Ribbonness on boundary surface-link

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ABSTRACT

It is shown that a boundary surface-link in the 4-sphere is a ribbon surface-link if the surface-knot obtained from it by surgery along a pairwise nontrivial 1-handle system is a ribbon surface-link. As a corollary, the surface-knot obtained from the anti-parallel surface-link of a non-ribbon surface-knot by surgery along a nontrivial fusion 1-handle is a non-ribbon surface-knot. This result answers Cochran's conjecture on non-ribbon sphere-knots in the affirmative.

Keywords: Boundary surface-link, Ribbon surface-link, Anti-parallel surface-link, Cochran's conjecture.
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1. Introduction

A *surface-link* is a closed oriented (possibly, disconnected) surface F smoothly embedded in the 4-sphere S^4. When F is connected, F is called a *surface-knot*. The components $F_i\,(i = 1, 2, \ldots, r)$ of F are 2-spheres, then F is called a *sphere-link* (or an S^2-*link*) of r components. A *1-handle system* on a surface-link F is a system h of disjoint 1-handles $h_j\,(j = 1, 2, \ldots, s)$ on F smoothly embedded in S^4. Let $F(h)$ be the surface-link obtained from F by surgery along a 1-handle system h. The 1-handle system h on F is a *fusion 1-handle system* if the number of connected components of $F(h)$ is $r - s(\geq 1)$, where the 1-handles $h_j\,(j = 1, 2, \ldots, s)$ of h are called *fusion 1-handles*. A surface-link F is a *boundary surface-link* if there is a system V of disjoint compact connected oriented 3-manifolds $V_i\,(i = 1, 2, \ldots, r)$ smoothly embeded in S^4 with $\partial V_i = F_i\,(i = 1, 2, \ldots, r)$. Assume that $r \geq 2$. A *ribbon surface-link* is a surface-link F which is obtained from a trivial S^2-link O by surgery along a 1-handle system h, [5], [10]. For a bounadry surface-link F, let D be a disk system of r disks $D_i(\subset F_i)$ $(i=1,2,\dots,r)$, and a disjoint 3-ball system B of 3-balls B_i $(i=1,2,\dots,r)$ in S^4 with $B_i \cap V_i = D_i$ $(i=1,2,\dots,r)$. Note that the surface-link F is equivalent to the union $\cup_{i=1}^r \partial(V_i \cup B_i)$ since $V_i \cup B_i$ is cell-move equivalent to V_i $(i=1,2,\dots,r)$. A *local* S^2-link of a bounadry surface-link F is a trivial S^2-link system $L=\partial B$. Every 1-handle system h on a boundary surface-link F is moved into a 1-handle system to attach on the disk system $D' = \mathrm{cl}(\partial B \setminus D)$ and to meet transversely the interior of B with a trasversal disk system. Thus, every 1-handle system h on a boundary surface-link F is understood as a 1-handle system on a local S^2-link L of F in S^4, so that $L(h)$ is a ribbon S^2-link. A 1-handle h_j in a 1-handle system h on a boundary surface-link F is *trivial* if the core arc c_j of h_j is ∂-relatively isotopic to a simple arc with interior disjoint from V in S^4. Otherwise, h_j is *nontrivial.* A 1-handle system h on a boundary surface-link F is *pairwise nontrivial* if every 1-handle h_j is nontrivial. The following theorem is a main result of this paper which is a revised version of an earlier result, [8, Theorem 1.4], [9].

Theorem 1.1. Let F be a boundary surface-link of $r(\geq 2)$ components in S^4. If the surface-knot $F(h)$ obtained from F by surgery along a pairwise nontrivial fusion 1-handle system h is a ribbon surface-link, then the surface-link F is a ribbon surface-link with h belonging to the 1-handle system of the ribbon surface-knot $F(h)$.

