Vanishing phenomena on Trudinger-Moser inequalities with scaling parameter

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Abstract

We investigate the asymptotic behavior of maximizers for the Trudinger-Moser inequalities with a scaling parameter, both in the Neumann and Dirichlet cases. In particular, we focus on the vanishing phenomena of maximizers when the exponent in the Trudinger-Moser functional is sufficiently small. We show that, after suitable normalization, sequences of maximizers converge to solutions of elliptic equations with polynomial nonlinearities, which are characterized as maximizers of explicit variational problems. Moreover, we derive asymptotic expansions of the best constants.

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1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. It is well known that there is a Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{2p/(2-p)}(\Omega)$ for $p \in [1, 2)$. If we look at the limiting case $p = 2$, then $H_0^1(\Omega) := W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \geq 1$, but $H_0^1(\Omega) \not\hookrightarrow L^\infty(\Omega)$. To fill this gap, it is natural to seek the maximal growth function $g : \mathbb{R} \to \mathbb{R}_+$ such that

$$\sup_{\substack{u \in H_0^1(\Omega) \\ \|\nabla u\|_2 \leq 1}} \int_\Omega g(u)dx < \infty,$$

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where $\|\nabla u\|_2^2 = \int_\Omega |\nabla u|^2 dx$ denotes the Dirichlet norm of u. Pohozaev [21] and Trudinger [24] proved independently that the maximal growth is of exponential type, more precisely that there exists a constant α such that

$$\sup_{\substack{u \in H_0^1(\Omega) \\ \|\nabla u\|_2 \le 1}} \int_\Omega e^{\alpha u^2} dx < \infty.$$

Later, Moser [18] sharpened this inequality as follows:

$$\sup_{\substack{u \in H_0^1(\Omega) \\ \|\nabla u\|_2 \le 1}} \int_\Omega e^{\alpha u^2} dx \begin{cases} < C|\Omega| & \text{if} \quad \alpha \le 4\pi, \\ = \infty & \text{if} \quad \alpha > 4\pi. \end{cases} \tag{1}$$

Lions [16] showed that for (1) there is a loss of compactness at the limiting exponent $\alpha = 4\pi$. Despite this loss of compactness, the existence of an extremal function attaining the supremum in (1) for $\alpha = 4\pi$ was established by Carleson and Chang [1] in the case where Ω is a unit ball, and this result was extended to arbitrary bounded domains in $\mathbb{R}^2$ by Flucher [6].

In this paper, we study the Trudinger-Moser functional

$$E_\alpha(u) := \int_\Omega \left(e^{\alpha u^2} - 1\right) dx, \quad \alpha > 0,$$

subject to either of the constraints

$$\Sigma_\lambda := \left\{ u \in H^1(\Omega) \;\middle|\; \int_\Omega (|\nabla u|^2 + \lambda u^2) dx = 1 \right\}$$

or

$$\Sigma_\lambda^0 := \left\{ u \in H_0^1(\Omega) \;\middle|\; \int_\Omega (|\nabla u|^2 + \lambda u^2) dx = 1 \right\},$$

where $\lambda > 0$ is a parameter. The parameter λ represents the scaling of the domain Ω. Indeed, by considering the transformation $u_\lambda(x) = u((x-p)/\sqrt{\lambda})$ for $u \in H^1(\Omega)$, $\lambda > 0$, and $p \in \mathbb{R}^2$, one has that the functional $E_\alpha(u)$ on Ω coincides with $E_\alpha(u_\lambda)$ on $\Omega_\lambda := \left\{ \sqrt{\lambda}x + p \middle| x \in \Omega \right\}$, and similarly, $u \in \Sigma_\lambda$ on Ω if and only if $u_\lambda \in \Sigma_1$ on Ω_λ. Thus, the existence of a maximizer for $\sup_{u \in \Sigma_\lambda} E_\alpha(u)$ on Ω is equivalent to that for $\sup_{u \in \Sigma_1} E_\alpha(u)$ on Ω_λ. The same equivalence holds in the Dirichlet case.

Concerning the existence of maximizers, it is known that for any $\lambda > 0$ there exists a maximizer for $\sup_{u \in \Sigma_\lambda} E_\alpha(u)$ provided $\alpha \in (0, 2\pi]$, and for

$\sup_{u\in\Sigma_\lambda^0} E_\alpha(u)$ provided $\alpha \in (0, 4\pi]$. In particular, the existence of a maximizer for $\sup_{u\in\Sigma_\lambda} E_{2\pi}(u)$ and for $\sup_{u\in\Sigma_\lambda^0} E_{4\pi}(u)$ has been proved by Yang [25] and Ruf [22], respectively.

We focus on the asymptotic behavior of maximizers for $\sup_{u\in\Sigma_\lambda} E_\alpha(u)$ and $\sup_{u\in\Sigma_\lambda^0} E_\alpha(u)$ as $\lambda \to \infty$. In [7], it was shown that the asymptotic behavior of maximizers for $\sup_{u\in\Sigma_\lambda} E_\alpha(u)$ depends on the exponent α. Namely, there exists a threshold $\alpha_* \in (0, 2\pi)$ such that if $\alpha \in (\alpha_*, 2\pi)$, then any maximizer of $\sup_{u\in\Sigma_\lambda} E_\alpha(u)$ develops a single spike layer with its unique peak located on the boundary of Ω, while if $\alpha \in (0, \alpha_*)$, then any sequence of maximizers converges to zero in $C(\overline{\Omega})$ as $\lambda \to \infty$. Moreover, in [8], it was shown that the situation in the critical case $\alpha = 2\pi$ is the same as in the case $\alpha \in (\alpha_*, 2\pi)$. Furthermore, for any $\alpha \in (\alpha_*, 2\pi]$ maximizers of $\sup_{u\in\Sigma_\lambda} E_\alpha(u)$ concentrate around a point where the mean curvature is maximized as $\lambda \to \infty$. A similar phenomenon holds for $\sup_{u\in\Sigma_\lambda^0} E_\alpha(u)$. In [8], it was also shown that there exists a threshold $\beta_* \in (0, 4\pi)$ such that if $\alpha \in (\beta_*, 4\pi]$, then maximizers of $\sup_{u\in\Sigma_\lambda^0} E_\alpha(u)$ concentrate at a single point, whereas if $\alpha \in (0, \beta_*)$, then any sequence of maximizers vanishes in $C(\overline{\Omega})$ as $\lambda \to \infty$. However, the concentration point of maximizers for $\sup_{u\in\Sigma_\lambda^0} E_\alpha(u)$ differs from that in the case of $\sup_{u\in\Sigma_\lambda} E_\alpha(u)$. Maximizers for $\sup_{u\in\Sigma_\lambda^0} E_\alpha(u)$ necessarily concentrate around a "most centered point" of the domain Ω, namely a point of maximum distance to the boundary. These concentration results follow from asymptotic expansions of the best constants $\sup_{u\in\Sigma_\lambda} E_\alpha(u)$ and $\sup_{u\in\Sigma_\lambda^0} E_\alpha(u)$ as $\lambda \to \infty$. On the other hand, a precise description of the vanishing behavior of maximizers for small α remains open in both cases.

