THE PARALLEL TRANSPORT MAP OVER REDUCTIVE HOMOGENEOUS SPACE

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Abstract. We show that the parallel transport map over a reductive homogeneous space with the natural torsion-free connection becomes an affine submersion with horizontal distribution. This generalizes one of the main results in the author's previous paper in the case of affine symmetric spaces. Moreover we consider its relation to weakly reflective submanifolds and generalize some results obtained in the author's previous papers.

Keywords. parallel transport map; reductive homogeneous space; affine submersion with horizontal distribution; weakly reflective submanifold

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1. Introduction

The parallel transport map is a natural fibration defined over a homogeneous space N. More precisely, it is defined as a submersion $\Phi_N : V \to N$ from an infinite dimensional Hilbertable space V onto N and becomes a principal fiber bundle whose structure group is a path group. For a submanifold M of N its inverse image $\hat{M}$ under Φ_N is a submanifold of V. Thus, one can obtain examples of submanifolds in V. Furthermore, the parallel transport map is a useful tool to reduce a problem of a submanifold M in N to a problem of the submanifold $\hat{M}$ in the *flat* space V.

The parallel transport map was originally introduced by R. S. Palais and C.-L. Terng [9, 11] in the case that N is a connected compact Lie group with a bi-invariant Riemannian metric. Later, C.-L. Terng and G. Thorbergsson [12] studied it when N is a compact Riemannian symmetric space and showed that it is a Riemannian submersion. Afterward, N. Koike [4] studied it when N is a Riemannian symmetric space of non-compact type and showed that it becomes a pseudo-Riemannian submersion. Recently, the author [7] studied it when N is a symmetric space with the canonical connection (i.e. N is an affine symmetric space) and showed that it is an affine submersion with horizontal distribution in the sense of Abe and Hasegawa [1].

The main purpose of the present paper is to generalize the author's previous result to the case that N is a reductive homogeneous space. Although the canonical connection is still defined for a reductive homogeneous space, it is no longer torsion-free, which implies that the previous result does not hold when N is a reductive homogeneous space with the canonical connection. Thus we consider another affine

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connection on N, namely the *natural torsion-free connection* [8, 3]. The canonical connection and the natural torsion-free connection share the same geodesics. Those connections coincide when N is a symmetric space. By equipping N with the natural torsion-free connection, we generalize the previous result to the case of reductive homogeneous spaces (Theorem 3.3). Moreover we consider its relation to weakly reflective submanifolds and generalize some results obtained in the author's previous papers (cf. Section 4). To do these, we start with reviewing invariant affine connections on reductive homogeneous spaces.

2. Invariant affine connections on reductive homogeneous spaces

In this section, we give a brief survey on invariant affine connections on reductive homogeneous spaces. We essentially follow the paper of Nomizu [8] where a simple and direct formulation is given. We also refer to Kobayashi-Nomizu [3] for a formulation based on the theory of connections on principal fiber bundles.

Let G be a connected Lie group and K a closed subgroup of G. Denote by e the identity element. The coset manifold $N := G/K$ is called a *homogeneous space*. The projection $\pi : G \to N$ becomes a principal K-bundle. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. G/K is called *reductive* if there exists an $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{p}$ of $\mathfrak{g}$ satisfying $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Note that $\mathfrak{p}$ is isomorphic to $T_{eK}N$ and the representation $\mathrm{Ad} : K \to GL(\mathfrak{p})$ is equivalent to the isotropy representation $K \to GL(T_{eK}N)$. When we speak of a reductive homogeneous space, we always fix such a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. For $X \in \mathfrak{g}$ we denote by $X_{\mathfrak{k}}$ and $X_{\mathfrak{p}}$ the $\mathfrak{k}$- and $\mathfrak{p}$-components respectively.

Let $N = G/K$ be a reductive homogeneous space with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The following lemma is elementary and not explicitly stated in [8]. However this is useful in understanding the formulation.

Lemma 2.1. *There exists a local trivialization $\phi : \pi^{-1}(U) \to U \times K$ around eK such that $\tilde{U} := \phi^{-1}(U \times \{e\})$ is equal to $\exp W$ for an open neighborhood W of 0 in $\mathfrak{p}$.*

Proof. By the inverse function theorem there exists an open neighborhood W of $0 \in \mathfrak{p}$ such that $\pi : \exp W \to \pi(\exp W) =: U$ is a diffeomorphism. This defines the local smooth section of $\pi : G \to N$ which defines the desired local trivialization. □

Fix the above local trivialization. Then $\pi : \tilde{U} \to U$ is a diffeomorphism. For each $X \in \mathfrak{p}$ the vector field $X^{\#}$ on U is defined by $X^{\#}_{aK} := d\pi(X_a)$ where $a \in \tilde{U}$. Then

$$X^{\#}_{aK} = dL_a(X^{\#}_{eK}) \tag{2.1}$$

for $a \in \tilde{U}$. Here L_a denotes the left translation on G/K by a. The following fact is essentially shown in [8, p. 42] (where $X^{\#}$ is written as X^*).

