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ON EXAMPLES OF GEODESIC ORBIT PSEUDO-RIEMANNIAN MANIFOLDS

KENRO FURUTANI, IRINA MARKINA, AND YURII NIKONOROV

Abstract. We present examples of geodesic orbit pseudo-Riemannian manifolds, that is, pseudo-Riemannian manifolds in which each geodesic is an orbit of some 1-parameter isometry group. The examples are special 15-dimensional pseudo H-type Lie groups, i.e., 2-step nilpotent Lie groups of Heisenberg type equipped with a left invariant pseudo-Riemannian metric.

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On the 80th anniversary of Professor Semën Samsonovich Kutateladze's birth

1. Introduction

A pseudo-Riemannian manifold (M, g) is called a geodesic orbit manifold if any geodesic γ of M is an orbit of a 1-parameter subgroup of the full isometry group of (M, g). This terminology in the case of Riemannian manifolds was introduced in [18] by O. Kowalski and L. Vanhecke, who initiated a systematic study of spaces $(M = G/H, g)$, where G is an isometry group and H is an isotropy subgroup. We refer to [1, 3, 21, 22, 23] for expositions on general properties of geodesic orbit Riemannian manifolds and historical surveys. Some important results on geodesic orbit pseudo-Riemannian spaces were obtained in [2, 6, 7, 9, 20]. It should be noted that symmetric spaces, weakly symmetric spaces, naturally reductive homogeneous spaces, normal homogeneous spaces, generalized normal homogeneous spaces (but not only) are subclasses of geodesic orbit pseudo-Riemannian spaces.

The pseudo-Riemannian H-type nilmanifold is a 2-step nilpotent H-type Lie group endowed with a left invariant pseudo-Riemannian metric. The Lie algebra $\mathfrak{n}$ of a pseudo-Riemannian H-type nilmanifold carries a non-degenerate scalar product $\langle \cdot\,, \cdot \rangle$, whose restriction $\langle \cdot\,, \cdot \rangle_{\mathfrak{z}}$ on the centre $\mathfrak{z}$ of the Lie algebra $\mathfrak{n}$ is also non-degenerate. The orthogonal complement $\mathfrak{v}$ to the center is a representation space for the Clifford algebra $\mathrm{Cl}(\mathfrak{z}, \langle \cdot\,, \cdot \rangle_{\mathfrak{z}})$ such that the representation map is the dual to the adjoint map on the Lie algebra, see Section 2.

The problem of classification of geodesic orbit and naturally reductive pseudo-Riemannian nilmanifolds of H-type is very relevant. The results on Riemannian H-type nilmanifolds (the metric is positive definite) was obtained by A. Kaplan in [17] and C. Riehm in [25] (see Theorem 1) and the results on pseudo-Riemannian H-type nilmanifolds when the space $\mathfrak{v}$ is a minimal dimensional admissible Clifford module was obtained in [12].

Theorem 1 ([17, 25])**.** *Let N be H-type Lie group (supplied with a left invariant Riemannian metric) with the H-type algebra $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, $m = \dim(\mathfrak{z})$, $n = \dim(\mathfrak{v})$, and the center $\mathfrak{z}$. Then N is geodesic orbit if and only if one of the following three conditions holds:*

1) *$m = 1, 2, 3$ and n is any possible;*
2) *$m = 5, 6$ and $n = 8$;*

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3) $m = 7$, $n = 8, 16, 24$ *and* $\mathfrak{v}$ *is an isotypic Clifford module (in this case it is equivalent to the following property: if* $Z_1, Z_2, \dots, Z_7$ *is an orthonormal basis of* $\mathfrak{z}$*, the linear transformation* $X \mapsto J_{Z_1} J_{Z_2} \cdots J_{Z_7}(X)$ *of* $\mathfrak{v}$ *is either* Id *or* $-\mathrm{Id}$*).*

Moreover, N *is naturally reductive if and only if* $m = 1$ *or* $m = 3$*.*

Recently, an analogue of this theorem was obtained for pseudo-Riemannian metrics.

