THE PARALLEL TRANSPORT MAP OVER AFFINE SYMMETRIC SPACE

MASAHIRO MORIMOTO

Abstract. In the 1990s, C.-L. Terng and G. Thorbergsson investigated a natural Riemannian submersion from an infinite dimensional Hilbert space onto a compact Riemannian symmetric space G/K. This map is called the parallel transport map over G/K. Later, N. Koike extended their theory to the case that G/K is a Riemannian symmetric space of non-compact type. In this paper, more generally, we define the parallel transport map over an affine symmetric space and show that it is an affine submersion with horizontal distribution in the sense of Abe and Hasegawa. Based on this result, we prove the Fredholm property of affine immersions into a Hilbertable space lifted by the parallel transport map. Furthermore, we greatly extend the author's previous result on weakly reflective submanifolds from the case of compact Riemannian symmetric spaces to the case of affine symmetric spaces.

1. Introduction

Palais and Terng [23, 28] began the study of submanifolds in Hilbert spaces in the 1980s. In order to use results in the infinite dimensional differential topology and Morse theory [27, 21, 26], they introduced a suitable class of submanifolds in Hilbert spaces of finite codimension, namely *proper Fredholm* (PF) submanifolds. Among other things, they showed examples of PF submanifolds which are orbits of a Lie group action. To be more precise, for a connected compact Lie group G with a bi-invariant Riemannian metric, they essentially considered the Hilbert space $V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g})$ of all L^2-maps from $[0,1]$ to the Lie algebra $\mathfrak{g}$ of G, defined an isometric action on $V_{\mathfrak{g}}$ by an infinite dimensional Lie group and showed that its orbits are PF submanifolds in $V_{\mathfrak{g}}$. To study those examples, they considered an equivariant submersion $\Phi : V_{\mathfrak{g}} \to G$ which is nowadays called the *parallel transport map*.

In 1995, Terng and Thorbergsson [30] showed that the parallel transport map is a Riemannian submersion. Moreover they gave an interesting application to the submanifold geometry in symmetric spaces. For a Riemannian symmetric space G/K of compact type with projection $\pi : G \to G/K$ they considered the composition $\Phi_{G/K} := \pi \circ \Phi : V_{\mathfrak{g}} \to G \to G/K$ which is also a Riemannian submersion called the parallel transport map over G/K. They showed that for a submanifold M of G/K its inverse image $\hat{M}$ under $\Phi_{G/K}$ is a PF submanifold of $V_{\mathfrak{g}}$. Moreover they showed that under some assumptions, $\hat{M}$ is isoparametric if and only if M is equifocal. Based on

---

2020 *Mathematics Subject Classification.* 53C40, 53C42, 53A15, 58B15.

Key words and phrases. parallel transport map, affine symmetric space, affine submersion with horizontal distribution, Fredholm submanifold, weakly reflective submanifold.

This work was partly supported by Grant-in-Aid for JSPS Research fellow (No. 23KJ1793) and by MEXT Promotion of Distinctive Joint Research Center Program JPMXP0723833165.

this result, they investigated equifocal submanifolds in G/K by taking advantage of isoparametric PF submanifolds in the *flat* space $V_{\mathfrak{g}}$.

In 2004, Koike [11] studied the parallel transport map over a Riemannian symmetric space G/K of non-compact type. Since there are no bi-invariant Riemannian metrics on G we cannot define the Hilbert space together with the *isometric* action on it similarly to the compact case. Thus he equipped G with a bi-invariant *pseudo-Riemannian* metric and considered the *pseudo-Hilbert space* $V_{\mathfrak{g}}$. Roughly speaking, a pseudo-Hilbert space is a Hilbert space with a continuous indefinite inner product. He showed that the parallel transport map $\Phi : V_{\mathfrak{g}} \to G$ is a pseudo-Riemannian submersion relative to those indefinite metrics. Then $\Phi_{G/K} = \pi \circ \Phi : V_{\mathfrak{g}} \to G \to G/K$ becomes a pseudo-Riemannian submersion. Based on these results, he extended the result of Terng and Thorbergsson on equifocal and isoparametric submanifolds to the case of Riemannian symmetric spaces of non-compact type (see also [12]).

Note that his formulation of the parallel transport map cannot be applied to the non-semisimple case, because he uses the semisimplicity of G to ensure the existence of a bi-invariant pseudo-Riemannian metric on G. Note also that his formulation is based on several metrics. In fact, there are essentially two inner products on $V_{\mathfrak{g}}$; one is positive definite and generates its topology and the other is indefinite. Besides, although G is equipped with a pseudo-Riemannian metric, we also need another Riemannian metric on G to define the infinite dimensional Lie group acting isometrically on $V_{\mathfrak{g}}$, namely the Hilbert Lie group $H^1([0,1], G)$ of Sobolev H^1-paths in G. Due to those metrics, one may find it difficult to understand the essentials of the parallel transport map. In view of these, it is reasonable to study the parallel transport map in the framework of *affine differential geometry.* There always exists a bi-invariant torsion-free affine connection ∇^G on G, called the *canonical connection*, even when G is not semisimple. The affine geometric framework gives us a better understanding of Riemannian symmetric spaces [10]. We can expect that the same is also true in the case of the parallel transport map.

One of the purposes of this paper is to show affine geometric properties of the parallel transport map over a symmetric space which is not necessarily semisimple. Before stating our results, we make several remarks: First, from now on we always regard $V_{\mathfrak{g}} = L^2([0,1], \mathfrak{g})$ as a *Hilbertable* space, that is, a topological vector space whose topology is induced by a complete inner product. In fact, since $\mathfrak{g}$ is finite dimensional, the topology on $V_{\mathfrak{g}}$ induced by the L^2-inner product does not depend on the choice of an inner product on $\mathfrak{g}$. Second, we will make sure that the Hilbert Lie group $H^1([0,1], G)$ does not depend on the choice of a Riemannian metric on G (Section 5). Third, the parallel transport map $\Phi : V_{\mathfrak{g}} \to G$ is defined via the solution of an ordinary differential equation and its unique existence was originally proved by using compactness of G (cf. Remark 6.3). We will modify that proof so that Φ can be defined for any connected Lie group G. Note that Φ is defined without any use of a metric. Fourth, we define the parallel transport map $\Phi_{G/K} := \pi \circ \Phi : V_{\mathfrak{g}} \to G \to G/K$ over a homogeneous space G/K and it turns out that Φ is a special case of $\Phi_{G/K}$ (Remark 7.1). Thus we only state the results about $\Phi_{G/K}$ here.

The following theorem shows affine geometric properties of the parallel transport map over a symmetric space which is not necessarily compact or semisimple. This extends the above-mentioned results of Terng-Thorbergsson [30, Theorem 4.5] and Koike [11, Proposition 4.4] to the affine case (Theorem 7.4):

Theorem 1.1. *Let G/K be a symmetric space with the canonical connection $\nabla^{G/K}$. Denote by D the flat connection on $V_{\mathfrak{g}}$. Then the parallel transport map $\Phi_{G/K} : (V_{\mathfrak{g}}, D) \to (G/K, \nabla^{G/K})$ is an affine submersion with horizontal distribution $\mathcal{H}$. Moreover the tensor $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating.*

Here the concept of an affine submersion with horizontal distribution was introduced by Abe and Hasegawa [1] in the finite dimensional case. This is a generalized concept of a pseudo-Riemannian submersion and a dual concept of an affine immersion with transversal bundle (cf. Section 3). In Theorem 1.1 the horizontal distribution $\mathcal{H}$ of $\Phi_{G/K}$ is naturally defined by the structure of G/K as a reductive homogeneous space, without any use of a metric. The tensor $\mathcal{A}$ is one of the fundamental tensors introduced by O'Neill [20] in the case of Riemannian submersions. When the tensor $\mathcal{A}$ restricted to $\mathcal{H} \times \mathcal{H}$ is alternating, a smooth curve in G/K is a geodesic if and only if its horizontal lift is a geodesic in $V_{\mathfrak{g}}$.

We also generalize the concept of isometric Fredholm immersions into Hilbert spaces [23, 28] to the affine case and show the following fact (Proposition 4.6), which generalizes some result of Heintze-Liu-Olmos [4, Lemma 5.1 (i)] to the affine case and can be applied to Theorem 1.1.

Proposition 1.2. *Let (V, D) be a Hilbertable space with the flat connection, (N, ∇^N) a finite dimensional affine manifold and $\phi : (V, D) \to (N, \nabla^N)$ an affine submersion with horizontal distribution $\mathcal{H}$ such that $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating. If $\varphi : (M, \nabla) \to (N, \nabla^N)$ is an affine immersion with transversal bundle $\mathcal{W}$, then its lift $\hat{\varphi} : (\hat{M}, \hat{\nabla}) \to (V, D)$ is an affine Fredholm immersion of Fredholm index 0 with transversal bundle $\hat{\mathcal{W}}$, where $\hat{\mathcal{W}}$ denotes the horizontal lift of $\mathcal{W}$.*

Here an affine immersion into (V, D) with transversal bundle $\hat{\mathcal{W}}$ is said to be *Fredholm* if the end point map $\hat{\eta} : \hat{\mathcal{W}} \to V$, $(p, \xi) \mapsto p + \xi$ is a Fredholm map and *of Fredholm index* 0 if in addition the Fredholm index of $\hat{\eta}$ is 0. In the Riemannian case, the property of Fredholm index 0 always holds since the shape operators are self-adjoint (Remark 4.5). In the affine case, this does not hold in general and is expected to be a clue to study the spectrum of the non-self-adjoint shape operators.

The goal of this paper is to extend the author's previous results [16, 17] on weakly reflective submanifolds to the affine case. Those submanifolds were originally introduced by Ikawa, Sakai and Tasaki [5] in the finite dimensional Riemannian case and constitute a special class of minimal submanifolds. To be more precise, a submanifold M of a Riemannian manifold N is called *weakly reflective* if for each normal vector ξ at each $p \in M$ there exists an isometry ν of N satisfying

$$\nu(M) = M, \qquad \nu(p) = p, \qquad d\nu(\xi) = -\xi. \tag{1.1}$$

Such a ν is called a *weak reflection* of M with respect to (p, ξ). If every ν can be chosen from a particular group S, then M is called *S-weakly reflective.* It follows that weakly reflective submanifolds are austere [3], that is, for each normal vector ξ the set of eigenvalues with multiplicities of the shape operator A^M_ξ is invariant under the multiplication by (-1). A typical example of a weakly reflective submanifold is a singular orbit of a cohomogeneity one action on a Riemannian manifold [25, 5]. The author [16, 17] defined weakly reflective PF submanifolds in Hilbert spaces by the same way as in finite dimensions and studied their relations to the parallel transport map over a compact normal homogeneous space. In this paper, more generally, we define a weakly reflective submanifold M of an affine manifold N with transversal bundle $\mathcal{W}$ by the condition that for each $\xi \in \mathcal{W}$ at each $p \in M$ there exists an affine transformation ν of N satisfying (1.1) and $d\nu(\mathcal{W}) = \mathcal{W}$. Then we study its relation to the parallel transport map over a reductive homogeneous space which is not necessarily compact or Riemannian. Despite the lack of Riemannian structure, we can greatly extend the author's previous results, as we will now explain.

The following theorem (Theorem 8.3) shows examples of infinite dimensional weakly reflective submanifolds in Hilbertable spaces which are not totally geodesic (cf. Remark 8.4). This generalizes Corollary 5 of [16] in the case that G/K is a compact normal homogeneous space.

Theorem 1.3. *Let G/K be a reductive homogeneous space. Then each fiber of the parallel transport map $\Phi_{G/K} : V_{\mathfrak{g}} \to G/K$ is a weakly reflective submanifold of the Hilbertable space $(V_{\mathfrak{g}}, D)$ where its transversal bundle is defined as the restriction of the horizontal distribution of $\Phi_{G/K}$ to it.*

The above theorem together with Theorem 1.1 and Proposition 1.2 implies the following corollary. From this we know that our extensions of Fredholm submanifolds and weakly reflective submanifolds to the affine cases are meaningful.

Corollary 1.4. *Let G/K be a symmetric space. Then each fiber of the parallel transport map $\Phi_{G/K} : V_{\mathfrak{g}} \to G/K$ is a weakly reflective Fredholm submanifold of $(V_{\mathfrak{g}}, D)$ of Fredholm index 0 where its transversal bundle is defined as above.*

The following theorem gives a characterization of weakly reflective submanifolds in affine symmetric spaces. This extends Theorem 8 of [16] in the case that N is an (irreducible) Riemannian symmetric space of compact type and Theorem 1 of [17] in the case that N is a compact isotropy irreducible Riemannian homogeneous space (see Theorem 8.6 and Remarks 8.7–8.9 for more details).

Theorem 1.5. *Let (N, ∇^N) be an affine symmetric space, $\tilde{G}$ the affine transformation group of N, G its identity component, $\tilde{K}$ the isotropy subgroup of $\tilde{G}$ at a fixed $p \in N$ and $K := G \cap \tilde{K}$. Denote by $\Phi_{G/K} = \pi \circ \Phi : V_{\mathfrak{g}} \to G \to G/K = N$ the parallel transport map. Let M be a submanifold of (N, ∇^N) with transversal bundle $\mathcal{W}$. Denote by $\bar{\mathcal{W}}$ (resp. $\hat{\mathcal{W}}$) the horizontal lift of $\mathcal{W}$ with respect to π (resp. $\Phi_{G/K}$). Then the following conditions are equivalent:*

(i) *M is a weakly reflective submanifold of (N, ∇^N) with transversal bundle $\mathcal{W}$.*
(ii) *$\bar{M} := \pi^{-1}(M)$ is a $(\tilde{G} \times \tilde{K})$-weakly reflective submanifold of (G, ∇^G) with transversal bundle $\bar{\mathcal{W}}$.*
(iii) *$\hat{M} := \Phi_{G/K}^{-1}(M)$ is a $P(\tilde{G}, \tilde{G} \times \tilde{K})$-weakly reflective Fredholm submanifold of $(V_{\mathfrak{g}}, D)$ of Fredholm index 0 with transversal bundle $\hat{\mathcal{W}}$.*

Here $\tilde{G} \times \tilde{K}$ acts on $\tilde{G}$ affinely by $(b, c) \cdot a := bac^{-1}$ and $P(\tilde{G}, \tilde{G} \times \tilde{K}) := \{g \in H^1([0,1], \tilde{G}) \mid (g(0), g(1)) \in \tilde{G} \times \tilde{K}\}$ acts on $V_{\mathfrak{g}}$ affinely by $g * u := \mathrm{Ad}(g)u - dr_g^{-1}(g')$.

Note that when N is an *irreducible* Riemannian symmetric space of compact or non-compact type, the affine transformation group coincides with the isometry group and thus a weakly reflective submanifold (with normal bundle) in the affine sense is equivalent to a weakly reflective submanifold in the Riemannian sense (Remark 8.1). Thus, Theorem 1.5 completely generalizes the case of irreducible Riemannian symmetric spaces of compact or non-compact type (Corollary 8.12 with $S = \tilde{G}$, see also [16, Corollary 8]). This generalization is meaningful because, for example, a single point of an affine symmetric space is a weakly reflective submanifold in the affine sense.

The proofs of Theorems 1.3 and 1.5 are different from the proofs in the previous cases [16, 17]. In fact the previous proofs depends on the property of a Riemannian submersion ([16, Lemma 2], [17, Lemma 1]). To prove Theorem 1.3 we essentially use the *canonical reflection* on $V_{\mathfrak{g}}$ which was introduced in the previous Riemannian case [16]. In the affine case we need a further argument on its relation to the horizontal distribution of $\Phi_{G/K}$. To prove Theorem 1.5 we rely on the structure of the symmetric space N (Lemma 8.10) to verify that the lifted weak reflection leaves the lifted transversal bundle invariant.