Lemma 2.2 (Nomizu [8])**.** *For each $k \in K$ there exists a neighborhood $\tilde{U}_1$ of $e \in \tilde{U}$ satisfying $k\tilde{U}_1k^{-1} \subset \tilde{U}$. Moreover,*

(i) *$L_k(U_1) \subset U$ where $U_1 := \pi(\tilde{U}_1)$,*
(ii) *$dL_k(X^{\#}) = (\mathrm{Ad}(k)X)^{\#}$ holds on $L_k(U_1)$ for any $X \in \mathfrak{p}$.*

In fact, since $\mathrm{Ad}(k)^{-1}\mathfrak{p} \subset \mathfrak{p}$ we can take an open neighborhood W_1 of $0 \in \mathfrak{p}$ satisfying $0 \in W_1 \subset W \cap \mathrm{Ad}(k)^{-1}W$. Then $\tilde{U}_1 := \exp W_1$ is the desired neighborhood. Then (i) is obvious and (ii) can be proven by use of (2.1).

For a vector bundle E over a reductive homogeneous space $N = G/K$ we denote by $\Gamma(E)$ the set of all smooth sections of E. $\Gamma(TN)$ is also denoted by $\mathfrak{X}(N)$. An affine connection $\nabla : \Gamma(TN) \to \Gamma(TN \otimes T^*N)$ on N is called *G-invariant* if

$$\nabla_{dL_a(v)} dL_a(Z) = dL_a(\nabla_v Z)$$

for $Z \in \Gamma(TN)$, $v \in TN$ and $a \in G$. A bilinear map $\alpha : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ is called $\mathrm{Ad}(K)$*-invariant* if

$$\alpha(\mathrm{Ad}(k)X, \mathrm{Ad}(k)Y) = \mathrm{Ad}(k)\alpha(X, Y)$$

for $X, Y \in \mathfrak{p}$ and $k \in K$. Such an α is also called a *connection function.*

Proposition 2.3 (Nomizu [8]). *Let G/K be a reductive homogeneous space with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then there exists a one-to-one correspondence between the set of all G-invariant affine connections on G/K and the set of all $\mathrm{Ad}(K)$-invariant bilinear maps $\mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$. The correspondence is given by*

$$\nabla_{X^{\#}_{eK}} Y^{\#} = \alpha(X, Y). \tag{2.2}$$

In fact, by taking a basis $Y_1, \cdots, Y_n$ of $\mathfrak{p}$ we can express each $Z \in \mathfrak{X}(U)$ as $Z = \sum_{i=1}^n \varphi_i Y_i^{\#}$ where $\varphi_i \in C^\infty(U)$. Then the Leibniz rule shows

$$\nabla_{X^{\#}_{eK}} Z = \sum_{i=1}^n (\varphi_i(eK)\nabla_{X^{\#}_{eK}} Y_i^{\#} + (X^{\#}_{eK}\varphi_i)(Y_i^{\#})_{eK}), \tag{2.3}$$

which implies that ∇ is uniquely determined by α. The G-invariance of ∇ is equivalent to the $\mathrm{Ad}(K)$-invariance of α, which can be seen by using Lemma 2.2 (ii).

For $X \in \mathfrak{p}$ we denote by X^* the fundamental vector field on G/K, namely

$$X^*_{aK} := \left.\frac{d}{dt}\right|_{t=0} (\exp tX)aK.$$

Since $X^*_{eK} = X^{\#}_{eK}$ and $[X^*, Y_i^{\#}]_{eK} = 0$ it follows from (2.3) that:

Corollary 2.4. $\nabla_{X^*_{eK}} Z = [X^*, Z]_{eK} + \alpha(X, Z_{eK})$ *for $X \in \mathfrak{p}$ and $Z \in \mathfrak{X}(G/K)$.*

Let ∇ be an affine connection on a reductive homogeneous space $N = G/K$ with connection function α. Then:

(i) ∇ is called the *canonical connection* (or the canonical connection of the second kind) if $\alpha(X, Y) = 0$ for any $X, Y \in \mathfrak{p}$.

(ii) ∇ is called the *natural torsion-free connection* (or the canonical connection of the first kind) if $\alpha(X, Y) = \frac{1}{2}[X, Y]_{\mathfrak{p}}$ for $X, Y \in \mathfrak{p}$.