Theorem 2 ([12]). *Let* $N_{r,s}$ *be an* H*-type Lie group, where* (r, s)*,* $s \geq 1$*, is the signature of the left invariants pseudo-Riemannian metric restricted to the center of the group. Let* $\mathfrak{n}_{r,s} = \mathfrak{z} \oplus \mathfrak{v}$ *be the Lie algebra of* $N_{r,s}$*, where* $\mathfrak{z}$ *is the centre and* $\mathfrak{v}$ *is a minimal admissible module for the Clifford algebra* $\mathrm{Cl}(\mathbb{R}^{r,s})$ *orthogonal to* $\mathfrak{z}$*. Then the following four assertions hold:*

1) $N_{r,s}$ *is naturally reductive (hence, geodesic orbit) if and only if* $(r, s) \in \{(0, 1), (1, 2)\}$*;*
2) *If* $N_{r,s}$ *is geodesic orbit but not naturally reductive, then* $(r, s) = (3, 4)$*;*
3) $N_{3,4}$ *is a geodesic orbit pseudo-Riemannian manifold;*
4) $N_{r,s}$ *with* $(r, s) \notin \{(0, 1), (1, 2), (3, 4)\}$ *is not geodesic orbit pseudo-Riemannian manifold.*

It is important that item 3) of the above theorem is proved in [12] using a system of symbolic computations that allows one to calculate and evaluate the ranks of a number of auxiliary matrices to prove that each geodesic can be represented as an orbit of a 1-parameter isometry group of the manifold under study. This kind of reasoning proves the existence of the required one-parameter group, but doesn't always allow one to specify it explicitly.

The main goal of this note is to provide an alternative proof of the corresponding result, which also provides explicit expressions for the corresponding one-parameter isometry groups, see Theorem 4.

The paper is organized as follows. In Section 2, we recall some important results on pseudo-Riemannian geodesic orbit metrics on nilpotent Lie groups. The main role here is played by the notion of the transitive normalizer condition, which Riemannian version was used by C. Gordon in order to study geodesic orbit Riemannian metric on nilpotent Lie groups. We also recall some important properties of pseudo H-type Lie groups, their isometry and automorphism groups. In Section 3, we show the geodesic orbit property for the 15-dimensional pseudo Riemannian H-type nilmanifold $N_{3,4}(\mathfrak{v}_{\min})$. Section 4 is dedicated to the same property for the Riemannian H-type nilmanifold $N_{7,0}(\mathfrak{v}_{\min})$.

2. Definitions and notation

2.1. Geodesic orbit pseudo-Riemannian manifolds.

Definition 1. A pseudo-Riemannian homogeneous reductive manifold $(G/H, g)$ is called geodesic orbit if any geodesic through the point eH is an orbit of some 1-parameter isometry group of $(G/H, g)$.

Let (N, g) be a 2-step pseudo-Riemannian nilpotent Lie group with the Lie algebra $\mathfrak{n}$ and $\langle \cdot, \cdot \rangle$ the scalar product (a symmetric non-degenerate bilinear form) generating the pseudo-Riemannian left invariant metric g. We call such groups *pseudo-Riemannian nilmanifolds*. If the centre $\mathfrak{z}$ of $\mathfrak{n}$ is non-degenerate with respect to $\langle \cdot, \cdot \rangle$, then we write $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{v} = \mathfrak{z}^{\perp}$ relative to $\langle \cdot, \cdot \rangle$. In this case, $\mathfrak{v}$ is also non-degenerate, see [19, Lemma 2.60].