The asymptotic behavior of maximizers for $\sup_{u\in\Sigma_\lambda} E_\alpha(u)$ and $\sup_{u\in\Sigma_\lambda^0} E_\alpha(u)$ is closely related to the following variational problem:

$$d_\beta := \sup_{\substack{u\in H^1(\mathbb{R}^2) \\ \int_{\mathbb{R}^2}(|\nabla u|^2+u^2)dx\le 1}} \int_{\mathbb{R}^2} \left(e^{\beta u^2} - 1\right) dx.$$

The variational problem d_β exhibits a loss of compactness in $H^1(\mathbb{R}^2)$ caused not only by concentration but also by vanishing phenomena. In [10], it has been shown that for sufficiently small β, there does not exist a function that attains the supremum d_β, due to vanishing loss of compactness. This nonexistence result causes the vanishing behavior of maximizers for $\sup_{u\in\Sigma_\lambda} E_\alpha(u)$ and $\sup_{u\in\Sigma_\lambda^0} E_\alpha(u)$ when α is small. In fact, their behavior corresponds to the vanishing behavior of maximizing sequences for d_β. Thus, maximizers of

$\sup_{u\in\Sigma_\lambda} E_\alpha(u)$ and $\sup_{u\in\Sigma_\lambda^0} E_\alpha(u)$ can be regarded as "normalized vanishing sequences" introduced in [10], in the following sense.

Let $\lambda > 0$, and let u_λ and v_λ be maximizers of $\sup_{u\in\Sigma_\lambda} E_\alpha(u)$ and $\sup_{u\in\Sigma_\lambda^0} E_\alpha(u)$, respectively. Consider $\Omega_\lambda := \left\{ \sqrt{\lambda}(x-p)+p \;\middle|\; x\in\Omega \right\},\ p\in\mathbb{R}^2$, and define the scaled functions

$$\tilde{u}_\lambda(x) := u_\lambda\left(\frac{x-p}{\sqrt{\lambda}}+p\right), \quad \tilde{v}_\lambda(x) := v_\lambda\left(\frac{x-p}{\sqrt{\lambda}}+p\right),$$

which satisfy $\|\tilde{u}_\lambda\|_{H^1(\Omega_\lambda)} = \|\tilde{v}_\lambda\|_{H^1(\Omega_\lambda)} = 1$ for all $\lambda > 0$. The results in [7, 8] show that

$$\lim_{\lambda\to\infty} \|\nabla\tilde{u}_\lambda\|_{L^2(\Omega_\lambda)} = \lim_{\lambda\to\infty} \|\nabla\tilde{v}_\lambda\|_{L^2(\Omega_\lambda)} = 0,$$
$$\lim_{\lambda\to\infty} \|\tilde{u}_\lambda\|_{L^2(\Omega_\lambda)} = \lim_{\lambda\to\infty} \|\tilde{v}_\lambda\|_{L^2(\Omega_\lambda)} = 1,$$

and

$$\lim_{\lambda\to\infty} \int_{\Omega_\lambda\cap B_R(q)} \left(e^{\alpha\tilde{u}_\lambda^2}-1\right) dx = \lim_{\lambda\to\infty} \int_{\Omega_\lambda\cap B_R(q)} \left(e^{\alpha\tilde{v}_\lambda^2}-1\right) dx = 0$$

for each $R > 0$ and $q \in \mathbb{R}^2$, where q may depend on λ.

The vanishing loss of compactness of maximizing sequences for d_β arises from the fact that the embedding $H^1_{rad}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$ is continuous but not compact, where $H^1_{rad}(\mathbb{R}^2)$ consists of radially symmetric functions in $H^1(\mathbb{R}^2)$. For further studies on vanishing phenomena, their relation to Trudinger-Moser type inequalities, and related variational problems, we refer the reader to [2, 3, 4, 5, 9, 11, 14, 15, 17, 23] and references therein.

In this paper, we clarify the profiles of the normalized vanishing sequences $\tilde{u}_\lambda$ and $\tilde{v}_\lambda$ by investigating the asymptotic behavior of u_λ and v_λ.

To state our results, we first define the constant

$$\beta_* := \inf\{\beta\in(0,4\pi) \mid d_\beta > \beta\},$$

and set $\alpha_* := \beta_*/2$. We also define the best constant of the Gagliardo-Nirenberg inequality by

$$A := \sup_{\substack{\phi\in H^1(\mathbb{R}^2)\\ \phi\not\equiv 0}} \frac{\int_{\mathbb{R}^2}\phi^4 dx}{\left(\int_{\mathbb{R}^2}|\nabla\phi|^2 dx\right)\left(\int_{\mathbb{R}^2}\phi^2 dx\right)}.$$

Note that $\beta_* \in (0, 4\pi)$, and hence $\alpha_* \in (0, 2\pi)$. More precisely, it holds that $\beta_* \le 2/A$, and therefore $\alpha_* \le 1/A$, by Theorem 1.1 in [10]. For the relation between α_* and the variational problem on the Trudinger-Moser inequalities in the half space, see Appendix in [7].

For $\alpha \in (0, 2\pi]$ and $\lambda > 0$, we write $I(\alpha, \lambda) := \sup_{u \in \Sigma_\lambda} E_\alpha(u)$, and for $\alpha \in (0, 4\pi]$ and $\lambda > 0$, $d(\alpha, \lambda) := \sup_{u \in \Sigma_\lambda^0} E_\alpha(u)$. For $\gamma > 0$ we further define

$$B_\gamma := \sup_{\substack{u \in H^1(\Omega) \\ \int_\Omega u^2 dx = 1}} \left(\int_\Omega u^4 dx - \frac{2}{\gamma} \int_\Omega |\nabla u|^2 dx \right),$$

and

$$B_\gamma^0 := \sup_{\substack{v \in H_0^1(\Omega) \\ \int_\Omega v^2 dx = 1}} \left(\int_\Omega v^4 dx - \frac{2}{\gamma} \int_\Omega |\nabla v|^2 dx \right).$$

In Section 4, we will prove that $B_\gamma < \infty$ if $\gamma < 1/A$, and that $B_\gamma^0 < \infty$ if $\gamma < 2/A$, as well as the existence of maximizers.

Within this framework, we obtain the following results.

Theorem 1.1. *Assume that $\alpha \in (0, \alpha_*)$. Let u_λ be a maximizer of $I(\alpha, \lambda)$ for large λ. Then*

$$\sqrt{\lambda} u_\lambda(x) \to u_0 \quad in \quad C^2(\overline{\Omega})$$

as $\lambda \to \infty$, where $u_0 \in H^1(\Omega)$ is a maximizer of B_α.

Theorem 1.2. *Assume that $\alpha \in (0, \beta_*)$. Let v_λ be a maximizer of $d(\alpha, \lambda)$ for large λ. Then*

$$\sqrt{\lambda} v_\lambda(x) \to v_0 \quad in \quad C^2(\overline{\Omega})$$

as $\lambda \to \infty$, where $v_0 \in H^1(\Omega)$ is a maximizer of B_α^0.

From the viewpoint of normalized vanishing sequences $\tilde{u}_\lambda$ and $\tilde{v}_\lambda$, it follows that

$$\sqrt{\lambda}\tilde{u}_\lambda \left(\sqrt{\lambda}(x - p) + p \right) \to u_0, \qquad \sqrt{\lambda}\tilde{v}_\lambda \left(\sqrt{\lambda}(x - p) + p \right) \to v_0$$

as $\lambda \to \infty$. This transformation preserves the L^2-norm and was introduced in [10] as an explicit form of vanishing sequences.

