Revised note on surface-link of trivial components

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Abstract

It is shown that a surface-link of ribbon surface-knot components is a ribbon surface-link if and only if it is a surface-link producing a ribbon surface-link by surgery along a self-trivial 1-handle system. This corrects an earlier statement. This result makes a corrected proof for the claim that every surface-link of trivial surface-knot components with at most one aspheric component is a ribbon surface-link. For non-ribbon surface-links of trivial components with at least two aspheric components constructed in a previous note, it adds the new property that non-ribbonability continues through surgery along any self-trivial 1-handle system.

Keywords: Ribbon surface-link, Surgery, Self 1-handle, Self-trivial 1-handle.
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1. Introduction

Let $\mathbf{F}$ be a (possibly disconnected) closed surface. An $\mathbf{F}$-*link* in the 4-sphere S^4 is the image of a smooth embedding $\mathbf{F} \to S^4$. When $\mathbf{F}$ is connected, it is also called an $\mathbf{F}$-*knot*. An $\mathbf{F}$-link or $\mathbf{F}$-knot for an $\mathbf{F}$ is called a *surface-link* or *surface-knot* in S^4, respectively. If $\mathbf{F}$ consists of some copies of the 2-sphere S^2, then it is also called an S^2-*link* and an S^2-*knot* for $\mathbf{F} = S^2$. A *trivial surface-link* is a surface-link F which bounds disjoint handlebodies smoothly embedded in S^4. A *1-handle system* on a surface-link F in S^4 is a system h of disjoint 1-handles h_j $(j = 1, 2, \dots, s)$ on F in S^4. Let $F(h)$ be the surface-link obtained from F by surgery along h. A 1-handle system h on a surface-link F is a *self 1-handle system* on F if the number of connected components of the union $F \cup h$ is equal to the number of connected components of F. A self 1-handle system h on a surface-link F is a *self-trivial 1-handle system* on F if every 1-handle h_j in h is a trivial 1-handle on the attached connected component of F in S^4. For a disconnected surface-link F, note that a self-trivial 1-handle system on F need not be a trivial 1-handle system on F. A *ribbon surface-link* is a surface-link $F = O(h)$ obtained from a trivial S^2-link O by surgery along a 1-handle system

h on O, [5], [11]. A *semi-unknotted multi-punctured handlebody system* or simply a *SUPH system* for a surface-link F in S^4 is a multi-punctured handlebody system W (smoothly embedded) in S^4 such that $\partial W = F \cup O$ for a trivial S^2-link O in S^4. A typical SUPH system W is constructed from a ribbon surface-link defined from a trivial S^2-link O and a 1-handle system h as the union $O \times [0,1] \cup h$ for a normal collar $O \times [0,1]$ of O in S^4 with $O \times 0 = O$ and the 1-handle system h attaching to $O = O \times 0$. For a SUPH system W with $\partial W = F \cup O$, there is a proper arc system α in W spanning O such that a regular neighborhood $N(O \cup \alpha)$ of the union $O \cup \alpha$ in W is diffeomorphic to the closed complement $\mathrm{cl}(W \setminus c(F \times [0,1]))$ of a boundary collar $c(F \times [0,1])$ of F in W. This pair (O, α) is called a sphere-chord system of the SUPH system W. By replacing α with a 1-handle system h attaching to O with core arc system α, the surface-link F is a ribbon surface-link defined by O and h. In other words, to give a SUPH system W with $\partial W = F \cup O$ is the same as to say that the surface-link F is a ribbon surface-link with sphere system O. A 1-handle system h of 1-handles $h_j\, (j = 1, 2, \ldots, s)$ on a surface-link F of $r(\geq 2)$ components is a *fusion 1-handle system* if the number of connected components of the union $F \cup h$ has just $r - s$ connected components, where the inequality $s \leq r - 1$ must hold. As it is explained at the end of this section, an earlier claimed characterization of when a surface-link F of ribbon surface-knot components is a ribbon surface-link is not true in general, [9, Theorem 1.4]. The following theorem gives a new characterization.