This paper is organized as follows. In Section 2 we review affine connections on Hilbert manifolds and give a modern definition equivalent to others. In Section 3 we formulate affine immersions and affine submersions in the framework of Hilbert manifolds. In Section 4 we study focal points and the Fredholm property of affine immersions lifted by some affine submersion. In Section 5 we review the Hilbert Lie group $H^1([0,1], G)$. In Section 6 we study the parallel transport map over a Lie group. In Section 7 we study parallel transport maps over homogeneous and symmetric spaces. In Section 8 we define weakly reflective submanifolds in the affine case and study their relations to the parallel transport map.

2. Affine connections on Hilbert manifolds

In this section, we review affine connections on Hilbert manifolds. We refer to Lang [14], Eliasson [2] and Klingenberg [8] for details.

Recall that a *Hilbert manifold* is a smooth manifold locally modeled on a Hilbert space. Strictly speaking, the Hilbert space here is interpreted as a *Hilbertable* space, that is, a topological vector space (over the real numbers $\mathbb{R}$) whose topology is induced by a complete inner product. In fact, Lang [14] treats such a Hilbert manifold or more generally, a Banach manifold, which is a smooth manifold locally modeled on a *Banachable* space (i.e. a topological vector space whose topology is induced by a complete norm). Note that in his book, the distinction between a Banach space and a Banachable space is not emphasized [14, p. 6]. However we will distinguish them since we deal with affine differential geometry. We always assume that Hilbert spaces and Hilbertable spaces are separable.

Let M be a Hilbert manifold. We denote by $C^\infty(M)$ the set of all smooth, $\mathbb{R}$-valued functions on M and by $\mathfrak{X}(M)$ the set of all smooth vector fields on M. When we speak of a *vector bundle* over M, it means a smooth vector bundle over M whose fibers are Banachable spaces. For a vector bundle E over M we denote by E_p the fiber at $p \in M$ and by $\Gamma(E)$ the set of all smooth sections of E. For two vector bundles E and E' over M we denote by $\mathrm{VBHom}(E, E')$ the set of all vector bundle homomorphisms from E to E' (acting on M identically) and by $L(E, E')$ the vector bundle over M whose fiber at each $p \in M$ is the Banachable space $L(E_p, E'_p)$ of all continuous linear maps from E_p to E'_p. In particular $L(E, E)$ is denoted by $\operatorname{End} E$. We have a canonical isomorphism between modules over $C^\infty(M)$:

$$\mathrm{VBHom}(E, E') \cong \Gamma(L(E, E')).$$

Moreover we denote by $\mathrm{Hom}_{C^\infty(M)}(\Gamma(E), \Gamma(E'))$ the set of all $C^\infty(M)$-linear maps from $\Gamma(E)$ to $\Gamma(E')$. In the finite dimensional case, it is well-known that this set is identified with $\mathrm{VBHom}(E, E')$ [15, Lemma 10.29]. However, this fact is not clear in the infinite dimensional case. More precisely, it is clear that each $F \in \mathrm{VBHom}(E, E')$ defines an element $\bar{F} \in \mathrm{Hom}_{C^\infty(M)}(\Gamma(E), \Gamma(E'))$ by $\bar{F}(s) := F \circ s$. However, regarding the converse, we only know:

(i) If $\operatorname{rank} E < \infty$, then for each $\mathcal{F} \in \mathrm{Hom}_{C^\infty(M)}(\Gamma(E), \Gamma(E'))$ there exists a map $F : E \to E'$ such that its restriction to each fiber E_p is a continuous linear map into E'_p and $\mathcal{F} = \bar{F}$.

(ii) If in addition $\operatorname{rank} E' < \infty$, then the map F is smooth and thus belongs to $\mathrm{VBHom}(E, E')$. Therefore, in this case, we have the isomorphism

$$\mathrm{VBHom}(E, E') \cong \mathrm{Hom}_{C^\infty(M)}(\Gamma(E), \Gamma(E')). \tag{2.1}$$

These facts can be proven by the same arguments as in the finite dimensional case. See also [14, Lemma 2.3 in Chapter VIII] for discussions in the infinite dimensional case. Note that the existence of cut off functions over Hilbert manifolds [14, Theorem 3.7 in Chapter II] is used in the proof.

For vector bundles $E_1, \cdots, E_k$ and E' over M we denote by $L(E_1, \cdots, E_k; E')$ the vector bundle over M whose fiber at each $p \in M$ is the Banachable space of all continuous multi-linear maps from $(E_1)_p \times \cdots \times (E_k)_p$ to E'_p. If $E_1 = \cdots = E_k (=: E)$ then it is denoted by $L^k(E; E')$. We denote by $L^k_{sym}(E, E')$ (resp. $L^k_{alt}(E, E')$) the subbundle of $L^k(E; E')$ whose fiber at each $p \in M$ is the subspace of symmetric (resp. alternating) multi-linear ones. Set $\Omega^k(E) := \Gamma(L^k_{alt}(TM; E))$ and $\Omega^0(E) := \Gamma(E)$ where TM denotes the tangent bundle of M.

Definition. Let E be a vector bundle over a Hilbert manifold M and $\nabla : \Omega^0(E) \to \Omega^1(E)$ an $\mathbb{R}$-linear map. For $s \in \Omega^0(E)$ and $v \in T_pM$ we write $\nabla_v s$ for $(\nabla s)_p(v)$. ∇ is called a *connection* on E if it satisfies:

(a) $\nabla_v(fs) = f(p)(\nabla_v s) + (vf)s(p)$ where $f \in C^\infty(M)$,

(b) for each local trivialization $E|_U \cong U \times \mathbb{F}$ there exists $\omega_U \in \Omega^1(\mathrm{End}(U \times \mathbb{F}))$ satisfying $\nabla|_U = D + \omega_U$.

Here D denotes the trivial connection over the trivial bundle $U \times \mathbb{F}$ where $\mathbb{F}$ is a Banachable space and ω_U is called the *(local) connection* 1*-form*.

Note that the restriction $\nabla|_U : \Omega^0(E|_U) \to \Omega^1(E|_U)$ is well-defined since (a) holds and Hilbert manifolds admit cut off functions. Note also that if M and E are finite dimensional, then the condition (b) is automatically satisfied, owing to (2.1).

The above definition of a connection is equivalent to that of Eliasson [2] and Klingenberg [8] who define it as the connection map $K : TE \to E$. (Note that Eliasson [2] considers the more general case that M is a Banach manifold.) The equivalence of the definitions can be seen by observing that a connection over E is characterized by the formula

$$\omega_\beta = g_{\alpha\beta}^{-1} \circ \omega_\alpha \circ g_{\alpha\beta} + g_{\alpha\beta}^{-1} \circ D g_{\alpha\beta} \tag{2.2}$$

where $\{U_\alpha\}_{\alpha \in A}$ are the trivializing coverings, $\{\omega_\alpha\}_{\alpha \in A}$ the local connection 1-forms and $\{g_{\alpha\beta}\}_{\alpha,\beta \in A}$ the transition functions. Note that a connection ∇ induces an $\mathbb{R}$-bilinear map $\nabla : \Gamma(E) \times \mathfrak{X}(M) \to \Gamma(E)$ satisfying

$$\nabla_X fs = f\nabla_X s + (Xf)s \quad \text{and} \quad \nabla_{fX} s = f\nabla_X s.$$

Lang [14, p. 375] calls such a map a *derivative on E relative to TM*. It is not clear whether such a map conversely induces a connection on E or not, due to lack of (2.1) in infinite dimensions. Lang [14, p. 201] also considers a class of *covariant derivatives associated with sprays*, which is equivalent to our class of torsion-free connections on TM, where torsion tensors are explained below.

Despite the lack of (2.1) in infinite dimensions, we can develop the theory of connections on vector bundles over Hilbert manifolds, especially the theory of affine connections on Hilbert manifolds, owing to the condition (b) above. To be more precise, let E be a vector bundle over a Hilbert manifold M and ∇ a connection on E with the local connection 1-forms $\{\omega_\alpha\}_{\alpha\in A}$. The curvature tensor $R \in \Omega^2(\operatorname{End} E)$ is defined via the local formula

$$R_\alpha := D\omega_\alpha + \omega_\alpha \wedge \omega_\alpha$$

and it satisfies $R(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s$. If $E = TM$, a connection ∇ is called an *affine connection* on M and the pair (M, ∇) is called an *affine Hilbert manifold*. In this case the torsion tensor $T \in \Omega^2(TM)$ is defined via the local formula

$$T_\alpha := \omega_\alpha \wedge \mathrm{id}$$

and it satisfies $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. If $T = 0$ then ∇ is called *torsion-free*. Geodesics and exponential maps for affine Hilbert manifolds are defined by the same way as in finite dimensions.

Let g be a pseudo-Riemannian metric on a Hilbert manifold M. The pair (M, g) is called a *pseudo-Riemannian Hilbert manifold*. There exists a unique torsion-free affine connection on M satisfying $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, called the *Levi-Civita connection*. In fact, the local expression of the Koszul formula

$$2g_\alpha((\omega_\alpha)_{X_\alpha}Y_\alpha, Z_\alpha) = (Dg_\alpha)_{X_\alpha}(Y_\alpha, Z_\alpha) + (Dg_\alpha)_{Y_\alpha}(Z_\alpha, X_\alpha) - (Dg_\alpha)_{Z_\alpha}(X_\alpha, Y_\alpha),$$

uniquely determines the local connection 1-forms, which define the Levi-Civita connection. For more details about affine and pseudo-Riemannian geometries of Hilbert manifolds, see [14] and [8].

3. Affine immersions and affine submersions

In this section, we formulate affine immersions and affine submersions in the framework of Hilbert manifolds. We refer to Nomizu-Sasaki [19] and Abe-Hasegawa [1] for details in the finite dimensional case.

3.1. **Affine immersions.** Let M and N be Hilbert manifolds. A smooth map $\varphi : M \to N$ is called an *immersion* if the differential $(d\varphi)_p : T_pM \to T_{\varphi(p)}N$ at each $p \in M$ is injective and the image $d\varphi_p(T_pM)$ is a closed subspace of $T_{\varphi(p)}N$. Then $d\varphi(TM)$ is a subbundle of the pullback bundle φ^*TN. A *transversal bundle* $\mathcal{W}$ of φ is a subbundle of φ^*TN satisfying

$$\varphi^*TN = d\varphi(TM) \oplus \mathcal{W}.$$

Note that there is no natural choice of $\mathcal{W}$ in general. The projections onto $d\varphi(TM)$ and $\mathcal{W}$ are denoted by the superscripts $\top$ and $\mathcal{W}$ respectively. We often write an element of $\mathcal{W}$ as the pair (p, ξ) of a vector ξ and its foot point p.

Let (M, ∇^M) and (N, ∇^N) be affine Hilbert manifolds, $\varphi : M \to N$ an immersion and $\mathcal{W}$ a transversal bundle of φ. Then $\varphi : (M, \nabla^M) \to (N, \nabla^N)$ is called an *affine immersion with transversal bundle* $\mathcal{W}$ if

$$d\varphi(\nabla^M_X Y) = (\nabla^{\varphi^*TN}_X d\varphi(Y))^\top \tag{3.1}$$

holds for $X, Y \in \mathfrak{X}(M)$. Here ∇^{φ^*TN} denotes the induced connection on φ^*TN.

Let $\varphi : M \to N$ be an immersion, $\mathcal{W}$ a transversal bundle of φ and ∇^N be an affine connection on N. Then (3.1) defines an affine connection ∇^M on M so that $\varphi : (M, \nabla^M) \to (N, \nabla^N)$ is an affine immersion with transversal bundle $\mathcal{W}$. Moreover the *affine fundamental form* $\alpha^\varphi \in \Gamma(L^2(TM; \mathcal{W}))$, the *shape operator* $A^\varphi \in \Gamma(L(\mathcal{W}, \operatorname{End} TM))$ and the *transversal connection* $\nabla^{\mathcal{W}}$ which is a connection on $\mathcal{W}$, are defined by the Gauss and Weingarten formulas

$$\nabla^{\varphi^*TN}_X d\varphi(Y) = d\varphi(\nabla^M_X Y) \oplus \alpha^\varphi(X, Y), \tag{3.2}$$

$$\nabla^{\varphi^*TN}_X Z = -d\varphi(A^\varphi_Z(X)) \oplus \nabla^{\mathcal{W}}_X Z, \tag{3.3}$$

where $X, Y \in \mathfrak{X}(M)$ and $Z \in \Gamma(\mathcal{W})$. In fact these are locally expressed as

$$D_X d\varphi(Y) + \omega^{\varphi^*TN}_X d\varphi(Y) = (D_X d\varphi(Y) + (\omega^{\varphi^*TN}_X d\varphi(Y))^\top) \oplus (\omega^{\varphi^*TN}_X d\varphi(Y))^{\mathcal{W}},$$

$$D_X Z + \omega^{\varphi^*TN}_X Z = (\omega^{\varphi^*TN}_X Z)^\top \oplus (D_X Z + (\omega^{\varphi^*TN}_X Z)^{\mathcal{W}}),$$

where ω^{φ^*TN} denotes the local connection 1-form of ∇^{φ^*TN}. If ∇^N is torsion-free, then ∇^M is also torsion-free and α^φ is symmetric.

Any isometric immersion becomes an affine immersion by defining its transversal bundle as the normal bundle and the connections as the Levi-Civita connections.

3.2. **Affine submersions.** Let $\hat{N}$ and N be Hilbert manifolds and $\pi : \hat{N} \to N$ a submersion. Then $\mathcal{V} := \operatorname{Ker} d\pi$ is a subbundle of $T\hat{N}$, which is called the *vertical distribution* of π. The vertical subspace $\mathcal{V}(u)$ at $u \in \hat{N}$ is equal to the tangent space $T_u F$ of the fiber F through u. A *horizontal distribution* $\mathcal{H}$ of π is a subbundle of $T\hat{N}$ satisfying

$$T\hat{N} = \mathcal{V} \oplus \mathcal{H}.$$

Note that there is no natural choice of $\mathcal{H}$ in general. The projections onto $\mathcal{V}$ and $\mathcal{H}$ are denoted by the superscripts $\mathcal{V}$ and $\mathcal{H}$ respectively.

Let $(\hat{N}, \hat{\nabla})$ and (N, ∇) be affine Hilbert manifolds, $\pi : \hat{N} \to N$ a submersion and $\mathcal{H}$ a horizontal distribution of π. Then $\pi : (\hat{N}, \hat{\nabla}) \to (N, \nabla)$ is called an *affine submersion with horizontal distribution* $\mathcal{H}$ if

$$(\hat{\nabla}_{\hat{X}} \hat{Y})^{\mathcal{H}} = \widehat{(\nabla_X Y)}$$

holds for $X, Y \in \mathfrak{X}(N)$. Here $\hat{X}$ denotes the horizontal lift of X, that is, $\hat{X} \in \Gamma(\mathcal{H})$ satisfying $d\pi(\hat{X}_u) = X_{\pi(u)}$ for all $u \in \hat{N}$.

