\rangle_V + \langle u, J_z v\rangle_V = 0 \quad \text{for any} \quad z \in \mathbb{R}^{r,s} \quad \text{and} \quad u, v \in V. \tag{2.1}$$

We call such a module $V = (V, \langle . \, , . \rangle_V)$ an *admissible module* of the Clifford algebra $\mathrm{Cl}_{r,s}$. We write $V_{min} = (V_{min}, \langle . \, , . \rangle_V)$ or simply V_{min} for an admissible $\mathrm{Cl}_{r,s}$-module of the minimal dimension and call it a *minimal admissible module*. The reader can find more about analogous constructions of 2 step nilpotent Lie algebras, not related to representation of Clifford algebras in [Ebe04].

We emphasise the difference between an irreducible Clifford module and a minimal admissible module. Not all irreducible modules can be equipped with a non-degenerate bilinear symmetric form, satisfying (2.1). For instance, lack of dimension of an irreducible module can make any bilinear symmetric form degenerate. An admissible module V of $\mathrm{Cl}_{r,s}$ has an even dimension $\dim(V) = 2n = N$. It is isometric to $\mathbb{R}^{n,n}$ if $s > 0$ and it is isometric to $\mathbb{R}^{\pm N,0}$ if $s = 0$, see [Cia00, Theorem 3.1] and [FM17, Proposition 1]. Any admissible $\mathrm{Cl}_{r,s}$-module can be decomposed into an orthogonal direct sum of minimal admissible modules [FM19, Proposition 2.3 (2)].

2.2. **Pseudo H-type Lie algebras and Lie groups.**

Definition 2.1. *Let $(V, \langle . \, , . \rangle_V)$ be an admissible module of a Clifford algebra $Cl_{r,s}$ with the representation map J. Define the Lie bracket on $V \times \mathbb{R}^{r,s}$ by*

$$\langle J_z u, v\rangle_V = \langle z, [u, v]\rangle_{r,s}, \quad z \in \mathbb{R}^{r,s}, \quad u, v \in V. \tag{2.2}$$

The pseudo H-type Lie algebra $\mathfrak{n}_{r,s}(V) = (V \oplus \mathbb{R}^{r,s}, [. \, , .])$ is a Lie algebra whose non-vanishing Lie bracket is defined in (2.2)*.*

Note that the Lie algebra $\mathfrak{n}_{r,s}(V)$ is 2-step nilpotent where $\mathbb{R}^{r,s}$ is the centre. Property (2.1) and the representation property $J_z^2 v = -\langle z, z\rangle_{r,s} v$ for $v \in V$ imply

$$\langle J_z u, J_z v\rangle_{r,s} = \langle z, z\rangle_{r,s}\langle u, v\rangle_V, \quad \langle J_z u, J_w u\rangle_{r,s} = \langle z, w\rangle_{r,s}\langle u, u\rangle_V. \tag{2.3}$$

The connected simply connected Lie group $\mathbb{N}_{r,s}(V)$ of the Lie algebra $\mathfrak{n}_{r,s}(V)$ is called the pseudo H-type Lie group. The exponential map $\exp\colon \mathfrak{n}_{r,s}(V) \to \mathbb{N}_{r,s}(V) \cong V \times \mathbb{R}^{r,s}$ is a global analytic diffeomorphism [CG90, Theorem 1.2.1]. It allows to induce the coordinates on the Lie group from the Lie algebra by means of Backer-Campbell-Hausdroff formula. Points $g \in \mathbb{N}_{r,s}(V)$ are considered as vectors $g = u \oplus z \in V \oplus \mathbb{R}^{r,s} = \mathfrak{n}_{r,s}(V)$. The group product $*$ on $\mathbb{N}_{r,s}(V)$ is given by

$$\begin{aligned} &*\colon \mathbb{N}_{r,s}(V) \times \mathbb{N}_{r,s}(V) \to \mathbb{N}_{r,s}(V), \\ &(u_1, z_1) * (u_2, z_2) = \Big(u_1 + u_2, \ z_1 + z_2 + \frac{1}{2}[u_1, u_2]\Big). \end{aligned}$$

2.3. **Automorphisms of pseudo H-type Lie algebras.** Since automorphisms of a Lie algebra define the automorphisms of its connected simply connected Lie group, we consider only the automorphisms of Lie algebras. The complete description of the group of automorphisms of pseudo H-type Lie algebras can be found in [Rie82, Saa96, FM21], see also [AS14].

The automorphisms of pseudo H-type Lie algebras are generated by the following ones:

[1] The dilations $\delta_\lambda(u,z) = (\lambda u, \lambda^2 z)$.

[2] Let $A\colon V \to V$ be a nonsingular linear map and $C \in \mathrm{O}(r,s)$ an orthogonal transformation of $\mathbb{R}^{r,s}$. Then the map $A \oplus C$ is a pseudo H-type Lie algebra automorphism, if and only if

$$A^\tau \circ J_z \circ A = J_{C^\tau(z)}, \quad z \in \mathbb{R}^{r,s}, \tag{2.4}$$

where A^τ, C^τ are transpose maps defined as

$$\langle A^\tau u, v\rangle_V = \langle u, Av\rangle_V, \quad \langle C^\tau z, w\rangle_{r,s} = \langle z, Cw\rangle_{r,s}.$$

[3] Let $B\colon V \to \mathbb{R}^{r,s}$ be a linear map, then $(v,z) \mapsto \big(v, z+Bv\big)$ is an automorphism.

Remark 2.1. *If $A \oplus C$ is a pseudo H-type Lie algebra automorphism, then not only $A^{-1} \oplus C^{-1}$, but $A^\tau \oplus C^{-1}$ is also a pseudo H-type Lie algebra automorphism.*

2.4. **Rational structures, uniform discrete subgroups, lattices.** We refer to works [Rag72, CG90] for the details discussed in this section.

Definition 2.2. *A Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ over rational numbers $\mathbb{Q}$ is called the rational structure of a real Lie algebra $\mathfrak{g}$ if $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$.*

A real Lie algebra $\mathfrak{g}$ has a rational structure if and only if there is a basis for $\mathfrak{g}$ such that the structure constants of the Lie algebra are rational numbers.

Definition 2.3. *Let G be a Lie group. A subgroup Γ is called uniform subgroup if Γ is discrete and G/Γ is a compact space.*

Definition 2.4. *Let G be a Lie group with a measure μ. A subgroup Λ is called lattice if $\mu(G/\Lambda) < \infty$.*

Let G be a nilpotent Lie group and μ the Haar measure on it. Then a discrete subgroup Γ is lattice if and only if it is a uniform subgroup, i.e $\mu(G/\Gamma) < \infty$ implies that G/Γ is compact. From now on we will not distinguish the lattices and uniform subgroups. A result from [Mc49] can be formulated as follows.

- If Γ is a uniform subgroup of G, then $\mathfrak{g}$ has a rational structure $\mathfrak{g}_{\mathbb{Q}}$ such that $\mathfrak{g}_{\mathbb{Q}} = \mathrm{span}_{\mathbb{Q}}\{\log(\Gamma)\}$.
- If $\mathfrak{g}$ has a rational structure $\mathfrak{g}_{\mathbb{Q}}$, then G has a uniform subgroup Γ such that $\log(\Gamma) \subseteq \mathfrak{g}_{\mathbb{Q}}$.

Theorem 2.5. [Rag72] *Let $\Gamma_i \subset G_i$, $i = 1, 2$, be uniform subgroups of simply connected nilpotent Lie groups G_i. An isomorphism $\varphi\colon \Gamma_1 \to \Gamma_2$ of discrete subgroups, can be extended to the smooth isomorphism $\tilde{\varphi}\colon G_1 \to G_2$ of the Lie groups.*

3. Invariant bases and groups of positive involutions

3.1. **Definition of invariant integral structure and uniform subgroups.** From now on we will consider only **minimal admissible modules** of Clifford algebras $\mathrm{Cl}_{r,s}$, denoting them either by $V^{r,s}$ or simply by V. Let $\mathfrak{n}_{r,s}(V) = (V \oplus \mathbb{R}^{r,s}, [.\,,.])$ be a pseudo H-type Lie algebra with $B_{r,s}$ a basis for $\mathbb{R}^{r,s}$ and $\mathfrak{B}(V)$ a basis for V. We write the structure constants c_{ij}^l for $\mathfrak{n}_{r,s}(V)$ with respect to bases $\mathfrak{B}(V)$ and $B_{r,s}$ by

$$[v_i, v_j] = \sum_{l=1}^{r+s} c_{ij}^l z_l. \tag{3.1}$$

Definition 3.1. *A basis $\{\mathfrak{B}(V), B_{r,s}\}$ for $\mathfrak{n}_{r,s}(V)$ is called integral if the structure constants c_{ij}^l in* (3.1) *take the values in $\{-1, 0, 1\}$.*

We want to study a special class of integral bases of $\mathfrak{n}_{r,s}(V)$. To describe it, we fix an orthonormal basis $B_{r,s} = \{z_1, \dots, z_r, z_{r+1}, \dots, z_{r+s}\}$ of $\mathbb{R}^{r,s}$, where

$$\begin{cases} z_1, \dots, z_r \quad \text{are positive, i.e., } \langle z_i,\, z_i\rangle_{r,s} = 1, \quad i = 1, \dots, r, \\ z_{r+1}, \dots, z_{r+s} \text{ are negative, i.e., } \langle z_i,\, z_i\rangle_{r,s} = -1, \quad j = r+1, \dots, r+s. \end{cases} \tag{3.2}$$

Consider a finite subgroup $G(B_{r,s})$ of the Pin group in $\mathrm{Cl}_{r,s}$ defined by

$$G(B_{r,s}) = \{ \ \pm\mathbf{1},\ \pm z_1,\ \dots,\ \pm z_{r+s},\ \dots,\ \pm z_{i_1} \cdots z_{i_k} \mid \\ 1 \le i_1 < \cdots < i_k \le r+s, \quad k = 2, \dots, r+s \}.$$

Thus the generators of the group $G(B_{r,s})$ are $\{-\mathbf{1}, B_{r,s}\}$. Elements $\sigma \in G(B_{r,s})$ satisfy the properties: either $\sigma^2 = \mathbf{1}$ or $\sigma^2 = -\mathbf{1}$.

We proceed to the construction of bases $\mathfrak{B}(V^{r,s})$ for the minimal admissible module $V^{r,s}$. In Table 1 the reader finds dimensions of $V^{r,s}$ for $0 \le r, s \le 8$, which is extended by periodicity. We marked by red colour the Clifford algebras, where the minimal admissible modules differ from the irreducible modules. With the subscript $_{\times 2}$ we indicated the presence of two non-equivalent minimal admissible modules.

Table 1. Dimensions of minimal admissible modules

8	16	32	64	$64_{\times 2}$	128	128	128	$128_{\times 2}$	256
7	16	32	64	64	128	128	128	128	256
6	16	$16_{\times 2}$	32	32	64	$64_{\times 2}$	128	128	256
5	16	16	16	16	32	64	128	128	256
4	8	8	8	$8_{\times 2}$	16	32	64	$64_{\times 2}$	128
3	8	8	8	8	16	32	64	64	128
2	4	$4_{\times 2}$	8	8	16	$16_{\times 2}$	32	32	64
1	2	4	8	8	16	16	16	16	32
0	1	2	4	$4_{\times 2}$	8	8	8	$8_{\times 2}$	16
s/r	0	1	2	3	4	5	6	7	8

(1) If a minimal admissible module $V^{r,s}$ is irreducible, then the set

$$O_v = G(B_{r,s}).v := \{J_\sigma v \mid \ \sigma \in G(B_{r,s})\} \tag{3.3}$$

contains a basis $\mathfrak{B}(V^{r,s})$ for any non-zero vector $v \in V^{r,s}$.

(2) If a minimal admissible module $V^{r,s}$ is reducible, then set (3.3) contains a basis $\mathfrak{B}(V^{r,s})$ for any non-zero and non-null vector $v \in V^{r,s}$.

Thus, $V^{r,s} = \mathrm{span}_{\mathbb{R}}\{O_v\} = \mathrm{span}_{\mathbb{R}}\{\mathfrak{B}(V^{r,s})\}$ for a non-null vector $v \neq 0$. If $v \in V^{r,s}$ is a null vector, then the orbit O_v depends on the choice of v, but even in this case, one can make a special choice of a null vector $v \in V^{r,s}$, that generates an entire orbit O_v including $\mathfrak{B}(V^{r,s})$. From the other side if $V^{r,s} = V_1^{r,s} \oplus V_2^{r,s}$ is a decomposition of a minimal admissible module on irreducible modules, then the bilinear form $\langle . , . \rangle_{V^{r,s}}$ vanishes identically on $V_i^{r,s}$, $i = 1, 2$. In this case only the union $\bigcup_{i=1}^2 \big\{ J_\sigma v_i \mid \sigma \in G(B_{r,s}) \big\}$ contains a basis $\mathfrak{B}(V^{r,s})$, where one needs to choose two non-zero vectors $v_i \in V_i^{r,s}$.

Based on the latter discussions we restrict ourselves at bases $\mathfrak{B}(V^{r,s})$ consisting of non-null vectors and make the following definition.

Definition 3.2. *Fix an orthonormal basis $B_{r,s}$ of $\mathbb{R}^{r,s}$. An orthonormal basis $\mathfrak{B}(V^{r,s})$ of a minimal admissible module $V^{r,s}$ is called invariant basis if it is invariant under the action of $G(B_{r,s})$; that is for any $v_i \in \mathfrak{B}(V^{r,s})$ and $z_j \in B_{r,s}$, there exists $v_k \in \mathfrak{B}(V^{r,s})$ such that $J_{z_j} v_i = v_k$ or $J_{z_j} v_i = -v_k$.*

According to Definition 3.2 the maps J_{z_j}, $z_j \in B_{r,s}$ act on an invariant basis $\mathfrak{B}(V^{r,s})$ by permutations up to the sign $\pm$.

Remark 3.1. *We emphasise that Definition 3.2 requires bases $\mathfrak{B}(V^{r,s})$ to be both orthonormal and invariant.*

Example A. *Consider the Heisenberg Lie algebra $\mathfrak{n}_{1,0}(V)$ with the normalized basis $B_{1,0} = \{z\}$ for the centre and $V^{1,0} \cong \mathbb{R}^{2,0}$. Set $v_1 \in V^{1,0}$, $v_2 = J_z v_1$, and*

$$u_1 = A v_1, \quad u_2 = A v_2,$$

where A is an orthogonal transformation of $V^{1,0}$. Then the basis $(V^{1,0}) = \{u_1, u_2\}$ is orthonormal. It is invariant under the action of $G(B_{1,0})$ if and only if J_z commutes with A. Thus a basis $\mathfrak{B}(V^{1,0})$ can be orthonormal, but not invariant.

Example B. *Consider the Lie algebra $\mathfrak{n}_{0,3}(V)$ with an orthonormal basis $B_{0,3} = \{z_1, z_2, z_3\}$ for the centre and a minimal admissible module $V^{0,3} \cong \mathbb{R}^{4,4}$ of the Clifford algebra $Cl_{0,3}$. We take $v \in V^{0,3}$, such that $\langle v, v \rangle_{V^{0,3}} = 1$. The eight vectors*

$$v,\ J_{z_1}v,\ J_{z_2}v,\ J_{z_3}v,\ J_{z_1}J_{z_2}v,\ J_{z_1}J_{z_3}v,\ J_{z_2}J_{z_3}v,\ J_{z_1}J_{z_2}J_{z_3}v \tag{3.4}$$

are linearly independent, have square of the norm equal to ± 1, and invariant under the action of $G(B_{0,3})$. Nevertheless, the value $\langle v, J_{z_1}J_{z_2}J_{z_3}v \rangle_{V^{0,3}}$ depends on the choise of $v \in V$, see [FM14, Lemmas 2.8, 2.9]. *The basis $\mathfrak{B}(V^{0,3})$ is invariant, but not necessary orthonormal.*

Proposition 3.3. *Let $\mathfrak{B}(V^{r,s})$ be an invariant basis. Then it is an integral basis.*

Proof. We claim that for any $v \in V^{r,s}$ with $\langle v, v \rangle_{V^{r,s}} \neq 0$ we have:

$$J_{z_i} v = \pm J_{z_j} v, \quad \Longrightarrow \quad z_i = z_j. \tag{3.5}$$

Indeed, (3.5) implies $J_{z_i}J_{z_j}v = \pm v$ and therefore $\big(J_{z_i}J_{z_j}\big)^2 v = v$. Assume by contrary that $z_i \neq z_j$. Suppose first that both z_i and z_j are positive or negative. Then $\big(J_{z_i}J_{z_j}\big)^2 = -J_{z_i}^2 J_{z_j}^2 = -\mathrm{Id}$, which is a contradiction. From the other side, if z_i and z_j are opposite, then

$$\langle \pm v, \pm v\rangle_{V^{r,s}} = \langle J_{z_i}J_{z_j}v, J_{z_i}J_{z_j}v\rangle_{V^{r,s}} = \langle z_i, z_i\rangle_{r,s}\langle z_j, z_j\rangle_{r,s}\langle v, v\rangle_{V^{r,s}} = -\langle v, v\rangle_{V^{r,s}}$$

by (2.3), and v must be a null vector, which is again a contradiction.

Assume now that $\mathfrak{B}(V^{r,s})$ is an invariant basis for $V^{r,s}$ and that $J_{z_\ell}v_i = \pm v_k$ for $v_i, v_k \in \mathfrak{B}(V^{r,s})$. Then, by definition of the Lie bracket (2.2), we obtain

$$\langle z_\ell, [v_i, v_j]\rangle_{r,s} = \langle J_{z_\ell}v_i, v_j\rangle_{V^{r,s}} = \langle \pm v_k, v_j\rangle_{V^{r,s}} = \pm\delta_{kj}.$$

If $k = j$, then the orthonormality of $B_{r,s}$ and $\langle z_\ell, [v_i, v_j]\rangle_{r,s} = \pm 1$ imply that $[v_i, v_j] = \pm z_\ell$, and the structure constants in (3.1) are such that $c_{ij}^{\ell} = \pm 1$. If $k \neq j$ then $c_{ij}^{\ell} = 0$. □

The definition of an invariant basis leads to the definition of an invariant integral structure on pseudo H-type Lie algebras and (invariant) integral uniform subgroup on the respective pseudo H-type Lie groups.

Definition 3.4. *Let $B_{r,s} = \{z_k\}_{k=1}^{r+s}$ be an orthonormal basis for $\mathbb{R}^{r,s}$ and $\mathfrak{B}(V^{r,s}) = \{v_i\}_{i=1}^{N}$ an invariant basis for a minimal admissible module $V^{r,s}$. An invariant integral structure on the pseudo H-type Lie algebra $\mathfrak{n}_{r,s}(V)$ is the vector space over $\mathbb{Z}$ given by*

$$\mathrm{span}_{\mathbb{Z}}\{\mathfrak{B}(V^{r,s})\} \oplus \mathrm{span}_{\mathbb{Z}}\{B_{r,s}\} = \Big\{ \sum_{i=1}^{N} n_i v_i \oplus \sum_{k=1}^{r+s} m_k z_k \;\Big|\; n_i, m_k \in \mathbb{Z}\Big\}.$$

An (invariant) integral uniform subgroup on the pseudo H-type Lie group $\mathbb{N}_{r,s}(V) = \{(v, z) \mid v \in V^{r,s}, z \in \mathbb{R}^{r,s}\}$ is given by the coordinates

$$\Bigg(\Big(\sum_{i=1}^{N} n_i v_i \mid n_i \in \mathbb{Z}\Big),\quad \Big(\frac{1}{2}\sum_{k=1}^{r+s} m_k z_k \mid m_k \in \mathbb{Z}\Big)\Bigg).$$

The main goal of the present work is the classification of invariant integral structures on pseudo H-type Lie algebras that give rise to a classification of integral uniform subgroups on the corresponding pseudo H-type Lie groups. Note that invariant integral structures is a subclass of integral (not necessary invariant and/or orthonormal) structures on pseudo H-type Lie algebras. In the present work we make a first step and classify only *invariant integral structures.* Classification of general integral structures and more general rational structures is postponed for the future works. In the article [GW86] the authors made a classification of rational uniform subgroups on the Heisenberg groups, where the starting point was a *unique* invariant integral basis of the Heisenberg algebra. Thus, in an essence, we make a first step towards the full classification of rational structures on two step nilpotent Lie algebras related to Clifford algebras.

