RELATIVE KOSZUL CORESOLUTIONS AND RELATIVE BETTI NUMBERS

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Abstract. Let G be a generator and a cogenerator in the category of finitely generated right A-modules for a finite-dimensional algebra A over a fieled $\Bbbk$, and $\mathscr{I}$ the additive closure of G. We will define a $\mathscr{I}$-relative Koszul coresolution $\mathscr{K}^{\bullet}(V)$ of an indecomposable direct summand V of G, and show that for a finitely generated A-module M, the $\mathscr{I}$-relative i-th Betti number for M at V is given as the $\Bbbk$-dimension of the i-th homology of the $\mathscr{I}$-relative Koszul complex $\mathscr{K}_V(M)_{\bullet} := \operatorname{Hom}_A(\mathscr{K}^{\bullet}(V), M)$ of M at V for all $i \geq 0$. This is applied to investigate the minimal interval resolution/coresolution of a persistence module M, e.g., to check the interval decomposability of M, and to compute the interval approximation of M.

1. Introduction

In topological data analysis (TDA for short), persistent homology plays an important role in examining the topological property of the data [20], in which context the data are usually given in the form of a point cloud (a finite subset of a finite-dimensional Euclidean space). Given one-parameter filtrations arising from the data, the persistent homology yields representations of a Dynkin quiver Q of type A, thus modules over the path category of Q, that are sometimes called 1-*dimensional persistence modules* [27, 11, 14]. The product quiver of d Dynkin quiver of type A with full commutativity relations for some $d \geq 1$ is called dD-grid. By considering multi-parameter filtrations, representations of dD-grid naturally arise in practical settings, which are called d-dimensional persistence modules [14, 19]. Since the linear category defined by this quiver with relations can be regarded as the incidence category of a poset, persistence modules are understood as modules over the incidence category $\Bbbk[\mathbf{P}]$ of a poset $\mathbf{P}$ in general, or equivalently, functors from $\mathbf{P}$ (regarded as a category) to the category $\operatorname{mod}\Bbbk$ of finite-dimensional vector spaces over a field $\Bbbk$. In this paper we restrict ourselves to finite posets. In this case, the incidence category $\Bbbk[\mathbf{P}]$ can be regarded as the incidence algebra $\Bbbk\mathbf{P}$, and persistence modules are nothing but modules over $\Bbbk\mathbf{P}$. We assume that persistence modules are finite-dimensional

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over $\Bbbk$ unless otherwise stated. The category of (right) modules over an algebra A is denoted by $\operatorname{mod} A$.

For a persistence module $M \in \operatorname{mod} \Bbbk\mathbf{P}$, recall that if

$$\cdots \to \bigoplus_{a\in\mathbf{P}} P_a^{\beta_M^1(a)} \to \bigoplus_{a\in\mathbf{P}} P_a^{\beta_M^0(a)} \to M \to 0$$

is a minimal projective resolution of M, where P_a is the projective indecomposable $\Bbbk\mathbf{P}$-module corresponding to each $a \in \mathbf{P}$, then the number $\beta_M^i(a)$ is uniquely determined by M, and is called the i-th *Betti number* for M at a ([22]). A minimal projective resolution of M gives important (homological) information of M using projective modules and morphisms between them, and the Betti numbers give us all the terms of the resolution. Thus Betti numbers are useful to investigate the structure of M.

Let $\{V_I \mid I \in \mathbb{I}\}$ be a finite set of indecomposable modules in $\operatorname{mod} A$ and $\mathscr{I}$ the full subcategory of $\operatorname{mod} A$ consisting of all finite direct sums of modules in this set. Analogous to projective resolutions of a module M over an algebra A in homological algebra, there is a notion of $\mathscr{I}$-resolutions of M in $\mathscr{I}$-relative homological algebra, which measures M by using modules in $\mathscr{I}$ instead of projective modules (see e.g., [9, 21, 23], also [12, 13] for applications to TDA). Analogous to a minimal projective resolution, a minimal $\mathscr{I}$-resolution

$$\cdots \to \bigoplus_{I\in\mathscr{I}} V_I^{\beta_M^1(a)} \to \bigoplus_{I\in\mathscr{I}} V_I^{\beta_M^0(a)} \to M \to 0$$

of M is defined by $\mathscr{I}$-ralative homological algebra (Definition 2.8). Similar to the case of the standard homological algebra, the numbers $\beta_M^i(a)$ are uniquely determined by M, and is called the $\mathscr{I}$-relative i-th *Betti number* for M at a ([16]).

In the paper [18], Chachólski et. al. gave an effective way to compute relative Betti numbers for a persistence module M, using the so-called Koszul complexes, which is a complex $\mathscr{K}_a(M)_\bullet$

$$\cdots \to \mathscr{K}_a(M)_d \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathscr{K}_a(M)_1 \xrightarrow{\partial} \mathscr{K}_a(M)_0 \to 0$$

of vector spaces that satisfies

$$\beta_M^d(a) = \dim H_d(\mathscr{K}_a(M)_\bullet).$$

This property of the Koszul complex suggests us the following problem.

Problem 1.1. *Let A be a finite-dimensional algebra, M a right A-module, V_I indecomposable A-modules for all $I \in \mathbb{I}$, where $\mathbb{I}$ is a finite set. Set $\mathscr{I}$ to be the class of A-modules that are isomorphic to finite direct sums of V_I $(I \in \mathbb{I})$. Let*

$$\cdots \to \bigoplus_{I\in\mathscr{I}} V_I^{\beta_M^1(a)} \to \bigoplus_{I\in\mathscr{I}} V_I^{\beta_M^0(a)} \to M \to 0$$

be a minimal $\mathscr{I}$-resolution of M. In this setting, define a complex $\mathscr{K}_I(M)_\bullet$ such that

$$\beta_M^d(I) = \dim H_d(\mathscr{K}_I(M)_\bullet). \tag{1.1}$$

We will solve this problem in the case where $G := \bigoplus_{I \in \mathbb{I}} V_I$ is both a generator and a cogenerator in mod A, that is, where the set $\{V_I \mid I \in \mathbb{I}\}$ contains all projective indecomopsables and all injective indecomposables up to isomorphisms, by developing an idea used in [18, Theorem 3.8]. Analogous to the above, we call the defined complex $\mathscr{K}_I(M)_\bullet$ the *Koszul complex* of M at I.

To define it, we introduce the *$\mathscr{I}$-relative Koszul coresolution* of V, which corresponds to the augmented chain complex of the standard simplex considered in Observation 2 in the proof of [18, Theorem 3.8]. Observation 2 gives a canonical projective resolution of each simple left module over the incidence algebra of a poset. Similar to this, our $\mathscr{I}$-relative Koszul coresolution is a sequence that yields also the minimal projective resolution of the simple left module corresponding to V_I over the endomorphism algebra of G. Instead of following Observation 3 in the paper that uses induction, we gave an alternative proof using the fact that the extension groups are defined in two ways: (i) using a projective resolution of the first variable, and (ii) using an injective resolution of the second variable, which led us to a definition (Definition 3.2) of the $\mathscr{I}$-relative Koszul coresolution $\mathscr{K}^\bullet(V_I)$ of V_I, which defines the desired $\mathscr{I}$-ralative Koszul complex $\mathscr{K}_I(M)_\bullet$ of M at V_I (again Definition 3.2). Our main theorem (Theorem 3.7) verifies the equality (1.1). As an application, this will be used to check whether M is in $\mathscr{I}$ or not.

In application, we set A to be the incidence algebra of a finite poset $\mathbf{P}$, $\mathbb{I}$ to be the set of all intervals in $\mathbf{P}$, and V_I to be the interval module defined by I for all $I \in \mathbb{I}$ (Definition 4.3 (2)). To compute a minimal right/left interval approximation in examples, we give a handy criterion for a homomorphism to be a right/left interval approximation (Propositions 4.9/4.9′), which immediately yields a way to construct a minimal right/left interval approximation (Corollary 4.11/4.11′). This gives us a way to check whether a given module M is interval decomposable or not (Definition 4.3 (3)) by using the 0-th interval Betti numbers $\beta^0_M(I)$ (Convention 4.4) only for intervals I such that V_I appears both as a submodule and a factor module of M (Corollary 4.14), where the main theorem is used to compute $\beta^0_M(I)$. This problem was also considered in [3] and [4] that give different ways to check the interval decomposability. The interval resolutions were also investigated in [12].

The paper is organized as follows. In Sect. 2, we collect necessary terminologies, such as minimal right (left) $\mathscr{I}$-approximations. In Sect. 3, we introduce a concept of relative Koszul coresolution of each indecomposable direct summand V_I of G, which defines relative Koszul complex of an A-module M at I, and prove our main theorem. In Sect. 4 we apply our result to interval resolutions of persistence modules as explained above, and give some examples. Finally, in Sect. 5, we apply our main theorem to the formulas of the "compressed multiplicities" and the "interval replacement" given in [5, Theorem 5.5 and Corollary 5.7].

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2. Preliminaries

Throughout this paper, $\Bbbk$ is a field, A is a finite-dimensional $\Bbbk$-algebra. We denote by $D = \mathrm{Hom}_{\Bbbk}(\text{-}, \Bbbk)$ the usual self-duality of A, and by $\mathrm{mod}\, A$ the category of finite-dimensional right A-modules. In particular, the category of finite-dimensional $\Bbbk$-vector spaces is denoted by $\mathrm{mod}\, \Bbbk$. Moreover, $\{V_I \mid I \in \mathbb{I}\}$ is a finite set of mutually non-isomorphic indecomposable modules in $\mathrm{mod}\, A$, and we set $G := \bigoplus_{I \in \mathbb{I}} V_I$. Then the endomorphism algebra $\Lambda := \mathrm{End}_A(G)$ of G turns out to be a *basic* finite-dimensional algebra, namely, as a right Λ-module, Λ is the direct sum of mutually non-isomorphic indecomposable modules. We regard G as a Λ-A-bimodule.

For a collection $\mathscr{S}$ of right A-modules, we denote by $\mathrm{add}\, \mathscr{S}$ the full subcategory of $\mathrm{mod}\, A$ consisting of all direct summands of the finite direct sums of modules in $\mathscr{S}$, and set $\mathscr{I} := \mathrm{add}\{V_I \mid I \in \mathbb{I}\} = \mathrm{add}\{G\}$. A right A-module M is called a *generator* (resp. *cogenerator*) if the right A-module A (resp. DA) is contained in $\mathrm{add}\{M\}$. We are interested in the case where G is a generator and a cogenerator, that is, where $A, DA \in \mathscr{I}$.

We sometimes use the abbreviation ${}_A(\text{-}, ?) := \mathrm{Hom}_A(\text{-}, ?)$ for variables -, ? in $\mathrm{mod}\, A$. We denote by $\ell(M)$ the composition length of an A-module M. For each morphism $f\colon X \to Y$ in $\mathrm{mod}\, A$, we set $\mathrm{dom}(f) := X$, $\mathrm{cod}(f) := Y$. For a positive integer n, we set $[n] := \{1, \dots, n\}$.

We refer the reader to [7] for fundamental facts on quivers, algebras, and the representation theory of them.

Convention 2.1. Let (Q, ρ) be a *bound quiver*, i.e., a pair of a finite quiver Q and an admissible ideal ρ of the path algebra $\Bbbk Q$ of Q, and suppose that A is defined as $A = \Bbbk(Q, \rho) := \Bbbk Q/\rho$. Here, we write the composite of paths in Q *from the left to the right* (call it *left-to-right notation*), but that of linear maps between vector spaces *from the right to the left* (call it *right-to-left notation*) so that representations of the bound quiver (Q, ρ) present *right* A-modules as the convention used in [7].

For example the composite of the paths a_1 from a vertex 1 to 2 and a_2 from 2 to 3 in Q is denoted by $a_1 a_2$, which is a path from 1 to 3. If M is a representation of (Q, ρ), then the composite of linear maps $M(1) \xrightarrow{M(a_1)} M(2) \xrightarrow{M(a_2)} M(3)$ is denoted by $M(a_2) \circ M(a_1)$ defined by $(M(a_2) \circ M(a_1))(x) = M(a_2)(M(a_1)(x))$ for all $x \in M(1)$, which is equal to $M(a_1 a_2)$. Therefore, if we use the abbreviation $M(a)(x) = xa$ for a path $a\colon u \to v$ in Q and $x \in M(u)$ (this is used in the proof of Lemma 4.8), then the equality $M(a_2)(M(a_1)(x)) = M(a_1 a_2)(x)$ shows that $(xa_1)a_2 = x(a_1 a_2)$ for all $x \in M(1)$. Thus M can be viewed as a right A-module, and is naturally extended to a contravariant functor $\Bbbk[Q, \rho] \to \mathrm{mod}\, \Bbbk$, where $\Bbbk[Q, \rho]$ (see Remark 5.1 for details) is a category version of $\Bbbk(Q, \rho)$ whose composition is also written by left-to-right notation.