Let $\pi : (\hat{N}, \hat{\nabla}) \to (N, \nabla)$ be an affine submersion with horizontal distribution $\mathcal{H}$. Then the *fundamental tensors* $\mathcal{A}, \mathcal{T} \in \Gamma(L^2(T\hat{N}; T\hat{N}))$ are defined by

$$\mathcal{T}_Z W = (\hat{\nabla}_{Z^{\mathcal{V}}} W^{\mathcal{V}})^{\mathcal{H}} + (\hat{\nabla}_{Z^{\mathcal{V}}} W^{\mathcal{H}})^{\mathcal{V}},$$

$$\mathcal{A}_Z W = (\hat{\nabla}_{Z^{\mathcal{H}}} W^{\mathcal{H}})^{\mathcal{V}} + (\hat{\nabla}_{Z^{\mathcal{H}}} W^{\mathcal{V}})^{\mathcal{H}},$$

where $Z, W \in \mathfrak{X}(\hat{N})$. In fact, these are locally expressed in terms of connection 1-forms of $\hat{\nabla}$ and thus tensors. Note that if $\hat{\nabla}$ is torsion-free, then so is ∇.

By using the tensor $\mathcal{A}$, we can write

$$\hat{\nabla}_{\hat{X}}\hat{Y} = \widehat{(\nabla_X Y)} + \mathcal{A}_{\hat{X}}\hat{Y}$$

where $X, Y \in \mathfrak{X}(N)$. This implies that, *when the restriction* $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}} \in \Gamma(L^2(\mathcal{H};\mathcal{V}))$ *is alternating, a smooth curve* γ *in* N *is a geodesic in* (N, ∇) *if and only if its horizontal lift* $\hat{\gamma}$ *is a geodesic in* $(\hat{N}, \hat{\nabla})$.

Any pseudo-Riemannian submersion becomes an affine submersion by defining its horizontal distribution as the orthogonal complement of the vertical distribution. In this case, the tensor $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating.

3.3. **Lifts.** Let $\varphi : M \to N$ be an immersion and $\pi : \hat{N} \to N$ a submersion. The *lift* of φ (with respect to π) is an immersion $\hat{\varphi} : \hat{M} \to \hat{N}$ defined by $\hat{M} := \{(p, u) \in M \times \hat{N} \mid \varphi(p) = \pi(u)\}$ and $\hat{\varphi}(p, u) := u$. We have the commutative diagram:

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{\varphi}} & \hat{N} \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\varphi} & N. \end{array}$$

Here $pr : \hat{M} \to M$ denotes the submersion $(p, u) \mapsto p$. Note that φ is an injective immersion if and only if $\hat{\varphi}$ is an injective immersion, and in this case $\hat{M}$ is identified with the inverse image of $\varphi(M)$ under π. Let $\mathcal{H}$ be a horizontal distribution of π. Then the horizontal distribution of *pr* is induced. If $\mathcal{W}$ is a transversal bundle of φ, then its horizontal lift $\hat{\mathcal{W}}$ is a transversal bundle of $\hat{\varphi}$. In particular, for each fiber F of π, the restriction of $\mathcal{H}$ to F is a transversal bundle of $F \subset \hat{N}$.

Let $\pi : (\hat{N}, \nabla^{\hat{N}}) \to (N, \nabla^N)$ be an affine submersion with horizontal distribution $\mathcal{H}$ and $\varphi : (M, \nabla^M) \to (N, \nabla^N)$ an affine immersion with transversal bundle $\mathcal{W}$. We equip $\hat{M}$ with the affine connection $\nabla^{\hat{M}}$ so that $\hat{\varphi} : (\hat{M}, \nabla^{\hat{M}}) \to (\hat{N}, \nabla^{\hat{N}})$ is an affine immersion with the transversal bundle $\hat{\mathcal{W}}$ (the horizontal lift of $\mathcal{W}$). By straightforward computations, we have

$$\begin{aligned} (\nabla^{\hat{M}}_{\hat{X}}\hat{Y})^{\mathcal{H}} &= \widehat{(\nabla^M_X Y)}, & \alpha^{\hat{\varphi}}(\hat{X}, \hat{Y}) &= \widehat{\alpha^{\varphi}(X, Y)}, \\ (A^{\hat{\varphi}}_{\hat{Z}}\hat{X})^{\mathcal{H}} &= \widehat{A^{\varphi}_Z(X)}, & \nabla^{\hat{\mathcal{W}}}_{\hat{X}}\hat{Z} &= \widehat{(\nabla^{\mathcal{W}}_X Z)}, \end{aligned} \tag{3.4}$$

where $X, Y \in \mathfrak{X}(M)$ and $Z \in \Gamma(\mathcal{W})$.

4. Focal points and Fredholm properties

In this section, we define focal points of affine immersions in the torsion-free case, study their relations to affine submersions and show the Fredholm property of affine immersions lifted by some affine submersion. We refer to Kobayashi-Nomizu [10] for Jacobi fields on affine manifolds, to Lang [14] and Klingenberg [8] for Jacobi-fields on (Riemannian) Hilbert manifolds and to Heintze-Liu-Olmos [4] for focal points and Riemannian submersions.

Let N be a Hilbert manifold and $\gamma : [a, b] \to N$ a smooth curve. A *variation* of γ is a smooth map $F : [a, b] \times (-\epsilon, \epsilon) \to N$, $(t, s) \mapsto F(t, s)$ satisfying $F(t, 0) = \gamma(t)$. We set $\gamma_s(t) := F(t, s)$ and often write $\{\gamma_s\}_s$ for F. The *variational vector field* $X \in \Gamma(\gamma^*TN)$ is defined by $X(t) := \frac{\partial F}{\partial s}(t, 0)$. Let ∇ be an affine connection on N

and γ a geodesic in (N, ∇). If $\{\gamma_s\}_s$ is a variation of γ and each γ_s is a geodesic of (N, ∇), then it is called a variation *through geodesics*. We write X' for $\nabla^{\gamma^*TN}_{\frac{d}{dt}} X$.

Let (N, ∇) be an affine Hilbert manifold with a torsion-free connection and $\gamma : [a, b] \to N$ a geodesic. Set $p := \gamma(a)$. Then $J \in \Gamma(\gamma^*TN)$ is called a *Jacobi field* along γ if it is the variational vector field of a variation of γ through geodesics. An equivalent definition is that J satisfies the Jacobi equation

$$(J')' + R(J, \gamma')\gamma' = 0 \tag{4.1}$$

where R denotes the curvature tensor of ∇^{γ^*TN}. The set of all Jacobi fields along γ is denoted by $\mathfrak{J}_\gamma$. Since (4.1) is locally a linear ordinary differential equation we have the linear isomorphism

$$\mathfrak{J}_\gamma \to T_pN \oplus T_pN, \qquad J \mapsto (J(a), J'(a)). \tag{4.2}$$

Let (M, ∇^M) and (N, ∇^N) be affine Hilbert manifolds with torsion-free connections and $\varphi : (M, \nabla^M) \to (N, \nabla^N)$ an affine immersion with transversal bundle $\mathcal{W}$. Let $p \in M$ and $\gamma : [0, 1] \to N$ be a geodesic satisfying $\gamma(0) = \varphi(p)$ and $\xi := \gamma'(0) \in \mathcal{W}_p$. Then $J \in \Gamma(\gamma^*TN)$ is called a *φ-Jacobi field* along γ if J is the variational vector field of a variation $\{\gamma_s\}_s$ through geodesics such that there exists a smooth curve $\delta : (-\epsilon, \epsilon) \to M$ satisfying $\gamma_s(0) = \varphi(\delta(s))$ and $\gamma_s'(0) \in \mathcal{W}_{\delta(s)}$. An equivalent definition is that J is a Jacobi field along γ satisfying

$$J(0) \in d\varphi(T_pM) \quad \text{and} \quad J'(0)^\top = -d\varphi(A^\varphi_\xi(d\varphi^{-1}J(0))). \tag{4.3}$$

The set of all φ-Jacobi fields along γ is denoted by $\mathfrak{J}^\varphi_\gamma$. We have the linear isomorphism

$$\mathfrak{J}^\varphi_\gamma \to d\varphi(T_pM) \oplus \mathcal{W}_p, \qquad J \mapsto (J(0), J'(0)^{\mathcal{W}}). \tag{4.4}$$

Note that $\mathfrak{J}^\varphi_\gamma$ is also isomorphic to $T_{(p,\xi)}\mathcal{W}$ because $\nabla^{\mathcal{W}}$ decomposes $T_{(p,\xi)}\mathcal{W}$ into the horizontal and vertical subspaces which are isomorphic to T_pM and $\mathcal{W}_p$ respectively. More explicitly the isomorphism between $\mathfrak{J}^\varphi_\gamma$ and $T_{(p,\xi)}\mathcal{W}$ is described as follows. For $X \in T_{(p,\xi)}\mathcal{W}$ we take a smooth curve $\omega : (-\epsilon, \epsilon) \to \mathcal{W}$ satisfying $\omega'(0) = X$. Define the variation of γ by $F(t, s) := \operatorname{Exp}^N t\omega(s)$. Then its variational vector field is the corresponding φ-Jacobi field along γ. Under the identification by this isomorphism, the differential of the *transversal exponential map* $\eta : \mathcal{W} \to N$, $(p, \xi) \mapsto \operatorname{Exp}^N_{\varphi(p)} \xi$ is expressed as $(d\eta)_{(p,\xi)}(J) = J(1)$.

Definition. Let $\varphi : (M, \nabla^M) \to (N, \nabla^N)$ be an affine immersion with transversal bundle $\mathcal{W}$ where ∇^M and ∇^N are torsion-free. If $(p, \xi) \in \mathcal{W}$ lies in the domain of η and if $\operatorname{Ker}(d\eta)_{(p,\xi)}$ has dimension m, then (p, ξ) is called a *multiplicity-m focal direction* of φ and $\eta(p, \xi)$ is called a *multiplicity-m focal point* of φ.

The following lemma extends Lemma 6.1 of [4] in the case of an arbitrary Riemannian submersion $\pi : \hat{N} \to N$ (see also [30, Lemmas 5.12]) and includes the case of the parallel transport map over an affine symmetric space (Theorem 7.4). (See Remark 4.2 for the difference between our proof and the original proof.)

Lemma 4.1. *Let (V, D) be a Hilbertable space with the flat connection, $\pi : (V, D) \to (N, \nabla^N)$ be an affine submersion with horizontal distribution $\mathcal{H}$ such that $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating and $\varphi : (M, \nabla^M) \to (N, \nabla^N)$ be an affine immersion with transversal bundle $\mathcal{W}$. Denote by $\hat{\varphi} : (\hat{M}, \nabla^{\hat{M}}) \to (V, D)$ the lift of φ. Let $(\hat{p}, \hat{\xi}) \in \hat{\mathcal{W}}$ and set $(p, \xi) := (\pi(\hat{p}), d\pi(\hat{\xi})) \in \mathcal{W}$. Then the following conditions are equivalent:*

(i) $(\hat{p}, \hat{\xi})$ *is a multiplicity-m focal direction of* $\hat{\varphi}$,
(ii) (p, ξ) *is a multiplicity-m focal direction of* φ.

Proof. Since $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating, a smooth curve in N is a geodesic if and only if its horizontal lift is a geodesic. Thus we have the commutative diagram

$$\begin{array}{ccc} \hat{\mathcal{W}} & \xrightarrow{\hat{\eta}} & V \\ {\scriptstyle d\pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathcal{W} & \xrightarrow{\eta} & N \end{array}$$

where η and $\hat{\eta}$ are the transversal exponential maps of φ and $\hat{\varphi}$ respectively. Set $\gamma(t) := \operatorname{Exp} t\xi$ and $\hat{\gamma}(t) := \operatorname{Exp} t\hat{\xi}$. Write $\mathfrak{J}$ for $\mathfrak{J}^{\varphi}_{\gamma}$ and $\hat{\mathfrak{J}}$ for $\mathfrak{J}^{\hat{\varphi}}_{\hat{\gamma}}$. Set $q := \gamma(1)$ and $\hat{q} := \hat{\gamma}(1)$. Differentiating the above diagram we have

$$\begin{array}{ccc} \hat{\mathfrak{J}} & \xrightarrow{\hat{f}} & T_{\hat{q}}V \\ {\scriptstyle \tilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle (d\pi)_{\hat{q}}} \\ \mathfrak{J} & \xrightarrow{f} & T_qN \end{array} \tag{4.5}$$

where f and $\hat{f}$ are the evaluation maps at $t = 1$ and $\tilde{\pi}(\hat{J})(t) = d\pi(J(t))$. By commutativity, $\tilde{\pi}$ induces a map from $\operatorname{Ker}\hat{f}$ to $\operatorname{Ker} f$. It suffices to show that this is an isomorphism.

We denote by $\hat{\mathfrak{J}}_1 := \operatorname{Ker}\tilde{\pi}$ the subspace of $\hat{\mathfrak{J}}$ consisting of everywhere vertical ones and by $\hat{\mathfrak{J}}_2$ the subspace of $\hat{\mathfrak{J}}$ whose elements are induced by the variation $\{\hat{\gamma}_s\}_s$ of $\hat{\gamma}$ where $\hat{\gamma}_s$ is a horizontal lift of γ. We claim:

$$\text{(a) } \hat{\mathfrak{J}}_1 = \hat{\mathfrak{J}}_2 \quad \text{and} \quad \text{(b) } \hat{f} : \hat{\mathfrak{J}}_2 \to \operatorname{Ker}(d\pi)_{\hat{q}} \text{ is an isomorphism.}$$

In fact $\hat{\mathfrak{J}}_1 \supset \hat{\mathfrak{J}}_2$ is clear. To see the converse we let $J \in \hat{\mathfrak{J}}_1$ and suppose $J(0) = 0$. Then $J'(0)^{\top} = 0$ and thus $J'(0) \in \mathcal{H}(\hat{p})$. Since V is a Hilbertable space we have $J(t) = tJ'(0)$. Since J is everywhere vertical, $J'(0)$ must be zero. (More precisely, define a section s of $\hat{\gamma}^*TV = \hat{\gamma}^*\mathcal{H} \oplus \hat{\gamma}^*\mathcal{V}$ by $s(t) := J'(0)$. Then $s(t) \in \mathcal{V}(\hat{\gamma}(t))$ for all $t > 0$. Since $\hat{\gamma}^*\mathcal{V}$ is closed in $\hat{\gamma}^*TV$, it follows by continuity that $s(0) \in \mathcal{V}(\hat{p})$ and hence $s(0) = 0$.) This shows that each $J \in \hat{\mathfrak{J}}_1$ is uniquely determined by the value $J(0)$. Thus $\hat{\mathfrak{J}}_1 = \hat{\mathfrak{J}}_2$. (b) follows by an observation that the horizontal lifts of γ define a diffeomorphism between $\pi^{-1}(p)$ and $\pi^{-1}(q)$.

The injectivity of $\tilde{\pi} : \operatorname{Ker}\hat{f} \to \operatorname{Ker} f$ follows immediately from (a) and (b). The surjectivity follows from (a) and (b) together with the surjectivity of $\tilde{\pi} : \hat{\mathfrak{J}} \to \mathfrak{J}$. More precisely, for $J \in \operatorname{Ker} f$ we take $\hat{J} \in \hat{\mathfrak{J}}$ satisfying $\tilde{\pi}(\hat{J}) = J$. By commutativity of (4.5) we have $\hat{J}(1) \in \operatorname{Ker}(d\pi)_q$. By (a) and (b) there exists $K \in \hat{\mathfrak{J}}_1$ satisfying $K(1) = \hat{J}(1)$. Then $\hat{J} - K$ is the desired Jacobi-field. This completes the proof. □

Remark 4.2. An essential point in the above proof is to show $\hat{\mathfrak{J}}_1 \subset \hat{\mathfrak{J}}_2$. In the proof of [4, Lemma 6.1] this was shown by using a Riemannian metric of $\hat{N}$. In the above proof, we showed it by supposing that $\hat{N}$ is a Hilbertable space (without a metric) and by using its linearity. Except for this point, our proof is similar to theirs.

