Original Trudinger-Moser inequality revisited

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Abstract

We give a new proof of the original Trudinger-Moser inequality. Our proof relies on the techniques developed in [1], which were used to obtain the exact growth condition of the Trudinger-Moser inequality in $\mathbb{R}^2$.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a smooth bounded domain. It is well known that there is a Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{Np/(N-p)}(\Omega)$ for $p \in [1, N)$. If we look at the limiting case $p = N$, then $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \geq 1$, but $W_0^{1,N}(\Omega) \not\hookrightarrow L^\infty(\Omega)$. Actually, Yudovich [10], Pohozaev [8] and Trudinger [9] independently proved that $W_0^{1,N}(\Omega) \hookrightarrow L_\phi(\Omega)$, where $L_\phi(\Omega)$ denotes the Orlicz space associated with the Young function $\phi(t) := \exp(t^{\frac{N}{N-1}}) - 1$. This result was improved by Moser [7], obtaining the sharp inequality as follows:

Theorem 1.1. *It holds that*

$$\sup_{\substack{u \in W_0^{1,N}(\Omega) \\ \int_\Omega |\nabla u|^N dx \leq 1}} \int_\Omega e^{\alpha_N |u|^{\frac{N}{N-1}}} dx \leq C|\Omega|, \tag{1}$$

where $\alpha_N := N\omega_{N-1}^{\frac{1}{N-1}}$ *and* ω_{N-1} *is the surface measure of the unit sphere* $\mathbb{S}^{N-1} \subset \mathbb{R}^N$.

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The inequality (1) is sharp in the sense that the supremum in (1) is not bounded if α_N is replaced by any larger constant. The proof of Theorem 1.1 in [7] is based on symmetric decreasing rearrangement and rather technical arguments with exact estimates. Jodeit [2] proposed an alternative approach for a one dimensional problem to which (1) is reduced through the symmetric rearrangement in [7]. Moreover, the method can be applied to more general contexts.

In this paper, we give a new proof of Theorem 1.1 different from those in [7, 2]. Our arguments are inspired by [1]. To state the main proposition precisely, we introduce some notation. In the following, $B_R \subset \mathbb{R}^N$ denotes the open ball of radius R centered at the origin and B denotes the unit ball in $\mathbb{R}^N$. For a radially symmetric function in $W_0^{1,N}(B)$ and $x \in B$ with $|x| = r$, we shall write $u(x) = u(r)$.

We will prove the following proposition:

Proposition 1.2. *There exist positive constants ε_0 and C_0 such that for any $\varepsilon \in (0, \varepsilon_0)$ and nonnegative radially decreasing function $u \in W_0^{1,N}(B)$ satisfying $\|\nabla u\|_{L^N(B)} = 1$, it holds that*

$$\int_{B_{R_\varepsilon}} e^{\alpha_N |u|^{\frac{N}{N-1}}} dx \leq C_0 e^{\alpha_N \left(\frac{u(R_\varepsilon)}{1-\varepsilon}\right)^{\frac{N}{N-1}}} R_\varepsilon^N,$$

where $R_\varepsilon \in (0,1)$ satisfying $\|\nabla u\|_{L^N(B \setminus B_{R_\varepsilon})} = 1 - \varepsilon$.

Proposition 1.2 is a higher dimensional extension of an estimate in [1], which plays a crucial role in determining the exact growth condition of the Trudinger-Moser inequality in $\mathbb{R}^2$. The proof of Proposition 1.2 relies on the method used to obtain this estimate. Although the estimate and main results in [1] have been extended to higher dimensional case in [5] and to Adams type inequalities in [4, 6, 3], these arguments cannot be applied for our settings since they require uniform estimates such as Theorem 1.1.

In the next section, we first prove Proposition 1.2, and then, we prove Theorem 1.1.

2. Proof of Theorem 1.1

For simplicity, we write N' as $N/(N-1)$. We prepare the following lemma which follows from the Hölder inequality.

Lemma 2.1. *For any radially symmetric function $u \in W_0^{1,N}(B)$ and $r_1 \le r_2$, it holds that*

$$|u(r_1)| \le |u(r_2)| + \left(\frac{N}{\alpha_N}\right)^{\frac{1}{N'}} \|\nabla u\|_{L^N(B_{r_2} \setminus B_{r_1})} \left(\log \frac{r_2}{r_1}\right)^{\frac{1}{N'}}$$

We first prove Proposition 1.2.