Note that this convention is different from that used in our earlier papers [3, 4, 5], where the composite of paths in Q is written by *right-to-left* notation. In particular, with this convention, the computation of almost split sequences starting from (or ending in) interval modules are explained in detail in [3, Sect. 5]. Therefore, care is needed when this is applied in examples in Sect. 4.

To well understand the notion of right (resp. left) minimal morphisms that is reviewed below, we first recall the following well-known category constructions.

Definition 2.2 (Comma categories)**.** Let M be in $\operatorname{mod} A$.

(1) We define a category $\mathscr{C}_M := (\operatorname{mod} A)\downarrow M$ as follows. The set of objects is given by $(\mathscr{C}_M)_0 := \bigsqcup_{X \in \operatorname{mod} A} \operatorname{Hom}_A(X, M)$, and for any $f, f' \in (\mathscr{C}_M)_0$, we set $\mathscr{C}_M(f, f') := \{g \in \operatorname{Hom}_A(\operatorname{dom}(f), \operatorname{dom}(f')) \mid f = f'g\}$. The composition of $\mathscr{C}_M$ is given by that of $\operatorname{mod} A$.

Note that for any $g \in \mathscr{C}_M(f, f')$, g is an isomorphism in $\mathscr{C}_M$ if and only if g is an isomorphism in $\operatorname{mod} A$, and that $\mathscr{C}_M(f, f') \neq \emptyset$ if and only if f factors through f' in $\operatorname{mod} A$.

(2) We define a category $\mathscr{C}^M := M\downarrow(\operatorname{mod} A)$ as follows. The set of objects is given by $\mathscr{C}^M_0 := \bigsqcup_{X \in \operatorname{mod} A} \operatorname{Hom}_A(M, X)$, and for any $f, f' \in \mathscr{C}^M_0$, we set $\mathscr{C}^M(f, f') := \{g \in \operatorname{Hom}_A(\operatorname{cod}(f), \operatorname{cod}(f')) \mid gf = f'\}$. The composition of $\mathscr{C}^M$ is given by that of $\operatorname{mod} A$.

Note that for any $g \in \mathscr{C}^M(f, f')$, g is an isomorphism in $\mathscr{C}$ if and only if g is an isomorphism in $\operatorname{mod} A$, and that $\mathscr{C}^M(f, f') \neq \emptyset$ if and only if f' factors through f in $\operatorname{mod} A$.

We cite the following terminologies from [8] and [10].

Definition 2.3 (Right minimal morphisms)**.** Let M be in $\operatorname{mod} A$.

(1) A morphism $f \colon X \to M$ in $\operatorname{mod} A$ is said to be *right minimal* if for each $h \in \operatorname{End}_A(X)$, the condition $f = fh$ implies that h is an automorphism, or equivalently, if $\mathscr{C}_M(f, f)$ consists only of automorpisms of f in $\mathscr{C}_M$.
(2) Morphisms $f, f' \in \mathscr{C}_M$ are said to be *equivalent* if $\mathscr{C}_M(f, f') \neq \emptyset$ and $\mathscr{C}_M(f', f) \neq \emptyset$. The equivalence class containing f is denoted by $[f]$.
(3) Let $f, f' \in \mathscr{C}_M$. Then f' is called a *right minimal version* of f if f' is right minimal and $[f] = [f']$.

Remark 2.4. The following facts are well-known for an $M \in \operatorname{mod} A$.

(1) By [10, Proposition 2.1], a morphism $f \colon X \to M$ in $\operatorname{mod} A$ is right minimal if and only if $\ell(\operatorname{dom}(f))$ is the smallest among $\{\ell(\operatorname{dom}(g)) \mid g \in [f]\}$. In particular, any $f \in \mathscr{C}_M$ has its right minimal version, which is unique up to isomorphism in $\mathscr{C}_M$.
(2) [10, Theorem 2.2]: For any $f \colon X \to M$ in $\operatorname{mod} A$, there exists a decomposition $X = X_1 \oplus X_2$ such that $f|_{X_1}$ is right minimal and $f|_{X_2} = 0$. Moreover, $f|_{X_1}$ is a right minimal version of f.
(3) [10, Corollary 2.3]: A morphism $f \colon X \to M$ is right minimal if and only if $\operatorname{Ker} f$ contains no nonzero direct summands of X. The latter condition is equivalent to saying that a section $s \colon X' \to X$ is 0 if $fs = 0$.

Definition 2.5 (Right approximations)**.** Let M be in $\operatorname{mod} A$.

(1) A *right $\mathscr{I}$-approximation* of M is a morphism $f\colon X \to M$ in $\operatorname{mod} A$ with $X \in \mathscr{I}$ such that for any $Z \in \mathscr{I}$, ${}_A(Z, f)\colon {}_A(Z, X) \to {}_A(Z, M)$ is surjective. Note that if G is a generator, this f necessarily is an epimorphism.

(2) A *minimal right $\mathscr{I}$-approximation* of M is a right $\mathscr{I}$-approximation of M that is right minimal.

Remark 2.6. The following facts are well-known for an $M \in \operatorname{mod} A$.

(1) Since $\mathbb{I}$ is a finite set, there always exist right $\mathscr{I}$-approximations of M.

(2) Let $f, f' \in \mathscr{C}_M$. In the case where $\operatorname{dom}(f), \operatorname{dom}(f') \in \mathscr{I}$ and $\mathscr{C}(f, f') \neq \emptyset$, if f is a right $\mathscr{I}$-approximation of M, then so is f'. In particular, moreover when $[f] = [f']$, we see that f is a right $\mathscr{I}$-approximation of M if and only if so is f'.

(3) By (2) above and Remark 2.4(2), among right $\mathscr{I}$-approximations of M, we can always choose a minimal right $\mathscr{I}$-approximation of M, which is uniquely determined by M up to isomorphism in $\mathscr{C}_M$ by Remark 2.4(1).

For a later use in Sect. 4, we here give a way how to compute a minimal right $\mathscr{I}$-approximation from a right $\mathscr{I}$-approximation (the procedure in the statement (3) below for $\sigma = \mathbb{1}_{[n]}$), which would also be helpful to make a computer program.

Lemma 2.7. *Assume that a morphism $f = (f_i)_{i\in[n]}\colon \bigoplus_{i\in[n]} X_i \to M$ in $\operatorname{mod} A$ is a right $\mathscr{I}$-approximation with n a positive integer and X_i indecomposable for all $i \in [n]$, and let $S \subseteq [n]$. For each $T \subseteq [n]$, we set $X_T := \bigoplus_{i\in T} X_i$ and $f_T := (f_i)_{i\in T}$ for short. Then the following are equivalent.*

(1) *$f_S\colon X_S \to M$ is a minimal right $\mathscr{I}$-approximation of M.*

(2) *f_S is a right $\mathscr{I}$-approximation of M, and for each $j \in S$, $f_{S\setminus\{j\}}$ is not a right $\mathscr{I}$-approximation of M.*

(3) *There exists some permutation σ of $[n]$ such that $S = I^\sigma_n$, where for each $0 \le m \le n$, I^σ_m is defined inductively as follows: For $m = 0$, we set $I^\sigma_0 := [n]$; and for each $m \in [n]$, we set*

$$I^\sigma_m := \begin{cases} I^\sigma_{m-1} \setminus \{\sigma(m)\} & \text{if } f_{I^\sigma_{m-1}\setminus\{\sigma(m)\}} \text{ is a right } \mathscr{I}\text{-approximation of } M,\\ I^\sigma_{m-1} & \text{otherwise.}\end{cases}$$

In particular, there exists some $S \subseteq [n]$ such that the statement (1) *holds.*

Proof. (3) $\Rightarrow$ (2). Assume that (3) holds, but (2) does not. Then there exists some $m \in [n]$ with $\sigma(m) \in I^\sigma_n$ such that $f_{I^\sigma_n\setminus\{\sigma(m)\}}$ is a right $\mathscr{I}$-approximation of M. Then since $I^\sigma_n \subseteq I^\sigma_{m-1}$, we see that $f_{I^\sigma_{m-1}\setminus\{\sigma(m)\}}$ is a right $\mathscr{I}$-approximation of M by Remark 2.6(2). Hence $I^\sigma_n \subseteq I^\sigma_m = I^\sigma_{m-1} \setminus \{\sigma(m)\}$. Thus $\sigma(m) \notin I^\sigma_n$, a contradiction.

(2) $\Rightarrow$ (1). Assume that (2) holds. By [10, Theorem 2.2], there exists a decomposition $X_S = Y_1 \oplus Y_2$ such that the matrix expression of f_S for this decomposition has the form $f_S = (g_1, 0)$ with $g_1\colon Y_1 \to M$ a right minimal

version of f. Thus g_1 is a minimal right approximation of M. It is enough to show that $Y_2 = 0$ because if this is the case, then $f_S = g_1$, and (1) holds. Assume contrarily that $Y_2 \neq 0$. Let $\sigma_i \colon Y_i \to X_S$ be the inclusions, and $\pi_i \colon X_S \to Y_i$ the canonical projections for each $i = 1, 2$. Then $f_S = (g_1, 0)$ shows that

$$g_1 = f_S\sigma_1,\ f_S\sigma_2 = 0.$$

By [1, 12.7 Corollary], the decomposition $X_S = \bigoplus_{i \in S} X_i$ of X_S complements Y_2, namely, we have

$$X_S = X_T \oplus Y_2 \tag{2.2}$$

for some $T \subseteq S$. Since $Y_2 \neq 0$, we have $T \subsetneq S$. Thus there exists some $j \in S \setminus T$. The matrix expression of f_S for the decomposition (2.2) is given by $f_S = (f_T, 0)$. Let $\sigma' \colon X_T \to X_S$ be the inclusion. Then $f_T = f_S\sigma'$ by definition. Here the direct sum (2.2) shows that $\ell := \pi_1\sigma' \colon X_T \to Y_1$ is an isomorphism. Moreover, the diagram

$$\begin{array}{ccc} X_T & \xrightarrow{\quad\ell\quad} & Y_1 \\ & {\scriptstyle f_T}\searrow \quad \swarrow{\scriptstyle g_1} & \\ & M & \end{array}$$

is commutative. Indeed, $f_T - g_1\ell = f_S\sigma' - f_S\sigma_1\pi_1\sigma' = f_S(\mathbb{1}_V - \sigma_1\pi_1)\sigma' = f_S\sigma_2\pi_2\sigma' = 0$. Hence f_T is also a (minimal) right $\mathscr{I}$-approximation of M. Since $T \subseteq S \setminus \{j\}$, $f_{S\setminus\{j\}}$ is also a right $\mathscr{I}$-approximation of M, a contradiction.

(1) $\Rightarrow$ (3). Assume that (1) holds. Let t be the number of elements of S. Then $0 \leq t \leq n$. Choose any permutation σ of $[n]$ such that for each $i \in [n]$, $\sigma(i) \in S$ if and only if $n - t < i$. Then for each $m \in [n-t]$, we have $I^\sigma_m = I^\sigma_{m-1} \setminus \{\sigma(m)\} \supseteq S$, and $I^\sigma_{n-t} = S$. We show that $S = I^\sigma_m$ for all $n - t \leq m \leq n$ by induction on m. Indeed, this is trivial for $m = n - t$. For m with $n - t < m \leq t$, $S = I^\sigma_{m-1}$ by induction hypothesis. If $S \neq I^\sigma_m$, then $I^\sigma_m = I^\sigma_{m-1} \setminus \{\sigma(m)\} = S \setminus \{\sigma(m)\}$, and hence $f_{S\setminus\{\sigma(m)\}}$ is a right $\mathscr{I}$-approximation of M. Thus f_S factors through $f_{S\setminus\{\sigma(m)\}}$. It is trivial that $f_{S\setminus\{\sigma(m)\}}$ factors through f_S, and hence we have $[f_S] = [f_{S\setminus\{\sigma(m)\}}]$. Then by (1) and Remark 2.4(1), $\ell(X_S) \leq \ell(X_{S\setminus\{\sigma(m)\}})$, a contradiction. □

A *left minimal morphism*, an *equivalence* relation on the object set of $\mathscr{C}^M$ with $M \in \operatorname{mod} A$ and a *left minimal version* are defined as dual to those in Definition 2.3. A *left $\mathscr{I}$-approximation* and a *minimal left $\mathscr{I}$-approximation* are defined as dual versions of Definition 2.5. The dual versions of Remark 2.4, 2.6 and Lemma 2.7 also hold, which we omit to state, but refer to them as Remark 2.4′, 2.6′ and Lemma 2.7′, respectively.