The following corollary generalizes Corollary 6.2 of [4] to the affine case.

Corollary 4.3. *Let the notation be as in Lemma* 4.1. *Then the following conditions are equivalent for* $\lambda \in \mathbb{R}\backslash\{0\}$*:*

(i) λ *is a multiplicity-m eigenvalue of the shape operator* $A_{\hat{\xi}}$ *of* $\hat{\varphi}$*,*

(ii) $(p, \frac{1}{\lambda}\xi) \in \mathcal{W}$ *is a multiplicity-m focal direction of* φ*.*

Proof. A Jacobi field J on (V, D) is expressed as $J(t) = J(0) + tJ'(0)$. Thus (i) is equivalent to the condition that $(\hat{p}, \frac{1}{\lambda}\hat{\xi})$ is a multiplicity-m focal direction of $\hat{\varphi}$. Hence the assertion follows from Lemma 4.1 □

Remark 4.4. The above corollary relates the focal points of φ and the *real* eigenvalues of $A_{\hat{\xi}}$. In the case of Riemannian submersions, this suffices because all the eigenvalues of $A_{\hat{\xi}}$ are real. In the affine case, $A_{\hat{\xi}}$ may have complex eigenvalues and hence Corollary 4.3 does not suffice. This is one of the difficulties to study this area.

To describe the Fredholm property of lifted affine immersions, we make the following definition (see [23, 28] for the Riemannian case).

Definition. Let (V, D) be a Hilbertable space and $\varphi : (M, \nabla^M) \to (V, D)$ an affine immersion of finite codimension with transversal bundle $\mathcal{W}$. Then φ is called *Fredholm* if the transversal exponential map $\eta : \mathcal{W} \to V$, $(p, \xi) \mapsto p + \xi$ is a Fredholm map. If in addition η has Fredholm index 0, that is,

$$\dim \operatorname{Ker}(d\eta)_{(p,\xi)} = \dim \operatorname{Coker}(d\eta)_{(p,\xi)}$$

holds for any $(p, \xi) \in \mathcal{W}$, then φ is said to be *of Fredholm index* 0.

Remark 4.5. One can see that $\operatorname{Ker}(d\eta)_{(p,\xi)}$ and $\operatorname{Coker}(d\eta)_{(p,\xi)}$ are isomorphic to $\operatorname{Ker}(\mathrm{id} - A^{\varphi}_{\xi})$ and $\operatorname{Coker}(\mathrm{id} - A^{\varphi}_{\xi})$ respectively. Thus, in the Riemannian case, isometric Fredholm immersions are always of Fredholm index 0.

The following proposition generalizes Lemma 5.1 (i) of [4] to the affine case (see also [30, Lemma 5.8]). In the proof, we will essentially use Lemma 4.1.

Proposition 4.6. *Let the notation be as in Lemma* 4.1. *Suppose that* N *has finite dimension. Then* $\hat{\varphi}$ *is an affine Fredholm immersion of Fredholm index* 0.

Proof. It is clear that $\operatorname{codim} \hat{\varphi}$ is finite and equal to $\operatorname{codim} \varphi$. To see the Fredholm property, we consider the diagram (4.5). $d\pi$ induces a surjection from $\operatorname{Im} \hat{f}$ onto $\operatorname{Im} f$ whose kernel is the vertical subspace $\mathcal{V}_{\hat{q}}$. Thus $\operatorname{Coker} \hat{f} = T_{\hat{q}}\hat{N}/\operatorname{Im} \hat{f} \cong T_q N/\operatorname{Im} f = \operatorname{Coker} f$. This together with Lemma 4.1 implies that $\dim \operatorname{Coker} \hat{f} = \dim \operatorname{Coker} f = \dim \operatorname{Ker} f = \dim \operatorname{Ker} \hat{f}$. □

Remark 4.7. An isometric immersion into a Hilbert space of finite codimension is called *proper* [23, 28] if the normal exponential map restricted to the disc bundle of any finite radius is a proper map. There seems to be no natural generalization of this concept for affine immersions because a metric on the ambient space is necessary.

5. The Hilbert Lie group of Sobolev H^1-paths

In this section, we review the Hilbert Lie group $H^1([0, 1], G)$ of Sobolev H^1-paths in a Lie group G, make sure that it does not depend on the choice of a Riemannian metric on G and show a fundamental fact which will be used later. We refer to Klingenberg [8] (see also [7]) for the Hilbert manifold $H^1([0, 1], N)$ of Sobolev H^1-paths in a smooth manifold N and its analytic foundations.

Let E be a finite dimensional vector bundle over $[0,1]$. Choose a fiber metric $\langle\cdot,\cdot\rangle$ on E and a connection ∇ on E. On $\Gamma(E)$ we define the L^2-inner product and the Sobolev H^1-inner product by

$$\langle u,v\rangle_{L^2} := \int_0^1 \langle u(t), v(t)\rangle dt, \qquad \langle u,v\rangle_{H^1} := \langle u,v\rangle_{L^2} + \langle \nabla u, \nabla v\rangle_{L^2}$$

respectively. The completions of $\Gamma(E)$ with respect to these inner products are denoted by $L^2(E)$ and $H^1(E)$ respectively and they are Hilbert spaces. Since E has finite rank and $[0,1]$ is compact, the topologies on those spaces do not depend on the choice of a metric or a connection. Thus, without fixing a metric or a connection on E, we can consider topological vector spaces $L^2(E)$ and $H^1(E)$, which are *Hilbertable* spaces. By the Sobolev embedding theorem, H^1-sections are continuous.

Let N be a finite dimensional smooth manifold. A continuous map $c : [0,1] \to N$ is of *Sobolev class* H^1 if for each local coordinate (U, ϕ) of N the path

$$I' := [0,1] \cap c^{-1}(U) \xrightarrow{c} U \xrightarrow{\phi} \phi(U) \subset \mathbb{R}^n$$

belongs to the Sobolev space $H^1(I', \mathbb{R}^n)$ where $n := \dim N$. An equivalent definition is that c is absolutely continuous and the derivative c' (defined almost everywhere in $[0,1]$) satisfies $\int_0^1 \langle c'(t), c'(t)\rangle dt < \infty$ for a Riemannian metric on N. This definition does not depend on the choice of a Riemannian metric since N has finite dimension and $[0,1]$ is compact. The set of all maps of Sobolev class H^1 is denoted by $H^1([0,1], N)$. The following fact was originally found by Eliasson [2] and Palais [22] with more generality:

Lemma 5.1. *$\mathcal{N} := H^1([0,1], N)$ becomes a Hilbert manifold.*

Here the manifold structure of $\mathcal{N}$ is defined so that for each $c \in \mathcal{N}$ there is an open neighborhood U of c in $\mathcal{N}$ which is diffeomorphic to an open neighborhood of the zero section in the Hilbertable space $H^1(c^*TN)$. To define this explicitly, one has to take a Riemannian metric on N. Nevertheless one can see that *the manifold structure of $\mathcal{N}$ does not depend on the choice of a Riemannian metric on N*. This can be seen by the arguments similar to those for the smoothness of the transition functions using Lemma 2.3.9 of [8] (see also Lemma 1.2.5 of [7]). If N is a submanifold of some Euclidean space $\mathbb{R}^m$, then $\mathcal{N}$ is regarded as a subset of $H^1([0,1], \mathbb{R}^m)$ and the distance d on $\mathcal{N}$ is defined by $d(c_1, c_2) := \|c_1 - c_2\|_{H^1}$. The topology on $\mathcal{N}$ induced by d coincides with the topology on $\mathcal{N}$ explained above.

Suppose that $N = G$ is a Lie group with Lie algebra $\mathfrak{g}$. Then the group structure on the Hilbert manifold $H^1([0,1], G)$ is defined by the pointwise multiplication $(c \cdot d)(t) := c(t) \cdot d(t)$. Such a group is called a *path group*. Denote by $e \in G$ the identity element of G. The identity element of $\mathcal{G}$ is the constant path $\hat{e}$ with value e.

Corollary 5.2. *$\mathcal{G} := H^1([0,1], G)$ becomes a Hilbert Lie group.*

Here a *Hilbert Lie group* is a Hilbert manifold with a smooth group structure. The smoothness of the group structure of $\mathcal{G}$ can be shown by using Lemma 2.3.9 of [8]. The Lie algebra of $\mathcal{G}$ is the Hilbertable space $H^1([0,1], \mathfrak{g})$ with pointwise bracket. The exponential map $\exp^{\mathcal{G}} : H^1([0,1], \mathfrak{g}) \to \mathcal{G}$ is defined by $(\exp^{\mathcal{G}} X)(t) := \exp^G X(t)$ where $\exp^G : \mathfrak{g} \to G$ denotes the exponential map of G. By using Lemma 2.3.9 of [8] one can see that $\exp^{\mathcal{G}}$ is a smooth map. Moreover, if one takes neighborhoods W of $0 \in \mathfrak{g}$ and U of $e \in G$ such that $\exp^G : W \to U$ is a diffeomorphism, then

$\exp^{\mathcal{G}} : H^1([0,1], W) \to H^1([0,1], U)$ gives a chart at $\hat{e} \in \mathcal{G}$ which is compatible with the manifold structure of $\mathcal{G}$. Extending this chart by left (resp. right) translations on $\mathcal{G}$, one can obtain a left (resp. right) invariant atlas on $\mathcal{G}$.

The following proposition is fundamental in the theory of the parallel transport map. Here l_g and r_g denote the left and right translations by $g \in \mathcal{G}$ respectively and $V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g})$ denotes the Hilbertable space of all L^2-maps from $[0,1]$ to $\mathfrak{g}$.

Proposition 5.3. *Let G be a Lie group. Then for each $u \in V_{\mathfrak{g}}$ there exists a unique $g \in \mathcal{G}$ satisfying $dl_g^{-1}(g') = u$ and $g(0) = e$. Similarly, for each $u \in V_{\mathfrak{g}}$ there exists a unique $g \in \mathcal{G}$ satisfying $dr_g^{-1}(g') = u$ and $g(0) = e$.*

The above proposition was proved in [6, Proposition 3.2] when G is compact and the proof depends on the compactness of G (cf. Remark 6.3). We will now give another proof which does not depend on the compactness of G. We denote by $GL(n, \mathbb{R})$ the general linear group of degree n and by $\mathfrak{gl}(n, \mathbb{R})$ its Lie algebra.

Lemma 5.4. *Let $X : [0,1] \to \mathfrak{gl}(n, \mathbb{R})$ be an L^2-map and $x_0 \in \mathbb{R}^n$. Then there exists a unique solution $\varphi \in H^1([0,1], \mathbb{R}^n)$ to the linear ordinary differential equation*

$$\frac{d\varphi}{dt} = X\varphi, \qquad \varphi(0) = x_0.$$

Proof. Set $\kappa := \int_0^1 |X(s)|^2 ds$. Let $[a,b]$ be a subintervbal of $[0,1]$ satisfying $b - a < \kappa^{-1}$. For $y_0 \in \mathbb{R}^n$ we define the map $\mathcal{F}_{y_0} : L^2([a,b], \mathbb{R}^n) \to L^2([a,b], \mathbb{R}^n)$ by $\mathcal{F}_{y_0}(\varphi)(t) = y_0 + \int_a^t X(s)\varphi(s) ds$ for $\varphi \in L^2([a,b], \mathbb{R}^n)$. Note that $\mathcal{F}_{y_0}(\varphi) \in H^1([0,1], \mathbb{R}^n)$. For $t \in [a,b]$ we have

$$\begin{aligned}
|\mathcal{F}_{y_0}(\varphi)(t) - \mathcal{F}_{y_0}(\psi)(t)|^2 &= \left| \int_a^t X(s)(\varphi(s) - \psi(s)) ds \right|^2 \\
&\leq \left(\int_a^t |X(s)||\varphi(s) - \psi(s)| ds \right)^2 \\
&\leq \left(\int_a^t |X(s)|^2 ds \right) \left(\int_a^t |\varphi(s) - \psi(s)|^2 ds \right) \\
&\leq \kappa \int_a^b |\varphi(s) - \psi(s)|^2 ds.
\end{aligned}$$

Thus $\int_a^b |\mathcal{F}_{y_0}(\varphi)(t) - \mathcal{F}_{y_0}(\psi)(t)|^2 dt \leq (b-a)\kappa \int_a^b |\varphi(s) - \psi(s)|^2 ds$. This shows that $\mathcal{F}_{y_0}$ is a contraction map. Thus there exists a unique $\varphi \in L^2([a,b], \mathbb{R}^n)$ satisfying

$$\varphi(t) = y_0 + \int_a^t X(s)\varphi(s) ds.$$

This is the unique H^1-solution on $[a,b]$ to the equation $\frac{d\varphi}{dt} = X\varphi$, $\varphi(a) = y_0$.

Let $0 = a_0 < a_1 < \cdots < a_l = 1$ be a refinement of $[0,1]$ satisfying $a_{i+1} - a_i < \kappa$. By the above result there exists the unique solution $\varphi_0 \in H^1([a_0, a_1], \mathbb{R}^n)$ to the equation $\frac{d\varphi_0}{dt} = X\varphi_0$, $\varphi_0(a_0) = x_0$. For each $i \geq 1$ there exists the unique solution $\varphi_i \in H^1([a_i, a_{i+1}], \mathbb{R}^n)$ to the equation $\frac{d\varphi_i}{dt} = X\varphi_i$, $\varphi_i(a_i) = \varphi_{i-1}(a_i)$. Connecting these solutions continuously we obtain the desired unique H^1-solution φ on $[0,1]$. $\square$

Proof of Proposition 5.3. If G is a Lie subgroup of $GL(n, \mathbb{R})$ for some n, then the assertion follows from Lemma 5.4. If G is not a Lie subgroup of $GL(n, \mathbb{R})$ for any

n, then we use Ado's theorem which ensures that $\mathfrak{g}$ is isomorphic to a subalgebra $\mathfrak{g}'$ of $\mathfrak{gl}(n,\mathbb{R})$ for some n. By Lemma 5.4 the assertion follows for the connected Lie subgroup G' of $GL(n,\mathbb{R})$ with Lie algebra $\mathfrak{g}'$. By lifting the solution to the universal covering group $\tilde{G}$ the assertion follows for $\tilde{G}$ and thus for G. □

6. The parallel transport map over Lie group

In this section, we study the parallel transport map over a Lie group.

Let G be a (finite dimensional) Lie group with Lie algebra $\mathfrak{g}$. We denote by $\mathcal{G} := H^1([0,1],G)$ the Hilbert Lie group of Sobolev H^1-paths in G and by $V_{\mathfrak{g}} := L^2([0,1],\mathfrak{g})$ the Hilbertable space of all L^2-maps from $[0,1]$ to $\mathfrak{g}$. For $g \in \mathcal{G}$ the continuous linear map $\mathrm{Ad}(g) : V_{\mathfrak{g}} \to V_{\mathfrak{g}}$ is defined by $(\mathrm{Ad}(g)u)(t) := \mathrm{Ad}^G(g(t))u(t)$ where Ad^G denotes the adjoint representation of G. We define the $\mathcal{G}$-action on $V_{\mathfrak{g}}$ by the affine transformation

$$g * u := \mathrm{Ad}(g)u - dr_g^{-1}(g'). \tag{6.1}$$

This action is smooth and the differential of the orbit map $\omega_u : \mathcal{G} \to V_{\mathfrak{g}}$, $g \mapsto g * u$ at $\hat{e} \in \mathcal{G}$ is given by

$$d\omega_u : H^1([0,1],\mathfrak{g}) \to V_{\mathfrak{g}}, \qquad Z \mapsto [Z,u] - Z'. \tag{6.2}$$

This is a Fredholm operator and thus the $\mathcal{G}$-action on $V_{\mathfrak{g}}$ is Fredholm (cf. [23]). From Proposition 5.3 we know that the $\mathcal{G}$-action on $V_{\mathfrak{g}}$ is transitive.