3.2. **Subgroups $\mathcal{S} \subset G(B_{r,s})$ of positive involutions.** In the present section we study subgroups $\mathcal{S}$ of $G(B_{r,s}) \subset \mathrm{Cl}_{r,s}$ which will be a core for the construction of invariant bases $\mathfrak{B}(V^{r,s})$. Some of the properties of $\mathcal{S}$ can be learned from the definition of the subgroups $\mathcal{S}$, but some of them became clear by considering their action on minimal admissible modules $V^{r,s}$. The representation map $J\colon S \to End(V^{r,s})$ is not injective only for $r - s = 3 \mod 4$. In most of the proofs, we will concentrate on the module, where the map $J\colon S \to \mathrm{End}(V^{r,s})$ is injective.

Proofs for another non-equivalent module follow after a light modification of arguments.

Recall that the group $\mathrm{Pin}(r, s)$ consists of elements of the Clifford algebra $\mathrm{Cl}_{r,s}$ of the form

$$\sigma = x_{i_1} \cdots x_{i_k}, \quad \langle x_{i_j}, x_{i_j} \rangle_{r,s} = \pm 1. \tag{3.6}$$

The subgroup $\mathrm{Spin}(r, s) \subset \mathrm{Pin}(r, s)$ is generated by the even number of elements in (3.6). Thus the group $G(B_{r,s})$ is a finite subgroup of $\mathrm{Pin}(r, s)$.

Definition 3.5. *We denote by $\mathcal{S}$ a subgroup of $G(B_{r,s})$ satisfying the conditions*
- $(S1)$ $-\mathbf{1} \notin \mathcal{S}$;
- $(S2)$ $p \in \mathrm{Pin}(r, 0) \times \mathrm{Spin}(0, s)$;
- $(S3)$ $p^2 = \mathbf{1}$.

Elements $p \in \mathcal{S}$ are called positive involutions.

The name *positive involution* refers to the action of $p \in \mathcal{S}$ on $V^{r,s}$: if $\langle v, v\rangle_{V^{r,s}} > 0$ $\big(\langle v, v\rangle_{V^{r,s}} < 0\big)$ then $\langle J_p v, J_p v\rangle_{V^{r,s}} > 0$ $\big(\langle J_p v, J_p v\rangle_{V^{r,s}} < 0\big)$. We denote by $\mathbb{S}_{r,s}$ (or just $\mathbb{S}$), the set of all subgroups of $G(B_{r,s})$ satisfying Definition 3.5. This set is a partially ordered set with respect to the inclusion relation among subsets.

Remark 3.2. *The groups $\mathcal{S} \in \mathbb{S}_{r,s}$ are necessarily commutative.*

Example 3.1. *Consider $G(B_{4,0})$. Then the example of possible subgroups $\mathcal{S}$ are*

$$\mathcal{S}_1 = \{\mathbf{1}, z_1z_2z_3\},\ \mathcal{S}_2 = \{\mathbf{1}, z_1z_2z_4\},\ \mathcal{S}_3 = \{\mathbf{1}, z_1z_3z_4\},\ \mathcal{S}_4 = \{\mathbf{1}, -z_1z_2z_4\}$$

and

$$\mathcal{S}_5 = \{\mathbf{1}, z_1z_2z_3z_4\}.$$

The first four groups are isomorphic under the action of the orthogonal group $\mathrm{O}(4)$. A map $C \in \mathrm{O}(4)$ permutes the basis vectors $\{z_i\}$, $i = 1, 2, 3, 4$ or change their sign. All five groups are isomorphic as abelian groups of order 2. However, the roles of the first four and the last one are different in construction of an invariant basis for $\mathfrak{B}(V^{4,0})$.

To avoid the ambiguity occurring with the very similar groups $\mathcal{S}_2$ and $\mathcal{S}_4$, we define a bigger group.

Definition 3.6. *Let $\mathcal{S}$ be a group satisfying Definition 3.5. We denote by $\widehat{\mathcal{S}} \subset G(B_{r,s})$ the extended group*

$$\widehat{\mathcal{S}} = \mathcal{S} \cup \{-\sigma :\ \sigma \in \mathcal{S}\}.$$

In Example 3.1 $\mathcal{S}_2, \mathcal{S}_4$ are subgroups of $G(B_{4,0})$, where we fix the basis $\{z_1, z_2, z_3, z_4\}$. The subgroups $\mathcal{S}_2, \mathcal{S}_4$ are different, nevertheless

$$\widehat{\mathcal{S}_4} = \widehat{\mathcal{S}_2} = \{\pm\mathbf{1}, \pm z_1 z_2 z_4\}.$$

3.3. Generators for $\mathcal{S}$.

Definition 3.7. *We denote by $PI = \{p_i\}_{i=1}^{\ell}$, where $\ell = \#[PI]$ is the cardinality of the set PI, a subset in $G(B_{r,s})$ satisfying the conditions:*

$(PI1)$ $\mathbf{1} \notin PI$, *$p_i p_j = p_j p_i$ for $i \neq j$, and $p_i \in PI$ satisfy $(S2) - (S3)$ in Definition 3.5 for all $i = 1, \ldots, \ell$.*

$(PI2)$ *The vectors*

$$\{\mathbf{1}, p_1, \ \ldots, \ p_\ell, \ \ p_{i_1} \cdots p_{i_k} \mid \ 1 \leq i_1 < \ldots < i_k \leq \ell, \ k = 2, \ldots, \ell\} \tag{3.7}$$

are linearly independent in the vector space $\mathrm{Cl}_{r,s}$.

Proposition 3.8. *Condition $(PI2)$ is equivalent to the condition*

$(PI2)'$ *non of the products $p_{i_1} \cdots p_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq \ell$, $k = 1, \ldots, \ell$, is equal to* $\pm\mathbf{1}$.

Proof. The elements

$$\{\epsilon_0 \mathbf{1}, \ \epsilon_{i_1,\ldots,i_k} z_{i_1} \cdots z_{i_k}\} \ \subset \mathrm{Cl}_{r,s}, \tag{3.8}$$

form a basis for $\mathrm{Cl}_{r,s}$. Here $1 \leq i_1 < \cdots < i_k \leq r+s$, $k = 1, \ldots, r+s$, where ϵ_0 and $\epsilon_{i_1,\ldots,i_k}$ are equal to 1 or -1.

It is obvious that $(PI2)$ implies $(PI2)'$. Assume that condition $(PI2)'$ is fulfild for a collection $\mathcal{C}$. Then the collection $\mathcal{C}$ is a subfamily of linearly independent basis vectors from (3.8), and therefore they are linearly independent. □

As an example of a set PI we present the minimal length positive involutions, which can be classified in the following types:

$$T_1 \begin{cases} p = z_{i_1} z_{i_2} z_{i_3} z_{i_4}, & \text{where all } z_{i_k} \text{ are positive basis vectors;} \\ p = z_{i_1} z_{i_2} z_{i_3} z_{i_4}, & \text{where all } z_{i_k} \text{ are negative basis vectors;} \\ p = z_{i_1} z_{i_2} z_{i_3} z_{i_4}, & \text{where two } z_{i_k} \text{ are positive and two } z_{i_l} \\ & \qquad \text{are negative basis vectors;} \end{cases}$$
$$T_2 \begin{cases} q = z_{i_1} z_{i_2} z_{i_3}, & \text{where all } z_{i_k} \text{ are positive basis vectors;} \\ q = z_{i_1} z_{i_2} z_{i_3}, & \text{where one } z_{i_k} \text{ is positive and two } z_{i_l} \\ & \qquad \text{are negative basis vectors.} \end{cases} \tag{3.9}$$

An easy combinatorial computation shows that generally positive involutions can contain either $3 \mod 4$ or $4 \mod 4$ basis vectors. This observation inspires us to make a more general definition.

Definition 3.9. *A positive involution containing $4 \mod 4$ basis vectors is called a type T_1 involution. A positive involution containing $3 \mod 4$ basis vectors is called a type T_2 involution.*

Notation 3.1. *For an element $\sigma = \pm z_{i_1}\cdots z_{i_k} \in G(B_{r,s})$, we denote by $\mathfrak{b}(\sigma) = \{z_{i_1},\ldots,z_{i_k}\}$ the set of the vectors in the product σ, and by $|\mathfrak{b}(\sigma)|$ the number of the vectors in $\mathfrak{b}(\sigma)$. Analogously, $\mathfrak{b}^+(\sigma)$ $\big(\mathfrak{b}^-(\sigma)\big)$ is the set of positive (negative) vectors in σ and $|\mathfrak{b}^+(\sigma)|$ $(|\mathfrak{b}^-(\sigma)|)$ is the cardinality of the respective sets.*

Proposition 3.10. *The following properties can be easily verified.*

(A) *Two type T_1 involutions p_1 and p_2 commute if the number $|\mathfrak{b}(p_1)\cap\mathfrak{b}(p_2)|$ is even. The product p_1p_2 is an involution of type T_1.*
(B) *A type T_1 involution p and a type T_2 involution q commute if the number $|\mathfrak{b}(p)\cap \mathfrak{b}(q)|$ is even. The product pq is an involution of type T_2.*
(C) *Two type T_2 involutions q_1 and q_2 commute if the number $|\mathfrak{b}(q_1)\cap\mathfrak{b}(q_2)|$ is odd. The product q_1q_2 is an involution of type T_1.*

Proof. The proof is based on the Clifford algebra property

$$z_1z_2 + z_2z_1 = -2\langle z_1, z_2\rangle_{r,s}\mathbf{1}, \quad z_1, z_2 \in \mathbb{R}^{r,s},$$

which for orthogonal vectors z_1 and z_2 leads to $z_1z_2 = -z_2z_1$. □

Notation 3.2. *We denote by $\mathbb{PI}_{r,s}$ the collection of sets PI satisfying Definition 3.7. The set $\mathbb{PI}_{r,s}$ is partially ordered by the inclusion relation similar to $\mathbb{S}_{r,s}$. If $PI \in \mathbb{PI}_{r,s}$, then we denote by $\mathcal{S}(PI)$ a group generated by the set PI.*

Proposition 3.11. (1) *Let $PI \in \mathbb{PI}$. Then*

$$\mathcal{S}(PI) = \{\mathbf{1},\ p_1,\ldots,p_\ell,\ \ldots,\ p_{i_1}\cdots p_{i_k} |\ 1 \le i_1 < \cdots < i_k \le \ell = \#[PI] \tag{3.10}$$

is a group of order $\#[\,\mathcal{S}(PI)\,] = 2^\ell$ in $G(B_{r,s})$ and $\mathcal{S}(PI) \in \mathbb{S}$.

(2) *Conversely, let $\mathcal{S} \in \mathbb{S}$. Then there is a (non unique) set $PI \in \mathbb{PI}$ such that $\mathcal{S}(PI) = \mathcal{S}$.*
(3) *Let $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_\ell)$ be a tuple consisting of ±1, and $PI = \{p_i\}_{i=1}^{\ell} \in \mathbb{PI}_{r,s}$. Then $\varepsilon\cdot PI = \{\varepsilon_1p_1,\ldots,\varepsilon_\ell p_\ell\} \in \mathbb{PI}_{r,s}$ and $\widehat{\mathcal{S}(PI)} = \mathcal{S}\widehat{(\varepsilon\cdot PI)}$.*

Proof. Set in (3.7) is linearly independent and coincides with $\mathcal{S}(PI)$ in (3.10), therefore $\#[\mathcal{S}(PI)] = 2^\ell$. If p is in the set (3.7), then $-p$ is not in the set (3.7), which implies that $-\mathbf{1} \notin \mathcal{S}(PI)$. Any $p \in \mathcal{S}(PI)$ is a positive involution by definition of the set PI. We showed (1).

The second property will be proved by induction arguments with respect to the order of the group $\mathcal{S}$. Let $\mathcal{S} \in \mathbb{S}_{r,s}$ be given. Assume $p_1 \in \mathcal{S}$ and if there are no elements in $\mathcal{S}$ other than $\mathbf{1}, p_1$, then we can put $PI = \{p_1\}$ and $\mathcal{S}(PI) = \mathcal{S}$.

Assume now that there is a set $PI' = \{p_1,\ldots,p_\ell\}_{\ell\ge2}$ satisfying Definition 3.7. If

$$\mathcal{S}(PI') = \{\mathbf{1},\ p_1,\ldots,p_\ell,\ \ldots,\ p_{i_1}\cdots p_{i_k} \mid\ 1 \le i_1 < \cdots < i_k \le \ell,\ k = 1,\ldots,\ell\},$$

is a proper subset of $\mathcal{S}$, then there is a positive involution $q \in \mathcal{S}$ such that $q \notin \mathcal{S}(PI')$, and $q \neq \pm\mathbf{1}$. Consider the set of commuting involutions

$$\mathcal{S}(PI')\cdot q = \{q,\ p_1q,\ldots,p_\ell q,\ \ldots,\ p_{i_1}\cdots p_{i_k}q \mid\ 1 \le i_1 < \cdots < i_k \le \ell,\ k = 1,\ldots,\ell\}.$$

If $p_{i_1}\cdots p_{i_m}=p_{j_1}\cdots p_{j_{m'}}q$, then $q\in\mathcal{S}(PI')$, as a product of involutions $p_{j_1}\cdots p_{j_{m'}}$ and $p_{i_1}\cdots p_{i_m}$ from $\mathcal{S}(PI')$. Thus non of the elements in $\mathcal{S}(PI')$ can be written in the form $p_{j_1}\cdots p_{j_{m'}}q$ for $p_{j_1}\cdots p_{j_{m'}}\in\mathcal{S}(PI')$. If

$$p_{i_1}\cdots p_{i_k}\neq p_{j_1}\cdots p_{j_{k'}}\quad\text{for}\quad p_{i_1}\cdots p_{i_k},\ p_{j_1}\cdots p_{j_{k'}}\in\mathcal{S}(PI'),$$

then $p_{i_1}\cdots p_{i_k}q\neq p_{j_1}\cdots p_{j_{k'}}q$. So the set $PI''=PI'\cup\{q\}$ satisfies Definition 3.7.

Continuing the procedure, we find in finitely many steps a set PI satisfying Definition 3.7 such that $\mathcal{S}(PI)=\mathcal{S}$.

The proof of the last assertion is easily follows from Definition 3.7. □

3.4. **Relation of $\mathcal{S}$ and an isotropy subgroup $\mathcal{S}_v$.** Now we relate groups $\mathcal{S}$ of positive involutions with the isotropy subgroups $\mathcal{S}_v$ for $v\in V^{r,s}$ and show that they are in a close relation.

Proposition 3.12. *Let $v\in V^{r,s}$ be a non-null vector and let $\mathcal{S}_v$ denote the isotropy subgroup in $G(B_{r,s})$ of the vector v:*

$$\mathcal{S}_v=\{\sigma\in G(B_{r,s})\mid J_\sigma v=v\}.$$

Then $\mathcal{S}_v$ satisfies Definition 3.5.

Proof. Let us assume that $V^{r,s}$ is the Clifford module, where the volume for acts as identity. Then it is clear that $-\mathbf{1}\notin\mathcal{S}_v$. To check the second property we take $\sigma\in\mathcal{S}_v\subset G(B_{r,s})$ and assume by contrary that σ is a product containing an odd number of negative basis vectors from $B_{r,s}$. Then for $v\in V^{r,s}$ with $\langle v,v\rangle_{V^{r,s}}>0$ we obtain

$$0<\langle v,v\rangle_{V^{r,s}}=\langle J_\sigma v,J_\sigma v\rangle_{V^{r,s}}<0$$

by (2.3), which is a contradiction. Similar argument is applied for a vector $v\in V^{r,s}$ with $\langle v,v\rangle_{V^{r,s}}<0$. Hence $\sigma\in\mathrm{Pin}(r,0)\times\mathrm{Spin}(0,s)$.

The square of every element in $G(B_{r,s})$ equal either $\mathbf{1}$ or $-\mathbf{1}$. If $\sigma\in\mathcal{S}_v$, then $J_\sigma^2=\mathrm{Id}_{V^{r,s}}$. Hence $\sigma^2=\mathbf{1}$.

If $r-s=3\mod 4$ and S_v includes the volume form ω which acts as minus identity on $V^{r,s}$, then we change ω to $-\omega$ and the proof will follow. □

The relation of an arbitrary $\mathcal{S}$ to an isotropy group $\mathcal{S}_v$ for some $v\in V^{r,s}$ is given in the following statement.

Proposition 3.13. *Let $\mathcal{S}\in\mathbb{S}_{r,s}$ and $PI=\{p_1,\ldots,p_\ell\}\in\mathbb{PI}_{r,s}$ be such that $\mathcal{S}(PI)=\mathcal{S}$. Let $E^{+1}(p_k)=\{u\in V^{r,s}\mid J_{p_k}u=u\}$. Then the intersection $\bigcap_{k=1}^{\ell}E^{+1}(p_k)$ contains a non-null vector v. Moreover, the group $\mathcal{S}(PI)$ is the isotropy subgroup $\mathcal{S}_v$ of the vector v, and $\#[\mathcal{S}]=\#[\mathcal{S}_v]=2^{\#[PI]}$.*

Proof. Let $r-s\neq 3\mod 4$ and let $E^{+1}(p_k)$, $E^{-1}(p_k)$ be the eigenspaces of an involution J_{p_k} with eigenvalue 1 and -1, respectively. If one of the spaces $E^{\pm1}(p_k)$ is trivial, then the symmetric bi-linear form $\langle .\,,.\rangle_{V^{r,s}}$ on the non-trivial subspace is non-degenerate. If both of $E^{\pm1}(p_k)$ are non-trivial spaces, then they are orthogonal with respect to $\langle .\,,.\rangle_{V^{r,s}}$ and the restriction of $\langle .\,,.\rangle_{V^{r,s}}$ onto $E^{\pm1}(p_k)$ is non-degenerate too.

Assume $E^{+1}(p_1) \neq \{0\}$. Then the space $E^{+1}(p_1)$ is invariant under the action of the involution J_{p_2}. Therefore, $E^{+1}(p_1) \bigcap E^{+1}(p_2) \neq \{0\}$. By repeating the procedures we get that $E = \bigcap_{k=1}^{\ell} E^{+1}(p_k) \neq \{0\}$ and the restriction of $\langle . , . \rangle_{V^{r,s}}$ onto E is non-degenerate. Thus there is a non-null vector $v \in E$ such that $J_{p_k} v = v$ for all $k = 1, \ldots, \ell$. Hence $\mathcal{S}(PI) = \mathcal{S}_v$.

If $r - s = 3 \mod 4$, then without loss of generality we can assume that J_{p_1} acts as $-\mathrm{Id}$. We change p_1 to $-p_1$ to get $E^{+1}(p_1) = \{u \in V^{r,s} \mid J_{p_1} u = u\}$ and continue the proof as above. □

Corollary 3.14. *Let $\mathcal{S} \in \mathbb{S}_{r,s}$, and let $\mathcal{S}_v = \mathcal{S}$ be an isotropy subgroup of v as in Proposition 3.13. The orbit $O_v = G(B_{r,s}).v$, defined in* (3.3)*, contains an invariant basis $\mathfrak{B}(V^{r,s})$ of the minimal admissible module $V^{r,s}$. There is no canonical way to prescribe the direction u or $-u$ for a basis vector in $\mathfrak{B}(V^{r,s})$. Therefore O_v is a set of basis vectors counted with signes $\pm$. Hence $G(B_{r,s})/S_v \cong G(B_{r,s}).v$ and $\dim(V^{r,s}) = \frac{1}{2}\#[G(B_{r,s}).v]$.*

Proof. If the group $\mathcal{S}_v$ is an isotropy subgroup of an invariant basis, then

$$\#[\mathcal{S}_v] \cdot \#[G(B_{r,s}).v] = 2^{r+s+1} = \#[G(B_{r,s})]. \tag{3.11}$$

Since the module is minimal admissible and the basis vectors are counted twice (with $\pm$ signs), we conclude $\#[G(B_{r,s}).v] = 2\dim(V^{r,s})$. Therefore $\dim(V^{r,s}) = 2^{r+s-\ell}$, where $\ell = \#[PI]$ for $\mathcal{S} = \mathcal{S}(PI)$. □

Notation 3.3. *Let $\mathbb{S}^M_{r,s}$ and $\mathbb{PI}^M_{r,s}$ be subsets of $\mathbb{S}_{r,s}$, respectively $\mathbb{PI}_{r,s}$ consisting of maximal number of the involutions. Then $PI \in \mathbb{PI}^M_{r,s}$ if and only if $\mathcal{S}(PI) \in \mathbb{S}^M_{r,s}$, although the correspondence $PI \mapsto \mathcal{S}(PI)$ is not injective. We denote by $\ell(r,s)$ the maximal number of involutions in $\mathbb{PI}^M_{r,s}$. The value $\ell(r,s)$ depends only on the signature (r,s) and it satisfies $2^{\ell(r,s)} = \frac{2^{r+s}}{\dim(V^{r,s})}$ by Corollary 3.14.*

Proposition 3.15. *The number $\ell(r,s)$ has three periodicities:*

$$\begin{aligned} \ell(r+8,s) &= \ell(r,s+8) = \ell(r+4,s+4) = \ell(r,s)+4 \\ &= \ell(r,s)+\ell(8,0) = \ell(r,s)+\ell(0,8) = \ell(r,s)+\ell(4,4). \end{aligned}$$

Proof. The number $\ell(r,s)$ is determined by $2^{\ell(r,s)} \cdot \dim(V^{r,s}) = 2^{r+s}$. Hence,

$$2^{\ell(r+8,s)} \cdot \dim(V^{r+8,s}) = 2^{r+8+s} = 2^{r+s}2^8 = 2^{\ell(r,s)} \cdot \dim(V^{r,s}) \cdot 2^8.$$

We know that $\dim(V^{r+8,s}) = 2^4 \dim(V^{r,s})$, see [FM17, Section 4.1]. Hence it holds $\ell(r+8,s) = \ell(r,s)+4$.