Definition 2.8. Let M be in $\operatorname{mod} A$.

(1) A sequence

$$\cdots \xrightarrow{f_{r+1}} X_r \xrightarrow{f_r} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \xrightarrow{f_{-1}} 0 \tag{2.3}$$

is called a *minimal $\mathscr{I}$-resolution* of M if f_i restricts to a minimal right $\mathscr{I}$-approximation $X_i \to \operatorname{Ker} f_{i-1}$ of $\operatorname{Ker} f_{i-1}$ for all $i \geq 0$. As is easily seen, this sequence is a complex, and is uniquely determined by M up to isomorphism (of sequences of modules) by Remark 2.6(3). For each $i \geq 0$, X_i has a direct sum decomposition $X_i \cong \bigoplus_{I \in \mathbb{I}} V_I^{\beta^i_M(I)}$ with the unique non-negative integers $\beta^i_M(I)$ ($I \in \mathbb{I}$), which is called the *$\mathscr{I}$-relative i-th Betti number* of M at I.

Note that by the definition of right $\mathscr{I}$-approximations, the functor ${}_A(Z,\text{-})$ sends the sequence (2.3) to an exact sequence of vector spaces for all $Z \in \mathscr{I}$, and that the sequence (2.3) turns out to be an exact sequnce if G is a generator.

(2) Dually, a sequence

$$0 \xrightarrow{g^{-1}} M \xrightarrow{g^0} Y^0 \xrightarrow{g^1} Y^1 \xrightarrow{g^2} \cdots \xrightarrow{g^r} Y^r \xrightarrow{g^{r+1}} \cdots \tag{2.4}$$

is called a *minimal $\mathscr{I}$-coresolution* of M if g^i induces a minimal left $\mathscr{I}$-approximation $\operatorname{Coker} g^{i-1} \to Y^i$ of $\operatorname{Coker} g^{i-1}$ for all $i \geq 0$. As is easily seen, this sequence is a cocomplex, and is uniquely determined by M up to isomorphism (of sequences of modules) by Remark 2.6′(3). For each $i \geq 0$, Y^i has a direct sum decomposition $Y^i \cong \bigoplus_{I \in \mathbb{I}} V_I^{\overline{\beta}^i_M(I)}$ with the unique non-negative integers $\overline{\beta}^i_M(I)$ ($I \in \mathbb{I}$), which is called the *$\mathscr{I}$-relative i-th co-Betti number* of M at I.

Note that by the definition of left $\mathscr{I}$-approximations, the functor ${}_A(\text{-}, Z)$ sends the sequence (2.4) to an exact sequence of vector spaces for all $Z \in \mathscr{I}$, and that the sequence (2.4) turns out to be an exact sequence if G is a cogenerator.

3. Relative Koszul coresolutions

Throughout this section, we fix an $I \in \mathbb{I}$, and denote by $e_I \in \Lambda$ the idempotent given by the composite $G \to V_I \to G$ of canonical maps.

Notation 3.1. We denote the radical of $\operatorname{mod} A$ by rad_A ([7, A3.3.3 Definition]). Then $\operatorname{rad}_A(G, G)$ is the Jacobson radical of Λ ([7, A3.3.5(a) Proposition]), and the radical of the left Λ-module ${}_A(V_I, G)$ is given by $\operatorname{rad}_A(V_I, G)$, namely we have

$$\operatorname{rad}_A(V_I, G) = \operatorname{rad}_A(G, G) \cdot {}_A(V_I, G)$$

because the LHS is identified with $\operatorname{rad}_A(G, G) \cdot e_I$ and the RHS is identified with $\operatorname{rad}_A(G, G) \cdot \Lambda e_I$. Similarly, the radical of the right Λ-module ${}_A(G, V_I)$ is given by

$$\operatorname{rad}_A(G, V_I) = {}_A(G, V_I) \cdot \operatorname{rad}_A(G, G).$$

We set ${}_IS := {}_A(V_I, G)/\operatorname{rad}_A(V_I, G)$ (resp. $S_I := {}_A(G, V_I)/\operatorname{rad}_A(G, V_I)$), which is the simple left (resp. right) Λ-module corresponding to I.

From now on, we freely use Auslander–Reiten theory, for which we refer the reader to [7, Ch. 4]. A left (resp. right) almost split morphism that is left (resp.

right) minimal is called a *source map* (resp. *sink map*) for short as used in [25, p.55].

Definition 3.2. An *$\mathscr{I}$-relative Koszul coresolution* $\mathscr{K}^{\bullet}(V_I)$ of V_I is a sequence

$$0 \to V_I \xrightarrow{\eta} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \cdots$$

of modules in $\mathscr{I}$ such that

(1) $\eta = f_0\mu_0$, where $\mu_0 \colon V_I \to E_I$ is a source map from V_I and $f_0 \colon E_I \to X^1$ is a minimal left $\mathscr{I}$-approximation of E_I; and
(2) $d_1 = f_1\mu_1$, where $\mu_1 \colon X^1 \to \operatorname{Coker}\eta$ is the canonical epimorphism and the sequence

$$0 \to \operatorname{Coker}\eta \xrightarrow{f_1} X^2 \xrightarrow{d^2} X^3 \xrightarrow{d^3} \cdots \tag{3.5}$$

is a minimal left $\mathscr{I}$-coresolution of $\operatorname{Coker}\eta$.

Note that $\mathscr{K}^{\bullet}(V_I)$ is uniquely determined by V_I up to isomorphism (of sequences of modules). Let M be in $\operatorname{mod} A$. Then the complex $\mathscr{K}_I(M)_{\bullet} := \operatorname{Hom}_A(\mathscr{K}^{\bullet}(V_I), M)$ in $\operatorname{mod}\Bbbk$ is called the *$\mathscr{I}$-relative Koszul complex* of M at I.

Notation 3.3. We set $C^1 := \operatorname{Coker}\eta$, and for each $i \geq 2$, decompose d^i as the composite $d^i = f_i\mu_i$ as in the diagram below, where $\mu_i \colon X^i \to C^i := \operatorname{Coker} f_{i-1}$ is the canonical epimorphism, and $f_i \colon C^i \to X^{i+1}$ is a minimal left $\mathscr{I}$-approximation of C^i:

$$\begin{array}{ccccccccccccc}
0 \to V_I & & \xrightarrow{\quad\eta\quad} & & X^1 & & \xrightarrow{\quad d^1\quad} & & X^2 & & \xrightarrow{\quad d^2\quad} & & X^3 \xrightarrow{\quad d^3\quad} \cdots \\
& {\scriptstyle\mu_0}\searrow & & \nearrow{\scriptstyle f_0} & & {\scriptstyle\mu_1}\searrow & & \nearrow{\scriptstyle f_1} & & {\scriptstyle\mu_2}\searrow & & \nearrow{\scriptstyle f_2} & \quad {\scriptstyle\mu_3}\searrow \quad \nearrow{\scriptstyle f_3} \\
& & E_I & & & & C^1 & & & & C^2 & & \quad C^3
\end{array}.$$

Remark 3.4. In Definition 3.2, the sequence (3.5) is not always exact. Assume that G is a cogenerator. Then the sequence (3.5) is exact, and hence the sequence $\mathscr{K}^{\bullet}(V_I)$ is exact at X^i for all $i \geq 1$, but not at V_I in general. More precisely, if V_I is non-injective, then μ_1 is a monomorphism as a left map of the almost split sequence starting from V_I, and f_1 is also a monomorphism from the beginning. Thus the sequence is exact, i.e., the complex $\mathscr{K}^{\bullet}(V_I)$ is acyclic.

On the contrary if V_I is injective, then μ_1 is the canonical epimorphism $V_I \to V_I/\operatorname{soc} V_I$, and hence $\operatorname{Ker}\eta = \operatorname{Ker}\mu_1 = \operatorname{soc} V_I$, and the sequence $\mathscr{K}^{\bullet}(V_I)$ is not exact at V_I.

Recall the following fundamental properties of a generator and a cogenerator.

Lemma 3.5. *Let $f \colon X \to Y$ be in* $\operatorname{mod} A$.

(1) *If C is a generator in* $\operatorname{mod} A$*, then* ${}_A(C, f) = 0$ *implies* $f = 0$.
(2) *If C is a cogenerator in* $\operatorname{mod} A$*, then* ${}_A(f, C) = 0$ *implies* $f = 0$.

Proof. We only prove the statement (2) because (1) is proved dually. Assume that $f\colon X \to Y$ satisfies the property that ${}_A(f,C) = 0$. Since Y is finite-dimensional, it is finitely cogenerated by C. Namely, there exists a monomorphism $g = (g_i)_{i=1}^m\colon Y \to C^m$ in $\operatorname{mod} A$ for some positive integer m. Since ${}_A(f,C) = 0$, $g_i \circ f = {}_A(f,C)(g_i) = 0$ for all $i = 1,\dots,m$. Hence $g \circ f = 0$, which shows that $f = 0$ because g is a monomorphism. □

When G is a generator or a cogenerator, we have the following.

Lemma 3.6. *Let $M \in \operatorname{mod} A$.*

(1) *Assume that G is a generator, and let $X \xrightarrow{f} M \to 0$ be a minimal right $\mathscr{I}$-approximation of M. Then ${}_A(G,X) \xrightarrow{{}_A(G,f)} {}_A(G,M) \to 0$ is a projective cover of ${}_A(G,M)$ in $\operatorname{mod}\Lambda$. Hence if (2.3) is a minimal right $\mathscr{I}$-resolution of M, then the sequence*

$$\cdots \to {}_A(G,X_r) \xrightarrow{{}_A(G,f_r)} \cdots \xrightarrow{{}_A(G,f_2)} {}_A(G,X_1) \xrightarrow{{}_A(G,f_1)} {}_A(G,X_0) \xrightarrow{{}_A(G,f_0)} {}_A(G,M) \to 0$$

is a minimal projective resolution of ${}_A(G,M)$ in $\operatorname{mod}\Lambda$.

(2) *Assume that G is a cogenerator, and let $0 \to M \xrightarrow{g} Y$ be a minimal left $\mathscr{I}$-approximation of M. Then ${}_A(Y,G) \xrightarrow{{}_A(g,G)} {}_A(M,G) \to 0$ is a projective cover of ${}_A(M,G)$ in $\operatorname{mod}\Lambda^{\mathrm{op}}$. Hence if (2.4) is a minimal left $\mathscr{I}$-coresolution of M, then the sequence*

$$\cdots \to {}_A(Y^r,G) \xrightarrow{{}_A(g^r,G)} \cdots \xrightarrow{{}_A(g^2,G)} {}_A(Y^1,G) \xrightarrow{{}_A(g^1,G)} {}_A(Y^0,G) \xrightarrow{{}_A(g^0,G)} {}_A(M,G) \to 0$$

is a minimal projective resolution of ${}_A(M,G)$ in $\operatorname{mod}\Lambda^{\mathrm{op}}$.

Proof. We prove only the statement (2) because (1) is proved dually. Assume that a section $s\colon U \to {}_A(Y,G)$ satisfies ${}_A(g,G) \circ s = 0$. Then since U is a direct summand of a finitely generated projective Λ-module ${}_A(Y,G)$, U is a finitely generated projective Λ-module and has the form $U = {}_A(U',G)$ for some $U' \in \mathscr{I}$. Since s is a section, there exists an $r\colon {}_A(Y,G) \to U$ such that $rs = \mathbb{1}_U$. Here, $U', Y \in \mathscr{I}$. Therefore, both s and r have the form $s = {}_A(s',G), r = {}_A(r',G)$ for some $s'\colon Y \to U', r'\colon U' \to Y$. Then we have

$${}_A(\mathbb{1}_{U'},G) = \mathbb{1}_U = {}_A(r',G) \circ {}_A(s',G) = {}_A(s' \circ r',G),$$

which shows that ${}_A(\mathbb{1}_{U'} - s' \circ r',G) = 0$. Since G is a cogenerator, we have $\mathbb{1}_{U'} - s' \circ r = 0$, and thus, s' is a retraction. Moreover, $0 = {}_A(g,G) \circ s = {}_A(g,G) \circ {}_A(s',G) = {}_A(s' \circ g,G)$ shows that $s' \circ g = 0$ because again G is a cogenerator. Since g is left minimal, we have to have $s' = 0$ by Remark 2.4′(3), and hence $s = 0$. Therefore, ${}_A(g,G)$ is right minimal by Remark 2.4(3). □

The $\mathscr{I}$-relative Koszul complex of M gives a way to compute the $\mathscr{I}$-relative Betti numbers of M at I as follows.