Theorem 1.1. Let F be a surface-link in S^4 of ribbon surface-knot components $F_i\, (i = 1, 2, \ldots, r)$. Then the following statements on (1)-(3) on F are mutually equivalent.

(1) F is a ribbon surface-link.

(2) The surface-link obtained from $F(h)$ by surgery along every 1-handle system h is a ribbon surface-link.

(3) The surface-link $F(h)$ obtained from F by surgery along a self-trivial 1-handle system h on F is a ribbon surface-link.

It will be explained elsewhere how a certain surface-link F can be determined to be a ribbon surface-link by the ribbonness of the surface-knot $F(h)$ obtained from F by applying a certain fusion h, [10]. There exist many non-ribbon surface-links F such that the surface-link $F(h)$ for a self 1-handle system h on F is a ribbon surface-link. The following lemma is implicitely used in the proofs of [9, Theorem 1.4] and [8, Theorem 1] although the full proof is given in this paper for convenience.

Lemma 1.2. For every surface-link F in S^4 with at most one aspheric component, there is a self 1-handle system h on F such that the surface-link $F(h)$ is a ribbon surface-link.

On the other hand, there exist non-ribbon surface-links F with at least two aspheric components such that the surface-link $F(h)$ obtained from F by surgery along any self-trivial 1-handle system h on F is a non-ribbon surface-link (see Theorem 1.4 later). The following corollary to Theorem 1.1 and Lemma 1.2 can be viewed as providing a corrected proof of [8, Theorem 1].[1]

Corollary 1.3. Every surface-link F of trivial surface-knot components with at most one aspheric component is a ribbon surface-link.

Proof of Corollary 1.3. Every self 1-handle on every surface-link F of trivial surface-knot components is a self-trivial 1-handle on F, [3]. Thus, if F has at most one aspheric component, then there is a self 1-handle system h on F such that the surface-link $F(h)$ is a ribbon surface-link by Lemma 1.2. Hence there is a self-trivial 1-handle system h on F such that the surface-link $F(h)$ is a ribbon surface-link. By Theorem 1.1, F is a ribbon surface-link. This completes the proof of Corollary 1.3.

The following result slightly strengthens an earlier result, [8, Theorem 2].

Theorem 1.4. Let $\mathbf{F}$ be any closed oriented disconnected surface with at least two aspheric components. Then there are pairs (K, K') of $\mathbf{F}$-links K, K' in S^4 both of trivial components with the same fundamental groups up to meridian-preserving isomorphisms such that K is a ribbon surface-link and K' is a non-ribbon surface-link. Further, there exists a canonical correspondence between the self-trivial 1-handle systems on K and the self-trivial 1-handle systems on K' so that every self-trivial 1-handle surgery transforms a pair (K, K') into a pair (L, L') of a ribbon $\mathbf{F}'$-link L and a non-ribbon $\mathbf{F}'$-link L' with the same fundamental group up to meridian-preserving isomorphisms for a closed oriented surface $\mathbf{F}'$.

The continuation of non-ribbonability from K' to L' in Theorem 1.4 explains why non-ribbonability continues through surgery along every self 1-handle system. A non-ribbon surface-link K' of two components in Theorem 1.4 has the free abelian fundamental group of rank 2. Then by van Kampen theorem, the surface-knot $K'(h)$ obtained from K' by surgery along any fusion 1-handle h on K' has the infinite cyclic fundamental group, so that $K'(h)$ is a trivial surface-knot in S^4 by smooth unknotting result of a surface-knot, [6, 7]. This example shows that there is a non-ribbon surface-link F of ribbon surface-knot components such that the surface-knot obtained from F by surgery along any fusion is a ribbon surface-knot. This means that the condition that the surface-knot obtained from F by fusion is a ribbon surface-knot does not characterize that F is a ribbon surface-link, although this condition is claimed to characterize the ribbonness of F, [9, Theorem 1.4].

because

[1] The result for an S^2-link of trivial components is contrary to a previously believed result, [12].