Maximizers of $I(\alpha,\lambda)$ and $d(\alpha,\lambda)$ correspond to solutions of the following elliptic equations:

$$\begin{cases} -\Delta u + \lambda u = \frac{ue^{\alpha u^2}}{\int_\Omega u^2 e^{\alpha u^2}dx} & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta u + \lambda u = \frac{ue^{\alpha u^2}}{\int_\Omega u^2 e^{\alpha u^2}dx} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

respectively. The proofs of Theorems 1.1 and 1.2 are based on an analysis of these equations, under the assumption that both u_λ and v_λ vanish in $C(\overline{\Omega})$ as $\lambda \to \infty$. Precise asymptotic properties of u_λ and v_λ, together with the Taylor expansion

$$e^s - 1 = s + \frac{s^2}{2} + O(s^4) \quad \text{as} \quad s \to 0,$$

lead to the following asymptotic expansions:

$$I(\alpha,\lambda) = \sup_{u\in\Sigma_\lambda} E_\alpha(u) = \frac{1}{\lambda}\left[\alpha + \frac{\alpha^2}{2}B_\alpha\frac{1}{\lambda} + o(\lambda^{-1})\right],$$

and

$$d(\alpha,\lambda) = \sup_{u\in\Sigma_\lambda^0} E_\alpha(u) = \frac{1}{\lambda}\left[\alpha + \frac{\alpha^2}{2}B_\alpha^0\frac{1}{\lambda} + o(\lambda^{-1})\right],$$

as $\lambda \to \infty$. It is worth noting that the difference between the behaviors of $I(\alpha,\lambda)$ and $d(\alpha,\lambda)$ is that $\lambda I(\alpha,\lambda)$ approaches α from above, whereas $\lambda d(\alpha,\lambda)$ approaches α from below, since $B_\alpha > 0$ while $B_\alpha^0 < 0$.

This paper is organized as follows. In Section ,2 we prove Theorem 1.1, and in Section 3, we prove Theorem 1.2. In Section 4, we establish several results concerning B_α and B_α^0.

2. Proof of Theorem 1.1

Assume that $\alpha \in (0,\alpha_*)$. Let $\{\lambda_n\}$ be a sequence with $\lambda_n \to \infty$ as $n \to \infty$, and let u_n be a maximizer of $\sup_{u\in\Sigma_{\lambda_n}} E_\alpha(u)$ for sufficiently large n. Without loss of generality, we may assume that u_n is nonnegative in Ω. For simplicity, set $\mu_n := \|u_n\|_{L^\infty(\Omega)}$. Let $x_n \in \overline{\Omega}$ be a point where u_n attains its maximum, i.e., $u_n(x_n) = \mu_n$. Then u_n satisfies the elliptic equation

$$\begin{cases} -\Delta u_n + \lambda_n u_n = L_n u_n e^{\alpha u_n^2} & \text{in} \quad \Omega, \\ \frac{\partial u_n}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

where L_n is the Lagrange multiplier expressed as

$$L_n = \frac{1}{\int_\Omega u_n^2 e^{\alpha u_n^2} dx}.$$

Moreover, we have $L_n < \lambda_n$ for each $n \in \mathbb{N}$. By elliptic regularity theory, it follows that $u_n \in C^2(\Omega) \cap C^1(\overline{\Omega})$, and by the maximum principle, we obtain $u_n > 0$ in $\overline{\Omega}$.

The following results have been proved in [7].

Proposition 2.1. *We have*

$$\mu_n \to 0, \quad \int_\Omega |\nabla u_n|^2 dx \to 0, \quad \textit{and} \quad \lambda_n \int_\Omega u_n^2 dx \to 1$$

as $n \to \infty$.

Concerning the behavior of L_n, Lemma 2.5 in [7] yields

$$\lim_{n\to\infty} \frac{L_n}{\lambda_n} = 1. \tag{3}$$

We first prove the following lemma.

Lemma 2.2. *It holds that*

$$\liminf_{n\to\infty} \lambda_n \mu_n^2 \geq \frac{1}{|\Omega|},$$

where $|\Omega|$ *denotes the Lebesgue measure of the domain* Ω.

Proof. By Proposition 2.1, we obtain

$$\lambda_n \mu_n^2 |\Omega| \geq \lambda_n \int_\Omega u_n^2 dx = 1 + o(1).$$

This directly implies the desired lower bound in Lemma 2.2. □

Next, we establish the following estimates.

Lemma 2.3. *We have*

$$\limsup_{n\to\infty} \frac{1}{\mu_n^2} \int_\Omega |\nabla u_n|^2 dx \le \alpha \tag{4}$$

and

$$\limsup_{n\to\infty} \left(1 - \frac{L_n}{\lambda_n}\right) \frac{1}{\mu_n^2} \le \alpha. \tag{5}$$

Proof. Multiplying (2) by u_n and integrating over Ω, we have

$$\int_\Omega |\nabla u_n|^2 dx + \lambda_n \int_\Omega u_n^2 dx = L_n \int_\Omega u_n^2 e^{\alpha u_n^2} dx.$$

For the right hand side, using Proposition 2.1 and (3), we estimate

$$\begin{aligned} L_n \int_\Omega u_n^2 e^{\alpha u_n^2} dx &\le \left(1 + \alpha\mu_n^2\right) L_n \int_\Omega u_n^2 dx + O\left(L_n \mu_n^4 \int_\Omega u_n^2 dx\right) \\ &= L_n \int_\Omega u_n^2 dx + \alpha\mu_n^2 + o(\mu_n^2). \end{aligned}$$

Hence,

$$\int_\Omega |\nabla u_n|^2 dx + \left(1 - \frac{L_n}{\lambda_n}\right) \lambda_n \int_\Omega u_n^2 dx \le \alpha\mu_n^2 + o(\mu_n^2).$$

Applying Proposition 2.1 once again, we deduce

$$\frac{1}{\mu_n^2} \left[\int_\Omega |\nabla u_n|^2 dx + \left(1 - \frac{L_n}{\lambda_n}\right)(1 + o(1))\right] \le \alpha + o(1),$$

which yields the desired estimates (4) and (5). □

By the positivity of $\lambda_n - L_n$ and estimate (5) in Lemma 2.3, we may, after passing to a subsequence, assume that $\omega = \lim_{n\to\infty} \left(1 - L_n/\lambda_n\right)/\mu_n^2$. Then, we obtain the following upper bound for $\lambda_n \mu_n^2$.

Lemma 2.4. *There exists a positive constant K such that*

$$\limsup_{n\to\infty} \lambda_n \mu_n^2 \le K.$$

Proof. Assume by contradiction that, after passing to a subsequence if necessary, $\lambda_n \mu_n^2 \to \infty$ as $n \to \infty$. We may also assume that $x_n \to x_0 \in \overline{\Omega}$ if necessary. For simplicity, set $R_n := \sqrt{\lambda_n}\mu_n$.

We first consider the case $\lim_{n\to\infty} \operatorname{dist}(x_n, \partial\Omega) R_n = \infty$. Define $\Omega_n := \{R_n(x - x_n) \mid x \in \Omega\}$ and consider a scaled function

$$w_n(z) = \frac{1}{\mu_n} u_n \left(\frac{z}{R_n} + x_n \right).$$

The function w_n satisfies $w_n(z) \le 1$ for any $z \in \overline{\Omega_n}$, $\|w_n\|_{L^\infty(\Omega_n)} = 1$ and

$$\begin{cases} -\Delta w_n + \left(1 - \frac{L_n}{\lambda_n}\right) \frac{1}{\mu_n^2} w_n = \frac{L_n}{\lambda_n} \frac{1}{\mu_n^2} w_n \left(e^{\alpha \mu_n^2 w_n^2} - 1 \right) & \text{in} \quad \Omega_n, \\ \frac{\partial w_n}{\partial \nu} = 0 & \text{on} \quad \partial\Omega_n. \end{cases}$$

Since $\lim_{n\to\infty} \operatorname{dist}(x_n, \partial\Omega) R_n = \infty$, for any $R > 0$ there exists $N \in \mathbb{N}$ such that $B_R(x_n) \subset \Omega_n$ for any $n \ge N$. Then, by elliptic regularity theory, there exists a function w_0 such that

$$w_n \to w_0 \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2)$$

and

$$-\Delta w_0 + \omega w_0 = \alpha w_0^3, \quad w_0 > 0 \quad \text{in} \quad \mathbb{R}^2. \tag{6}$$

Since $\lambda_n \int_\Omega u_n^2 dx = 1 + o(1)$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} w_0^2 dx &= \lim_{R\to\infty} \int_{B_R} w_0^2 dx \\ &= \lim_{R\to\infty} \lim_{n\to\infty} \int_{B_R} w_n^2 dx \\ &= \lim_{R\to\infty} \lim_{n\to\infty} \lambda_n \int_{B_{R/R_n}(x_n)} u_n^2 dx \\ &\le 1. \end{aligned} \tag{7}$$

Combining this with (4), we deduce $w_0 \in H^1(\mathbb{R}^2)$. Then, by the Harnack inequality, we derive that $w_0(x) \to 0$ as $|x| \to \infty$, so w_0 is uniquely determined by the uniqueness result for (6) with the decay condition at infinity (see for instance, Kwong [12]).