Other equalities can be shown by the same arguments. □

Proposition 3.16.

$$\ell(r,s) \leq \ell(r+s,0), \quad \ell(r,s) \leq \ell(r+1,s), \quad \ell(r,s) \leq \ell(r,s+1). \tag{3.12}$$

Proof. The inequalities follow from the structure of involutions in (3.9). □

The orbit $O_v = G(B_{r,s}).v$ gives the invariant basis for $V^{r,s}$ up to a sign. Since the elements in $G(B_{r,s})$ either commute or anti-commute with elements in $\mathcal{S}_v$, we can describe the construction of an invariant basis for a minimal admissible module $V^{r,s}$.

Theorem 3.17. *Let $v \in V^{r,s}$ be a unit vector from Proposition 3.13. There is a set $\Sigma \subset G(B_{r,s})$ such that the family $\{J_\sigma v\}_{\sigma\in\Sigma}$ is an invariant basis of $V^{r,s}$.*

Proof. Let $\mathcal{S}_v \in \mathbb{S}^M_{r,s}$. We fix a maximal set $PI_{r,s} = \{p_i\}_{i=1}^{\ell(r,s)}$ such that $\mathcal{S}(PI_{r,s}) = \mathcal{S}_v$ and write $E^{\varepsilon_i}(p_i) = \{v \in V^{r,s} \mid J_{p_i}v = \varepsilon_i v\}$, where ε_i is either $+1$ or -1. We denote $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{\ell(r,s)})$ and define

$$E = \bigcap_{i=1}^{\ell(r,s)} E^{+1}(p_i), \quad E^{\varepsilon_1,\dots,\varepsilon_{\ell(r,s)}} = \bigcap_{i=1}^{\ell(r,s)} E^{\varepsilon_i}(p_i). \tag{3.13}$$

Before we continue the proof we note that $\dim(E) \in \{1, 2, 4, 8\}$, and either $\dim(V^{r,s}) = \dim(E) \times 2^{\ell(r,s)}$ or $\dim(V^{r,s}) = \dim(E) \times 2^{\ell(r,s)-1}$. In the latter case, one involution J_{p_i} acts as Id or $-$Id on $V^{r,s}$, which happens if $r - s = 3 \mod 4$, see details in [FM21]. Thus

$$\dim(E) = 2^{r+s-2\ell(r,s)} \quad \text{or} \quad \dim(E) = 2^{r+s-2(\ell(r,s)-1)}.$$

Let $\mathbf{C}_{G(B_{r,s})}(\mathcal{S}(PI_{r,s}))$ be the centralizer of $\mathcal{S}(PI_{r,s})$ in $G(B_{r,s})$ and $v \in E$ a unit vector. Then we can find representatives $\{\sigma_i\}_{i=1}^{\dim(E)} \in \mathbf{C}_{G(B_{r,s})}\big(\mathcal{S}(PI_{r,s})\big)/\widehat{\mathcal{S}(PI_{r,s}}),$ and $\{\tau_j\}_{j=1}^{2^{\ell(r,s)}} \in G(B_{r,s})/\mathbf{C}_{G(B_{r,s})}\big(\mathcal{S}(PI_{r,s})\big)$ such that

$$\text{the vectors} \quad \{J_{\sigma_i}v\}_{i=1}^{\dim(E)} \quad \text{form an orthonormal basis for } E,$$
$$\text{the vectors} \quad \{J_{\tau_j}J_{\sigma_i}v\}_{i=1}^{\dim(E)}{}_{j=1}^{2^{\ell(r,s)}} \quad \text{form an orthonormal basis for } V^{r,s}.$$

These $\{\sigma_i\}_{i=1}^{\dim(E)}$ and $\{\tau_j\}_{j=1}^{2^{\ell(r,s)}}$ form the set Σ. □

Theorem 3.18. *Fix the group $\mathcal{S}(PI_{r,s})$ and the representatives*

$$\{\sigma_i\}_{i=1}^{\dim(E)} \in \mathbf{C}_{G(B_{r,s})}\big(\mathcal{S}(PI_{r,s})\big)/\widehat{\mathcal{S}(PI_{r,s}}),$$

$$\{\tau_j\}_{j=1}^{2^{\ell(r,s)}} \in G(B_{r,s})/\mathbf{C}_{G(B_{r,s})}\big(\mathcal{S}(PI_{r,s})\big).$$

Assume that $v_1, v_2 \in E$ generate two sets of invariant bases

$$\mathfrak{B}_{v_k}(V^{r,s}) = \{v_k,\ J_{\sigma_i}v_k,\ J_{\tau_j}v_k,\ J_{\tau_j}J_{\sigma_i}v_k\}_{i=1}^{\dim(E)}{}_{j=1}^{2^{\ell(r,s)}}, \quad k = 1, 2,$$

as in Theorem 3.17. Then the invariant integral structures

$$\begin{aligned} &\text{span}_{\mathbb{Z}}\{\mathfrak{B}_{v_1}(V^{r,s})\} \oplus \text{span}_{\mathbb{Z}}\{B_{r,s}\} \\ &\text{span}_{\mathbb{Z}}\{\mathfrak{B}_{v_2}(V^{r,s})\} \oplus \text{span}_{\mathbb{Z}}\{B_{r,s}\} \end{aligned} \tag{3.14}$$

are isomorphic.

Proof. We define the correspondence $A\colon \mathfrak{B}_{v_1}(V^{r,s}) \to \mathfrak{B}_{v_2}(V^{r,s})$ by

$$
\begin{array}{ll}
v_1 \mapsto v_2, & J_{\sigma_i} v_1 \mapsto J_{\sigma_i} v_2, \\
J_{\tau_j} v_1 \mapsto J_{\tau_j} v_2, & J_{\tau_j} J_{\sigma_i} v_1 \mapsto J_{\tau_j} J_{\sigma_i} v_2,
\end{array}
\tag{3.15}
$$

and extend it by linearity over $\mathbb{Z}$. Then the map $A \oplus \mathrm{Id}$ is an automorphism of invariant integral structures (3.14). To show that $A \oplus \mathrm{Id}$ is an isomoprhism, we denote the basis vectors from $\mathfrak{B}_{v_1}(V^{r,s})$ by $\{u_\alpha\}_{\alpha=1}^{\dim(V^{r,s})}$ and the basis vectors from $\mathfrak{B}_{v_2}(V^{r,s})$ by $\{w_\alpha\}_{\alpha=1}^{\dim(V^{r,s})}$, where $w_\alpha = Au_\alpha$. Then we note that the bases $\mathfrak{B}_{v_1}(V^{r,s})$ and $\mathfrak{B}_{v_2}(V^{r,s})$ are invariant, which means that for any $u_\alpha \in \mathfrak{B}_{v_1}(V^{r,s})$ and any $z_k \in B_{r,s}$ there is $u_\beta \in \mathfrak{B}_{v_1}(V^{r,s})$ such that

$$
J_{z_k} u_\alpha = \pm u_\beta = \pm J_\varkappa v_1, \quad \text{for some} \quad \varkappa \in \Sigma = \{\sigma_i, \tau_j, \tau_j \sigma_i\}. \tag{3.16}
$$

The correspondence (3.15) and (3.16) imply that for chosen $u_\alpha \in \mathfrak{B}_{v_1}(V^{r,s})$ and $z_k \in B_{r,s}$ we have

$$
J_{z_k} A u_\alpha = J_{z_k} w_\alpha = \pm w_\beta = \pm J_\varkappa v_2 = \pm A J_\varkappa v_1 = A J_{z_k} u_\alpha .
$$

Note also that $A^\tau A = \mathrm{Id}_{V^{r,s}}$ since it maps an othonormal basis to an orthonormal basis. Then we have

$$
\begin{aligned}
\langle [Au_\alpha, Au_\beta], z_k \rangle_{r,s} &= \langle J_{z_k} A u_\alpha, A u_\beta \rangle_{V^{r,s}} = \langle A J_{z_k} u_\alpha, A u_\beta \rangle_{V^{r,s}} \\
&= \langle A^\tau A J_{z_k} u_\alpha, u_\beta \rangle_{V^{r,s}} = \langle J_{z_k} u_\alpha, u_\beta \rangle_{V^{r,s}} \\
&= \langle [u_\alpha, u_\beta], z_k \rangle_{r,s}.
\end{aligned}
\tag{3.17}
$$

□

3.5. **Equivalence of groups $\mathcal{S}$.** We define an equivalence relation between groups $\mathcal{S} \subset G(B_{r,s})$ that will descend to the equivalence of their generating sets $PI_{r,s}$. We also introduce signatures to distinguish sets $PI_{r,s}$ for a fixed value (r,s). Different signatures will lead to non-equivalent generating sets $PI_{r,s}$ and the groups $\mathcal{S} = \mathcal{S}(PI_{r,s})$. Our aim is to show that equivalent groups $\mathcal{S} \subset G(B_{r,s})$ lead to the isomorphic invariant integral structures on the Lie algebras $\mathfrak{n}_{r,s}$.

We recall Notation 3.1 and extend it to the sets PI.

Notation 3.4. *Let $PI \in \mathbb{PI}_{r,s}$. We denote*

$$
\mathfrak{b}^+(PI) = \{z_i \mid \ z_i \text{ is a positive vector in some } p_i \in PI\},
$$
$$
\mathfrak{b}^-(PI) = \{z_i \mid \ z_i \text{ is a negative vector in some } p_i \in PI\}.
$$

We set also $|\mathfrak{b}^+(PI)|$, $|\mathfrak{b}^-(PI)|$ for the cardinality of the respective set, and $|\mathfrak{b}(PI)| = |\mathfrak{b}^+(PI)| + |\mathfrak{b}^-(PI)|$.

Definition 3.19. *A set PI consisting only of the involutions of type T_1 will be called $(T1)$-type set. A set PI consisting of the involutions of type T_1 and having at least one involution of type T_2 will be called $(T2)$-type set.*

Proposition 3.20. *Any $(T2)$-type set can be reduced to $(T2)$-type set containing at most one involution of type T_2 and the rest of involutions will be of type T_1.*

Proof. The proof follows directly from Proposition 3.10. □

Notation 3.5. *If $C \in \mathrm{O}(r,s)$, then we denote by the same letter C its natural extension $C\colon Cl^*_{r,s} \to Cl^*_{r,s}$ to the action on the group of invertible elements $Cl^*_{r,s} \subset Cl_{r,s}$.*

Let $B_{r,s}$ be a basis as in (3.2). Let $C \in \mathrm{O}(r,s)$. Then C is a signed permutation matrix for $B_{r,s}$ having only one nonzero component " ± 1" in each column. We call such a map *(signed) re-ordering* of $B_{r,s}$. If $\sigma = z_{i_1}\cdots z_{i_k} \in G(B_{r,s})$, then C defines an element $C(\sigma) := C(z_{i_1})\cdots C(z_{i_k}) \in G(B_{r,s})$. Since a re-ordering matrix C maps positive basis vectors to positive vectors and negative basis vectors to negative basis vectors, it induces a map $C\colon \mathbb{PI}_{r,s} \to \mathbb{PI}_{r,s}$. For the particular case (r,r) the map C can be chosen also to map positive basis vectors to negative vectors and vice versa. The changes for (r,r) will be discussed separately in a forthcoming paper.

Definition 3.21. *We say that the groups $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent, writing $\mathcal{S}_1 \sim \mathcal{S}_2$, if there is a map $C \in \mathrm{O}(r,s)$ such that its natural extention to $Cl^*_{r,s} \subset Cl_{r,s}$ gives the isomorphism between the extended groups $\widehat{\mathcal{S}_1}$ and $\widehat{\mathcal{S}_2}$; that is $C(\widehat{\mathcal{S}_1}) = \widehat{\mathcal{S}_2}$.*

Definition 3.22. *Let PI_1 and PI_2 be two sets of involutions. Then we say that PI_1 and PI_2 are equivalent, writing $PI_1 \sim PI_2$, if $\mathcal{S}(PI_1)$ is equivalent to $\mathcal{S}(PI_2)$ in the sense of Definition 3.21.*

Example 3.2. *Recall Example 3.1 and consider $G(B_{4,0})$. Then $PI_1 = \{z_1z_2z_3\}$ and $PI_2 = \{z_1z_2z_4\}$ are equivalent, nevertheless PI_1 is not equivalent to $PI_5 = \{z_1z_2z_3z_4\}$.*

Example 3.3. *In this example we present a construction of a sequence of subgroups that will be important in Section 4. We call these subgroups standard. Let $B_{r,s}$ be an orthonormal basis of $\mathbb{R}^{r,s}$ as in (3.2). We form a set of mutually different pairs*

$$\pi_{i,j} = z_iz_j, \quad i<j, \quad i,j \in \begin{cases} \{1,\dots,r\} & \text{if } r \text{ is even} \\ \{1,\dots,r-1\} & \text{if } r \text{ is odd} \end{cases}, \tag{3.18}$$

$$\nu_{k,l} = z_kz_l, \quad k<l, \quad k,l \in \begin{cases} \{r+1,\dots,s\} & \text{if } s \text{ is even} \\ \{r+1,\dots,s-1\} & \text{if } s \text{ is odd} \end{cases}, \tag{3.19}$$

and

$$\mathfrak{b}(\pi_{i_1,j_1}) \cap \mathfrak{b}(\pi_{i_2,j_2}) = \emptyset, \quad \mathfrak{b}(\nu_{k_1,l_1}) \cap \mathfrak{b}(\nu_{k_2,l_2}) = \emptyset,$$

The cardinalities of the sets of pairs are

$$\mathbf{p} = \#\{\pi_{i,j}\} = \begin{cases} \frac{r}{2} & \text{if } r \text{ is even} \\ \frac{r-1}{2} & \text{if } r \text{ is odd} \end{cases}, \quad \mathbf{n} = \#\{\nu_{kl}\} = \begin{cases} \frac{s}{2} & \text{if } s \text{ is even} \\ \frac{s-1}{2} & \text{if } s \text{ is odd} \end{cases}.$$

Now we form a set of involutions of type T_1, which from now on will be denoted always by p_i. For any positive integers $\bar{p} \in \{1,\dots,\mathbf{p}\}$ and $\bar{n} \in \{1,\dots,\mathbf{n}\}$ we make a product of pairs:

$$\pi_{i_\alpha,j_\alpha}\pi_{i_\beta,j_\beta}, \quad \pi_{i_\alpha,j_\alpha}\nu_{k_\gamma,l_\gamma}, \quad \nu_{k_\gamma,l_\gamma}\nu_{k_\delta,l_\delta}, \quad \alpha,\beta \in \{1,\dots,\bar{p}\},\ \gamma,\delta \in \{1,\dots,\bar{n}\}. \tag{3.20}$$

We denote by $\mathcal{S}^{\bar{p},\bar{n}}$ the group generated by involutions (3.20).

Proposition 3.23. *In the notation above the groups $\mathcal{S}^{\bar p,\bar n}$ have the following properties.*

(i) *$\mathcal{S}^{\bar p,\bar n}$ is a subgroup of $G(B_{r,s})$ for any $\bar p\in\{1,\dots,\mathbf{p}\}$ and $\bar n\in\{1,\dots,\mathbf{n}\}$;*
(ii) *$\mathcal{S}^{\bar p-k_1,\bar n-k_2}$ is a subgroup of $\mathcal{S}^{\bar p,\bar n}$ for any $k_1=0,1,\dots,\bar p$ and $k_2=0,1,\dots,\bar n$;*
(ii) *The standard groups $\mathcal{S}^{\bar p,\bar n}$ are equivalent for fixed $(\bar p,\bar n)$ in the sense of Definition 3.21;*
(iv) *Any set $PI_{r,s}$ satisfying Definition 3.7 and such that $\mathcal{S}^{\mathbf{p},\mathbf{n}}=\mathcal{S}(PI_{r,s})$ will be equivalent in the sense of Definition 3.22;*
(v) *Pairs $\pi_{i,j}$ and $\nu_{k,l}$ commute with all elements in $\mathcal{S}^{\mathbf{p},\mathbf{n}}$;*
(vi) *Let $\theta=z_{i_1}\cdots z_{i_{\mathbf{p}+\mathbf{n}}}$ be a product such that each z_{i_t}, $t=1,\dots,\mathbf{p}+\mathbf{n}$ belongs only to one pair from (3.18) or (3.19). Then θ commutes with all elements in $\mathcal{S}^{\mathbf{p},\mathbf{n}}$.*

Proof. Properties (i)-(ii) are obvious. Statements (iii) and (iv) follows from the fact the pairs can be chosen up to a sign permutation of the basis in $\mathbb{R}^{r,s}$. Properties (v) and (vi) are the consequence of the facts that pairs $\pi_{i,j}$, $\nu_{k,l}$, and the product θ will have even number of common elements and that the number of vectors z_i in any element of the group $\mathcal{S}^{\mathbf{p},\mathbf{n}}\subset G(B_{r,s})$ is also even. □

Example 3.4. *Consider $\mathbb{R}^{6,3}$ with the basis $B_{6,3}=\{z_1,\dots,z_9\}$. The first six elements of the basis are positive and the last three are negative. We can choose the pairs*

$$\pi_{1,2}=z_1z_2,\quad \pi_{3,4}=z_3z_4,\quad \pi_{5,6}=z_5z_6,\quad \nu_{78}=z_7z_8, \tag{3.21}$$

up to the sign permutation. They generate a group $\mathcal{S}^{\mathbf{3},\mathbf{1}}\subset G(B_{6,3})$ of cardinality $\#\mathcal{S}^{\mathbf{3},\mathbf{1}}=8$. A possible choice of $(T1)$-type set of involutions PI generating $\mathcal{S}^{\mathbf{3},\mathbf{1}}$ is

$$PI_{6,3}=\{p_1=\pi_{1,2}\pi_{3,4},\quad p_2=\pi_{1,2}\pi_{5,6},\quad p_3=\pi_{1,2}\pi_{7,8}\}. \tag{3.22}$$

Any pair from (3.21) will commute with involutions in (3.22) and therefore with all elements in the group $\mathcal{S}^{\mathbf{3},\mathbf{1}}\subset G(B_{6,3})$. Furthermore, $\theta=z_1z_3z_5z_7$, which is chosen up to a sign permutation, commutes with elements in the group $\mathcal{S}^{\mathbf{3},\mathbf{1}}\subset G(B_{6,3})$ as well. The pairs

$$\pi_{1,2},\quad \pi_{3,4},\quad \pi_{5,6}\quad \text{generates the subgroup } \mathcal{S}^{\mathbf{3},\mathbf{0}}\subset\mathcal{S}^{\mathbf{3},\mathbf{1}}.$$

Likewise the pairs

$$\pi_{1,2},\quad \pi_{3,4},\quad \pi_{7,8}\quad \text{generates the subgroup } \mathcal{S}^{\mathbf{2},\mathbf{1}}\subset\mathcal{S}^{\mathbf{3},\mathbf{1}}.$$

Each of the subgroups $\mathcal{S}^{\mathbf{3},\mathbf{0}}$ and $\mathcal{S}^{\mathbf{2},\mathbf{1}}$ is a representative in its class of equivalence. Nevertheless, the groups $\mathcal{S}^{\mathbf{3},\mathbf{0}}$ and $\mathcal{S}^{\mathbf{2},\mathbf{1}}$ are not equivalent.