Whenever (N, g) is connected simply connected, we do not distinguish between the group of automorphisms of the nilmanifold (N, g) and of its Lie algebra $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$. Note that each skew-symmetric derivation of $\mathfrak{n}$ leaves each of $\mathfrak{v}$ and $\mathfrak{z}$ invariant. For any $Z \in \mathfrak{z}$, we consider the operator

$$J_Z \colon \mathfrak{v} \to \mathfrak{v}, \quad \text{such that } \langle J_Z(X), Y \rangle = \langle [X, Y], Z \rangle, \quad X, Y \in \mathfrak{v}. \tag{1}$$

The map J_Z is skew-symmetric and $J_Z(Y) = (\operatorname{ad} Y)'(Z)$, where $(\operatorname{ad} Y)'$ is dual to $\operatorname{ad} Y$ with respect to $\langle\cdot\,,\cdot\rangle$. The map $J\colon Z \to \mathfrak{so}(\mathfrak{v}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}})$, sending $Z \mapsto J_Z$ is linear. The group of isometries of the nilmanifold (N, g) is given by

$$H = \{(\varphi, \psi) \in O(\mathfrak{z}, \langle\cdot\,,\cdot\rangle_{\mathfrak{z}}) \times O(\mathfrak{v}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}}) \mid \psi J_Z \psi^{-1} = J_{\varphi(Z)},\ Z \in \mathfrak{z}\}, \tag{2}$$

while its Lie algebra is

$$\mathfrak{h} = \operatorname{Der}(\mathfrak{n}) \cap \mathfrak{so}(\mathfrak{n}, \langle\cdot\,,\cdot\rangle) = \{D = (C, A) \in \mathfrak{so}(\mathfrak{z}, \langle\cdot\,,\cdot\rangle_{\mathfrak{z}}) \times \mathfrak{so}(\mathfrak{v}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}}) \mid [A, J_Z] = J_{C(Z)},\ Z \in \mathfrak{z}\}. \tag{3}$$

The next result is well known, see e.g. Corollary 3.5 in [5].

Proposition 1. *Let (N, g) be a pseudo-Riemannian nilmanifold with non-degenerate center. Then the connected isometry group of (N, g) is $N \rtimes H$, where N is the set of left translations by elements of N and the isotropy subgroup H is given by the isometric automorphisms (2) with Lie algebra $\mathfrak{h}$ as in (3).*

In this case, the isotropy group of (N, g) at the identity element e is exactly H with the embedding $a \in H \mapsto (e, a) \in N \rtimes H$. The following Proposition summarizes well known Geodesic Lemma [9], its reformulation in the spirit of [14], and consideration done in [12] for the setting of H-type nilmanifolds, see definition in Section 2.2.

Proposition 2. *A pseudo-Riemannian H-type nilmanifold (N, g) is geodesic orbit if and only if for any $z_0 \in \mathfrak{z}$ and $x_0 \in \mathfrak{v}$ there is $D = (C, A) \in \mathfrak{h}$ such that*

$$A(x_0) = J_{z_0}(x_0), \quad C(z_0) = 0, \quad [A, J_w] = J_{C(w)} \ \textit{for all} \ w \in \mathfrak{z}. \tag{4}$$

A special case of geodesic orbit pseudo-Riemannian spaces are naturally reductive homogeneous pseudo-Riemannian spaces. We refer to [12] and references therein for more detailed study.

Definition 2. Let $(M = G/H, g)$ be a homogeneous reductive pseudo-Riemannian manifold. Then M is said to be a naturally reductive if there is a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that

$$\langle [T, Q]_{\mathfrak{m}}, R\rangle = \langle Q, [T, R]_m\rangle \tag{5}$$

for the corresponding scalar product and any $T, Q, R \in \mathfrak{m}$.

Theorem 3 (Theorem 3.2 in [24])**.** *Let (N, g) denote a 2-step pseudo-Riemannian nilmanifold with a non-degenerate center. Assume that the map $J\colon \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ is injective, see (1). Then the metric is naturally reductive with respect to $G = N \rtimes H$ (see Proposition 1), if and only if*

(i) $\mathbf{V} = J(\mathfrak{z})$ *is a Lie subalgebra of* $\mathfrak{so}(\mathfrak{v})$ *and*
(ii) $[J(Z_1), J(Z_2)] = J\big(\tau_{Z_1}(Z_2)\big)$ *where* $\tau_{Z_1} \in \mathfrak{so}(\mathfrak{z})$ *for any* $Z_1 \in \mathfrak{z}$.