Let L be a closed subgroup of $G \times G$ with Lie algebra $\mathfrak{l}$. Then

$$P(G,L) := \{g \in \mathcal{G} \mid (g(0),g(1)) \in L\}$$

is a Lie subgroup of $\mathcal{G}$ and its Lie algebra is

$$\mathrm{Lie}\, P(G,L) := \{Z \in H^1([0,1],\mathfrak{g}) \mid (Z(0),Z(1)) \in \mathfrak{l}\}. \tag{6.3}$$

Note that $P(G,L)$ is the inverse image of L under the submersion $\Psi : \mathcal{G} \to G \times G$, $g \mapsto (g(0),g(1))$. Thus $P(G,L)$ is closed and of finite codimension. In particular the $P(G,L)$-action on $V_{\mathfrak{g}}$ is Fredholm. From Proposition 5.3 we know that $P(G,\{e\}\times G)$ and $P(G,G\times\{e\})$ act on $V_{\mathfrak{g}}$ simply transitively.

Remark 6.1. If G is compact, then the $\mathcal{G}$-action on $V_{\mathfrak{g}}$ is proper and thus the $P(G,L)$-action on $V_{\mathfrak{g}}$ is proper [29, p. 132]. This is no longer true when G is non-compact: Consider the case $G = SL(2,\mathbb{C})$. Set

$$g_n(t) := \begin{bmatrix} 1+nt & 0 \\ 0 & (1+nt)^{-1} \end{bmatrix}, \qquad u_n(t) := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then $g_n \in P(G,\{e\}\times G)$ and

$$g_n * u_n = \begin{bmatrix} 1 - \frac{n}{1+nt} & 0 \\ 0 & -1 + \frac{n}{1+nt} \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 - \frac{1}{t} & 0 \\ 0 & -1 + \frac{1}{t} \end{bmatrix} \quad (n \to \infty).$$

However g_n cannot have a convergent subsequence. Therefore, in this case, the action of $P(G,\{e\}\times G)$ on $V_{\mathfrak{g}}$ is not proper.

In order to define the parallel transport map over G, we need the following lemma.

Lemma 6.2. *The map $F : P(G,\{e\}\times G) \to V_{\mathfrak{g}}$ defined by $F(g) := dl_g^{-1}(g') = g^{-1} * \hat{0}$ is a diffeomorphism.*

Remark 6.3. Lemma 6.2 was proved by King and Terng [6, Proposition 3.2] when G is compact. An essential point is the bijectivitiy of F, which was proved by using the properness of the $P(G, \{e\} \times G)$-action on $V_{\mathfrak{g}}$, which depends on the compactness of G (Remark 6.1). Hence their arguments do not apply in the non-compact case and should be modified. (Such a modification is not made in [11].) In the last section, we have modified their arguments (Proposition 5.3) and therefore we know that F is a bijection. The smoothness can be seen by the same arguments as theirs.

Let G be a connected Lie group. The *parallel transport map* $\Phi : V_{\mathfrak{g}} \to G$ is the submersion defined by

$$\Phi := p \circ E : V_{\mathfrak{g}} \to P(G, \{e\} \times G) \to G$$

where E denotes the inverse map of F in Lemma 6.2 and p the projection $g \mapsto g(1)$. In other words, if $g \in \mathcal{G}$ is the unique solution to the ordinary differential equation

$$dl_g^{-1}(g') = u, \quad g(0) = e$$

then $\Phi(u) = g(1)$. The following two propositions can be proven by the same arguments as in the compact case ([29, Proposition 1.1], [30, Corollary 4.4]).

Proposition 6.4 (Terng [29])**.**

(i) $\Phi(\hat{x}) = \exp x$ *where* $\hat{x} \in V_{\mathfrak{g}}$ *denotes the constant path with value* $x \in \mathfrak{g}$.

(ii) $\Phi(g * u) = \Psi(g) \cdot \Phi(u)$ *for* $g \in \mathcal{G}$ *and* $u \in V_{\mathfrak{g}}$.

(iii) $P(G, L) * u = \Phi^{-1}(L \cdot \Phi(u))$ *for any closed subgroup* L *of* $G \times G$.

Here $G \times G$ *(and* L*) acts on* G *by* $(b, c) \cdot a := bac^{-1}$.

Proposition 6.5 (Terng-Thorbergsson [30])**.**

(i) $P(G, \{e\} \times \{e\})$ *acts on each fiber of* Φ *simply transitively.*

(ii) $\Phi : V_{\mathfrak{g}} \to G$ *is a principal* $P(G, \{e\} \times \{e\})$*-bundle.*

(iii) *Any two fibers of* Φ *are congruent under* $g*$ *for some* $g \in P(G, G \times \{e\})$.

The Lie algebra of $P(G, \{e\} \times G)$ is given by

$$\operatorname{Lie} P(G, \{e\} \times G) = \{Z \in H^1([0, 1], \mathfrak{g}) \mid Z(0) = 0\}.$$

From (6.2) we see that the differential $(dF)_{\hat{e}} : \operatorname{Lie} P(G, \{e\} \times G) \to T_{\hat{0}} V_{\mathfrak{g}} = V_{\mathfrak{g}}$ is given by $(dF)_{\hat{e}} Z = Z'$. Thus $(dE)_{\hat{0}} X = \int_0^t X(t)dt$ for $X \in V_{\mathfrak{g}}$. Therefore

$$(d\Phi)_{\hat{0}}(X) = \int_0^1 X(t)dt. \tag{6.4}$$

Thus we have the direct sum decomposition

$$T_{\hat{0}} V_{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \operatorname{Ker}(d\Phi)_{\hat{0}}, \quad X = \int_0^1 X(t)dt \oplus \left(X - \int_0^1 X(t)dt\right) \tag{6.5}$$

where $\hat{\mathfrak{g}}$ denotes the set of constant paths with values in $\mathfrak{g}$. From Proposition 6.4 (ii) we have

$$T_{g*\hat{0}} V_{\mathfrak{g}} = \operatorname{Ad}(g)\hat{\mathfrak{g}} \oplus \operatorname{Ker}(d\Phi)_{g*\hat{0}}. \tag{6.6}$$

Thus the horizontal distribution $\mathcal{H}$ of Φ is defined by $\mathcal{H}(g * \hat{0}) := \operatorname{Ad}(g)\hat{\mathfrak{g}}$.

Let ∇^G denote the *canonical connection* on G, which is the $(G \times G)$-invariant, torsion-free affine connection determined by

$$\nabla^G_X Y = \frac{1}{2}[X, Y]$$

for $X, Y \in \mathfrak{g}$. We denote by D the flat connection on the Hilbertable space $V_{\mathfrak{g}}$.

Theorem 6.6. *Let G be a connected Lie group. Then the parallel transport map $\Phi : (V_{\mathfrak{g}}, D) \to (G, \nabla^G)$ is an affine submersion with horizontal distribution $\mathcal{H}(g*\hat{0}) := \mathrm{Ad}(g)\hat{\mathfrak{g}}$. Moreover the tensor $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating.*

Proof. Proposition 6.4 (ii) shows that the diagram

$$\begin{array}{ccc} V_{\mathfrak{g}} & \xrightarrow{g*} & V_{\mathfrak{g}} \\ {\scriptstyle\Phi}\Big\downarrow & & {\scriptstyle\Phi}\Big\downarrow \\ G & \xrightarrow{(g(0),g(1))} & G \end{array} \tag{6.7}$$

commutes for any $g \in \mathcal{G}$. Note that $g*$ and $(g(0), g(1))$ are affine transformations of $(V_{\mathfrak{g}}, D)$ and (G, ∇^G) respectively and $\mathcal{H}(g * \hat{0}) = d(g*)\mathcal{H}(\hat{0})$. Thus, in order to show that Φ is an affine submersion with horizontal distribution $\mathcal{H}$, it suffices to show

$$(D_{\hat{X}}\hat{Y})^{\mathcal{H}}_{\hat{0}} = \widehat{((\nabla^G_X Y)_e)}$$

for $X, Y \in \mathfrak{X}(G)$. We can assume $X \in \mathfrak{g}$ and identify $\hat{X} \in \hat{\mathfrak{g}}$ with X. Set $g_s(t) := \exp((1-t)sX)$. Then $g_s \in P(G, G \times \{e\})$ and $g_s * \hat{0} = sX$. By (6.7) we have

$$\hat{Y}(sX) = \widehat{(Y_{\exp sX})} = \mathrm{Ad}(g_s)(dl^{-1}_{\exp sX}(Y_{\exp sX})).$$

Then we have

$$\begin{aligned}(D_{\hat{X}}\hat{Y})_{\hat{0}} &= \frac{d}{ds}\bigg|_{s=0} \mathrm{Ad}(g_s)(dl^{-1}_{\exp sX}(Y_{\exp sX})) \\ &= (1-t)[X, Y^L] + \frac{d}{ds}\bigg|_{s=0} dl^{-1}_{\exp sX}(Y_{\exp sX})\end{aligned}$$

where Y^L denotes the left invariant vector field satisfying $Y^L_e = Y_e$. By (6.5) we have

$$(D_X\hat{Y})^{\mathcal{H}}_{\hat{0}} = \frac{1}{2}[X, Y^L] + \frac{d}{ds}\bigg|_{s=0} dl^{-1}_{\exp sX}(Y_{\exp sX}).$$

Take a basis $\{\eta_1, \cdots, \eta_d\}$ of $\mathfrak{g}$ and write $Y = \sum_i f_i\eta_i$ where $f_i \in C^\infty(G)$. Then

$$\begin{aligned}2(D_X\hat{Y})^{\mathcal{H}}_{\hat{0}} &= \sum_i \left(f_i(e)[X, \eta_i] + 2\frac{d}{ds}\bigg|_{s=0} f_i(\exp sX)\eta_i) \right) \\ &= \sum_i (f_i(e)[X, \eta_i] + 2(X_e f_i)\eta_i) = 2\widehat{(\nabla^G_X Y)}_e.\end{aligned}$$

Moreover, from the above calculations, we have $(\mathcal{A}_{\hat{X}}\hat{Y})(t) = (\frac{1}{2} - t)[X, Y]$ for $X, Y \in \mathfrak{g}$. Thus $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating. This completes the proof. □

Suppose that G is semisimple. Take a bi-invariant pseudo-Riemannian metric ρ on G and denote by $\langle\cdot,\cdot\rangle^\rho$ the corresponding non-degenerate, $\mathrm{Ad}(G)$-invariant, bi-linear form on $\mathfrak{g}$. (For example, the Killing form on $\mathfrak{g}$ defines such a metric.) Then

$$\langle u, v\rangle^\rho_{L^2} := \int_0^1 \langle u(t), v(t)\rangle^\rho dt$$

defines a continuous, non-degenerate, symmetric bi-linear form on the Hilbertable space $V_{\mathfrak{g}}$ [11, p. 180]. It is easy to see that the $\mathcal{G}$-action on $(V_{\mathfrak{g}}, \langle\cdot,\cdot\rangle^\rho_{L^2})$ is isometric.

The following lemma is essentially due to Terng and Thorbergsson [30, Theorem 4.5] who proved it in the case that G is compact.

Lemma 6.7 ([30]). *The decomposition* (6.6) *is orthogonal with respect to* $\langle\cdot,\cdot\rangle^\rho_{L^2}$.

Proof. Since the $\mathcal{G}$-action on $(V_\mathfrak{g}, \langle\cdot,\cdot\rangle^\rho_{L^2})$ is isometric, it suffices to show that the decomposition (6.5) is orthogonal. By Proposition 6.5 (i) and (6.2) we have

$$\mathrm{Ker}(d\Phi)_{\hat{0}} = \{Z' \mid Z \in H^1([0,1],\mathfrak{g}),\ Z(0) = Z(1) = 0\}. \tag{6.8}$$

For $x \in \mathfrak{g}$ we have $\langle \hat{x}, Z'\rangle^\rho_{L^2} = \int_0^1 \langle \hat{x}, Z'(t)\rangle^\rho dt = \langle \hat{x}, \int_0^1 Z'(t)dt\rangle^\rho = \langle \hat{x}, Z(1) - Z(0)\rangle^\rho = 0$. This proves the lemma. $\square$

The following fact was shown by Koike [11, Proposition 4.4]. For convenience, we give a simpler proof here.

Proposition 6.8 (Koike [11]). *Let G be a connected semisimple Lie group with a bi-invariant pseudo-Riemannian metric ρ. Then the parallel transport map $\Phi : (V_\mathfrak{g}, \langle\cdot,\cdot\rangle^\rho_{L^2}) \to (G, \rho)$ is a pseudo-Riemannian submersion.*

Proof. Owing to Lemma 6.7 we have only to show that $\langle\cdot,\cdot\rangle^\rho_{L^2}$ induces a pseudo-Riemannian metric on each fiber of Φ. Since the $\mathcal{G}$-action is isometric and Φ an equivariant submersion (Proposition 6.4 (ii)) it suffices to show that the restriction of $\langle\cdot,\cdot\rangle^\rho_{L^2}$ to the subspace (6.8) is non-degenerate. Consider the decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ where $\langle\cdot,\cdot\rangle^\rho$ is positive (resp. negative) definite on $\mathfrak{g}^+$ (resp. $\mathfrak{g}^-$). Denote by $\langle\cdot,\cdot\rangle^{\rho,+}$ (resp. $\langle\cdot,\cdot\rangle^{\rho,-}$) the restriction of $\langle\cdot,\cdot\rangle^\rho$ to $\mathfrak{g}^+$ (resp. $\mathfrak{g}^-$). Then

$$\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle^{\rho,+} \oplus (-\langle\cdot,\cdot\rangle^{\rho,-})$$

defines a positive definite inner product on $\mathfrak{g}$. The corresponding L^2-inner product is denoted by $\langle\cdot,\cdot\rangle_{L^2}$. Consider the linear map from $\mathrm{Ker}(d\Phi)_{\hat{0}}$ to its continuous dual:

$$\mathrm{Ker}(d\Phi)_{\hat{0}} \to (\mathrm{Ker}(d\Phi)_{\hat{0}})^*, \qquad Z' \mapsto \langle Z', \cdot\rangle^\rho_{L^2}. \tag{6.9}$$

By continuity of $\langle\cdot,\cdot\rangle^\rho$ this map is continuous. By the Riesz representation theorem, for each $\omega \in (\mathrm{Ker}(d\Phi)_{\hat{0}})^*$ there exists a unique $W' \in \mathrm{Ker}(d\Phi)_{\hat{0}}$ satisfying $\omega = \langle W', \cdot\rangle_{L^2}$. Decomposing $W = W^+ \oplus W^-$ we define $\bar{W} := W^+ \oplus (-W^-)$. Then $\bar{W}' \in \mathrm{Ker}(d\Phi)_{\hat{0}}$ and $\omega = \langle \bar{W}', \cdot\rangle^\rho_{L^2}$. Thus $\omega \mapsto \bar{W}'$ defines the inverse map of (6.9). Hence (6.9) is a topological linear isomorphism. This completes the proof. $\square$

Suppose that G is compact. Then we can assume ρ to be a (positive definite) bi-invariant Riemannian metric on G. Thus Lemma 6.7 implies ([30, Theorem 4.5]):

Proposition 6.9 (Terng-Thorbergsson [30]). *Let G be a connected compact Lie group with a bi-invariant Riemannian metric ρ. Then the parallel transport map $\Phi : (V_\mathfrak{g}, \langle\cdot,\cdot\rangle^\rho_{L^2}) \to (G, \rho)$ is a Riemannian submersion.*

7. The parallel transport map over symmetric space

In this section, we define the parallel transport map over a homogeneous space and investigate it especially over a symmetric space.