3.6. **Connectivity of groups $\mathcal{S}$.** Here we introduce another tool of detecting non-equivalent subgroups $\mathcal{S}\subset G(B_{r,s})$, that we call "connectedness" for $\mathcal{S}=\mathcal{S}(PI_{r,s})$.

Definition 3.24. *A group $\mathcal{S}\in\mathbb{S}_{r,s}$ is called connected if there is no two subgroups $\mathcal{S}_{(1)},\mathcal{S}_{(2)}\subset\mathcal{S}$, such that $\mathcal{S}$ is isomorphic to $\mathcal{S}_{(1)}\times\mathcal{S}_{(2)}$ with $\mathfrak{b}(\mathcal{S}_{(1)})\cap\mathfrak{b}(\mathcal{S}_{(2)})=\emptyset$. We write in this case $\pi_0(\mathcal{S})=1$.*

If a group $\mathcal{S}\in\mathbb{S}_{r,s}$ admits the decomposition into subgroups $\mathcal{S}=\mathcal{S}_{(1)}\times\ldots\times\mathcal{S}_{(k)}$ with $\pi_0(\mathcal{S}_{(i)})=1$ and $\mathfrak{b}(\mathcal{S}_{(i)})\cap\mathfrak{b}(\mathcal{S}_{(j)})=\emptyset$ for any $i\neq j$, then we say that $\mathcal{S}$ is disconnected, has k connected components and we write $\pi_0(\mathcal{S})=k$.

Lemma 3.25. *Let $PI = \{p_i\}_{i=1}^{\ell(r,s)} \in \mathbb{PI}_{r,s}^M$, and $|\mathfrak{b}(PI)| = r + s$. Assume that there is $z_\alpha \in G(B_{r,s})$ such that $z_\alpha \in \bigcap_{i=1}^{\ell(r,s)} \mathfrak{b}(p_i)$, and moreover, there is no $\sigma \in \mathcal{S}(PI)$ such that $\mathfrak{b}(\sigma) \subset \mathfrak{b}(p_i)$ for any $p_i \in PI$. Then $\pi_0(\mathcal{S}(PI)) = 1$.*

Proof. Note that any product $\prod_j^{2k+1} p_j$ of odd number contains z_α. Let us assume that $\mathcal{S} = \mathcal{S}_{(1)} \times \mathcal{S}_{(2)}$ is a non-trivial decomposition.

If both subgroups include a product of odd number of involutions $\prod_j^{2l+1} p_j$, $p_j \in PI$, then $z_\alpha \in \mathfrak{b}(\mathcal{S}_{(1)}) \bigcap \mathfrak{b}(\mathcal{S}_{(2)})$. Therefore $\mathcal{S}$ should be connected.

Assume the subgroup $\mathcal{S}_{(1)}$ consists of only even products $\eta = \prod_j^{2k} p_j$ of involutions in PI. We write one of these products in the form $\eta = p_{i_0} \cdot \sigma \in \mathcal{S}_{(1)}$, where p_{i_0} is one of the generators from the set PI and σ is a product of odd number of some involutions in PI. It implies that $\sigma \in \mathcal{S}_{(2)}$. By the assumption $\mathfrak{b}(\sigma) \not\subset \mathfrak{b}(p_i)$ for any $p_i \in PI$, there exists a basis vector $z_\beta \in \mathfrak{b}(\sigma)$ such that $z_\beta \notin \mathfrak{b}(p_{i_0})$. This implies that $z_\beta \in \mathfrak{b}(p_{i_0} \cdot \sigma)$ and therefore $z_\beta \in \mathfrak{b}(\sigma) \cap \mathfrak{b}(p_{i_0} \cdot \sigma) \subset \mathfrak{b}(\mathcal{S}_{(2)}) \bigcap \mathfrak{b}(\mathcal{S}_{(1)})$. This shows that the group $\mathcal{S}$ is connected. □

Example 3.5. *The standard subgroups $\mathcal{S}^{\mathbf{p},\mathbf{0}} \in \mathbb{S}_{r,0}$ constructed in Example 3.3 are connected for any $r \geq 0$.*

Proposition 3.26. *Let $PI_1, PI_2 \in \mathbb{PI}_{r,s}^M$ be two generating sets. If $PI_1 \sim PI_2$, then $\pi_0(PI_1) = \pi_0(PI_2)$.*

Proof. We write $PI_1 = \{p_k\}_{k=1}^{\ell(r,s)}$, $PI_2 = \{q_m\}_{m=1}^{\ell(r,s)}$ and $|\mathfrak{b}(PI_j)| = t$, $j = 1, 2$. By the assumption there exists an orthogonal map C which performs the re-ordering of the basis $B_{r,s}$ and such that $C\Big(\mathcal{S}\big(\widehat{\{p_k\}}_{k=1}^{\ell(r,s)}\big)\Big) = \mathcal{S}\big(\widehat{\{q_m\}}_{m=1}^{\ell(r,s)}\big)$. If

$$\mathcal{S}(PI_1) = \mathcal{S}_{(1)} \times \mathcal{S}_{(2)} = \mathcal{S}_{(1)}(PI_{11}) \times \mathcal{S}_{(2)}(PI_{12}),$$

with

$$PI_{11} = \{p_{i_k}\}_{k=1}^{a}, \quad |\mathfrak{b}(\{p_{i_k}\}_{k=1}^{a})| = \beta,$$

and

$$PI_{12} = \{p_{j_k}\}_{k=a+1}^{\ell(r,s)}, \quad |\mathfrak{b}(\{p_{j_k}\}_{k=a+1}^{\ell(r,s)})| = t - \beta,$$

then $\mathfrak{b}(\{p_{i_k}\}_{k=1}^{a}) \cap \mathfrak{b}(\{p_{j_k}\}_{k=a+1}^{\ell(r,s)}) = \emptyset$. The re-ordering map C will map the disjoint sets $\mathfrak{b}(\{p_{i_k}\}_{k=1}^{a})$ and $\mathfrak{b}(\{p_{j_k}\}_{k=a+1}^{\ell(r,s)})$ onto disjoint sets $\mathcal{Z}_1 = \{z_{i_1}, \dots, z_{i_\beta}\}$ and $\mathcal{Z}_2 = \{z_{j_{\beta+1}}, \dots, z_{i_t}\}$. The set $\mathcal{Z}_1$ (with possible change of signs) will form the set $PI_{21} = \{q_{i_k}\}_{k=1}^{a}$ and the set $\mathcal{Z}_2$ (again with possible change of signs) will form the set $PI_{22} = \{q_{j_k}\}_{k=a+1}^{t}$. Thus we obtain $\mathcal{S}(PI_2) = \mathcal{S}(PI_{21}) \times \mathcal{S}(PI_{22})$. □

We describe how the $\mathbb{Z}_2$-graded product of Clifford algebras can lead to the construction of disconnected subgroups $\mathcal{S} \subset G(B_{r,s})$. Consider the following decompositions of an orthonormal basis $B_{r,s} = \{z_1, \dots, z_r, z_{r+1}, \dots, z_{r+s}\}$:

$$\underbrace{z_1, \dots, z_{r_1}}_{\text{positive}}, \underbrace{z_{r+1}, \dots, z_{r+s_1}}_{\text{negative}}, \quad \text{and} \quad \underbrace{z_{r_1+1}, \dots, z_r}_{\text{positive}}, \underbrace{z_{r+s_1+1}, \dots, z_{r+s}}_{\text{negative}},$$

for $r_1 \le r$ and $s_1 \le s$. We put $r_2 = r - r_1$ and $s_2 = s - s_1$ and consider the decomposition $\mathbb{R}^{r,s} \cong \mathbb{R}^{r_1,s_1} \oplus \mathbb{R}^{r_2,s_2}$, where we assume $r_1 + s_1 \ge (r - r_1) + (s - s_1) = r_2 + s_2$. This decomposition leads to the isomorphism $\mathrm{Cl}_{r_1,s_1} \widehat{\otimes} \mathrm{Cl}_{r_2,s_2} \cong \mathrm{Cl}_{r_1+r_2,s_1+s_2} = \mathrm{Cl}_{r,s}$, where $\widehat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product of Clifford algebras, see [LM89, Proposition 1.5]. For each of the Clifford algebras Cl_{r_k,s_k}, $k = 1, 2$, we consider the minimal admissible modules V^{r_k,s_k} and the corresponding sets PI_{r_k,s_k}. For $r = r_1 + r_2$ and $s = s_1 + s_2$, we have $\ell(r_1, s_1) \le \ell(r, s)$. Let $PI_{r_1,s_1} \in \mathbb{PI}^M_{r_1,s_1}$ and $PI_{r_2,s_2} \in \mathbb{PI}^M_{r_2,s_2}$ satisfy

$$|\mathfrak{b}^+(PI_{r_1,s_1})| = r_1, \quad |\mathfrak{b}^-(PI_{r_1,s_1})| = s_1,$$
$$|\mathfrak{b}^+(PI_{r_2,s_2})| = r_2, \quad |\mathfrak{b}^-(PI_{r_2,s_2})| = s_2,$$

and $PI_{r_1,s_1} \bigcap PI_{r_2,s_2} = \emptyset$. We assume also that each set contains at most one type T_2 involution $q_k \in PI_{r_k,s_k}$, $k = 1, 2$. Then by non-commutativity of q_1 and q_2 it is easy to see the following properties:

If at least one of the sets PI_{r_1,s_1} or PI_{r_2,s_2} is $(T1)$-type set, then

$$PI_{r_1,s_1} \bigcup PI_{r_2,s_2} \in \mathbb{PI}_{r,s}.$$

This implies

$$\ell(r_1, s_1) + \ell(r_2, s_2) \le \ell(r, s). \tag{3.23}$$

If both PI_{r_1,s_1} and PI_{r_2,s_2} are $(T2)$-type sets, containing type T_2 involutions $q_1 \in PI_{r_1,s_1}$ and $q_2 \in PI_{r_2,s_2}$, then

$$\big(PI_{r_1,s_1} \backslash \{q_1\}\big) \bigcup PI_{r_2,s_2} \in \mathbb{PI}_{r,s} \quad \text{and} \quad PI_{r_1,s_1} \bigcup \big(PI_{r_2,s_2} \backslash \{q_2\}\big) \in \mathbb{PI}_{r,s}.$$

This implies

$$\ell(r_1, s_1) + \ell(r_2, s_2) - 1 \le \ell(r, s). \tag{3.24}$$

One can state similar properties for any number of components in a decomposition $PI = \cup_k PI_{r_k,s_k}$.

Remark 3.3. *If the equalities in* (3.23) *or* (3.24) *hold, then non-connected subgroups* $\mathcal{S}(PI_{r_1,s_1})$ *and* $\mathcal{S}(PI_{r_2,s_2})$ *can be constructed from lower dimensions and*

$$\mathcal{S}(PI_{r,s}) = \mathcal{S}(PI_{r_1,s_1}) \times \mathcal{S}(PI_{r_2,s_2}).$$

Particularly, if $r \le 9$ *and* $s \in \{0, 1\}$*, then all the groups are connected. It follows by showing that the inequalities* (3.23) *and* (3.24) *are always strict.*

4. Construction of subgroups in $\mathbb{S}^M_{r,s}$, $r \in \{3, \dots, 16\}$, $s \in \{0, 1\}$

4.1. **General method of the construction.** In this section we apply the previous theory for the classification of groups $\mathcal{S} \subset G(B_{r,s})$ and perform the exact construction of non-equivalent subgroups for $0 \le r + s \le 16$. We restrict ourself to $0 \le r + s \le 16$ because we want to illustrate the main features that appear in classification without diving into technical details. The classification for arbitrary $\mathcal{S} \subset G(B_{r,s})$ is postponed for the forthcoming paper.

We start from $s = 0$. We classify groups $\mathcal{S} \subset \mathbb{S}^M_{r,0}$ according to parameters: $\pi_0(\mathcal{S})$, $|\mathfrak{b}(PI_{r,0})|$, and the type $(T1)$ or $(T2)$ of the set PI generating the group $\mathcal{S} \in \mathbb{S}^M_{r,s}$. We use the standard groups and notations introduced in Example 3.3. For a standard group we will add from none to two additional involutions, see Step 1 below for details. To distinguish the groups, where all previous parameters coincide, we assign the following information about (TI)-type sets, $I = 1, 2$:

(4.1)
- (i) We use the signature (TI, π) if an additional involution is related to product $\pi_{1,2}$;
- (ii) We use the signature (TI, θ) if an additional involution is related to product θ;
- (iii) We use the signature (TI, π, θ) if there are two additional involutions, which are related to both products $\pi_{1,2}$ and θ;
- (iv) Finally we just write (TI) if there are no involutions, except of standards;

For each set of involutions $PI_{r,0}$ we write the signature

$$(\ell(r,0), (T1, \bullet, \bullet), |\mathfrak{b}(PI_{r,0})|) \tag{4.2}$$

We summarise the results in Table 2. We list the set of generators $PI_{r,0}$ for each group. The group itself and the set of generators will be given up to a sign permutation. The word *unique* is understood in the sense of equivalence relation of Definition 3.21 or Definition 3.22.

4.1.1. *Main steps of the construction of $\mathcal{S} \in \mathbb{S}^M_{r,0}$ for a fixed $r > 0$.* We divide the construction into three steps.

Step 1. We start from a group satisfying $\pi_0(\mathcal{S}) = 1$ and $|\mathfrak{b}(PI_{r,0})| = r$. First we find standard subgroup $\mathcal{S}^{\mathbf{p},\mathbf{0}} \subset \mathcal{S}$ and complement it (if necessary) by involutions to reach the maximal number $\ell(r,0)$ of involutions in $PI_{r,0}$ generating $\mathcal{S}(PI_{r,0}) \in \mathbb{S}^M_{r,0}$. The additional involutions will be formed by checking whether the product of $\pi_{1,2}$ and/or θ by z_r are involutions commuting with $\mathcal{S}^{\mathbf{p},\mathbf{0}}$. Then we consider a smaller standard subgroup $\mathcal{S}^{\mathbf{p-1},\mathbf{0}} \subset \mathcal{S}^{\mathbf{p},\mathbf{0}}$ and complement it by a *careful choice of* involutions to reach the maximal number $\ell(r,0)$ for $\mathcal{S}(PI_{r,0})$, checking whether the connectivity $\pi_0(\mathcal{S}(PI_{r,0})) = 1$ is not violated. We can repeat the last step several times if the condition $\pi_0(\mathcal{S}(PI_{r,0})) = 1$ still holds.

Step 2. We continue to look on $\pi_0(\mathcal{S}) = 1$ and $|\mathfrak{b}(PI_{r,0})| = r - 1$. In most cases it will be a simple step back from $(r,0)$ to $(r-1,0)$ as, for example, for reduction from $PI_{4,0}$ to $PI_{3,0}$.

Step 3. Next we check $\pi_0(\mathcal{S}) = 2$ and $\mathcal{S} = \mathcal{S}_{(1)} \times \mathcal{S}_{(2)}$. This step is reduced to combinations of the previous 2 steps. If needs, we can proceed to higher number of connected components.

The equivalence of the groups constructed in the previous three steps is summarised in the following proposition.

Proposition 4.1. *Let $\mathcal{S}=\mathcal{S}(PI_{r,0})\in\mathbb{S}^M_{r,0}$, with $|\mathfrak{b}(PI_{r,0})|=r$ and $\pi_0(\mathcal{S})=1$. Then, the maximal standard subgroups $\mathcal{S}^{\mathbf{p},\mathbf{0}}$, included in a given group $\mathcal{S}\in\mathbb{S}^M_{r,0}$, are equivalent modulo reordering by induction arguments with respect to the dimension $(r,0)$, see also Proposition 3.23, item (v).*

Moreover, once we fix a standard group $\mathcal{S}^{\mathbf{p},\mathbf{0}}$ with its generators of the form (3.20), the maximally complemented sets $PI_{r,0}$ obtained by adding involutions as in Step 1, will be equivalent in the sense of Definition 3.22 if they have the same signature described in (4.1) and $\pi_0(\mathcal{S}(PI_{r,0}))=1$.

Lemma 4.2. *If $r=3+8k,5+8k,6+8k,7+8k$ for $k\geq 0$, then sets $PI_{r,0}\in\mathbb{PI}^M_{r,0}$ satisfying $\pi_0(\mathcal{S}(PI_{r,0}))=1$ and $|\mathfrak{b}(PI_{r,0})|=r$ are always of $(T2)$-type.*

Proof. We start from $r=3+8k$. For the case $r=3$ there is only one type T_2 involution. Let $k\geq 1$ and assume, by contrary, that there is a $(T1)$-type set $PI_{r,0}\in\mathbb{PI}^M_{r,0}$. We have $\ell(r,0)=\ell(3+8k,0)=1+4k$. The standard subgroup $\mathcal{S}^{\mathbf{p},\mathbf{0}}\subset\mathcal{S}(PI_{r,0})$, $\mathbf{p}=1+4k$, does not contain z_r, since r is odd. Let $p_1,\ldots,p_{4k}$ will be involutions generating $\mathcal{S}^{\mathbf{p},\mathbf{0}}$, then $z_r\in\mathfrak{b}(p_{1+4k})$. It implies

$$\{p_1,\ldots,p_{4k},\ z_r\cdot p_{1+4k}\}\in\mathbb{PI}^M_{r-1,0}.$$

This contradicts to $\ell(r-1,0)=\ell(2+8k,0)=\ell(3+8k,0)-1=\ell(r,0)-1$.

The arguments for the cases $r=5+8k$, and $r=7+8k$ are similar to the case $r=3+8k$.

Let $r=6+8k$. We assume that there is a $(T1)$-type set $PI_{r,0}\in\mathbb{PI}^M_{r,0}$. We have $\ell(r,0)=\ell(6+8k,0)=3+4k$. The standard subgroup $\mathcal{S}^{\mathbf{p},\mathbf{0}}\subset\mathcal{S}(PI_{r,0})$, $\mathbf{p}=3+4k$, contains z_r. Let $p_1,\ldots,p_{2+4k}$ be involutions generating $\mathcal{S}^{\mathbf{p},\mathbf{0}}$, where we can assume that $z_r\in\mathfrak{b}(p_{2+4k})$ and $p_{3+4k}\in PI_{6+8k,0}$ is the last type T_1 involution.

(1) If $z_r\notin\mathfrak{b}(p_{3+4k})$, then

$$\{p_1,\ldots,p_{1+4k},\ z_r\cdot p_{2+4k},\ p_{3+4k}\}\in\mathbb{PI}^M_{r-1,0}.$$

This contradicts to $\ell(r-1,0)=\ell(5+8k,0)=\ell(6+8k,0)-1=\ell(r,0)-1$.

(2) If $z_r\in\mathfrak{b}(p_{3+4k})$, then we replace $p_{3+4k}\in PI_{r,0}$ by another type T_1 involution $\tilde{p}_{3+4k}=p_{2+4k}p_{3+4k}\in\widetilde{PI_{r,0}}$. In this case $z_r\notin\mathfrak{b}(\hat{p}_{3+4k})$ and the situation is reduced to the previous step (1). Note that the group $\mathcal{S}(PI_{r,0})$ is equivalent $\mathcal{S}(\widetilde{PI_{r,0}})$.

We also note that for $r=3+8k$ and $r=7+8k$ the volume forms $\Omega_r=\prod_{i=1}^r z_i$ which are type T_2 involutions can be included to $PI_{r,0}$. It justifies the $(T2)$-type set of PIs in cases $r=3+8k$ and $r=7+8k$. □

4.2. **Construction of connected groups $\mathcal{S}\in\mathbb{S}^M_{r,0}$ for $r\in\{3,\ldots,16\}$.** In Table 2 we collect the nonisomorphic connected groups $\mathcal{S}\in\mathbb{S}^M_{r,0}$ for $r\in\{3,\ldots,16\}$; that is $\pi_0(\mathcal{S})=1$. Note that if there is a group $\mathcal{S}\in\mathbb{S}^M_{r,0}$ with signature

$$\mathcal{S}=(\ell(r,0),(Tk,\bullet,\bullet),|\mathfrak{b}(PI_{r,0})|),$$

where

$$\ell(r,0)=\ell(r-1,0), \quad |\mathfrak{b}(PI_{r,0})|=r-1, \quad k=1,2,$$

then

$$\mathcal{S}=(\ell(r-1,0),(Tk,\bullet,\bullet),r-1)\in\mathbb{S}^M_{r-1,0},$$

as for instance in Table 2 for $r=4,8,9$ and some others values of r.