Here, recalling that $\alpha < \alpha_* \le 1/A$ and the Gagliardo-Nirenberg inequality, we have

$$\frac{\int_{\mathbb{R}^2} w_0^4 dx}{\left(\int_{\mathbb{R}^2} |\nabla w_0|^2 dx\right)\left(\int_{\mathbb{R}^2} w_0^2 dx\right)} \le A < \frac{1}{\alpha}. \tag{8}$$

On the other hand, the Pohozaev identity gives

$$\omega \int_{\mathbb{R}^2} w_0^2 dx = \frac{\alpha}{2} \int_{\mathbb{R}^2} w_0^4 dx,$$

which leads to $\omega > 0$ and

$$\int_{\mathbb{R}^2} w_0^4 dx = \frac{2\omega}{\alpha} \int_{\mathbb{R}^2} w_0^2 dx. \tag{9}$$

Moreover, multiplying (6) by w_0, integrating over $\mathbb{R}^2$ and using (9), we have

$$\int_{\Omega} |\nabla w_0|^2 dx = \omega \int_{\mathbb{R}^2} w_0^2 dx. \tag{10}$$

Thus, combining (7), (9) and (10), we obtain

$$\frac{\int_{\mathbb{R}^2} w_0^4 dx}{\left(\int_{\mathbb{R}^2} |\nabla w_0|^2 dx\right)\left(\int_{\mathbb{R}^2} w_0^2 dx\right)} = \frac{2}{\alpha \int_{\Omega} w_0^2 dx} \ge \frac{1}{\alpha}.$$

However, the inequality contradicts (8). Consequently, Lemma 2.4 holds in the case $\lim_{n\to\infty} \mathrm{dist}(x_n, \partial\Omega) R_n = \infty$.

We next consider the case $\mathrm{dist}(x_n, \partial\Omega) = O(R_n^{-1})$. We recall a diffeomorphism straightening a boundary portion around a point P on $\partial\Omega$, introduced in [13, 19, 20].

Fix $P \in \partial\Omega$. We may assume that P is the origin and the inner normal to $\partial\Omega$ at P points in the direction of the positive x_2-axis, where $x = (x_1, x_2) \in \mathbb{R}^2$. In a neighborhood N of P, $\partial\Omega \cap N$ can be represented by

$$x_2 = \psi(x_1) = \frac{1}{2} H x_1^2 + o(x_1^2),$$

where H is the curvature of $\partial\Omega$ at P. Define a map $x = \Phi(y) = (\Phi_1(y), \Phi_2(y))$ by

$$\Phi_1(y) = y_1 - y_2 \frac{\partial \psi}{\partial x_1}(y_1), \quad \Phi_2(y) = y_2 + \psi(y_1).$$

Since $\psi'(0) = 0$, the differential map $D\Phi$ of Φ satisfies $D\Phi(0) = I$, the identity map. Thus, Φ has the inverse mapping $y = \Phi^{-1}(x)$ for small $|x|$. We write $\Psi(x) = (\Psi_1(x), \Psi_2(x))$ instead of $\Phi^{-1}(x)$.

Since $x_0 \in \partial\Omega$, we apply this diffeomorphism around x_0. We may assume that $\Phi = \Psi^{-1}$ is defined in an open set containing the closed ball $\overline{B_{2\kappa}}, \kappa > 0$, and that $P_n := \Psi(x_n) \in B_\kappa^+$ for all n, where $B_\kappa^+ := \{y \in B_\kappa \mid y_2 > 0\}$. Define

$$\hat{u}_n(y) := u_n(\Phi(y)) \quad \text{for} \quad y \in \overline{B_{2\kappa}^+}$$

and extend it to $\overline{B_{2\kappa}}$ by reflection:

$$\tilde{u}_n(y) := \begin{cases} \hat{u}_n(y) & \text{if} \quad y \in \overline{B_{2\kappa}^+}, \\ \hat{u}_n((y_1, -y_2)) & \text{if} \quad y \in B_{2\kappa}^-, \end{cases}$$

where $B_{2\kappa}^- := \left\{y \in \overline{B_{2\kappa}} \;\middle|\; y_2 < 0\right\}$. Now, set

$$w_n(z) := \frac{1}{\mu_n}\tilde{u}_n\left(\frac{z}{R_n} + P_n\right) \quad \text{for} \quad z \in \overline{B_{\kappa R_n}}.$$

Let $P_n = (p_n, q_n/R_n)$. The condition $\operatorname{dist}(x_n, \partial\Omega) = O(R_n^{-1})$ implies $q_n < \infty$. By (2), w_n satisfies the following elliptic equations:

$$\sum_{i,j=1}^{2} a_{ij}^n(z)\frac{\partial^2 w_n}{\partial z_i \partial z_j} + \frac{1}{R_n}\sum_{j=1}^{2} b_j^n(z)\frac{\partial w_n}{\partial z_j} + \left(1 - \frac{L_n}{\lambda_n}\right)\frac{1}{\mu_n^2}w_n = \frac{L_n}{\lambda_n}\frac{1}{\mu_n^2}w_n\left(e^{\alpha\mu_n^2 w_n^2} - 1\right),$$

where a_{ij}^n, b_j^n are defined as follows: First, put

$$\begin{aligned} a_{ij}(y) &= \sum_{\ell=1}^{2}\frac{\partial\Psi_i}{\partial x_\ell}(\Phi(y))\frac{\partial\Psi_j}{\partial x_\ell}(\Phi(y)) \quad 1 \le i, j \le 2 \\ b_j(y) &= (\Delta\Psi_j)(\Phi(y)) \quad 1 \le j \le 2. \end{aligned}$$

Then, set

$$a_{ij}^n(z) = \begin{cases} a_{ij}(P_n + z/R_n) & z_2 \ge -q_n, \\ (-1)^{\delta_{i2}+\delta_{j2}}a_{ij}((p_n + z_1/R_n, -(q_n + z_2)/R_n) & z_2 < q_n, \end{cases}$$

$$b_j^n(z) = \begin{cases} b_j(P_n + z/R_n) & z_2 \ge -q_n, \\ (-1)^{\delta_{j2}}b_j((p_n + z_1/R_n), -(q_n + z_2)/R_n) & z_2 < -q_n, \end{cases}$$

where δ_{ij} is the Kronecker symbol.

By elliptic regularity theory, we have

$$w_n \to w_0 \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2),$$

and

$$-\Delta w_0 + \omega w_0 = \alpha w_0^3, \quad w_0 > 0 \quad \text{in} \quad \mathbb{R}^2.$$

As in (7), we have

$$\int_{\mathbb{R}^2} w_0^2 dx \le 2,$$

so $w_0 \in H^1(\mathbb{R}^2)$, and thus $w_0(x) \to 0$ as $|x| \to \infty$. Hence w_0 is uniquely determined.

