GEODESIC ORBIT PSEUDO-RIEMANNIAN H-TYPE NILMANIFOLDS: CASE OF MINIMAL ADMISSIBLE CLIFFORD MODULES

KENRO FURUTANI, IRINA MARKINA, AND YURII NIKONOROV

Abstract. We investigate the geodesic orbit property of pseudo-Riemannian nilmanifolds, specifically those known in the literature as pseudo H-type Lie groups – i.e., 2-step nilpotent Lie groups of Heisenberg type equipped with a left invariant pseudo-Riemannian metric. The study of homogeneous geodesics on Riemannian H-type Lie groups was completed by C. Riehm in 1984. In this work, we extend these results to the pseudo-Riemannian H-type Lie groups and provide a complete characterization of the geodesic orbit property for the case where the underlying Lie algebras are constructed from the admissible Clifford modules of minimal dimension.

Contents

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1. Introduction and main results

A Riemannian manifold (M, g) is called *a manifold with homogeneous geodesics* or *a geodesic orbit manifold* (shortly, *GO-manifold*) if any geodesic γ of M is an orbit of a 1-parameter subgroup of the full isometry group of (M, g). A Riemannian manifold $(M = G/H, g)$, where H is a compact subgroup of a Lie group G and g is a G-invariant Riemannian metric, is called *a geodesic orbit space* (shortly, *GO-space*) if any geodesic γ of M is an orbit of a 1-parameter subgroup of the group G. Hence, a Riemannian manifold (M, g) is a geodesic orbit Riemannian manifold, if it is a geodesic orbit space with respect to its full connected isometry group. This terminology was introduced in [39] by O. Kowalski and L. Vanhecke, who initiated a systematic study of such spaces. In the same paper, O. Kowalski and L. Vanhecke classified all geodesic orbit Riemannian manifolds of dimension ≤ 6.

We refer to [39], [2], [18], [45], and [9] for expositions on general properties of geodesic orbit Riemannian manifolds and historical surveys. One can find many interesting results about geodesic orbit Riemannian spaces and its subclasses in [32, 57, 54, 33, 11, 12, 46, 44], and in the references therein. It should be noted that symmetric spaces, weakly symmetric spaces, naturally reductive homogeneous spaces, normal homogeneous spaces, generalized normal homogeneous spaces (but not only) are subclasses of the class of geodesic orbit Riemannian spaces.

This paper is devoted to the study of one special and important class of geodesic orbit pseudo-Riemannian spaces, namely, pseudo H-type nilpotent Lie groups. On the other hand, many of the results obtained below can also be used for more general classes of geodesic orbit pseudo-Riemannian nilmanifolds.

Some important results on geodesic orbit pseudo-Riemannian spaces were obtained in [19, 20, 21, 5, 6, 56, 14, 43, 13]. Here we recall some important results related to homogeneous geodesic of pseudo-Riemannian manifolds. We note that weakly symmetric pseudo-Riemannian spaces and naturally reductive homogeneous pseudo-Riemannian spaces are important subclasses of geodesic orbit pseudo-Riemannian spaces [48, 56].

Definition 1. A pseudo-Riemannian homogeneous reductive manifold $(G/H, g)$ is called geodesic orbit (GO) if a geodesic through the point eH with the initial vector ξ is an orbit of some 1-parameter isometry group of $(G/H, g)$ for any initial vector.

We recall the following useful criterion for the property to be a geodesic orbit pseudo-Riemannian manifold.

Proposition 1 (Geodesic Lemma [19])**.** *Let $(M = G/H, g)$ be a homogeneous reductive pseudo-Riemannian manifold, with the corresponding reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then M is a G-geodesic orbit space if and only if, for any $T \in \mathfrak{m}$, there exist $P = P(T) \in \mathfrak{h}$ and $k = k(T) \in \mathbb{R}$ such that if $Q \in \mathfrak{m}$ then*

$$\langle [T + P, Q]_{\mathfrak{m}}, T\rangle = k \, \langle T, Q\rangle, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ *denotes the inner product on* $\mathfrak{m}$ *defined by* g*, and the subscript* $\mathfrak{m}$ *in* (1) *means taking the* $\mathfrak{m}$*-component in* $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$*. Note that* $k(T)=0$ *unless* T *is a null vector* (*it suffices to substitute* $Q=T$ *in* (1)).

We recall also the following definition

Definition 2. Let $(M=G/H,g)$ be a homogeneous reductive pseudo-Riemannian manifold. Then M is said to be a naturally reductive if there is a reductive decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ such that

$$\langle [T,Q]_{\mathfrak{m}},R\rangle=\langle Q,[T,R]_m\rangle \tag{2}$$

for the corresponding scalar product and any $T,Q,R\in\mathfrak{m}$.

All naturally reductive H-type Lie groups, endowed with left invariant Riemannian metrics were classified by A. Kaplan in [36]. In the same paper, the first examples of geodesic orbit but not naturally reductive Riemannian manifolds were obtained: these are H-type groups with 2-dimensional center (a minimal dimension of such groups is 6). The complete classification of geodesic orbit H-type groups with left invariant Riemannian metrics was obtained by C. Riehm in [50]:

Theorem 1 ([50])**.** *Let* N *be* H*-type group* (*supplied with a left invariant Riemannian metric*) *with the* H*-type algebra* $\mathfrak{n}=\mathfrak{z}\oplus\mathfrak{v}$*,* $m=\dim(\mathfrak{z})$*,* $n=\dim(\mathfrak{v})$*. Then* N *is geodesic orbit if and only if one of the following three conditions holds:*

1) $m=1,2,3$ *and* n *is any possible;*
2) $m=5,6$ *and* $n=8$*;*
3) $m=7$*,* $n=8,16,24$ *and* $\mathfrak{v}$ *is an isotypic Clifford module* (*in this case it is equivalent to the following property: if* $Z_1,Z_2,\dots,Z_7$ *is an orthonormal basis of* $\mathfrak{z}$*, the linear transformation* $X\mapsto J_{Z_1}J_{Z_2}\cdots J_{Z_7}(X)$ *of* $\mathfrak{v}$ *is either* Id *or* $-\mathrm{Id}$).

Moreover, N *is naturally reductive if and only if* $m=1$ *or* $m=3$*.*

Our main result is as follows.

Theorem 2. *Let* $N_{r,s}$ *be an* H*-type group, where* (r,s)*,* $s\geq 1$*, is the signature of the left invariants pseudo-Riemannian metric restricted to the centre of the group. Let* $\mathfrak{n}_{r,s}=\mathfrak{z}\oplus\mathfrak{v}$ *be the Lie algebra of* $N_{r,s}$*, where* $\mathfrak{z}$ *is the centre and* $\mathfrak{v}$ *is a minimal admissible module for the Clifford algebra* $\mathrm{Cl}(\mathbb{R}^{r,s})$*. Then the following four assertions hold:*

1) $N_{r,s}$ *is naturally reductive* (*hence, geodesic orbit*) *if and only if* $(r,s)\in\{(0,1),(1,2)\}$*;*
2) *If* $N_{r,s}$ *is geodesic orbit but not naturally reductive, then* $(r,s)=(3,4)$*;*
3) $N_{3,4}$ *is a geodesic orbit pseudo-Riemannian manifold;*
4) $N_{r,s}$ *with* $(r,s)\notin\{(0,1),(1,2),(3,4)\}$ *is not geodesic orbit pseudo-Riemannian manifold.*

It is clear that the last assertion of this theorem easily follows from the three first assertions.

The paper is organized as follows. In Section 2, we recall some important results on pseudo-Riemannian geodesic orbit metrics on nilpotent Lie groups. The main role here is played by the notion of the transitive normalizer condition, which Riemannian version was used by C. Gordon in order to study geodesic orbit Riemannian metric on nilpotent Lie groups. In Section 3, we recall some important properties of pseudo H-type nilpotent Lie groups, their isometry groups and automorphism groups. In Section 4, we check the geodesic orbit property for pseudo-Riemannian H-type groups with small dimension of the center. More exactly, the groups $N_{0,1}$, $N_{1,1}$, $N_{0,2}$, $N_{1,2}$, $N_{2,1}$, and $N_{0,3}$ are studied, after which the classification of naturally reductive pseudo H-type nilpotent Lie groups is obtained (see Proposition 7). Section 5 is devoted to the description of some important properties of geodesic orbit pseudo-Riemannian manifolds. As in the case of Riemannian manifolds, it is proved that any geodesically complete totally geodesic submanifold of a given geodesic orbit pseudo-Riemannian manifold is geodesic orbit itself (Theorem 4). This result is further refined for the case of two-step nilpotent pseudo-Riemannian groups (Theorem 5 and Corollary 3). In Section 6, we obtain some general results on the geodesic orbit property for pseudo H-type Riemannian manifolds, that allow us to consistently check all pseudo-Riemannian H-type manifolds, except $N_{3,4}$, for the property to be geodesic orbit. Finally, in Section 7, we prove that the 15-dimensional pseudo H-type nilmanifold $N_{3,4}$ is geodesic orbit (Theorem 14).

2. Auxiliary results

Here we recall some important results related to 2-step nilpotent groups supplied by left-invariant pseudo-Riemannian metrics and their isometry groups. We call such groups 2-step pseudo-Riemannian nilmanifolds. Finally, we formulate some useful statements about 2-step pseudo-Riemannian nilmanifolds that are geodesic orbit.

Let (N, g) be a 2-step pseudo-Riemannian nilmanifold with the Lie algebra $\mathfrak{n}$ and $\langle \cdot\,, \cdot \rangle$ the scalar product (a symmetric nondegenerate bilinear form) generating the pseudo-Riemannian left invariant metric g. If the center $\mathfrak{z}$ of $\mathfrak{n}$ is non-degenerate with respect to $\langle \cdot\,, \cdot \rangle$, then we write $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{v} = \mathfrak{z}^{\perp}$ relative to $\langle \cdot\,, \cdot \rangle$. In this case, $\mathfrak{v}$ is also non-degenerate, see [40, Lemma 2.60].

Whenever (N, g) is connected simply connected, we do not distinguish between the group of automorphisms of the nilmanifold (N, g) and of its Lie algebra $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$. Note that each skew-symmetric derivation of $\mathfrak{n}$ leaves each of $\mathfrak{v}$ and $\mathfrak{z}$ invariant. For any $Z \in \mathfrak{z}$, we consider the operator

$$J_Z : \mathfrak{v} \to \mathfrak{v}, \quad \text{such that} \ \ \langle J_Z(X), Y \rangle = \langle [X, Y], Z \rangle, \quad X, Y \in \mathfrak{v}. \tag{3}$$

The map J_Z is skew-symmetric and $J_Z(Y) = (\operatorname{ad} Y)'(Z)$, where $(\operatorname{ad} Y)'$ is adjoint to $\operatorname{ad} Y$ with respect to $\langle \cdot\,, \cdot \rangle$. The map $J\colon Z \to \mathfrak{so}(\mathfrak{v}, \langle \cdot\,, \cdot \rangle_{\mathfrak{v}})$, sending $Z \mapsto J_Z$ is linear. We denote $\mathbf{V} = J(\mathfrak{z})$ the linear subspace in $\mathfrak{so}(\mathfrak{v}, \langle \cdot\,, \cdot \rangle_{\mathfrak{v}})$. The group of isometries of the nilmanifold (N, g) is given by

$$H = \{(\varphi, \psi) \in O(\mathfrak{z}, \langle \cdot\,, \cdot \rangle_{\mathfrak{z}}) \times O(\mathfrak{v}, \langle \cdot\,, \cdot \rangle_{\mathfrak{v}}) \,|\, \psi J_Z \psi^{-1} = J_{\varphi(Z)},\ Z \in \mathfrak{z}\}, \tag{4}$$

while its Lie algebra is

$$\mathfrak{h} = \operatorname{Der}(\mathfrak{n}) \cap \mathfrak{so}(\mathfrak{n}, \langle \cdot\,, \cdot \rangle) = \{D = (C, A) \in \mathfrak{so}(\mathfrak{z}, \langle \cdot\,, \cdot \rangle_{\mathfrak{z}}) \times \mathfrak{so}(\mathfrak{v}, \langle \cdot\,, \cdot \rangle_{\mathfrak{v}}) \,|\, [A, J_Z] = J_{C(Z)},\ Z \in \mathfrak{z}\}. \tag{5}$$

The next result is well known, see e.g. Corollary 3.5 in [10] or Proposition 2.3 in [48].

Proposition 2. *Let (N, g) be a pseudo-Riemannian nilmanifold with non-degenerate center. Then the connected isometry group of (N, g) is $N \rtimes H$, where N is the set of left translations by elements of N and the isotropy subgroup H is given by the isometric automorphisms (4) with Lie algebra $\mathfrak{h}$ as in (5).*

In this case, the isotropy group of (N,g) at the identity element e is exactly H with the embedding $a \in H \mapsto (e,a) \in N \rtimes H$.

Applying Proposition 1 we note the following. If a nilmanifold (N,g) is geodesic orbit, then for any $X \in \mathfrak{v}$ and any $Z \in \mathfrak{z}$, such that $X+Z$ is not a null vector, there exists $D \in \mathfrak{h}$ such that

$$\langle [X+Z+D, \widetilde{X}+\widetilde{Z}], X+Z\rangle = 0$$

for all $\widetilde{X} \in \mathfrak{v}$ and all $\widetilde{Z} \in \mathfrak{z}$. It is easy to see that

$$[X+Z+D, \widetilde{X}+\widetilde{Z}] = [X,\widetilde{X}] + [D,\widetilde{X}] + [D,\widetilde{Z}],$$

where $[D,\widetilde{X}] \in \mathfrak{v}$ and $[X,\widetilde{X}]+[D,\widetilde{Z}] \in \mathfrak{z}$. Hence,

$$0 = \langle [D,\widetilde{X}], X\rangle + \langle [X,\widetilde{X}]+[D,\widetilde{Z}], Z\rangle = -\langle \widetilde{X}, [D,X]\rangle + \langle J_Z X, \widetilde{X}\rangle - \langle \widetilde{Z}, [D,Z]\rangle.$$

Since $\widetilde{X} \in \mathfrak{v}$ and $\widetilde{Z} \in \mathfrak{z}$ are arbitrary, then

$$[D,X] = J_Z X, \qquad [D,Z] = 0. \tag{6}$$

Note, that $[D,Z]=0$ implies $[D,J_Z]=0$ according to (5). If $X+Z$ is a null vector, then we can approach it by non-null vectors of the type $\widetilde{X}+Z$, where $\widetilde{X} \in \mathfrak{v}$. Therefore, (6) is valid also for null vectors, see also Proposition 5.

On the other hand, equalities (6) imply Proposition 1 for all $X \in \mathfrak{v}$ and $Z \in \mathfrak{z}$. These observations allow us to rewrite Proposition 1 for 2-step nilpotent pseudo-Riemannian groups in the spirit of work [32], where this idea was used for Riemannian metrics on nilpotent Lie groups.

Proposition 3. *In the above notations, (N,g) is a geodesic orbit pseudo-Riemannian nilmanifold if and only if for any $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$ there is $D \in \mathfrak{h}$ such that $[D,Z] = D(Z) = 0$, $[D,X] = D(X) = J_Z(X)$.*

Consider an action ρ of the isotropy algebra $\mathfrak{h}$ in (5) on $\mathfrak{so}(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$; that is $\rho\colon \mathfrak{h} \to \mathfrak{so}(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$. We may reformulate the condition of Proposition 3 as follows. We know that $\mathbf{V} = J(\mathfrak{z})$ is a linear subspace in $\mathfrak{so}(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$. Further, for every $D \in \mathfrak{h}$ and $Z \in \mathfrak{z}$ we get $J_{[D,Z]} = [\rho(D), J_Z]$ (it easy follows from the condition on D to be a skew-symmetric derivation), hence, the subspace $\mathbf{V} = J(\mathfrak{z})$ is normalized by the subalgebra $\mathbf{N} := \rho(\mathfrak{h})$ in $\mathfrak{so}(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$ by the fact that

$$[\rho(D), J_Z] = J_{[D,Z]} \in \mathbf{V}.$$

The equality $J_{[D,Z]} = [\rho(D), J_Z]$ implies that the representation $\rho\colon \mathfrak{h} \to \mathfrak{so}(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$ is faithful (otherwise, some non-trivial $D \in \mathfrak{h}$ acts trivially both on $\mathfrak{v}$ and on $\mathfrak{z}$, hence, on $\mathfrak{n}$). Moreover, any element of the normalizer $\mathbf{N}$ of $\mathbf{V} = J(\mathfrak{z})$ in $\mathfrak{so}(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$ can be considered as an image of some element $D \in \mathfrak{h}$ under the map $\rho\colon \mathfrak{h} \to \mathfrak{so}(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$, which follows from (5). Therefore, we have the following statement.

Corollary 1. *If (N,g) is a geodesic orbit pseudo-Riemannian nilmanifold, then the Lie algebra $\mathfrak{h}$ is isomorphic to the normalizer $\mathbf{N}$ of $\mathbf{V} = J(\mathfrak{z})$ in $\mathfrak{so}(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$ under the representation $\rho\colon \mathfrak{h} \to \mathfrak{so}(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$.*

Therefore, if (N,g) is a geodesic orbit pseudo-Riemannian nilmanifold, then we have a Lie subalgebra $\mathbf{N} \subset \mathfrak{so}(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$ and an $\mathrm{ad}(\mathbf{N})$-invariant module $\mathbf{V}$ in $\mathfrak{so}(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}}))$, such that for every $X \in \mathfrak{v}$ and $W \in \mathbf{V}$ there is $B \in \mathbf{N}$ with the following properties: $[B,W]=0$ and $B(X) = W(X)$.

Remark 1. Let $\mathfrak{h}$ be as in (5) and $\mathbf{V} = J(\mathfrak{z})$. Then every element of $\mathfrak{h}$ is determined by a skew-symmetric operator $A \in \mathfrak{so}(\mathfrak{v}, \langle \cdot\,, \cdot \rangle_{\mathfrak{v}})$ with the property $[A, \mathbf{V}] \subset \mathbf{V}$ (i.e., A is from the normalizer of $\mathbf{V}$ in $\mathfrak{so}(\mathfrak{v}, \langle \cdot\,, \cdot \rangle_{\mathfrak{v}})$). In this case the operator $C \in \mathfrak{so}(\mathfrak{z}, \langle \cdot\,, \cdot \rangle_{\mathfrak{z}})$ can be recovered from the equality $J_{C(Z)} = [A, J_Z]$ for all $Z \in \mathfrak{z}$.

A special case of geodesic orbit pseudo-Riemannian spaces are naturally reductive homogeneous pseudo-Riemannian spaces (Definition 2), see also [48] and references therein. The naturally reductive Riemannian homogeneous spaces are generalizations of normal homogeneous Riemannian spaces and symmetric spaces, see, e.g., [17], [38], [31], [1], [52], [53]. It should be noted that a complement $\mathfrak{m}$ in the definition of naturally reductive (pseudo) Riemannian manifold is not unique in general. For instance, any invariant Riemannian metric on the Ledger–Obata space $(F \times F \times F)/\operatorname{diag}(F)$, where F is any simple compact Lie group, is naturally reductive with respect a suitable reductive complement [42]. The following result gives us a useful criterion of a 2-step Lie group with pseudo-Riemannian left-invariant metric to be naturally reductive.

Theorem 3. [48, Theorem 3.2] *Let (N, g) denote a 2-step pseudo-Riemannian nilmanifold with a non-degenerate center. Assume that the map $J\colon \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ is injective, see (3). Then the metric is naturally reductive with respect to $G = N \rtimes H$ (see Proposition 2), if and only if*

(i) $\mathbf{V} = J(\mathfrak{z})$ *is a Lie subalgebra of* $\mathfrak{so}(\mathfrak{v})$ *and*
(ii) $[J(Z_1), J(Z_2)] = J\big(\tau_{Z_1}(Z_2)\big)$ *where* $\tau_{Z_1} \in \mathfrak{so}(\mathfrak{z})$ *for any* $Z_1 \in \mathfrak{z}$.

Since the map J is supposed to be injective, the map τ can be easily recovered from (ii).

3. Pseudo H-type nilpotent Lie groups

Let (N, g) be a 2-step pseudo-Riemannian nilmanifold and $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ be its Lie algebra endowed with a scalar product $\langle .\,, . \rangle$ making the center non-degenerate. We identify $(\mathfrak{z}, \langle .\,, . \rangle_{\mathfrak{z}})$ with the pseudo Euclidean vector space $\mathbb{R}^{r,s} = (\mathbb{R}^{r+s}, \langle .\,, . \rangle_{r,s})$, where

$$\langle Z, W \rangle_{r,s} = \sum_{i=1}^{r} z_i\, w_i - \sum_{j=1}^{s} z_{r+j}\, w_{r+j}, \quad Z, W \in \mathbb{R}^{r+s}.$$

If the linear operator $J\colon \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ is defined by

$$\langle J_Z(X), Y \rangle_{\mathfrak{v}} = \langle [X, Y], Z \rangle_{r,s}, \quad X, Y \in \mathfrak{v},\ Z \in \mathfrak{z} = \mathbb{R}^{r,s}, \tag{7}$$

and satisfies $J_Z^2(X) = -\langle Z, Z \rangle_{r,s} X$ for any $Z \in \mathbb{R}^{r,s}$ and all $X \in \mathfrak{v}$, then $\mathfrak{n} = \mathfrak{n}_{r,s}$ is called the pseudo H(eisenberg)-type Lie algebra. It is easy to check that this definition implies

$$\langle J_Z(X), J_W(X) \rangle_{\mathfrak{v}} = \langle Z, W \rangle_{r,s} \langle X, X \rangle_{\mathfrak{v}}. \tag{8}$$

We denote by $N_{r,s}$ the connected simply connected Lie group, whose Lie algebra is the pseudo H-type Lie algebra $\mathfrak{n}_{r,s}$. The H-type Lie algebras $N_{r,0}$ with a positive definite scalar product were introduced in [34] and with an arbitrary indefinite scalar product in [15], see also [29].

These Lie algebras are related to the representations of the Clifford algebras in the following way. Let $J\colon \mathrm{Cl}(\mathbb{R}^{r,s}) \to \mathrm{End}(\mathfrak{v})$ be a representation of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{r,s})$ generated by the pseudo Euclidean vector space $\mathbb{R}^{r,s}$. If there is a scalar product

$\langle\cdot\,,\cdot\rangle_{\mathfrak{v}}$ on the representation space $\mathfrak{v}$ (Clifford module) such that the linear map J_Z is skew-symmetric for any $Z\in\mathbb{R}^{r,s}$; that is

$$\langle J_Z(X),Y\rangle_{\mathfrak{v}}=-\langle X,J_Z(Y)\rangle_{\mathfrak{v}},\quad Z\in\mathbb{R}^{r,s},\quad X,Y\in\mathfrak{v},$$

then we get a pseudo H-type Lie algebra with the commutators defined in (7), see details in [35, 15, 37, 30, 24, 25, 26, 27]. The scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{v}}$ in this case is called admissible and $\mathfrak{v}$ is called admissible (Clifford) module.