Theorem 3.7. *Let $M \in \operatorname{mod}$. Then the following statements hold.*

(1) *Assume that G is a cogenerator. Then* $\operatorname{Hom}_A(\mathscr{K}^{\bullet}(V_I),G)$ *gives a minimal projective resolution*

$$\cdots \to {}_A(X^2,G) \to {}_A(X^1,G) \to {}_A(V_I,G) \to {}_IS \to 0$$

of the simple left Λ-module ${}_IS$; and

(2) *Assume that G is a generator and a cogenerator. Then for each $i \geq 0$,*

$$\dim_{\Bbbk} H_i(\mathscr{K}_I(M)_{\bullet}) = \beta^i_M(I).$$

Proof. (1) Since ${}_IS = {}_A(V_I,G)/\operatorname{rad}{}_A(V_I,G)$, we have an exact sequence

$$0 \to \operatorname{rad}{}_A(V_I,G) \xrightarrow{\sigma} {}_A(V_I,G) \xrightarrow{\pi} {}_IS \to 0 \tag{3.6}$$

in $\operatorname{mod}\Lambda^{\mathrm{op}}$, where σ is the inclusion and π is the canonical epimorphism, and π turns out to be a projective cover of ${}_IS$. Since ${}_A(\text{-},G)$ is left exact, and the sequence $V_I \xrightarrow{\eta} X^1 \xrightarrow{\mu_1} C^1 \to 0$ is exact, the first row in the following commutative diagram is exact:

$$\begin{array}{ccccccc}
0 & \longrightarrow & {}_A(C^1,G) & \xrightarrow{{}_A(\mu_1,G)} & {}_A(X^1,G) & \xrightarrow{\qquad {}_A(\eta,G) \qquad} & {}_A(V_I,G) \\
 & & & & \downarrow{\scriptstyle {}_A(f_0,G)} & & \Vert \\
 & & & & {}_A(E_I,G) & \xrightarrow{\qquad {}_A(\mu_0,G) \qquad} & {}_A(V_I,G) \\
 & & & & & {\scriptstyle \delta}\searrow \quad \nearrow{\scriptstyle \sigma} & \\
 & & & & & \operatorname{rad}{}_A(V_I,G) &
\end{array}.$$

Since μ_0 is a source map, the image of ${}_A(\mu_0,G)$ coincides with $\operatorname{rad}{}_A(V_I,G)$. Thus ${}_A(\mu_0,G)$ is written as ${}_A(\mu_0,G) = \sigma\delta$, where δ is the restriction of ${}_A(\mu_0,G)$. Since G is a cogenerator and f_0 is left $\mathscr{I}$-approximation, $\delta \circ {}_A(f_0,G)$ is an epimorphism. Hence the equality

$$\operatorname{Im}{}_A(\mu_1,G) = \operatorname{Ker}{}_A(\eta,G) = \operatorname{Ker}(\sigma \circ \delta \circ {}_A(f_0,G)) = \operatorname{Ker}(\delta \circ {}_A(f_0,G))$$

shows the exactness of the sequence

$$0 \to {}_A(C^1,G) \xrightarrow{{}_A(\mu_1,G)} {}_A(X^1,G) \xrightarrow{\delta \circ {}_A(f_0,G)} \operatorname{rad}{}_A(V_I,G) \to 0 \tag{3.7}$$

in $\operatorname{mod}\Lambda^{\mathrm{op}}$. On the other hand, since again G is a cogenerator, the minimal left $\mathscr{I}$-coresolution (3.5) of C^1 yiels a minimal projective resolution

$$\cdots \xrightarrow{{}_A(d^3,G)} {}_A(X^3,G) \xrightarrow{{}_A(d^2,G)} {}_A(X^2,G) \xrightarrow{{}_A(f^1,G)} {}_A(C^1,G) \to 0 \tag{3.8}$$

of ${}_A(C^1,G)$ in $\operatorname{mod}\Lambda^{\mathrm{op}}$ by Lemma 3.6(2). By combining the exact sequences (3.6), (3.7) and (3.8), we obtain the following commutative diagram with the first row exact:

$$\begin{array}{ccccccccc}
\cdots \to {}_A(X^3,G) & \xrightarrow{\qquad {}_A(d^2,G) \qquad} & {}_A(X^2,G) & \xrightarrow{\qquad {}_A(d^1,G) \qquad} & {}_A(X^1,G) & \xrightarrow{\qquad {}_A(\eta,G) \qquad} & {}_A(V_I,G) \xrightarrow{\pi} {}_IS \to 0 \\
{\scriptstyle {}_A(f_2,G)}\searrow & \nearrow{\scriptstyle {}_A(\mu_2,G)} & {\scriptstyle {}_A(f_1,G)}\searrow & \nearrow{\scriptstyle {}_A(\mu_1,G)} & {\scriptstyle \delta\circ{}_A(f_0,G)}\searrow & \nearrow{\scriptstyle \sigma} & \\
 & {}_A(C^2,G) & & {}_A(C^1,G) & & \operatorname{rad}{}_A(V_I,G) &
\end{array}.$$

Since V_I and X^i $(i \geq 1)$ are in $\mathscr{I}$, the first row is a projective resolution of ${}_IS$. It remains to show the minimality of this sequence. Let $\cdots \to P_2 \to P_1 \to P_0 \to {}_IS \to 0$ be a minimal projective resolution. Then there exist projective modules P'_i $(i \geq 0)$ such that ${}_A(V_I, G) \cong P_0 \oplus P'_0$, ${}_A(X^i, G) \cong P_i \oplus P'_i \oplus P'_{i-1}$ for all $i \geq 1$. Here, since (3.8) is a minimal projective resolution, we have $P'_i \oplus P'_{i-1} = 0$ for all $i \geq 2$. Thus $P'_i = 0$ for all $i \geq 1$. Moreover, since π in (3.6) is a projective cover of ${}_IS$, we have $P'_0 = 0$. As a consequence, we see that the first row above is a minimal projective resolution of ${}_IS$.

(2) Since G is a generator, ${}_A(G, \text{-})$ sends a minimal $\mathscr{I}$-resolution (2.3) of M with $X_i \cong \bigoplus_{J \in \mathbb{I}} V_J^{\beta^i_M(J)}$ for all $i \geq 0$ to the minimal projective resolution

$$\cdots \to {}_A(G, X_1) \xrightarrow{{}_A(G,f_1)} {}_A(G, X_0) \xrightarrow{{}_A(G,f_0)} {}_A(G, M) \to 0 \tag{3.9}$$

of $F := {}_A(G, M)$ in $\operatorname{mod} \Lambda$ by Lemma 3.6(1). Then by the minimality of this sequence, the functor $\operatorname{Hom}_\Lambda(\text{-}, S_I)$ sends the complex ${}_A(G, X_\bullet)$ to the complex $\operatorname{Hom}_\Lambda({}_A(G, X_\bullet), S_I)$ having the form

$$0 \to {}_\Lambda({}_A(G, X_0), S_I) \xrightarrow{0} {}_\Lambda({}_A(G, X_1), S_I) \xrightarrow{0} \cdots .$$

Then noting that $\operatorname{Hom}_\Lambda({}_A(G, V_J), S_I) \cong \begin{cases} \Bbbk & (\text{ if } J = I) \\ 0 & (\text{ if } J \neq I) \end{cases}$ for all $J \in \mathbb{I}$, we have

$$\operatorname{Ext}^i_\Lambda(F, S_I) \cong H^i \operatorname{Hom}_\Lambda({}_A(G, X_\bullet), S_I) \cong \Bbbk^{\beta^i_M(I)}.$$

Hence we have $\beta^i_M(I) = \dim_\Bbbk \operatorname{Ext}^i_\Lambda(F, S_I)$. On the other hand, by the statement (1), $D({}_A(\mathscr{K}^\bullet(V_I), G))$ turns out to be an injective coresolution of $S_I \cong D({}_IS)$. Hence by setting $(\text{-})^t := \operatorname{Hom}_\Lambda(\text{-}, \Lambda)$ to be the Λ-dual functor, we have isomorphisms

$$\begin{aligned} \operatorname{Ext}^i_\Lambda(F, S_I) &\cong H^i \operatorname{Hom}_\Lambda({}_A(G, M), D({}_A(\mathscr{K}^\bullet(V_I), G))) \\ &\cong H^i D(\operatorname{Hom}_\Lambda(({}_A(\mathscr{K}^\bullet(V_I), G))^t, {}_A(G, M)) \\ &\cong D H_i \operatorname{Hom}_\Lambda({}_A(G, \mathscr{K}^\bullet(V_I)), {}_A(G, M)) \\ &\cong D H_i \operatorname{Hom}_A({}_A(G, \mathscr{K}^\bullet(V_I)) \otimes_\Lambda G, M) \\ &\cong D H_i \operatorname{Hom}_A(\mathscr{K}^\bullet(V_I), M) \\ &= D H_i(\mathscr{K}_I(M)_\bullet). \end{aligned}$$

As a consequence, we have $\beta^i_M(I) = \dim_\Bbbk D H_i(\mathscr{K}_I(M)_\bullet) = \dim_\Bbbk H_i(\mathscr{K}_I(M)_\bullet)$. □

We needed the assumption that G is a cogenerator to guarantee the minimality of the projective resolution (3.8) of ${}_A(C^1, G)$. If there would be another way to insure it, then the theorem would be valid under the weaker condition that G is a generator.

The following is immediate from the theorem above.

Corollary 3.8. *Assume that G is a generator and a cogenerator. Then the following are equivalent for all $M \in \operatorname{mod} A$:*

(1) $M \in \mathscr{I}$;
(2) $H_1(\mathscr{K}_I(M)_\bullet) = 0$ *for all* $I \in \mathbb{I}$;
(3) $\dim_{\Bbbk} M = \sum_{I\in\mathbb{I}} \dim_{\Bbbk} H_0(\mathscr{K}_I(M)_\bullet) \dim_{\Bbbk} V_I$; *and*
(4) $M \cong \bigoplus_{I\in\mathbb{I}} V_I^{\dim_{\Bbbk} H_0(\mathscr{K}_I(M)_\bullet)}$.

Proof. Each statement is equivalent to the fact that f_0 in (2.3) is an isomorphism. □

The following is useful in applications.

Remark 3.9. To compute $H_0(\mathscr{K}_I(M)_\bullet)$ in (3) and (4) above, we only need to know the beginning part (call it the *0-1-part*)

$$0 \to V_I \xrightarrow{\eta} X^1$$

of $\mathscr{K}^\bullet(V_I)$ with $\eta = f_0\mu_0$, which gives us the formula

$$\beta^0_M(I) = \dim \operatorname{Coker}{}_A(\eta, M) = \dim{}_A(V_I, M) - \dim \operatorname{Im}{}_A(f_0\mu_0, M). \qquad (3.10)$$

Therefore, to apply Corollary 3.8 in the setting of Sect. 4, it would be convenient first to compute the 0-1-parts of Koszul coresolutions for all interval modules V_I ($I \in \mathbb{I}$) (Definition 4.3). (Note that we can further restrict $I \in \mathbb{I}$ as in Corollary 4.14 and in Remark 4.13.) To this end, we need to compute the source map $\mu_0\colon V_I \to E_I$ and the minimal left interval approximation $f_0\colon E_I \to X^1$ for all $I \in \mathbb{I}$. These can be done by using a recipe explained in [3, Sect. 5] and Corollary 4.11′ in the next section, respectively.

4. Applications and examples

Throughout this section, Q is an acyclic quiver without multiple arrows, ρ is the ideal of the path-algebra $\Bbbk Q$ generated by the full commutativity relations, and set $A := \Bbbk Q/\rho$. Then the vertex set Q_0 turns out to be a poset $(Q_0, \preceq)$ by the partial order $\preceq$ defined by $x \preceq y$ if and only if there exists a path from x to y in Q for all $x, y \in Q_0$.

Definition 4.1. Let $S = (S, \le)$ be a finite poset.