For a surface-knot F in S^4, let $F \times [0,1]$ be a normal $[0,1]$-bundle over F in S^4 such that the natural homomorphism $H_1(F \times 1; Z) \to H_1(S^4 \setminus F \times 0; Z)$ is the zero map. In other words, take $F \times [0,1]$ a boundary collar of a compact connected oriented 3-manifold V smoothly embedded in S^4 with $\partial V = F$, [2]. The surface-link $P(F) = \partial(F \times [0,1]) = F \times 0 \cup F \times 1$ in S^4 is called the *anti-parallel surface-link* of F, where by convention $F \times 0$ and $F \times 1$ are identified with $-F$ (i.e., the orientation-reversed F) and F, respecrtively. The anti-parallel surface-link $P(F)$ is a boundary surface-link, because $P(F)$ is the boundary of $V \times 0 \cup V \times 1$ for a normal $[0,1]$-bundle $V \times [0,1]$ of a compact connected oriented 3-manifold V with $\partial V = F$ smoothly embedded in S^4. The half parti of the following theorem is a direct consequence of Theorem 1.1.

Theorem 1.2. Let $P(F)$ be the anti-parallel surface-link of a non-ribbon surface-knot F in S^4, and $P(F)(h)$ the surface-knot obtained from $P(F)$ by surgery along a fusion 1-handle h. According to whether h is a trivial or nontrivial 1-handle, the surface-knot $P(F)(h)$ is a trivial or non-ribbon surface-knot, respectively.

Therem 1.2 positively answers Cochran's conjecture on non-ribbonability of the S^2-knot $P(F;h)$ for a non-ribbon S^2-knot F and any sufficiently complicated fusion1-handle h, [1].

2. Proofs of Theorem 1.1 and 1.2

The proof of Theorem 1.1 is done as follows.

2.1: Proof of Theorem 1.1. If $F(h)$ is a disconnected ribbon surface-link, then there is a pairwise nontrivial fusion 1-handle system h^+ on F extending h such that $F(h^+)$ is a ribbon surface-knot. Thus, assume that $F(h)$ is a ribbon surface-knot. First, the proof of the case $r = 2$ is given. Let $i = 1$ or 2. For the 3-manifold V_i with $\partial V_i = F_i$, let β_i be a 1-handle system on F_i embedded in V_i and disjoint from h and B such that $V_i' = \mathrm{cl}(V_i \setminus \beta_i)$ is a handlebody, [4]. Then the surface-link $F(\beta) = F_1(\beta_1) \cup F_2(\beta_2)$ for the 1-handle system $\beta = \beta_1 \cup \beta_2$ bounds the disjoint handlebody system $V' = V_1' \cup V_2'$ and hence is a trivial surface-link in S^4. The surface-knot $F(\beta)(h)$ is a ribbon surface-knot which is equivalent to the connected sum of the non-trivial ribbon S^2-knot $L(h)$ and the trivial surface-knots $F_i(\beta_i)\,(i = 1, 2)$ attaching along the disks $D_i\,(i = 1, 2)$. The ribbon S^2-knot $L(h)$ has a canonical SUPH system $W(Lh) = B^{(0)} \cup h$, where $B^{(0)} = B_1^{(0)} \cup B_2^{(0)}$ for a once-punctured 3-ball $B_i^{(0)}$ of the 3-ball B_i in the 3-ball system $B = B_1 \cup B_2$. Then the ribbon surface-knot $F(\beta)(h)$ has a SUPH system $W = W(Lh) \cup V'$ which is a disk sum of $W(Lh)$ and V' pasting along the disk system $D = D_1 \cup D_2$. On the other hand, the surface-knot $F(h)$ is a ribbon surface-link and hence has a SUPH system $W(Fh)$. If necessary, by replacing $W(Fh)$ with a multi-punctured $W(Fh)$, the union $W' = W(Fh) \cup \beta$ is a SUPH system for the surface-knot $F(\beta)(h)$. By replacing W and W' with multi-punctured W and W', respectively, there is an orientation-preserving diffeomorphism f of S^4 sending W and W'. The following property is used here.

(2.1.1) The diffeomorphism f of S^4 is isotopically deformed so that the restriction of f to $F(\beta)(h)$ is the identity map.