It is important that, see e.g. [15], or [27, Propososition 2.2.2]) the signature of the scalar product space $(\mathfrak{v},\langle\cdot\,,\cdot\rangle_{\mathfrak{v}})$ is neutral and $(\mathfrak{v},\langle\cdot\,,\cdot\rangle_{\mathfrak{v}})$ is isometric to $\mathbb{R}^{l,l}$ for some $l\in\mathbb{N}$ if $s>0$, whereas the corresponding signature is either $(l,0)$ or $(0,l)$ for some $l\in 2\mathbb{N}$ if $s=0$.

We use the symbol ^{t} for the transposition according to the standard Euclidean product on $\mathfrak{v}=\mathbb{R}^{2l}$. On the other hand, we use the symbol $^\tau$ for the transposition according to scalar product $\langle\cdot\,,\cdot\rangle_{l,l}$ on $\mathfrak{v}=\mathbb{R}^{l,l}$ or $\langle\cdot\,,\cdot\rangle_{r,s}$ on $\mathfrak{z}=\mathbb{R}^{r,s}$. In particular, for any operator A on $\mathfrak{v}$, we get $\langle A(X),Y\rangle=\langle X,A^\tau(Y)\rangle$, $X,Y\in\mathfrak{v}$, and $J_Z^\tau=-J_Z$ for any $Z\in\mathfrak{z}$. If $\eta=\operatorname{diag}(\mathrm{Id}_l,-\mathrm{Id}_l)$, then $J_Z^\tau=\eta J_Z^t\eta$ and $A^\tau=\eta A^t\eta$.

Thus we use the identification $\mathbf{V}=J(\mathfrak{z})\subset\mathfrak{so}(\mathfrak{v})=\mathfrak{so}(l,l)$. Recall that $\dim\mathbf{V}=\dim\mathfrak{z}=r+s$. In the next proposition we collect some useful results.

Proposition 4. *Let $\mathfrak{n}_{r,s}$ be a pseudo H-type Lie algebra, $\mathbf{N}$ and $\mathbf{Z}$ the normalizer and the centralizer of $\mathbf{V}=J(\mathfrak{z})$ in $\mathfrak{so}(l,l)$. Then we have the following properties:*

1. *$[\mathbf{V},\mathbf{V}]$ and $\mathbf{L}:=[\mathbf{V},\mathbf{V}]+\mathbf{V}$ are Lie subalgebras in $\mathfrak{so}(l,l)$;*
2. *the Lie algebra $[\mathbf{V},\mathbf{V}]$ is isomorphic to $\mathfrak{so}(r,s)$;*
3. *$[\mathbf{V},\mathbf{V}]\subset\mathbf{N}$ and $\dim\mathbf{N}\geq(r+s)(r+s-1)/2$;*
4. *$\Big(\mathbf{L},[\mathbf{V},\mathbf{V}]\Big)$ is a symmetric pair, i.e., $\mathbf{V}$ is a Lie triple system;*
5. *the Lie algebra $\mathbf{L}$ is commutative if $(r,s)\in\{(1,0),(0,1)\}$, it is simple if $(r,s)\notin\{(1,0),(0,1),(3,0),(1,2)\}$, and it is semisimple if $(r,s)\in\{(3,0),(1,2)\}$;*
6. *$\mathbf{N}=[\mathbf{V},\mathbf{V}]\oplus\mathbf{Z}$ (a direct sum of Lie algebras).*

Proof. We give an outline of the proof. For every pair of orthogonal vectors $Z',Z''\in\mathfrak{z}$, the map $\Phi_{Z',Z''}$, defined by

$$\Phi_{Z',Z''}(X+Z)=J_{Z'}J_{Z''}(X)+2\langle Z',Z\rangle Z''-2\langle Z'',Z\rangle Z',\quad Z\in\mathfrak{z},\quad X\in\mathfrak{v},\tag{9}$$

is a skew-symmetric derivation of $(\mathfrak{n}_{r,s},\langle\cdot\,,\cdot\rangle)$, see e.g. Lemma 2.2 in [16] or [49]. Since $[J_{Z'},J_{Z''}]=2J_{Z'}J_{Z''}$, then $[\mathbf{V},\mathbf{V}]\subset\mathbf{N}$, due to the fact that $\Phi_{Z',Z''}$ and, hence, $[J_{Z'},J_{Z''}]$ is in the isotropy Lie algebra $\mathfrak{h}$, see (5) and Proposition 2. Moreover, $[\mathbf{V},\mathbf{V}]$ is a Lie subalgebra in $\mathbf{N}$, that is isomorphic to $\mathfrak{so}(r,s)$. Indeed, by Lemma 5.1 in [4], we can choose an orthonormal basis Z_i, $i=1,\dots,r+s$, for $\mathfrak{z}=\mathbb{R}^{r,s}$, such that $\mathbf{V}=J(\mathfrak{z})=J(\mathbb{R}^{r,s})\subset\mathfrak{so}(l,l)$ has a basis of the following type: $\{J_{Z_i}\}$, $i=1,\dots,r+s$, while $\{J_{Z_j}J_{Z_k}\}$, $j,k=1,\dots,r+s$, $j<k$, constitute a basis in $[\mathbf{V},\mathbf{V}]\subset\mathbf{N}$. Therefore, $[\mathbf{V},\mathbf{V}]$ is isomorphic to $\mathfrak{so}(r,s)$, then $\dim\mathbf{N}\geq\dim[\mathbf{V},\mathbf{V}]=\dim\mathfrak{so}(r,s)=(r+s)(r+s-1)/2$. This result also follows from [27, Proposition 3.2.4].

One can easily check that $[[\mathbf{V},\mathbf{V}],\mathbf{V}]\subset\mathbf{V}$, which means that $[\mathbf{V},\mathbf{V}]\subset\mathbf{N}$. Moreover, $\mathbf{L}=[\mathbf{V},\mathbf{V}]+\mathbf{V}$ is a Lie algebra and $\mathbf{V}$ is a Lie triple system (see Proposition 5.2 in [4]).

Finally, the Lie algebra $\mathbf{L}$ is commutative and 1-dimensional if $(r,s)\in\{(1,0),(0,1)\}$, it is simple if $(r,s)\notin\{(1,0),(0,1),(3,0),(1,2)\}$, and it is semisimple if $(r,s)\in\{(3,0),(1,2)\}$ by Theorem 5.1 in [4].

It is known that each $B \in \mathbf{N}$ (i.e., each skew-symmetric derivation of $\mathfrak{z}$) decomposes as a sum B_0+B_1, where $B_0 \in \mathbf{Z}$ and $B_1 \in [\mathbf{V}, \mathbf{V}]$, see detail in [16, Corollary 2.6]. Therefore, $\mathbf{N} = [\mathbf{V}, \mathbf{V}] \oplus \mathbf{Z}$. On the level of automorphism groups, similar results were obtained in [27, Subsection 3.2]. ■

It should be noted that the intersection of $\mathbf{V}$ and $[\mathbf{V}, \mathbf{V}]$ can be nontrivial. We have the following result.

Lemma 1 (see e.g. Lemma 6 in [8]). *Suppose that $K := \mathbf{V} \cap [\mathbf{V}, \mathbf{V}]$ is non-trivial. Then K is an ideal in $\mathbf{L} = \mathbf{V} + [\mathbf{V}, \mathbf{V}]$. If, in addition, $\mathbf{L}$ is simple, then $K = \mathbf{V} = [\mathbf{V}, \mathbf{V}] = \mathbf{L}$.*

Proof. We see that

$$[K, \mathbf{V}] \subset [\mathbf{V}, \mathbf{V}], \quad [K, \mathbf{V}] \subset [[\mathbf{V}, \mathbf{V}], \mathbf{V}] \subset \mathbf{V},$$
$$[K, [\mathbf{V}, \mathbf{V}]] \subset [\mathbf{V}, [\mathbf{V}, \mathbf{V}]] \subset \mathbf{V}, \quad [K, [\mathbf{V}, \mathbf{V}]] \subset [[\mathbf{V}, \mathbf{V}], [\mathbf{V}, \mathbf{V}]] \subset [\mathbf{V}, \mathbf{V}].$$

Therefore,

$$[K, \mathbf{V}] \subset \mathbf{V} \cap [\mathbf{V}, \mathbf{V}] = K, \quad [K, [\mathbf{V}, \mathbf{V}]] \subset \mathbf{V} \cap [\mathbf{V}, \mathbf{V}] = K, \quad [K, \mathbf{L}] \subset K,$$

hence, K is an ideal of $\mathbf{L}$. ■

Note that the constant $k = k(T)$ in Proposition 1 can be different from zero for pseudo-Riemannian manifolds, see [5], where it is shown that k depends on the reparametrization of a homogeneous geodesic and it could be $k(T) \neq 0$ for a null initial velocity T of such a geodesic. We show that such a reparametrization in case of pseudo H-type Lie groups $N_{r,s}$ is a multiplication by a non-vanishing real constant, that is, it is a linear rescaling, which implies that $k = 0$.

Recall that the isometry group of the pseudo H-type Lie group $N_{r,s}$ with a given left invariant pseudo Riemannian metric is identified with the group $G = N_{r,s} \rtimes H$, where H is defined in (4) and its Lie algebra $\mathfrak{h}$ is given by (5).

Proposition 5. *Let $\gamma \colon l \mapsto \gamma(l) \in N_{r,s}(\cong G/H)$, be a geodesic passing through the identity element $e = [H] \in N_{r,s}$, which is homogeneous with respect to the isometry group $G \cong N_{r,s} \rtimes H$; that is there exists a one parameter subgroup $\{\phi(t)\}$ of G, such that $\gamma(l(t)) = \phi(t) \cdot e$, for a diffeomorphism $l \colon \mathbb{R} \to I$, $l(0) = 0$. Then $l(t) = t$.*

Proof. The Lie algebra of the pseudo H-type Lie group $N_{r,s}$ is $n_{r,s} = \mathfrak{z} \oplus \mathfrak{v} \cong \mathbb{R}^{r,s} \oplus \mathbb{R}^{n,n}$, where $\mathfrak{v}$ is a minimal admissible module of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{r,s})$. We use coordinates of the first kind for

$$N_{r,s} \ni (v, w) = \exp_N(vZ + wX) \quad \text{for} \quad Z \in \mathfrak{z}, \ X \in \mathfrak{v},$$

that is, in this case we identify the group and its Lie algebra as sets. Then $v = (v_1, \ldots, v_{r+s})$, $w = (w_1, \ldots, w_{2n})$.

Without loss of generality, we can consider a geodesic $\gamma \colon I \to G/H = N_{r,s}$, $I = (-a, a) \subset \mathbb{R}$,

$$\gamma(l) = (z(l), x(l)) = (z_1(l), \ldots, z_{r+s}(l), x_1(l), \ldots, x_{2n}(l)) \quad \gamma(0) = e, \quad \dot\gamma(0) = (\dot z(0), \dot x(0)), \quad l \in I.$$

Then

$$z(l) = l\dot z(0) + \frac{1}{2}\int_0^l [x(s), \dot x(s)]\, ds, \quad x(l) = \int_0^l \exp_H(sJ_{\dot z(0)}) \cdot \dot x(0)\, ds, \tag{10}$$

where $J_{\dot z(0)}$ is the Clifford multiplication by the element $\dot z(0) \in \mathfrak{z}$, $J_{\dot z(0)} : \mathfrak{v} \to \mathfrak{v}$ (see [6]). Assume that γ is a homogeneous geodesic for a one-parameter subgroup ϕ of isometry group $G = N_{r,s} \rtimes H$. We write $\phi(t) = (v(t), w(t), c(t), a(t))$, where we use the coordinate representation $(v(t), w(t)) \in N_{r,s}$ and $(c(t), a(t)) \in H$ for $t \in \mathbb{R}$. Then there is $D = (C, A) \in \mathfrak{h}$, such that coordinates of $\phi(t)$ have the form

$$w(t) = \int_0^t \exp_H(sA)\dot{w}(0)\,ds,\ \ v(t) = \int_0^t \exp_H(sC)\dot{v}(0) \,+\, \frac{1}{2}[(w(s), \dot{w}(s)]\,ds, \tag{11}$$

$$c(t) = \exp_H(tC), \quad a(t) = \exp_H(tA). \tag{12}$$

Let assume a geodesic $\gamma(l)$ in (10) is homogeneous under a rescaling $l = l(t)$ through a diffeomorphism $\mathbb{R} \ni t \mapsto l(t) \in \mathbb{R}$, then $D = (C, A) \in \mathfrak{h}$ are such that

$$[J_z, A] = J_{C^\tau(z)}, \quad z \in \mathfrak{z}, \quad C(\dot{z}(0)) = 0, \quad A(\dot{x}(0)) = J_{\dot{z}(0)}(\dot{x}(0)),$$

and the orbit through $e \in N_{r,s}$ is expressed as

$$\begin{aligned} z(l(t)) &= l(t)\dot{z}(0) + \frac{1}{2}\int_0^{l(t)} [x(s), \dot{x}(s)]\,ds \\ &= \int_0^t \exp_H(sC)\dot{z}(0) \,+\, \frac{1}{2}[w(s), \dot{w}(s)]\,ds = v(t) \end{aligned} \tag{13}$$

and

$$x(l(t)) = \int_0^l \exp_H(sJ_{\dot{z}(0)})\dot{x}(0)\,ds = \int_0^t \exp_H(sA)\dot{x}(0)\,ds = w(t). \tag{14}$$

By differentiating equation (13), we obtain

$$\dot{l}(t)\dot{z}(0) = \exp_H(tC) \cdot \dot{z}(0), \qquad \ddot{l}(t)\dot{z}(0) = \exp_H(tC) \cdot C(\dot{z}(0)) = 0.$$

If $\dot{z}(0) \neq 0$, then we obtain that $\ddot{l}(t) = 0$, and $l(t) = ct$, $c \in \mathbb{R} \setminus \{0\}$. In fact $c = 1$, since

$$\dot{l}(0)\dot{z}(0) = c\dot{z}(0) = \dot{z}(0).$$

If $\dot{z}(0) = 0$, then we may assume $\dot{x}(0) \neq 0$, since we are considering non-trivial initial conditions. Then by differentiating equation (14) we can deduce

$$0 = J_{\dot{z}(0)}(\dot{x}(0)) = A(\dot{x}(0)).$$

Then (10) and (11) imply

$$x(l(t)) = l(t)\dot{x}(0) = t\dot{x}(0).$$

Therefore again $l(t) = t$. Thus, we see, that for any initial velocity $\dot{\gamma}(0) = (\dot{x}(0), \dot{z}(0)) \neq 0$ the reparametrization $l(t)$ of a homogeneous geodesic is the trivial one; that is $l(t) = t$. ■

Corollary 2. *In the notation of Proposition 1 for a pseudo H-type Lie group $N_{r,s}$, for any $T \in \mathfrak{n}_{r,s}$ there is $P = P(T) \in \mathfrak{h}$ such that*

$$\langle [T + P, Q]_{\mathfrak{n}_{r,s}}, T\rangle = 0 \tag{15}$$

for any $Q \in \mathfrak{n}_{r,s}$.

Proof. We write $X = T + P$, $\mathfrak{n}_{r,s} = \mathfrak{z} \oplus \mathfrak{v}$. The general form of the geodesic lemma states that there is $k \in \mathbb{R}$ such that,

$$\langle [X, Q], T\rangle = k(l^{-1}(t))\,\langle T, Q\rangle$$

holds, where k is related to the reparametrization $l(t)$. A geodesic γ is homogeneous, if and only if the left invariant vector field X^* defined by X satisfies the equation

$$\nabla_{X*}(X^*) = -k(l^{-1}(t))X^*, \quad k(l^{-1}(t)) = -\frac{\ddot{l}(t)}{\dot{l}(t)} \quad \text{is constant} \tag{16}$$

where ∇ is the Levi-Civita connection on $N_{r,s}$, see [5].

Since we showed that $l(t) = t$ in Proposition 5 we deduce $k(l^{-1}(t)) = 0$. ■

3.1. **Integral basis, periodicity, and the automorphism groups of the H-type Lie algebras.** The pseudo H-type Lie algebras are closely related to Clifford algebras $\mathrm{Cl}(\mathbb{R}^{r,s})$ and their representation spaces $\mathfrak{v}$. Now we describe a convenient basis for pseudo H-type Lie algebras. We fix an orthonormal basis $B_{r,s} = \{Z_1, \dots, Z_r, Z_{r+1}, \dots, Z_{r+s}\}$ for $\mathbb{R}^{r,s}$, where

$$\begin{cases} Z_1, \dots, Z_r \quad \text{are positive, i.e., } \langle Z_i,\, Z_i\rangle_{r,s} = 1, \quad i = 1, \dots, r, \\ Z_{r+1}, \dots, Z_{r+s} \text{ are negative, i.e., } \langle Z_i,\, Z_i\rangle_{r,s} = -1, \quad j = r+1, \dots, r+s. \end{cases} \tag{17}$$

Consider a finite subgroup $G(B_{r,s})$ of the Pin group in $\mathrm{Cl}(\mathbb{R}^{r,s})$ generated by the basis $B_{r,s}$:

$$G(B_{r,s}) = \big\{ \pm \mathbf{1},\ \pm Z_{i_1} \cdots Z_{i_k} \ \big|\ 1 \le i_1 < \cdots < i_k \le r+s, \quad k = 1, \dots, r+s \big\}.$$

In the present work we will consider only **minimal admissible modules**, which are pairs $(\mathfrak{v}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}})$ of the representation space $\mathfrak{v}$ of minimal dimension, where an (non-degenerate) admissible scalar product can be constructed. The construction of admissible scalar products and their description can be found in [15] and [26, Section 2].

Definition 3. Fix an orthonormal basis $B_{r,s}$ of $\mathbb{R}^{r,s}$. An orthonormal basis $\mathfrak{B}(\mathfrak{v})$ of a minimal admissible module $\mathfrak{v}$ is called invariant basis if it is invariant under the action of $G(B_{r,s})$; that is for any $X_i \in \mathfrak{B}(\mathfrak{v})$ and $Z_j \in B_{r,s}$, there exists $X_k \in \mathfrak{B}(\mathfrak{v})$ such that $J_{Z_j}(X_i) = \pm X_k$.

According to Definition 3 the maps J_{Z_j}, $Z_j \in B_{r,s}$ act on an invariant basis $\mathfrak{B}(\mathfrak{v})$ by permutations up to the sign $\pm$. To construct an invariant basis for $\mathfrak{v}$ we consider a maximal subgroup $\mathcal{S}$ of $G(B_{r,s})$ consisting of elements $p \in G(B_{r,s})$ satisfying

P1. $p^2 = 1 \in \mathrm{Cl}(\mathbb{R}^{r,s})$;
P2. if $\langle X, X\rangle_{\mathfrak{v}} > 0$ $\big(\langle X, X\rangle_{\mathfrak{v}} < 0\big)$ then $\langle J_p(X), J_p(X)\rangle_{\mathfrak{v}} > 0$ $\big(\langle J_p(X), J_p(X)\rangle_{\mathfrak{v}} < 0\big)$.

Elements $p \in \mathcal{S}$ are called *positive involutions*. We denote the generating set for the maximal subgroup $\mathcal{S}$ by PI and number of elements in PI by $\ell = \ell(r,s)$. The set of generators is not unique. As an example of a set PI for the purpose of the present work we list the minimal length positive involutions, which can be classified in the following types.

$$\begin{aligned}
&\text{type } T_1 \begin{cases} p = Z_{i_1}Z_{i_2}Z_{i_3}Z_{i_4}, & \text{where all } Z_{i_k} \text{ are positive basis vectors;}\\ p = Z_{i_1}Z_{i_2}Z_{i_3}Z_{i_4}, & \text{where all } Z_{i_k} \text{ are negative basis vectors;}\\ p = Z_{i_1}Z_{i_2}Z_{i_3}Z_{i_4}, & \text{where two } Z_{i_k} \text{ are positive and two } Z_{i_l}\\ & \qquad\qquad \text{are negative basis vectors;} \end{cases}\\
&\text{type } T_2 \begin{cases} q = Z_{i_1}Z_{i_2}Z_{i_3}, & \text{where all } Z_{i_k} \text{ are positive basis vectors;}\\ q = Z_{i_1}Z_{i_2}Z_{i_3}, & \text{where one } Z_{i_k} \text{ is positive and two } Z_{i_l}\\ & \qquad\qquad \text{are negative basis vectors.} \end{cases}
\end{aligned} \tag{18}$$

Here we always assume that $i_k \neq i_m$ for $k \neq m$. A combinatorial computation shows that generally positive involutions can contain either $3 \mod 4$ or $4 \mod 4$ basis vectors, [28].