Notation 4.1. *We write $\theta_{\overline{i,j}}$ to indicate that product in θ starts from z_i and ends with z_j containing all z_k for odd k between i and j. We have $|\mathfrak{b}(\theta_{\overline{i,j}})|=\frac{j-i}{2}+1$.*

TABLE 2. Connected groups in $\mathbb{S}^M_{r,0}$ for $r=3,\dots,16$

r	Signatures	$PI_{r,0}$
12	$\mathcal{S}^{(1)}_{12}=\big(5,(T1,\pi),12\big)$	$\mathcal{S}^{(1)}_{12}=\begin{cases}p_1=\pi_{1,2}\pi_{3,4}\\p_2=\pi_{1,2}\pi_{5,6}\\p_3=\pi_{1,2}\pi_{7,8}\\p_4=\pi_{1,2}\pi_{9,10}\\p_5=\pi_{1,2}\pi_{11,12}\end{cases}$
	$\mathcal{S}^{(2)}_{12}=\big(5,(T2,\theta),12\big)$	$\mathcal{S}^{(2)}_{12}=\begin{cases}p_1=\pi_{1,2}\pi_{3,4}\\p_2=\pi_{1,2}\pi_{5,6}\\p_3=\pi_{1,2}\pi_{7,8}\\p_4=\pi_{1,2}\pi_{9,10}\\q=\theta_{\overline{1,9}}\pi_{11,12}\end{cases}$
	$\mathcal{S}^{(3)}_{12}=\big(5,(T2,\pi),11\big)=\mathcal{S}_{11}$	
11	$\mathcal{S}_{11}=\big(5,(T2,\pi),11\big)$	$p_1=\pi_{1,2}\pi_{3,4}$ $p_2=\pi_{1,2}\pi_{5,6}$ $p_3=\pi_{1,2}\pi_{7,8}$ $p_4=\pi_{1,2}\pi_{9,10}$ $q=\pi_{1,2}z_{11}$
10	$\mathcal{S}^{(1)}_{10}=\big(4,(T1,\pi),10\big)$ $\mathcal{S}^{(2)}_{10}=\mathcal{S}^{(1)}_9$ $\mathcal{S}^{(3)}_{10}=\mathcal{S}^{(2)}_9=\mathcal{S}^{(1)}_8$ $\mathcal{S}^{(4)}_{10}=\mathcal{S}^{(3)}_9=\mathcal{S}^{(2)}_8=\mathcal{S}_7$	$\mathcal{S}^{(1)}_{10}=\begin{cases}p_1=\pi_{1,2}\pi_{3,4}\\p_2=\pi_{1,2}\pi_{5,6}\\p_3=\pi_{1,2}\pi_{7,8}\\p_4=\pi_{1,2}\pi_{9,10}\end{cases}$
9	$\mathcal{S}^{(1)}_9=\big(4,(T2,\pi),9\big)$ $\mathcal{S}^{(2)}_9=\mathcal{S}^{(1)}_8$ $\mathcal{S}^{(3)}_9=\mathcal{S}^{(2)}_8=\mathcal{S}_7$	$\mathcal{S}^{(1)}_9=\begin{cases}p_1=\pi_{1,2}z_3z_4\\p_2=\pi_{1,2}z_5z_6\\p_3=\pi_{1,2}z_7z_8\\q=\pi_{1,2}z_9\end{cases}$
8	$\mathcal{S}^{(1)}_8=\big(4,(T1,\theta),8\big)$ $\mathcal{S}^{(2)}_8=\mathcal{S}_7$	$\mathcal{S}^{(1)}_8=\begin{cases}p_1=\pi_{1,2}z_3z_4\\p_2=\pi_{1,2}z_5z_6\\p_3=\pi_{1,2}z_7z_8\\p_4=\theta_{\overline{1,7}}\end{cases}$
7	$\mathcal{S}_7=\big(4,(T2,\pi,\theta),7\big)$	$p_1=\pi_{1,2}z_3z_4$ $p_2=\pi_{1,2}z_5z_6$ $p_3=\theta_{\overline{1,5}}z_7$ $q=\pi_{1,2}z_7$
6	$\mathcal{S}_6=\big(3,(T2,\theta),6\big)$	$p_1=\pi_{1,2}z_3z_4$ $p_2=\pi_{1,2}z_5z_6$ $q=\theta_{\overline{1,5}}$
5	$\mathcal{S}_5=\big(2,(T2,\theta),5\big)$	$p=\pi_{1,2}z_3z_4$ $q=\theta_{\overline{1,5}}=z_1z_3z_5$
4	$\mathcal{S}^{(1)}_4=\big(1,(T1),4\big)$ $\mathcal{S}^{(2)}_4=\big(1,(T2,\pi),3\big)=\mathcal{S}_3$	$p=\pi_{1,2}z_3z_4$ $q=\pi_{1,2}z_3$
3	$\mathcal{S}_3=\big(1,(T2,\pi),3\big)$	$q=\pi_{1,2}z_3$

r	Signatures	$PI_{r,0}$
16	$\mathcal{S}^{(1)}_{16}=\big(8,(T1,\theta),16\big)$ $\mathcal{S}^{(2)}_{16}=\mathcal{S}_{15}$	$\mathcal{S}^{(1)}_{16}=\begin{cases}p_1=\pi_{1,2}\pi_{3,4}\\p_2=\pi_{1,2}\pi_{5,6}\\p_3=\pi_{1,2}\pi_{7,8}\\p_4=\pi_{1,2}\pi_{9,10}\\p_5=\pi_{1,2}\pi_{11,12}\\p_6=\pi_{1,2}\pi_{13,14}\\p_7=\pi_{1,2}\pi_{15,16}\\p_8=\theta_{\overline{1,13}}z_{15}\end{cases}$
15	$\mathcal{S}_{15}=\big(8,(T2,\pi),15\big)$	$p_1=\pi_{1,2}\pi_{3,4}$ $p_2=\pi_{1,2}\pi_{5,6}$ $p_3=\pi_{1,2}\pi_{7,8}$ $p_4=\pi_{1,2}\pi_{9,10}$ $p_5=\pi_{1,2}\pi_{11,12}$ $p_6=\pi_{1,2}\pi_{13,14}$ $p_7=\theta_{\overline{1,13}}z_{15}$ $q=\pi_{1,2}z_{15}$
14	$\mathcal{S}^{(1)}_{14}=\big(7,(T2,\theta),14\big)$	$p_1=\pi_{1,2}\pi_{3,4}$ $p_2=\pi_{1,2}\pi_{5,6}$ $p_3=\pi_{1,2}\pi_{7,8}$ $p_4=\pi_{1,2}\pi_{9,10}$ $p_5=\pi_{1,2}\pi_{11,12}$ $p_6=\pi_{1,2}\pi_{13,14}$ $q=\theta_{\overline{1,13}}$
	$\mathcal{S}^{(2)}_{14}=\big(7,(T2,\pi,\theta),14\big)$	$p_1=\pi_{1,2}\pi_{3,4}$ $p_2=\pi_{1,2}\pi_{5,6}$ $p_3=\pi_{1,2}\pi_{7,8}$ $p_4=\pi_{1,2}\pi_{9,10}$ $p_5=\pi_{1,2}\pi_{11,12}$ $p_6=\theta_{\overline{1,11}}\pi_{13,14}$ $q=\pi_{1,2}z_{14}$
13	$\mathcal{S}^{(1)}_{13}=\big(6,(T2,\pi),13\big)$	$p_1=\pi_{1,2}\pi_{3,4}$ $p_2=\pi_{1,2}\pi_{5,6}$ $p_3=\pi_{1,2}\pi_{7,8}$ $p_4=\pi_{1,2}\pi_{9,10}$ $p_5=\pi_{1,2}\pi_{11,12}$ $q=\pi_{1,2}z_{13}$
	$\mathcal{S}^{(2)}_{13}=\big(6,(T2,\theta),13\big)$	$p_1=\pi_{1,2}\pi_{3,4}$ $p_2=\pi_{1,2}\pi_{5,6}$ $p_3=\pi_{1,2}\pi_{7,8}$ $p_4=\pi_{1,2}\pi_{9,10}$ $p_5=\pi_{1,2}\pi_{11,12}$ $q=\theta_{\overline{1,13}}$
	$\mathcal{S}^{(3)}_{13}=\big(6,(T2,\pi,\theta),13\big)$	$p_1=\pi_{1,2}\pi_{3,4}$ $p_2=\pi_{1,2}\pi_{5,6}$ $p_3=\pi_{1,2}\pi_{7,8}$ $p_4=\pi_{1,2}\pi_{9,10}$ $p_5=\theta_{\overline{1,9}}z_{11}z_{12}z_{13}$ $q=\pi_{1,2}z_{12}$

We explain construction only for $r=7$, since it is the most illustrative.

The standard subgroup $\mathcal{S}^{\mathbf{3,0}} \subset \mathcal{S}$ is generated by two involutions

$$p_1 = \pi_{1,2}\pi_{3,4}, \quad p_2 = \pi_{1,2}\pi_{5,6}. \tag{4.3}$$

We need to add two involutions since $\ell(7,0) = 4$, at least one of which must contain z_7. We observe that the products $\pi_{1,2}z_7$ and $\theta_{\overline{1,5}}z_7 = z_1z_3z_5z_7$ are both involutions commuting with generators (4.3) and with each other. We append them both to reach $\ell(7,0) = 4$. The reductions to $|\mathfrak{b}(PI_{7,0})| = 6$ is not possible due to $\ell(6,0) < \ell(7,0)$.

4.3. **Constructions of disconnected groups $\mathcal{S} \in \mathbb{S}^M_{r,0}$ for $r \in \{10,\dots,16\}$.** We show in Table 3 the disconnected groups with $\pi_0(\mathcal{S}_{r,0}) = 2$ for $r \in \{10,\dots,16\}$.

We explain in detailes only the case for $r = 11$. To construct the disconnected subgroup $\mathcal{S}^{(1)}_{11} = \mathcal{S}^{(2)}_{(1)} \times \mathcal{S}^{(2)}_{(2)}$ corresponding to the $\mathbb{Z}_2$-graded tensor product of the Clifford algebras $\mathrm{Cl}_{11,0} \cong \mathrm{Cl}_{8,0}\hat{\otimes}\mathrm{Cl}_{3,0}$ we start from the standard subgroup $\mathcal{S}^{\mathbf{4,0}}_{(1)} \subset \mathcal{S}^{(1)}_{(1)}$ generated by

$$p_1 = \pi_{1,2}\pi_{3,4}, \quad p_2 = \pi_{1,2}\pi_{5,6}, \quad p_3 = \pi_{1,2}\pi_{7,8}. \tag{4.4}$$

and add type T_1 involution $\theta_{\overline{1,7}} = z_1z_3z_5z_7$. The group $\mathcal{S}^{(1)}_{(1)}$ has the folowing signature $(4,(T1,\theta),8)$. Then the signature of $\mathcal{S}^{(2)}_{(1)} = \{\mathbf{1}, \pi_{9,10}z_{11}\}$ is $(1,(T2,\pi),3)$.

To obtain $\mathcal{S}^{(2)}_{11} = \mathcal{S}^{(2)}_{(1)} \times \mathcal{S}^{(2)}_{(2)}$ corresponding to the $\mathbb{Z}_2$-graded tensor product of the Clifford algebras $\mathrm{Cl}_{11,0} \cong \mathrm{Cl}_{7,0}\hat{\otimes}\mathrm{Cl}_{4,0}$ we consider standard subgroup $\mathcal{S}^{\mathbf{3,0}}_{(1)} \subset \mathcal{S}^{(2)}_{(1)}$ generated by (4.3) and add type T_1 involution $\theta_{\overline{1,7}}$ and type T_2 involution $\pi_{1,2}z_7$. The group $\mathcal{S}^{(2)}_{(1)}$ obtain the signature $(4,(T2,\pi,\theta),7)$. Then $\mathcal{S}^{(2)}_{(2)} = \{\mathbf{1}, \pi_{8,9}\pi_{10,11}\}$ has the signature $(1,(T1),4)$.

Table 3. Disconnected groups in $\mathbb{S}^M_{r,0}$ for $r = 10,\dots,16$

r	Signatures	PI
12	$\mathcal{S}^{(1)}_{12} = (4,(T1,\theta),8) \times (1,(T1),4)$	$\mathcal{S}^{(1)}_{12}$ = $(p_1)_1 = \pi_{1,2}\pi_{3,4}$ $(p_1)_2 = \pi_{9,10}\pi_{11,12}$ $(p_2)_1 = \pi_{1,2}\pi_{5,6}$ $(p_3)_1 = \pi_{1,2}\pi_{7,8}$ $(p_4)_1 = \theta_{\overline{1,7}}$
12	$\mathcal{S}^{(2)}_{12} = (3,(T1,\theta),7) \times (2,(T2,\pi),5)$	$\mathcal{S}^{(2)}_{12}$ = $(p_1)_1 = \pi_{1,2}\pi_{3,4}$ $(p_1)_2 = \pi_{8,9}\pi_{10,11}$ $(p_2)_1 = \pi_{1,2}\pi_{5,6}$ $(q)_2 = \pi_{8,9}z_{12}$ $(p_3)_1 = \theta_{\overline{1,7}}$
12	$\mathcal{S}^{(3)}_{12} = (3,(T2,\theta),6) \times (2,(T1),6)$	$\mathcal{S}^{(3)}_{12}$ = $(p_1)_1 = \pi_{1,2}\pi_{3,4}$ $(p_1)_2 = \pi_{7,8}\pi_{9,10}$ $(p_2)_1 = \pi_{1,2}\pi_{5,6}$ $(p_1)_2 = \pi_{7,8}\pi_{11,12}$ $(q)_1 = \theta_{\overline{1,5}}$
12	$\mathcal{S}^{(4)}_{12} = \mathcal{S}^{(1)}_{11}$	
12	$\mathcal{S}^{(5)}_{12} = \mathcal{S}^{(2)}_{11}$	
11	$\mathcal{S}^{(1)}_{11} = (4,(T1,\theta),8) \times (1,(T2,\pi),3)$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(q)_{(2)} = \pi_{9,10}z_{11}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_3)_{(1)} = \pi_{1,2}\pi_{7,8}$, $(p_4)_{(1)} = \theta_{\overline{1,7}}$,
11	$\mathcal{S}^{(2)}_{11} = (4,(T2,\pi,\theta),7) \times (1,(T1),4)$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)} = \pi_{8,9}\pi_{10,11}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_3)_{(1)} = \theta_{\overline{1,7}}$, $(q)_{(1)} = \pi_{1,2}z_7$,
10	$\mathcal{S}^{(1)}_{10} = (3,(T1,\theta),7) \times (1,(T2,\pi),3)$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(q)_{(2)} = \pi_{8,9}z_{10}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_3)_{(1)} = \theta_{\overline{1,7}}$,
10	$\mathcal{S}^{(2)}_{10} = (3,(T2,\theta),6) \times (1,(T1),4)$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)} = \pi_{7,8}\pi_{9,10}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(q)_{(1)} = \theta_{\overline{1,5}}$,
16	$\mathcal{S}^{(1)}_{16} = (4,(T1,\theta),8) \times (4,(T1,\theta),8)$	$\mathcal{S}^{(1)}_{16}$ = $(p_1)_1 = \pi_{1,2}\pi_{3,4}$, $(p_1)_2 = \pi_{9,10}\pi_{11,12}$ $(p_2)_1 = \pi_{1,2}\pi_{5,6}$, $(p_2)_2 = \pi_{9,10}\pi_{13,14}$ $(p_3)_1 = \pi_{1,2}\pi_{7,8}$, $(p_3)_2 = \pi_{9,10}\pi_{15,16}$ $(p_4)_1 = \theta_{\overline{1,7}}$, $(p_4)_2 = \theta_{\overline{9,15}}$
16	$\mathcal{S}^{(2)}_{16} = \mathcal{S}_{15}$	
15	$\mathcal{S}_{15} = (4,(T1,\theta),8) \times (4,(T2,\pi,\theta),7)$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)} = \pi_{9,10}\pi_{11,12}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_2)_{(2)} = \pi_{9,10}\pi_{13,14}$ $(p_3)_{(1)} = \pi_{1,2}\pi_{7,8}$, $(p_3)_{(2)} = \theta_{\overline{9,15}}$ $(p_4)_{(1)} = \theta_{\overline{1,7}}$, $(q)_{(2)} = \pi_{9,10}z_{15}$
14	$\mathcal{S}^{(1)}_{14} = (4,(T1,\theta),8) \times (3,(T2,\theta),6)$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)} = \pi_{9,10}\pi_{11,12}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_2)_{(2)} = \pi_{9,10}\pi_{13,14}$ $(p_3)_{(1)} = \pi_{1,2}\pi_{7,8}$, $(q)_{(2)} = \theta_{\overline{9,13}}$ $(p_4)_{(1)} = \theta_{\overline{1,7}}$,
14	$\mathcal{S}^{(2)}_{14} = (4,(T2,\pi,\theta),7) \times (3,(T1,\theta),7)$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)} = \pi_{8,9}\pi_{10,11}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_2)_{(2)} = \pi_{8,9}\pi_{12,13}$ $(p_3)_{(1)} = \theta_{\overline{1,7}}$, $(p_3)_{(2)} = \theta_{\overline{9,13}}z_{14}$ $(q_4)_{(1)} = \pi_{1,2}z_7$,
13	$\mathcal{S}^{(1)}_{13} = (4,(T1,\theta),8) \times (2,(T2,\pi),5)$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)} = \pi_{9,10}\pi_{11,12}$, $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(q)_{(2)} = \pi_{9,10}z_{13}$ $(p_3)_{(1)} = \pi_{1,2}\pi_{7,8}$, $(p_4)_{(1)} = \theta_{\overline{1,7}}$,
13	$\mathcal{S}^{(2)}_{13} = (4,(T2,\pi,\theta),7) \times (2,(T1),6)$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)} = \pi_{8,9}\pi_{10,11}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_2)_{(2)} = \pi_{8,9}\pi_{12,13}$ $(p_3)_{(1)} = \theta_{\overline{1,7}}$, $(q)_{(1)} = \pi_{1,2}z_7$,
13	$\mathcal{S}^{(3)}_{13} = (3,(T1,\theta),7) \times (3,(T2,\theta),6)$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)} = \pi_{8,9}\pi_{10,11}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_2)_{(2)} = \pi_{8,9}\pi_{12,13}$ $(p_3)_{(1)} = \theta_{\overline{1,7}}$, $(q)_{(2)} = z_8z_{10}z_{12}$

Proposition 4.3. *Table 2 and Table 3 are the same for H-type Lie algebras* $\mathfrak{n}_{r,1}$, $r \in \{3, \ldots, 16\}$.

Proof. For $s = 1$, the negative basis vector plays no role in forming the involutions, see Definition 3.5. □

TABLE 4. Number of non-equivalent groups

r	1	2	3	4	5	6	7	8
$\pi_0(\mathcal{S}) = 1$	0	0	1	2	1	1	1	2
$\pi_0(\mathcal{S}) = 2$	0	0	0	0	0	0	0	0
r	9	10	11	12	13	14	15	16
$\pi_0(\mathcal{S}) = 1$	3	4	1	3	3	2	1	2
$\pi_0(\mathcal{S}) = 2$	0	2	2	5	3	2	1	2

4.4. **Construction of connected groups** $\mathcal{S} \in \mathbb{S}^M_{r,s}$ **for** $0 < r + s < 16$**.** We show in Table 6 the possible maximal sets $PI \in \mathbb{PI}^M_{r,s}$ with $\pi_0(PI_{r,s}) = 1$ for some for $r+s \leq 16$. The different sets of involutions are determined based on the data of $\mathbb{PI}^M_{r+s,0}$ by a recurrent procedure. Note that $|\mathfrak{b}(PI_{r,s})| = |\mathfrak{b}^+(PI_{r,s})| + |\mathfrak{b}^-(PI_{r,s})| \leq r + s$ because some of the basis vectors z_j are not used when $s \geq 0$, as for instance the $(T2)$-type set in $\mathbb{PI}^M_{4,0}$ consists only of product of three vectors z_j. We use the signature

$$\ell(r,s),\ \text{Type } (T1 \text{ or } T2),\ (|\mathfrak{b}^+(PI)|, |\mathfrak{b}^-(PI)|)$$

to indicate the non-equivalent sets of involutions $\mathbb{PI}^M_{r,s}$ in Table 6.

The values $\ell(r,s)$ of the maximal number of involutions for $r + s \leq 16$ are collected in Table 5.