Let G be a (finite dimensional) connected Lie group and K a closed subgroup of G. The coset manifold G/K is called a *homogeneous* space. The Lie algebras of G and K are denoted by $\mathfrak{g}$ and $\mathfrak{k}$ respectively. A homogeneous space G/K is called *reductive* if there exists an $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{p}$ of $\mathfrak{g}$ satisfying $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. This decomposition is called the *reductive decomposition.* Note that $T_{eK}(G/K)$ and

$\mathfrak{p}$ are identified. We have $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. For $x \in \mathfrak{g}$ we denote by $x_{\mathfrak{k}}$ and $x_{\mathfrak{p}}$ the $\mathfrak{k}$- and $\mathfrak{p}$-components respectively.

A homogeneous space G/K is called a *symmetric space* if K is a symmetric subgroup of G, that is, there exists an involutive automorphism θ of G satisfying $G_0^\theta \subset K \subset G^\theta$ where G^θ denotes the fixed-point subgroup of G and G_0^θ its identity component. The pair (G, K) is called a *symmetric pair*. Denote by $\mathfrak{p}$ the (-1)-eigenspace of $d\theta : \mathfrak{g} \to \mathfrak{g}$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the *canonical decomposition*. Since $\mathfrak{p}$ is $\mathrm{Ad}(K)$-invariant, any symmetric space is a reductive homogeneous space. We have $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

Definition. Let G/K be a homogeneous space with projection $\pi : G \to G/K$. Denote by $\Phi : V_{\mathfrak{g}} \to G$ the parallel transport map. The composition $\Phi_{G/K} := \pi \circ \Phi : V_{\mathfrak{g}} \to G \to G/K$ is called the *parallel transport map over G/K*.

Remark 7.1. The above definition generalizes the definition in the Lie group case in the last section. There are two meanings about this. One is that Φ and $\Phi_{G/\{e\}}$ are naturally identified. The other is that Φ and $\Phi_{(G\times G)/\Delta G}$ are identified via natural isomorphisms: There exists an injective homomorphism $\Omega : \mathcal{G} \to H^1([0,1], G \times G)$ with image $P(G \times G, G \times G \times \Delta G)$ and a linear isomorphism $\Upsilon : V_{\mathfrak{g}} \to V_{\mathfrak{g}\oplus\mathfrak{g}}$ such that Υ is equivariant with respect to the $\mathcal{G}$-action on $V_{\mathfrak{g}}$ and the $\Omega(\mathcal{G})$-action on $V_{\mathfrak{g}\oplus\mathfrak{g}}$ via Ω and the diagram

$$\begin{array}{ccc} V_{\mathfrak{g}} & \xrightarrow{\Upsilon} & V_{\mathfrak{g}\oplus\mathfrak{g}} \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi_{(G\times G)/\Delta G}} \\ G & \xleftarrow{\phi} & (G\times G)/\Delta G \end{array}$$

commutes, where $\phi(a, b) := ab^{-1}$. For details, see [24, 18].

From Propositions 6.4 and 6.5, we have:

Proposition 7.2. *Let G/K be a homogeneous space. Then*

(i) $\Phi_{G/K}(\hat{x}) = \pi(\exp x)$ *for* $x \in \mathfrak{g}$.
(ii) $\Phi_{G/K}(g * u) = g(0) \cdot \Phi_{G/K}(u)$ *for* $g \in P(G, G \times K)$ *and* $u \in V_{\mathfrak{g}}$.
(iii) $P(G, H \times K) * u = \Phi_{G/K}^{-1}(H \cdot \Phi_{G/K}(u))$ *for any closed subgroup H of G.*

Here G (and H) acts on G/K by $b \cdot aK := (ba)K$.

Proposition 7.3. *Let G/K be a homogeneous space. Then*

(i) *$P(G, \{e\} \times K)$ acts on each fiber of $\Phi_{G/K}$ simply transitively.*
(ii) *If G/K is reductive, then $\Phi_{G/K}$ is a principal $P(G, \{e\} \times K)$-bundle.*
(iii) *Any two fibers of $\Phi_{G/K}$ are congruent under $g*$ for some $g \in P(G, G \times \{e\})$.*

Let G/K be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. From (6.4) we have

$$(d\Phi_{G/K})_{\hat{0}}(X) = \int_0^1 X(t)_{\mathfrak{p}} dt \tag{7.1}$$

where $X \in V_{\mathfrak{g}}$. Thus we have the direct sum decomposition

$$T_{\hat{0}} V_{\mathfrak{g}} = \hat{\mathfrak{p}} \oplus \mathrm{Ker}(d\Phi_{G/K})_{\hat{0}}, \quad X = \int_0^1 X(t)_{\mathfrak{p}} dt \oplus \left(X - \int_0^1 X(t)_{\mathfrak{p}} dt \right) \tag{7.2}$$

where $\hat{\mathfrak{p}}$ denotes the set of constant paths with values in $\mathfrak{p}$. From Proposition 7.2 (ii) we have

$$T_{g*\hat{0}}V_{\mathfrak{g}} = \mathrm{Ad}(g)\hat{\mathfrak{p}} \oplus \mathrm{Ker}(d\Phi_{G/K})_{g*\hat{0}} \tag{7.3}$$

where $g \in P(G, G \times K)$. This together with $\mathrm{Ad}(K)$-invariance of $\mathfrak{p}$ implies that the horizontal distribution $\mathcal{H}$ of $\Phi_{G/K}$ is well-defined by

$$\mathcal{H}(g * \hat{0}) = \mathrm{Ad}(g)\hat{\mathfrak{p}}. \tag{7.4}$$

Let G/K be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The *canonical connection* $\nabla^{G/K}$ is the G-invariant affine connection on G/K determined by

$$(\nabla^{G/K}_{X^*}Y)_{eK} = [X^*, Y]_{eK} \tag{7.5}$$

where $X \in \mathfrak{p}$ and $Y \in \mathfrak{X}(G/K)$ ([10, p. 192], [13, p. 30]). Here X^* denotes the associated vector field on G/K defined by $X^*_{aK} := \frac{d}{dt}\big|_{t=0} (\exp tX) \cdot aK$. A symmetric space G/K equipped with the canonical connection $\nabla^{G/K}$ is called an *affine symmetric space*. Equivalently an affine symmetric space is a connected affine manifold (N, ∇^N) such that each $p \in N$ is an isolated fixed point of an involutive affine transformation s_p (called the *symmetry* at p) of (N, ∇^N).

The following theorem generalizes Theorem 6.6. In fact, if we consider the symmetric space $(G \times G)/\Delta G$ then it is equivalent to Theorem 6.6 (cf. Remark 7.1).

Theorem 7.4. *Let $(G/K, \nabla^{G/K})$ be an affine symmetric space with canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then the parallel transport map $\Phi_{G/K} : (V_{\mathfrak{g}}, D) \to (G/K, \nabla^{G/K})$ is an affine submersion with horizontal distribution $\mathcal{H}(g * \hat{0}) := \mathrm{Ad}(g)\hat{\mathfrak{p}}$ where $g \in P(G, G \times K)$. Moreover the tensor $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating.*

Owing to Theorem 6.6, we have only to prove:

Proposition 7.5. *Let $(G/K, \nabla^{G/K})$ be an affine symmetric space with canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then the projection $\pi : (G, \nabla^G) \to (G/K, \nabla^{G/K})$ is an affine submersion with horizontal distribution $\mathcal{H}(a) := dl_a(\mathfrak{p})$ where $a \in G$. Moreover the tensor $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating.*

Proof. Let l_a and L_a denote the left translations on G and G/K by $a \in G$ respectively. It is clear that the diagram

$$\begin{array}{ccc} G & \xrightarrow{l_a} & G \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ G/K & \xrightarrow{L_a} & G/K \end{array} \tag{7.6}$$

commutes for any $a \in G$. Note that l_a and L_a are affine transformations of (G, ∇^G) and $(G/K, \nabla^{G/K})$ respectively and $\mathcal{H}(a) = dl_a(\mathcal{H}(\hat{0}))$. Thus, in order to show that π is an affine submersion with horizontal distribution $\mathcal{H}$, it suffices to show

$$(\nabla^G_X \hat{Y})^{\mathcal{H}}_e = \widehat{((\nabla^{G/K}_{X^*} Y)_{eK})} \tag{7.7}$$

for $X \in \mathfrak{p}$ and $Y \in \mathfrak{X}(G/K)$. We may assume that Y is a vector field defined on an open neighborhood of $eK \in G/K$. Let $\eta_1, \cdots, \eta_d$ be a basis of $\mathfrak{p}$. Denote by $\eta^*_1, \cdots, \eta^*_d$ the associated vector fields on G/K. Then $\{\eta^*_1|_{eK}, \cdots, \eta^*_d|_{eK}\}$ is a basis of $T_{eK}(G/K)$. By continuity there exists an open neighborhood U of eK such that

$\{\eta_1^*|_{aK}, \cdots, \eta_d^*|_{aK}\}$ is a basis of $T_{aK}(G/K)$ at each $aK \in U$. We write $Y = \sum_i f_i \eta_i^*$ for $f_i \in C^\infty(U)$. Identifying $T_{eK}(G/K)$ with $\mathfrak{p}$ we have

$$\begin{aligned}\hat{Y}_a &= dl_a \circ dL_a^{-1}(Y_{aK}) \\ &= \sum_i f_i(aK) dl_a \circ dL_a^{-1}(\eta_i^*|_{aK}) \\ &= \sum_i f_i(aK) dl_a (\mathrm{Ad}(a)^{-1}\eta_i)_{\mathfrak{p}}. \\ &= \sum_i f_i(aK)(dr_a(\eta_i|_e))^{\mathcal{H}(a)}.\end{aligned}$$

Thus $\hat{Y} = \sum_i (f_i \circ \pi)(\eta_i^R)^{\mathcal{H}}$ where η_i^R denotes the right invariant vector field on $\pi^{-1}(U)$ satisfying $\eta_i^R|_e = \eta_i|_e$ and $(\eta_i^R)^{\mathcal{H}}$ its horizontal component. Then we have

$$(\nabla_X^G \hat{Y})_e = \sum_i \left(f_i(eK)(\nabla_X^G (\eta_i^R)^{\mathcal{H}})_e + (X_{eK}^* f_i)\eta_i|_e \right). \tag{7.8}$$

Take a basis $\{\kappa_1, \cdots, \kappa_n\}$ of $\mathfrak{k}$ and express $\eta_i^R = \sum_j \psi_{ij}\kappa_j \oplus \sum_k \varphi_{ik}\eta_k$ where $\psi_{ij}, \varphi_{ik} \in C^\infty(\pi^{-1}(U))$. Then $(\eta_i^R)^{\mathcal{H}} = \sum_k \varphi_{ik}\eta_k$. Note that $\psi_{ij}(e) = 0$ for all j and $\varphi_{ik}(e)$ is equal to the Kronecker delta δ_{ik}. Thus

$$0 = [X, \eta_i^R]_e = \sum_j (X_e \psi_{ij})\kappa_j|_e + \sum_k (X_e \varphi_{ik})\eta_k|_e + [X, \eta_i]_e.$$

Hence

$$(\nabla_X^G (\eta_i^R)^{\mathcal{H}})_e = \sum_k (\varphi_{ik}(e)(\nabla_X^G \eta_k)_e + (X_e\varphi_{ik})\eta_k|_e) = -\frac{1}{2}[X, \eta_i]_e - \sum_j (X_e\psi_{ij})\kappa_j|_e.$$

Thus, by projecting (7.8) onto $\mathfrak{p} \cong T_{eK}(G/K)$, we have

$$(\nabla_X^G \hat{Y})_e^{\mathcal{H}} = \sum_i \left(-\frac{1}{2} f_i(eK)[X, \eta_i]_{eK}^* + (X_{eK}^* f_i)\eta_i^*|_{eK} \right).$$

On the other hand, by (7.5) we have

$$\begin{aligned}(\nabla_{X^*}^{G/K} Y)_{eK} &= \sum_i (f_i(eK)[X^*, \eta_i^*]_{eK} + (X_{eK}^* f_i)\eta_i^*|_{eK}) \\ &= \sum_i (-f_i(eK)[X, \eta_i]_{eK}^* + (X_{eK}^* f_i)\eta_i^*|_{eK}).\end{aligned}$$

Since G/K is a symmetric space we have $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and thus

$$(\nabla_X^G \hat{Y})_e^{\mathcal{H}} = \sum_i (X_{eK}^* f_i)\eta_i^*|_{eK} = \widehat{(\nabla_{X^*}^{G/K} Y)}_{eK}.$$

This proves (7.7). Moreover

$$\mathcal{A}(X(e), Y(e)) = (\nabla_X^G Y)_e^{\mathcal{V}} = \frac{1}{2}[X, Y]_{\mathfrak{k}} = \frac{1}{2}[X, Y]$$

for $X, Y \in \mathfrak{p}$ and therefore $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating. This completes the proof. $\square$

Let N be a semisimple Riemannian symmetric space. Denote by $G = I_0(N)$ the identity component of the isometry group $I(N)$ of N and by $K = G_p$ the isotropy subgroup of G at fixed $p \in N$. Then K is a compact symmetric subgroup of G. The Riemannian metric on N induces an $\mathrm{Ad}(K)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{p}$. Then there exists a bi-invariant pseudo-Riemannian metric ρ on G invariant under θ such that the restriction of $\langle \cdot, \cdot \rangle^\rho$ to $\mathfrak{p}$ coincides with $\langle \cdot, \cdot \rangle$. In fact, by decomposing the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ into irreducible ones $(\mathfrak{g}_i, \mathfrak{k}_i)$ we see that the restriction of $\langle \cdot, \cdot \rangle$ to each $\mathfrak{p}_i$ is the scalar multiplication of the Killing form on $\mathfrak{g}_i$ [10, p. 257].

Conversely, let G be a semisimple Lie group and K a compact symmetric subgroup of G. Suppose that ρ a bi-invariant pseudo-Riemannian metric on G invariant under θ and the restriction of $\langle \cdot, \cdot \rangle_\rho$ to $\mathfrak{p}$ is positive definite. Then it defines a G-invariant Riemannian metric $\rho^{G/K}$ on G/K so that $(G/K, \rho^{G/K})$ is a semisimple Riemannian symmetric space and $\pi : (G, \rho) \to (G/K, \rho^{G/K})$ is a pseudo-Riemannian submersion. Thus Proposition 6.8 implies:

Proposition 7.6 (Koike [11]). *Let $(G/K, \rho^{G/K})$ be a semisimple Riemannian symmetric space. Then the parallel transport map $\Phi_{G/K} : (V_\mathfrak{g}, \langle \cdot, \cdot \rangle^\rho_{L^2}) \to (G/K, \rho^{G/K})$ is a pseudo-Riemannian submersion.*

If G is compact, we can assume that $\langle \cdot, \cdot \rangle^\rho$ is positive definite on $\mathfrak{g}$. Thus Proposition 6.9 implies:

Proposition 7.7 (Terng-Thorbergsson [30]). *Let $(G/K, \rho^{G/K})$ be a Riemannian symmetric space of compact type. Then the parallel transport map $\Phi_{G/K} : (V_\mathfrak{g}, \langle \cdot, \cdot \rangle^\rho_{L^2}) \to (G/K, \rho^{G/K})$ is a Riemannian submersion.*

8. Weakly reflective submanifolds

In this section, we extend the concept of weakly reflective submanifolds [5] to the affine case and study their relations to the parallel transport map.