Proposition 6. [28, Section 3] *Let $\mathcal{S}$, $PI = \{p_1, \dots, p_\ell\}$ and $E^{+1}(p_k) = \{X \in \mathfrak{v} \mid J_{p_k}(X) = X\}$ be given. Then the intersection $E^{+1}_{r,s} = \bigcap_{k=1}^{\ell} E^{+1}(p_k)$ over $p_k \in \mathcal{S}$ contains a non-null vector v. Moreover, there is a set $\Sigma \subset G(B_{r,s})$ such that the family $\{J_\sigma v\}_{\sigma\in\Sigma}$, $\|v\|^2 = 1$, is an orthogonal invariant basis for $\mathfrak{v}$.*

Consider the following example of pseudo H-type Lie algebras $\mathfrak{n}_{\mu,\nu}$, $(\mu,\nu) \in \{(8,0),(0,8),(4,4)\}$ with the minimal admissible module $\mathfrak{v}_{\mu,\nu}$. Let us choose the orthonormal basis $B_{\mu,\nu} = \{\zeta_k\}_{k=1}^8$ on the center of $\mathfrak{n}_{\mu,\nu}$ such that

$$\begin{aligned}
&\langle\zeta_k,\zeta_k\rangle_{8,0} = -\langle\zeta_k,\zeta_k\rangle_{0,8} = 1, && k = 1,\dots,8,\\
&\langle\zeta_k,\zeta_k\rangle_{4,4} = -\langle\zeta_{k+4},\zeta_{k+4}\rangle_{4,4} = 1, && k = 1,\dots,4.
\end{aligned} \tag{19}$$

The set $PI_{\mu,\nu}$ generating the maximal subgroup $\mathcal{S} \subset G(B_{\mu,\nu})$ of positive involutions consists of four elements and it is given by

$$p_1 = \zeta_1\zeta_2\zeta_3\zeta_4, \quad p_2 = \zeta_1\zeta_2\zeta_5\zeta_6, \quad p_3 = \zeta_1\zeta_2\zeta_7\zeta_8, \quad p_4 = \zeta_1\zeta_3\zeta_5\zeta_7.$$

The dimension of minimal admissible modules $\mathfrak{v}_{\mu,\nu}$ equals 16 and the modules are decomposed into 16 one dimensional common eigenspaces of four involutions p_k, $k = 1,2,3,4$ under the action of the representation maps J_{p_k}. We denote

$$E^{+1}_{\mu,\nu} = \{X \in \mathfrak{v}_{\mu,\nu} : \ J_{p_k}(X) = X, \ k = 1,2,3,4\}.$$

Let $v \in E^{+1}_{\mu,\nu}$ be such that $\langle v,v\rangle_{\mathfrak{v}_{\mu,\nu}} = 1$. Then other common eigenspaces are spanned by $J_{\zeta_i}(v)$, $i = 1,\dots,8$, and $J_{\zeta_1}J_{\zeta_j}(v)$, $j = 2,\dots,8$. Hence we have

$$\mathfrak{v}_{\mu,\nu} = E^{+1}_{\mu,\nu} \bigoplus_{i=1}^{8} J_{\zeta_i}(E^{+1}_{\mu,\nu}) \bigoplus_{j=2}^{8} J_{\zeta_1}J_{\zeta_j}(E^{+1}_{\mu,\nu}). \tag{20}$$

The basis

$$\begin{array}{llll}
v_1 = v, & v_2 = J_{\zeta_1}v, & v_3 = J_{\zeta_2}v, & v_4 = J_{\zeta_3}v,\\
v_5 = J_{\zeta_4}v, & v_6 = J_{\zeta_5}v, & v_7 = J_{\zeta_6}v, & v_8 = J_{\zeta_7}v,\\
v_9 = J_{\zeta_8}v, & v_{10} = J_{\zeta_1}J_{\zeta_2}v, & v_{11} = J_{\zeta_1}J_{\zeta_3}v, & v_{12} = J_{\zeta_1}J_{\zeta_4}v,\\
v_{13} = J_{\zeta_1}J_{\zeta_5}v, & v_{14} = J_{\zeta_1}J_{\zeta_6}v, & v_{15} = J_{\zeta_1}J_{\zeta_7}v, & v_{16} = J_{\zeta_1}J_{\zeta_8}v.
\end{array} \tag{21}$$

is an orthonormal invariant basis for $\mathfrak{v}_{\mu,\nu}$. The set Σ mentioned in Proposition 6 is the following

$$\Sigma = \{\zeta_k, \ k = 1,\dots,8, \ \ \zeta_1\zeta_i, \ i = 2,\dots,8\}.$$

It is well known that the Clifford algebras $\mathrm{Cl}(\mathbb{R}^{r,s})$ admit the Atiyah–Boot periodicity [3]:

$$\mathrm{Cl}(\mathbb{R}^{r+8,s}) = \mathrm{Cl}(\mathbb{R}^{r,s}) \otimes \mathrm{Cl}(\mathbb{R}^{8,0}), \quad \mathrm{Cl}(\mathbb{R}^{r,s+8}) = \mathrm{Cl}(\mathbb{R}^{r,s}) \otimes \mathrm{Cl}(\mathbb{R}^{0,8}).$$
$$\mathrm{Cl}(\mathbb{R}^{r+4,s+4}) = \mathrm{Cl}(\mathbb{R}^{r,s}) \otimes \mathrm{Cl}(\mathbb{R}^{4,4}).$$

This periodicity has the following affect on the structure of the H-type Lie algebras. Let $\mathfrak{v}_{r,s}$ be a minimal admissible module in $\mathfrak{n}_{r,s} = \mathfrak{v}_{r,s} \oplus \mathfrak{z}$ and $\ell(r,s)$ the number of positive involutions in the generating set for the maximal subgroup $\mathcal{S} \subset G(B_{r,s})$. Then

$$\dim(\mathfrak{v}_{r+\mu,s+\nu}) = 16\dim(\mathfrak{v}_{r,s}), \quad (\mu,\nu) \in \{(8,0),(0,8),(4,4)\},$$
$$\ell(r+\mu,s+\nu) = \ell(r,s) + \ell(\mu,\nu) = \ell(r,s) + 4.$$

The tensor product

$$\mathfrak{v}_{r,s} \otimes \mathfrak{v}_{\mu,\nu} = \big(\mathfrak{v}_{r,s} \otimes E^{+1}_{\mu,\nu}\big) \bigoplus_{i=1}^{8} \big(\mathfrak{v}_{r,s} \otimes J_{\zeta_i}(E^{+1}_{\mu,\nu})\big) \bigoplus_{j=2}^{8} \big(\mathfrak{v}_{r,s} \otimes J_{\zeta_1} J_{\zeta_j}(E^{+1}_{\mu,\nu})\big) \tag{22}$$

is a minimal admissible module $\mathfrak{v}_{r+\mu,s+\nu}$ of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{r+\mu,s+\nu})$.

Conversely, if $\mathfrak{v}_{r+\mu,s+\nu}$ is a minimal admissible module of $\mathrm{Cl}(\mathbb{R}^{r+\mu,s+\nu})$, then the common 1-eigenspace $E^{+1} \subset \mathfrak{v}_{r+\mu,s+\nu}$ of the involutions J_{p_k}, $k = 1,2,3,4$ from the example above can be considered as a minimal admissible module $\mathfrak{v}_{r,s}$ of the algebra $\mathrm{Cl}(\mathbb{R}^{r,s})$. The action of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{r,s})$ on E^{+1} is the restricted action of $\mathrm{Cl}(\mathbb{R}^{r+\mu,s+\nu})$ obtained by the natural inclusion $\mathrm{Cl}(\mathbb{R}^{r,s}) \subset \mathrm{Cl}(\mathbb{R}^{r+\mu,s+\nu})$.

The group of automorphisms and the isometries of the H-type Lie algebras also has periodic structure, see [49, 51, 27].

4. The groups $N_{r,s}$ with $1 \le r+s \le 3$

For a given pseudo H-type group $N_{r,s}$, $1 \le r+s \le 3$, we check whether $N_{r,s}$ is geodesic orbit or not. For this goal we apply Propositions 3 to the corresponding Lie algebra $\mathfrak{n}_{r,s} = \mathfrak{z} \oplus \mathfrak{v}$. We choose an orthonormal basis $Z_1, Z_2, \dots, Z_{r+s}$ for $\mathfrak{z}$ and compute the operators J_{Z_k}, $1 \le k \le r+s$, which constitute the basis for $\mathbf{V} = J(\mathfrak{z}) \subset \mathfrak{so}(l,l)$. Then the products $J_{Z_k} J_{Z_l}$, $1 \le k < l \le r+s$, form a basis for the subalgebra $[\mathbf{V},\mathbf{V}]$. We find also a basis for the centralizer $\mathbf{Z}$ of $\mathbf{V}$ in $\mathfrak{so}(l,l)$ and obtain the base for the normalizer $\mathbf{N} = [\mathbf{V},\mathbf{V}] \oplus \mathbf{Z}$ of $\mathbf{V}$ in $\mathfrak{so}(l,l)$. After this auxiliary computation we check the *transitive normalizer condition* for $\mathbf{V}$, i.e. for any $X \in \mathfrak{v}$ and any $Z \in \mathbf{V}$, we are looking for $B \in \mathbf{N}$ such that $[B,Z] = 0$ and $B(X) = Z(X)$.

Note that up to similarity and the action of the isotropy subgroup, we have only three classes in the center $\mathfrak{z}$ of $\mathfrak{n}$. They are represented by some Z_1 (a positive vector with $\langle Z_1, Z_1\rangle > 0$), Z_2 (a negative vector with $\langle Z_2, Z_2\rangle < 0$), and Z_3 (a null vector $\langle Z_3, Z_3\rangle = 0$).

Indeed, the group $O(r,s)$ with the Lie algebra $\mathfrak{so}(r,s)$, that is a linear span of the operators $[J_{Z_i}, J_{Z_j}]$, $1 \le i < j \le r+s$, acts naturally by automorphisms on the center $\mathfrak{z}$. It is well known that $O(r,s)$ acts transitively on every hyperquadric $Q(r) = \{Z \in \mathfrak{z} \,|\, \langle Z, Z\rangle_{r,s} = \rho\}$ with $\rho \neq 0$, see e.g. P. 239 in [47] or Theorem 2.4.4 in [55].

Let $\mathfrak{z} = \mathfrak{z}_p \oplus \mathfrak{z}_n$ be an orthogonal decomposition, where the restriction of the scalar product on $\mathfrak{z}_p$ and $\mathfrak{z}_n$ are positive and negative definite, respectively. Let us suppose that $Z, \overline{Z} \in \mathfrak{z}$ and $\langle Z, Z\rangle_{r,s} = \langle \overline{Z}, \overline{Z}\rangle_{r,s} = 0$. If $\langle Z|_{\mathfrak{z}_p}, Z|_{\mathfrak{z}_p}\rangle_{r,s} = \langle \overline{Z}|_{\mathfrak{z}_p}, \overline{Z}|_{\mathfrak{z}_p}\rangle_{r,s} = \rho \neq 0$ or, equivalently, $\langle Z|_{\mathfrak{z}_n}, Z|_{\mathfrak{z}_n}\rangle_{r,s} = \langle \overline{Z}|_{\mathfrak{z}_n}, \overline{Z}|_{\mathfrak{z}_n}\rangle_{r,s} = -\rho \neq 0$, where $Z|_{\mathfrak{z}_p}$ and $Z|_{\mathfrak{z}_p}$ are the components of vector Z in $\mathfrak{z}_p$ and $\mathfrak{z}_n$, then it easy to see there there is $Q \in O(r) \cdot O(s) \subset O(r,s)$ such that $Q(Z|_{\mathfrak{z}_p}) = \overline{Z}|_{\mathfrak{z}_p}$ and $Q(Z|_{\mathfrak{z}_n}) = \overline{Z}|_{\mathfrak{z}_n}$.

Hence, up to a similarity, it suffices to consider only one point in every of the following hyperquadrics: $Q(1)$, $Q(-1)$, $Q(0)$ (non-trivial in the last case).

Suppose that we fix $Z \in \mathbf{V}$, then we need not to check all $X \in \mathfrak{v}$. Indeed, if $Q \in \exp(\mathbf{N})$ (or, more generally, Q is an isometric automorphism of the Lie algebra $\mathfrak{n}$) such that $[Q, Z] = 0$, then the equality $B(X) = Z(X)$ is equivalent to the following one: $QBQ^{-1}(QX) = QZQ^{-1}(QX) = Z(QX)$. Note that $[B, Z] = 0$ implies $[QBQ^{-1}, Z] = 0$, therefore, we can find $B \in \mathbf{N}$ such that $[B, Z] = 0$ and $B(X) = Z(X)$ if and only if we can find $\widetilde{B} \in \mathbf{N}$ such that $[B, Z] = 0$ and $B(QX) = Z(QX)$.

Consider a Lie subalgebra $\mathbf{N}^Z = \{B \in \mathbf{N} \,|\, [B, Z] = 0\}$ of $\mathbf{N}$. Any orbit Orb in $\mathfrak{v}$ under the action of $\mathbf{N}^Z$ consists of equivalent elements in the sense that if $X_1, X_2 \in \mathrm{Orb}$, then there is $B_1 \in \mathbf{N}$ such that $[B_1, Z] = 0$ and $B_1(X_1) = Z(X_1)$ if and only if there is $B_2 \in \mathbf{N}$ such that $[B_2, Z] = 0$ and $B_2(X_2) = Z(X_2)$.

In what follows, we omit all classical cases with the signature $(r, 0)$.

4.1. ***H*-type nilmanifold $N_{0,1}$.** We denote by $N_{0,1}$ the H-type nilmanifold with the Lie algebra isometric to $\mathbb{R}^{0,1} \oplus \mathbb{R}^{n,n}$. This will be the only example of H-type Lie algebra having the module $(\mathfrak{v}, \langle .\,,.\rangle_{n,n}) \cong \mathbb{R}^{n,n}$ which is not minimal dimensional, but rather the direct sum of n minimal dimensional modules isometric to $\mathbb{R}^{1,1}$. We choose an orthonormal basis $\{X_1, \dots X_n, Y_1 \dots, Y_n, Z\}$ satisfying

$$\langle X_i, X_i\rangle_{n,n} = -\langle Y_i, Y_i\rangle_{n,n} = 1, \quad \langle Z, Z\rangle_{0,1} = -1.$$

Then

$$J_Z = \begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix} \in \mathfrak{so}(n,n), \quad J_Z^2 = -\langle Z, Z\rangle_{1,0} \,\mathrm{Id}_{2n} = \mathrm{Id}_{2n}\,.$$

It is clear that J_Z spans the one dimensional subspace $\mathbf{V} = J(\mathfrak{z}) \subset \mathfrak{so}(n,n)$. The space $\mathbf{V}$ is an abelian subalgebra of $\mathfrak{so}(n,n)$ and

$$[J(Z_1), J(Z_2)] = [J(aZ), J(bZ)] = ab[J(Z), J(Z)] = 0 = J\big(\tau_{Z_1}(Z_2)\big), \quad a, b \in \mathbb{R},\ a \neq 0,\ b \neq 0,$$

for the operator $\mathfrak{so}(\mathfrak{z}) \ni \tau_{Z_1} \equiv 0$ for any $Z_1 \in \mathfrak{z}$. Therefore, we get naturally reductive (see Theorem 3), hence, geodesic orbit nilmanifold.

4.2. ***H*-type nilmanifold $N_{1,1}$.** The H-type nilmanifold $N_{1,1}$ is the H-type group of dimension 6 with the Lie algebra isometric to $\mathbb{R}^{1,1} \oplus \mathbb{R}^{2,2}$ and satisfying

$$[X_1, X_2] = [X_3, X_4] = Z_1, \quad [X_1, X_3] = [X_2, X_4] = -Z_2$$

with respect to an orthonormal basis $\{X_1, \dots, X_4, Z_1, Z_2\}$ such that

$$\langle X_k, X_k\rangle_{2,2} = -\langle X_i, X_i\rangle_{2,2} = 1, \quad k = 1,2, \quad i = 3,4, \quad \langle Z_1, Z_1\rangle_{1,1} = -\langle Z_2, Z_2\rangle_{1,1} = 1.$$

The maps

$$J_{z_1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad J_{Z_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

span a 2-dimensional subspace $\mathbf{V} \subset \mathfrak{so}(2,2)$. We calculate

$$[aJ_{Z_1}+bJ_{Z_2}, cJ_{Z_1}+dJ_{Z_2}] = (ad-bc)[J_{Z_1}, J_{Z_2}] = (ad-bc)\begin{pmatrix}0&0&0&2\\0&0&2&0\\0&2&0&0\\2&0&0&0\end{pmatrix} \notin \operatorname{span}\{J_{Z_1}, J_{Z_2}\}.$$

Thus the vector space $\mathbf{V}$ is not a Lie subalgebra of $\mathfrak{so}(2,2)$ and therefore $N_{1,1}$ is not a naturally reductive pseudo-Riemannian nilmanifold. It is easy to check that the normalizer $\mathbf{N} = \mathbf{Z} \oplus [\mathbf{V}, \mathbf{V}]$ consists of matrices of the following type:

$$\begin{pmatrix}0 & -b_3 & b_2 & 2a-b_1\\ b_3 & 0 & 2a+b_1 & b_2\\ b_2 & 2a+b_1 & 0 & b_3\\ 2a-b_1 & b_2 & -b_3 & 0\end{pmatrix}, \qquad a, b_1, b_2, b_3 \in \mathbb{R}.$$

Here, the variable a corresponds to $[\mathbf{V}, \mathbf{V}]$, whereas b_1, b_2, b_3 parameterize $\mathbf{Z}$.

Let us consider $Y = X_1 + X_2 + X_3 + X_4 \in \mathfrak{v}$ and $Z = J_{Z_1} \in \mathbf{V}$. Assume that $N_{1,1}$ is geodesic orbit. Then there is $B \in \mathbf{N}$ such that $[B, Z] = 0$ and $B(Y) = Z(Y)$. The first condition implies $a = 0$. The second condition is equivalent to the linear system of equations:

$$\begin{pmatrix}2 & -1 & 1 & -1\\ 2 & 1 & 1 & 1\\ 2 & 1 & 1 & 1\\ 2 & -1 & 1 & -1\end{pmatrix}\begin{pmatrix}a\\ b_1\\ b_2\\ b_3\end{pmatrix} = \begin{pmatrix}-1\\ 1\\ -1\\ 1\end{pmatrix}.$$

Since the system has no solution (it suffices to compare the second and the third equations in the system), $N_{1,1}$ is not geodesic orbit.

4.3. ***H*-type nilmanifold** $N_{0,2}$**.** The H-type nilmanifold $N_{0,2}$ is the H-type group of dimension 6 whose Lie algebra is isometric $\mathbb{R}^{0,2} \times \mathbb{R}^{2,2}$ and has the commutation relations

$$[X_1, X_3] = [X_2, X_4] = -Z_1, \quad [X_1, X_4] = -[X_2, X_3] = -Z_2$$

with respect to an orthonormal basis $\{X_1, \dots, X_4, Z_1, Z_2\}$ such that

$$\langle X_k, X_k\rangle_{2,2} = -\langle X_i, X_i\rangle_{2,2} = 1, \quad k = 1, 2, \quad i = 3, 4, \quad \langle Z_1, Z_1\rangle_{0,2} = \langle Z_{2,2}\rangle_{0,2} = -1.$$

We calculate

$$J_{Z_1} = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix} \in \mathfrak{so}(2,2) \quad \text{and} \quad J_{Z_2} = \begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix} \in \mathfrak{so}(2,2),$$

$$[aJ_{Z_1}+bJ_{Z_2}, cJ_{Z_1}+dJ_{Z_2}] = (ad-bc)\begin{pmatrix}0&-2&0&0\\2&0&0&0\\0&0&0&2\\0&0&-2&0\end{pmatrix} \notin \operatorname{span}\{J_{Z_1}, J_{Z_2}\}.$$

Thus $N_{0,2}$ is not a naturally reductive manifold. The normalizer $\mathbf{N} = \mathbf{Z} \oplus [\mathbf{V}, \mathbf{V}]$ consists of:

$$\begin{pmatrix} 0 & -2a+b_3 & -b_2 & b_1 \\ 2a-b_3 & 0 & b_1 & b_2 \\ -b_2 & b_1 & 0 & 2a+b_3 \\ b_1 & b_2 & -2a-b_3 & 0 \end{pmatrix}, \qquad a, b_1, b_2, b_3 \in \mathbb{R}.$$

Here, the variable a corresponds to $[\mathbf{V}, \mathbf{V}]$, whereas b_1, b_2, b_3 parameterize $\mathbf{Z}$.

Assume $N_{0,2}$ is geodesic orbit and pick up $Y = y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4 \in \mathfrak{v}$. Moreover, without loss of generality, we take $Z = J_{Z_1} \in \mathbf{V}$. The condition $[B, Z] = 0$ for some $X \in \mathbf{B}$ implies $a = 0$. Then the condition $B(Y) = Z(Y)$ is equivalent to the linear system of equations:

$$\begin{pmatrix} y_2 & -y_1 & y_4 \\ y_1 & y_2 & -y_3 \\ y_4 & -y_3 & y_2 \\ y_3 & y_4 & -y_1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

If $y_1^2 + y_2^2 \neq y_3^2 + y_4^2$, then we obtain the following solution of this system:

$$b_1 = 2\frac{y_1y_2 - y_3y_4}{y_1^2 + y_2^2 - y_3^2 - y_4^2}, \quad b_2 = -\frac{y_1^2 - y_2^2 - y_3^2 + y_4^2}{y_1^2 + y_2^2 - y_3^2 - y_4^2}, \quad b_3 = -2\frac{y_1y_4 - y_2y_3}{y_1^2 + y_2^2 - y_3^2 - y_4^2}.$$

On the other hand, if $(y_1, y_2, y_3, y_4) = (3, 4, 5, 0)$, then we obtain the following system:

$$\begin{pmatrix} 4 & -3 & 0 \\ 3 & 4 & -5 \\ 0 & -5 & 4 \\ 5 & 0 & -3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 0 \end{pmatrix},$$

that have no solution (the rank of the basic matrix of the system is 2, and the rank of the extended matrix of the system is 3). Hence $N_{0,2}$ is not geodesic orbit.

4.4. ***H*-type nilmanifold $N_{1,2}$.** The H-type nilmanifold $N_{1,2}$ is the H-type group of dimension 7, the Lie algebra is isometric to $\mathbb{R}^{1,2} \times \mathbb{R}^{2,2}$ and has non-vanishing commutation relations

$$[X_1, X_2] = -[X_3, X_4] = Z_1, \quad [X_1, X_3] = -[X_2, X_4] = Z_2, \quad [X_1, X_4] = [X_2, X_3] = Z_3$$

in an orthonormal basis satisfying

$$\langle X_k, X_k \rangle_{2,2} = -\langle X_i, X_i \rangle_{2,2} = 1, \quad k = 1, 2, \quad i = 3, 4,$$
$$\langle Z_1, Z_1 \rangle_{1,2} = -\langle Z_2, Z_2 \rangle_{1,2} = -\langle Z_3, Z_3 \rangle_{1,2} = 1.$$

Then, in this basis we obtain the matrix representations

$$J_{Z_1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_{Z_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_{Z_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Due to the commutation relations

$$[J_{Z_1}, J_{Z_2}] = 2J_{Z_3}, \quad [J_{Z_1}, J_{Z_3}] = -2J_{Z_2}, \quad [J_{Z_2}, J_{Z_3}] = -2J_{Z_1},$$

we conclude that $\mathbf{V} = \operatorname{span}\{J_{Z_1}, J_{Z_2}, J_{Z_3}\} \subset \mathfrak{so}(2,2)$ is a Lie subalgebra by Theorem 3. Moreover, the operators $\tau_{Z_i} \in \mathfrak{so}(\mathfrak{z})$ for $i,j = 1,2,3$ (recall that $[J(Z_i), J(Z_j)] = J\big(\tau_{Z_i}(Z_j)\big)$, are given by:

$$\tau_{Z_1} = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau_{Z_2} = 2\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \tau_{Z_3} = 2\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We conclude that H-type nilmanifold $N_{1,2}$ is naturally reductive, hence, geodesic orbit.

4.5. H**-type nilmanifold** $N_{2,1}$**.** The H-type nilmanifold $N_{2,1}$ is the H-type group of dimension 11 with a Lie algebra $\mathfrak{n}_{2,1} = \mathfrak{z} \oplus \mathfrak{v}$ isometric to $\mathbb{R}^{2,1} \times \mathbb{R}^{4,4}$. We define an orthonormal basis $\{V_1, \dots, V_8, Z_1, Z_2, Z_3\}$ by taking $v \in \mathbb{R}^{4,4}$ with $\|v\|^2 = 1$ and setting

$$\begin{array}{llll} V_1 = v, & V_2 = J_{Z_1}v, & V_3 = J_{Z_2}v, & V_4 = J_{Z_1}J_{Z_2}v, \\ V_5 = J_{Z_3}v, & V_6 = J_{Z_1}J_{Z_3}v, & V_7 = J_{Z_2}J_{Z_3}v, & V_8 = J_{Z_1}J_{Z_2}J_{Z_3}v. \end{array}$$

Writing the operators

$$J_{Z_1} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_{Z_2} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_{Z_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and calculating their commutators, we see that $\mathbf{V} = \operatorname{span}\{J_{Z_1}, J_{Z_2}, J_{Z_3}\} \subset \mathfrak{so}(4,4)$ is not a Lie subalgebra of $\mathfrak{so}(4,4)$ and, therefore, $N_{2,1}$ is not a naturally reductive manifold.