In both cases, $\gamma(t) := \pi(\exp tX)$ is a geodesic through eK where $X \in \mathfrak{p}$. In the case (i), the parallel translation of $v \in T_{eK}N$ along γ is equal to $dL_{\gamma(t)}(v)$. In the case (ii), the torsion tensor vanishes. When G/K is a symmetric space, (i) and (ii) are equivalent.

Let G be a connected Lie group. Denote by ΔG the diagonal of $G \times G$. Then $\rho : (G \times G)/\Delta G \to G$, $(a, b) \mapsto ab^{-1}$ is an isomorphism between symmetric spaces.

Corollary 2.5. *The canonical connection ∇^G of $G \cong (G \times G)/\Delta G$ is given by $(\nabla^G_X Z)_e = \frac{1}{2}[X + X^R, Z]_e$ where $X \in \mathfrak{g}$ and $Z \in \mathfrak{X}(G)$. Here X^R is the right invariant vector field satisfying $X^R_e = X_e$. In particular $\nabla^G_X Y = \frac{1}{2}[X, Y]$ for $X, Y \in \mathfrak{g}$.*

Proof. The decomposition associated to $(G\times G)/\Delta G$ is given by $\mathfrak{g}\oplus\mathfrak{g} = \Delta\mathfrak{g}\oplus(\Delta\mathfrak{g})^\perp$ where $(\Delta\mathfrak{g})^\perp := \{(X,-X) \mid X \in \mathfrak{g}\}$. With respect to the $(G\times G)$-action on G defined by $(b,c)\cdot a := bac^{-1}$ we have $(X,-X)^*_a = \frac{d}{dt}\big|_{t=0} \exp t(X,-X)\cdot a = X_a + X^R_a$. In particular $(X,-X)^*_e = 2X_e$. Since ρ is equivariant with respect to the $(G\times G)$-actions, we have $2(\nabla_X Z)_e = (\nabla_{(X,-X)^*}Z)_e \cong (\nabla_{(X,-X)^*}Z)_{(e,e)} = [(X,-X)^*, Z]_{(e,e)} \cong [X + X^R, Z]_e$ by Corollary 2.4. □

From Corollary 2.5 we see that the canonical connection of $G \cong (G\times G)/\Delta G$ is the same as the natural torsion-free connection of $G \cong G/\{e\}$.

3. The parallel transport map

In this section we study the parallel transport map over a reductive homogeneous space. We assume basic facts about affine immersions and affine submersions formulated in [7, Section 3]. We also refer to [7, Section 6] and its references for the parallel transport map over a Lie group.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Denote by $\mathcal{G} := H^1([0,1], G)$ the Hilbert Lie group of Sobolev H^1-maps from $[0,1]$ to G and by $V_\mathfrak{g} := L^2([0,1],\mathfrak{g})$ the Hilbertable space of all L^2-maps from $[0,1]$ to $\mathfrak{g}$. The adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ induces the representation $\mathcal{G} \to GL(V_\mathfrak{g})$ which is still denoted by Ad. Write l_g and r_g for the left and right translations by $g \in \mathcal{G}$ respectively. Denote by g' the weak derivative of g. The affine action of $\mathcal{G}$ on $V_\mathfrak{g}$ is defined by

$$g * u := \mathrm{Ad}(g)u - dr_g^{-1}(g') \tag{3.1}$$

where $g \in \mathcal{G}$ and $u \in V_\mathfrak{g}$. This action is transitive. For a submanifold U of $G\times G$,

$$P(G,U) := \{g \in \mathcal{G} \mid (g(0), g(1)) \in U\}$$

is a submanifold of $\mathcal{G}$. If U is a Lie subgroup of $G\times G$, then $P(G,U)$ is a Lie subgroup of $\mathcal{G}$ and acts on $V_\mathfrak{g}$ by (3.1). If $U = \{e\}\times G$ or $G\times\{e\}$, then the $P(G,U)$-action on $V_\mathfrak{g}$ is simply transitive.

The *parallel transport map* $\Phi : V_\mathfrak{g} \to G$ *over* G is a submersion defined by $\Phi(u) := g_u(1)$ where $g_u \in \mathcal{G}$ is the unique solution to the ordinary differential equation

$$dl_g^{-1}(g') = u, \quad g(0) = e.$$

By definition we have $\Phi(\hat{X}) = \exp X$ where $\hat{X}$ denotes the constant path with value $X \in \mathfrak{g}$. Consider the $(G\times G)$-action on G defined by $(b,c)\cdot a := bac^{-1}$. Then

$$\Phi(g * u) = (g(0), g(1))\cdot\Phi(u) \tag{3.2}$$

where $g \in \mathcal{G}$ and $u \in V_\mathfrak{g}$. Moreover, for a Lie subgroup U of $G\times G$,

$$P(G,U) * u = \Phi^{-1}(U\cdot\Phi(u)). \tag{3.3}$$

Furthermore Φ becomes a principal $P(G, \{e\}\times\{e\})$-bundle.