By (2.1.1), assume that the restriction $f|F(\beta)(h)$ is the identity. Let $D(h)$ be a transversal disk of the 1-handle h, and $D(\beta)$ a transversal disk system of the 1-handle system β with one disk for each 1-handle of β. Let (k, k') be a loop basis of $F(\beta)(h)$ such that k is the boundary loop system of a meridian disk system $D(k)$ of V' and k' is the boundary loop system of a disk system $D(k')$ with $D(k') \cap V' = k'$, so that $(D(k), D(k'))$ is an O2-handle pair on $F(\beta)(h)$. Since W is a disk sum of $W(Lh)$ and V' with open 3-balls removed, it is assumed that the loop system k is equal to the loop system $\partial D(\beta)$. The interior of the disk $f(D(h))$ transversely meets the interior of the disk system $D(\beta)$ in the multi-punctured handlebody W'. Since any smoothly embedded 2-sphere in W' bounds a multi-punctured 3-ball in W', the interior of the disk $f(D(h))$ is isotopically deformed in W' so that $f(D(h)) \cap D(\beta) = \emptyset$. By cutting W' along the disk union $f(D(h)) \cup D(\beta)$, a SUPH system W'' for F is obtained. Thus, F is a ribbon surface-link with h belonging to the 1-handle system of the ribbon surface-knot $F(h)$, completing the proof for the case $r = 2$.

In general, if F has the $r(\geq 3)$ components $F_i\,(i = 1, 2, \ldots, r)$, then assume that $h' = h \backslash h_1$ is a pairwise nontrivial fusion 1-handle system on the boundary surface-link $F' = F \setminus F_1$ with $F'(h')$ a surface-knot. Then h_1 is a nontrivial fusion 1-handle on the boundary surface-link $F'' = F_1 \cup F'(h')$ with $F''(h_1) = F(h)$ a ribbon surface-knot. By the argument of $r = 2$ above, F'' is a ribbon surface-link with h_1 belonging to the ribbon 1-handle system of the ribbon surface-link F''. In particular, $F'(h')$ is a ribbon surface-knot. By inductive assumption on r, the boundary surface-link F' is a ribbon surface-link with h' belonging to the ribbon 1-handle system of the ribbon surface-knot $F'(h')$. This means that $F = F_1 \cup F'$ is a ribbon surface-link with h belonging to the ribbon 1-handle system of the ribbon surface-knot $F(h)$. This completes the proof of Theorem 1.1.

(2.1.1) is proved as follows.

Proof of (2.1.1). For $i = 1$ or 2, let (k_i, k_i') be a loop basis of $F_i(\beta_i)$ such that k_i is the boundary loop system of a meridian disk system $D(k_i)$ of the handlebody V_i' and k_i' is the boundary loop system of a disk system $D(k_i')$ with $D(k_i') \cap V_i' = k_i'$. Let (k, k') be a loop basis of $F(\beta)(h)$ consisting of $(k_i, k_i')(i = 1, 2)$ such that k is the boundary loop system of a meridian disk system $D(k)$ consisting of $D(k_i)\,(i = 1, 2)$ and k' is the boundary loop system of a disk system $D(k')$ consisting of $D(k_i')\,(i = 1, 2)$ with $D(k') \cap V' = k'$, so that $(D(k), D(k'))$ is an O2-handle basis on $F(\beta)(h)$. Let $F(\beta)(h)_*$ be the ribbon S^2-knot obtained from $F(\beta)(h)$ by surgery along the O2-handle basis $(D(k), D(k'))$, which is isotopic to the ribbon S^2-knot $L(h)$, [7]. The image $f(F(\beta)(h)_*)$ is a ribbon S^2-knot obtained from the ribbon surface-knot $f(F(\beta)(h))$ by surgery along the O2-handle basis $(f(D(k)), f(D(k')))$, which is isotoipic to the ribbon S^2-knot $f(L(h))$. Any S^2-knot K equivalent to $L(h)$ is isotopic to $L(h)$. In fact, K is written as a ribbon S^2-knot $L(h')$ for a 1-handle h' on the trivial S^2-link system $L = \partial B_1 \cup \partial B_2$ (since a trivial S^2-link is isotopically unique). Then the 1-handle h' is isotopically and attaching-part-relatively deformed into the 1-handle h. This is because equivalent ribbon S^2-knots $L(h)$ and $L(h')$ are faithfully equivalent, [6]. Thus, the ribbon S^2-knot $f(L(h))$ is isotopic to $L(h)$, so that $f(F(\beta)(h)_*)$ is isotopic to $F(\beta)(h)_*$. The uniqueness of an O2-handle pair in the soft sense, the image