The normalizer $\mathbf{N} = \mathbf{Z} \oplus [\mathbf{V}, \mathbf{V}]$ consists of matrices of the following type:

$$\begin{pmatrix} 0 & b_6 & -b_5 & -2a_1 + b_4 & b_3 & 2a_2 + b_2 & 2a_3 - b_1 & 0 \\ -b_6 & 0 & -2a_1 + b_4 & -b_5 & 2a_2 - b_2 & b_3 & 0 & 2a_3 - b_1 \\ b_5 & 2a_1 - b_4 & 0 & -b_6 & 2a_3 + b_1 & 0 & b_3 & -2a_2 - b_2 \\ 2a_1 + b_4 & b_5 & b_6 & 0 & 0 & 2a_3 + b_1 & -2a_2 + b_2 & b_3 \\ b_3 & 2a_2 - b_2 & 2a_3 + b_1 & 0 & 0 & -b_6 & b_5 & -2a_1 - b_4 \\ 2a_2 + b_2 & b_3 & 0 & 2a_3 + b_1 & b_6 & 0 & -2a_1 + b_4 & 2b_5 \\ 2a_3 - b_1 & 0 & b_3 & -2a_2 + b_2 & -b_5 & 2a_1 - b_4 & 0 & b_6 \\ 0 & 2a_3 - b_1 & -2a_2 - b_2 & b_3 & 2a_1 + b_4 & -b_5 & -b_6 & 0 \end{pmatrix},$$

Here, the variables a_1, a_2, a_3 correspond to $[\mathbf{V}, \mathbf{V}]$, whereas $b_1, b_2, b_3, b_4, b_5, b_6$ parameterize $\mathbf{Z}$.

Assume that $N_{2,1}$ is geodesic orbit and choose $Y = V_1 + V_5 \in \mathfrak{v}$ and $Z = J_{Z_1} \in \mathbf{V}$. If $B \in \mathbf{N}$ then $[B, Z] = 0$ implies $a_1 = a_2 = 0$. The second condition $B(Y) = Z(Y)$ gives

$$(b_3, -b_2 - b_6, b_1 + b_5 + 2a_3, b_4, b_2, b_2 + b_6, -b_1 - b_5 + 2a_3, b_4)^t = (0, 1, 0, 0, 0, 1, 0, 0, 0)^t,$$

which leads to a contradiction: it suffices to compare the second and and the sixth entries in these columns. Therefore, $N_{2,1}$ is not geodesic orbit.

4.6. H**-type nilmanifold** $N_{0,3}$**.** The H-type Lie algebra $\mathfrak{n}_{0,3}$ isometric to $\mathbb{R}^{0,3} \times \mathbb{R}^{4,4}$ and has the following non-vanishing commutation relations

$$[V_1, V_5] = [V_2, V_6] = [V_3, V_7] = [V_4, V_8] = Z_1, \quad [V_1, V_6] = -[V_2, V_5] = -[V_3, V_8] = [V_4, V_7] = Z_2,$$

$$[V_1, V_7] = [V_2, V_8] = -[V_3, V_5] = [V_4, V_6] = Z_3.$$

It implies that the matrices

$$J_{Z_1} = \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{pmatrix}, \quad J_{Z_2} = \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0 \end{pmatrix}, \quad J_{Z_3} = \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{pmatrix}$$

form the basis of the space $\mathbf{V} \subset \mathfrak{so}(4,4)$. Calculating their commutators, we show that $\mathbf{V}$ is not a Lie subalgebra of $\mathfrak{so}(4,4)$, and therefore $N_{0,3}$ is not a naturally reductive manifold. The normalizer $\mathbf{N} = \mathbf{Z} \oplus [\mathbf{V}, \mathbf{V}]$ consists of matrices of the following type:

$$\begin{pmatrix} 0 & -2a_1+b_6 & -2a_2-b_5 & -2a_3+b_4 & b_3 & -b_2 & b_1 & 0\\ 2a_1-b_6 & 0 & 2a_3+b_4 & -2a_2+b_5 & -b_2 & -b_3 & 0 & b_1\\ 2a_2+b_5 & -2a_3-b_4 & 0 & 2a_1+b_6 & b_1 & 0 & -b_3 & b_2\\ 2a_3-b_4 & 2a_2-b_5 & -2a_1-b_6 & 0 & 0 & b_1 & b_2 & b_3\\ b_3 & -b_2 & b_1 & 0 & 0 & 2a_1+b_6 & 2a_2-b_5 & -2a_3+b_4\\ -b_2 & -b_3 & 0 & b_1 & -2a_1-b_6 & 0 & 2a_3+b_4 & 2a_2+b_5\\ b_1 & 0 & -b_3 & b_2 & -2a_2+b_5 & -2a_3-b_4 & 0 & -2a_1+b_6\\ 0 & b_1 & b_2 & b_3 & 2a_3-b_4 & -2a_2-b_5 & 2a_1-b_6 & 0 \end{pmatrix},$$

where the variables a_1, a_2, a_3 correspond to $[\mathbf{V}, \mathbf{V}]$, and $b_1, b_2, b_3, b_4, b_5, b_6$ parameterize $\mathbf{Z}$.

Assuming that $N_{0,3}$ is geodesic orbit, and taking $Y = \sum_{i=1}^{8} y_i V_i \in \mathfrak{v}$ and a $Z \in \mathbf{V}$ we obtain that for an arbitrary $B \in \mathbf{N}$ the condition $[B, Z] = 0$ implies $a_1 = a_2 = 0$. Now, we choose $(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = (3, 4, 0, 0, 5, 0, 0, 0)$ and the condition $B(Y) = Z(Y)$ (assuming $a_1 = a_2 = 0$) is equivalent to the linear system of equations:

$$\begin{pmatrix} 0 & 3 & 0 & 0 & 0 & 5 & 0\\ 0 & 0 & 5 & 0 & 0 & 0 & 3\\ -6 & 0 & 0 & 0 & 3 & 4 & 0\\ 10 & 4 & 0 & 0 & -5 & 0 & 0\\ 8 & -5 & 0 & 0 & 4 & -3 & 0\\ 0 & 0 & -4 & 3 & 0 & 0 & 0\\ 0 & 0 & -3 & -4 & 0 & 0 & -5\\ 0 & 0 & 0 & 5 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} a_3\\ b_1\\ b_2\\ b_3\\ b_4\\ b_5\\ b_6 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\3\\4\\5 \end{pmatrix},$$

that does not have solutions (the rank of the basic matrix of the system is 5, and the rank of the extended matrix of the system is 6). Therefore, $N_{0,3}$ is not geodesic orbit.

Remark 2. We note that there are shorter arguments to prove that $N_{2,1}$ and $N_{0,3}$ are not geodesic orbit. One can use Corollary 3 and the ideas of Section 5 to check that $N_{1,1}$ is a totally geodesic submanifold in $N_{2,1}$ as well as $N_{0,2}$ is a totally geodesic submanifold in $N_{0,3}$. Recall that $N_{1,1}$ and $N_{0,2}$ are not geodesic orbit by the above considerations. Nevertheless, we decided to show the details of working with these pseudo H-type groups to prepare readers for a more sophisticated study of the H-type group $N_{3,4}$ in Section 7.

4.7. **On naturally reductive pseudo H-type nilpotent Lie group.** The above examples allow us to obtain a complete classification of naturally reductive pseudo H-type nilmanifolds $N_{r,s}$.

Proposition 7. *An H-type nilmanifolds $N_{r,s}$ naturally reductive if and only if* $(r,s) \in \{(1,0), (0,1), (3,0), (1,2)\}$.

Proof. We know that $N_{1,0}$, $N_{0,1}$, $N_{3,0}$, and $N_{1,2}$ are naturally reductive (see Theorem 1 and Section 4).

Now, suppose that $N_{r,s}$ is naturally reductive and $r+s>1$ (if $r+s=1$, then it is isometric to $N_{1,0}$ or to $N_{0,1}$). Then the linear space $\mathbf{V}=J(\mathfrak{z})$ is a Lie subalgebra in $\mathbf{L}=\mathbf{V}+[\mathbf{V},\mathbf{V}]$. Since $[\mathbf{V},[\mathbf{V},\mathbf{V}]]\subset\mathbf{V}$, then $\mathbf{V}$ is an ideal in $\mathbf{L}$. Therefore, either $\mathbf{L}$ is not simple, or $\mathbf{V}=[\mathbf{V},\mathbf{V}]$ is a simple Lie algebra. If $(r,s)\notin\{(1,0),(0,1),(3,0),(1,2)\}$, then $\mathbf{L}$ is simple by Proposition 4 and $\mathbf{V}=[\mathbf{V},\mathbf{V}]$ by Lemma 1.

Since $\dim(\mathbf{V})=r+s$ and $\dim([\mathbf{V},\mathbf{V}])=(r+s)(r+s-1)/2$ (see Proposition 4), then $r+s=3$. Now, it suffices to note that $N_{2,1}$ and $N_{0,3}$ are not geodesic orbit, hence, are not naturally reductive. ■

It will be proved in Section 7, that the pseudo H-type nilmanifold $N_{3,4}$ is geodesic orbit, although it is not naturally reductive.

5. On totally geodesic submanifolds of geodesic orbit pseudo-Riemannian manifolds

In this section we consider the relation between the geodesic orbit submanifolds and totally geodesic submanifolds and apply this to pseudo H-type Lie groups. P. Eberlein [22, 23] studied totally geodesic subalgebras and totally geodesic subgroups in nonsingular 2-step nilpotent Lie groups endowed with left-invariant Riemannian metrics. The pseudo H-type Lie algebras are examples of nonsingular 2-step nilpotent Lie algebras. It was proved in [7, Theorem 11] that every closed totally geodesic submanifold of a GO Riemannian manifold is GO itself. Let us consider a version of this result for pseudo-Riemannian manifolds. Note that any GO pseudo-Riemannian manifold is geodesically complete.

Theorem 4. *Every geodesically complete totally geodesic submanifold of a GO pseudo-Riemannian manifold is geodesic orbit itself.*

Proof. Let N be a geodesically complete totally geodesic submanifold of a GO pseudo-Riemannian manifold M (which is also geodesically complete). Let $U\neq 0$ be a tangent vector at some point $x\in N$. It is enough to prove that there is a Killing vector field Y on N with the following properties:

1) the value $Y(x)=U(x)$;

2) x is a critical point of $\|Y\|^2$ on N.

Indeed, in this case a geodesic passing through x to the direction U is an orbit of an one-dimensional isometry group generated by the Killing field Y (this one-parameter group is correctly defined because of the geodesic completeness of N), see e.g. [47, Exercise 10, page 259].

It is known that a point $z\in M$ is a critical point of the square of the length of a Killing vector field Z on M if and only if the integral curve of Z through the point z is a geodesic in M, see e.g. [47, Exercises 9 and 10, page 259]. Since M is a GO manifold, there is a Killing vector field X on M, such that $X(x)=U(x)$ with x being a critical point of $\|X\|^2$. Now we define Y to be the tangent component $\widetilde{X}$ of the Killing vector field X to N. According to Exercise 7 in [47, page 259]), $\widetilde{X}$ is a Killing vector field on N, and, moreover, $\widetilde{X}(x)=X(x)$.

Now we need to prove only that x is a critical point of $\|\widetilde{X}\|^2$ on N. Let $Z=X-\widetilde{X}$ be the normal component of the vector field X on the manifold N. It is clear that $\|\widetilde{X}\|^2=\|X\|^2-\|Z\|^2$ on M. The point x is a zero point for $\|Z\|^2$, therefore, x is a critical point of $\|Z\|^2$ on N. Consequently, x is a critical point for both functions: $\|X\|^2$ and $\|Z\|^2$ on the manifold N. But in this case x is a critical point for $\|\widetilde{X}\|^2$, since $\widetilde{X}(x) = U(x) \neq 0$. Theorem is proved. ∎

Theorem 5. *Let (N, g) be a 2-step pseudo-Riemannian nilmanifold with a non degenerate centre. Let us consider non-degenerate subspaces $\mathfrak{z}_1 \subset \mathfrak{z}$ and $\mathfrak{v}_1 \subset \mathfrak{v}$ and the corresponding orthogonal decompositions $\mathfrak{z} = \mathfrak{z}_1 \oplus \mathfrak{z}_2$ and $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$. Then the following assertions hold:*

1) If $J_Z(\mathfrak{v}_1) \subset \mathfrak{v}_2$ for any $Z \in \mathfrak{z}_2$, then $\mathfrak{n}_1 := \mathfrak{z}_1 \oplus \mathfrak{v}_1$ is a Lie subalgebra of $\mathfrak{n}$.

2) If $J_Z(\mathfrak{v}_1) \subset \mathfrak{v}_2$ for any $Z \in \mathfrak{z}_2$ and $J_Z(\mathfrak{v}_1) \subset \mathfrak{v}_1$ for any $Z \in \mathfrak{z}_1$, then the Lie subalgebra $\mathfrak{n}_1$ (with the induced scalar product) generates a totally geodesic submanifold (N_1, g_1) of (N, g).

Proof. The first assertion is almost obvious. Indeed, $[\mathfrak{z}_1, \mathfrak{n}_1] \subset [\mathfrak{z}, \mathfrak{n}] = 0$ and we have $\langle [X, Y], Z\rangle = \langle J_Z(X), Y\rangle = 0$ for all $X, Y \in \mathfrak{v}_1$ and any $Z \in \mathfrak{z}_2$ by the condition in 1). Hence, $[\mathfrak{n}_1, \mathfrak{n}_1] = [\mathfrak{v}_1, \mathfrak{v}_1] \subset \mathfrak{z}_1 \subset \mathfrak{n}_1$, that proves 1).

Let us show the second assertion. If ∇ is the Levi-Civita connection on (N, g), then we need to prove that $\nabla_X Y \in \mathfrak{n}_1$ for all $X, Y \in \mathfrak{n}_1$ (X, Y are identified with left-invariant vector fields). Recall the Koszul formula

$$2\langle \nabla_X Y, W\rangle = \langle [X, Y], W\rangle + \langle [W, X], Y\rangle + \langle X, [W, Y]\rangle, \quad X, Y, W \in \mathfrak{n}.$$

If $U = Z_1 + X_1$ and $V = Z_2 + X_2$, with $Z_1, Z_2 \in \mathfrak{z}$, $X_1, X_2 \in \mathfrak{v}$, then

$$2\nabla_U V = [X_1, X_2] - J_{Z_2}(X_1) - J_{Z_1}(X_2),$$

see details, e.g. [23, page 813].

Now, if $Z_1, Z_2 \in \mathfrak{z}_1$, $X_1, X_2 \in \mathfrak{v}_1$, then $[X_1, X_2] \in \mathfrak{n}_1$ by 1) and $J_{Z_2}(X_1), J_{Z_1}(X_2) \in \mathfrak{v}_1$ by the condition of 2). Therefore, $\nabla_U V \in \mathfrak{n}_1$ for all $U, V \in \mathfrak{n}_1$. This implies 2). ∎

Theorems 4 and 5 will be useful for us in the following reformulation.

Corollary 3. *Let N be a 2-step pseudo H-type Lie group with the Lie algebra $\mathfrak{n}$. Let $\mathfrak{n}_1 \subset \mathfrak{n}$ be a subalgebra which generates a totally geodesic submanifold N_1 of N. If N_1 is not a geodesic orbit manifold, then N is neither a geodesic orbit manifold.*

6. The groups $N_{r,s}$ with $r + s > 3$, $(r, s) \neq (3, 4)$

6.1. **Pseudo H-type Lie groups $N_{r,s}$ with $r+s = 0 \mod 4$.** We extend a result of [36] to the pseudo H-type Lie algebras $\mathfrak{n}_{r,s}$.

Lemma 2. *Let $D \in \mathfrak{h}$ be a skew-symmetric derivation of a pseudo H-type Lie algebra $\mathfrak{n}_{r,s}$, $r + s = 0 \mod 4$, see (5). If we write $D = (C, A)$, then A satisfies*

$$A \prod_{i=1}^{r+s} J_{Z_i} = \prod_{i=1}^{r+s} J_{Z_i} A \quad \textit{for} \ \ r + s = 0 \mod 4$$

and any orthonormal basis $\{Z_1, \ldots, Z_{r+s}\}$ as in (17).

Proof. Note that if $D = (C, A)$ is a skew-symmetric derivation as in (5), then

$$A \prod_{i=1}^{r+s} J_{Z_i} = \prod_{i=1}^{r+s} J_{Z_i} A - \prod_{i=1}^{r+s} J_{Z_1} \ldots J_{C(Z_i)} \ldots J_{Z_{r+s}}.$$

Thus we need to show that the last product on the right hand side vanishes.

Let $r+s=2$ and let $D=(C,A)$ be as in (5). Assume that $\{Z_1,Z_2\}$ is an orthonormal basis for $\mathfrak{z}$ and consider 3 possibilities.

If $\|Z_1\|^2=\|Z_2\|^2=\pm1$ then any skew-symmetric map C is given by $\pm a\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, $a\in\mathbb{R}$. It implies

$$\begin{aligned}AJ_{Z_1}J_{Z_2} &= J_{Z_1}J_{Z_2}A-J_{C(Z_1)}J_{Z_2}-J_{Z_1}J_{C(Z_2)}\\ &= J_{Z_1}J_{Z_2}A-a\Big(\mp J_{Z_1}^2\pm J_{Z_2}^2\Big)=J_{Z_1}J_{Z_2}A-a\Big(\operatorname{Id}-\operatorname{Id}\Big).\end{aligned}$$

If $\|Z_1\|^2=-\|Z_2\|^2=1$ then any skew-symmetric map C is given by $a\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $a\in\mathbb{R}$, and analogous calculations show $AJ_{Z_1}J_{Z_2}=J_{Z_1}J_{Z_2}A$.

We perform now the proof by induction of the dimension of the center. Let write a matrix $C=\{c_{ij}\}\subset\mathfrak{so}(r,s)$ with $r+s=4$. Then we will obtain

$$\begin{aligned}A\prod_{i=1}^{4}J_{Z_i} &= \prod_{i=1}^{4}J_{Z_i}A\\ &+ \Big(c_{12}J_{Z_2}^2+c_{21}J_{Z_1}^2\Big)J_{Z_3}J_{Z_4}-\Big(c_{13}J_{Z_3}^2+c_{31}J_{Z_1}^2\Big)J_{Z_2}J_{Z_4}+\Big(c_{14}J_{Z_4}^2+c_{41}J_{Z_1}^2\Big)J_{Z_2}J_{Z_3}\\ &+ \Big(c_{23}J_{Z_3}^2+c_{32}J_{Z_2}^2\Big)J_{Z_1}J_{Z_4}-\Big(c_{24}J_{Z_4}^2+c_{42}J_{Z_2}^2\Big)J_{Z_1}J_{Z_3}+\Big(c_{34}J_{Z_4}^2+c_{43}J_{Z_3}^2\Big)J_{Z_1}J_{Z_2}\\ &= \prod_{i=1}^{4}J_{Z_i}A,\end{aligned}$$

since $c_{ij}J_{Z_j}^2+c_{ji}J_{Z_i}^2=0$ due to the skew symmetry in $\mathfrak{so}(r,s)$. By the induction of this arguments we obtain for any $r+s=4k$, $k=1,2,\dots$

$$\prod_{i=1}^{4k}J_{Z_1}\dots J_{C(Z_i)}\dots J_{Z_{4k}}=\sum_{i<j}(-1)^{j-i+1}\big(c_{ij}J_{Z_j}^2+c_{ji}J_{Z_i}^2\big)J_{Z_1}\dots\hat{J}_{Z_i}\dots\hat{J}_{Z_j}\dots J_{Z_{4k}}=0,$$

where $\hat{J}_{Z_i}$ denotes the omitted term in the product and $C=\{c_{ij}\}\in\mathfrak{so}(r,s)$, $r+s=0 \mod 4$. ■

Proposition 8. [41, Proposition 3.3] *The volume element $\omega=\prod_{i=1}^{r+s}Z_i$ in $\mathrm{Cl}(\mathbb{R}^{r,s})$ has the following basic properties. Let $n=r+s$. Then*

$$\omega^2=(-1)^{\frac{n(n+1)}{2}+s}. \tag{23}$$

In particular, if $n=r+s$ is odd, then $Z\omega=\omega Z$ for all $Z\in\mathbb{R}^{r,s}$, and if $n=r+s$ is even, then $Z\omega=-\omega Z$ for all $Z\in\mathbb{R}^{r,s}$. Formula (23) can be also written as

$$\omega^2=\begin{cases}(-1)^s & \text{if}\quad r+s=0 \text{ or } 3 \mod 4,\\ (-1)^{s+1} & \text{if}\quad r+s=1 \text{ or } 2 \mod 4.\end{cases} \tag{24}$$

Theorem 6. *If $r+s=0 \mod 4$, $s=0 \mod 2$, then the pseudo H-type Lie group $N_{r,s}$ is not a geodesic orbit nilmanifold.*

Proof. Let $Z_1,\dots,Z_{r+s}$ be an orthonormal basis of the centre as in (17) of the pseudo H-type Lie algebra $\mathfrak{n}_{r,s}=\mathfrak{z}\oplus\mathfrak{v}$. Consider the volume form $\omega=\prod_{i=1}^{r+s}Z_i$ in $\mathrm{Cl}(\mathbb{R}^{r,s})$. Then

$\omega^2 = 1$ by Proposition 8 since $\frac{n(n+1)}{2} + s$ is even. Then operator $J_\omega \colon \mathfrak{v} \to \mathfrak{v}$ decomposes the module $\mathfrak{v}$ into the eigenspaces of ω:

$$\mathfrak{v} = \mathfrak{v}^+ \oplus \mathfrak{v}^-.$$

Moreover, since $A\omega = \omega A$ by Lemma 2, any A in $D = (C, A)$ will leave the spaces $\mathfrak{v}^\pm$ invariant.

Take any $X \in \mathfrak{v}^+$ and $Z \in \mathfrak{z}$ as initial vector of a geodesic in the group $N_{r,s}$ (we can assume that $Z = Z_1$). Note that since $n = r + s$ is even we have $J_{Z_1}\omega = -\omega J_{Z_1}$ by Proposition 8. It implies that

$$J_{Z_1} \colon \mathfrak{v}^+ \to \mathfrak{v}^-. \tag{25}$$

Thus if a skew-symmetric derivation $D = (C, A)$ exists, then it should leave the spaces $\mathfrak{v}^\pm$ invariant. This contradicts to the behavior of the restriction of $D = (C, A)$ to the set $\mathfrak{x} = \operatorname{span}\{X, J_{Z_1}X, Z_1\}$ by (25). ■

Corollary 4. *The pseudo H-type Lie groups $N_{\mu,\nu}$, $(\mu, \nu) \in \{(8, 0), (0, 8), (4, 4)\}$ are not geodesic orbit.*

Remark 3. In the proof of Theorem 6 we implicitly used that for any non-null vector $Z \in \mathfrak{z}$ the map $J_Z = (0, A)$ is a skew-symmetric derivation acting on the space $\mathfrak{x} = \operatorname{span}_{\mathbb{R}}\{X, J_ZX, Z\}$ for any $X \in \mathfrak{v}$. Note also that for $\|X\|^2 = \|Z\|^2 = 1$ the space $\mathfrak{x}$ is isometric to the Heisenberg algebra $\mathfrak{n}_{1,0}$, and for $\|X\|^2 = \pm 1$, $\|Z\|^2 = -1$ the space $\mathfrak{x}$ is isometric to the pseudo H-type algebra $\mathfrak{n}_{0,1}$.