Proof of Proposition 1.2. We slightly refine the arguments in [1] to apply to higher dimensional case. Let $u \in W_0^{1,N}(B)$ be a nonnegative radially decreasing function satisfying $\int_B |\nabla u|^N dx = 1$. For small $\varepsilon > 0$ which will be determined later, choose $R_\varepsilon > 0$ such that $\|\nabla u\|_{L^N(B \setminus B_{R_\varepsilon})} = 1 - \varepsilon$. Let

$$R_j := R_\varepsilon e^{-j}, \quad A_j := B_{R_{j-1}} \setminus B_{R_j}, \quad t_j := u(R_j),$$

$$D_j := \|\nabla u\|_{L^N(A_j)}, \quad E_j := \|\nabla u\|_{L^N(B \setminus B_{R_j})}^{N'}.$$

By the monotone convergence theorem, we may assume that u is constant for small $|x|$. Thus, there exits $K \in \mathbb{N}$ such that

$$\int_{B_{R_\varepsilon}} e^{\alpha_N |u|^{N'}} dx \le e^N |B| R_\varepsilon^N \sum_{j=1}^{K} e^{\alpha_N t_j^{N'} - Nj},$$

where $|B|$ denotes the Lebesgue measure of the unit ball. Then, it suffices to show that

$$\sum_{j=1}^{K} e^{\alpha_N t_j^{N'} - Nj} \le C e^{\alpha_N \frac{t_0^{N'}}{E_0}}.$$

We prepare several main inequalities on $\alpha_N t_j^{N'}$. By Lemma 2.1, we have

$$t_j \le t_{j-1} + \left(\frac{N}{\alpha_N}\right)^{\frac{1}{N'}} D_j. \tag{2}$$

Applying the following inequality

$$(a+b)^p \le (1+\gamma)a^p + \left(1 - \frac{1}{(1+\gamma)^{1/(p-1)}}\right)^{1-p} b^p \quad \text{for } a, b \ge 0,\ p \ge 1,\ \gamma > 0,$$

we estimate $t_j^{N'}$ as follows:

$$t_j^{N'} \le (1+\gamma) t_{j-1}^{N'} + \left(1 - \frac{1}{(1+\gamma)^{N-1}}\right)^{-\frac{1}{N-1}} \frac{N}{\alpha_N} D_j^{N'}$$

for any $\gamma > 0$. Now from the relation

$$E_j^{N-1} - E_{j-1}^{N-1} = D_j^N,$$

we deduce that $E_j = E_{j-1}$ and $t_j = t_{j-1}$ are equivalent. Thus, we may assume that $E_j \neq E_{j-1}$ and replacing $1+\gamma$ with E_j/E_{j-1}, we derive

$$t_j^{N'} \leq \frac{E_j}{E_{j-1}} t_{j-1}^{N'} + \frac{N}{\alpha_N} E_j.$$

Hence, defining

$$\xi_j := \alpha_N \frac{t_j^{N'}}{E_j},$$

we have

$$\xi_j \leq \xi_{j-1} + N. \tag{3}$$

Going back to (2), we also have

$$t_j^{N'} \leq t_{j-1}^{N'} + N' \left(\frac{N}{\alpha_N}\right)^{\frac{1}{N'}} D_j t_{j-1}^{\frac{1}{N-1}} + \frac{N}{\alpha_N} D_j^{N'},$$

and thus,

$$\xi_j \leq \xi_{j-1} + \frac{1}{E_0}\left(\alpha_N^{\frac{1}{N}} N' N^{\frac{1}{N'}} D_j t_{j-1}^{\frac{1}{N-1}} + N D_j^{N'}\right). \tag{4}$$

Moreover, we note that

$$\alpha_N t_j^{N'} \leq \xi_j \tag{5}$$

for any j.

Now, let $J = \{1, \ldots, K\}$ and define the sets $\mathcal{A}$ and $\mathcal{B}$ by

$$\mathcal{A} := \left\{ j \in J \;\middle|\; \alpha_N^{\frac{1}{N}} N' N^{\frac{1}{N'}} D_j t_{j-1}^{\frac{1}{N-1}} \leq \frac{N}{2} \right\}, \quad \mathcal{B} := J \setminus \mathcal{A}.$$

We first consider the case $j \in \mathcal{A}$. By (4), we can choose ε sufficiently small so that

$$\xi_j \leq \xi_{j-1} + \frac{1}{E_0}\left(\frac{N}{2} + N D_j^{N'}\right) \leq \xi_{j-1} + N\delta_1 j$$

for some $\delta_1 \in (1/2, 1)$. Hence, for sum over $\mathcal{A}$ it follows from the estimate, (3) and (5) that

$$\sum_{j\in\mathcal{A}} e^{\alpha_N t_j^{N'} - Nj} \leq \sum_{j\in\mathcal{A}} e^{\xi_j - Nj} \leq \sum_{j=1}^{\#\mathcal{A}} e^{\xi_0 - N(1-\delta_1)j} \leq C_1 e^{\xi_0}. \tag{6}$$