TABLE 5. The value $\ell(r,s)$ for $r + s \leq 16$

16	8																
15	7	7															
14	6	7	7														
13	5	6	7	8													
12	5	6	7	8	8												
11	4	5	6	7	7	7											
10	4	5	5	6	6	7	7										
9	4	4	4	5	5	6	7	8									
8	4	4	4	5	5	6	7	8	8								
7	3	3	3	4	4	5	6	7	7	7							
6	2	3	3	4	4	5	5	6	6	7	7						
5	1	2	3	4	4	4	4	5	5	6	7	8					
4	1	2	3	4	4	4	4	5	5	6	7	8	8				
3	0	1	2	3	3	3	3	4	4	5	6	7	7	7			
2	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7		
1	0	0	0	1	1	2	3	4	4	4	4	5	5	6	7	8	
0	0	0	0	1	1	2	3	4	4	4	4	5	5	6	7	8	8
s/r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

We mentioned here the facts that allow us to complete Tables 6 and 7.

(F1) $\ell(r+s,0) \geq \ell(r,s) \geq \ell(r+s,0) - 1$, see Table 5.

(F2) Assume that $r \geq 2$, $s \geq 0$, and $\ell(r,s) = \ell(r-1,s)$. There is a $(T1)$-type set $PI \in \mathbb{PI}^M_{r,s}$, if and only if there is a $(T2)$-type set $PI' \in \mathbb{PI}^M_{r-1,s}$. Through the natural inclusion $\mathrm{Cl}_{r-1,s} \subset \mathrm{Cl}_{r,s}$, we can regard $PI' \in \mathbb{PI}^M_{r,s}$. For instance the $(T1)$-type set of involutions $PI_{4,0} = \{z_1z_2z_3z_4\} \in \mathbb{PI}^M_{4,0}$ exists if and only if exists the $(T2)$-type set $PI' = \{z_1z_2z_3\} \in \mathbb{PI}^M_{3,0}$.

(F3) If $p = z_{i_1} \cdots z_{i_a} \in PI_{r,0}$ is an involution, where z_{i_k}, $k = 1, \ldots, a$, are all positive basis vectors, then $p' = z_{i_1} \cdots z_{i_l} z^*_{i_{l+1}} \cdots z^*_{i_k} \in PI_{r',s'}$, $r' + s' = r$ is an involution where we replaced the even number of positive basis vectors $z_{i_{l+1}}, \ldots, z_{i_k}$ by negative basis vectors $z^*_{i_{l+1}}, \ldots, z^*_{i_k}$, see (3.9).

To complete Tables 6 and 7 we do the following steps

(S1) We determine the cases $(r, 0)$ for all $r > 0$. This was done in Table 2.

(S2) For any given $(r, 0)$, we determine all the equivalence classes of $PI_{r',s'}$ with $r' + s' = r$ successively for $r = 3, 4, \ldots$ as follows.

(S3) Let us assume that (r', s') with $r' + s' = r$ are such that the all equivalent sets of $PI_{r'-1,s'}$ and $PI_{r',s'-1}$ are already determined. There are possible cases which define the rest of the steps:

$$\ell(r', s') \geq \ell(r'-1, s'), \tag{4.5}$$

$$\ell(r', s') \geq \ell(r', s'-1), \tag{4.6}$$

$$\ell(r', s') \leq \ell(r, 0). \tag{4.7}$$

(S4) If $\ell(r', s') > \ell(r'-1, s')$, then we do nothing. If $\ell(r', s') = \ell(r'-1, s')$, then we include all classes of involutions $PI_{r'-1,s'}$ to $\mathbb{PI}^M_{r',s'}$. We check whether we can add classes of involutions satisfying $(F2)$.

(S5) If $\ell(r', s') > \ell(r', s'-1)$, then we do nothing. If $\ell(r', s') = \ell(r', s'-1)$, then we include all classes of involutions $PI_{r',s'-1}$ to $\mathbb{PI}^M_{r',s'}$. We remove the classes of equivalence of involutions which coincide in steps $(S4)$ and $(S5)$. Note that due to the induction the steps $(S4)$ and $(S5)$ allow us to include all the classes of involutions $PI_{r'',s''} \in \mathbb{PI}^M_{r',s'}$ with $r'' + s'' < r' + s' = r$.

(S6) If $\ell(r', s') < \ell(r, 0)$, then all the involutions were included in steps $(S4)$ and $(S5)$. If $\ell(r', s') = \ell(r, 0)$, then we check all $PI_{r,0} \in \mathbb{PI}^M_{r,0}$ with $r = r' + s'$, whether we can apply property $(F3)$ to the involutions $p \in PI_{r,0}$.

4.5. **Constructions of disconnected groups $\mathcal{S} \in \mathbb{S}^M_{r,s}$ for $0 < r + s < 16$.** Let $\pi_0(PI_{r,s}) = 2$ for $10 \leq r + s \leq 16$.

If $\ell(r, s) = \ell(r + s, 0)$, then the different sets of involutions are determined based on the data of $\mathbb{PI}^M_{r+s,0}$ with $\pi_0(PI) = 2$. We apply the rules for connected sets to each collection of involution in the disconnected set listed in Table 3 for $r = 10, \ldots 16$. We summarize the possible maximal sets $PI \in \mathbb{PI}^M_{r,s}$ in Tables 8 and 9.

If $\ell(r, s) < \ell(r + s, 0)$ then we proceed as in steps (S4)-(S6) of Section 4.4 for each connected subgroup $\mathcal{S}_{(1)}$ in the direct product decomposition $\mathcal{S} = \mathcal{S}_{(1)} \times \mathcal{S}_{(2)}$. We will

TABLE 6. Connected groups for $r+s\leq 16$. Part 1

16	$8,T1,(0,16)$									
15	$7,T1,(0,15)$	$7,T1,(0,15)$ $7,T2,(1,14)$								
14	$6,T1,(0,14)^1$ $6,T1,(0,14)^2$	$7,T2,(1,14)$	$7,T1,(2,14)$ $7,T2,(1,14)$ $7,T2,(2,12)^1$ $7,T2,(2,12)^2$							
13	$5,T1,(0,12)$	$6,T2,(1,12)^1$ $6,T2,(1,12)^2$	$7,T2,(2,12)^1$ $7,T2,(2,12)^2$	$8,T2,(3,12)$						
12	$5,T1,(0,12)$	$6,T2,(1,12)^1$ $6,T2,(1,12)^2$	$7,T2,(2,12)^1$ $7,T2,(2,12)^2$	$8,T2,(3,12)$	$8,T1,(4,12)$ $8,T2,(3,12)$					
11	$4,T1,(0,10)$ $4,T1,(0,8)$	$5,T2,(1,10)$ $5,T2,(1,11)$	$6,T2,(2,11)^2$ $6,T2,(2,11)^3$	$7,T2,(3,11)^2$	$7,T1,(4,11)^2$ $7,T2,(3,11)^2$	$7,T1(4,11)^2$ $7,T2,(3,11)^2$ $7,T2(5,10)$				
10	$4,T1,(0,10)$ $4,T1,(0,8)$	$5,T2,(1,10)$	$5,T1,(2,10)$ $5,T2,(1,10)$ $5,T2,(2,10)$	$6,T2,(3,10)^1$ $6,T2,(3,10)^3$	$6,T1,(4,10)^1$ $6,T2,(3,10)^1$ $6,T1,(4,10)^3$ $6,T2,(3,10)^3$	$7,T2,(5,10)$	$7,T1,(6,10)$ $7,T2,(5,10)$ $7,T2,(6,8)^1$ $7,T2,(6,8)^2$			
9	$4,T1,(0,8)$	$4,T2,(1,8)$ $4,T1,(0,8)$	$4,T1,(2,8)$ $4,T2,(1,8)$ $4,T1,(0,8)$	$5,T2,(3,8)$	$5,T1,(4,8)$ $5,T2,(3,8)$ $5,T2,(4,8)$	$6,T2,(5,8)^1$ $6,T2,(5,8)^2$ $6,T2,(5,8)^3$	$7,T2,(6,8)^1$ $7,T2,(6,8)^2$	$8,T2,(7,8)$		
8	$4,T1,(0,8)$	$4,T2,(1,8)$ $4,T1,(0,8)$	$4,T1,(2,8)$ $4,T2,(1,8)$ $4,T1,(0,8)$	$5,T2,(3,8)$	$5,T1,(4,8)$ $5,T2,(3,8)$ $5,T2,(4,8)$	$6,T2,(5,8)^1$ $6,T2,(5,8)^2$ $6,T2,(5,8)^3$	$7,T2,(6,8)^1$ $7,T2,(6,8)^2$	$8,T2,(7,8)$	$8,T1,(8,8)$ $8,T2,(7,8)$	
7	$3,T1,(0,7)$	$3,T1,(0,7)$ $3,T2,(1,6)$	$3,T1,(0,7)$ $3,T1,(2,6)$ $3,T2,(1,6)$ $3,T2,(2,4)$	$4,T2,(3,6)$ $4,T2,(3,4)$	$4,T1,(4,6)$ $4,T2,(3,6)$ $4,T1,(4,4)$ $4,T2,(3,4)$	$5,T2,(5,6)$ $5,T2,(5,7)$	$6,T2,(6,7)^1$ $6,T2,(6,7)^2$ $6,T2,(6,7)^3$	$7,T2,(7,7)^1$	$7,T1,(8,7)^1$ $7,T2,(7,7)^1$	$7,T2,(9,6)$ $7,T1,(8,7)$ $7,T2,(7,7)$
6	$2,T1,(0,6)$	$3,T2,(1,6)$	$3,T1,(2,6)$ $3,T2,(1,6)$ $3,T2,(2,4)$	$4,T2,(3,6)$ $4,T2,(3,4)$	$4,T1,(4,6)$ $4,T2,(3,6)$ $4,T1,(4,4)$ $4,T2,(3,4)$ $4,T2,(4,5)$	$5,T2,(5,6)$	$5,T1,(6,6)$ $5,T2,(5,6)$ $5,T2,(6,6)$	$6,T2,(7,6)^1$ $6,T2,(7,6)^2$ $6,T2,(7,6)^3$	$6,T1,(8,6)^1$ $6,T2,(7,6)^1$ $6,T1,(8,6)^2$ $6,T2,(7,6)^2$ $6,T1,(8,6)^3$ $6,T2,(7,6)^3$	$7,T2,(9,6)$
5	$1,T1,(0,4)$	$2,T2,(1,4)$	$3,T2,(2,4)$	$4,T2,(3,4)$	$4,T2,(4,5)$ $4,T1,(4,4)$ $4,T2,(3,4)$	$4,T2,(5,4)$ $4,T1,(5,5)$ $4,T2,(4,5)$ $4,T1,(4,4)$ $4,T2,(3,4)$	$4,T1,(6,4)$ $4,T2,(5,4)$ $4,T1,(5,5)$ $4,T2,(4,5)$ $4,T1,(4,4)$ $4,T2,(3,4)$	$5,T2,(7,5)$ $5,T2,(7,4)$	$5,T1,(8,5)$ $5,T2,(7,5)$ $5,T1,(8,4)$ $5,T2,(7,4)$	$6,T2,(9,4)^1$ $6,T2,(9,4)^2$ $6,T2,(9,4)^3$
4	$1,T1,(0,4)$	$2,T2,(1,4)$	$3,T2,(2,4)$	$4,T2,(3,4)$	$4,T1,(4,4)$ $4,T2,(3,4)$	$4,T2,(5,4)$ $4,T1,(4,4)$ $4,T2,(3,4)$	$4,T1,(6,4)$ $4,T2,(5,4)$ $4,T1,(4,4)$ $4,T2,(3,4)$	$5,T2,(7,4)$	$5,T1,(8,4)$ $5,T2,(7,4)$ $5,T2,(8,4)$	$6,T2,(9,4)^1$ $6,T2,(9,4)^2$ $6,T2,(9,4)^3$
3	$\ell=0$	$1,T2,(1,2)$	$2,T2,(2,3)$	$3,T2,(3,3)$	$3,T1,(4,3)$ $3,T2,(3,3)$	$3,T2,(5,2)$ $3,T1,(4,3)$ $3,T2,(3,3)$	$3,T1,(6,2)$ $3,T2,(5,2)$ $3,T2,(6,0)$ $3,T1,(4,3)$ $3,T2,(3,3)$	$4,T2,(7,2)$ $4,T2,(7,0)$	$4,T1,(8,2)$ $4,T2,(7,2)$ $4,T1,(8,0)$ $4,T2,(7,0)$	$5,T2,(9,3)$ $5,T2,(9,2)$
2	$\ell=0$	$1,T2,(1,2)$	$1,T1,(2,2)$ $1,T2,(1,2)$	$2,T2,(3,2)$	$2,T1,(4,2)$ $2,T2,(3,2)$	$3,T2,(5,2)$	$3,T1,(6,2)$ $3,T2,(5,2)$ $3,T2,(6,0)$	$4,T2,(7,2)$ $4,T2,(7,0)$	$4,T1,(8,2)$ $4,T2,(7,2)$ $4,T1,(8,0)$ $4,T2,(7,0)$	$5,T2,(9,2)$
1	$\ell=0$	$\ell=0$	$\ell=0$	$1,T2,(3,0)$	$1,T1,(4,0)$ $1,T2,(3,0)$	$2,T2,(5,0)$	$3,T2,(6,0)$	$4,T2,(7,0)$	$4,T1,(8,0)$ $4,T2,(7,0)$	$4,T2,(9,0)$ $4,T1,(8,0)$ $4,T2,(7,0)$
0	$\ell=0$	$\ell=0$	$\ell=0$	$1,T2,(3,0)$	$1,T1,(4,0)$ $1,T2,(3,0)$	$2,T2,(5,0)$	$3,T2,(6,0)$	$4,T2,(7,0)$	$4,T1,(8,0)$ $4,T2,(7,0)$	$4,T2,(9,0)$ $4,T1,(8,0)$ $4,T2,(7,0)$
s/r	0	1	2	3	4	5	6	7	8	9

not write this cases into Tables 8 and 9, since they can be easily obtained by applying steps (S4)-(S6).

We explain in details the case $r+s=10$ to illustrate the procedure for the situation $\ell(r,s)=\ell(r+s,0)$. In Table 3 there are two disconnected subgroups

$$\mathcal{S}_{10}^{(1)} \quad \text{with the signature} \quad (3,(T1,\theta),(7,0))\times(1,(T2,\pi),(3,0)),$$

$$\mathcal{S}_{10}^{(2)} \quad \text{with the signature} \quad (3,(T2,\theta),(6,0))\times(1,T1,(4,0)).$$

TABLE 7. Connected groups for $r+s \le 16$. Part 2

s/r	10	11	12	13	14	15	16
6	$7,T1,(10,6)$ $7,T2,(9,6)$ $7,T2,(10,4)^1$ $7,T2,(10,4)^2$						
5	$7,T2,(10,4)^1$ $7,T2,(10,4)^2$	$8,T2,(11,4)$					
4	$7,T2,(10,4)^1$ $7,T2,(10,4)^2$	$8,T2,(11,4)$	$8,T1,(12,4)$ $8,T2,(11,4)$				
3	$6,T2,(10,3)^2$ $6,T2,(10,3)^3$	$7,T1,(12,3)^2$	$7,T1,(12,3)^2$ $7,T2,(11,3)^2$	$7,T2,(13,3)^2$ $7,T2,(11,3)^2$ $7,T2,(13,2)$			
2	$5,T1,(10,2)$ $5,T2,(9,2)$ $5,T2,(10,2)$	$6,T2,(11,2)^1$ $6,T2,(11,2)^2$	$6,T2,(11,2)^1$ $6,T1,(12,2)^1$ $6,T2,(11,2)^3$ $6,T1,(12,2)^3$	$7,T2,(13,2)$	$7,T1,(14,2)$ $7,T2,(13,2)$ $7,T2,(14,0)^1$ $7,T2,(14,0)^2$		
1	$4,T1,(10,0)$ $4,T2,(9,0)$ $4,T1,(8,0)$ $4,T2,(7,0)$	$5,T2,(11,0)$	$5,T1,(12,0)$ $5,T2,(11,0)$ $5,T2,(12,0)$	$6,T2,(13,0)^1$ $6,T2,(13,0)^2$ $6,T2,(13,0)^3$	$7,T2,(14,0)^1$ $7,T2,(14,0)^2$	$8,T2,(15,0)$	$8,T1,(16,0)$ $8,T2,(15,0)$
0	$4,T1,(10,0)$ $4,T2,(9,0)$ $4,T1,(8,0)$ $4,T2,(7,0)$	$5,T2,(11,0)$	$5,T1,(12,0)$ $5,T2,(11,0)$ $5,T2,(12,0)$	$6,T2,(13,0)^1$ $6,T2,(13,0)^2$ $6,T2,(13,0)^3$	$7,T2,(14,0)^1$ $7,T2,(14,0)^2$	$8,T2,(15,0)$	$8,T1,(16,0)$ $8,T2,(15,0)$
s/r	10	11	12	13	14	15	16

Consider the case $\mathcal{S}^{(1)}_{10}$. We analyse Tables 6 and 7 and find all possible $(T1)$-type sets of involutions having the signature

$$(3,T1,(r,s)) \quad \text{with} \quad r+s=7.$$

We obtain

$$\begin{array}{llll} & (3,T1,(7,0)) & \text{which comes from } (3,T2,(6,0)) & \text{by using property } (F2) \\ (4.8) & (3,T1,(3,4)) & \text{which comes from } (3,T2,(2,4)) & \text{by using property } (F2) \\ & (3,T1,(4,3)) & \text{and } \ (3,T1,(0,7)) & \text{listed in Tables 6 and 7.} \end{array}$$

Next, we find all possible $(T2)$-type sets of involutions having the signature

$$(1,T2,(r,s)) \quad \text{with} \quad r+s=3.$$

They are the following

$$(1,T2,(3,0)) \quad \text{and} \quad (1,T2,(1,2)). \tag{4.9}$$

At the end we make all possible products of two groups of involutions, where the first one belongs to (4.8) and the second one belongs to (4.9). For instance, we obtain the disconnected subgroups in $\mathbb{PI}^M_{r,s}$ with $r+s=10$:

$$\mathcal{S}^{(1)}_{10} = \begin{array}{l}(3,T1,(7,0)) \\ \times(1,T2,(3,0))\end{array} \in \mathbb{PI}^M_{10,0}, \quad \mathcal{S}^(1)_{6,4} = \begin{array}{l}(3,T1,(3,4)) \\ \times(1,T2,(3,0))\end{array} \in \mathbb{PI}^M_{6,4}, \quad \dots$$

These disconnected groups and all others are listed in Tables 8 and 9.

We do analogous calculations for the disconnected group $\mathcal{S}^{(2)_{10}} \in \mathbb{PI}^M_{10,0}$ and write the results in Tables 8 and 9.