Before we proceed to show that $N_{r,s}$ is not geodesic orbit for $r + s = 0 \mod 4$, $s = 1 \mod 2$ we formulate a generalization of [50, Theorem 6] for the pseudo H-type Lie algebras. Let us write $\mathfrak{h} = \mathfrak{h}_{r,s}$ in (5) as

$$\mathfrak{h}_{r,s} = (\mathfrak{h}_{r,s})_1 \oplus (\mathfrak{h}_{r,s})_0,$$

where

$$(\mathfrak{h}_{r,s})_0 = \{D_0 = (0, A),\ [A, J_{Z_j}] = 0, \quad j = 1, \dots, r + s\}$$

is the Lie algebra of automorphisms of $\mathfrak{n}_{r,s}$ acting as identity on the center, and

$$(\mathfrak{h}_{r,s})_1 = \{D_1 = (C, A),\ AJ_Z - J_ZA = J_{C^\tau(Z)}\}.$$

According to formula (9) for an orthonormal basis $\{Z_1, \dots, Z_{r+s}\}$ for $\mathfrak{z}$ we have that a corresponding skew-symmetric derivation $D_1 = (C, A) \in (\mathfrak{h}_{r,s})_1$ can be written as

$$A(X) = J_{Z_i}J_{Z_j}(X), \quad C(Z) = 2(\langle Z_i, Z\rangle_{r,s}Z_j - \langle Z_j, Z\rangle_{r,s}Z_i) = \operatorname{ad}_{Z_iZ_j} Z = [Z_iZ_j, Z].$$

Let $Z_0 \in \mathfrak{z}$ be a non-null vector and $\mathfrak{z}_0 = \operatorname{span}\{Z_0\}^\perp$ be the orthogonal complement in $\mathfrak{z}$. We also write $\operatorname{Cl}(\mathbb{R}^{r',s'}) = \operatorname{Cl}(\mathfrak{z}_0)$, $r' + s' = r + s - 1$ for the Clifford algebra generated by the space $(\mathfrak{z}_0, \langle\cdot\,,\cdot\rangle_{r,s})|_{\mathfrak{z}_0}$ and acting on the module $\mathfrak{v}$.

Lemma 3. *Any extension of the skew-symmetric derivation $(0, J_{Z_0})$ to a skew-symmetric derivation $D = (C, A) \in \mathfrak{h}_{r,s}$ must belong to*

$$\mathfrak{h}_{r,s} = (\mathfrak{h}_{r',s'})_1 + (\mathfrak{h}_{r,s})_0, \quad r' + s' = r + s - 1.$$

Proof. Let us assume that $D = (C, A) \in \mathfrak{h}_{r,s}$ is an extension of $(0, J_{Z_0})$. Then $C(Z_0) = 0$. Any $D \in (\mathfrak{h}_{r,s})_0$ will satisfy it. Let $Z_0, Z_1, \dots, Z_{r+s-1}$ be an orthonormal basis of $\mathfrak{z}_{r,s}$. We write $\mathfrak{z}_{r',s'} = \operatorname{span}\{Z_1, \dots, Z_{r+s-1}\}$ then the set

$$\{Z_kZ_l, \quad 0 < k < l \leq r + s - 1\}$$

form a basis of $\mathfrak{so}(\mathfrak{z}_{r',s'})$ and $D_1 = D_1(Z_kZ_l) = (C, A)$ generated by Z_kZ_l satisfies

$$A(X) = J_{Z_k}J_{Z_l}(X), \quad C(Z_0) = \begin{cases} 0 & \text{if} \quad 0 < k < l \le r+s-1 \\ 2\|Z_0\|^2 Z_l & \text{if} \quad 0 = k < l \le r+s-1. \end{cases}$$

Thus $D_1(Z_kZ_l) = (C, A) \in (\mathcal{D}_{r',s'})_1$ and $C(Z_0) = 0$. ■

Theorem 7. *If $r+s = 0 \mod 4$, $s = 1 \mod 2$, then the pseudo H-type Lie group $N_{r,s}$ is not geodesic orbit manifold.*

Proof. Let us write $s = s'+1$ and consider the last vector Z_{r+s}, $\|Z_{r+s}\|^2 = -1$ in the orthonormal basis $\{Z_1, \dots, Z_{r+s}\}$ for $\mathfrak{z}_{r,s}$. Note that $\omega^2 = 1$, where $\omega = \prod_{i=1}^{r+s} Z_i$ by (24). The module $\mathfrak{v}_{r,s}$ of $\mathrm{Cl}(\mathbb{R}^{r,s})$ is decomposed into the direct sum of two subspaces

$$\mathfrak{v}_{r,s} = \mathfrak{v}^+_{r,s'} + \oplus \mathfrak{v}^-_{r,s'},$$

that are the eigenspaces of the volume form ω. The spaces $\mathfrak{v}^\pm_{r,s'}$ are the non-equivalent modules of $\mathrm{Cl}(\mathbb{R}^{r,s'}) = \mathrm{Cl}(\mathfrak{z}^\perp_{r,s})$, where $\mathfrak{z}^\perp_{r,s}$ is the orthogonal complement of $\operatorname{span}\{Z_{r,s}\}$ in $\mathfrak{z}_{r,s}$. Due to $J_{Z_{r+s}}\omega = -\omega J_{z_{r+s}}$, the map $J_{Z_{r+s}} \colon \mathfrak{v}^+_{r,s'} \to \mathfrak{v}^-_{r,s'}$ is an isomorphism of vectors spaces.

If $D = (0, A) \in (\mathfrak{h}_{r,s})_0$, then

$$[A, J_{Z_i}] = 0 \quad \Longrightarrow \quad A\omega = \omega A$$

by Lemma 2 and therefore A must preserve the spaces $\mathfrak{v}^\pm_{r,s'}$. If $D = (C, A) \in (\mathfrak{h}_{r,s'})_1$, then A also leaves spaces $\mathfrak{v}^\pm_{r,s'}$ invariant since $\mathfrak{v}^\pm_{r,s'}$ are submodules of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{r,s'}) = \mathrm{Cl}(\mathfrak{z}^\perp_{r,s})$.

The extension $D = (0, A)$ of $(0, J_{Z_{r+s}})$ acting on the $\mathfrak{x} = \operatorname{span}\{X, J_{Z_{r+s}}(X), Z_{r+s}\}$, $X \in \mathfrak{v}^+_{r,s'}$ must belong to $(\mathfrak{h}_{r,s'})_1 + (\mathfrak{h}_{r,s})_0$ by Lemma 3. By the above arguments A should preserve the submodule $\mathfrak{v}^+_{r,s'}$. But this contradicts to the fact that the restriction $A|_{\mathfrak{x}} = J_{Z_{r+s}}$ of A to $\mathfrak{x}$ has the property $J_{Z_{r+s}}(X) \in \mathfrak{v}^-_{r,s'}$. ■

6.2. **The geodesic orbit property and the periodicity property.** As another illustration of Theorem 5 and Corollary 3 and the structure of admissible modules under the periodicity property, we show the following theorem.

Theorem 8. *If a pseudo H-type Lie group $N_{r,s}$ is not geodesic orbit, then $N_{r+\mu,s+\nu}$ with $(\mu, \nu) \in \{(8,0), (0,8), (4,4)\}$ is neither geodesic orbit.*

Proof. Consider a pseudo H-type Lie algebra $\mathfrak{n}_{r+\mu,s+\nu}$ with the minimal admissible module $\mathfrak{v}_{r+\mu,s+\nu} = \mathfrak{v}_{r,s} \otimes \mathfrak{v}_{\mu,\nu}$, the scalar product

$$\langle \cdot\,,\cdot \rangle_{\mathfrak{v}_{r+\mu,s+\nu}} = \langle \cdot\,,\cdot \rangle_{\mathfrak{v}_{r,s}} \cdot \langle \cdot\,,\cdot \rangle_{\mathfrak{v}_{\mu,\nu}},$$

and the basis $\{u_i \otimes v_k,\ i = 1, \dots, \dim(\mathfrak{v}_{r,s}),\ k = 1, \dots, 16\}$. Here $\{u_i\}_{i=1}^{\dim(\mathfrak{v}_{r,s})}$ is an orthonormal invariant basis for $\mathfrak{v}_{r,s}$ generated by a vector $u \in E^{+1}_{r,s}$ with $\langle u, u \rangle_{\mathfrak{v}_{r,s}} = 1$, see Proposition 6. Analogously, $\{v_k\}_{k=1}^{16}$ is the orthonormal invariant basis for $\mathfrak{v}_{\mu,\nu}$ from (21) generated by a unit vector $v \in E^{+1}_{\mu,\nu}$. The orthonormal basis

$$Z_1, \dots, Z_{r+s}, \zeta_1, \dots, \zeta_{\mu+\nu} \tag{26}$$

for $\mathbb{R}^{r+\mu,s+\nu}$ is the union of the basis (17) for $\mathbb{R}^{r,s}$ and the basis (19) for $\mathbb{R}^{\mu,\nu}$. The basis (26) acts on $\mathfrak{v}_{r+\mu,s+\nu} = \mathfrak{v}_{r,s} \otimes \mathfrak{v}_{\mu,\nu}$ by

$$\tilde{J}_{Z_l} = J_{Z_l} \otimes \mathrm{Id},\ l = 1, \dots, \dim \mathfrak{v}_{r,s}, \qquad \tilde{J}_{\zeta_m} = \mathrm{Id} \otimes J_{\zeta_m},\ m = 1, \dots, 16.$$

The commutators in $\mathfrak{n}_{r+\mu,s+\nu}$ satisfy the following relations

$$[u_i\otimes v_k, u_j\otimes v_k]_{\mathfrak{v}_{r+\mu,s+\nu}} = [u_i,u_j]_{\mathfrak{v}_{r,s}}\|v_k\|^2,\quad i,j=1,\ldots,\dim\mathfrak{v}_{r,s},\tag{27}$$

for any $k=1,\ldots,16$. Indeed, let $Z\in\mathbb{R}^{r,s}$, then

$$\begin{aligned}\langle Z,[u_i\otimes v_k,u_j\otimes v_k]_{\mathfrak{v}_{r+\mu,s+\nu}}\rangle_{r+\mu,s+\nu} &= \langle \tilde{J}_Z(u_i\otimes v_k),u_j\otimes v_k\rangle_{\mathfrak{v}_{r+\mu,s+\nu}}\\ &= \langle J_Z(u_i)\otimes v_k,u_j\otimes v_k\rangle_{\mathfrak{v}_{r+\mu,s+\nu}}\\ &= \langle J_Z(u_i),u_j\rangle_{\mathfrak{v}_{r,s}}\cdot\langle v_k,v_k\rangle_{\mathfrak{v}_{\mu,\nu}}\\ &= \langle Z,[u_i,u_j]_{\mathfrak{v}_{r,s}}\rangle_{r,s}\|v_k\|^2.\end{aligned}$$

since $\langle Z,\bullet\rangle_{r+\mu,s+\nu}=\langle Z,\bullet\rangle_{r,s}$ for any $Z\in\mathbb{R}^{r,s}$. Analogously

$$[u_k\otimes v_i,u_k\otimes v_j]_{\mathfrak{v}_{r+\mu,s+\nu}} = [v_i,v_j]_{\mathfrak{v}_{\mu,\nu}}\|u_k\|^2\quad i,j=1,\ldots,16,\tag{28}$$

and any $k=1,\ldots,\dim\mathfrak{v}_{r,s}$.

We denote

$$\mathbb{R}^{r,s}=\mathfrak{z}_1=\operatorname{span}\{Z_1,\ldots,Z_{r+s}\},\qquad \mathbb{R}^{\mu,\nu}=\mathfrak{z}_2=\operatorname{span}\{\zeta_1,\ldots,\zeta_{\mu+\nu}\},$$

Equality (27) and Theorem 5 show that the pseudo H-type Lie algebra $\mathfrak{n}_{r,s}=\mathfrak{v}_{r,s}\oplus\mathbb{R}^{r,s}$ is a subalgebra of $\mathfrak{n}_{r+\mu,s+\nu}=\mathfrak{v}_{r+\mu,s+\nu}\oplus\mathbb{R}^{r+\mu,s+\nu}$. We have 16 such subalgebras, which have the form

$$\mathfrak{v}_{r,s}\otimes\operatorname{span}\{v_k\}\oplus\mathbb{R}^{r,s},\quad v_k\in\mathfrak{v}_{\mu,\nu},\quad k=1,\ldots,16.$$

Analogously, (28) implies that there are $m=\dim(\mathfrak{v}_{r,s})$ subalgebras isomorphic to $\mathfrak{n}_{\mu,\nu}$ inside the Lie algebra $\mathfrak{n}_{r+\mu,s+\nu}=\mathfrak{v}_{r+\mu,s+\nu}\oplus\mathbb{R}^{r+\mu,s+\nu}$. Analogously to the above, they have the form

$$\operatorname{span}\{u_k\}\otimes\mathfrak{v}_{\mu,\nu}\oplus\mathbb{R}^{\mu,\nu},\quad u_k\in\mathfrak{v}_{r,s},\quad k=1,\ldots,\dim(\mathfrak{v}_{r,s}).$$

Now, we want to apply Corollary 3. We set

$$\mathfrak{v}_1=\mathfrak{v}_{r,s}\otimes E^{+1}_{\mu,\nu},\quad \mathfrak{v}_2=\bigoplus_{j=1}^{8}\Big(\mathfrak{v}_{r,s}\otimes J_{\zeta_j}E^{+1}_{\mu,\nu}\Big)\bigoplus_{j=2}^{8}\Big(\mathfrak{v}_{r,s}\otimes J_{\zeta_1}J_{\zeta_j}E^{+1}_{\mu,\nu}\Big).$$

Under this notation we obtain from (27) for $v_1=v\in E^{+1}_{\mu,\nu}$

$$\mathfrak{v}_{r+\mu,s+\nu}=\mathfrak{v}_1\oplus\mathfrak{v}_2,\quad \mathfrak{n}_1=\mathfrak{v}_1\oplus\mathfrak{z}_1\cong\mathfrak{n}_{r,s}$$

and moreover

$$J_{Z_i}(\mathfrak{v}_1)\subset\mathfrak{v}_1\ \text{ for all }\ Z_i\in\mathfrak{z}_1,\qquad J_{\zeta_j}(\mathfrak{v}_1)\subset\mathfrak{v}_2\ \text{ for all }\ \zeta_j\in\mathfrak{z}_2.$$

This implies that the pseudo H-type Lie algebra $\mathfrak{n}_1\cong\mathfrak{n}_{r,s}$ generates a totally geodesic subgroup $N_{r,s}$ in $N_{r+\mu,s+\nu}$. By the hypothesis of the theorem $N_{r,s}$ is not geodesic orbit and Corollary 3 implies that $N_{r+\mu,s+\nu}$ is neither geodesic orbit. ■

Theorem 9. *Pseudo H-type Lie groups $N_{r,s}$ with $\max\{r,s\}\geq 8$ and $\min\{r,s\}\geq 4$ are not geodesic orbit.*

Proof. Let $N_{r',s'}$ be a pseudo H-type Lie group with $\max\{r,s\}\geq 8$ and $\min\{r,s\}\geq 4$. Then write $r'=r+\mu$, $s'=s+\nu$ for some $(\mu,\nu)\in\{(8,0),(0,8),(4,4)\}$. Arguing as in the proof of Theorem 8, we show that for

$$\mathbb{R}^{r,s}=\mathfrak{z}_1=\operatorname{span}\{Z_1,\ldots,Z_{r+s}\},\qquad \mathbb{R}^{\mu,\nu}=\mathfrak{z}_2=\operatorname{span}\{\zeta_1,\ldots,\zeta_{\mu+\nu}\},$$

and

$$\mathfrak{v}_1 = \mathrm{span}\{u\} \otimes \mathfrak{v}_{\mu,\nu}, \quad \mathfrak{v}_2 = \mathfrak{v}_{r,s} \otimes E^{+1}_{\mu,\nu},$$

where $u \in E^{+1}_{r,s}$ with $\langle u, u\rangle_{\mathfrak{v}_{r,s}} = 1$, the Lie algebra

$$\mathfrak{n}_2 = \mathfrak{v}_2 \oplus \mathfrak{z}_2 \cong \mathfrak{n}_{\mu,\nu}$$

is a subalgebra of $\mathfrak{n}_{r+\mu,s+\nu}$. Moreover

$$J_{Z_i}(\mathfrak{v}_2) \subset \mathfrak{v}_2 \ \text{ for all } \ Z_i \in \mathfrak{z}_1, \qquad J_{\zeta_j}(\mathfrak{v}_2) \subset \mathfrak{v}_1 \ \text{ for all } \ \zeta_j \in \mathfrak{z}_2.$$

It implies that the pseudo H-type Lie algebra $\mathfrak{n}_2 \cong \mathfrak{n}_{\mu,\nu}$ generates a totally geodesic subgroup $N_{\mu,\nu}$ in $N_{r+\mu,s+\nu}$. By Corollary 4 the group $N_{\mu,\nu}$ is not geodesic orbit and Corollary 3 implies that $N_{r+\mu,s+\nu}$ is neither geodesic orbit. ■

6.3. Pseudo H-type groups $N_{r,s}$, for $(r, 1)$, $r \geq 3$ and $(0, s)$, $s > 0$.

Theorem 10. *The pseudo H-type Lie groups $N_{0,s}$ with $s \geq 4$ are not geodesic orbit.*

Proof. Let $\{Z_i\}_{i=1}^{s}$ be an orthonormal basis of the center $\mathbb{R}^{0,s}$. Consider a positive involution $p = Z_1Z_2Z_3Z_4$ and a vector $X \in \mathfrak{v}$, $\langle X, X\rangle_{\mathfrak{v}} = 1$ such that $J_p(X) = X$.

Assume that a geodesic defined by an initial vector $(X, Z_1) \in \mathfrak{n}_{0,s}$ is homogeneous. Then there is $D = (C, A) \in \mathfrak{h}$ such that

$$A(X) = J_{Z_1}(X), \quad C(z_1) = 0, \quad [A, J_Z] = J_{C(Z)}, \quad Z \in \mathbb{R}^{0,s}.$$

Then we obtain

$$\begin{aligned} J_{Z_1}(X) &= A(X) = A(J_p(X)) && (29)\\ &= J_pA(X) - J_{Z_1}J_{C(Z_2)}J_{Z_3}J_{Z_4}(X) - J_{Z_1}J_{Z_2}J_{C(Z_3)}J_{Z_4}(X) - J_{Z_1}J_{Z_2}J_{Z_3}J_{C(Z_4)}(X)\\ &= -J_{Z_1}(X) - J_{Z_1}J_{C(Z_2)}J_{Z_3}J_{Z_4}(X) - J_{Z_1}J_{Z_2}J_{C(Z_3)}J_{Z_4}(X) - J_{Z_1}J_{Z_2}J_{Z_3}J_{C(Z_4)}(X), \end{aligned}$$

where in the last step we used

$$J_pA(X) = J_pJ_{Z_1}(X) = -J_{Z_1}J_p(X) = -J_{Z_1}(X). \tag{30}$$

Hence $2J_{Z_1}(X) = Y$ with

$$Y = -J_{Z_1}J_{C(Z_2)}J_{Z_3}J_{Z_4}(X) - J_{Z_1}J_{Z_2}J_{C(Z_3)}J_{Z_4}(X) - J_{Z_1}J_{Z_2}J_{Z_3}J_{C(Z_4)}(X).$$

From one side Y satisfies $\langle Y, Y\rangle > 0$ by (8). From the other side $\langle 2J_{Z_1}(X), 2J_{Z_1}(X)\rangle = -4$, which is a contradiction. ■

Theorem 11. *The pseudo H-type Lie groups $N_{r,1}$, $r \geq 2$, are not geodesic orbit.*

Proof. Fix a basis $\{Z_i\}_i^{r+1}$ as in (17). Let $\mathcal{S}$ be a maximal subgroup of $G(B_{r,1})$ of positive involutions and let $PI_{r,1}$ be its generating set. Note that $PI_{r,1} = PI_{r,0}$ by the structure of involutions, see (18), which says that none of positive involutions in $PI_{r,1}$ contains the basis vector Z_{r+1}. We assume that the set $PI_{r,1}$ contains an involution $q = Z_{i_1} \dots J_{Z_{i_a}}$ of Type 2 as was defined in (18). This assumption always can be achieved by removing one of the basis vectors or multiplying some of the involutions.

Let us choose a geodesic $\gamma(t) = (z(t), x(t))$ with an initial vector (Z_{r+1}, X) such that $\langle X, X\rangle_{\mathfrak{v}} = 1$, and $J_q(X) = X$ and assume that the geodesic is homogeneous. Then there is $D = (C, A) \in \mathfrak{h}$ such that

$$A(X) = J_{Z_{r+1}}(X), \quad C(Z_{r+1}) = 0, \quad [A, J_Z] = J_{C^{\tau}(Z)}, \quad Z \in \mathbb{R}^{r,0}.$$

The condition $C(Z_{r+1}) = 0$ implies that C has vanishing last column. Now as in (29) of Theorem 10 we have an equality

$$J_{Z_{r+1}}(X) = A(X) = A(J_q(X)) = J_q A(X) - J_{\dot{q}}(X) = -J_{Z_{r+1}}(X) - J_{\dot{q}}(X)$$

with

$$\dot{q} = C(Z_{i_1}) \cdots Z_{i_a} + Z_{i_1} C(Z_{i_2}) \cdots Z_{i_a} + \ldots + Z_{i_1} Z_{i_2} \cdots C(Z_{i_a}).$$

In the last equality we used $J_q J_{Z_{r+1}}(X) = -J_{Z_{r+1}} J_q(X)$ since q contains odd number of basis vectors and does not contain Z_{r+1}. The rest of calculations as in (30).

$$2J_{Z_{r+1}}(X) = -J_{\dot{q}}(X).$$

Note that $\dot{q}$ does not include Z_{r+1} because $C(Z_i) = \sum_{j=1}^{r} c_{ij} Z_j$ since $c_{ij} = 0$ for $j = r+1$. Therefore $\langle J_{\dot{q}}(X), J_{\dot{q}}(X)\rangle_{\mathfrak{v}} \geq 0$ by (8). From the other side $\langle 2J_{Z_{r+1}}(X), 2J_{Z_{r+1}}(X)\rangle_{\mathfrak{v}} = -4$, which is a contradiction. ■

We summarize the results of Sections 4, 5 and 6 and show in Table 1 the cases of pseudo H-type Lie groups $N_{r,s}$ that need to be studied.