We next consider the case $j \in \mathcal{B}$. Let $\tilde{\mathcal{B}}$ be any maximal consecutive subset of $\mathcal{B}$, namely,

$$\tilde{\mathcal{B}} := \{b, b+1, \cdots, b+M-1, b+M\},$$

$$b-1 \in \mathcal{A} \quad \text{or} \quad b-1=0, \qquad \text{and} \qquad b+M+1 \in \mathcal{A} \quad \text{or} \quad b+M=K.$$

For $j \in \tilde{\mathcal{B}} \setminus \{b+M\}$, by the definition of $\mathcal{B}$, (3) and (5), there exists $\delta_2 > 0$ such that

$$D_{j+1}^N \geq \frac{\delta_2}{\xi_b + N(j-b)}. \tag{7}$$

Let

$$\zeta_j^b := E_j\left(\xi_b + N(j-b)\right).$$

Note that, by (3),

$$\alpha_N t_j^{N'} \leq \zeta_j^b. \tag{8}$$

Using (7), we have

$$\begin{aligned} \zeta_j^b &= \zeta_{j+1}^b - \frac{D_{j+1}^N}{\sum_{k=0}^{N-2} E_j^{N-2-k} E_{j-1}^k}\left(\xi_b + N(j-b)\right) - NE_{j+1} \\ &\leq \zeta_{j+1}^b - N\left(E_0 + \frac{\delta_2}{N(N-1)}\right) \end{aligned}$$

Now we choose ε sufficiently small so that $E_0 + \delta_2/N(N-1) \geq 1 + \delta_3$ for some $\delta_3 > 0$. Thus, replacing $j = b+m$, we have

$$\begin{aligned} \zeta_{b+m}^b &\leq \zeta_{b+M}^b - N(1+\delta_3)(M-m) \\ &\leq \xi_b - Nb + N(m+b) - N\delta_3(M-m). \end{aligned}$$

The estimate and (8) yield

$$\alpha_N t_{b+m}^{N'} - N(b+m) \leq \xi_b - Nb - N\delta_3(N-m).$$

Hence, for sum over $\tilde{\mathcal{B}}$ we have

$$\sum_{j \in \tilde{\mathcal{B}}} e^{\alpha_N t_j^{N'} - Nj} \leq e^{\xi_b - Nb} \sum_{m=0}^{M} e^{-N\delta_3(M-m)} \leq C_2 e^{\xi_{b-1} - N(b-1)}. \tag{9}$$

Consequently, recalling that $b-1 \in \mathcal{A}$ or $b-1=0$, we have

$$\sum_{j=1}^{K} e^{\alpha_N t_j^{N'} - Nj} \le (1+C_2) \sum_{j\in\mathcal{A}} e^{\xi_j - Nj} \le (1+C_2)\, C_1 e^{\xi_0}$$

by (6) and (9), which completes the proof.

□

Proof of Theorem 1.1. By the symmetric decreasing rearrangement as in [7], it suffices to prove the existence of $C > 0$ such that

$$\int_B e^{\alpha_N |u|^{N'}} dx \le C$$

for any nonnegative radially decreasing function $u \in W_0^{1,N}(B)$ satisfying $\int_B |\nabla u|^N dx = 1$. Let ε_0 be the constant given in Proposition 1.2. Fix small $\varepsilon \in (0, \varepsilon_0)$. For nonnegative radially decreasing function $u \in W_0^{1,N}(B)$ satisfying $\int_B |\nabla u|^N dx = 1$, define $R_\varepsilon > 0$ so that $\|\nabla u\|_{L^N(B\setminus B_{R_\varepsilon})} = 1 - \varepsilon$. We estimate

$$\int_{B\setminus B_{R_\varepsilon}} e^{\alpha_N |u|^{N'}} dx, \quad \int_{B_{R_\varepsilon}} e^{\alpha_N |u|^{N'}} dx.$$

On $B \setminus B_{R_\varepsilon}$, using Lemma 2.1, we have

$$\int_{B\setminus B_{R_\varepsilon}} e^{\alpha_N |u|^{N'}} dx \le \int_{B\setminus B_{R_\varepsilon}} |x|^{-N(1-\varepsilon)^{N'}} dx \le C_3.$$

On B_{R_ε}, by Proposition 1.2 and Lemma 2.1, we have

$$\int_{B_{R_\varepsilon}} e^{\alpha_N |u|^{N'}} dx \le C_0.$$

Consequently, the proof of Theorem 1.1 completes. □

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