To derive a contradiction, arguing as in (9) and (10), we see

$$\int_{\mathbb{R}^2} w_0^4 dx = \frac{2\omega}{\alpha} \int_{\mathbb{R}^2} w_0^2 dx$$

and

$$\int_{\mathbb{R}^2} |\nabla w_0|^2 dx = \omega \int_{\mathbb{R}^2} w_0^2 dx.$$

Hence, the Gagliardo-Nirenberg inequality yields

$$A \ge \frac{\int_{\mathbb{R}^2} w_0^4 dx}{\left(\int_{\mathbb{R}^2} |\nabla w_0|^2 dx\right)\left(\int_{\mathbb{R}^2} w_0^2 dx\right)} = \frac{2}{\alpha \int_{\mathbb{R}^2} w_0^2 dx} \ge \frac{1}{\alpha},$$

which contradicts the assumption $\alpha < \alpha_* \le 1/A$. Consequently, Lemma 2.4 also holds in the case $\mathrm{dist}(x_n, \partial\Omega) = O(R_n^{-1})$, completing the proof. □

By Lemmas 2.2 and 2.4, we may, up to a subsequence, assume without loss of generality that $K = \lim_{n\to\infty} \lambda_n \mu_n^2$. Thus, we suppose that

$$\lim_{n\to\infty} (\lambda_n - L_n) = \omega K.$$

Define $u_n^* := \sqrt{\lambda_n} u_n$. By (2), the function u_n^* satisfies

$$\begin{cases} -\Delta u_n^* + (\lambda_n - L_n) u_n^* = L_n u_n^* \left(e^{\alpha \lambda_n^{-1} |u_n^*|^2} - 1\right) & \text{in} \quad \Omega, \\ \frac{\partial u_n^*}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Since $\|u_n^*\|_{L^\infty(\Omega)} \to \sqrt{K}$ as $n \to \infty$, using elliptic regularity theory, we observe that there exists $u_0 \in C^2(\overline{\Omega})$ such that

$$u_n^* \to u_0 \quad \text{in} \quad C^2(\overline{\Omega}),$$

$\|u_0\|_{L^\infty(\Omega)} = \sqrt{K}$, $\int_\Omega u_0^2 dx = 1$, and

$$\begin{cases} -\Delta u_0 + \omega K u_0 = \alpha u_0^3 & \text{in} \quad \Omega, \\ \frac{\partial u_0}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Finally, to complete the proof, we establish the following proposition.

Lemma 2.5. *The function u_0 is a maximizer of B_α.*

Proof. We compute

$$\begin{aligned} \lambda_n E_\alpha(u_n) &= \lambda_n \int_\Omega \left(\alpha u_n^2 + \frac{\alpha^2}{2} u_n^4 \right) dx + O\left(\lambda_n \int_\Omega u_n^6 dx \right) \\ &= \alpha + \frac{\alpha}{\lambda_n} \left(\frac{\alpha}{2} \lambda_n^2 \int_\Omega u_n^4 dx - \lambda_n \int_\Omega |\nabla u_n|^2 dx \right) + O(\mu_n^4) \\ &= \alpha + \frac{\alpha^2}{2\lambda_n} \left(\int_\Omega |u_n^*|^4 dx - \frac{2}{\alpha} \int_\Omega |\nabla u_n^*|^2 dx \right) + O(\lambda_n^{-2}) \\ &= \alpha + \frac{\alpha^2}{2\lambda_n} \left(\int_\Omega u_0^4 dx - \frac{2}{\alpha} \int_\Omega |\nabla u_0|^2 dx \right) + o(\lambda_n^{-1}). \end{aligned}$$

Since $\int_\Omega u_0^2 dx = 1$, it follows that

$$\lambda_n I(\alpha, \lambda_n) = \lambda_n E_\alpha(u_n) \le \alpha + \frac{\alpha^2}{2\lambda_n} B_\alpha + o(\lambda_n^{-1}). \tag{11}$$

On the other hand, for $\eta \in H^1(\Omega)$ set

$$\eta_n := \frac{\eta}{\sqrt{\lambda_n} \|\eta\|_{H^1_{\lambda_n}}},$$

where

$$\|\eta\|_{H^1_{\lambda_n}} := \left(\int_\Omega \left(\lambda_n^{-1} |\nabla \eta|^2 + \eta^2 \right) dx \right)^{\frac{1}{2}}.$$

Then $\eta_n \in \Sigma_{\lambda_n}$, and we obtain

$$\begin{aligned}
\lambda_n E_\alpha(\eta_n) &= \lambda_n \int_\Omega \left(e^{\alpha \eta_n^2} - 1 \right) dx \\
&= \alpha \int_\Omega \frac{\eta^2}{\|\eta\|^2_{H^1_{\lambda_n}}} dx + \frac{\alpha^2}{2\lambda_n} \int_\Omega \frac{\eta^4}{\|\eta\|^4_{H^1_{\lambda_n}}} dx + O(\lambda_n^{-2}) \\
&= \alpha - \frac{\alpha}{\lambda_n} \int_\Omega \frac{|\nabla \eta|^2}{\|\eta\|^2_{H^1_{\lambda_n}}} dx + \frac{\alpha^2}{2\lambda_n} \int_\Omega \frac{\eta^4}{\|\eta\|^4_{H^1_{\lambda_n}}} dx + O(\lambda_n^{-2}) \\
&= \alpha + \frac{\alpha^2}{2\lambda_n} \left(\int_\Omega \frac{\eta^4}{\|\eta\|^4_{H^1_{\lambda_n}}} dx - \frac{2}{\alpha} \int_\Omega \frac{|\nabla \eta|^2}{\|\eta\|^2_{H^1_{\lambda_n}}} dx \right) + O(\lambda_n^{-2}) \\
&= \alpha + \frac{\alpha^2}{2\lambda_n} \left(\int_\Omega \frac{\eta^4}{\|\eta\|^4_{L^2(\Omega)}} dx - \frac{2}{\alpha} \int_\Omega \frac{|\nabla \eta|^2}{\|\eta\|^2_{L^2(\Omega)}} dx \right) + O(\lambda_n^{-2}).
\end{aligned}$$

Choosing U as a maximizer of B_α, we deduce

$$\begin{aligned}
\lambda_n I(\alpha, \lambda_n) &\geq \lambda_n E_\alpha(U_n) \\
&= \alpha + \frac{\alpha^2}{2\lambda_n} \left(\int_\Omega \frac{U^4}{\|U\|^4_{L^2(\Omega)}} dx - \frac{2}{\alpha} \int_\Omega \frac{|\nabla U|^2}{\|U\|^2_{L^2(\Omega)}} dx \right) + O(\lambda_n^{-2}) \\
&= \alpha + \frac{\alpha^2}{2\lambda_n} B_\alpha + O(\lambda_n^{-2}).
\end{aligned}$$

The upper estimate (11) and the lower bound above together yield

$$\lambda_n I(\alpha, \lambda_n) = \lambda_n E_\alpha(u_n) = \alpha + \frac{\alpha^2}{2\lambda_n} B_\alpha + o(\lambda_n^{-1}),$$

which implies that u_0 is indeed a maximizer of B_α. This completes the proof of Lemma 2.5. □

Consequently, the proof of Theorem 1.1 is complete.

3. Proof of Theorem 1.2

Assume that $\alpha \in (0, \beta_*)$. Let $\{\lambda_n\}$ be a sequence with $\lambda_n \to \infty$ as $n \to \infty$, and let v_n be a maximizer of $\sup_{u \in \Sigma^0_{\lambda_n}} E_\alpha(u)$ for sufficiently large n. Without loss of generality, we may assume that v_n is nonnegative in Ω. For simplicity, set $\kappa_n := \|v_n\|_{L^\infty(\Omega)}$. Let $\tilde{x}_n \in \Omega$ be a point such that $v_n(\tilde{x}_n) = \kappa_n$. Then v_n satisfies the elliptic equation

$$\begin{cases} -\Delta v_n + \lambda_n v_n = M_n v_n e^{\alpha v_n^2} & \text{in} \quad \Omega, \\ v_n = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{12}$$

where the Lagrange multiplier M_n is expressed as

$$M_n = \frac{1}{\int_\Omega v_n^2 e^{\alpha v_n^2} dx}.$$

By elliptic regularity theory, we have $v_n \in C^2(\Omega) \cap C(\overline{\Omega})$, and by the maximum principle, $v_n > 0$ in Ω.