Table 1. Pseudo H-type Lie groups need to be studied

7			$N_{2,7}$	$N_{3,7}$				
6		$N_{1,6}$		$N_{3,6}$				
5		$N_{1,5}$	$N_{2,5}$					
4		$N_{1,4}$	$N_{2,4}$	$N_{3,4}$				
3			$N_{2,3}$	$N_{3,3}$	$N_{4,3}$		$N_{6,3}$	$N_{7,3}$
2				$N_{3,2}$	$N_{4,2}$	$N_{5,2}$		$N_{7,2}$
1								
0								
s/r	0	1	2	3	4	5	6	7

6.4. **Relation between $\mathfrak{n}_{r,s}$, $\mathfrak{n}_{r+1,s}$, and $\mathfrak{n}_{r,s+1}$.** In this section we present some of the arguments based on the structure of the generating set for the maximal group of positive involutions, which allows us to find out which of the groups in Table 1 are not geodesic orbit.

Let $\ell(r,s)$ be the number of the mutually commuting positive involutions in the generating set $PI_{r,s}$ of the maximal group of positive involutions $\mathcal{S}$. Recall that the number $\ell(r,s)$ is periodic with the three periods $(8,0)$, $(4,4)$ and $(0,8)$, that is,

$$\ell(r+8,s) = \ell(r,s+8) = \ell(r+4,s+4) = \ell(r,s) + 4.$$

Table 2 shows the values $\ell(r,s)$ which are interesting for us. In general there are two cases of the dimensions of minimal admissible modules $\mathfrak{v}_{r,s}$ according to the relations of the values $\ell(r,s)$, $\ell(r+1,s)$ and $\ell(r,s+1)$:

$$\ell(r,s) \leq \ell(r+1,s) \leq \ell(r,s) + 1 \text{ and } \ell(r,s) \leq \ell(r,s+1) \leq \ell(r,s) + 1.$$

TABLE 2. The value $\ell(r,s)$ for $r+s\le 16$

7	3	3	3	4				
6	2	3	3	4				
5	1	2	3	4				
4	1	2	3	4				
3	0	1	2	3	3	3	3	4
2	0	1	1	2	2	3	3	4
1	0	0	0	1	1	2	3	4
0	0	0	0	1	1	2	3	4
s/r	0	1	2	3	4	5	6	7

Then

$$\begin{aligned}
&[1]\quad \ell(r,s)=\ell(r+1,s) \text{ implies } \dim\mathfrak{v}_{r+1,s}=2\dim\mathfrak{v}_{r,s},\\
&[2]\quad \ell(r,s)+1=\ell(r+1,s) \text{ implies } \dim\mathfrak{v}_{r+1,s}=\dim\mathfrak{v}_{r,s},\\
&[3]\quad \ell(r,s)=\ell(r,s+1) \text{ implies } \dim\mathfrak{v}_{r,s+1}=2\dim\mathfrak{v}_{r,s},\\
&[4]\quad \ell(r,s)+1=\ell(r,s+1) \text{ implies } \dim\mathfrak{v}_{r,s+1}=\dim\mathfrak{v}_{r,s}.
\end{aligned}$$

In cases [1] and [3] a module $\mathfrak{v}_{r,s}$ is a submodule of $\mathfrak{v}_{r+1,s}$ ($\mathfrak{v}_{r,s+1}$), which is not true for cases [2] and [4]. This can be shown as follows.

We fix a set $PI_{r,s}$, construct an orthonormal basis $\{X_i\}_{i=1}^{\dim\mathfrak{v}_{r,s}}$ for $\mathfrak{v}_{r,s}$ as in Proposition 6 generated by a vector v, $\langle v,v\rangle_{\mathfrak{v}_{r,s}}=1$ which is 1-eigen vector for all the involutions in $PI_{r,s}$. We restrict the proof to the case $\mathfrak{n}_{r,s+1}$, since for $\mathfrak{n}_{r+1,s}$ the arguments are similar. The set $\mathfrak{v}_{r,s}=\operatorname{span}\{X_i,\ i=1,\dots,\dim\mathfrak{v}_{r,s}\}$ is a subset of $\mathfrak{v}_{r,s+1}$ and a module under the action of $\mathrm{Cl}(\mathbb{R}^{r,s})$. The map $J_{Z_{r+s+1}}$ is an orthogonal transformation on $\mathfrak{v}_{r,s+1}$ and therefore $J_{Z_{r+s+1}}(\mathfrak{v}_{r,s})\subset\mathfrak{v}_{r,s+1}$ and moreover $J_{Z_{r+s+1}}(\mathfrak{v}_{r,s})$ is a minimal admissible module for $\mathrm{Cl}(\mathbb{R}^{r,s})$. Thus

$$\mathfrak{v}_{r,s+1}=\mathfrak{v}_{r,s}\oplus J_{Z_{r+s+1}}(\mathfrak{v}_{r,s})$$

is an orthogonal decomposition in two minimal admissible modules for $\mathrm{Cl}(\mathbb{R}^{r,s})$. The set $\{X_i, J_{Z_{r+s+1}}(X_i),\ i=1,\dots,\dim(\mathfrak{v}_{r,s})\}$ is an orthonormal basis for $\mathfrak{v}_{r,s+1}$.

Theorem 12. *In the above notation in cases* [1] *and* [3] *the pseudo H-type Lie groups $N_{r,s}$ are totally geodesic submanifolds of $N_{r+1,s}$ and $N_{r,s+1}$, respectively.*

Proof. We write for $\mathfrak{n}_{r,s+1}$

$$\mathbb{R}^{r,s}=\mathfrak{z}_1=\operatorname{span}\{Z_1,\dots,Z_{r+s}\},\quad \mathfrak{z}_2=\operatorname{span}\{Z_{r+s+1}\},$$

and

$$\mathfrak{v}_1=\mathfrak{v}_{r,s}\quad \mathfrak{v}_2=J_{Z_{r+s+1}}(\mathfrak{v}_{r,s}).$$

Then we obtain that

$$J_{Z_k}(\mathfrak{v}_1)\subset\mathfrak{v}_1,\quad \text{for any}\quad Z_k\in\mathfrak{z}_1$$

since $\mathfrak{v}_1$ is a submodule of $\mathrm{Cl}(\mathbb{R}^{r,s})$. We also have

$$J_{Z_{r+s+1}}(\mathfrak{v}_1)\subset\mathfrak{v}_2.$$

Applying Theorem 5 and Corollary 3 we finish the proof. ∎

Corollary 5. *The pseudo H-type Lie groups*

$$N_{2,7}, \quad N_{3,6}, \quad N_{3,7} \quad N_{6,3}, \quad N_{7,2}, \quad N_{7,3}$$

are not geodesic orbit

Proof. Applying Theorem 12 to the groups in Table 1 and using Table 2 we finish the proof. For instance since $N_{7,1}$ is not geodesic orbit and

$$\ell(7,1) = \ell(7,2) = \ell(7,3).$$

We conclude that $N_{7,2}$ and $N_{7,3}$ are not geodesic orbit. ■

Remark 4. One can show the following. In the cases [2] and [4], the module $\mathfrak{v}_{r+1,s}$ (or $\mathfrak{v}_{r,s+1}$) is also a minimal admissible module of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{r,s})$. In these cases the natural inclusion map

$$\mathfrak{n}_{r,s}(\mathfrak{v}_{r,s}) = \mathfrak{n}_{r,s}(\mathfrak{v}_{r+1,s}) \subset \mathfrak{n}_{r+1,s}(\mathfrak{v}_{r+1,s})$$

is not a Lie algebra homomorphism

Theorem 13. *The pseudo H-type Lie groups $N_{r,s}$ for*

$$(r,s) \in \{(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(3,2),(3,3),(4,2),(4,3),(5,2)\}$$

are not geodesic orbit.

Proof. We start from *pseudo H-type Lie groups $N_{1,4}$ and $N_{3,2}$*.

We choose an orthonormal basis for centers as in (17) and the generating set $\{p_1, p_2\}$ for the maximal group of positive involutions, where

$$p_1 = Z_1Z_2Z_3, \quad p_2 = Z_2Z_3Z_4Z_5.$$

Let $v \in \{X \in \mathfrak{v} : \ J_{p_1}(X) = J_{p_2}(X) = X\}$, $\|v\|^2 = 1$. We construct an orthonormal invariant basis for $\mathfrak{v}$:

$$\begin{array}{llll} X_1 = v, & X_2 = J_{Z_2}(v), & X_3 = J_{Z_4}(v), & X_4 = J_{Z_2}J_{Z_4}(v), \\ X_5 = J_{Z_1}(v), & X_6 = J_{Z_3}(v), & X_7 = J_{Z_5}(v), & X_8 = J_{Z_3}J_{Z_4}(v). \end{array}$$

We denote

$$\begin{array}{c} \mathfrak{z}_1 = \mathrm{span}\{Z_2, Z_4\}, \quad \mathfrak{z}_2 = \mathrm{span}\{Z_1, Z_3, Z_5\}, \quad \mathfrak{z} = \mathfrak{z}_1 \oplus \mathfrak{z}_2, \\ \mathfrak{v}_1 = \mathrm{span}\{X_1, \dots X_4\}, \quad \mathfrak{v}_2 = \mathrm{span}\{X_5, \dots, X_8\}, \quad \mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2. \end{array}$$

It is obvious that $J_{Z_k}(\mathfrak{v}_1) \subset \mathfrak{v}_1$ for $k = 1, 4$. If $(r,s) = (1,4)$, then the pseudo H-type Lie algebra $\mathfrak{z}_1 \oplus \mathfrak{v}_1$ is isomorphic to $\mathfrak{n}_{0,2}$. If $(r,s) = (3,2)$, then the pseudo H-type Lie algebra $\mathfrak{z}_1 \oplus \mathfrak{v}_1$ is isomorphic to $\mathfrak{n}_{1,1}$.

To show that $J_{Z_k}(\mathfrak{v}_1) \subset \mathfrak{v}_2$ for $k = 2, 3, 5$ we observe that $p_1p_2 = -Z_1Z_4Z_5$ and $J_{Z_1Z_4Z_5}(v) = -v$. Then it is easy to see the following

$$\begin{array}{llll} J_{Z_1}(X_1) = \pm X_5, & J_{Z_1}(X_2) = \pm X_6, & J_{Z_1}(X_3) = \pm X_7, & J_{Z_1}(X_4) = \pm X_8, \\ J_{Z_3}(X_1) = \pm X_6, & J_{Z_3}(X_2) = \pm X_5, & J_{Z_3}(X_3) = \pm X_8, & J_{Z_3}(X_4) = \pm X_7, \\ J_{Z_5}(X_1) = \pm X_7, & J_{Z_5}(X_2) = \pm X_8, & J_{Z_5}(X_3) = \pm X_5, & J_{Z_5}(X_4) = \pm X_6. \end{array}$$

Theorem 5 and Corollary 3 imply that pseudo H-type Lie groups $N_{1,4}$ and $N_{3,2}$ are not geodesic orbit.

Consider *pseudo H-type Lie groups $N_{1,6}$ and $N_{5,2}$*.

We choose an orthonormal basis for the center as in (17) and the generating set $\{p_1, p_2, p_3\}$ for the maximal group of positive involutions, where

$$p_1 = Z_1Z_2Z_3, \quad p_2 = Z_2Z_3Z_4Z_5, \quad p_2 = Z_2Z_3Z_6Z_7.$$

Let $v \in \{X \in \mathfrak{v} : J_{p_1}(X) = J_{p_2}(X) = J_{p_3}(X) = X\}$, $\|v\|^2 = 1$. We construct an orthonormal invariant basis for $\mathfrak{v}$:

$$\begin{array}{llll}
X_1 = v, & X_2 = J_{Z_2}(v), & X_3 = J_{Z_4}(v), & X_4 = J_{Z_2}J_{Z_4}(v), \\
X_5 = J_{Z_1}(v), & X_6 = J_{Z_3}(v), & X_7 = J_{Z_5}(v), & X_8 = J_{Z_3}J_{Z_4}(v), \\
X_9 = J_{Z_6}(v), & X_{10} = J_{Z_7}(v), & X_{11} = J_{Z_2}J_{Z_6}(v), & X_{12} = J_{Z_2}J_{Z_7}(v), \\
X_{13} = J_{Z_4}J_{Z_6}(v), & X_{14} = J_{Z_4}J_{Z_7}(v), & X_{15} = J_{Z_2}J_{Z_4}J_{Z_6}(v), & X_6 = J_{Z_2}J_{Z_4}J_{Z_7}(v).
\end{array}$$

We denote

$$\mathfrak{z}_1 = \mathrm{span}\{Z_2, Z_4\}, \quad \mathfrak{z}_2 = \mathrm{span}\{Z_1, Z_3, Z_5, Z_6, Z_7\}, \quad \mathfrak{z}_{1,6} = \mathfrak{z}_1 \oplus \mathfrak{z}_2$$

$$\mathfrak{v}_1 = \mathrm{span}\{X_1, \dots X_4\}, \quad \mathfrak{v}_2 = \mathrm{span}\{X_5, \dots, X_{16}\}, \quad \mathfrak{v}_{1,6} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$$

As in the previous case we show that if $(r, s) = (1, 6)$, then $\mathfrak{z}_1 \oplus \mathfrak{v}_1$ is isomorphic to $\mathfrak{n}_{0,2}$ and if $(r, s) = (5, 2)$, then $\mathfrak{z}_1 \oplus \mathfrak{v}_1$ is isomorphic to $\mathfrak{n}_{1,1}$. Theorem 5 and Corollary 3 imply that the groups $N_{1,6}$ and $N_{5,2}$ are not geodesic orbit.

Pseudo H-type Lie group $N_{2,3}$. We choose an orthonormal basis for the center as in (17) and the generating set $PI_{2,3} = \{p_1, p_2\}$ for the maximal group of positive involutions, where

$$p_1 = Z_1Z_4Z_5, \quad p_2 = Z_1Z_2Z_3Z_4.$$

Let $v \in \{X \in \mathfrak{v}_{2,3} : J_{p_1}(X) = J_{p_2}(X) = X\}$, $\|v\|^2 = 1$. We construct an orthonormal invariant basis as in the case $N_{3,2}$. All other calculations are analogous to the case $N_{3,2}$.

Pseudo H-type Lie groups $N_{2,4}$ and $N_{3,3}$. We choose an orthonormal basis for the center as in (17) and the generating set $\{p_1, p_2, p_3\}$ for the maximal group of positive involutions, where

$$p_1 = Z_1Z_4Z_5, \quad p_2 = Z_1Z_2Z_3Z_4, \quad p_3 = Z_1Z_2Z_5Z_5$$

Let $v \in \{X \in \mathfrak{v} : J_{p_1}(X) = J_{p_2}(X) = J_{p_3}(X) = X\}$, $\|v\|^2 = 1$. We construct an orthonormal invariant basis for $\mathfrak{v}$:

$$\begin{array}{llll}
X_1 = v, & X_2 = J_{Z_5}(v), & X_3 = J_{Z_6}(v), & X_4 = J_{Z_5}J_{Z_6}(v), \\
X_5 = J_{Z_1}(v), & X_6 = J_{Z_2}(v), & X_7 = J_{Z_3}(v), & X_8 = J_{Z_4}(v).
\end{array}$$

We denote

$$\mathfrak{z}_1 = \mathrm{span}\{Z_5, Z_6\}, \quad \mathfrak{z}_2 = \mathrm{span}\{Z_1, Z_2, Z_3, Z_4\}, \quad \mathfrak{z} = \mathfrak{z}_1 \oplus \mathfrak{z}_2,$$

$$\mathfrak{v}_1 = \mathrm{span}\{X_1, \dots X_4\}, \quad \mathfrak{v}_2 = \mathrm{span}\{X_5, \dots, X_8\}, \quad \mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2.$$

It is obvious that $J_{Z_k}(\mathfrak{v}_1) \subset \mathfrak{v}_1$ for $k = 5, 6$. The pseudo H-type Lie algebra $\mathfrak{z}_1 \oplus \mathfrak{v}_1$ is isomorphic to $\mathfrak{n}_{0,2}$. To show that $J_{Z_k}(\mathfrak{v}_1) \subset \mathfrak{v}_2$ for $k = 1, 2, 3, 4$ one can check the following

$$\begin{array}{llll}
J_{Z_1}(X_1) = \pm X_5, & J_{Z_1}(X_2) = \pm X_8, & J_{Z_1}(X_3) = \pm X_7, & J_{Z_1}(X_4) = \pm X_6, \\
J_{Z_2}(X_1) = \pm X_6, & J_{Z_2}(X_2) = \pm X_7, & J_{Z_2}(X_3) = \pm X_8, & J_{Z_2}(X_4) = \pm X_5, \\
J_{Z_3}(X_1) = \pm X_7, & J_{Z_3}(X_2) = \pm X_6, & J_{Z_3}(X_3) = \pm X_5, & J_{Z_3}(X_4) = \pm X_8, \\
J_{Z_4}(X_1) = \pm X_8, & J_{Z_4}(X_2) = \pm X_5, & J_{Z_4}(X_3) = \pm X_6, & J_{Z_4}(X_4) = \pm X_7.
\end{array}$$

Theorem 5 and Corollary 3 imply that pseudo H-type Lie groups $N_{2,4}$ and $N_{3,3}$ are not geodesic orbit.

To finish the proof we apply Theorem 12 to the cases

$$\ell(1, 4) = \ell(1, 5), \quad \ell(3, 2) = \ell(4, 2), \quad \ell(2, 4) = \ell(2, 5), \quad \ell(3, 3) = \ell(4, 3).$$

■

7. Pseudo H-type nilmanifold $N_{3,4}$

It is known that the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{3,4})$ has two non-equivalent minimal admissible 8-dimensional modules. Moreover, the corresponding 15-dimensional pseudo H-type Lie algebras are isomorphic and isometric, see [25, Theorem 12]. Therefore, it sufficient to check only one such pseudo H-type nilmanifold $N_{3,4}$.

The pseudo H-type nilmanifold $N_{3,4}$ has dimension 15, and the corresponding left invariant metric is generated by the scalar product $\langle\cdot\,,\cdot\rangle_{3,4} + \langle\cdot\,,\cdot\rangle_{4,4}$ on the Lie algebra $\mathfrak{n}_{3,4} = \mathbb{R}^{3,4} \oplus \mathbb{R}^{4,4}$. We define the orthonormal basis $\{V_1, \dots, V_8, Z_1, \dots, Z_7\}$ by taking $v \in \mathbb{R}^{4,4}$ with $\|v\|^2 = 1$ and setting $\|Z_1\|^2 = \|Z_2\|^2 = \|Z_3\|^2 = 1$, $\|Z_4\|^2 = \|Z_5\|^2 = \|Z_6\|^2 = \|Z_7\|^2 = -1$. We use involutions p_i, $i = 1, 2, 3, 4$, where

$$\begin{array}{ll} J_{p_1}(v) = J_{Z_1Z_2Z_4Z_5}(v) = v, & J_{p_2}(v) = J_{Z_1Z_2Z_6Z_7}(v) = v, \\ J_{p_3}(v) = J_{Z_1Z_3Z_5Z_7}(v) = v, & J_{p_4}(v) = J_{Z_1Z_2Z_3}(v) = v, \end{array} \tag{31}$$

with

$$J_{Z_1}^2 = J_{Z_2}^2 = J_{Z_3}^2 = -\,\mathrm{Id}, \quad J_{Z_4}^2 = J_{Z_5}^2 = J_{Z_6}^2 = J_{Z_7}^2 = \mathrm{Id}\ .$$

Note that

$$\begin{array}{lllllll} J_{Z_1}(v) & = & -J_{Z_2}J_{Z_3}(v) & = & J_{Z_4}J_{Z_7}(v) & = & J_{Z_5}J_{Z_6}(v), \\ J_{Z_2}(v) & = & J_{Z_1}J_{Z_3}(v) & = & J_{Z_4}J_{Z_6}(v) & = & -J_{Z_5}J_{Z_7}(v), \\ J_{Z_3}(v) & = & -J_{Z_1}J_{Z_2}(v) & = & J_{Z_4}J_{Z_5}(v) & = & J_{Z_6}J_{Z_7}(v), \\ J_{Z_4}(v) & = & J_{Z_1}J_{Z_7}(v) & = & J_{Z_2}J_{Z_6}(v) & = & J_{Z_3}J_{Z_5}(v), \\ J_{Z_5}(v) & = & -J_{Z_2}J_{Z_7}(v) & = & -J_{Z_3}J_{Z_4}(v) & = & J_{Z_1}J_{Z_6}(v), \\ J_{Z_6}(v) & = & -J_{Z_1}J_{Z_5}(v) & = & -J_{Z_2}J_{Z_4}(v) & = & J_{Z_3}J_{Z_7}(v), \\ J_{Z_7}(v) & = & -J_{Z_1}J_{Z_4}(v) & = & J_{Z_2}J_{Z_5}(v) & = & -J_{Z_3}J_{Z_6}(v). \end{array}$$

We construct a basis for $\mathfrak{v}$ by setting $V_1 = v$ and $V_i = J_{Z_{i-1}}(v)$.

Then we have

$$\langle V_k, V_k\rangle_{4,4} = -\langle V_l, V_l\rangle_{4,4} = 1, \quad k = 1, \dots, 4, \quad l = 5, \dots, 8,$$

and the operators J_{Z_k}, $k = 1, \dots, 8$ with respect to this basis take the form

$$\begin{gathered} J_{Z_1} = \begin{pmatrix} 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&-1&0&0&0 \end{pmatrix}, \quad J_{Z_2} = \begin{pmatrix} 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{pmatrix}, \quad J_{Z_3} = \begin{pmatrix} 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0 \end{pmatrix}, \\ J_{Z_4} = \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{pmatrix}, \quad J_{Z_5} = \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0 \end{pmatrix}, \quad J_{Z_6} = \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&-1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{pmatrix}, \\ J_{Z_7} = \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}. \end{gathered} \tag{32}$$

The operators J_{Z_k}, $k = 1, \dots, 7$, span a 7-dimensional subspace in $\mathbf{V} \subset \mathfrak{so}(4,4)$, nevertheless the vector space $\mathbf{V}$ is not a Lie subalgebra of $\mathfrak{so}(4,4)$. Hence, $N_{3,4}$ is not a naturally reductive manifold.