2. Proofs of Theorem 1.1, Lemma 1.2 and Theorem 1.4

The proof of Theorem 1.1 is done as follows.

Proof of Theorem 1.1. The assertions (1) $\to$ (2) and (2) $\to$ (3) are obvious by definitions. The assertion (3) $\to$ (1) is shown as follows. For a surface-link F in S^4 of ribbon surface-knot components $F_i\,(i=1,2,\dots,r)$, assume that $F(h)$ is a ribbon surface-link for a system h of self-trivial 1-handles $h_j\,(j=1,2,\dots,s)$ on F. Assume that the 1-handle h_1 attaches to F_1. Let $h'=h\setminus\{h_1\}$. Then h' is a self-trivial 1-handle system on F and h_1 is a self-trivial 1-handle on the surface-link $F(h')$. If it is shown that the ribbonness of $F(h_1)$ implies the ribbonness of F, then the ribbonness of $F(h)$ implies the ribbonness of $F(h')$ by replacing F with $F(h')$. By inductive argument, the assertion (3) $\to$ (1) is obtained. Thus, it sufficies to show that if $F(h_1)$ is a ribbon surface-link, then the surface-link F is a ribbon surface-link. Assume that $F(h_1)$ is a ribbon surface-link. Since h_1 is a trivial 1-handle on F_1, the ribbon surface-knot $F_1(h_1)$ admits an O2-handle pair $(D\times I, D'\times I)$ where the core disk D of $D\times I$ is a transverse disk of the 1-handle h_1 and the interior of the core disk D' of $D'\times I$ may transversely meet the ribbon surface-link $F'=F\setminus F_1$ with finite points. Let $W_1(h_1)\cup W'$ be a SUPH system for $F(h_1)$ where $W_1(h_1)$ is a SUPH system for $F_1(h_1)$ with $\partial W_1(h_1)=F(h_1)\cup O(h_1)$ and W' is a SUPH system for F' with $\partial W'=F'\cup O'$. Let W_1 be a SUPH system for the ribbon surface-knot F_1 with $\partial W_1=F_1\cup O_1$. Since h_1 is a trivial 1-handle on F_1, the union $W_1\cup h_1$ with the 1-handle h_1 trivially attached to F_1 is considered as a SUPH system for a surface-knot G_1 equivalent to$F_1(h_1)$ with $\partial(W_1\cup h_1)=G_1\cup O_1$. Equivalent ribbon surface-links are faithfully equivalent and they are moved into each other by the moves M0, M1, M2, [5]. This means that after replacing multi-punctured manifolds of $W_1(h_1)$ and W_1 with $W_1(h_1)$ and W_1, respectively, there is an orientation-preserving diffeomorphism f of S^4 sending $W_1(h_1)$ to $W_1\cup h_1$. Then $G=G_1\cup f(F')=fF(h_1)$ is a ribbon surface-link. Since the 1-handle h_1 is trivially attached to W_1, there is an O2-handle pair $(E\times I, E'\times I)$ for G_1 such that the core disk E of $E\times I$ is a transverse disk of the 1-handle h_1 and the core disk E' of $E'\times I$ has $E'\cap(W_1\cup h_1)=\partial E'$. By Uniqueness of an O2-handle pair in the soft sense, there is an orientation-preserving diffeomorphism g of S^4 such that $gG_1=G_1$ and $(gfD\times I, gfD'\times I)=(E\times I, E'\times I)$, [9]. Let (O',α') be a sphere-chord system for F' in W'. Then $(gf(O'),gf(\alpha'))$ is a sphere-chord system for $gf(F')$ in $gf(W')$. By general position, the arc system $gf(\alpha')$ is deformed into an arc system $gf(\alpha')^*$ disjoint from the interior of the core disk E' of $E'\times I$ by an isotopy of $gf(\alpha')$ keeping the boundary point system fixed. On the other hand, the sphere system $gf(O')$ may transversely meet the interior of E' with finite points, but which is also deformed to a sphere system $gf(O')^*$ disjoint from the interior of E' by Finger Move Canceling, [7]. Let $gf(F')^*$ be a ribbon surface-link given by the sphere-chord system $(gf(O')^*, gf(\alpha')^*)$. The surface-link $G^*=G_1\cup gf(F')^*$ is still a ribbon surface-link because there is a SUPH system $g(W_1\cup h_1)$ for G_1 disjoint from the sphere-chord system $(gf(O')^*, gf(\alpha')^*)$. The O2-handle pair $(E\times I, E'\times I)$ for G_1 is an