We first recall the following results, which correspond to Theorem 1.3 (II) in [8].

Proposition 3.1. *It holds that*

$$\kappa_n \to 0, \quad \int_\Omega |\nabla v_n|^2 dx \to 0, \quad \textit{and} \quad \lambda_n \int_\Omega v_n^2 dx \to 1$$

as $n \to \infty$.

By the same argument as in the proof of Lemma 2.2, we obtain:

Lemma 3.2.

$$\liminf_{n\to\infty} \lambda_n \kappa_n^2 \geq \frac{1}{|\Omega|}.$$

Assume that $\Lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ with the zero Dirichlet boundary condition and ϕ_1 is the eigenfunction associated to $\Lambda_1(\Omega)$. Multiplying (12) by ϕ_1 and integrating over Ω by parts, we have

$$\Lambda_1(\Omega) \int_\Omega v_n \phi_1 dx = \int_\Omega v_n \phi_1 \left(M_n e^{\alpha v_n^2} - \lambda_n \right) dx \geq (M_n - \lambda_n) \int_\Omega v_n \phi_1 dx,$$

which implies that $\lambda_n - M_n \geq -\Lambda_1(\Omega)$. Moreover, by Hopf's lemma and Proposition 3.1, it follows that

$$0 < \int_\Omega v_n \left(M_n e^{\alpha v_n^2} - \lambda_n \right) dx \leq \left(M_n (1 + 2\alpha\kappa_n^2) - \lambda_n \right) \int_\Omega v_n dx.$$

Thus, we have

$$-\Lambda_1(\Omega) \le \lambda_n - M_n \le 2\alpha M_n \kappa_n^2. \tag{13}$$

Furthermore, using Proposition 3.1 again, we obtain

$$\lambda_n \int_\Omega v_n^2 e^{v_n^2} dx = 1 + o(1),$$

and hence

$$\lim_{n\to\infty} \frac{M_n}{\lambda_n} = 1.$$

Combining Lemma 3.2, (13), and the above equality, we derive

$$-\Lambda_1(\Omega)|\Omega| \left(1 + o(1)\right) \le \left(1 - \frac{M_n}{\lambda_n}\right) \frac{1}{\kappa_n^2} \le 2\alpha \left(1 + o(1)\right) \tag{14}$$

as $n \to \infty$.

By (14), we may assume, after passing to a subsequence, that $\omega_0 = \lim_{n\to\infty} \left(1 - M_n/\lambda_n\right)/\kappa_n^2$. Then we obtain the following lemma.

Lemma 3.3. *There exists a constant $D > 0$ such that*

$$\limsup_{n\to\infty} \frac{1}{\kappa_n^2} \int_\Omega |\nabla v_n|^2 dx \le D.$$

Proof. Multiplying (12) by v_n, integrating over Ω by parts, and using Proposition 3.1, we obtain

$$\int_\Omega |\nabla v_n|^2 dx + \left(1 - \frac{M_n}{\lambda_n}\right) \lambda_n \int_\Omega v_n^2 dx \le \alpha\kappa_n^2 + o(\kappa_n^2).$$

This estimate together with (14) yields the existence of a constant $D > 0$ such that

$$\frac{1}{\kappa_n^2} \int_\Omega |\nabla v_n|^2 dx \le D + o(1)$$

as $n \to \infty$, which proves the desired estimate. □

Then, we prove the following lemma.

Lemma 3.4. *There exists a positive constant K_0 such that*

$$\limsup_{n\to\infty} \lambda_n \kappa_n^2 \le K_0.$$

Proof. The proof follows the strategy used to prove Lemma 2.4. Assume by contradiction that, up to a subsequence, $\lambda_n \kappa_n^2 \to \infty$ as $n \to \infty$. Define $\Omega_n := \left\{ \sqrt{\lambda_n}\kappa_n(x - \tilde{x}_n) \;\middle|\; x \in \Omega \right\}$, and consider a scaled function

$$w_n(z) = \frac{1}{\kappa_n} v_n \left(\frac{z}{\sqrt{\lambda_n}\kappa_n} + \tilde{x}_n \right).$$

Then w_n satisfies $w_n(z) \le 1$ for $z \in \overline{\Omega_n}$, $\|w_n\|_{L^\infty(\Omega_n)} = 1$ and

$$\begin{cases} -\Delta w_n + \left(1 - \frac{M_n}{\lambda_n}\right) \frac{1}{\kappa_n^2} w_n = \frac{M_n}{\lambda_n} \frac{1}{\kappa_n^2} w_n \left(e^{\alpha \kappa_n^2 w_n^2} - 1 \right) & \text{in} \quad \Omega_n, \\ w_n = 0 & \text{on} \quad \partial\Omega_n. \end{cases}$$

By elliptic regularity theory, there exists w_0 such that

$$w_n \to w_0 \quad \text{in} \quad C^2_{\text{loc}}(X),$$

where w_0 satisfies

$$-\Delta w_0 + \omega w_0 = \alpha w_0^3, \quad w_0 > 0 \quad \text{in} \quad X,$$

and $w_0 \in H_0^1(X)$. Here, $X = \mathbb{R}^2$ if $\text{dist}(\tilde{x}_n, \partial\Omega)\sqrt{\lambda_n}\kappa_n \to \infty$, and a half space if $\text{dist}(\tilde{x}_n, \partial\Omega) = O((\sqrt{\lambda_n}\kappa_n)^{-1})$.

However, computing as in the proof of Lemma 2.4, we obtain

$$A \ge \frac{\int_X w_0^4 dx}{\left(\int_X |\nabla w_0|^2 dx\right)\left(\int_X w_0^2 dx\right)} \ge \frac{2}{\alpha},$$

which contradicts the assumption $\alpha < \beta_* \le 2/A$. This completes the proof of Lemma 3.4. □

By Lemmas 3.2 and 3.4, we may assume without loss of generality that $K_0 = \lim_{n\to\infty} \lambda_n \kappa_n^2$, up to a subsequence. Then, we also assume that $\lim_{n\to\infty}(\lambda_n - M_n) = \omega_0 K_0$.

Set $v_n^* := \sqrt{\lambda_n} v_n$. Since v_n is a solution of (12), the function v_n^* satisfies

$$\begin{cases} -\Delta v_n^* + (\lambda_n - M_n) v_n^* = M_n v_n^* \left(e^{\alpha \lambda_n^{-1} |v_n^*|^2} - 1 \right) & \text{in} \quad \Omega, \\ v_n^* = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Since $\lim_{n\to\infty} \|v_n^*\|_{L^\infty(\Omega)} = \sqrt{K_0}$, by elliptic regularity theory, there exists $v_0 \in C^2(\overline{\Omega})$ such that

$$v_n^* \to v_0 \quad \text{in} \quad C^2(\overline{\Omega}),$$

where v_0 satisfies $\|v_0\|_{L^\infty(\Omega)} = \sqrt{K_0}$, $\int_\Omega v_0^2 dx = 1$, and

$$\begin{cases} -\Delta v_0 + \omega_0 K_0 v_0 = \alpha v_0^3 & \text{in} \quad \Omega, \\ v_0 = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Thus, to complete the proof of Theorem 1.2, we establish the following lemma.