TABLE 3. Commutators in $\mathfrak{n}_{3,4}$

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
V_1	0	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7
V_2	$-Z_1$	0	$-Z_3$	Z_2	$-Z_7$	$-Z_6$	Z_5	Z_4
V_3	$-Z_2$	Z_3	0	$-Z_1$	$-Z_6$	Z_7	Z_4	$-Z_5$
V_4	$-Z_3$	$-Z_2$	Z_1	0	$-Z_5$	Z_4	$-Z_7$	Z_6
V_5	$-Z_4$	Z_7	Z_6	Z_5	0	Z_3	Z_2	Z_1
V_6	$-Z_5$	Z_6	$-Z_7$	$-Z_4$	$-Z_3$	0	Z_1	$-Z_2$
V_7	$-Z_6$	$-Z_5$	$-Z_4$	Z_7	$-Z_2$	$-Z_1$	0	Z_3
V_8	$-Z_7$	$-Z_4$	Z_5	$-Z_6$	$-Z_1$	Z_2	$-Z_3$	0

The direct computations shows that the centralizer $\mathbf{Z}$ of $\mathbf{V}$ in the isotropy subalgebra is trivial. Hence, the normalizer $\mathbf{N} = \mathbf{Z} \oplus [\mathbf{V}, \mathbf{V}]$ of $\mathbf{V}$ is the linear span of all matrices of the following type: $J_{ik} := [J_{Z_i}, J_{Z_k}]$, $1 \leq i \leq k \leq 7$. In particular, $\dim \mathbf{N} = 21$ and any operator $B \in \mathbf{N}$ has the form $B = \sum_{i,k} x_{ik} J_{ik}$ for some $x_{ik} \in \mathbb{R}$, $1 \leq i < k \leq 7$. An explicit form of the operator B is as follows:

$$
2\begin{pmatrix}
0 & x_{23}-x_{47}-x_{56} & -x_{13}-x_{46}+x_{57} & x_{12}-x_{45}-x_{67} & x_{17}+x_{26}+x_{35} & x_{16}-x_{27}-x_{34} & -x_{15}-x_{24}+x_{37} & -x_{14}+x_{25}-x_{36} \\
-x_{23}+x_{47}+x_{56} & 0 & -x_{12}-x_{45}-x_{67} & -x_{13}+x_{46}-x_{57} & x_{14}+x_{25}-x_{36} & x_{15}-x_{24}+x_{37} & x_{16}+x_{27}+x_{34} & x_{17}-x_{26}-x_{35} \\
x_{13}+x_{46}-x_{57} & x_{12}+x_{45}+x_{67} & 0 & -x_{23}-x_{47}-x_{56} & -x_{15}+x_{24}+x_{37} & x_{14}+x_{25}+x_{36} & -x_{17}+x_{26}-x_{35} & x_{16}+x_{27}-x_{34} \\
-x_{12}+x_{45}+x_{67} & x_{13}-x_{46}+x_{57} & x_{23}+x_{47}+x_{56} & 0 & x_{16}-x_{27}+x_{34} & -x_{17}-x_{26}+x_{35} & -x_{14}+x_{25}+x_{36} & x_{15}+x_{24}+x_{37} \\
x_{17}+x_{26}+x_{35} & x_{14}+x_{25}-x_{36} & -x_{15}+x_{24}+x_{37} & x_{16}-x_{27}+x_{34} & 0 & x_{12}+x_{45}-x_{67} & -x_{13}+x_{46}+x_{57} & x_{23}+x_{47}-x_{56} \\
x_{16}-x_{27}-x_{34} & x_{15}-x_{24}+x_{37} & x_{14}+x_{25}+x_{36} & -x_{17}-x_{26}+x_{35} & -x_{12}-x_{45}+x_{67} & 0 & x_{23}-x_{47}+x_{56} & x_{13}+x_{46}+x_{57} \\
-x_{15}-x_{24}+x_{37} & x_{16}+x_{27}+x_{34} & -x_{17}+x_{26}-x_{35} & -x_{14}+x_{25}+x_{36} & x_{13}-x_{46}-x_{57} & -x_{23}+x_{47}-x_{56} & 0 & x_{12}-x_{45}+x_{67} \\
-x_{14}+x_{25}-x_{36} & x_{17}-x_{26}-x_{35} & x_{16}+x_{27}-x_{34} & x_{15}+x_{24}+x_{37} & -x_{23}-x_{47}+x_{56} & -x_{13}-x_{46}-x_{57} & -x_{12}+x_{45}-x_{67} & 0
\end{pmatrix}
$$

Remark 5. Recall that the matrices J_{Z_i}, $i = 1, \dots, 7$, as well as B are skew-symmetric with respect the inner product $\langle \cdot, \cdot \rangle_{4,4}$. Therefore, if A is one of such matrices and $Y \in \mathfrak{v} = \mathbb{R}^{4,4}$, then the vector $\overline{Y} = A(Y)$ is such that

$$\langle Y, \overline{Y} \rangle_{4,4} = y_1\overline{y}_1 + y_2\overline{y}_2 + y_3\overline{y}_3 + y_4\overline{y}_4 - y_5\overline{y}_5 - y_6\overline{y}_6 - y_7\overline{y}_7 - y_8\overline{y}_8 = 0.$$

This implies that for a given Y, the linear space $\mathbf{L}(Y)$ for $\mathbf{L} = \mathbf{V} \oplus [\mathbf{V}, \mathbf{V}]$ has dimension ≤ 7.

The following result is useful for various computations.

Lemma 4. *If a matrix Lie group G with the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(m, \mathbb{R})$ preserves the value $\sum_{ij} c_{ij} y_i y_j$ for some fixed $Y = (y_1, \dots, y_m)$, where $c_{ij} \in \mathbb{R}$, then for any matrix $A \in \mathfrak{g}$ we have $\sum_{ij} c_{ij}\Big((A(Y))_i y_j + (A(Y))_j y_i\Big) = 0$, where $A(Y)$ is the image of Y under the action of A.*

Proof. For any $Q \in G$ and any $Y \in \mathbb{R}^m$ we have $\sum_{ij} c_{ij}(Q(Y))_i(Q(Y))_j = \sum_{ij} c_{ij} y_i y_j$, where $G(Y)$ is the image of Y under the action of the matrix G. If $Q = \exp(tA) = \mathrm{Id} + tA + o(t)$ when $t \to 0$ for some $A \in \mathfrak{g}$, then

$$
\begin{aligned}
\sum_{ij} c_{ij}(Q(Y))_i(Q(Y))_j &= \sum_{ij} c_{ij}(Y + tA(Y))_i(Y + tA(Y))_j + o(t) \\
&= \sum_{ij} c_{ij} y_i y_j + t\sum_{ij} c_{ij}\Big(A(Y)_i y_j + A(Y)_j y_i\Big) + o(t)
\end{aligned}
$$

when $t \to 0$, that proves the lemma. ■

If $N_{3,4}$ is geodesic orbit, then for any $Y \in \mathfrak{v}$ and any $Z \in \mathbf{V}$, there is some $B \in \mathbf{N}$ such that $[B, Z] = 0$ and $B(Y) = Z(Y)$. We aim to find such matrix B. Without loss of generality we may take

$$Y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = y_1V_1 + y_2V_2 + y_3V_3 + y_4V_4 + y_5V_5 + y_6V_6 + y_7V_7 + y_8V_8 \in \mathfrak{v} \tag{33}$$

for any given $y_i \in \mathbb{R}$, $i = 1, \dots, 8$.

Theorem 14. *For any $Z \in \mathbf{V}$ and for any $Y \in \mathbb{R}^{4,4}$, it is possible to find $B \in \mathbf{N} = [\mathbf{V}, \mathbf{V}]$ such that $[B, Z] = 0$ and $B(Y) = Z(Y)$. Hence, $N_{3,4}$ is geodesic orbit pseudo nilmanifold.*

Remark 6. In particular, $N_{3,4}$ is the first example of pseudo H-type geodesic orbit manifolds that is not naturally reductive. Moreover, it is the first pseudo H-type geodesic orbit manifold such that the space $\mathbf{V}$ satisfies the *strong transitive normalizer condition*; that is for every $Y \in \mathfrak{v} = \mathbb{R}^{l,l}$ and every $Z \in \mathbf{V} = J(\mathfrak{z})$ there is some $B \in [\mathbf{V}, \mathbf{V}]$ such that $[B, Z]_{\mathfrak{so}(l,l)} = 0$ and $B(Y) = Z(Y)$.

The proof of Theorem 14 is based on Propositions 9, 10, and 6.

We also use the fact that we have only three classes in the center of $\mathfrak{n}_{3,4}$, up to similarity and the action of the isotropy subgroup. They are represented by Z_1 (positive vector), Z_4 (negative vector), and $Z_1 + Z_4$ (null vector).

Indeed, the group $O(3, 4)$ with the Lie algebra $\mathfrak{so}(3, 4)$, that is a linear span of the operators $[J_{Z_i}, J_{Z_j}]$, $1 \le i < j \le 7$, acts naturally by automorphisms on the center $\mathfrak{z}$. It is well known that $O(3, 4)$ acts transitively on every hyperquadric $Q(r) = \{z \in \mathfrak{z} \mid \langle z, z\rangle_{3,4} = r\}$ with $r \neq 0$, see e.g. [47, page 239] or [55, Theorem 2.4.4].

Now, let us suppose that $z, \overline{z} \in \mathfrak{z}$ and $\langle z, z\rangle_{3,4} = \langle \overline{z}, \overline{z}\rangle_{3,4} = 0$. If $\langle z_p, z_p\rangle_{3,4} = \langle \overline{z}_p, \overline{z}_p\rangle_{3,4} = \rho \neq 0$ or, equivalently, $\langle z_n, z_n\rangle_{3,4} = \langle \overline{z}_n, \overline{z}_n\rangle_{3,4} = -\rho \neq 0$, where the subscripts p and n mean the components of vectors in $\mathrm{span}(Z_1, Z_2, Z_3)$ and $\mathrm{span}(Z_4, Z_5, Z_6, Z_7)$ respectively. Then it easy to see that there is $Q \in O(3) \cdot O(4) \subset O(3, 4)$ such that $Q(z_p) = \overline{z}_p$ and $Q(z_n) = \overline{z}_n$.

Hence, up to a similarity, it suffices to consider only one point in every of the following hyperquadrics: $Q(1)$, $Q(-1)$, $Q(0)$ (non-trivial in the last case). For instance, we can take Z satisfying one of the following possibilities:

$$Z = J_{Z_1} \in \mathbf{V}, \quad Z = J_{Z_4} \in \mathbf{V}, \quad Z = J_{Z_1} + J_{Z_4} = J_{Z_1+Z_4} \in \mathbf{V},$$

which corresponds to positive, negative or zero length of Z.

Remark 7. It would be interesting to find shorter and more conceptual proof of Theorem 14. In our proof we have used some standard results on Lie algebras, on representations of Lie groups, as well as classical results in the linear algebra and properties of polynomial ideals.

It is clear that for trivial Y (i.e. $y_i = 0$ for all $i = 1, \dots, 8$) and Z as above, we can take the trivial (zero) operator B in order to get equalities $B(Y) = Z(Y)$ and $[B, Z] = 0$. In what follows, it suffices to check only non-trivial Y.

7.1. **The case $Z = J_{Z_1}$.** The first condition $[B, Z] = 0$ implies $x_{12} = x_{13} = x_{14} = x_{15} = x_{16} = x_{17} = 0$. Let us consider the condition $B(Y) = Z(Y)$ for a non-trivial B. Let us assume in addition that $x_{23} = x_{24} = x_{25} = x_{26} = x_{27} = x_{34} = x_{35} = x_{36} = x_{37} = 0$. Then we get the following explicit solutions

If $y_1^2+y_2^2+y_3^2+y_4^2 \neq 0$ and $y_5^2+y_6^2+y_7^2+y_8^2 = 0$, then we can take $x_{56}=x_{57}=x_{67}=0$ and

$$x_{45} = -\frac{y_1y_3-y_2y_4}{y_1^2+y_2^2+y_3^2+y_4^2}, \quad x_{46} = \frac{y_1y_4+y_2y_3}{y_1^2+y_2^2+y_3^2+y_4^2}, \quad x_{47} = \frac{y_1^2+y_2^2-y_3^2-y_4^2}{2(y_1^2+y_2^2+y_3^2+y_4^2)}.$$

If $y_5^2+y_6^2+y_7^2+y_8^2 \neq 0$ and $y_1^2+y_2^2+y_3^2+y_4^2 = 0$, then we can take $x_{56}=x_{57}=x_{67}=0$ and

$$x_{45} = -\frac{y_5y_7-y_6y_8}{y_5^2+y_6^2+y_7^2+y_8^2}, \quad x_{46} = \frac{y_5y_6+y_7y_8}{y_5^2+y_6^2+y_7^2+y_8^2}, \quad x_{47} = \frac{y_5^2-y_6^2-y_7^2+y_8^2}{2(y_5^2+y_6^2+y_7^2+y_8^2)}.$$

Finally, if $y_1^2+y_2^2+y_3^2+y_4^2 \neq 0$ and $y_5^2+y_6^2+y_7^2+y_8^2 \neq 0$, then we can take

$$\begin{aligned}
x_{45} &= -\frac{1}{2}\left(\frac{y_1y_3-y_2y_4}{y_1^2+y_2^2+y_3^2+y_4^2}+\frac{y_5y_7-y_6y_8}{y_5^2+y_6^2+y_7^2+y_8^2}\right),\\
x_{46} &= \frac{1}{2}\left(\frac{y_1y_4+y_2y_3}{y_1^2+y_2^2+y_3^2+y_4^2}+\frac{y_5y_6+y_7y_8}{y_5^2+y_6^2+y_7^2+y_8^2}\right),\\
x_{47} &= \frac{(y_1^2+y_2^2)(y_5^2+y_8^2)-(y_3^2+y_4^2)(y_6^2+y_7^2)}{2(y_1^2+y_2^2+y_3^2+y_4^2)(y_5^2+y_6^2+y_7^2+y_8^2)},\\
x_{56} &= \frac{(y_1^2+y_2^2)(y_6^2+y_7^2)-(y_3^2+y_4^2)(y_5^2+y_8^2)}{2(y_1^2+y_2^2+y_3^2+y_4^2)(y_5^2+y_6^2+y_7^2+y_8^2)},\\
x_{57} &= \frac{1}{2}\left(-\frac{y_1y_4+y_2y_3}{y_1^2+y_2^2+y_3^2+y_4^2}+\frac{y_5y_6+y_7y_8}{y_5^2+y_6^2+y_7^2+y_8^2}\right),\\
x_{67} &= \frac{1}{2}\left(\frac{-y_1y_3+y_2y_4}{y_1^2+y_2^2+y_3^2+y_4^2}+\frac{y_5y_7+y_6y_8}{y_5^2+y_6^2+y_7^2+y_8^2}\right).
\end{aligned}$$

Hence, we have the following result.

Proposition 9. *For any $Y \in \mathbb{R}^{4,4}$, it is possible to find $B \in \mathbf{N} = [\mathbf{V},\mathbf{V}]$ of $\mathbf{V}$ such that $[B,Z]=[B,J_{Z_1}]=0$ and $B(Y)=Z(Y)=J_{Z_1}(Y)$.*

7.2. **The case $Z = J_{Z_4}$.** The first condition $[B,Z]=0$ implies $x_{14}=x_{24}=x_{34}=x_{45}=x_{46}=x_{47}=0$. Let us consider a subalgebra $\mathbf{N}_1 \subset \mathbf{N}$ that is generated by the matrices of the type $[J_{Z_k}, J_{Z_l}]$, where $k,l \in \{1,2,3,5,6,7\}$. It is clear that $[J_{Z_4},\mathbf{N}_1]=0$.

Let us consider the condition $B(Y)=Z(Y)$. The above arguments show that we need to consider only $B \in \mathbf{N}_1$. It is more convenient to consider the linear system $2B(Y)=Z(Y)=J_{Z_4}(Y)$ which allows to find $B=\{x_{ij}\}$. Since $J_{Z_4}(Y)=(y_5,y_8,y_7,y_6,y_1,y_4,y_3,y_2)$, then we have a system of linear equations with respect to the variables

$$x_{12}, x_{13}, x_{15}, x_{16}, x_{17}, x_{23}, x_{25}, x_{26}, x_{27}, x_{35}, x_{36}, x_{37}, x_{56}, x_{57}, x_{67}$$

with the following extended matrix (the last column is the column of free terms of our system of linear equations):

$$ME := \begin{pmatrix}
-y_6 & y_7 & y_3 & -y_4 & -y_1 & -y_8 & -y_2 & -y_1 & y_4 & -y_1 & y_2 & -y_3 & y_8 & -y_7 & y_6 & -y_1\\
y_7 & y_6 & -y_4 & -y_3 & -y_2 & y_5 & -y_1 & y_2 & -y_3 & y_2 & y_1 & -y_4 & -y_5 & y_6 & y_7 & -y_2\\
-y_8 & -y_5 & y_1 & -y_2 & y_3 & y_6 & -y_4 & -y_3 & -y_2 & y_3 & -y_4 & -y_1 & y_6 & y_5 & -y_8 & -y_3\\
y_5 & -y_8 & -y_2 & -y_1 & y_4 & -y_7 & -y_3 & y_4 & y_1 & -y_4 & -y_3 & -y_2 & -y_7 & -y_8 & -y_5 & -y_4\\
-y_4 & y_3 & y_7 & -y_6 & -y_5 & -y_2 & -y_8 & -y_5 & y_6 & -y_5 & y_8 & -y_7 & y_2 & -y_3 & y_4 & -y_5\\
y_1 & -y_2 & -y_8 & -y_5 & y_6 & -y_3 & -y_7 & y_6 & y_5 & -y_6 & -y_7 & -y_8 & -y_3 & -y_2 & -y_1 & -y_6\\
-y_2 & -y_1 & y_5 & -y_8 & y_7 & y_4 & -y_6 & -y_7 & -y_8 & y_7 & -y_6 & -y_5 & y_4 & y_1 & -y_2 & -y_7\\
y_3 & y_4 & -y_6 & -y_7 & -y_8 & y_1 & -y_5 & y_8 & -y_7 & y_8 & y_5 & -y_6 & -y_1 & y_4 & y_3 & -y_8
\end{pmatrix}.$$

By the Kronecker–Capelli theorem, this system has a solution if and only if the rank of ME coincides with the rank of the matrix M, which is obtained from ME by deleting the last column (the column of free terms).

It is easy to see that the product of the vector $(-y_5, -y_8, -y_7, -y_6, y_1, y_4, y_3, y_2)$ and ME is a vector with 16 zero entries. Hence, $\operatorname{rank}(ME) \leq 7$ (see also Remark 5). Therefore, if $\operatorname{rank}(M) = 7$, then the system $2B(Y) = J_{Z_4}(Y)$ has a solution. By symbol $M^{[j_1,j_2,\dots,j_s]}_{[i_1,i_2,\dots,i_s]}$, we denote the minor of M which is determined by the rows with the numbers $i_1, i_2, \dots, i_s$ and columns with the numbers $j_1, j_2, \dots, j_s$. It is easy to check that

$$M^{[9,10,11,12,13,14,15]}_{[1,2,3,4,6,7,8]} = y_1 \cdot (y_1^2 + y_2^2 - y_3^2 - y_4^2 - y_5^2 + y_6^2 + y_7^2 - y_8^2) \times$$
$$\times \Big((y_1^2 + y_2^2 + y_3^2 + y_4^2 - y_5^2 - y_6^2 - y_7^2 - y_8^2)^2 + 4(y_1y_6 - y_2y_7 + y_3y_8 - y_4y_5)^2 \Big).$$

Hence, for *almost all* $Y \in \mathfrak{v}$, we obtain $M^{[1,2,3,4,6,7,8]}_{[1,2,3,4,6,7,8]} \neq 0$ and $\operatorname{rank}(M) = \operatorname{rank}(ME) = 7$, that implies that we have a solution of the corresponding system.

It is possible to compute all other minors of M of order 7. For example,

$$M^{[1,3,7,8,10,11,15]}_{[1,2,3,4,6,7,8]} = -8y_1 \cdot (y_1y_3 + y_2y_4 - y_5y_7 - y_6y_8) \times$$
$$\Big((y_1^2 + y_2^2 + y_5^2 + y_8^2)(y_3y_7 + y_4y_6) - (y_3^2 + y_4^2 + y_6^2 + y_7^2)(y_1y_5 + y_2y_8) \Big).$$

The set of $Y \in \mathfrak{v}$ with the property $\operatorname{rank}(M) < 7$ is the zero set of several polynomials, hence, it determines a polynomial ideals. It is interesting to describe this ideal completely.

More precisely, let $M(7)$ be the set of all minors of size 7 of the matrix M (any such minor is a 7-form in coordinates of Y). There are $8 \cdot C^7_{15} = 51480$ minors of size 7 of the matrix M. Now, let us consider the following set:

$$ZM(7) = \left\{ Y \in \mathbb{R}^8 \,|\, f(Y) = 0 \text{ for any } f \in M(7) \right\}.$$

If $Y \notin ZM(7)$, then there is a minor f of size 7 of the matrix M such that $f(y) \neq 0$. Therefore, $\operatorname{rank}(M) = 7$, hence, the corresponding linear system for the vector Y has a solution. If $Y \in ZM(7)$ then $\operatorname{rank}(M) \leq 6$, but this does not mean that the corresponding linear system has no solution! But all possible "bad" vectors Y are in the set $ZM(7)$. Note that $ZM(7)$ is a closed subset of zero measure in $\mathbb{R}^8$. Moreover, $ZM(7)$ is an algebraic set. On the other hand, this approach demands a lot of computations. Hence, we are going to apply some other ideas.

Let us show that we may assume (without loss of generality) that $y_1 = \langle Y, V_1 \rangle_{4,4} \neq 0$. Indeed, if $y_1 = 0$, then there is $y_i \neq 0$ for some $i = 2, \dots, 8$. We can choose a matrix of the type $Q = \exp(A)$, where $A \in \mathbf{N}_1$ such that $\langle QY, V_1 \rangle_{4,4} \neq 0$. Indeed, if $i \in \{2,3,4,5,6,7,8\}$, then we can take $Q = \exp\big(t[J_{Z_k}, J_{Z_l}]\big)$ such that $(k,l) \in \{(2,3),(1,3),(1,2),(1,7),(1,6),(1,5),(2,5)\}$ for respective i, and t is a small positive number. For instance, $\langle QY, V_1 \rangle_{4,4} = \cos(2t)y_1 + \sin(2t)y_2$ if $Q = \exp\big(t[J_{Z_2}, J_{Z_3}]\big)$ and $\langle QY, V_1 \rangle_{4,4} = \cosh(2t)y_1 + \sinh(2t)y_5 = \frac{e^{2t}+e^{-2t}}{2}y_1 + \frac{e^{2t}-e^{-2t}}{2}y_5$ if $Q = \exp(t[J_{Z_1}, J_{Z_7}])$.

Recall that $k \neq 4$ and $l \neq 4$ in the above pairs. In particular, $[J_{Z_3}, [J_{Z_k}, J_{Z_l}]] = [J_{Z_4}, Q] = 0$ for all these pairs (k,l) and $QJ_{Z_4}Q^{-1} = J_{Z_4}$. The equality $B(Y) = J_{Z_4}(Y)$ is equivalent to $QBQ^{-1}(QY) = QJ_{Z_4}Q^{-1}(QY) = J_{Z_4}(QY)$. Since $[QBQ^{-1}, J_{Z_4}] = 0$ if $[B, J_{Z_4}] = 0$, we can find a suitable $B \in \mathbf{N}$ for the pair (J_{Z_4}, Y) if and only if we can find a suitable $\widetilde{B} \in \mathbf{N}$ for the pair (J_{Z_4}, QY), where $Q \in \mathbf{N}$ such that $[Q, J_{Z_4}] = 0$.