O2-handle pair for the ribbon surface-link G^*. It is known that the ribbonness of G^* implies the ribbonness of the surgery surface-link $G^*(E \times I, E' \times I)$, [9]. The surgery surface-link $G^*(E \times I, E' \times I)$ is equivalent to the surgery surface-link $G^*(E \times I)$, [6]. By construction, $G^*(E \times I)$ is equivalent to the surface-link $G(E \times I) = gfF(h_1)(gfD \times I)$, which is equivalent to $F(h_1)(D \times I) = F$, for D is a transverse disk of the 1-handle h_1. Thus, the surface-link F is a ribbon surface-link. This completes the proof of Theorem 1.1.

The proof of Lemma 1.2 is done as follows.

Proof of Lemma 1.2. Let F be a surface-link in S^4 of a possibly non-sphere surface-knot component K and the remaining S^2-link $L = F \setminus K$. Since the second homology class $[K] = 0$ in $H_2(S^4 \setminus L; Z) = 0$, there is a compact connected oriented 3-manifold V_K smoothly embedded in S^4 with $\partial V_K = K$ and $V_K \cap L = \emptyset$. Let h_K be a 1-handle system on K in V_K such that the closed complement $V(K) = \mathrm{cl}(V_K \setminus h_K)$ is a handlebody given by a decomposition into a 3-ball $B(K)$ and an attaching 1-handle system $h(K)$. The surface-knot $K(h_K)$ obtained from K by surgery along h is a trivial surface-knot bounding the handlebody $V(K)$. Let S be any S^2-knot component in L, which bounds a compact connected oriented 3-manifold V_S smoothly embedded in S^4 such that $V_S \cap (L \setminus S) = \emptyset$. The 3-ball $B(K)$ and the 1-handle system $h(K)$ are deformed in S^4 so that $V_S \cap B(K) = \emptyset$ and the 1-handle system $h(K)$ transversely meets V_S with transversal disks in the interior of V_S. Then there is a 1-handle system h_S on S in V_S such that the closed complement $V(S) = \mathrm{cl}(V_S \setminus h_S)$ is a handlebody given by a decomposition into a 3-ball $B(S)$ and an attaching 1-handle system $h(S)$ such that the transversal disks of $h(K)$ in the interior of V_S are in the interior of $B(S)$. Then the surface-link $K(h_K) \cup S(h_S)$ is a ribbon surface-link. Then the 1-handle systems h_K and h_S are made disjoint. Next, let T be any S^2-knot component in $L \setminus S$, which bounds a compact connected oriented 3-manifold V_T smoothly embedded in S^4 such that $V_T \cap (L \setminus (S \cup T)) = \emptyset$. The 3-balls $B(K)$, $B(S)$ and the 1-handle systems $h(K)$ and $h(S)$ are deformed in S^4 so that $V_T \cap (B(K) \cup B(S)) = \emptyset$ and the 1-handle systems $h(K)$ and $h(S)$ transversely meet V_T with transversal disks in the interior of V_T. Then there is a 1-handle system h_T on T in V_T such that the closed complement $V(T) = \mathrm{cl}(V_T \setminus h_T)$ is a handlebody given by a decomposition into a 3-ball $B(T)$ and an attaching 1-handle system $h(T)$ such that the transversal disks of $h(K)$ and $h(S)$ in the interior of V_T are in the interior of $B(T)$. Then the surface-link $K(h_K) \cup S(h_S) \cup T(h_S)$ is a ribbon surface-link given by the trivial S^2-link $\partial B(K) \cup \partial B(S) \cup \partial B(T)$ and the 1-handle system $h(K) \cup h(S) \cup h(T)$. Then the 1-handle systems h_K, h_S and h_T are made disjoint. By continuing this process, it is shown that there is a 1-handle system $\bar{h}$ on F such that the surface-link $F(\bar{h})$ obtained from F by surgery along $\bar{h}$ is a ribbon surface link. This completes the proof of Lemma 1.2.