Let (N, ∇^N) be an affine Hilbert manifold, M a submanifold of N and $\mathcal{W}$ a transversal bundle of the inclusion map $i : M \to N$. Then there exists a unique affine connection ∇^M on M such that $i : (M, \nabla^M) \to (N, \nabla^N)$ is an affine immersion with transversal bundle $\mathcal{W}$. We call M an (affine) submanifold of (N, ∇^N) with transversal bundle $\mathcal{W}$.

Definition. Let M be a submanifold of (N, ∇^N) with transversal bundle $\mathcal{W}$. Then M is called *weakly reflective* if for each $(p, \xi) \in \mathcal{W}$ there exists an affine transformation $\nu = \nu_{(p,\xi)}$ of (N, ∇^N) satisfying

$$\nu(M) = M, \qquad \nu(p) = p, \qquad d\nu(\mathcal{W}) = \mathcal{W}, \qquad d\nu(\xi) = -\xi.$$

We call ν a *weak reflection* of M with respect to (p, ξ). If every ν can be chosen from a particular group S, then M is called *S-weakly reflective*.

Remark 8.1. In the Riemannian case [5], ν is required to be an isometry and the condition $d\nu(T^\perp M) = T^\perp M$ is automatically satisfied. Thus, our definition applied to the Riemannian case (with Levi-Civita connection, $\mathcal{W} = T^\perp M$) is weaker than the original one. Nevertheless an important property of weakly reflective submanifolds, namely austere property, still holds in the affine case (Proposition 8.2). Note that if N is a complete irreducible Riemannian manifold and not a 1-dimensional Euclidean space, then the affine transformation group coincides with the isometry group [9,

p. 242]. Thus, our definition applied to such an irreducible Riemannian case is equivalent to the original one.

The following proposition shows that weakly reflective submanifolds are austere [3]. This was originally shown in [5] in the Riemannian case by using a Riemannian metric and so we give a metric-free proof here.

Proposition 8.2. *Let M be a submanifold of an affine Hilbert manifold (N, ∇^N) with transversal bundle $\mathcal{W}$. If M is weakly reflective, then for each $(p, \xi) \in \mathcal{W}$ there exists a topological linear automorphism ϕ on T_pM such that $\phi \circ A^M_\xi \circ \phi^{-1} = -A^M_\xi$.*

Proof. Let ν be a weak reflection of M with respect to (p, ξ). Denote by $\tilde{\xi} \in \Gamma(\mathcal{W})$ an extension of ξ. Since ν is an affine transformation, we have

$$\nabla^N_{d\nu(v)} d\nu(\tilde{\xi}) = d\nu(\nabla^N_v \tilde{\xi})$$

where $v \in T_pM$. Since $d\nu(\mathcal{W}) = \mathcal{W}$ we have $d\nu(\tilde{\xi}) \in \Gamma(\mathcal{W})$. Thus (3.3) implies

$$A^M_{d\nu(\xi)}(d\nu(v)) = d\nu(A^M_\xi(v)).$$

Since $d\nu(\xi) = -\xi$, this proves the proposition. $\square$

The following theorem shows examples of infinite dimensional weakly reflective submanifolds in Hilbertable spaces which are not totally geodesic (cf. Remark 8.4). This generalizes Corollary 5 of [16] in the case that G/K is a compact normal homogeneous space.

Theorem 8.3. *Let G/K be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then each fiber of the parallel transport map $\Phi_{G/K} : V_\mathfrak{g} \to G/K$ is a weakly reflective submanifold of $(V_\mathfrak{g}, D)$ where its transversal bundle is defined as the restriction of the horizontal distribution* (7.4) *of $\Phi_{G/K}$ to it.*

Proof. If F_{aK} is the fiber at aK, then $g * F_{eK} = F_{aK}$ holds for $g \in P(G, G \times \{e\})$ satisfying $g(0) = a$. Moreover the horizontal distribution $\mathcal{H}$ of $\Phi_{G/K}$ is invariant under the $P(G, G \times K)$-action. Thus the transversal bundles of F_{eK} and F_{aK} are identified via $g*$. Hence F_{aK} is weakly reflective if and only if F_{eK} is weakly reflective. Therefore we have only to consider the fiber F_{eK} at eK.

For $g \in \mathcal{G}$ and $u \in V_\mathfrak{g}$ we define $g_\# \in \mathcal{G}$ and $u_\# \in V_\mathfrak{g}$ by $g_\#(t) := g(1-t)$ and $u_\#(t) := u(1-t)$. We consider the involutive linear transformation $\mathfrak{r} : V_\mathfrak{g} \to V_\mathfrak{g}$ defined by $\mathfrak{r}(u) := -u_\#$ which is called the *canonical reflection* [16]. Since $(g_\#)' = -(g')_\#$ we have $\mathfrak{r}(g * \hat{0}) = g_\# * \hat{0}$ for $g \in \mathcal{G}$. Thus we get the commutative diagram:

$$\begin{array}{ccc} V_\mathfrak{g} & \xrightarrow{\mathfrak{r}} & V_\mathfrak{g} \\ {\scriptstyle\Phi}\downarrow & & {\scriptstyle\Phi}\downarrow \\ G & \xrightarrow{inv} & G \end{array}$$

where inv denotes the inversion $a \mapsto a^{-1}$. Clearly $\mathfrak{r}(\hat{0}) = \hat{0}$, $\mathfrak{r}(F_{eK}) = F_{eK}$ and $\mathfrak{r}(\hat{\xi}) = -\hat{\xi}$ for any $\xi \in \mathfrak{p}$. Furthermore we have

$$\mathfrak{r}(\mathcal{H}(h * \hat{0})) = \mathfrak{r}(\mathrm{Ad}(h)\hat{\mathfrak{p}}) = \mathrm{Ad}(h_\#)\hat{\mathfrak{p}} = \mathcal{H}(h_\# * \hat{0}) = \mathcal{H}(\mathfrak{r}(h * \hat{0}))$$

for $h \in P(G, \{e\} \times K)$. This shows that $\mathfrak{r}$ leaves the transversal bundle $\mathcal{W}$ of F_{eK} invariant. Thus $\mathfrak{r}$ is a weak reflection of F_{eK} with respect to $(\hat{0}, \hat{\xi})$ for any $\xi \in \mathfrak{p}$.

Since $\mathcal{W}$ is invariant under the $P(G, \{e\} \times K)$-action, we get a weak reflection of F_{eK} with respect to any transversal vector at any point of F_{eK}. This completes the proof. $\square$

Remark 8.4. The arguments similar to the proof of Lemma 1 in [16] shows that the affine fundamental form of the fiber $F_{eK} := \Phi_{G/K}^{-1}(eK)$ is given by

$$\alpha^{F_{eK}}(-Q', -R') = \int_0^1 [Q(t), -R'(t)]_{\mathfrak{p}} dt$$

where $Q, R \in \mathrm{Lie}\, P(G, \{e\} \times K)$. From this formula one can see that F_{eK} is totally geodesic (i.e. an affine subspace of $V_{\mathfrak{g}}$) if and only if $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{k}$. In particular, the fiber $F_e := \Phi^{-1}(e)$ is totally geodesic if and only if $\mathfrak{g}$ is abelian. Note that if one fiber of $\Phi_{G/K}$ (or Φ) is totally geodesic, then any fiber of it is totally geodesic.

The above theorem together with Theorem 7.4 and Proposition 4.6 implies:

Corollary 8.5. *Let G/K be a symmetric space with canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then each fiber of the parallel transport map $\Phi_{G/K} : V_{\mathfrak{g}} \to G/K$ is a weakly reflective Fredholm submanifold of $(V_{\mathfrak{g}}, D)$ of Fredholm index 0 where its transversal bundle is defined by the same way as in Theorem 8.3.*

The following theorem gives a characterization of weakly reflective submanifolds in affine symmetric spaces. This extends Theorem 8 of [16] in the case that N is an (irreducible) Riemannian symmetric space of compact type and Theorem 1 of [17] in the case that N is a compact isotropy irreducible Riemannian homogeneous space. (Considering the case $S = \tilde{G}$ we get Theorem 1.5.)

Theorem 8.6. *Let (N, ∇^N) be an affine symmetric space, $\tilde{G}$ the affine transformation group of (N, ∇^N), G its identity component, $\tilde{K}$ the isotropy subgroup of $\tilde{G}$ at a fixed $p \in N$ and $K := G \cap \tilde{K}$. Denote by $\Phi_{G/K} = \pi \circ \Phi : V_{\mathfrak{g}} \to G \to G/K = N$ the parallel transport map. Let M be a submanifold of (N, ∇^N) with transversal bundle $\mathcal{W}$. Denote by $\bar{\mathcal{W}}$ (resp. $\hat{\mathcal{W}}$) the horizontal lift of $\mathcal{W}$ with respect to π (resp. $\Phi_{G/K}$). Then the following conditions are equivalent for a closed subgroup S of $\tilde{G}$ satisfying $aSa^{-1} = S$ for any $a \in G$:*

- (i) *M is an S-weakly reflective submanifold of (N, ∇^N) with transversal bundle $\mathcal{W}$.*
- (ii) *$\bar{M} := \pi^{-1}(M)$ is an $(S \times S_p)$-weakly reflective submanifold of (G, ∇^G) with transversal bundle $\bar{\mathcal{W}}$, where $S_p := \tilde{K} \cap S$.*
- (iii) *$\hat{M} := \Phi_{G/K}^{-1}(M)$ is a $P(S, S \times S_p)$-weakly reflective Fredholm submanifold of $(V_{\mathfrak{g}}, D)$ of Fredholm index 0 with transversal bundle $\hat{\mathcal{W}}$.*

Remark 8.7. In the above (ii) and (iii) we are considering the affine actions of $S \times S_p$ on $\tilde{G}$ defined by $(b, c) \cdot a := bac^{-1}$ and $P(S, S \times S_p)$ on $V_{\mathfrak{g}}$ defined by (6.1) respectively. Note that an element of $S \times S_p$ does not necessarily leave G invariant. (ii) claims that weak reflections of $\bar{M}$ can be chosen so that each one of them is expressed as the restriction of the affine transformation on $\tilde{G}$ caused by an element of $S \times S_p$.

Remark 8.8. In Theorem 8 of [16] the compact semisimple Lie group G is equipped with a bi-invariant Riemannian metric induced from the Killing form on $\mathfrak{g}$. This is to ensure that the weak reflection of $\pi^{-1}(M)$ (obtained as an automorphism of G) is an isometry of G. In Theorem 8.6 we do not need such an assumption at all.

Remark 8.9. In Theorem 1 of [17] the compact Riemannian homogeneous space N is assumed to be isotropy irreducible. This is to ensure that N is isometric to the homogeneous space G/K equipped with a normal homogeneous metric. In Theorem 8.6 we do not need such an assumption at all.

To prove the theorem we prepare two lemmas. The first lemma is related to the structure of the symmetric space N:

Lemma 8.10. *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the canonical decomposition associated with the symmetric pair (G, K). Then $\mathfrak{p}$ is invariant under (not only $\operatorname{Ad}(K)$ but also) $\operatorname{Ad}(\tilde{K})$.*

Proof. Let s denote the symmetry at p. Define an involutive automorphism $\tilde{\theta} : \tilde{G} \to \tilde{G}$ by $\tilde{\theta}(f) = s \circ f \circ s^{-1}$. The restriction $\theta := \tilde{\theta}|_G : G \to G$ is the involution satisfying $G_0^\theta \subset K \subset G^\theta$ and thus $d\theta = d\tilde{\theta}$ is $-\operatorname{id}$ on $\mathfrak{p}$. We claim that $\tilde{\theta}(k) = k$ for $k \in \tilde{K}$. In fact $\tilde{\theta}(k)$ and k are affine transformations on N with the same differentials at $p \in N$ and therefore they are equal [9, Lemma 4 in p. 254]. Thus, for $x \in \mathfrak{p}$ we have $d\tilde{\theta}(\operatorname{Ad}(k)x) = \operatorname{Ad}(\tilde{\theta}(k))d\tilde{\theta}(x) = -\operatorname{Ad}(k)x$ which shows $\operatorname{Ad}(k)x \in \mathfrak{p}$. $\square$

The following lemma generalizes Lemma 2 in [16] in the Riemannian case:

Lemma 8.11. *Let $\pi : \hat{N} \to N$ be a submersion with horizontal distribution $\mathcal{H}$ and M a submanifold of N with transversal bundle $\mathcal{W}$. Set $\hat{M} := \pi^{-1}(M)$ and denote by $\hat{\mathcal{W}}$ the horizontal lift of $\mathcal{W}$. Take $(\hat{p}, \hat{\xi}) \in \hat{\mathcal{W}}$ and set $(p, \xi) := (\pi(\hat{p}), d\pi(\hat{\xi})) \in \mathcal{W}$. Let $\nu : N \to N$ be a diffeomorphism fixing p and $\hat{\nu} : \hat{N} \to \hat{N}$ a diffeomorphism fixing $\hat{p}$ and leaving $\mathcal{H}$ invariant such that the diagram*

$$\begin{array}{ccc} \hat{N} & \xrightarrow{\hat{\nu}} & \hat{N} \\ {\scriptstyle \pi}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ N & \xrightarrow{\nu} & N \end{array}$$

commutes. Then the following conditions are equivalent:

(i) *$\hat{\nu}(\hat{M}) = \hat{M}$, $d\hat{\nu}(\hat{\mathcal{W}}) = \hat{\mathcal{W}}$ and $d\hat{\nu}(\hat{\xi}) = -\hat{\xi}$.*
(ii) *$\nu(M) = M$, $d\nu(\mathcal{W}) = \mathcal{W}$ and $d\nu(\xi) = -\xi$.*

Proof. It is easy to see that $\hat{\nu}(\hat{M}) = \hat{M}$ if and only if $\nu(M) = M$. Since $\hat{\nu}$ leaves $\mathcal{H}$ invariant, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{H}(\hat{q}) & \xrightarrow{d\hat{\nu}} & \mathcal{H}(\hat{\nu}(\hat{q})) \\ {\scriptstyle d\pi}\big\downarrow & & {\scriptstyle d\pi}\big\downarrow \\ T_qN & \xrightarrow{d\nu} & T_{\nu(q)}N \end{array}$$

where $\hat{q} \in \hat{N}$ and $q := \pi(\hat{q})$. Thus $d\hat{\nu}(\hat{\mathcal{W}}) = \hat{\mathcal{W}}$ if and only if $d\nu(\mathcal{W}) = \mathcal{W}$. Considering the case $\hat{q} = \hat{p}$ we see that $d\hat{\nu}(\hat{\xi}) = -\hat{\xi}$ if and only if $d\nu(\xi) = -\xi$. $\square$