The differential of $\Phi : V_\mathfrak{g} \to G$ at $\hat{0} \in V_\mathfrak{g}$ is given by

$$(d\Phi)_{\hat{0}}(X) = \int_0^1 X(t)dt \tag{3.4}$$

where $X \in V_\mathfrak{g} \cong T_{\hat{0}}V_\mathfrak{g}$. Denote by $\hat{\mathfrak{g}}$ the space of constant paths with values in $\mathfrak{g}$ and by F_u the fiber of Φ through $u \in V_\mathfrak{g}$. From (3.4) we have

$$T_{\hat{0}}V_\mathfrak{g} = \hat{\mathfrak{g}} \oplus T_{\hat{0}}(F_{\hat{0}}).$$

From (3.2) we know that $g*$ maps fibers of Φ to fibers of Φ. Thus

$$T_{g*\hat{0}}V_{\mathfrak{g}} = \mathrm{Ad}(g)\hat{\mathfrak{g}} \oplus T_{g*\hat{0}}(F_{g*\hat{0}})$$

for any $g \in \mathcal{G}$. Therefore the horizontal distribution $\mathcal{H}^{\Phi}$ of Φ is defined by

$$\mathcal{H}^{\Phi}(g * \hat{0}) := \mathrm{Ad}(g)\hat{\mathfrak{g}}.$$

We denote by D the flat connection on $V_{\mathfrak{g}}$ and by ∇^G the canonical connection on $G \cong (G \times G)/\Delta G$. The following fact was shown ([7, Theorem 6.6]):

Theorem 3.1 ([7])**.** *Let G be a connected Lie group. Then the parallel transport map $\Phi : (V_{\mathfrak{g}}, D) \to (G, \nabla^G)$ is an affine submersion with horizontal distribution $\mathcal{H}^{\Phi}$. Moreover the fundamental tensor $\mathcal{A}^{\Phi}$ restricted to $\mathcal{H}^{\Phi} \times \mathcal{H}^{\Phi}$ is alternating.*

Let G/K be a homogeneous space. Then the *parallel transport map $\Phi_{G/K}$ over G/K* is defined as the composition

$$\Phi_{G/K} := \pi \circ \Phi : V_{\mathfrak{g}} \to G \to G/K.$$

Note that if $K = \{e\}$ then $\Phi_{G/K} = \Phi$. Note also that $\Phi_{(G\times G)/\Delta G}$ is naturally identified with Φ (cf. [6]). By definition we have $\Phi_{G/K}(\hat{X}) = (\exp X)K$ for $X \in \mathfrak{g}$. Consider the G-action on G/K defined by $b \cdot aK := (ba)K$. By (3.2) we have

$$\Phi_{G/K}(g * u) = g(0) \cdot \Phi_{G/K}(u) \tag{3.5}$$

where $g \in P(G, G \times K)$ and $u \in V_{\mathfrak{g}}$. Moreover, by (3.3) we have

$$P(G, H \times K) * u = \Phi_{G/K}^{-1}(H \cdot \Phi_{G/K}(u)) \tag{3.6}$$

where H is a Lie subgroup of G. Furthermore the following fact holds. This is stated in [7, Proposition 7.3] without proof. For convenience we give its proof here.

Proposition 3.2 ([7])**.** *$\Phi_{G/K}$ becomes a principal $P(G, \{e\} \times K)$-bundle.*

Proof. The map $P(G, \{e\}\times G) \to V_{\mathfrak{g}}$, $g \mapsto dl_g^{-1}(g')$ is diffeomorphism [7, Lemma 6.2]. Under this identification we have $\Phi_{G/K}(g) = g(1)K$. Moreover the $P(G, \{e\} \times G)$-action on $V_{\mathfrak{g}}$ is identified with the $P(G, \{e\} \times G)$-action on itself by $h \cdot g := gh^{-1}$. Take a subspace $\mathfrak{p}$ of $\mathfrak{g}$ satisfying $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. ($\mathfrak{p}$ need not be $\mathrm{Ad}(K)$-invariant.) By the inverse function theorem there exist open neighborhoods W of $0 \in \mathfrak{p}$ and U of $eK \in G/K$ such that the map $W \to U$, $X \mapsto (\exp X)K$ is a diffeomorphism. For each $g \in P(G, \{e\} \times \pi^{-1}(U))$ we define $X_g \in W$ by $g(1)K = (\exp X_g)K$. Then