TABLE 8. $\pi_0(PI_{r,s}) = 2$, $r+s \leq 16$, Part 1

16	$\mathcal{S}^{(1)}_{0,16} = (4,T1,(0,8)) \times(4,T1,(0,8))$								
15									
14									
13									
12	$\mathcal{S}^{(1)}_{0,12} = (4,T1,(0,8)) \times(1,T1,(0,4))$	$\mathcal{S}^{(1)}_{1,12} = (4,T1,(0,8)) \times(2,T2,(2,3))$	$\mathcal{S}^{(1)}_{2,12} = (4,T1,(0,8)) \times(3,T2,(2,4))$	$\mathcal{S}_{3,12} = (4,T1,(0,8)) \times(4,T2,(3,4))$	$\mathcal{S}^{(1)}_{4,12} = (4,T1,(4,4)) \times(2,T1,(0,8))$ $\mathcal{S}^{(1)}_{4,12} = \mathcal{S}_{3,12}$				
11		$\mathcal{S}^{(1)}_{1,11} = (3,T1,(0,7)) \times(2,T2,(1,4))$	$\mathcal{S}^{(1)}_{2,11} = (4,T1,(0,8)) \times(2,T2,(2,3))$ $\mathcal{S}^{(3)}_{2,11} = (3,T1,(0,7)) \times(3,T2,(2,4))$	$\mathcal{S}^{(1)}_{3,11} = (4,T1,(0,8)) \times(3,T2,(3,3))$ $\mathcal{S}^{(2)}_{3,11} = (2,T2,(3,4)) \times(3,T1,(0,7))$					
10		$\mathcal{S}^{(1)}_{1,10} = (4,T1,(0,8)) \times(1,T2,(1,2))$	$\mathcal{S}^{(1)}_{2,10} = (4,T1,(0,8)) \times(1,T1,(2,2))$ $\mathcal{S}^{(2)}_{2,10} = (3,T1,(0,7)) \times(2,T2,(2,3))$ $\mathcal{S}^{(3)}_{2,10} = (3,T2,(2,4)) \times(2,T1,(0,6))$ $\mathcal{S}^{(4)}_{2,10} = \mathcal{S}^{(1)}_{1,10}$	$\mathcal{S}^{(1)}_{3,10} = (4,T1,(0,8)) \times(2,T2,(3,2))$ $\mathcal{S}^{(1)}_{3,10} = (4,T2,(3,4)) \times(2,T1,(0,6))$ $\mathcal{S}^{(1)}_{3,10} = (3,T1,(0,7)) \times(3,T2,(3,3))$					
9		$\mathcal{S}^{(1)}_{1,9} = (3,T1,(0,7)) \times(1,T2,(1,2))$		$\mathcal{S}^{(2)}_{3,9} = (3,T1,(0,7)) \times(2,T2,(3,2))$ $\mathcal{S}^{(3)}_{3,9} = (3,T2,(3,3)) \times(2,T1,(0,6))$					
8			$\mathcal{S}^{(2)}_{2,8} = (3,T2,(2,4)) \times(1,T1,(0,4))$	$\mathcal{S}^{(1)}_{3,8} = (4,T1,(0,8)) \times(1,T2,(3,0))$ $\mathcal{S}^{(2)}_{3,8} = (4,T2,(3,4)) \times(1,T1,(0,4))$	$\mathcal{S}^{(1)}_{4,8} = (4,T1,(4,4)) \times(1,T1,(0,4))$ $\mathcal{S}^{(1)}_{4,8} = (4,T1,(0,8)) \times(1,T1,(4,0))$ $\mathcal{S}^{(2)}_{4,8} = (3,T1,(3,4)) \times(2,T2,(1,4))$ $\mathcal{S}^{(3)}_{4,8} = (3,T2,(2,4)) \times(2,T1,(2,4))$ $\mathcal{S}^{(4)}_{4,8} = \mathcal{S}^{(1)}_{3,8}$ $\mathcal{S}^{(5)}_{4,8} = \mathcal{S}^{(2)}_{3,8}$	$\mathcal{S}^{(1)}_{5,8} = (4,T1,(4,4)) \times(2,T2,(1,4))$ $\mathcal{S}^{(1)}_{5,8} = (4,T1,(0,8)) \times(2,T2,(5,0))$ $\mathcal{S}^{(2)}_{5,8} = (4,T2,(3,4)) \times(2,T1,(2,4))$ $\mathcal{S}^{(3)}_{5,8} = (3,T1,(3,4)) \times(3,T2,(2,4))$	$\mathcal{S}^{(1)}_{6,8} = (4,T1,(4,4)) \times(3,T2,(2,4))$ $\mathcal{S}^{(1)}_{6,8} = (4,T1,(0,8)) \times(3,T2,(6,0))$ $\mathcal{S}^{(2)}_{6,8} = (2,T2,(3,4)) \times(3,T1,(3,4))$	$\mathcal{S}^{(1)}_{7,8} = (4,T1,(4,4)) \times(4,T2,(3,4))$ $\mathcal{S}^{(2)}_{7,8} = (4,T1,(0,8)) \times(4,T2,(7,0))$	$\mathcal{S}^{(1)}_{8,8} = (4,T1,(8,0)) \times(2,T1,(0,8))$ $\mathcal{S}^{(2)}_{8,8} = \mathcal{S}^{(1)}_{7,8}$ $\mathcal{S}^{(2)}_{8,8} = \mathcal{S}^{(2)}_{7,8}$
7				$\mathcal{S}^{(1)}_{3,7} = (3,T1,(0,7)) \times(1,T2,(3,0))$ $\mathcal{S}^{(2)}_{3,7} = (3,T2,(3,3)) \times(1,T1,(0,4))$		$\mathcal{S}^{(2)}_{5,7} = (3,T1,(4,3)) \times(2,T2,(1,4))$ $\mathcal{S}^{(2)}_{5,7} = (3,T1,(3,4)) \times(2,T2,(2,3))$ $\mathcal{S}^{(2)}_{5,7} = (3,T1,(0,7)) \times(2,T2,(5,0))$ $\mathcal{S}^{(3)}_{5,7} = (3,T2,(3,3)) \times(2,T1,(2,4))$ $\mathcal{S}^{(3)}_{5,7} = (3,T2,(2,4)) \times(2,T1,(3,3))$	$\mathcal{S}^{(1)}_{6,7} = (4,T1,(4,4)) \times(2,T2,(2,3))$ $\mathcal{S}^{(2)}_{6,7} = (4,T2,(3,4)) \times(2,T1,(3,3))$ $\mathcal{S}^{(3)}_{6,7} = (3,T1,(4,3)) \times(3,T2,(2,4))$ $\mathcal{S}^{(3)}_{6,7} = (3,T1,(3,4)) \times(3,T2,(3,3))$ $\mathcal{S}^{(3)}_{6,7} = (3,T1,(0,7)) \times(3,T2,(6,0))$	$\mathcal{S}^{(1)}_{7,7} = (4,T1,(4,4)) \times(3,T2,(3,3))$ $\mathcal{S}^{(2)}_{7,7} = (2,T2,(7,0)) \times(3,T1,(0,7))$ $\mathcal{S}^{(2)}_{7,7} = (2,T2,(3,4)) \times(3,T1,(4,2))$	
6					$\mathcal{S}^{(1)}_{4,6} = (3,T1,(3,4)) \times(1,T2,(1,2))$ $\mathcal{S}^{(2)}_{4,6} = (3,T2,(2,4)) \times(1,T1,(2,2))$	$\mathcal{S}^{(1)}_{5,6} = (4,T1,(4,4)) \times(1,T2,(1,2))$ $\mathcal{S}^{(2)}_{5,6} = (4,T2,(3,4)) \times(1,T1,(2,2))$	$\mathcal{S}^{(1)}_{6,6} = (4,T1,(4,4)) \times(1,T1,(2,2))$ $\mathcal{S}^{(2)}_{6,6} = (3,T1,(4,3)) \times(2,T2,(2,3))$ $\mathcal{S}^{(2)}_{6,6} = (3,T1,(3,4)) \times(2,T2,(3,2))$ $\mathcal{S}^{(3)}_{6,6} = (3,T2,(6,0)) \times(2,T1,(0,6))$ $\mathcal{S}^{(3)}_{6,6} = (3,T2,(3,3)) \times(2,T1,(3,3))$ $\mathcal{S}^{(3)}_{6,6} = (3,T2,(2,4)) \times(2,T1,(4,2))$ $\mathcal{S}^{(4)}_{6,6} = \mathcal{S}^{(1)}_{5,6}$ $\mathcal{S}^{(5)}_{6,6} = \mathcal{S}^{(2)}_{5,6}$	$\mathcal{S}^{(1)}_{7,6} = (4,T1,(4,4)) \times(2,T2,(3,2))$ $\mathcal{S}^{(2)}_{7,6} = (4,T2,(7,0)) \times(2,T1,(0,6))$ $\mathcal{S}^{(2)}_{7,6} = (4,T2,(3,4)) \times(2,T1,(4,2))$ $\mathcal{S}^{(3)}_{7,6} = (3,T1,(4,3)) \times(3,T2,(3,3))$	
5						$\mathcal{S}^{(1)}_{5,5} = (3,T1,(4,3)) \times(1,T2,(1,2))$ $\mathcal{S}^{(2)}_{5,5} = (3,T2,(3,3)) \times(1,T1,(2,2))$		$\mathcal{S}^{(1)}_{7,5} = (3,T1,(4,3)) \times(2,T2,(3,2))$ $\mathcal{S}^{(2)}_{7,5} = (3,T2,(3,3)) \times(2,T1,(4,2))$	
4							$\mathcal{S}^{(1)}_{6,4} = (3,T1,(3,4)) \times(1,T2,(3,0))$ $\mathcal{S}^{(2)}_{6,4} = (3,T2,(6,0)) \times(1,T1,(0,4))$ $\mathcal{S}^{(2)}_{6,4} = (3,T2,(2,4)) \times(1,T1,(4,0))$	$\mathcal{S}^{(1)}_{7,4} = (4,T1,(4,4)) \times(1,T2,(3,0))$ $\mathcal{S}^{(2)}_{7,4} = (4,T2,(7,0)) \times(1,T1,(0,4))$ $\mathcal{S}^{(2)}_{7,4} = (4,T2,(3,4)) \times(1,T1,(4,0))$	$\mathcal{S}^{(1)}_{8,4} = (4,T1,(8,0)) \times(1,T1,(0,4))$ $\mathcal{S}^{(1)}_{8,4} = (4,T1,(4,4)) \times(1,T1,(4,0))$ $\mathcal{S}^{(2)}_{8,4} = (3,T1,(7,0)) \times(2,T2,(1,4))$ $\mathcal{S}^{(2)}_{8,4} = (3,T1,(3,4)) \times(2,T2,(5,0))$ $\mathcal{S}^{(3)}_{8,4} = (3,T2,(6,0)) \times(2,T1,(2,4))$ $\mathcal{S}^{(3)}_{8,4} = (3,T2,(2,4)) \times(2,T1,(6,0))$ $\mathcal{S}^{(2)}_{8,4} = \mathcal{S}^{(1)}_{7,4}$ $\mathcal{S}^{(3)}_{8,4} = \mathcal{S}^{(2)}_{7,4}$ $\mathcal{S}^{(3)}_{8,4} = \mathcal{S}^{(2)}_{7,4}$
3								$\mathcal{S}^{(1)}_{7,3} = (3,T1,(4,3)) \times(1,T2,(3,0))$ $\mathcal{S}^{(2)}_{7,3} = (3,T2,(3,3)) \times(1,T1,(4,0))$	
2									$\mathcal{S}^{(1)}_{8,2} = (3,T1,(7,0)) \times(1,T2,(1,2))$ $\mathcal{S}^{(2)}_{8,2} = (3,T2,(6,0)) \times(1,T1,(2,2))$
1									
0									
s/r	0	1	2	3	4	5	6	7	8

TABLE 9. $\pi_0(PI_{r,s}) = 2$, $r + s \leq 16$, Part 2

4	$\mathcal{S}^{(1)}_{9,4} = (4,T1,(8,0)) \times(2,T2,(1,4))$ $\mathcal{S}^{(1)}_{9,4} = (4,T1,(4,4)) \times(2,T2,(5,0))$ $\mathcal{S}^{(2)}_{9,4} = (4,T2,(7,0)) \times(2,T1,(2,4))$ $\mathcal{S}^{(2)}_{9,4} = (4,T2,(3,4)) \times(2,T1,(6,0))$ $\mathcal{S}^{(3)}_{9,4} = (3,T1,(7,0)) \times(3,T2,(2,4))$ $\mathcal{S}^{(3)}_{9,4} = (3,T1,(3,4)) \times(3,T2,(6,0))$	$\mathcal{S}^{(1)}_{10,4} = (4,T1,(8,0)) \times(3,T2,(2,4))$ $\mathcal{S}^{(1)}_{10,4} = (4,T1,(4,4)) \times(3,T2,(6,0))$ $\mathcal{S}^{(2)}_{10,4} = (2,T2,(7,0)) \times(3,T1,(3,4))$ $\mathcal{S}^{(2)}_{10,4} = (2,T2,(3,4)) \times(3,T1,(7,0))$	$\mathcal{S}^{(1)}_{11,4} = (4,T1,(8,0)) \times(4,T2,(3,4))$ $\mathcal{S}^{(2)}_{11,4} = (4,T1,(4,4)) \times(4,T2,(7,0))$	$\mathcal{S}^{(1)}_{12,4} = (4,T1,(8,0)) \times(4,T1,(4,4))$ $\mathcal{S}^{(2)}_{12,4} = \mathcal{S}^{(1)}_{11,4}$ $\mathcal{S}^{(2)}_{12,4} = \mathcal{S}^{(2)}_{11,4}$				
3	$\mathcal{S}^{(2)}_{9,3} = (3,T1,(7,0)) \times(2,T2,(2,3))$ $\mathcal{S}^{(2)}_{9,3} = (3,T1,(4,3)) \times(2,T2,(5,0))$ $\mathcal{S}^{(3)}_{9,3} = (3,T2,(6,0)) \times(2,T1,(3,3))$ $\mathcal{S}^{(3)}_{9,3} = (3,T2,(3,3)) \times(2,T1,(6,0))$	$\mathcal{S}^{(1)}_{10,3} = (4,T1,(8,0)) \times(2,T2,(2,3))$ $\mathcal{S}^{(2)}_{10,3} = (4,T2,(7,0)) \times(2,T1,(3,3))$ $\mathcal{S}^{(3)}_{10,3} = (3,T1,(7,0)) \times(3,T2,(3,3))$ $\mathcal{S}^{(3)}_{10,3} = (3,T1,(4,3)) \times(3,T2,(6,0))$	$\mathcal{S}^{(1)}_{11,3} = (4,T1,(8,0)) \times(3,T2,(3,3))$ $\mathcal{S}^{(2)}_{11,3} = (2,T2,(7,0)) \times(3,T1,(4,3))$					
2	$\mathcal{S}^{(1)}_{9,2} = (4,T1,(8,0)) \times(1,T2,(1,2))$ $\mathcal{S}^{(2)}_{9,2} = (4,T2,(7,0)) \times(1,T1,(2,2))$	$\mathcal{S}^{(1)}_{10,2} = (4,T1,(8,0)) \times(1,T1,(2,2))$ $\mathcal{S}^{(2)}_{10,2} = (3,T1,(7,0)) \times(2,T2,(3,2))$ $\mathcal{S}^{(3)}_{10,2} = (3,T2,(6,0)) \times(2,T1,(4,2))$ $\mathcal{S}^{(4)}_{10,2} = \mathcal{S}^{(1)}_{9,2}$ $\mathcal{S}^{(5)}_{10,2} = \mathcal{S}^{(2)}_{9,2}$	$\mathcal{S}^{(1)}_{11,2} = (4,T1,(8,0)) \times(2,T2,(3,2))$ $\mathcal{S}^{(2)}_{11,1} = (2,T2,(7,0)) \times(2,T1,(4,2))$					
1								
0		$\mathcal{S}^{(1)}_{10} = (3,T1,(7,0)) \times(1,T2,(3,0))$ $\mathcal{S}^{(2)}_{10} = (3,T2,(6,0)) \times(1,T1,(4,0))$	$\mathcal{S}^{(1)}_{11} = (4,T1,(8,0)) \times(1,T2,(3,0))$ $\mathcal{S}^{(2)}_{11} = (4,T2,(7,0)) \times(1,T1,(4,0))$	$\mathcal{S}^{(1)}_{12} = (4,T1,(8,0)) \times(1,T1,(4,0))$ $\mathcal{S}^{(2)}_{12} = (3,T1,(7,0)) \times(2,T2,(5,0))$ $\mathcal{S}^{(3)}_{12} = (3,T2,(6,0)) \times(2,T1,(6,0))$ $\mathcal{S}^{(4)}_{12} = \mathcal{S}^{(1)}_{11}$ $\mathcal{S}^{(5)}_{12} = \mathcal{S}^{(2)}_{11}$	$\mathcal{S}^{(1)}_{13} = (4,T1,(8,0)) \times(2,T2,(5,0))$ $\mathcal{S}^{(2)}_{13} = (4,T2,(7,0)) \times(2,T1,(6,0))$ $\mathcal{S}^{(3)}_{13} = (3,T1,(7,0)) \times(3,T2,(6,0))$	$\mathcal{S}^{(1)}_{14} = (4,T1,(8,0)) \times(3,T2,(6,0))$ $\mathcal{S}^{(2)}_{14} = (4,T2,(7,0)) \times(3,T1,(7,0))$	$\mathcal{S}_{15} = (4,T1,(8,0)) \times(4,T2,(7,0))$	$\mathcal{S}^{(1)}_{16} = (4,T1,(8,0)) \times(4,T1,(8,0))$ $\mathcal{S}^{(2)}_{16} = \mathcal{S}_{15}$
s/r	9	10	11	12	13	14	15	16

5. ISOMORPHISM OF INVARIANT INTEGRAL STRUCTURES

Theorem 5.1. *If*

$$(r,s) \in \{(0,0),(1,0),(2,0),(0,1),(1,1),(2,1),(0,2)\}, \tag{5.1}$$

then for any orthonormal basis $B_{r,s} = \{z_j\}$ and $v \in V^{r,s}$, with $\langle v,v\rangle_{V^{1,0}} = \pm 1$ the invariant orthonormal structures spanned by bases as in Table 10 are isomorphic.

TABLE 10. Invariant integral structures for (r,s) in Theorem 5.1

2	$\{v, J_{z_1}v, J_{z_2}v, J_{z_1}J_{z_2}v, z_1, z_2\}$		
1	$\{v, J_{z_1}v, z_1\}$	$\{v, J_{z_1}v, J_{z_2}v, J_{z_1}J_{z_2}v, z_1, z_2\}$	$\{v, J_{z_1}v, J_{z_2}v, J_{z_1}J_{z_2}v, z_1, z_2\}$
0	v	$\{v, J_{z_1}v, z_1\}$	$\{v, J_{z_1}v, J_{z_2}v, J_{z_1}J_{z_2}v, z_1, z_2\}$
s/r	0	1	2

Proof. There are only trivial groups $\mathcal{S} \subset \mathbb{S}^M_{r,s}$ for (r,s) as in (5.1) since there are no involutions. The proof of uniqueness is literally repeats the proof of Theorem 3.18. $\square$

We fix an orthonormal basis $B_{r,s} = \{z_1, \ldots, z_{r+s}\}$ and a group $\mathcal{S} = \mathcal{S}(PI_{r,s})$. Recall the construction of an invariant basis $\mathcal{B}_v(V^{r,s})$ on the minimal admissible module $V^{r,s}$ from Theorem 3.17, which used the centraliser of the isotropy group $\mathcal{S} = \mathcal{S}(PI_{r,s}) = \mathcal{S}_v$ of a unit vector $v \in V^{r,s}$. The invariant integral structure on the Lie algebra $\mathfrak{n}_{r,s}(V^{r,s})$ given by $\mathcal{S}$ will be denoted by

$$\mathcal{L}(\mathcal{S}) = \operatorname{span}_{\mathbb{Z}}\{\mathcal{B}_v(V^{r,s})\} \oplus \operatorname{span}_{\mathbb{Z}}\{B_{r,s}\}.$$

Theorem 5.2. *If two groups $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent; that is there exists a map $C \in O(r,s)$ such that $C(\widehat{S}_1) = \widehat{S}_2$, then the invariant integral structures $\mathcal{L}(\mathcal{S}_1)$ and $\mathcal{L}(\mathcal{S}_2)$ are isomorphic under a map $A \oplus C$, where $A\colon V^{r,s} \to V^{r,s}$ is an orthogonal map with respect to $\langle .\,,.\rangle_{V^{r,s}}$; that is $A^\tau A = \mathrm{Id}_{V^{r,s}}$.*

Proof. The proof is a light generalisation of Theorem 3.18. Let $\mathcal{S}_1 = \mathcal{S}(PI_1)$ and $\mathcal{S}_2 = \mathcal{S}(PI_2)$ be equivalent groups. It implies that there is $C \in \mathrm{O}(r,s)$ such that $C(\widehat{S}_1) = \widehat{S}_2$ where we denoted by the same letter C the extention of the orthogonal map to the group $\mathrm{Cl}^*_{r,s} \subset \mathrm{Cl}_{r,s}$ of invertible elements of the Clifford algebra $\mathrm{Cl}_{r,s}$. Let

$$\mathcal{B}_v(V^{r,s}) = \Big\{v, J_{\sigma_i}(v), J_{\tau_j}(v), J_{\tau_j}J_{\sigma_i}(v) \mid \ \sigma_i, \tau_j, \sigma_i\tau_j \in \Sigma(\mathcal{S}_1)\Big\} \tag{5.2}$$

be the invariant basis, constructed in Theorem 3.17 by making use the eigenspaces of involutions from PI_1. The set PI_1 is equivalent to PI_2 under C. We use the method of Theorem 3.17 and obtain a basis

$$\begin{aligned}\mathcal{B}_w(V^{r,s}) \ &= \ \Big\{w, J_{C(\sigma_i)}(w), J_{C(\tau_j)}(w), J_{C(\tau_j)}J_{C(\sigma_i)}(w) \mid \\ &\qquad C(\sigma_i), C(\tau_j), C(\sigma_i)C(\tau_j) \in \Sigma(\mathcal{S}_2)\Big\},\end{aligned} \tag{5.3}$$

where $\mathcal{S}_2 \cong \mathcal{S}(PI_2) \cong \mathcal{S}(C(PI_1))$ and the set PI_2 was replaced by $C(PI_1)$. Note that since $C(B_{r,s}) = B_{r,s}$ we also have $G(B_{r,s}) = G\big(C(B_{r,s})\big)$.