Before proving Theorem 1.4, some explanations of the null-homotopic Gauss sum invariant of a surface-link are made, [4]. For a surface-link F in S^4, let $\Delta(F; Z_2)$ be the subgroup of $H_1(F;Z_2)$ of an element represented a loop ℓ in F which bounds an immersed disk d in S^4 with $d \cap F = \ell$. Let $\xi : \Delta(F;Z_2) \to Z_2$ be the function defined by $\xi([\ell])$ to be the Z_2-self-trivial -intersection number $\mathrm{Int}(d,d) \mod 2$ in S^4 with respect to the framing of the surface F, which defines a possibly singular Z_2-quadratic function

$$\xi(x+y) = \xi(x) + \xi(y) + x \cdot y \quad (x, y \in \Delta(F;Z_2)),$$

where $x \cdot y$ denotes the z_2-intersection number of x and y in F and called the *null-homotopic quadratic function* of the surface-link F. The *null-homotopic Gauss sum* of F is the Gauss sum $GS_0(F)$ of ξ defined by

$$GS_0(F) = \sum_{x \in \Delta(F;Z_2)} \exp(2\pi\sqrt{-1}\frac{\xi(x)}{2}).$$

This number $GS_0(F)$ is an invariant of a surface-link F and calculable, [4]. In particular, it is known that if F is a ribbon surface-link of total genus g, then $GS_0(F) = 2^g$. The following result in the case that $\mathbf{F}$ has at least two aspheric components is obtained by using this invariant $GS_0(F)$ which strengthens an earlier result, [8, Theorem 2].

Proof of Theorem 1.4. Let $k \cup k'$ be a non-splittable link in the interior of a 3-ball B such that k and k' are trivial knots. For the boundary 2-sphere $S = \partial B$ and the disk D^2 with the boundary circle S^1, let K be the torus-link of the torus-components $T = k \times S^1$ and $T' = k' \times S^1$ in the 4-sphere S^4 with $S^4 = B \times S^1 \cup S \times D^2$, which is a ribbon torus-link in S^4, [5]. In particular, $GS_0(K) = 2^2$. Since k and k' are trivial knots in B, the torus-knots T and T' are trivial torus-knots in S^4 by construction. Since $k \cup k'$ is non-splittable in B, there is a simple loop $t(k)$ in T coming from the longitude of k in B such that $t(k)$ does not bound any disk not meeting T' in S^4, meaning that there is a simple loop c in T unique up to isotopies of T which bounds a disk d in S^4 not meeting T', where c and d are given by $c = p \times S^1$ and $d = a \times S^1 \cup q \times D^2$ for a simple arc a in B joining a point p of k to a point q in S with $a \cap (k \cup k') = \{p\}$ and $a \cap S = \{q\}$. Regard the 3-ball B as the product $B = B_1 \times [0,1]$ for a disk B_1. Let τ_1 is a diffeomorphism of the solid torus $B_1 \times S^1$ given by one full-twist rounding the meridian disk B_1 one time along the S^1-direction, and $\tau = \tau_1 \times 1$ the product diffeomorphism of $(B_1 \times S^1) \times [0,1] = B \times S^1$. Let $\partial\tau$ be the diffeomorphism of the boundary $S \times S^1$ of $B \times S^1$ obtained from τ by restricting to the boundary, and the 4-manifold M obtained from $B \times S^1$ and $S \times D^2$ by pasting the boundaries $\partial(B \times S^1) = S \times S^1$ and $\partial(S \times D^2) = S \times S^1$ by the diffeomorphism $\partial\tau$. Since the diffeomorphism $\partial\tau$ of $S \times S^1$ extends to the diffeomorphism τ of $B \times S^1$, the 4-manifold M is diffeomorphic to S^4. Let $K_M = T_M \cup T'_M$ be the torus-link in the 4-sphere M arising from $K = T \cup T'$ in $B \times S^1$. There is a meridian-preserving isomorphism $\pi_1(S^4 \setminus K, x) \to \pi_1(M \setminus K_M, x)$ by van Kampen theorem. The loop $t(k)$ in T_M does not bound any disk not meeting T'_M in M, so that the loop c in T_M is a unique simple loop up to isotopies of T_M which bounds a disk $d_M = a \times S^1 \cup D^2_M$ in M