$$P(G, \{e\} \times \pi^{-1}(U)) \to U \times P(G, \{e\} \times K), \quad g \mapsto (g(1)K, g(t)^{-1} \exp tX_g)$$

is an equivariant diffeomorphism. This proves the proposition. $\square$

Let G/K be a reductive homogeneous space with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. By (3.4), the differential of the parallel transport map $\Phi_{G/K}$ at $\hat{0} \in V_{\mathfrak{g}}$ is given by

$$(d\Phi_{G/K})_{\hat{0}}(X) = \int_0^1 X(t)_{\mathfrak{p}} dt. \tag{3.7}$$

where $X \in V_{\mathfrak{g}} \cong T_{\hat{0}}V_{\mathfrak{g}}$. Denote by $\hat{\mathfrak{p}}$ the space of constant paths with values in $\mathfrak{p}$ and by $\mathcal{F}_u$ the fiber of $\Phi_{G/K}$ through $u \in V_{\mathfrak{g}}$. From (3.7) we have

$$T_{\hat{0}}V_{\mathfrak{g}} = \hat{\mathfrak{p}} \oplus T_{\hat{0}}(\mathcal{F}_{\hat{0}}).$$

From (3.5) we know that $g*$ maps fibers of $\Phi_{G/K}$ to fibers of $\Phi_{G/K}$. Thus

$$T_{g*\hat{0}}V_{\mathfrak{g}} = \mathrm{Ad}(g)\hat{\mathfrak{p}} \oplus T_{g*\hat{0}}(\mathcal{F}_{g*\hat{0}})$$

where $g \in P(G, G\times K)$. Since $\mathfrak{p}$ is invariant under $\mathrm{Ad}(K)$ the horizontal distribution $\mathcal{H}^{\Phi_{G/K}}$ of $\Phi_{G/K}$ is well-defined by

$$\mathcal{H}^{\Phi_{G/K}}(g * \hat{0}) := \mathrm{Ad}(g)\hat{\mathfrak{p}}$$

where $g \in P(G, G \times K)$.

The following theorem generalizes Theorem 7.4 of [7] in the case of affine symmetric spaces and contains Theorem 3.1 as the case $K = \{e\}$.

Theorem 3.3. *Let G/K be a reductive homogeneous space with decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ and with natural torsion-free connection $\nabla^{G/K}$. Then the parallel transport map $\Phi_{G/K} : (V_{\mathfrak{g}}, D) \to (G/K, \nabla^{G/K})$ is an affine submersion with horizontal distribution $\mathcal{H}^{\Phi_{G/K}}$. Moreover the fundamental tensor $\mathcal{A}^{\Phi_{G/K}}$ restricted to $\mathcal{H}^{\Phi_{G/K}} \times \mathcal{H}^{\Phi_{G/K}}$ is alternating.*

To prove Theorem 3.3 we show the following lemma, which generalizes Proposition 7.5 of [7]. Here l_a denotes the left translation by $a \in G$.

Lemma 3.4. *Let $(G/K, \nabla^{G/K})$ be as in Theorem 3.3 and ∇^G the canonical connection of $G \cong (G \times G)/\Delta G$. Then $\pi : (G, \nabla^G) \to (G/K, \nabla^{G/K})$ is an affine submersion with horizontal distribution $\mathcal{H}^\pi(a) := dl_a(\mathfrak{p})$. Moreover the fundamental tensor $\mathcal{A}^\pi$ restricted to $\mathcal{H}^\pi \times \mathcal{H}^\pi$ is alternating.*

Proof. It is clear that the following diagram commutes:

$$\begin{array}{ccc} (G, \nabla^G) & \xrightarrow{l_a} & (G, \nabla^G) \\ \pi\Big\downarrow & & \pi\Big\downarrow \\ (G/K, \nabla^{G/K}) & \xrightarrow{L_a} & (G/K, \nabla^{G/K}). \end{array} \tag{3.8}$$

Note that l_a and L_a are affine transformations and $\mathcal{H}^\pi$ is invariant under l_a. Thus, to show that $\pi : (G, \nabla^G) \to (G/K, \nabla^{G/K})$ is an affine submersion with horizontal distribution $\mathcal{H}^\pi$, we have only to show

$$(\nabla^G_{X_e}\bar{Z})^{\mathcal{H}} = \overline{(\nabla^{G/K}_{X^*_{eK}}Z)} \tag{3.9}$$

for $X \in \mathfrak{p}$ and $Z \in \mathfrak{X}(G/K)$. Here $\bar{Z}$ denotes the horizontal lift of Z and the superscript $\mathcal{H}$ in the left term denotes the projection onto $\mathcal{H}^\pi(e) = \mathfrak{p}$.