We construct a correspondence $A\colon \mathcal{B}_v(V^{r,s}) \to \mathcal{B}_w(V^{r,s})$ by

$$\begin{aligned}&v \longmapsto w, \quad J_{\sigma_i}(v) \longmapsto J_{C(\sigma_i)}(w), \quad J_{\tau_j}(v) \longmapsto J_{C(\tau_j)}(w),\\ &J_{\tau_j}(v)J_{\sigma_i}(v) \longmapsto J_{C(\tau_j)}(w)J_{C(\sigma_i)}(w),\end{aligned}$$

and $C\colon z_k \longmapsto C(z_k)$. The correspondence $A \oplus C$ extended to a linear map over $\mathbb{R}$ or $\mathbb{Z}$ is an orthogonal map on $V^{r,s}$ since it maps orthonormal basis (5.2) to orthonormal basis (5.3). To show that the linear map $A \oplus C$ is an isomorphism of invariant integral structures, we argue as in Theorem 3.18. By the invariance of the bases $\mathcal{B}_v(V^{r,s})$ and $\mathcal{B}_w(V^{r,s})$ we have

$$J_{C(z_k)}Au_\alpha = \pm J_{C(\varkappa)}w = \pm AJ_\varkappa v = AJ_{z_k}u_\alpha$$

for any $u_\alpha \in \mathcal{B}_v(V^{r,s})$, $z_k \in B_{r,s}$, and for some $\varkappa \in \Sigma = \{\sigma_i, \tau_j, \tau_j\sigma_i\}$. It implies

$$\begin{aligned}\langle [Au_\alpha, Au_\beta], C(z_k)\rangle_{r,s} \ &= \ \langle J_{C(z_k)}Au_\alpha, Au_\beta\rangle_{V^{r,s}} = \langle AJ_{z_k}u_\alpha, Au_\beta\rangle_{V^{r,s}}\\ &= \ \langle A^\tau AJ_{z_k}u_\alpha, u_\beta\rangle_{V^{r,s}} = \langle J_{z_k}u_\alpha, u_\beta\rangle_{V^{r,s}}\\ &= \ \langle [u_\alpha, u_\beta], z_k\rangle_{r,s}.\end{aligned}$$

for any $u_\alpha, u_\beta \in \mathcal{B}_v(V^{r,s})$ and $z_k \in B_{r,s}$. □

Theorem 5.3. *Let $\mathcal{S}_1, \mathcal{S}_2 \in \mathbb{S}^M$ and $\mathcal{L}(\mathcal{S}_1)$, $\mathcal{L}(\mathcal{S}_2)$ be the corresponding invariant integral structures. If there is an isomorphism*

$$A \oplus C\colon \mathcal{L}(\mathcal{S}_1) \to \mathcal{L}(\mathcal{S}_2) \tag{5.4}$$

with $A\colon V^{r,s} \to V^{r,s}$ such that $A^\tau A = \mathrm{Id}_{V^{r,s}}$, then $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent in the sense of Definition 3.21.

Proof. Let

$$\mathcal{L}(\mathcal{S}_1) = \operatorname{span}_{\mathbb{Z}}\{\mathcal{B}_v(V^{r,s})\} \oplus \operatorname{span}_{\mathbb{Z}}\{B_{r,s}\} = L_1 \oplus \operatorname{span}_{\mathbb{Z}}\{B_{r,s}\}$$

$$\mathcal{L}(\mathcal{S}_2) = \operatorname{span}_{\mathbb{Z}}\{\mathcal{B}_u(V^{r,s})\} \oplus \operatorname{span}_{\mathbb{Z}}\{B_{r,s}\} = L_2 \oplus \operatorname{span}_{\mathbb{Z}}\{B_{r,s}\}$$

be the invariant integral srtuctures generated by the groups $\mathcal{S}_1$ and $\mathcal{S}_2$. Here we also assume that $\mathcal{S}_1 = \mathcal{S}_v$ is the isotropy subgroup of a unit vector $v \in V^{r,s}$ and $\mathcal{S}_2 = \mathcal{S}_u$ is the isotropy subgroup of a unit vector $u \in V^{r,s}$. Since $A \oplus C$ is an isomorphism, we obtain $A(L_1) = L_2$. By noting that $A^{-1}(L_2) = A^{\tau}(L_2) = L_1$, we deduce that $A^{\tau}A(L_1) = L_1$.

We denote by the same letter $A \oplus C \in \operatorname{Aut}(\mathfrak{n}_{r,s})$ the automorphism of $\mathfrak{n}_{r,s}(V^{r,s})$ which restriction to $\mathcal{L}(\mathcal{S}_1)$ gives map (5.4). The properties $A^{\tau}A = \operatorname{Id}_{V^{r,s}}$ and $A^{\tau}J_{C(z)}A = J_z$ imply $AJ_zx = J_{C(z)}Ax$ for $x \in L_1$ and $C \in \mathrm{O}(r,s)$, the latter one being an orthogonal transformation over $\mathbb{Z}$ as well. For $v \in \mathfrak{B}_v(V^{r,s})$ we find a basis vector $u_j \in \mathfrak{B}_u(V^{r,s})$ such that $Av = u_j$. If there holds $Av = -u_j$, then the proof is similar. By renumbering the basis vectors $\{u_j\}$ we can assume that $Av = u$. We have for the stationary group of Av

$$\begin{aligned} \mathcal{S}_{Av} &= \{\tilde{\sigma} \in G\big(C(B_{r,s})\big) \mid \ J_{\tilde{\sigma}}Av = Av\} \\ &= \{\tilde{\sigma} \in G\big(C(B_{r,s})\big) \mid \ J_{\tilde{\sigma}}u = u\} = \mathcal{S}_u \end{aligned} \tag{5.5}$$

Since $\tilde{\sigma} = C(z_{i_1})\dots C(z_{i_k})$, and $AJ_zx = J_{C(z)}Ax$, $x \in L_1$ we have

$$Av = J_{\tilde{\sigma}}Av = J_{C(z_{i_1})}\dots J_{C(z_{i_k})}Av = AJ_{z_{i_1}}\dots J_{z_{i_k}}v = AJ_{\sigma}v.$$

This implies $v = J_{\sigma}v$ for any $\sigma \in G(B_{r,s})$. Thus we conclude that if $\tilde{\sigma} \in \mathcal{S}_{Av}$, for $\tilde{\sigma} = C(z_{i_1})\dots C(z_{i_k}) \in G(C(B_{r,s}))$ then $\sigma = z_{i_1}\dots z_{i_k} \in \mathcal{S}_v$. Thus the groups $\mathcal{S}_{Av}$ and $\mathcal{S}_v$ are equivalent. The equalities (5.5) shows that $\mathcal{S}_2 = \mathcal{S}_u = \mathcal{S}_{Av}$ and $\mathcal{S}_1 = \mathcal{S}_v$ are equivalent. $\square$

Table 11 shows the classical groups $\mathbb{A}$ such that the map $A \oplus \operatorname{Id}$ with $A \in \mathbb{A}$ is the automorphism of H-type Lie algebras $\mathfrak{n}_{r,s}(V^{r,s})$, see also [FM21, Table 3] for non-minimal admissible modules. The groups $\mathrm{Sp}(n), \mathrm{O}(n,\mathbb{C}), \mathrm{U}(n), \mathrm{O}^*(n)$ are subgroups of orthogonal transformations.

Table 11. Groups $\mathbb{A}$

8	$\mathrm{GL}(1,\mathbb{R})$								
7	$\mathrm{O}(1,1,\mathbb{R})$	$\mathrm{U}(1,1)$	$\mathrm{Sp}(1,1)$	$\mathrm{Sp}(1)\times\mathrm{Sp}(1)$					
6	$\mathrm{O}(2,\mathbb{C})$	$\mathrm{O}^*(2)$	$\mathrm{GL}(1,\mathbb{H})$	$\mathrm{Sp}(1)$					
5	$\mathrm{O}^*(4)$	$\mathrm{O}^*(2)\times\mathrm{O}^*(2)$	$\mathrm{O}^*(2)$	$\mathrm{U}(1)$					
4	$\mathrm{GL}(1,\mathbb{H})$	$\mathrm{O}^*(2)$	$\mathrm{O}(1,\mathbb{C})$	$\mathrm{O}(1,\mathbb{R})$	$\mathrm{GL}(1,\mathbb{R})$				
3	$\mathrm{Sp}(1,1)$	$\mathrm{U}(1,1)$	$\mathrm{O}(1,1,\mathbb{R})$	$\mathrm{O}(1,\mathbb{R})\times\mathrm{O}(1,\mathbb{R})$	$\mathrm{O}(1,1,\mathbb{R})$	$\mathrm{U}(1,1)$	$\mathrm{Sp}(1,1)$	$\mathrm{Sp}(1)\times\mathrm{Sp}(1)$	
2	$\mathrm{Sp}(2,\mathbb{C})$	$\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{GL}(2,\mathbb{R})$	$\mathrm{O}(2,\mathbb{R})$	$\mathrm{O}(2,\mathbb{C})$	$\mathrm{O}^*(2)$	$\mathrm{GL}(1,\mathbb{H})$	$\mathrm{Sp}(1)$	
1	$\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{Sp}(2,\mathbb{R})\times\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{Sp}(4,\mathbb{R})$	$\mathrm{U}(2)$	$\mathrm{O}^*(4)$	$\mathrm{O}^*(2)\times\mathrm{O}^*(2)$	$\mathrm{O}^*(2)$	$\mathrm{U}(1)$	
0		$\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{Sp}(2,\mathbb{C})$	$\mathrm{Sp}(1)$	$\mathrm{GL}(1,\mathbb{H})$	$\mathrm{O}^*(2)$	$\mathrm{O}(1,\mathbb{C})$	$\mathrm{O}(1,\mathbb{R})$	$\mathrm{GL}(1,\mathbb{R})$
	0	1	2	3	4	5	6	7	8

Theorem 5.4. *Let (r,s) be such that the groups $\mathbb{A}$ in Table 11 is a subgroup of orthogonal transformations. The groups $\mathcal{S}_1,\mathcal{S}_2\in\mathbb{S}^M_{r,s}$ are equivalent in sense of Definition* (3.22)*, if and only if the corresponding invariant integral structures $\mathcal{L}(\mathcal{S}_1)$ and $\mathcal{L}(\mathcal{S}_2)$ are isomorphic.*

Proof. If (r,s) as in the statement of Theorem 5.4, then for an automorphism $\tilde{A}\oplus\mathrm{Id}$ of $\mathfrak{n}_{r,s}(V^{r,s})$ we have $\tilde{A}^\tau\tilde{A}=\mathrm{Id}_{V^{r,s}}$. It implies that the general automorphisms $A\oplus C$ of $\mathfrak{n}_{r,s}(V^{r,s})$ also satisfies $A^\tau A=\mathrm{Id}_{V^{r,s}}$, see [FM21, Section 3.2].

Thus if the invariant integral structures $\mathcal{L}(\mathcal{S}_1)$ and $\mathcal{L}(\mathcal{S}_2)$ are isomorphic, then they will be isomorphic under a map $A\oplus C$ with $A^\tau A=\mathrm{Id}_{V^{r,s}}$. It implies that the group $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent by Theorem 5.3.

Conversely, if we assume now that the groups $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent, then by Theorem 5.2 the corresponding invariant integral structures will be isomorphic. □

Note that in the proof of Theorem 5.3 the crucial assumption was $A^\tau A=\mathrm{Id}_{V^{r,s}}$. The following theorem shows that it is enough to find a subset $E\subset V^{r,s}$, which is invariant under the action $A^\tau A$. It allows to prove the general theorem.

Theorem 5.5. *The groups $\mathcal{S}_1,\mathcal{S}_2\in\mathbb{S}^M_{r,s}$ are equivalent in sense of Definition* (3.22)*, if and only if the corresponding invariant integral structures $\mathcal{L}(\mathcal{S}_1)$ and $\mathcal{L}(\mathcal{S}_2)$ are isomorphic.*

Proof. If $\mathcal{S}_1$ is equivalent to $\mathcal{S}_2$, then the corresponding invariant integral structures $\mathcal{L}(\mathcal{S}_1)$ and $\mathcal{L}(\mathcal{S}_2)$ are isomorphic by Theorem 5.2.

Suppose that invariant integral structures $\mathcal{L}(\mathcal{S}_1)$ and $\mathcal{L}(\mathcal{S}_2)$ are isomorphic. By contrary we assume that the groups $\mathcal{S}_1=\mathcal{S}(PI_1)\in\mathbb{S}^M_{r,s}$ and $\mathcal{S}_2=\mathcal{S}(PI_2)\in\mathbb{S}^M_{r,s}$ are not equivalent. Then there are $q_1\in PI_1$ and $q_2\in PI_2$ such that $q_1\cdot q_2=-q_2\cdot q_1$. For if $q_1\cdot p=p\cdot q_1$ for all $p\in PI_2$, then $q_1\in PI_2$, which would contradict to the maximality of $\mathcal{S}(PI_2)$.

Without loss of generality we can assume that the groups $\mathcal{S}(PI_1)$ and $\mathcal{S}(PI_2)$ are written in the standard form as in Example 3.3. Let k be a maximal number of type T_1 involutions p_j satisfying $p_j\in PI_1\cap PI_2$, $j=1,\dots k$. Note that $k<\ell(r,s)$ since $\mathcal{S}(PI_1)$ and $\mathcal{S}(PI_1)$ are not equivalent.

Let $\mathfrak{n}_{r,s}(V^{r,s})$ be a pseudo H-type Lie algebra and

$$E=\{x\in V^{r,s}\mid\ J_{p_j}x=x,\ p_j\in PI_1\cap PI_2,\ j=1,\dots,k\}.$$

Since $q_1p_j=p_jq_1$ and $q_2p_j=p_jq_2$ the subspace $E\subset V^{r,s}$ is invariant under the action of both J_{q_1} and J_{q_2}.

For an isomorphism $A\oplus C\colon\mathcal{L}(\mathcal{S}_1)\to\mathcal{L}(\mathcal{S}_2)$ we set

$$F=A(E)=\{Ax\in V^{r,s}\mid\ AJ_{p_j}x=J_{C(p_j)}Ax=Ax,\ p_j\in PI_1\cap PI_2,\ j=1,\dots,k\}.$$

The map C, extended to the Clifford algebra $\mathrm{Cl}_{r,s}$, satisfies $C(p_j)C(q_1)=C(q_1)C(p_j)$ and $C(p_j)C(q_2)=C(q_2)C(p_j)$, $j=1,\dots k$. These imply that $A(E)$ is invariant under the action of $J_{C(q_1)}$ and $J_{C(q_2)}$ by the same arguments as for E. Thus the direct sum

$$F=F_+\oplus F_-,\qquad F_+=\{y\in F\mid\ J_{C(q_2)}y=y\},\quad F_-=\{y\in F\mid\ J_{C(q_2)}y=y\}\tag{5.6}$$

is the orthogonal sum of non-trivial vector spaces.

Let $x \in E$ and put $Ax = y_+(x) + y_-(x)$, where $y_+(x) \in F_+$ and $y_-(x) \in F_-$. We have for the type T_1 involution $q_1 \in PI_1$ that

$$J_{C(q_1)}Ax = J_{C(q_1)}(y_+(x) + y_-(x)) = J_{C(q_1)}y_+(x) + J_{C(q_1)}y_-(x).$$

Since $C(q_1)C(q_2) = -C(q_2)C(q_1)$ we obtain

$$J_{C(q_1)}\colon F_+ \to F_-, \quad \text{and} \quad J_{C(q_1)}y_+(x) \in F_-, \quad J_{C(q_1)}y_-(x) \in F_+,$$

and therefore $y_+(x) = J_{C(q_1)}y_-(x)$ and $y_-(x) = J_{C(q_1)}y_+(x)$ by the uniqueness of the decomposition into a direct sum of vector spaces. We conclude

$$J_{C(q_1)}Ax = y_+(x) + J_{C(q_1)}y_+(x). \tag{5.7}$$

Since p_j are T_1-type involutions, we obtain

$$AJ_{p_j} = J_{C(p_j)}A, \quad J^{\tau}_{p_j} = J_{p_j}, \quad J^{\tau}_{C(p_j)} = J_{C(p_j)}, \quad A^{\tau}J_{C(p_j)} = J_{p_j}A^{\tau}, \quad j = 1, \dots, k$$

It implies $A^{\tau}A(E) = E$. Let $\{v_i\}$ be an orthonormal basis of the space E, which is a part of the invariant basis for $V^{r,s}$ defined by the $\mathcal{S}_1 = \mathcal{S}(PI_1)$. The matrix components a_{ij} of the operator $A^{\tau}A\colon E \to E$ with respect to the basis $\{v_i\}$ have the form

$$\begin{aligned}
a_{ij} &= \langle A^{\tau}Av_i, v_j\rangle_{V^{r,s}} = \langle Av_i, Av_j\rangle_{V^{r,s}} = \langle C(q_1), C(q_1)\rangle_{r,s}\langle Av_i, Av_j\rangle_{V^{r,s}} \\
&= \langle J_{C(q_1)}Av_i, J_{C(q_1)}Av_j\rangle_{V^{r,s}} \\
&= \langle y_+(v_i) + J_{C(q_1)}(y_+(v_i)), y_+(v_j) + J_{C(q_1)}y_+(v_j)\rangle_{V^{r,s}} \\
&= 2\langle y_+(v_i), y_+(v_j)\rangle_{V^{r,s}},
\end{aligned}$$

where we used (5.6) and (5.7). Hence the non-vanishing components of the matrix $A^{\tau}A$ restricted to E are always even numbers, so that $\det A^{\tau}A = 2^{\dim E} \cdot k$, $k \in \mathbb{Z}$.

Let us look on the structure of the map $A^{\tau}A$ acting on the entire minimal admissible module $V^{r,s}$. The space $V^{r,s}$ is an orthogonal sum of subspaces W_i of the form $J_{\varkappa_i}(E) = W_i$ with $\varkappa_i \in \Sigma$ from Theorem 3.17. Let $x \in V^{r,s}$. We write

$$x = x_E + x_1 + \ldots + x_m, \quad x_i \in W_i$$

and $A^{\tau}A|_{W_i}$ for the restriction of the map $A^{\tau}A$ on the set W_i. For any $x_i \in W_i$ there is $y_E \in E$ such that $J_{\varkappa_i}(y_E) = x_i$. Choose $i \in 1, \dots, m$ and assume that $\varkappa_i$ is a product of an even number of the basis vectors. Then we obtain

$$A^{\tau}A|_{W_i}(x_i) = A^{\tau}A|_{W_i}J_{\varkappa_i}(y_E) = J_{\varkappa_i}A^{\tau}A|_E(y_E) = J_{\varkappa_i}A^{\tau}A|_E J^{-1}_{\varkappa_i}(x_i).$$

Thus in this case $A^{\tau}A|_{W_i} = J_{\varkappa_i}A^{\tau}A|_E J^{-1}_{\varkappa_i}$. If τ_i is a product of an odd number of the basis vectors, then the isomorphism condition (2.4) for $A \oplus C$ implies $A^{\tau}AJ_{\varkappa_i}A^{\tau}A = J_{\varkappa_i}$. This leads to

$$A^{\tau}A|_{W_i}(x_i) = A^{\tau}A|_{W_i}J_{\varkappa_i}(y_E) = J_{\varkappa_i}(A^{\tau}A|_E)^{-1}(y_E) = J_{\varkappa_i}(A^{\tau}A|_E)^{-1}J^{-1}_{\varkappa_i}(x_i)$$

and therefore $A^{\tau}A|_{W_i} = J_{\varkappa_i}(A^{\tau}A|_E)^{-1}J^{-1}_{\varkappa_i}$.

The elements of the matrix A of the isomorphism of the lattices $\mathcal{L}(\mathcal{S}_1)$ and $\mathcal{L}(\mathcal{S}_2)$ are all integers. If all W_i are images $J_{\varkappa_i}(E)$ with $\varkappa_i$ being a product of an even number of the basis vectors, then it is clear that $A^{\tau}A \notin \mathrm{SL}(\dim V^{r,s}, \mathbb{Z})$. If there is

$W_i = J_{\varkappa_i}(E)$ with $\varkappa_i$ being a product of an odd number of the basis vectors, then $\det A^\tau A|_{W_i} = \frac{1}{2^{\dim E}.k}$ which contradicts to the fact that the terms of the matrix $A^\tau A$ are all integers. □

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