(1) For any $x, y \in S$ with $x \le y$, we set $[x, y] := \{i \in S \mid x \le i \le y\}$ and call it the *segment* from x to y. The set $\{[x, y] \mid x \le y, x, y \in S\}$ of all segments is denoted by $\operatorname{Seg}(S)$.
(2) The *incidence algebra* $\Bbbk S$ of S over $\Bbbk$ is defined to be the algebra having $\operatorname{Seg}(S)$ as a basis with the multiplication defined by $[x, y][u, v] := \delta_{y,u}[x, v]$ for all $[x, y], [u, v] \in \operatorname{Seg}(S)$.
(3) The *Hasse quiver* $H(S)$ of S is defined as follows: $H(S)_0 := S$, $H(S)_1 := \{a_{x,y} \mid x, y \in S, x < y, [x, y] = \{x, y\}\}$, and the source and the target of $a_{x,y}$ is x and y, respectively for all $a_{x,y} \in H(S)_1$. Then $H(S)$ is an acyclic quiver without multiple arrows. The *Hasse bound quiver* $(H(S), \rho)$ is the

pair of $H(S)$ and the ideal ρ of the path algebra $\Bbbk H(S)$ generated by the full commutativity relations.

Remark 4.2. Let $S = (S, \leq)$ be a finite poset, and $(H(S), \rho)$ the Hasse bound quiver of S.

(1) There exists a bijection from the set of paths in $H(S)$ modulo ρ to the set $\mathrm{Seg}(S)$ preserving the source and the target that induces an algebra isomorphism $\Bbbk(H(S), \rho) \to \Bbbk S$.
(2) In particular, for the poset $S := (Q_0, \preceq)$, note that (Q, ρ) is isomorphic to $(H(S), \rho)$, and hence A is isomorphic to the incidence algebra $\Bbbk S$.

Definition 4.3 (intervals, interval modules, interval decomposable modules)**.**

(1) A full subquiver I of Q is called an *interval* if it is connected and *convex* in the sense that if x, y are vertices in I_0 and p is a path in Q from x to y, then p is a path in I.
We denote by $\mathbb{I}$ the set of all interval subquivers of Q.
(2) Each interval subquiver $I \in \mathbb{I}$ defines an indecomposable representation V_I of Q by setting $V_I(x) = \Bbbk$ (resp. $V_I(x) = 0$) if $x \in I_0$ (resp. $x \notin I_0$) for all $x \in Q_0$, and $V_I(a) = \mathbb{1}_\Bbbk$ (resp. $V_I(a) = 0$) if $a \in I_1$ (resp. $a \notin I_1$) for all $a \in Q_1$.
For each $I \in \mathbb{I}$, note that V_I satisfies all the commutativity relations on Q, and hence can be seen as a right A-module. A right A-module is called an *interval module* if it is isomorphic to V_I for some $I \in \mathbb{I}$.
(3) A right A-module is said to be *interval decomposable* if it is the direct sum of some interval modules.
We set $G := \bigoplus_{I\in\mathbb{I}} V_I$ and $\mathscr{I} := \operatorname{add} G$. Then a right A-module is interval decomposable if and only if it is in $\mathscr{I}$.

Note that indecomposable projective modules and indecomposable injective modules are interval modules, and hence G is a generator and a cogenerator in $\operatorname{mod} A$. Therefore, we can apply all the results in Section 3 in this setting.

Convention 4.4. Here, "$\mathscr{I}$-relative" or "$\mathscr{I}$-" is replaced with the word "interval". For instance, a minimal $\mathscr{I}$-resolution and an $\mathscr{I}$-relative Koszul coresolution are called a *minimal interval resolution* and an *interval Koszul coresolution*, respectively.

To make the computations of minimal right (resp. left) interval approximations easier, we first give a handy criterion to check whether a homomorphism is a right (resp. left) interval approximation. By using this, we give examples of the interval Koszul coresolution of an interval module V_I, the interval Koszul complex $\mathscr{K}_I(M)_\bullet$ of a module M, and compute some interval Betti numbers $\beta^i_M(I)$.

4.1. **A criterion for right/left interval approximation.** We start with an easy remark.

Remark 4.5. Let $f\colon X \to M$ be in $\operatorname{mod} A$. It is clear from Definition 2.5 that the following are equivalent:

(1) f is a right interval approximation of M.
(2) For any $I \in \mathbb{I}$ and any $g \in \operatorname{Hom}_A(V_I, M)$, $g = fh$ for some $h\colon V_I \to X$.

We want to make the sets $\mathbb{I}$ and $\operatorname{Hom}_A(V_I, M)$ in the statement (2) above much smaller.

Notation 4.6. For any $X, Y \in \operatorname{mod} A$, we denote by $\operatorname{Mon}(X, Y)$ (resp. $\operatorname{Epi}(X, Y)$) the set of all monomorphisms (resp. epimorphisms) in $\operatorname{Hom}_A(X, Y)$.

Definition 4.7. Let M be in $\operatorname{mod} A$. Then we set

$$\mathrm{S}_{\mathrm{int}}(M) := \{I \in \mathbb{I} \mid \operatorname{Mon}(V_I, M) \neq \emptyset\},$$
$$\mathrm{F}_{\mathrm{int}}(M) := \{I \in \mathbb{I} \mid \operatorname{Epi}(M, V_I) \neq \emptyset\}.$$

The following is a special property of interval decomposable modules, the first half of which was used to show the global dimension of Λ is finite in [5]. The second half is used in the proof of Proposition 4.9.

Lemma 4.8. *Let $I \in \mathbb{I}$. Then all submodules and all factor*[1] *modules of V_I are interval decomposable.*

Proof. For the submodules this statement was already proved in [5]. The statement for factor modules is proved similarly as follows. Let W be a factor module of V_I. Then $W = V_I/V'$ for some $V' \leq V_I$. Denote $1 \in \Bbbk = V_I(x)$ by 1_x for all $x \in Q_0$.

We show that the full subquiver $\operatorname{supp} W$ of Q with the vertex set $(\operatorname{supp} W)_0 := \{x \in Q_0 \mid W(x) \neq 0\}$ is convex. Let $x, y \in (\operatorname{supp} W)_0$, and p a path from x to y in Q, and suppose that $p = p'p''$ for some paths p' from x to a vertex z, and p'' from z to y. To show the assertion, it is enough to show that $z \in (\operatorname{supp} W)_0$. Assume to the contrary that $z \notin (\operatorname{supp} W)_0$. Then $1_z \in V'(z)$, and $1_y = 1_z p'' \in V'(y)$, and hence $W(y) = 0$, a contradiction. Thus $\operatorname{supp} W$ must be convex. Let $C_1, \dots, C_t$ be the connected components of $\operatorname{supp} W$. Then they are connected and convex, and $W = W|_{C_1} \oplus \cdots \oplus W|_{C_t}$, where $W|_{C_i}$ are the restrictions of W on C_i for all $i = 1, \dots, t$. As is easily seen, $W|_{C_i}$ are interval modules for all i. Hence W is interval decomposable. □

Remark 4.5 and Lemma 4.8 immediately proves the following.

Proposition 4.9. *Let $f\colon X \to M$ be in $\operatorname{mod} A$. Then the following are equivalent.*

(1) *f is a right interval approximation of M.*
(2) *For any $I \in \mathrm{S}_{\mathrm{int}}(M)$ and any monomorphism $g\colon V_I \to M$ in $\operatorname{mod} A$, $g = fh$ for some $h\colon V_I \to X$.*

Proof. (1) $\Rightarrow$ (2). This is trivial.

[1]For a submodule N of an $M \in \operatorname{mod} A$, M/N is sometimes called a *factor* module of M.

(2) $\Rightarrow$ (1). Let $I \in \mathbb{I}$ and $g \colon V_I \to M$ be in $\operatorname{mod} A$. Since $\operatorname{Im} g$ is a factor module of V_I, it is interval decomposable by Lemma 4.8, thus we have an isomorphism $\operatorname{Im} g \cong \bigoplus_{i=1}^n V_{I_i}$ for some $I_i \in \mathbb{I}$ $(i = 1, \dots, n)$, by which we identify these modules. Consider the factorization $g = \sigma g'$ of g by the restriction $g' \colon V_I \to \operatorname{Im} g$ of g and the inclusion $\sigma \colon \operatorname{Im} f \to M$. Let $g' = {}^t(g'_i)_{i=1}^n$ and $\sigma = (\sigma_i)_{i=1}^n$ be the matrix presentations of g' and σ with this decomposition of $\operatorname{Im} g$, respectively. Then $\sigma_i \colon V_{I_i} \to M$ is a monomorphism with $I_i \in \mathrm{S}_{\mathrm{int}}(M)$ for all $i = 1, \dots, n$. Therefore by (2), for each i, there exists some $h_i \in {}_A(V_{I_i}, X)$ such that $\sigma_i = f h_i$. Hence $g = \sum_{i=1}^n \sigma_i g'_i = f(\sum_{i=1}^n h_i g'_i)$. □

Proposition 4.9 above immediately gives us a way to construct a right interval approximation as follows.

Corollary 4.10. *Let $M \in \operatorname{mod} A$. For each $I \in \mathrm{S}_{\mathrm{int}}(M)$, let W_I be the subspace of the finite-dimensional vector space $\operatorname{Hom}_A(V_I, M)$ spanned by $\operatorname{Mon}(V_I, M)$. Then $n_I := \dim W_I$ is finite, and hence there exists a finite subset $\{\bar{f}_I^{(1)}, \dots, \bar{f}_I^{(n_I)}\}$ of $\operatorname{Mon}(V_I, M)$ that is a basis of W_I, thus $W_I = \bigoplus_{i=1}^{n_I} \Bbbk \bar{f}_I^{(i)}$. Then*

$$\bar{f} := ((\bar{f}_I^{(1)}, \dots, \bar{f}_I^{(n_I)}))_{I \in \mathrm{S}_{\mathrm{int}}(M)} \colon \bigoplus_{I \in \mathrm{S}_{\mathrm{int}}(M)} V_I^{(n_I)} \to M$$

is a right interval approximation of M

Proof. Let $I \in \mathrm{S}_{\mathrm{int}}(M)$ and $g \in \operatorname{Mon}(V_I, M)$. Then by construction, we have $g = \sum_{i=1}^{n_I} k_i \bar{f}_I^{(i)}$ for some $k_i \in \Bbbk$ $(i = 1, \dots, n_I)$. By taking

$$h := {}^t({}^t(h_J^{(1)}, \dots, h_J^{(n_J)}))_{J \in \mathrm{S}_{\mathrm{int}}(M)} \in \operatorname{Hom}_A(V_I, \bigoplus_{I \in \mathrm{S}_{\mathrm{int}}(M)} V_I^{(n_I)}),$$

where $h_I^{(i)} := \delta_{J,I} k_i \mathbb{1}_{V_I}$, we have $g = \bar{f} h$. Therefore, $\bar{f}$ is a right interval approximation. □

Corollary 4.10 and Lemma 2.7 gives us a way to construct a minimal right interval approximation as follows.

Corollary 4.11. *Let $M \in \operatorname{mod} A$, and construct a right interval approximation $\bar{f}$ of M as in Corollary 4.10. Choose a finite subset $S \subseteq \mathrm{S}_{\mathrm{int}}(M)$ and a subset $\{f_I^{(1)}, \dots, f_I^{(m_I)}\}$ of $\{\bar{f}_I^{(1)}, \dots, \bar{f}_I^{(n_I)}\}$ for each $I \in S$ by the procedure described in Lemma 2.7(3) applied to the identity permutation σ. Then $\{f_I^{(i)}\}_{i \in [m_I]}$ is a linearly independent subset of $\operatorname{Mon}(V_I, M)$ such that*

$$f := ((f_I^{(1)}, \dots, f_I^{(m_I)}))_{I \in S} \colon \bigoplus_{I \in S} V_I^{(m_I)} \to M$$

is a minimal right interval approximation of M. □

Remark 4.12. After publishing the preprint version of this paper to arXiv, the preprint of the paper [2] appeared, in which Theorem 3.4 states the following: Let A be a finite-dimensional $\Bbbk$-algebra, $\mathscr{X}$ a full subcategory of $\operatorname{mod} A$ with $\mathscr{X} = \operatorname{add} \mathscr{X}$ that contains any quotients of the indecomposables in $\mathscr{X}$, and

$f\colon X \to M$ a minimal right $\mathscr{X}$-approximation. Then for any indecomposable direct summand X' of X, the restriction $f|_{X'}$ is injective.

Hence the existence of such an f in Corollary 4.11 follows from this theorem (except for the linear independence of $\{f_I^{(i)}\}_{i\in[m_I]}$, which follows also from Lemma 2.7), but this theorem (even its interval version [2, Corollary 3.11]) gives only necessary conditions for a morphism f to be a minimal right interval approximation of M and does not give a way to construct it. Conversely, note that also in this setting, Corollary 4.11 holds by the same argument used in its proof, and it gives a construction of a minimal right $\mathscr{X}$-approximation.