In what follows, we assume that $y_1 \neq 0$ (by the above arguments).

Lemma 5. *If a vector* $Y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$ *satisfies the equalities* $U_i = 0$, $i = 1, \dots, 8$, *where*

$$\begin{array}{rcl} U_1 & = & y_1y_8 - y_2y_5 - y_3y_6 + y_4y_7, \\ U_2 & = & y_1y_7 + y_2y_6 - y_3y_5 - y_4y_8, \\ U_3 & = & y_1y_4 + y_2y_3 - y_5y_6 - y_7y_8, \\ U_4 & = & y_1y_4 - y_2y_3 - y_5y_6 + y_7y_8, \\ U_5 & = & y_1y_3 - y_2y_4 - y_5y_7 + y_6y_8, \\ U_6 & = & y_1y_2 + y_3y_4 - y_5y_8 - y_6y_7, \\ U_7 & = & y_1^2 + y_2^2 - y_3^2 - y_4^2 - y_5^2 + y_6^2 + y_7^2 - y_8^2, \\ U_8 & = & y_1^2 - y_2^2 + y_3^2 - y_4^2 - y_5^2 + y_6^2 - y_7^2 + y_8^2, \end{array} \tag{34}$$

then one of the following two cases holds:

$$1)\ Y = (y_1, y_2, y_3, y_4, y_1, y_4, y_3, y_2), \quad 2)\ Y = (y_1, y_2, y_3, y_4, -y_1, -y_4, -y_3, -y_2),$$

for any $y_1, y_2, y_3, y_4 \in \mathbb{R}$. *In particular, in both cases we have* $y_1^2 + y_2^2 + y_3^2 + y_4^2 - y_5^2 - y_6^2 - y_7^2 - y_8^2 = 1$ *and* $y_1y_6 - y_2y_7 + y_3y_8 - y_4y_5 = 0$.

Proof. The polynomials U_i, $i = 1, \dots, 8$, generates an ideal $\mathfrak{I}$. Now, if we compute the elimination ideal $\mathfrak{I}_1$ from $\mathfrak{I}$ with respect to the variables $y_1, y_2, y_3, y_4, y_5, y_6, y_7$ (it can be done, say, by Maple), then we get that $\mathfrak{I}_1$ has two component:

$$\begin{array}{rcl} S_1 & = & \{(y_1, y_2, y_3, y_4, y_5, y_6, y_7) \,|\, y_1 = y_5, y_2 = y_8, y_3 = y_7, y_4 = y_6\}, \\ S_2 & = & \{(y_1, y_2, y_3, y_4, y_5, y_6, y_7) \,|\, y_1 = -y_5, y_2 = -y_8, y_3 = -y_7, y_4 = -y_6\}. \end{array}$$

This proves the lemma. ■

Lemma 6. *If* $\operatorname{rank}(M) < 7$, *then* $y_1^2 + y_2^2 + y_3^2 + y_4^2 = y_5^2 + y_6^2 + y_7^2 + y_8^2$ *and* $y_1y_6 + y_3y_8 = y_2y_7 + y_4y_5$.

Proof. For $i = 2, \dots, 9$, we denote by $D(i)$ the minor $M^{[i,10,11,12,13,14,15]}_{[1,2,4,5,6,7,8]}$. Then we get the following equalities:

$$\begin{array}{ll} D(2) = -2y_1 \cdot U_1 \cdot U_0, & D(6) = -2y_1 \cdot U_2 \cdot U_0, \\ D(3) = -y_1 \cdot U_6 \cdot U_0, & D(7) = 2y_1 \cdot U_5 \cdot U_0, \\ D(4) = -2y_1 \cdot U_8 \cdot U_0, & D(8) = 2y_1 \cdot U_3 \cdot U_0, \\ D(5) = 2y_1 \cdot U_4 \cdot U_0, & D(9) = y_1 \cdot U_7 \cdot U_0, \end{array}$$

where U_i, $1 \le i \le 8$, are given in (34) and

$$U_0 = (y_1^2 + y_2^2 + y_3^2 + y_4^2 - y_5^2 - y_6^2 - y_7^2 - y_8^2)^2 + 4(y_1y_6 - y_2y_7 + y_3y_8 - y_4y_5)^2.$$

If $U_0 = 0$, then the assertion of the theorem follows.

Let us suppose that $U_0 \neq 0$. If $U_i \neq 0$ for some $i = 1, \dots, 8$, then $\operatorname{rank}(M) = 7$. On the other hand, $U_1 = U_2 = U_3 = U_4 = U_5 = U_6 = U_7 = U_8 = 0$ implies $U_0 = 0$ by Lemma 5. Hence, the lemma is proved. ■

Note that all matrices from $\exp(\mathbf{N})$ preserves the quadratic form $y_1^2 + y_2^2 + y_3^2 + y_4^2 - y_5^2 - y_6^2 - y_7^2 - y_8^2$. On the other hand that, it is not true for the quadratic form $y_1y_6 - y_2y_7 + y_3y_8 - y_4y_5$.

Lemma 7. *If the vector $Y=(y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8)$ with $y_1\neq 0$ is such that there is no $B\in \mathbf{N}_1$ satisfying $B(Y)=J_{Z_4}(Y)$, then Y has one of the following two forms:*

$$1)\ Y=(y_1,y_2,y_3,y_4,y_1,y_4,y_3,y_2),\quad 2)\ Y=(y_1,y_2,y_3,y_4,-y_1,-y_4,-y_3,-y_2),$$

where $y_1,y_2,y_3,y_4\in\mathbb{R}$.

Proof. Let us consider any $Q\in\exp(\mathbf{N}_1)$. If Q does not preserve the equality $y_1y_6-y_2y_7+y_3y_8-y_4y_5=0$, then the matrix M for the vector $\overline{Y}=Q(Y)$ has rank 7 by Lemma 6. Therefore, we have some $\overline{B}\in\mathbf{N}_1$ such that $\overline{B}(\overline{Y})=J_{Z_4}(\overline{Y})$. Consequently, $B=Q^{-1}\overline{B}Q\in\mathbf{N}_1$ satisfies $B(Y)=J_{Z_4}(Y)$, that is impossible by the assumptions. Hence, any $Q\in\exp(\mathbf{N}_1)$ preserves the equality $y_1y_6+y_3y_8=y_2y_7+y_4y_5=0$.

By Lemma 4 we have

$$\overline{y}_1y_6+y_1\overline{y}_6+\overline{y}_3y_8+y_3\overline{y}_8-\overline{y}_2y_7-y_2\overline{y}_7-\overline{y}_4y_5-y_4\overline{y}_5=0, \tag{35}$$

where $\overline{Y}=A(Y)$ and A is any matrix in $\mathbf{N}_1$. Taking various $A=[J_{Z_k},J_{Z_l}]$ for $k,l\in\{1,2,3,5,6,7\}$, we obtain the equations $U_i=0$, $1\leq i\leq 8$, where U_i are given in (34). Now, it suffices to apply Lemma (5). ∎

Proposition 10. *For any $Y\in\mathbb{R}^{4,4}$, it is possible to find $B\in\mathbf{N}=[\mathbf{V},\mathbf{V}]$ such that*

$$[B,Z]=[B,J_{Z_4}]=0 \quad \text{and} \quad B(Y)=Z(Y)=J_{Z_4}(Y). \tag{36}$$

Proof. Suppose that for some $Y=(y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8)\in\mathbb{R}^8$, there is no $B\in\mathbf{N}$ satisfying (36). The discussion above shows that (without loss of generality) we may assume that $y_1\neq 0$. Lemma 7 implies that we have one of the following two possibility:

$$1)\ Y=(y_1,y_2,y_3,y_4,y_1,y_4,y_3,y_2),\quad 2)\ Y=(y_1,y_2,y_3,y_4,-y_1,-y_4,-y_3,-y_2),$$

where $y_1,y_2,y_3,y_4\in\mathbb{R}$. We will find an explicit form of $B\in\mathbf{N}$ satisfying (36) for both cases.

Let us start from the case 1).

If $y_1y_3-y_2y_4\neq 0$, then we have the solution B with the following entries:

$$x_{12}=-\frac{y_1y_2+y_3y_4}{y_1y_3-y_2y_4},\quad x_{13}=-\frac{y_1^2-y_2^2-y_3^2+y_4^2}{2(y_1y_3-y_2y_4)},\quad x_{15}=-\frac{y_1^2+y_2^2+y_3^2+y_4^2}{2(y_1y_3-y_2y_4)},$$

and $x_{16}=x_{17}=x_{23}=x_{25}=x_{26}=x_{27}=x_{35}=x_{36}=x_{37}=x_{56}=x_{57}=x_{67}=0$.

If $y_1y_3-y_2y_4=0$ and $y_4\neq 0$, then $y_3=\frac{y_2y_4}{y_1}$ and we can take the matrix B with the following entries:

$$x_{12}=\frac{(y_1^2-y_2^2)(y_1^2-y_4^2)}{2y_1y_4(y_1^2+y_2^2)},\quad x_{13}=-\frac{y_2(y_1^2-y_4^2)}{y_4(y_1^2+y_2^2)},\quad x_{15}=0,\quad x_{16}=\frac{y_1^2+y_4^2}{2y_1y_4},$$

and $x_{17}=x_{23}=x_{25}=x_{26}=x_{27}=x_{35}=x_{36}=x_{37}=x_{56}=x_{57}=x_{67}=0$.

If $y_1y_3-y_2y_4=0$ and $y_4=0$, then $y_3=0$. Therefore, we can take B with $x_{17}=1$, while $x_{ij}=0$ for all pairs $(i,j)\neq(1,7)$. It is easy to verify using the extended matrix ME (recall that $y_7=y_3$ and $y_6=y_4$, hence, $y_3=y_4=y_6=y_7=0$).

Now we are going to consider the case 2).

If $y_1y_4+y_2y_3\neq 0$, then we have the solution B with the following entries:

$$x_{12}=-\frac{y_1^2-y_2^2+y_3^2-y_4^2}{2(y_1y_4+y_2y_3)},\quad x_{13}=\frac{y_1y_2-y_3y_4}{y_1y_4+y_2y_3},\quad x_{15}=0,\quad x_{16}=\frac{y_1^2+y_2^2+y_3^2+y_4^2}{2(y_1y_4+y_2y_3)},$$

and $x_{17}=x_{23}=x_{25}=x_{26}=x_{27}=x_{35}=x_{36}=x_{37}=x_{56}=x_{57}=x_{67}=0$.

If $y_1y_4 + y_2y_3 = 0$ and $y_4 \neq 0$, then $y_2 \neq 0$, $y_3 \neq 0$, $y_3 = -\frac{y_2y_4}{y_1}$, and we can take B with the following entries:

$$x_{12} = -\frac{y_1(y_2^2 - y_4^2)}{y_4(y_1^2 + y_2^2)}, \quad x_{13} = -\frac{(y_1^2 - y_2^2)(y_2^2 - y_4^2)}{2y_2y_4(y_1^2 + y_2^2)}, \quad x_{15} = \frac{y_2^2 + y_4^2}{2y_2y_4},$$

and $x_{16} = x_{17} = x_{23} = x_{25} = x_{26} = x_{27} = x_{35} = x_{36} = x_{37} = x_{56} = x_{57} = x_{67} = 0$.

If $y_1y_4 + y_2y_3 = 0$ and $y_4 = 0$, then $y_2y_3 = 0$. If $y_3 = 0$, then we can take B with $x_{17} = 1$, while $x_{ij} = 0$ for all pairs $(i, j) \neq (1, 7)$. It is easy to verify using the extended matrix ME (recall that $y_7 = -y_3$ and $y_6 = -y_4$, hence, $y_3 = y_4 = y_6 = y_7 = 0$).

Finally, if $y_4 = 0$ and $y_2 = 0$, then we can take B with $x_{26} = 1$, while $x_{ij} = 0$ for all pairs $(i, j) \neq (2, 6)$. It is easy to verify using the extended matrix ME (recall that $y_8 = -y_2$ and $y_6 = -y_4$, hence, $y_2 = y_4 = y_6 = y_8 = 0$). The proposition is proved. ■

7.3. **The case $Z = J_{Z_1} + J_{Z_4}$.** The condition $[B, Z] = 0$ implies $x_{12} = -x_{24}$, $x_{13} = x_{34}$, $x_{14} = 0$, $x_{15} = x_{45}$, $x_{16} = x_{46}$, $x_{17} = x_{47}$. For the condition $B(Y) = Z(Y)$ we consider the liner equation system $2B(Y) = Z(Y) = J_{Z_1}(Y) + J_{Z_4}(Y)$ (any solution of this system is just one half of a solution of $B(Y) = Z(Y)$). Since $J_{Z_1}(Y)+J_{Z_4}(Y) = (y_5-y_2, y_1+y_8, y_4+y_7, y_6-y_3, y_1+y_8, y_4+y_7, y_3-y_6, y_2-y_5)$, then we have a system of linear equations with respect to the following variables

$$x_{23}, x_{25}, x_{24}, x_{26}, x_{27}, x_{34}, x_{35}, x_{36}, x_{37}, x_{45}, x_{46}, x_{47}, x_{56}, x_{57}, x_{67}$$

with the following extended matrices (the last column is the column of free terms of our system of linear equations):

$$\widetilde{ME} := \begin{pmatrix} y_1 & -y_3+y_6 & -y_5 & y_8 & -y_7 & -y_4-y_7 & y_8 & y_5 & -y_6 & y_3-y_6 & -y_4-y_7 & -y_1-y_8 & -y_1 & y_4 & y_3 & -y_1-y_8 \\ -y_8 & -y_3+y_6 & -y_2 & -y_1 & y_4 & -y_4-y_7 & -y_1 & y_2 & -y_3 & y_3-y_6 & -y_4-y_7 & -y_1-y_8 & y_8 & -y_7 & y_6 & -y_1-y_8 \\ -y_2 & y_4+y_7 & -y_8 & -y_5 & y_6 & -y_3+y_6 & -y_5 & y_8 & -y_7 & y_4+y_7 & y_3-y_6 & y_2-y_5 & y_2 & -y_3 & y_4 & y_2-y_5 \\ y_5 & -y_4-y_7 & -y_1 & y_2 & -y_3 & y_3-y_6 & y_2 & y_1 & -y_4 & -y_4-y_7 & -y_3+y_6 & -y_2+y_5 & -y_5 & y_6 & y_7 & -y_2+y_5 \\ -y_3 & -y_1-y_8 & -y_7 & y_6 & y_5 & y_2-y_5 & -y_6 & -y_7 & -y_8 & -y_1-y_8 & y_2-y_5 & -y_3+y_6 & -y_3 & -y_2 & -y_1 & y_3-y_6 \\ y_6 & y_1+y_8 & -y_4 & -y_3 & -y_2 & -y_2+y_5 & y_3 & -y_4 & -y_1 & y_1+y_8 & -y_2+y_5 & y_3-y_6 & y_6 & y_5 & -y_8 & -y_3+y_6 \\ -y_7 & y_2-y_5 & -y_3 & y_4 & y_1 & y_1+y_8 & -y_4 & -y_3 & -y_2 & -y_2+y_5 & -y_1-y_8 & y_4+y_7 & -y_7 & -y_8 & -y_5 & -y_4-y_7 \\ y_4 & y_2-y_5 & -y_6 & -y_7 & -y_8 & y_1+y_8 & y_7 & -y_6 & -y_5 & -y_2+y_5 & -y_1-y_8 & y_4+y_7 & y_4 & y_1 & -y_2 & -y_4-y_7 \end{pmatrix}.$$

By the Kronecker–Capelli theorem, this system has a solution if and only if the rank of $\widetilde{ME}$ coincides with the rank of the matrix $\widetilde{M}$, which is obtained from $\widetilde{ME}$ by deleting the last column (the column of free terms). It is easy to see that the product of the vector $(-y_2, y_5, -y_1, y_8, -y_4, y_7, y_6, -y_3)$ and the matrix $\widetilde{ME}$ is a zero vector with 16 entries. Hence, $\operatorname{rank}(\widetilde{ME}) \leq 7$ (see also Remark 5). Therefore, if $\operatorname{rank}(\widetilde{M}) = 7$, then the system $2B(Y) = J_{Z_1}(Y) + J_{Z_4}(Y)$ has a solution. By symbol $\widetilde{M}^{[j_1,j_2,\dots,j_s]}_{[i_1,i_2,\dots,i_s]}$, we denote the minor of $\widetilde{M}$ which is determined by the rows with the numbers $i_1, i_2, \dots, i_s$ and columns with the numbers $j_1, j_2, \dots, j_s$. It is easy to check that

$$\widetilde{M}^{[9,10,11,12,13,14,15]}_{[1,2,3,4,5,6,7]} = 2y_3 \cdot \big((y_1 + y_8)(y_4 + y_7) - (y_2 - y_5)(y_3 - y_6)\big) \times \\ \times \big((y_1 + y_8)^2 + (y_2 - y_5)^2 + (y_3 - y_6)^2 + (y_4 + y_7)^2\big)^2.$$

Hence, for *almost all* $Y \in \mathfrak{v}$ we have $\widetilde{M}^{[9,10,11,12,13,14,15]}_{[1,2,3,4,5,6,7]} \neq 0$ and $\operatorname{rank}(\widetilde{M}) = \operatorname{rank}(\widetilde{ME}) = 7$, that implies the existence of a solution of the corresponding system.

Moreover, we can change the above minor, removing any line instead of the 8th one, and this new minor is distinct from the given one exactly in the first multiple ($2y_3$ will change to $\pm 2y_i$ according to the entries of the vector $(-y_2, y_5, -y_1, y_8, -y_4, y_7, y_6, -y_3)$). Therefore, *if* $(y_1 + y_8)(y_4 + y_7) \neq (y_2 - y_5)(y_3 - y_6)$, *then* $\operatorname{rank}(\widetilde{M}) = \operatorname{rank}(\widetilde{ME}) = 7$, *hence, there is a solution of the corresponding linear system.* Furthermore, the following is true

Lemma 8. *If* $(y_1+y_8)^2+(y_2-y_5)^2+(y_3-y_6)^2+(y_4+y_7)^2 \neq 0$*, then* $\operatorname{rank}(\widetilde{M})=7$*. If* $(y_1+y_8)^2+(y_2-y_5)^2+(y_3-y_6)^2+(y_4+y_7)^2=0$*, then the last column of the matrix* $\widetilde{ME}$ *(the column of free terms) has only zero entries.*

Proof. Suppose (without loss of generality) that $y_1 \neq 0$. Now, we suppose that $(y_1+y_8)^2+(y_2-y_5)^2+(y_3-y_6)^2+(y_4+y_7)^2 \neq 0$. Let us prove that $\operatorname{rank}(\widetilde{M})=7$ in this case.

For $i=1,\dots,9$, we denote by $D(i)$ the minor $\widetilde{M}^{[i,10,11,12,13,14,15]}_{[1,2,4,5,6,7,8]}$. Then we get the following equalities:

$$
\begin{aligned}
D(3) &= -y_1\cdot W_1\cdot\big((y_1+y_8)^2+(y_2-y_5)^2+(y_3-y_6)^2+(y_4+y_7)^2\big)^2,\\
D(4) &= 2y_1\cdot W_3\cdot\big((y_1+y_8)^2+(y_2-y_5)^2+(y_3-y_6)^2+(y_4+y_7)^2\big)^2,\\
D(7) &= 2y_1\cdot W_4\cdot\big((y_1+y_8)^2+(y_2-y_5)^2+(y_3-y_6)^2+(y_4+y_7)^2\big)^2,\\
D(8) &= y_1\cdot W_2\cdot\big((y_1+y_8)^2+(y_2-y_5)^2+(y_3-y_6)^2+(y_4+y_7)^2\big)^2.
\end{aligned}
$$

where

$$
\begin{aligned}
W_1 &= (y_1+y_8)^2-(y_2-y_5)^2-(y_3-y_6)^2+(y_4+y_7)^2,\\
W_2 &= (y_1+y_8)^2-(y_2-y_5)^2+(y_3-y_6)^2-(y_4+y_7)^2,\\
W_3 &= (y_1+y_8)(y_2-y_5)-(y_3-y_6)(y_4+y_7),\\
W_4 &= (y_1+y_8)(y_2-y_5)+(y_3-y_6)(y_4+y_7).
\end{aligned}
$$

If $W_i \neq 0$ for any $i=1,\dots,4$, then $\operatorname{rank}(\widetilde{M})=7$. Let us suppose that $W_1=W_2=W_3=W_4=0$. Then $W_1=W_2=0$ implies $(y_1+y_8)^2=(y_2-y_5)^2$ and $(y_3-y_6)^2=(y_4+y_7)^2$. From $W_3=W_4=0$ we get $(y_1+y_8)(y_2-y_5)=(y_3-y_6)(y_4+y_7)=0$. Therefore, we easily obtain that $(y_1+y_8)=(y_2-y_5)=(y_3-y_6)=(y_4+y_7)=0$ that is impossible by our assumption. Hence, $\operatorname{rank}(\widetilde{M})=7$ if $(y_1+y_8)^2+(y_2-y_5)^2+(y_3-y_6)^2+(y_4+y_7)^2 \neq 0$.

The second assertion is obvious. The lemma is proved. ■

Corollary 6. *The linear system with the extended matrix* $\widetilde{ME}$ *has a solution for any* $Y \in \mathbb{R}^{4,4}$*. Hence, for any* $Y \in \mathbb{R}^{4,4}$*, it is possible to find* $B \in \mathbf{N}=[\mathbf{V},\mathbf{V}]$ *such that* $[B,Z]=[B,J_{Z_1}+J_{Z_4}]=0$ *and* $B(Y)=Z(Y)=J_{Z_1}(Y)+J_{Z_4}(Y)$*.*

Proof. If $\operatorname{rank}(\widetilde{M})=7$ then we get a solution due to $\operatorname{rank}(\widetilde{ME})=7$. If $\operatorname{rank}(\widetilde{M})<7$, then the column of free terms of the matrix $\widetilde{ME}$ (the column of free terms) has only zero entries by Lemma 8. This means $\operatorname{rank}(\widetilde{M})=\operatorname{rank}(\widetilde{ME})$ and we also have a solution. ■

It is possible to compute all minors of the matrix $\widetilde{M}$ of order 7. The set of Y with the property $\operatorname{rank}(\widetilde{M})<7$ is the zero set of several polynomials, hence, it determines a polynomial ideals. In this case, such a description follows from Lemma 8.

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K. Furutani. Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University, Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan
Email address: kf46089@gmail.com

I. Markina. Department of Mathematics, University of Bergen, P.O. Box 7803, Bergen N-5020, Norway
Email address: irina.markina@uib.no

Yu.G. Nikonorov. Southern Mathematical Institute of VSC RAS, 53 Vatutina St., Vladikavkaz, 362025, Russia
Email address: nikonorov2006@mail.ru