Lemma 3.5. *The function v_0 is a maximizer of B_α^0.*

Proof. Following the computation in the proof of Lemma 2.5, we obtain

$$\begin{aligned} \lambda_n d(\alpha, \lambda_n) &= \lambda_n E_\alpha(v_n) \\ &= \alpha + \frac{\alpha^2}{2\lambda_n}\left(\int_\Omega v_0^4 dx - \frac{2}{\alpha}\int_\Omega |\nabla v_0|^2 dx\right) + o(\lambda_n^{-1}) \\ &\leq \alpha + \frac{\alpha^2}{2\lambda_n} B_\alpha^0 + o(\lambda_n^{-1}), \end{aligned}$$

and

$$\lambda_n d(\alpha, \lambda_n) = \lambda_n E_\alpha(v_n) \geq \alpha + \frac{\alpha^2}{2\lambda_n} B_\alpha^0 + O(\lambda_n^{-2}).$$

Therefore, we conclude that v_0 is a maximizer of B_α^0. □

Consequently, the proof of Theorem 1.2 is complete.

4. Variational problems B_α and B_α^0

In this section, we investigate the variational problems B_α and B_α^0.

Proposition 4.1. *For any $\alpha < 1/A$, it holds that*

$$0 < B_\alpha < \infty,$$

and B_α is attained.

Proof. Step 1. Fix $\lambda > 0$ and set

$$B_{\alpha,\lambda} := \sup_{\substack{\zeta \in H^1(\Omega) \\ \|\zeta\|_{H^1_\lambda} = 1}} \left(\int_\Omega \zeta^4 dx - \frac{2}{\alpha}\int_\Omega |\nabla \zeta|^2 dx\right),$$

where

$$\|\zeta\|_{H^1_\lambda} := \left(\int_\Omega \left(\lambda^{-1} |\nabla \zeta|^2 + \zeta^2 \right) dx \right)^{\frac{1}{2}}.$$

We first show the existence of a maximizer of $B_{\alpha,\lambda}$. For any $\zeta \in H^1(\Omega)$ with $\|\zeta\|_{H^1_\lambda} = 1$, we observe that

$$\int_\Omega \zeta^4 dx - \frac{2}{\alpha} \int_\Omega |\nabla \zeta|^2 dx = \int_\Omega \zeta^4 dx + \frac{2\lambda}{\alpha} \int_\Omega \zeta^2 dx - \frac{2\lambda}{\alpha}.$$

Thus, it follows that

$$B_{\alpha,\lambda} = \tilde{B}_{\alpha,\lambda} - \frac{2\lambda}{\alpha},$$

and hence, the maximizers of $B_{\alpha,\lambda}$ coincide with those of $\tilde{B}_{\alpha,\lambda}$, where

$$\tilde{B}_{\alpha,\lambda} := \sup_{\substack{\rho \in H^1(\Omega) \\ \|\rho\|_{H^1_\lambda} \leq 1}} \left(\int_\Omega \rho^4 dx + \frac{2\lambda}{\alpha} \int_\Omega \rho^2 dx \right).$$

Let $\{\rho_n\} \subset H^1(\Omega)$ be a maximizing sequence for $\tilde{B}_{\alpha,\lambda}$. Since $\{\rho_n\}$ is bounded in $H^1(\Omega)$, there exists $\rho_0 \in H^1(\Omega)$ such that, up to a subsequence, $\rho_n \rightharpoonup \rho_0$ weakly in $H^1(\Omega)$, $\rho_n \to \rho_0$ strongly in $L^2(\Omega)$ and $\rho_n \to \rho_0$ strongly in $L^4(\Omega)$ as $n \to \infty$. Therefore, $\|\rho_0\|_{H^1_\lambda} \leq 1$ and

$$\int_\Omega \rho_0^4 dx + \frac{2\lambda}{\alpha} \int_\Omega \rho_0^2 dx = \lim_{n \to \infty} \left(\int_\Omega \rho_n^4 dx + \frac{2\lambda}{\alpha} \int_\Omega \rho_n^2 dx \right) = \tilde{B}_{\alpha,\lambda}.$$

This shows that ρ_0 is a maximizer of $\tilde{B}_{\alpha,\lambda}$, and hence also of $B_{\alpha,\lambda}$.

Step 2. Let $\{\lambda_n\}$ be a sequence such that $\lambda_n \to \infty$ as $n \to \infty$, and let $\{\zeta_n\} \subset H^1(\Omega)$ be a sequence of maximizers of B_{α,λ_n}. Without loss of generality, we may assume that $\zeta_n \geq 0$ in $\overline{\Omega}$. By the Lagrange multiplier theorem, there exists $\tau_n \in \mathbb{R}$ such that

$$\begin{cases} -\left(\frac{2}{\alpha} + \frac{\tau_n}{\lambda_n} \right) \Delta \zeta_n + \tau_n \zeta_n = 2\zeta_n^3 & \text{in} \quad \Omega, \\ \frac{\partial \zeta_n}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{15}$$

We observe that

$$\int_\Omega \left(2\zeta_n^2 - \tau_n \right) \zeta_n dx = 0,$$

and then, $\tau_n > 0$ and $\|\zeta_n\|_{L^\infty(\Omega)} \geq \sqrt{\tau_n/2}$.

We now show that there exist positive constants C_1 and C_2, independent of n, such that

$$C_1 \leq \tau_n \leq C_2. \tag{16}$$

Multiplying (15) by ζ_n and integrating over Ω, we obtain

$$\tau_n = B_{\alpha,\lambda_n} + \int_\Omega \zeta_n^4 dx. \tag{17}$$

Since the constant function $|\Omega|^{-1/2}$ satisfies $\||\Omega|^{-1/2}\|_{H^1_\lambda} = 1$ for any n, it follows that $B_{\alpha,\lambda_n} \geq |\Omega|^{-1}$. Thus, $\tau_n > |\Omega|^{-1}$, which provides a uniform lower bound.

For the upper bound, suppose by contradiction that $\tau_n \to \infty$ as $n \to \infty$. Under the hypothesis, we have $\|\zeta_n\|_{L^\infty(\Omega)} \to \infty$ as $n \to \infty$. By the Gagliardo-Nirenberg inequality in $H^1(\mathbb{R}^2)$ and the existence of an extension operator from $H^1(\Omega)$ to $H^1(\mathbb{R}^2)$, there exists a constant $A_\Omega > 0$ such that for any $\zeta \in H^1(\Omega)$, we derive

$$\int_\Omega \zeta^4 dx \leq A_\Omega \left(\int_\Omega \zeta^2 dx \right) \left[\int_\Omega \left(|\nabla \zeta|^2 + \zeta^2 \right) dx \right].$$

Using this inequality together with $\|\zeta_n\|_{H^1_{\lambda_n}} = 1$, we obtain

$$\tau_n = O(\lambda_n)$$

as $n \to \infty$. By the lower bound of B_{α,λ_n} and the expression (17), we deduce

$$\int_\Omega \zeta_n^4 dx = O(\tau_n), \quad \int_\Omega |\nabla \zeta_n|^2 dx = O(\tau_n).$$

Going back to (15), we see that the coefficient of $-\Delta\zeta_n$ is uniformly bounded in n. Therefore, recalling that $\alpha < 1/A$ and following the argument in the proof of Lemma 2.4, we reach a contradiction. Hence, τ_n is uniformly bounded from above.