Fix the local trivialization given in Lemma 2.1. Take a basis $\{Y_i\}_{i=1}^n$ of $\mathfrak{p}$ and write $Z|_U = \sum_{i=1}^n \varphi_i Y_i^\#$ where $\varphi_i \in C^\infty(U)$. By (2.3) we have

$$\nabla^{G/K}_{X^\#_{eK}}(Z|_U) = \sum_{i=1}^n \left(\frac{1}{2}\varphi_i(eK)[X, Y_i]_{\mathfrak{p}} + (X^\#_{eK}\varphi_i)(Y_i^\#)_{eK}\right). \tag{3.10}$$

Denote by W_i the horizontal lift of $Y_i^\#$. Note that W_i and Y_i are equal on $\tilde{U}$, but not necessarily equal on the whole $\pi^{-1}(U)$. We have $\bar{Z}|_{\pi^{-1}(U)} = \sum_{i=1}^n(\varphi_i \circ \pi)W_i$. By the Leibniz rule we have

$$\nabla^G_{X_e}(\bar{Z}|_{\pi^{-1}(U)}) = \sum_{i=1}^n \Big(\varphi_i(eK)\nabla^G_{X_e}W_i + (X^\#_{eK}\varphi_i)(Y_i)_e\Big)$$

Denote by $\iota : \tilde{U} \to G$ the inclusion map. Then

$$\nabla^G_{X_e} W_i = \nabla^{\iota^* TG}_{X_e}(\iota^* W_i) = \nabla^{\iota^* TG}_{X_e}(\iota^* Y_i) = \nabla^G_{X_e} Y_i = \frac{1}{2}[X, Y_i]_e$$

by Corollary 2.5. Here ι^* denotes the pullback. Thus (3.9) follows.

By Corollary 2.5 the fundamental tensor $\mathcal{A}^\pi$ is given by $\mathcal{A}^\pi(X,Y) = \frac{1}{2}[X,Y]_{\mathfrak{k}}$ where $X, Y \in \mathfrak{p}$. Thus $\mathcal{A}^\pi$ restricted to $\mathcal{H}^\pi \times \mathcal{H}^\pi$ is alternating. □

Remark 3.5. In [10, p. 3] it is stated that for a G-invariant affine connection $\nabla^{G/K}$ on a reductive homogeneous space G/K there exists a left invariant affine connection ∇^G on G such that $\pi : (G, \nabla^G) \to (G/K, \nabla^{G/K})$ is an affine submersion with horizontal distribution $\mathcal{H}^\pi$. This statement is true. However the argument is not accurate: In lines 2–3 of [10, p. 3] it is claimed that $\tilde{A}$ is the horizontal lift of A_*. This is not true. In fact, A_* cannot be defined globally in general. It is still not true locally: if the restriction of $\tilde{A}$ to K is the horizontal lift of $A_*(eK)$, then $(d\pi)_k(\tilde{A}(k))$ is independent of $k \in K$, which implies that the isotropy representation is trivial.

We are now in a position to prove Theorem 3.3.

Proof of Theorem 3.3. From (3.5) the diagram

$$\begin{array}{ccc} (V_{\mathfrak{g}}, D) & \xrightarrow{g*} & (V_{\mathfrak{g}}, D) \\ {\scriptstyle \Phi_{G/K}}\Big\downarrow & & {\scriptstyle \Phi_{G/K}}\Big\downarrow \\ (G/K, \nabla^{G/K}) & \xrightarrow{L_{g(0)}} & (G/K, \nabla^{G/K}) \end{array}$$

commutes for any $g \in P(G, G \times \{e\})$. Note that $g*$ and $L_{g(0)}$ are affine transformations and $\mathcal{H}^{\Phi_{G/K}}$ is invariant under $g*$. Thus, to show that $\Phi_{G/K}$ is an affine submersion with horizontal distribution $\mathcal{H}^{\Phi_{G/K}}$ we have only to show

$$(D_{\hat{X}} \hat{Z})^{\mathcal{H}} = (\nabla^{G/K}_{X^*_{eK}} Z)\widehat{\ } \tag{3.11}$$

for $X \in \mathfrak{p}$ and $Z \in \mathfrak{X}(G/K)$. Here $\hat{Z}$ denotes the horizontal lift of Z and the superscript $\mathcal{H}$ in the left term denotes the projection onto $\mathcal{H}^{\Phi_{G/K}}(\hat{0}) = \hat{\mathfrak{p}}$.