Remark 2.4(1) gives us an alternative proof of [2, Theorem 3.4] as follows: By assumption, we have a decomposition $X = X' \oplus X''$ with $X', X'' \in \mathscr{X}$. Set $(f', f'') := (f|_{X'}, f|_{X''}) = f$, and let $f'_2\colon f(X') \hookrightarrow M$ be the inclusion. Then $(f'_2, f'')\colon f(X') \oplus X'' \to M$ is equivalent to f. Indeed, since $f' = f'_2 f'_1$ with the restriction $f'_1\colon X' \to f(X')$ of f', we have $f = (f'_2, f'')\binom{f'_1}{1_{X''}}$. By assumption, we have $f(X')\oplus X'' \in \mathscr{X}$, and hence (f'_2, f'') factors through f because f is a right $\mathscr{X}$-approximation of M. Thus $[f] = [(f'_2, f'')]$ as desired. Then by Remark 2.4(1), we have $\ell(X) \leq \ell(f(X') \oplus X'')$, and hence $\ell(X') = \ell(f(X'))$. Thus $f|_{X'}$ is injective.

Both Proposition 4.9 and [2, Theorem 3.4] use the fact that $\mathscr{X}$ is closed under taking factor modules of indecomposables in $\mathscr{X}$ in the same way, but there is a difference between them: the former goes to a necessary and sufficient condition for a morphism f to be a right $\mathscr{X}$-approximation, and the latter goes to a necessary condition for f to be a minimal right $\mathscr{X}$-approximation.

Remark 4.13. Sometimes we may further make the set of necessary intervals I smaller than $\mathrm{S}_{\mathrm{int}}(M)$. For example, if for any $I \in \mathrm{S}_{\mathrm{int}}(M)$ that is not maximal in $\mathrm{S}_{\mathrm{int}}(M)$, any monomorphism $I \to M$ is extended to $J \to M$ for some $I \subsetneq J \in \mathrm{S}_{\mathrm{int}}(M)$, then we can replace $\mathrm{S}_{\mathrm{int}}(M)$ in Proposition 4.9 with the set $\max \mathrm{S}_{\mathrm{int}}(M)$ of maximal elements in $\mathrm{S}_{\mathrm{int}}(M)$. The dual remark works for Proposition 4.9′. These are used in Example 4.18.

The statements dual to Proposition 4.9 and Corollaries 4.10, 4.11 hold, to which we refer as Proposition 4.9′ and Corollaries 4.10′, 4.11′, respectively. For example, Proposition 4.9′ is stated as follows:

Proposition 4.9′. Let $f\colon X \to M$ be in $\operatorname{mod} A$. Then the following are equivalent.

(1) f is a left interval approximation of M.
(2) For any $I \in \mathrm{F}_{\mathrm{int}}(M)$ and any epimorphism $g\colon M \to V_I$ in $\operatorname{mod} A$, $g = hf$ for some $h\colon X \to V_I$.

The following is immediate from Propositions 4.9 and 4.9′.

Corollary 4.14. *Let $M \in \operatorname{mod} A$. Then the following are equivalent.*

(1) *M is interval decomposable;*
(2) $M \cong \bigoplus_{I\in \mathrm{S}_{\mathrm{int}}(M)} V_I^{\beta^0_M(I)}$;

(3) $\dim_{\Bbbk} M = \sum_{I \in \mathrm{S_{int}}(M)} \beta^0_M(I) \dim_{\Bbbk} V_I$;

(2′) $M \cong \bigoplus_{I \in \mathrm{F_{int}}(M)} V_I^{\overline{\beta}^0_M(I)}$;

(3′) $\dim_{\Bbbk} M = \sum_{I \in \mathrm{F_{int}}(M)} \overline{\beta}^0_M(I) \dim_{\Bbbk} V_I$;

(4) $M \cong \bigoplus_{I \in \mathrm{S_{int}}(M) \cap \mathrm{F_{int}}(M)} V_I^{\beta^0_M(I)} \cong \bigoplus_{I \in \mathrm{S_{int}}(M) \cap \mathrm{F_{int}}(M)} V_I^{\overline{\beta}^0_M(I)}$; *and*

(5) $\dim_{\Bbbk} M = \sum_{I \in \mathrm{S_{int}}(M) \cap \mathrm{F_{int}}(M)} \beta^0_M(I) \dim_{\Bbbk} V_I$
$\qquad = \sum_{I \in \mathrm{S_{int}}(M) \cap \mathrm{F_{int}}(M)} \overline{\beta}^0_M(I) \dim_{\Bbbk} V_I$.

The statement above can be used to check whether a given right A-module is interval decomposable or not.

Remark 4.15. In the above corollary, the values of $\beta^0_M(I)$ or $\overline{\beta}^0_M(I)$ can be computed by the formula (3.10) or its dual version. For example, we have the following.

Corollary 4.16. *Let $M \in \operatorname{mod} A$. Then M is interval decomposable if and only if the following holds:*

(3*) $\dim_{\Bbbk} M = \sum_{I \in \mathrm{S_{int}}(M)} \dim_{\Bbbk} H_0(\mathscr{K}_I(M)_\bullet) \dim_{\Bbbk} V_I$.

4.2. **Some examples.** Also for Auslander–Reiten quivers (*AR-quivers* for short), we refer the reader to [7, Ch. 4] (see also [25, 2.3]), but we treat them as translation quivers as in [25, p.61].

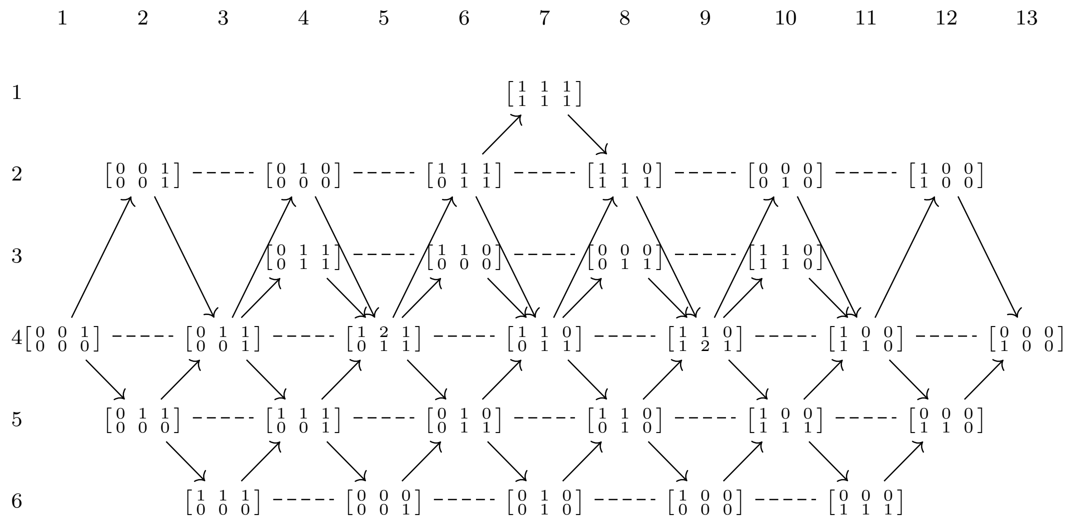
We define a bound quiver $(Q, \rho) := CL_n$ $(n \geq 2)$ to be the following quiver with full commutativity relations (call it a *commutative ladder*), which is used in the following examples:

$$
\begin{array}{ccccccc}
\overline{1} & \xrightarrow{\overline{a}_1} & \overline{2} & \xrightarrow{\overline{a}_2} & \cdots & \xrightarrow{\overline{a}_{n-1}} & \overline{n} \\
{\scriptstyle b_1}\uparrow & & {\scriptstyle b_2}\uparrow & & \cdots & & \uparrow{\scriptstyle b_n} \\
1 & \xrightarrow[a_1]{} & 2 & \xrightarrow[a_2]{} & \cdots & \xrightarrow[a_{n-1}]{} & n
\end{array} . \tag{4.11}
$$

In the first example (Example 4.17), we have the whole AR-quiver that has only 29 vertices (only 2 of them are not interval modules), and it is easy to control morphisms between indecomposables. In the second example (Example 4.18), since the algebra A is not representation-finite, we do not use the whole AR-eiten quiver. However, since the computation of almost split sequences starting from (or ending in) interval modules are relatively easy as explained in [3, Sect. 5], we can make full use of them in the computation. We will apply Corollary 4.11 (resp. 4.11′) to compute minimal right (resp. left) interval approximations.

In examples, each indecomposable module M is denoted by its dimension vector $\underline{\dim}\, M$. For each interval (subquiver) J of Q, we present J by $\underline{\dim}\, V_J$.

Example 4.17. Let $A := \Bbbk(CL_3)$. Then the AR-quiver of A is as follows:



We denote by $M_{i,j}$ the indecomposable module with the coordinate (i,j) $(1 \le i \le 6,\ 1 \le j \le 13)$. All these modules are interval modules except for $M_{4,5} = \left[\begin{smallmatrix}1&2&1\\0&1&1\end{smallmatrix}\right]$ and $M_{4,9} = \left[\begin{smallmatrix}1&1&0\\1&2&1\end{smallmatrix}\right]$. Irreducible morphisms starting from $M_{i,j}$ are denoted by $a_{i,j}, a'_{i,j}, a''_{i,j}$ from the top, e.g., the morphism $M_{4,3} \to M_{5,4}$ is denoted by $a''_{4,3}$.

(1) We now compute the interval Koszul coresolution $\mathscr{K}^{\bullet}(V_I)$ of $V_I = M_{5,4}$ for the interval subquiver $I = \left[\begin{smallmatrix}1&1&1\\0&0&1\end{smallmatrix}\right]$. First, the source map from V_I gives rise to the almost split sequence

$$0 \to M_{5,4} \xrightarrow{\mu_0} M_{4,5} \oplus M_{6,5} \xrightarrow{\varepsilon_0} M_{5,6} \to 0.$$

Here, $E_I = M_{4,5} \oplus M_{6,5}$. The minimal left interval coresolution of $M_{4,5}$ is given by the almost split sequence

$$0 \to M_{4,5} \xrightarrow{f_0} M_{2,6} \oplus M_{3,6} \oplus M_{5,6} \xrightarrow{g_0} M_{4,7} \to 0 \qquad (4.12)$$

starting from $M_{4,5}$ because both the middle term and the end term are interval decomposable, and that of $M_{6,5}$ is given by its identity. Thus the minimal left interval approximation of E_I is given by $f_0 \oplus \mathbb{1}_{M_{6,5}}$, which yields the short exact sequence

$$0 \to M_{4,5} \oplus M_{6,5} \xrightarrow{f_0 \oplus \mathbf{1}_{M_{6,5}}} (M_{2,6} \oplus M_{3,6} \oplus M_{5,6}) \oplus M_{6,5} \xrightarrow{(g_0,0)} M_{4,7} \to 0.$$

Therefore, we have the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & M_{5,4} & \xrightarrow{\mu_0} & M_{4,5} \oplus M_{6,5} & \xrightarrow{\varepsilon_0} & M_{5,6} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f_0 \oplus \mathbf{1}_{M_{6,5}}} & \text{po} & \downarrow & & \\
0 & \longrightarrow & M_{5,4} & \xrightarrow[\eta]{} & (M_{2,6} \oplus M_{3,6} \oplus M_{5,6}) \oplus M_{6,5} & \xrightarrow[\mu_1]{} & C^1 & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle (g_0,0)} & & & & \\
 & & & & M_{4,7} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

with exact raws and column, where the right square turns out to be a push out. Here, since $\mu_0 = \begin{bmatrix} a_{5,4} \\ a'_{5,4} \end{bmatrix}$ and $f_0 = \begin{bmatrix} a_{4,5} \\ a'_{4,5} \\ a''_{4,5} \end{bmatrix}$, we have $\eta = \begin{bmatrix} f_0 & 0 \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} a_{5,4} \\ a'_{5,4} \end{bmatrix} = \begin{bmatrix} f_0 a_{5,4} \\ a'_{5,4} \end{bmatrix}$, and hence

$$
\eta = \begin{bmatrix} a_{4,5}a_{5,4} \\ a'_{4,5}a_{5,4} \\ a''_{4,5}a_{5,4} \\ a'_{5,4} \end{bmatrix}.
$$

By the mesh relations, we have

$$
\begin{bmatrix} a'_{2,6} & a_{3,6} & a_{5,6} & 0 \\ 0 & 0 & \mathbf{1} & a_{6,5} \end{bmatrix} \begin{bmatrix} a_{4,5}a_{5,4} \\ a'_{4,5}a_{5,4} \\ a''_{4,5}a_{5,4} \\ a'_{5,4} \end{bmatrix} = \begin{bmatrix} (a'_{2,6}a_{4,5} + a_{3,6}a'_{4,5} + a_{5,6}a''_{4,5})a_{5,4} \\ a''_{4,5}a_{5,4} + a_{6,5}a'_{5,4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