The uniform bound (16) implies $B_{\alpha,\lambda_n} \leq C_2$ and $\int_\Omega \zeta_n^4 dx \leq C_2$, and consequently $\int_\Omega |\nabla \zeta_n|^2 dx$ is also uniformly bounded in n. By elliptic regularity theory, after passing to a subsequence, there exists $\zeta_0 \in C^2(\overline{\Omega})$ such that

$$\zeta_n \to \zeta_0 \quad \text{in} \quad C^2(\overline{\Omega}),$$

$\zeta_0 > 0$ in $\overline{\Omega}$, $\int_\Omega \zeta_0^2 dx = 1$, and

$$\begin{cases} -\frac{2}{\alpha}\Delta\zeta_0 + \tau_0\zeta_0 = 2\zeta_0^3 & \text{in} \quad \Omega, \\ \frac{\partial\zeta_0}{\partial\nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\tau_0 = \lim_{n\to\infty} \tau_n$.

Step 3. We show that ζ_0 is a maximizer of B_α. For $\zeta \in H^1(\Omega)$ with $\int_\Omega \zeta^2 dx = 1$ and $\lambda > 0$, define

$$\zeta_\lambda := \frac{\zeta}{\|\zeta\|_{H^1_\lambda}}.$$

Since $\|\zeta\|^2_{H^1_\lambda} = \lambda^{-1}\int_\Omega |\nabla\zeta|^2 dx + 1$, we have

$$\begin{aligned} & \int_\Omega \zeta^4 dx - \frac{2}{\alpha}\int_\Omega |\nabla\zeta|^2 dx \\ = \ & \|\zeta\|^4_{H^1_\lambda}\left(\int_\Omega \zeta_\lambda^4 dx - \frac{2}{\alpha}\int_\Omega |\nabla\zeta_\lambda|^2 dx\right) + \|\zeta\|^4_{H^1_\lambda}\frac{2}{\alpha}\int_\Omega |\nabla\zeta_\lambda|^2 dx - \frac{2}{\alpha}\int_\Omega |\nabla\zeta|^2 dx \\ \leq \ & \|\zeta\|^4_{H^1_\lambda} B_{\alpha,\lambda} + \frac{2}{\alpha\lambda}\left(\int_\Omega |\nabla\zeta|^2 dx\right)^2. \end{aligned} \tag{18}$$

Letting $\lambda \to \infty$, we obtain

$$\int_\Omega \zeta^4 dx - \frac{2}{\alpha}\int_\Omega |\nabla\zeta|^2 dx \leq \liminf_{\lambda\to\infty} B_{\alpha,\lambda}.$$

Since ζ is arbitrary subject only to $\int_\Omega \zeta^2 dx = 1$, we obtain

$$B_\alpha \leq \liminf_{\lambda\to\infty} B_{\alpha,\lambda}.$$

By Step 2, we have

$$\begin{aligned} B_\alpha &\leq \liminf_{n\to\infty} B_{\alpha,\lambda_n} \\ &= \liminf_{n\to\infty}\left(\int_\Omega \zeta_n^4 dx - \frac{2}{\alpha}\int_\Omega |\nabla\zeta_n|^2 dx\right) \\ &= \int_\Omega \zeta_0^4 dx - \frac{2}{\alpha}\int_\Omega |\nabla\zeta_0|^2 dx, \end{aligned}$$

which shows that ζ_0 is a maximizer of B_α. This completes the proof of Proposition 4.1. □

Proposition 4.2. *For any $\alpha < 2/A$, it holds that*

$$-\infty < B_\alpha^0 < 0$$

and B_α^0 is attained.

Proof. Similarly to the proof of Proposition 4.1, define

$$B_{\alpha,\lambda}^0 := \sup_{\substack{\zeta\in H_0^1(\Omega)\\ \|\zeta\|_{H_\lambda^1}=1}} \left(\int_\Omega \zeta^4 dx - \frac{2}{\alpha}\int_\Omega |\nabla\zeta|^2 dx \right).$$

Note that for any $\zeta \in H_0^1(\Omega)$ with $\|\zeta\|_{H_\lambda^1} = 1$, we have

$$\int_\Omega |\nabla\zeta|^2 dx \geq \frac{1}{\lambda^{-1} + \Lambda_1(\Omega)^{-1}},$$

where $\Lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. Then, by the Gagliardo-Nirenberg inequality and the assumption $\alpha < 2/A$, we obtain

$$\begin{aligned} \int_\Omega \zeta^4 dx - \frac{2}{\alpha}\int_\Omega |\nabla\zeta|^2 dx &< \left(\int_\Omega |\nabla\zeta|^2 dx\right)\left(\frac{\int_\Omega \zeta^4 dx}{\left(\int_\Omega |\nabla\zeta|^2 dx\right)\left(\int_\Omega \zeta^2 dx\right)} - \frac{2}{\alpha}\right) \\ &\leq \frac{\lambda\Lambda_1(\Omega)}{\lambda + \Lambda_1(\Omega)}\left(A - \frac{2}{\alpha}\right). \end{aligned}$$

Thus, $B_{\alpha,\lambda}^0 < 0$. Using the same estimate as in (18), we obtain

$$B_\alpha^0 \leq \liminf_{\lambda\to\infty} B_{\alpha,\lambda}^0.$$

For a lower bound of B_α^0, consider the eigenfunction ϕ_1 associated to $\Lambda_1(\Omega)$, normalized by $\int_\Omega \phi_1^2 dx = 1$. Then,

$$B_\alpha^0 \geq \frac{1}{|\Omega|} - \frac{2}{\alpha}\Lambda_1(\Omega).$$

Hence we obtain

$$\frac{1}{|\Omega|} - \frac{2}{\alpha}\Lambda_1(\Omega) \leq B_\alpha^0 \leq \liminf_{\lambda\to\infty} B_{\alpha,\lambda}^0 < 0. \tag{19}$$

For any $\lambda > 0$, the existence of a maximizer of $B^0_{\alpha,\lambda}$ can be shown in the same way as in Step 1 of the proof of Proposition 4.1. Let $\{\lambda_n\}$ be a sequence with $\lambda_n \to \infty$, and let $\{\zeta_n\} \subset H^1_0(\Omega)$ be maximizers of B^0_{α,λ_n}. Without loss of generality, we may assume $\zeta_n \geq 0$ in $\overline{\Omega}$. For each n, ζ_n satisfies

$$\begin{cases} -\left(\frac{2}{\alpha} + \frac{\sigma_n}{\lambda_n}\right) \Delta\zeta_n + \sigma_n\zeta_n = 2\zeta_n^3 & \text{in} \quad \Omega, \\ \zeta_n = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where the Lagrange multiplier $\sigma_n \in \mathbb{R}$ is given by

$$\sigma_n = B^0_{\alpha,\lambda_n} + \int_\Omega \zeta_n^4 dx.$$

If $\sigma_n \leq 0$, (19) implies

$$\int_\Omega \zeta_n^4 dx = \sigma_n - B^0_{\alpha,\lambda_n} \leq \frac{2}{\alpha}\Lambda_1(\Omega) - \frac{1}{|\Omega|},$$

so both $\int_\Omega \zeta_n^4 dx$ and $\int_\Omega |\nabla\zeta_n|^2 dx$ are bounded uniformly in n. If $\sigma_n > 0$, then

$$\int_\Omega \left(2\zeta_n^2 - \sigma_n\right) \zeta_n dx > 0,$$

which implies $\|\zeta_n\|_{L^\infty(\Omega)} \geq \sqrt{\sigma_n/2}$. Moreover, by (19) and the Gagliardo-Nirenberg inequality we have $\sigma_n \leq A\lambda_n$, and following Step 2 in the proof of Proposition 4.1, we conclude that σ_n is uniformly bounded. Consequently, both $\int_\Omega \zeta_n^4 dx$ and $\int_\Omega |\nabla\zeta_n|^2 dx$ are uniformly bounded as well.

Finally, applying Step 3 in the proof of Proposition 4.1, we conclude that B^0_α admits a maximizer, which completes the proof. □

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