From (3.2) we have the commutative diagram for any $g \in P(G, G \times \{e\})$:

$$\begin{array}{ccc} (V_{\mathfrak{g}}, D) & \xrightarrow{g*} & (V_{\mathfrak{g}}, D) \\ {\scriptstyle \Phi}\Big\downarrow & & {\scriptstyle \Phi}\Big\downarrow \\ (G, \nabla^{G}) & \xrightarrow{l_{g(0)}} & (G, \nabla^{G}). \end{array}$$

The horizontal distributions $\mathcal{H}^\Phi$ and $\mathcal{H}^\pi$ are invariant under $g*$ and $l_{g(0)}$ respectively. This implies $\hat{Z} = (\bar{Z})\widetilde{\ }$ where the tilde denotes the horizontal lift with respect to Φ. Thus, by Theorem 3.1 and Lemma 3.4 we have

$$(D_{\hat{X}} \hat{Z})^{\mathcal{H}} = (\nabla^G_{X_e} \bar{Z})\widetilde{\ }_{\mathfrak{p}} = (\nabla^{G/K}_{X^*_{eK}} Z)\widehat{\ }$$

which proves (3.11). Since $T_{\hat{0}}(\mathcal{F}_{\hat{0}}) = T_{\hat{0}}(F_{\hat{0}}) \oplus \hat{\mathfrak{k}}$ and Φ is an affine submersion with horizontal distribution $\mathcal{H}^\Phi$ we have

$$\mathcal{A}^{\Phi_{G/K}}(\hat{X}, \hat{Y}) = \mathcal{A}^{\Phi}(\hat{X}, \hat{Y}) \oplus \mathcal{A}^\pi(X, Y)$$

for $X, Y \in \mathfrak{p}$. Hence Theorem 3.1 and Lemma 3.4 imply that $\mathcal{A}^{\Phi_{G/K}}$ restricted to $\mathcal{H}^{\Phi_{G/K}} \times \mathcal{H}^{\Phi_{G/K}}$ is alternating. This completes the proof. □

4. Weakly reflective submanifolds

In this section we consider the relation between weakly reflective submanifolds and the parallel transport map over a reductive homogeneous space.

Recall that a submanifold M of an affine manifold (N, ∇^N) with transversal bundle $\mathcal{W}$ is called *weakly reflective* [2, 7] if for each $(p, \xi) \in \mathcal{W}$ there exists an affine transformation of (N, ∇^N) satisfying

$$\nu(M) = M, \qquad \nu(p) = p, \qquad d\nu(\mathcal{W}) = \mathcal{W}, \qquad d\nu(\xi) = -\xi.$$

Such a ν is called a *weak reflection* of M with respect to (p, ξ). If we can choose every ν from a particular group S, then M is said to be *S-weakly reflective.*

Owing to Theorem 3.3, we can use the general result on the Fredholm property of affine immersions lifted by some affine submersion [7, Proposition 1.2]. In particular, it turns out that each fiber of the parallel transport map $\Phi_{G/K}$ is an affine Fredholm submanifold of $(V_{\mathfrak{g}}, D)$ of Fredholm index 0. On the other hand, it was shown that each fiber of $\Phi_{G/K}$ is a weakly reflective submanifold [7, Theorem 1.3]. Combining these results, we get the following result which generalizes Corollary 1.4 of [7].

Theorem 4.1. *Let G/K be a reductive homogeneous space with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then each fiber of the parallel transport map $\Phi_{G/K} : V_{\mathfrak{g}} \to G/K$ is a weakly reflective Fredholm submanifold of $(V_{\mathfrak{g}}, D)$ of Fredholm index 0 where its transversal bundle is defined by the restriction of $\mathcal{H}^{\Phi_{G/K}}$ to it.*

In [7] a characterization of weakly reflective submanifolds in affine symmetric spaces was shown [7, Theorem 8.6]. Since its proof depends on the structure of a symmetric space, it cannot be generalized easily to the case of reductive homogeneous spaces. Nevertheless, we can extend it as follows:

Proposition 4.2. *Let G/K be a reductive homogeneous space with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and with natural torsion-free connection $\nabla^{G/K}$. Let M be a submanifold of $(G/K, \nabla^{G/K})$ with transversal bundle $\mathcal{W}$. Denote by $\bar{W}$ (resp. $\hat{W}$) the horizontal lift of $\mathcal{W}$ with respect to π (resp. $\Phi_{G/K}$). Then the following conditions are equivalent for any normal subgroup S of G:*

- (i) *M is an S-weakly reflective submanifold of $(G/K, \nabla^{G/K})$ with transversal bundle $\mathcal{W}$.*
- (ii) *$\pi^{-1}(M)$ is an $(S \times S_{eK})$-weakly reflective submanifold of (G, ∇^G) with transversal bundle $\bar{\mathcal{W}}$, where $S_{eK} := S \cap K$.*
- (iii) *$\Phi_{G/K}^{-1}(M)$ is a $P(S, S \times S_{eK})$-weakly reflective Fredholm submanifold of $(V_{\mathfrak{g}}, D)$ of Fredholm index 0 with transversal bundle $\hat{\mathcal{W}}$.*

Remark 4.3. In the above proposition, S must be contained in G and thus in the identity component of the affine transformation group of $(G/K, \nabla^{G/K})$. Hence the proposition does not characterize all the weakly reflective submanifolds in G/K, unlike in [7, Theorem 8.6]. However, we do not have to assume that G/K is a symmetric space or the pair (G, K) is effective, unlike in [7, Theorem 8.6].