This means that the epimorphism

$$
\begin{bmatrix} a'_{2,6} & a_{3,6} & a_{5,6} & 0 \\ 0 & 0 & \mathbf{1} & a_{6,5} \end{bmatrix} : M_{2,6} \oplus M_{3,6} \oplus M_{5,6} \oplus M_{6,5} \to M_{4,7} \oplus M_{5,6}
$$

is a cokernel morphism of the monomorphism η by counting the dimensions, and that we may take it as μ_1 and $C^1 = M_{4,7} \oplus M_{5,6}$. Since C^1 is already interval decomposable, we see that $\mathscr{K}^\bullet(V_I)$ is as follows:

$$
0 \to V_I \xrightarrow{\eta} \overbrace{M_{2,6} \oplus M_{3,6} \oplus M_{5,6} \oplus M_{6,5}}^{X^1} \xrightarrow{\mu_1} \overbrace{M_{4,7} \oplus M_{5,6}}^{X^2} \to 0.
$$

For the module $M = M_{4,5}$, we compute the interval Koszul complex $\mathscr{K}_I(M)_\bullet$. From the AR-quiver, we immediately see that ${}_A(X^1, M) = 0 = {}_A(X^2, M)$, which shows that $\mathscr{K}_I(M)_\bullet = {}_A(V_I, M) = \Bbbk a_{5,4}$ is a stalk complex concentrated in degree 0. Hence we have $\beta^i_M(I) = \delta_{i,0}$ for all $i \geq 0$. This coincides with the same result obtained by looking at the interval resolution of M that is given as the almost split sequence ending in M:

$$
0 \to M_{4,3} \to M_{2,4} \oplus M_{3,4} \oplus M_{5,4} \to M \to 0.
$$

Here, since $V_I = M_{5,4}$, we have $\beta^i_M(I) = \delta_{i,0}$ for all $i \geq 0$. This also shows that $\delta^\xi_M(I) = 1$ (see Definition 5.3 and Corollary 5.5).

(2) We next compute the interval Koszul coresolution $\mathscr{K}^\bullet(V_I)$ of $V_I = M_{4,3}$ for the interval subquiver $I = \left[\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right]$. The source map from V_I gives rise to the almost split sequence

$$0 \to M_{4,3} \xrightarrow{\mu_0} M_{2,4} \oplus M_{3,4} \oplus M_{5,4} \xrightarrow{\varepsilon_0} M_{4,5} \to 0. \tag{4.13}$$

Here, $E_I = M_{2,4} \oplus M_{3,4} \oplus M_{5,4}$ is interval decomposable, and hence the minimal left interval approximation of E_I is given by the identity $\mathbb{1}_{E_I}$ of E_I, and we see that $\eta = \mu_0$ and $C^1 = M_{4,5}$. The minimal left interval coresolution of $M_{4,5}$ is given by the almost split sequence (4.12). As a consequence, the interval Koszul coresolution $\mathscr{K}^\bullet(V_I)$ of V_I is given as follows:

$$0 \to M_{4,3} \xrightarrow{\mu_0} \overbrace{M_{2,4} \oplus M_{3,4} \oplus M_{5,4}}^{X^1} \xrightarrow{f_0 \varepsilon_0} \overbrace{M_{2,6} \oplus M_{3,6} \oplus M_{5,6}}^{X^2} \xrightarrow{g_0} \overbrace{M_{4,7}}^{X^3} \to 0.$$

Then for the module $M = M_{4,5}$, the interval Koszul complex $\mathscr{K}_I(M)_\bullet$ is given by

$$\cdots \to 0 \to 0 \to {}_A(X^1, M) \xrightarrow{{}_A(\mu_0, M)} {}_A(M_{4,3}, M) \to 0$$

because we see that ${}_A(X^2, M) = 0 = {}_A(X^3, M)$ by looking at the AR-quiver. Since ${}_A(\mu_0, M)$ is an epimorphism, we have $H_0(\mathscr{K}_I(M)_\bullet) = 0$, and thus $\beta^0_M(I) = 0$. Now, since ${}_A(X^1, M) = \Bbbk a_{2,4} \oplus \Bbbk a_{3,4} \oplus \Bbbk a_{5,4}$, each of its element f is uniquely written as $f = (ra_{2,4}, sa_{3,4}, ta_{5,4})$ for some $r, s, t \in \Bbbk$. Then since $\mu_0 = \begin{bmatrix} a_{4,3} \\ a'_{4,3} \\ a''_{4,3} \end{bmatrix}$, ${}_A(\mu_0, M)(f) = 0$ if and only if $ra_{2,4}a_{4,3} + sa_{3,4}a'_{4,3} + ta_{5,4}a''_{4,3} = 0$ if and only if $r = s = t$. Hence

$$H_1(\mathscr{K}_I(M)_\bullet) \cong \operatorname{Ker} {}_A(\mu_0, M) = \Bbbk(a_{2,4}, a_{3,4}, a_{5,4}),$$

and thus $\beta^1_M(I) = 1$. As a consequence, $\beta^i_M(I) = \delta_{i,1}$ for all $i \geq 0$, which can also be read from the interval resolution (4.13) of M. Moreover, we have $\delta^\xi_M(I) = -1$.

Example 4.18. Let $A := \Bbbk(CL_5)$, $I := \left[\begin{smallmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right]$, and $M \in \operatorname{mod} A$ be the indecomposable module

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \Bbbk & \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} & \Bbbk^2 & \xrightarrow{(0,1)} & \Bbbk \\ \uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & & \uparrow{\scriptstyle 1} \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \Bbbk & \xrightarrow{1} & \Bbbk \end{array}. \tag{4.14}$$

Then $\underline{\dim}\, M = \left[\begin{smallmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{smallmatrix}\right]$. We compute the interval Betti numbers $\beta^i_M(I)$. We first compute the interval Koszul coresolution $\mathscr{K}^\bullet(V_I)$ of V_I. The almost split sequence starting from V_I is given as follows:

$$0 \to \left[\begin{smallmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right] \xrightarrow{\mu_0} \overbrace{\left[\begin{smallmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right]}^{X^1} \xrightarrow{\varepsilon_0} M \to 0, \tag{4.15}$$

where $\mu_0 = {}^t(f_1, f_2, f_3)$ with f_1, f_3 canonical embeddings and f_2 a canonical epimorphism. Since the central term $E_I = X^1$ is already interval decomposable, the minimal left interval approximation of E_I is the identity $\mathbb{1}_{E_I}$ of E_I, and we have $\eta = \mu_0$. Next, we can compute the minimal interval coresolution of M by using Proposition 4.9′ and Remark 4.13, which is given as follows:

$$0 \to M \xrightarrow{\mu_1} \overbrace{\left[\begin{smallmatrix} 0&0&1&1&0\\ 0&0&0&0&0 \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} 0&0&0&1&0\\ 0&0&0&1&1 \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} 0&0&1&1&1\\ 0&0&0&1&1 \end{smallmatrix}\right]}^{X^2} \xrightarrow{d^2} \overbrace{\left[\begin{smallmatrix} 0&0&1&1&0\\ 0&0&0&1&1 \end{smallmatrix}\right]}^{X^3} \to 0, \tag{4.16}$$

where $\mu_1 = {}^t(g_1, g_2, g_3)$ with g_1, g_2, g_3 canonical epimorphisms having kernels that are interval modules of dimension vectors $\left[\begin{smallmatrix} 0&0&0&1&1\\ 0&0&0&1&1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0&0&1&1&1\\ 0&0&0&0&0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0&0&0&1&0\\ 0&0&0&0&0 \end{smallmatrix}\right]$, respectively. By combining these short exact sequences, we obtain the interval Koszul coresolution $\mathscr{K}^{\bullet}(V_I)$:

$$0 \to V_I \xrightarrow{\mu_0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} X^3 \to 0$$

with $d^1 := \mu_1 \varepsilon_0$. Hence the interval Koszul complex $\mathscr{K}_I(M)_{\bullet}$ of M at I is given by

$$0 \to {}_A(X^3, M) \xrightarrow{{}_A(d^2, M)} {}_A(X^2, M) \xrightarrow{{}_A(d^1, M)} {}_A(X^1, M) \xrightarrow{{}_A(\mu_0, M)} {}_A(V_I, M) \to 0.$$

By looking at the structure (4.14) of M and those of indecomposable direct summands of X^2, X^3 in (4.16), we easily see that ${}_A(X^3, M) = 0 = {}_A(X^2, M)$. Hence in particular, $\beta^i_M(I) = 0$ for all $i \geq 2$, and $\operatorname{Im} {}_A(d^1, M) = 0$. We now compute $\operatorname{Ker} {}_A(\mu_0, M)$. Since V_I is not a direct summand of M, the almost split sequence (4.15) is mapped to the following exact sequence by the functor ${}_A(\text{-}, M)$:

$$0 \to \operatorname{End}_A(M) \xrightarrow{{}_A(\varepsilon, M)} {}_A(X^1, M) \xrightarrow{{}_A(\mu_0, M)} {}_A(V_I, M) \to 0.$$

Thus $\operatorname{Ker} {}_A(\mu_0, M) \cong \operatorname{End}_A(M) \cong \Bbbk$. Hence

$$\beta^1_M(I) = \dim_{\Bbbk} H_1(\mathscr{K}_I(M)_{\bullet}) = \dim_{\Bbbk} \operatorname{Ker} {}_A(\mu_0, M) = 1,$$

and

$$\beta^0_M(I) = \dim_{\Bbbk} \operatorname{Coker} {}_A(\mu_0, M) = 0.$$

As a consequence, $\beta^i_M(I) = \delta_{i,1}$ for all $i \geq 0$, as the minimal interval resolution (4.15) of M suggests. We also have $\delta^{\xi}_M(I) = -1$.

5. Application to the interval approximation

In this section, we give a direct application to the computation of the *compressed multiplicity* $c^{\xi}_M(I)$ and the *interval approximation* $\delta^{\xi}_M(I)$ defined in [5, Definitions 5.1 and 5.6], respectively.

So far we used the language of algebras and modules over them because this makes it easier to state and prove the statements. However, in [5], we adopted the language of categories and modules over them because we need a functor between categories, which is not interpreted as a morphism between algebras preserving identity. To apply our theorem to results in [5], we now give a bridge in the following.

Remark 5.1. Let (Q, ρ) be a bound quiver. We here remark the relationship between the factor algebra $\Bbbk(Q, \rho)$ of the path algebra $\Bbbk Q$ and the factor category $\Bbbk[Q, \rho]$ of the path category $\Bbbk[Q]$.

(1) A $\Bbbk$-linear category $\Bbbk[Q]$, called the *path category* of Q is defined as follows. The set $\Bbbk[Q]_0$ of objects of $\Bbbk[Q]$ is equal to the vertex set Q_0 of Q, and for any vertices $x, y \in Q_0$, $\Bbbk[Q](x, y) = e_x \Bbbk Q e_y$ under Convention 2.1 (also $\Bbbk[Q, \rho]$ uses the left-to-right notation). The composite $p * q$ of $p \in \Bbbk[Q](x, y)$ and $q \in \Bbbk[Q](y, z)$ is defined to be the product pq in $\Bbbk Q$, and the identity of each $x \in Q_0$ is the path e_x of length 0 at x. Then the matrix algebra $\bigoplus_{x,y\in Q_0} \Bbbk[Q](x, y)$ is isomorphic to $\Bbbk Q$.

(2) Each ideal R of $\Bbbk Q$ corresponds the ideal $[R] := (R(x, y) := e_x R e_y)_{x,y\in Q_0}$ of $\Bbbk[Q]$, and the bound quiver (Q, ρ) defines a factor category $\Bbbk[Q, \rho] := \Bbbk[Q]/[\rho]$.

(3) Under Convention 2.1, the category $\operatorname{mod} \Bbbk[Q, \rho]$ of contravariant functors[2] from $\Bbbk[Q, \rho]$ to $\operatorname{mod} \Bbbk$ is isomorphic to the category $\operatorname{rep}(Q, \rho)$ of representations of (Q, ρ), and is equivalent to the category $\operatorname{mod} \Bbbk(Q, \rho)$ of right $\Bbbk(Q, \rho)$-modules .