Proof of Theorem 8.6. "(i) $\Leftrightarrow$ (ii)": Let $a \in \bar{M}$. Then $\bar{M}' := l_a^{-1}(\bar{M})$ is a submanifold of (G, ∇^G) with transversal bundle $\bar{\mathcal{W}}' := dl_a^{-1}(\bar{\mathcal{W}})$ and $M' := L_a^{-1}(M)$ is a submanifold of $(G/K, \nabla^{G/K})$ with transversal bundle $\mathcal{W}' := dL_a^{-1}(\mathcal{W})$. Since the diagram (7.6) commutes we have $\bar{M}' = \pi^{-1}(M')$. Since the horizontal distribution $\mathcal{H}$ of π (cf. Proposition 7.5) is invariant under l_a, $\bar{\mathcal{W}}'$ is the horizontal lift of $\mathcal{W}'$. Since $aSa^{-1} = S$ we have

$$S_{aK} = aS_{eK}a^{-1}, \qquad (S \times S_{eK})_a = (a,e)(S \times S_{eK})_e(a,e)^{-1}.$$

where S_{aK} and $(S \times S_{eK})_a$ denote the isotropy subgroups of S at aK and of $S \times S_{eK}$ at a respectively. These imply that it suffices to show a one-to-one correspondence between

(a) the set of weak reflections of M' with respect to $(eK, \eta) \in \mathcal{W}'$ which belong to S_{eK},
(b) the set of weak reflections of $\bar{M}'$ with respect to $(e, \bar{\eta}) \in \bar{\mathcal{W}}'$ which belong to $(S \times S_{eK})_e = \{(k,k) \mid k \in S_{eK}\}$,

where $\bar{\eta}$ is a horizontal lift of η. Let ν be in the set (a). Define $\bar{\nu} := (\nu, \nu)$, that is, $\bar{\nu}(b) := \nu b \nu^{-1}$. Under the identification $G/K = N$, $bK = b(p)$ we have $\nu(bK) = \nu(b(p)) = (\nu \circ b \circ \nu^{-1})(p) = (\nu b \nu^{-1})K$ where $b \in G$. Thus the diagram

$$\begin{array}{ccc} G & \xrightarrow{\bar{\nu}} & G \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ G/K & \xrightarrow{\nu} & G/K \end{array} \tag{8.1}$$

commutes. By Lemma 8.10 we have

$$\begin{aligned} d\bar{\nu}(\mathcal{H}(a)) &= d\bar{\nu}(dl_a(\mathfrak{p})) = d(\bar{\nu} \circ l_a)(\mathfrak{p}) = d(l_{\bar{\nu}(a)} \circ \bar{\nu})(\mathfrak{p}) \\ &= (dl_{\bar{\nu}(a)}) \operatorname{Ad}(\nu)\mathfrak{p} = (dl_{\bar{\nu}(a)})\mathfrak{p} = \mathcal{H}(\bar{\nu}(a)) \end{aligned}$$

for any $a \in G$. Thus $\bar{\nu}$ leaves the horizontal distribution $\mathcal{H}$ of π invariant. Therefore Lemma 8.11 shows that $\bar{\nu}$ is a weak reflection of $\bar{M}'$ with respect to $(e, \bar{\eta}) \in \bar{\mathcal{W}}'$. The converse is also true and thus (i) and (ii) are equivalent.

"(ii) $\Leftrightarrow$ (iii)": The Fredholm property of (iii) is a consequence of Theorem 7.4 and Proposition 4.6. It suffices to prove the equivalence of weakly reflective properties. Let $u \in \hat{M}$. Take $g \in P(G, G \times \{e\})$ satisfying $u = g * \hat{0}$. Set $a := g(0)$. Then $\hat{M}' := (g*)^{-1}(\hat{M})$ is a submanifold of $(V_{\mathfrak{g}}, D)$ with transversal bundle $\hat{\mathcal{W}}' := d(g*)^{-1}(\hat{\mathcal{W}})$ and $\bar{M}' := l_a^{-1}(\bar{M})$ is a submanifold of (G, ∇^G) with transversal bundle $\bar{\mathcal{W}}' := dl_a^{-1}(\bar{\mathcal{W}})$. Since the diagram (6.7) commutes we have $\hat{M}' = \Phi^{-1}(\bar{M}')$. Since the horizontal distribution $\mathcal{H}$ of $\Phi_{G/K}$ (cf. (7.4)) is invariant under $g*$, $\hat{\mathcal{W}}'$ is a horizontal lift of $\bar{\mathcal{W}}'$. Since $aSa^{-1} = S$ for any $a \in G$ we have

$$P(S, S \times S_{eK})_u = gP(S, S \times S_{eK})_{\hat{0}} g^{-1}$$

where $P(S, S \times S_{eK})_u$ denotes the isotropy subgroup of $P(S, S \times S_{eK})$ at u. These imply that it suffices to show a one-to-one correspondence between the set (b) and

(c) the set of weak reflections of $\hat{M}'$ with respect to $(\hat{0}, \hat{\eta}) \in \hat{\mathcal{W}}'$ which belong to $P(S, S \times S_{eK})_{\hat{0}} = \{\hat{k} \mid k \in S_{eK}\}$,

where $\hat{\eta}$ is a horizontal lift of $\bar{\eta}$ and $\hat{k}$ denotes the constant path with value k. Let $\bar{\nu} = (\nu, \nu)$ be in the set (b) where $\nu \in S_{eK}$. Let $\hat{\nu}$ denote the constant path with value $\nu \in \tilde{K}$, that is, $\hat{\nu} * u = \operatorname{Ad}(\hat{\nu})u$. Then $\hat{\nu} * (g * \hat{0}) = (\hat{\nu} g) * \hat{0} = (\hat{\nu} g \hat{\nu}^{-1}) * \hat{0}$ for any $g \in \mathcal{G}$. Since $\hat{\nu} g \hat{\nu}^{-1} \in \mathcal{G}$ it follows from Proposition 6.4 (ii) that the diagram

$$\begin{array}{ccc} V_{\mathfrak{g}} & \xrightarrow{\hat{\nu}} & V_{\mathfrak{g}} \\ {\scriptstyle \Phi}\big\downarrow & & {\scriptstyle \Phi}\big\downarrow \\ G & \xrightarrow{\bar{\nu}} & G \end{array} \tag{8.2}$$

commutes. By Lemma 8.10 we have

$$\begin{aligned} d\hat{\nu}(\mathcal{H}(g * \hat{0})) &= \mathrm{Ad}(\hat{\nu}) \circ \mathrm{Ad}(g)(\hat{\mathfrak{p}}) = \mathrm{Ad}(\hat{\nu}) \circ \mathrm{Ad}(g) \circ \mathrm{Ad}(\hat{\nu})^{-1}(\hat{\mathfrak{p}}) \\ &= \mathrm{Ad}(\hat{\nu} g \hat{\nu}^{-1})(\hat{\mathfrak{p}}) = \mathcal{H}((\hat{\nu} g \hat{\nu}^{-1}) * \hat{0})) = \mathcal{H}(\hat{\nu} * (g * \hat{0})) \end{aligned}$$

for any $g \in P(G, G \times K)$. Thus $\hat{\nu}$ leaves the horizontal distribution $\mathcal{H}$ of $\Phi_{G/K}$ invariant. Therefore Lemma 8.11 shows that $\hat{\nu}$ is a weak reflection of $\hat{M}'$ with respect to $(\hat{0}, \hat{\eta}) \in \hat{\mathcal{W}}'$. The converse is also true and thus (ii) and (iii) are equivalent. $\square$

As mentioned in Remark 8.1, in the irreducible Riemannian case, a weakly reflective submanifold (with normal bundle) in the affine sense is equivalent to a weakly reflective submanifold in the Riemannian sense. Therefore, from Theorem 8.6 we get the following corollary which generalizes Corollary 8 of [16].

Corollary 8.12. *Let (N, g) be an irreducible Riemannian symmetric space of compact (resp. non-compact) type, $\tilde{G}$ the isometry group of (N, g), G its identity component, $\tilde{K}$ the isotropy subgroup of $\tilde{G}$ at a fixed $p \in N$ and $K := G \cap \tilde{K}$. Denote by $\Phi_{G/K} = \pi \circ \Phi : V_{\mathfrak{g}} \to G \to G/K = N$ the parallel transport map. Equip $\mathfrak{g}$ with the Killing form multiplied by a negative (resp. positive) scalar and G with the corresponding bi-invariant Riemannian (resp. pseudo-Riemannian) metric ρ. Let M be a submanifold of (N, g). Then the following conditions are equivalent for a closed subgroup S of $\tilde{G}$ satisfying $aSa^{-1} = S$ for any $a \in G$:*

(i) *M is an S-weakly reflective submanifold of (N, g).*
(ii) *$\bar{M} := \pi^{-1}(M)$ is an $(S \times S_p)$-weakly reflective submanifold of (G, ρ).*
(iii) *$\hat{M} := \Phi_{G/K}^{-1}(M)$ is a $P(S, S \times S_p)$-weakly reflective, proper Fredholm submanifold (resp. Fredholm submanifold of Fredholm index 0) of $(V_{\mathfrak{g}}, \langle\cdot,\cdot\rangle_{L^2}^{\rho})$.*

Note that when N is of non-compact type, $\bar{M}$ and $\hat{M}$ are pseudo-Riemannian submanifolds, that is, ρ and $\langle\cdot,\cdot\rangle_{L^2}^{\rho}$ induce (non-degenerate) pseudo-Riemannian metrics on $\bar{M}$ and $\hat{M}$ respectively. This follows from the fact that π and Φ are pseudo-Riemannian submersions. Weakly reflective pseudo-Riemannian submanifolds are defined by the same way as in the Riemannian case.

Acknowledgements

The author would like to thank Professor Yoshihiro Ohnita for suggesting the proofs of Proposition 5.3 and Lemma 5.4. The author is also grateful to Professor Naoyuki Koike for his interest in this work and useful comments and to Professor Takashi Sakai for his constant support.

References

[1] N. Abe, K, Hasegawa, *An affine submersion with horizontal distribution and its applications*, Differential Geom. Appl. **14** (2001), no. 3, 235–250. https://doi.org/10.1016/S0926-2245(01)00034-1

[2] H. I. Eliasson, *Geometry of manifolds of maps*, J. Differential Geometry **1** (1967), 169–194. https://doi.org/10.4310/jdg/1214427887

[3] R. Harvey, H. B. Lawson, Jr., *Calibrated geometries*, Acta Math., **148** (1982), 47-157. https://doi.org/10.1007/BF02392726

[4] E. Heintze, X. Liu, C. Olmos, *Isoparametric submanifolds and a Chevalley-type restriction theorem*, Integrable systems, geometry, and topology, 151-190, AMS/IP Stud. Adv. Math., **36**, Amer. Math. Soc., Providence, RI, 2006. https://doi.org/10.1090/amsip/036/04

[5] O. Ikawa, T. Sakai, H. Tasaki, *Weakly reflective submanifolds and austere submanifolds*. J. Math. Soc. Japan **61** (2009), no. 2, 437-481. https://doi.org/10.2969/jmsj/06120437

[6] C. King, C.-L. Terng, *Minimal submanifolds in path space*, Global analysis in modern mathematics, 253–281, Publish or Perish, Houston, TX, 1993.

[7] W. Klingenberg, *Lectures on Closed Geodesics*, Grundlehren der Mathematischen Wissenschaften, Vol. **230**, Springer-Verlag, Berlin-New York, 1978, x+227 pp. https://doi.org/10.1007/978-3-642-61881-9

[8] W. Klingenberg, *Riemannian Geometry*, De Gruyter Stud. Math., 1, Walter de Gruyter & Co., Berlin, 1995, x+409 pp. https://doi.org/10.1515/9783110905120

[9] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Vol. I, Wiley Classics Lib. Wiley-Intersci. Publ. John Wiley & Sons, Inc., New York, 1996, xii+329 pp.

[10] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Vol. II, Wiley Classics Lib. Wiley-Intersci. Publ. John Wiley & Sons, Inc., New York, 1996, xvi+468 pp.

[11] N. Koike, *Submanifold geometries in a symmetric space of non-compact type and a pseudo-Hilbert space*, Kyushu J. Math. **58** (2004), no. 1, 167–202. https://doi.org/10.2206/kyushujm.58.167

[12] N. Koike, *Complex equifocal submanifolds and infinite dimensional anti-Kaehlerian isoparametric submanifolds*, Tokyo J. Math. **28** (2005), no. 1, 201–247. https://doi.org/10.3836/tjm/1244208289

[13] O. Kowalski, *Generalized Symmetric Spaces*, Lecture Notes in Math., **805**, Springer-Verlag, Berlin-New York, 1980, xii+187 pp. https://doi.org/10.1007/BFb0103324

[14] S. Lang, *Fundamentals of Differential Geometry*, Graduate Texts in Mathematics, **191**. Springer-Verlag, New York, 1999. https://doi.org/10.1007/978-1-4612-0541-8

[15] J. M. Lee, *Introduction to Smooth Manifolds*, Second edition, Grad. Texts in Math., **218**, Springer, New York, 2013. xvi+708 pp. https://doi.org/10.1007/978-1-4419-9982-5

[16] M. Morimoto, *On weakly reflective PF submanifolds in Hilbert spaces*, Tokyo J. Math. **44** (2021), no. 1, pp. 103–124. https://doi.org/10.3836/tjm/1502179323

[17] M. Morimoto, *On weakly reflective submanifolds in compact isotropy irreducible Riemannian homogeneous spaces*, Tokyo J. Math. **44** (2021), no. 2, 467–476. https://doi.org/10.3836/tjm/1502179344

[18] M. Morimoto, *The canonical isomorphism between path spaces and principal curvatures of PF submanifolds*, to appear in Tokyo J. Math. https://doi.org/10.3836/tjm/1502179418

[19] K. Nomizu, T. Sasaki, *Affine Differential Geometry*, Cambridge Tracts in Math., 111 Cambridge University Press, Cambridge, 1994, xiv+263 pp.

[20] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469. https://doi.org/10.1307/mmj/1028999604

[21] R. S. Palais, *Morse theory on Hilbert manifolds*, Topology **2** (1963), 299-340. https://doi.org/10.1016/0040-9383(63)90013-2

[22] R. S. Palais, *Foundations of Global Non-linear Analysis*, W. A. Benjamin, Inc., New York-Amsterdam, 1968, vii+131 pp.

[23] R. S. Palais, C.-L. Terng, *Critical Point Theory and Submanifold Geometry*, Lecture Notes in Mathematics, **1353**. Springer-Verlag, Berlin, 1988. https://doi.org/10.1007/BFb0087442

[24] U. Pinkall, G. Thorbergsson, *Examples of infinite dimensional isoparametric submanifolds*, Math. Z. **205** (1990), no. 2, 279–286. https://doi.org/10.1007/BF02571240

[25] F. Podestà, *Some remarks on austere submanifolds*, Boll. Un. Mat. Ital. B (7) **11** (1997), no. 2, suppl., 157–160.

[26] S. Smale, *Morse theory and a non-linear generalization of the Dirichlet problem*, Ann. of Math. (2)**80** (1964), 382-396. https://doi.org/10.2307/1970398

[27] S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–866. `https://doi.org/10.2307/2373250`

[28] C.-L. Terng, *Proper Fredholm submanifolds of Hilbert space.* J. Differential Geom. **29** (1989), no. 1, 9–47. `https://doi.org/10.4310/jdg/1214442631`

[29] C.-L. Terng, *Polar actions on Hilbert space.* J. Geom. Anal. **5** (1995), no. 1, 129–150. `https://doi.org/10.1007/BF02926445`

[30] C.-L. Terng, G. Thorbergsson, *Submanifold geometry in symmetric spaces.* J. Differential Geom. **42** (1995), no. 3, 665–718. `https://doi.org/10.4310/jdg/1214457552`

Department of Mathematical Sciences, Tokyo Metropolitan University, 1-1 Minami-Osawa, Hachioji-shi, Tokyo 192-0397, JAPAN

Email address: `morimoto.mshr@gmail.com`