$f(F(\beta)(h))$ is isotopic to the surface-knot $F(\beta)(h)$, [9]. This completes the proof of (2.1.1)

The proof of Theorem 1.2 is done as follows.

2.2: Proof of Theorem 1.2. Assume that the fusion 1-handle h on the anti-parallel surface-link $P(F)$ is nontrivial. Since the surface-knot $P(F)(h)$ is a ribbon surface-knot the surface-link $P(F)$ is a boundary surface-link, the surface-link $P(F)$ is a ribbon surface-link by Theorem 1.1, so that F is a ribbon surface-knot, contradicting that F is a non-ribbon surface-knot. Thus, $P(F)(h)$ is a non-ribbon surface-knot. Assume that h is a trivial fusion 1-handle on $P(F) = F_0 \cup F_1$ with $F_i = F \times i\,(i = 0, 1)$. Let $V = V_0 \cup V_1$ be a disconnected compact oriented 3-manifold without containing a closed 3-manifold such that $\partial V_i = F_i\,(i = 0, 1)$ and the 1-handle h on V does not meet V except for the attaching part. Let β_i be a 1-handle system on F_i embedded in V_i and disjoint from h and B such that $V_i' = \mathrm{cl}(V_i \setminus \beta_i)$ is a handlebody for $i = 0$ or 1, [4]. Then $\partial V_i' = F_i(\beta_i)\,(i = 0, 1)$. Let $h_0 = d \times [0, 1]$ in $F \times [0, 1]$ for a disk d in F which is a 1-handle of $P(F)$. Then the surface-knot $P(F)(h_0)$ is a trivial surface-knot which bounds a handlebody H containing h_0 as a thickenned meridian disk such that the union $W = H \cup \beta_0 \cup \beta_1$ is a handlebody. Let $H = h_0 \cup H_0 \cup H_1$ for two handlebodies $H_i\,(i = 0, 1)$ connected by h_0 so that $\partial H_i \supset F_i^{(0)}\,(i = 0, 1)$ and $W = h_0 \cup (H_0 \cup \beta_0) \cup (H_1 \cup \beta_1)$ with $H_i \cup \beta_i$ a handlebody containing $F_i(\beta_i)^{(0)}$ in the boundary for $i = 0, 1$. Since for any two spin loop bases (a, a'), (b, b') of a trivial surface-knot T in S^4, there is an orientation-preserving diffeomorphism of (S^4, F) sending (a, a') to (b, b'), there is an orientation-preserving diffeomorphism f of S^4 sending $W' = h \cup V_0' \cup V_1'$ to a handlebody $W_0' = h_0 \cup V_0' \cup V_1'$ such that the restriction of f to V_i' is the identity for $i = 0, 1$ by isotopically deforming h into h_0 together with V_1'. Furhter, there is an orientation-preserving diffeomorphism g of S^4 keeping $\partial W = \partial W_0'$ fixed and sending $W = h_0 \cup (H_0 \cup \beta_0) \cup (H_1 \cup \beta_1)$ to $W_0' = h_0 \cup V_0' \cup V_1'$ with $H_i \cup \beta_i$ sent to $V_i'\,(i = 0, 1)$. Then $F(h)$ bounds a handlebody $\mathrm{cl}(W' \setminus f^{-1}g(\beta_0 \cup \beta_1)$. Thus, $F(h)$ is a trivial surface-knot. This completes the proof of Theorem 1.2.

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