(4) This point of view is needed when we consider a morphism F from an algebra of the form $\Bbbk Q'$ with Q' a finite quiver to an algebra of the form $\Bbbk(Q, \rho)$ induced from a quiver morphism $Q' \to U(\Bbbk[Q, \rho])$ that does not need to send the identity element of $\Bbbk Q'$ to that of $\Bbbk(Q, \rho)$, where $U(\Bbbk[Q, \rho])$ is the underlying quiver of $\Bbbk[Q, \rho]$, namely the quiver obtained from it by forgetting the map $\mathbb{1}\colon x \mapsto \mathbb{1}_x$ $(x \in Q_0)$ and the composition of $\Bbbk[Q, \rho]$.

Even in that case, F can be seen as a functor from the category $\Bbbk[Q']$ to the category $\Bbbk[Q, \rho]$. Therefore, in that situation, it is convenient to treat the algebra $\Bbbk(Q, \rho)$ as the linear category $\Bbbk[Q, \rho]$. This is done in this section.

Throughout this section, we set $A := \Bbbk(CL_n) = \Bbbk(Q, \rho)$ (see (4.11)), and treat A as the linear category $\Bbbk[Q, \rho]$ as explained in Remark 5.1. To state our application of Theorem 3.7, we recall necessary definitions and facts from [5]. The set $\mathbb{I}$ of all intervals I in Q are given as follows. Since an interval I is a full subquiver of Q, we denote it by its vertex set I_0. For each $i, j \in \{1, \dots n\}$ with $i \le j$, we set

$$[i, j] := \{x \in \{1, \dots, n\} \mid i \le x \le j\} \subseteq Q_0, \text{ and}$$
$$[\bar{i}, \bar{j}] := \{\bar{x} \in \{\bar{1}, \dots, \bar{n}\} \mid i \le x \le j\} \subseteq Q_0.$$

Then by identifying I with I_0 for all $I \in \mathbb{I}$, [3, Proposition 21] states that

$$\mathbb{I} = \{[\bar{k}, \bar{l}], [i, j], [\bar{k}, \bar{l}] \sqcup [i, j] \mid 1 \le k \le i \le l \le j \le n\}.$$

[2] These are called right $\Bbbk[Q, \rho]$-modules

An interval I with $I_0 = [\overline{k}, \overline{l}] \sqcup [i, j]$ is illustrated as follows:

$$\begin{array}{ccccccccccc} \overline{k} & \xrightarrow{\overline{a}_k} & \cdots & \xrightarrow{\overline{a}_{i-1}} & \overline{i} & \xrightarrow{\overline{a}_i} & \cdots & \xrightarrow{\overline{a}_{l-1}} & \overline{l} & & & & \\ & & & & {\scriptstyle b_i}\uparrow & & \cdots & & \uparrow{\scriptstyle b_l} & & & & \\ & & & & i & \xrightarrow[a_i]{} & \cdots & \xrightarrow[a_{l-1}]{} & l & \xrightarrow[a_l]{} & \cdots & \xrightarrow[a_{j-1}]{} & j \end{array} \tag{5.17}$$

Now let Q' be the quiver:

$$\begin{array}{ccccccccc} & & 2 & & & & 4 & & \\ & \overset{\alpha_1}{\swarrow} & & \overset{\alpha_2}{\searrow} & & \overset{\alpha_3}{\swarrow} & & \overset{\alpha_4}{\searrow} & \\ 1 & & & & 3 & & & & 5 \end{array}$$

and set $B := \Bbbk[Q']$ to be the path category of the quiver Q'. We here regard Q'_0 as a totally ordered subset of the set of integers. For any $i, j \in Q'_0$ with $i \le j$, we set $[i, j] := \{x \in Q'_0 \mid i \le x \le j\}$. Then the set $\mathbb{I}(Q')$ of interval subquivers of Q' is given by

$$\mathbb{I}(Q') = \{[i, j] \mid 1 \le i \le j \le 5\}$$

that has 15 elements, where as before we identify each interval subquiver with its vertex set. By Gabriel's theorem the set $\mathscr{L} := \{V_I \mid I \in \mathbb{I}(Q')\}$ forms a complete set of representatives of the isoclasses of indecomposable right B-modules. Then each right B-module M is uniquely decomposed as

$$M = \bigoplus_{I \in \mathbb{I}(Q')} V_I^{d_M(V_I)} \tag{5.18}$$

with $d_M(V_I) \ge 0$ by the Krull–Schmidt theorem, which defines a function $d_M \colon \mathscr{L} \to \mathbb{Z}_{\ge 0}$.

For each morphism f in $\Bbbk[Q]$, $[f]$ denotes the image of f under the canonical functor $\Bbbk[Q] \to A$. For each interval I, we define a quiver morphism $\xi_I \colon Q' \to U(A)$ as follows, where $U(A)$ is the underlying quiver of the category A (see Remark 5.1 (4)).

(1) In the case that $I_0 = [\overline{k}, \overline{l}] \sqcup [i, j]$ with $1 \le k \le i \le l \le j \le n$. Define ξ_I by the following table:

x	1	2	3	4	5	α_1	α_2	α_3	α_4
$\xi_I(x)$	$\overline{l}$	$\overline{k}$	$\overline{i}$	i	j	$[\overline{a}_k \cdots \overline{a}_{l-1}]$	$[\overline{a}_k \cdots \overline{a}_{i-1}]$	$[b_i]$	$[a_i \cdots a_{j-1}]$

,

where if $k = i$, then $\overline{a}_k \cdots \overline{a}_{i-1}$ is replaced by e_i, (similar for the cases $i = l$ or $l = j$).

(2) In the case that $I_0 = [i, j]$ with $0 \le i \le j \le n$. Define ξ_I by the following table:

x	1	2	3	4	5	α_1	α_2	α_3	α_4
$\xi_I(x)$	j	i	i	i	j	$[a_i \cdots a_{j-1}]$	$[e_i]$	$[e_i]$	$[a_i \cdots a_{j-1}]$

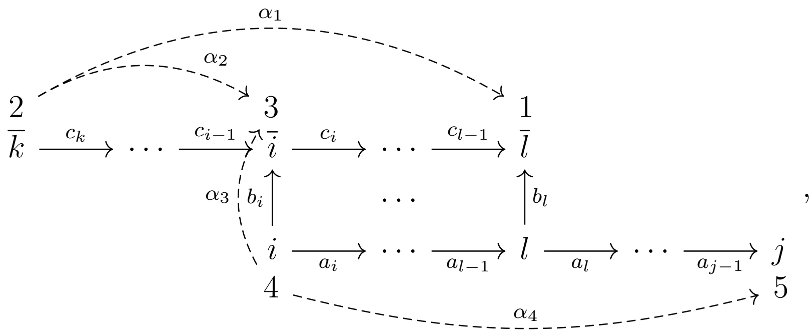
.

(3) In the case that $I_0 = [\overline{k}, \overline{l}]$ with $0 \le k \le l \le n$. Define ξ_I by the following table:

x	1	2	3	4	5	α_1	α_2	α_3	α_4
$\xi_I(x)$	$\bar{l}$	$\bar{k}$	$\bar{k}$	$\bar{k}$	$\bar{l}$	$[\bar{a}_k \cdots \bar{a}_{l-1}]$	$[e_k]$	$[e_k]$	$[\bar{a}_k \cdots \bar{a}_{l-1}]$

.

In the first case, ξ_I is visualized as follows:



,

where each broken arrow α_i represent an arrow in the quiver Q', and the corresponding solid path represents its image $\xi_I(\alpha_i)$ in the quiver $U(A)$.

In any case, ξ_I uniquely extends to a linear functor $F_I \colon B \to A$. By using F_I, we regard A to be the B-A-bimodule[3] ${}_BA_A = A(F_I(\cdot), \text{-})$, which gives us an adjoint pair

$$\mathrm{mod}\, A \underset{R}{\overset{L}{\rightleftarrows}} \mathrm{mod}\, B \ ,$$

where $L = L_{\xi_I} := \text{-} \otimes_B ({}_BA_A)$ is a left adjoint to $R = R_{\xi_I} := \mathrm{Hom}_A({}_BA_A, \text{-})$. Recall that a right A-module M is a contravariant functor $A \to \mathrm{mod}\, \Bbbk$ (Remark 5.1 (3)), and then $R(M) \cong M \circ F_I$. For instance, $R(V_I) \cong V_{[1,5]}$.

Definition 5.2 (Compressed multiplicity)**.** Let ξ be the map sending $I \in \mathbb{I}$ to ξ_I as defined above. We define the *compressed multiplicity with respect to* ξ of V_I in M as

$$c_M^{\xi}(I) := d_{R_{\xi_I}(M)}(R_{\xi_I}(V_I)) = d_{R_{\xi_I}(M)}(V_{[1,5]})$$

(see (5.18)), which defines a map $c_M^{\xi} \in \mathbb{R}^{\mathbb{I}}$, $I \mapsto c_M^{\xi}(I)$ $(I \in \mathbb{I})$.

We regard the set $\mathbb{I}$ as a poset by the partial order $\leq$ defined by $I \leq J$ if and only if $I_0 \subseteq J_0$ for all $I, J \in \mathbb{I}$. For each $I \in \mathbb{I}$, we set $\mathrm{Cov}(I) := \{J \in \mathbb{I} \mid I < J, [I, J] = \{I, J\}\}$ and call it the *cover set* of I. Finally we denote by $\bigvee S$ the join of S for all $S \subseteq \mathbb{I}$.

Consider the incidence algebra $R := \mathbb{R}(\mathbb{I}^{\mathrm{op}}) = \mathbb{R}(H(\mathbb{I}^{\mathrm{op}}), \rho)$ (see Definition 4.1 and Remark 4.2) of the opposite poset $\mathbb{I}^{\mathrm{op}}$ of $\mathbb{I}$. Then since $\mathbb{I}^{\mathrm{op}}$ itself is an interval, we can consider the interval module $V_{\mathbb{I}^{\mathrm{op}}}$, the underlying vector space of which is equal to $\mathbb{R}^{\mathbb{I}}$. By identifying these, $\mathbb{R}^{\mathbb{I}}$ can be seen as a right R-module. It is well known that the zeta function $\zeta := \sum_{[I,J] \in \mathrm{Seg}(\mathbb{I}^{\mathrm{op}})} [I, J] \in R$ has an inverse μ, called the Möbius function (see [26] for details).

[3]Note that by Convention 2.1, the bifunctor $(x, y) \mapsto A(x, y)$ is covariant in x and contravariant in y. Therefore, $(x, y) \mapsto A(F(x), y)$ is a B-A-bimodule.

Definition 5.3 (Interval approximation)**.** We define the *interval approximation* δ_M^ξ *with respect to* ξ to be the the Möbius inversion of c_M^ξ, i.e., $\delta_M^\xi := c_M^\xi \mu$, the explicit form of which is given by

$$\delta_M^\xi(J) := \sum_{S \subseteq \operatorname{Cov}(J)} (-1)^{\#S} c_M^\xi(\bigvee S)$$

for all $J \in \mathbb{I}$, where $\#S$ denotes the cardinality of S.

We refer the reader to [4, 5] for the meanings and properties of the compressed multiplicity and the interval approximation. The following is a main theorem in [5, Sect. 3]. The second equality follows from the first by the general theory of Möbius inversion.

Theorem 5.4 (Theorem 5.5 and Corollary 5.7 in [5])**.** *Let* $I \in \mathbb{I}$*, and* M *be in* $\operatorname{mod} A$ *with the minimal interval resolution* (2.3) *with* $X_{r+1} = 0$ *for some* $r \geq 0$*. (By* [5, Proposition 4.5]*, there always exists such an* r*.) Then we have*

$$c_M^\xi(I) = \sum_{I \leq J \in \mathbb{I}} \left(\sum_{i=0}^{r} (-1)^i \beta_M^i(J) \right), \text{ and}$$

$$\delta_M^\xi(I) = \sum_{i=0}^{r} (-1)^i \beta_M^i(I).$$

We are now in a position to state our application of Theorem 3.7. The following is a direct consequence of Theorems 3.7 and 5.4.

Corollary 5.5. *Let* $A = \Bbbk(CL_n)$ *for some* $n \geq 2$*,* $M \in \operatorname{mod} A$*, and* $I \in \mathbb{I}$*. Suppose that* M *has the minimal interval resolution* (2.3) *with* $X_{r+1} = 0$ *for some* $r \geq 0$*. Then we have*

$$c_M^\xi(I) = \sum_{I \leq J \in \mathbb{I}} \left(\sum_{i=0}^{r} (-1)^i \dim_\Bbbk H_i(\mathscr{K}_J(M)_\bullet) \right), \text{ and}$$

$$\delta_M^\xi(I) = \sum_{i=0}^{r} (-1)^i \dim_\Bbbk H_i(\mathscr{K}_I(M)_\bullet).$$

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