not meeting T'_M, where D^2_M denotes a proper disk in $S \times D^2$ bounded by the loop $\partial\tau(q \times S^1)$. An important observation is that the self-trivial -intersection number $\mathrm{Int}(d_M, d_M)$ in M with respect to the surface-framing on K_M is ± 1. This means that the loop c in T_M is a non-spin loop. Similarly, there is a unique non-spin loop c' in T'_M which bounds a disk d'_M with the self-trivial -intersection number $\mathrm{Int}(d'_M, d'_M) = \pm 1$ with respect to the surface-framing on K_M. Then it is calculated that $GS_0(K_M) = 0$ and the torus-link K_M in M is not any ribbon torus-link, [4]. Let $(S^4, K') = (M, K_M)$. If $\mathbf{F}$ consists of two tori, then the pair (K, K') forms a desired pair. If $\mathbf{F}$ is any surface of two aspheric components, then a desired $\mathbf{F}$-link pair is obtained from the pair (K, K') by taking connected sums of some trivial surface-knots, because every stabilization of a ribbon surface-link is a ribbon surface-link and every stable-ribbon surface-link is a ribbon surface-link, [8]. If $\mathbf{F}$ has some other surface $\mathbf{F}_1$ in addition to a surface $\mathbf{F}_0$ of two aspheric components, then a desired $\mathbf{F}$-link pair is obtained from a desired $\mathbf{F}_0$-link pair by adding the trivial $\mathbf{F}_1$-link as a split sum. Thus, a desired $\mathbf{F}$-link pair (K, K') is obtained. In particular, if $\mathbf{F}$ has total genus $g \leq 2$, then $GS_0(K) = 2^g$ and $GS_0(K') = 2^{g-2}$. Let A be a 4-ball in S^4 such that $A \cap L = A \cap K'$ is a trivial disk system in A with one disk component from one component of K and of K'. The self-trivial 1-handle system h used for every surgery of K in S^4 is deformed into A, so that h is also considered as a self-trivial 1-handle system used for a surgery of K' in S^4. Thus, the surface-links L and L' obtained from K and K' by surgery along the same self-trivial 1-handle system h in A are $\mathbf{F}'$-links for the same surface $\mathbf{F}'$ induced from $\mathbf{F}$ by the surgery along h. By van Kampen theorem, the fundamental groups of the $\mathbf{F}'$-links L and L' are the same group up to meridian-preserving isomorphisms. The null-homotopic Gauss sum invariant is independent of choices of a self-trivial 1-handle by calculations, [4]. Thus, if the self-trivial 1-handle system h consists of s self-trivial 1-handles, then L is a ribbon $\mathbf{F}'$-link of total genus $g + s$ with $GS_0(K) = 2^{g+s}$ and L' is a non-ribbon $\mathbf{F}'$-link with $GS_0(K') = 2^{g-2+s}$. This completes the proof of Theorem 1.4.

Note that the non-ribbon surface-link L' of two components starting from the Hopf link $k \cup k'$ in the interior of a 3-ball B has the free abelian fundamental group of rank 2. The diffeomorphism $\partial\tau$ of $S \times S^1$ in the proof of Theorem 1.4 coincides with Gluck's non-spin diffeomorphism of $S^2 \times S^1$, [2]. The torus-link (M, T_M) called a *turned torus-link* of a link $k \cup k'$ in B is an analogy of a turned torus-knot of a knot in B, [1].

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