In Proposition 4.2 the Fredholm property of (iii) is a consequence of Theorem 3.3 together with [7, Proposition 1.2]. The equivalence of weakly reflective properties can be proven almost similarly to the case of [7, Theorem 8.6]. The only difference lies in the argument to show that the lifted weak reflection leaves the horizontal distribution invariant: In the proof of [7, Theorem 8.6] this was shown by using the structure of symmetric spaces [7, Lemma 8.10]. In the case of Proposition 4.2 it follows easily from the condition $S \subset G$.

Finally we remark that Proposition 4.2 generalizes some results in the author's previous paper [5] in the Riemannian case. Let G be a connected compact Lie group with a bi-invariant Riemannian metric ρ. Denote by $\langle \cdot, \cdot \rangle^{\rho}$ the corresponding $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ and by $\langle \cdot, \cdot \rangle^{\rho}_{L^2}$ the L^2-inner product on $V_{\mathfrak{g}}$. Then $\Phi : (V_{\mathfrak{g}}, \langle \cdot, \cdot \rangle^{\rho}_{L^2}) \to (G, \rho)$ is a Riemannian submersion [12, Theorem 4.5]. Let K be a closed subgroup of G with Lie algebra $\mathfrak{k}$. Denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Restricting $\langle \cdot, \cdot \rangle^{\rho}$ to $\mathfrak{p}$ we equip G/K with the G-invariant Riemannian metric $\rho^{G/K}$. Then $(G/K, \rho^{G/K})$ is a compact normal homogeneous space. The Levi-Civita connection coincides with the natural torsion-free connection [3, p. 203]. Since $\pi : (G, \rho) \to (G/K, \rho^{G/K})$ is a Riemannian submersion, $\Phi_{G/K} : (V_{\mathfrak{g}}, \langle \cdot, \cdot \rangle^{\rho}_{L^2}) \to (G/K, \rho^{G/K})$ is also a Riemannian submersion. From Proposition 4.2 we get:

Corollary 4.4. *Let M be a submanifold of a compact normal homogeneous space $(G/K, \rho^{G/K})$. Then the following conditions are equivalent for any normal subgroup S of G:*

(i) *M is an S-weakly reflective submanifold of $(G/K, \rho^{G/K})$,*
(ii) *$\pi^{-1}(M)$ is an $(S \times S_{eK})$-weakly reflective submanifold of (G, ρ),*
(iii) *$\Phi^{-1}_{G/K}(M)$ is a $P(S, S \times S_{eK})$-weakly reflective PF submanifold of $(V_{\mathfrak{g}}, \langle \cdot, \cdot \rangle^{\rho}_{L^2})$.*

Considering the case $G = S$ we have:

Corollary 4.5 ([5, Theorem 6 (ii)])**.** *Let M be a submanifold of a compact normal homogeneous space $(G/K, \rho^{G/K})$. Then the following conditions are equivalent:*

(i) *M is a G-weakly reflective submanifold of $(G/K, \rho^{G/K})$,*
(ii) *$\pi^{-1}(M)$ is a $(G \times K)$-weakly reflective submanifold of (G, ρ),*
(iii) *$\Phi^{-1}_{G/K}(M)$ is a $P(G, G \times K)$-weakly reflective PF submanifold of $(V_{\mathfrak{g}}, \langle \cdot, \cdot \rangle^{\rho}_{L^2})$.*

Consider the group case $G \cong (G \times G)/\Delta G$. By using the canonical isomorphism between path spaces [6] we see that Corollary 4.5 implies:

Corollary 4.6 ([5, Theorem 6 (i)])**.** *Let M be a submanifold of a connected compact Lie group G with a bi-invariant Riemannian metric ρ. Then the following conditions are equivalent:*

(i) *M is a $(G \times G)$-weakly reflective submanifold of (G, ρ),*
(ii) *$\Phi^{-1}(M)$ is a $\mathcal{G}$-weakly reflective PF submanifold of $(V_{\mathfrak{g}}, \langle \cdot, \cdot \rangle^{\rho}_{L^2})$.*

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