Since the map J is supposed to be injective, the map τ can be easily recovered from (ii).

Remark 1. Using Theorem 3, it is easy to prove that the pseudo-Riemannian H-type nilmanifold $N_{3,4}(\mathfrak{v}_{\min})$, as well as the Riemannian H-type nilmanifold $N_{7,0}(\mathfrak{v}_{\min})$, that are studied further in this note, are not naturally reductive.

2.2. **Pseudo H-type Lie groups.** Now, we recall some important properties of pseudo H-type Lie groups. Let (N, g) be a 2-step pseudo-Riemannian nilmanifold and $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ be its Lie algebra endowed with a scalar product $\langle\cdot\,,\cdot\rangle$ making the center non-degenerate. We identify $(\mathfrak{z}, \langle\cdot\,,\cdot\rangle_{\mathfrak{z}})$ with the pseudo Euclidean vector space $\mathbb{R}^{r,s} = (\mathbb{R}^{r+s}, \langle\cdot\,,\cdot\rangle_{r,s})$, where

$$\langle z, w\rangle_{r,s} = \sum_{i=1}^{r} z_i\, w_i - \sum_{j=1}^{s} z_{r+j}\, w_{r+j}, \quad z, w \in \mathbb{R}^{r+s}.$$

If the linear operator $J\colon \mathfrak{z} \to \mathfrak{so}(\mathfrak{v}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}})$ is defined by for any $z \in \mathfrak{z}$ by (1), and satisfies $J_z^2(x) = -\langle z, z\rangle_{r,s} x$ for any $z \in \mathbb{R}^{r,s}$ and all $x \in \mathfrak{v}$, then $\mathfrak{n} = \mathfrak{n}_{r,s}$ is called the pseudo H(eisenberg)-type Lie algebra. We denote by $N_{r,s}$ the connected simply connected Lie group, whose Lie algebra is the pseudo H-type Lie algebra $\mathfrak{n}_{r,s}$. The H-type Lie algebras $N_{r,0}$ with a positive definite scalar product were introduced in [15] and with an arbitrary indefinite scalar product in [8], see also [13].

These Lie algebras are related to the representations of the Clifford algebras in the following way. Let $J\colon \mathrm{Cl}(\mathbb{R}^{r,s}) \to \mathrm{End}(\mathfrak{v})$ be a representation of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{r,s})$ generated by the pseudo Euclidean vector space $\mathbb{R}^{r,s}$. If there is a scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{v}}$ on the representation space $\mathfrak{v}$ (Clifford module) such that the linear map J_z is skew-symmetric for any $z \in \mathbb{R}^{r,s}$; that is

$$\langle J_z(x), y\rangle_{\mathfrak{v}} = -\langle x, J_z(y)\rangle_{\mathfrak{v}}, \quad z \in \mathbb{R}^{r,s}, \quad x, y \in \mathfrak{v},$$

then we get a pseudo H-type Lie algebra with the commutators defined in (1), see details in [16, 8, 11]. The scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{v}}$ in this case is called admissible and $\mathfrak{v}$ is called admissible (Clifford) module.

It is important that, see e.g. [8], or [11, Proposition 2.2.2], the signature of the scalar product space $(\mathfrak{v}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}})$ is neutral and $(\mathfrak{v}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}})$ is isometric to $\mathbb{R}^{l,l}$ for some $l \in \mathbb{N}$ if $s > 0$, whereas the corresponding signature is either $(l, 0)$ or $(0, l)$ for some $l \in 2\mathbb{N}$ if $s = 0$.

3. Pseudo-Riemannian H-type nilmanifold $N_{3,4} = N_{3,4}(\mathfrak{v}_{\min})$

Although the fact that a pseudo-Riemannian nilmanifold of H-type $N_{3,4}(\mathfrak{v}_{\min})$, where the complement of the centre is the minimal admissible modulus, is a geodesic orbit manifold is proved in [12], we give here an alternative proof of the following theorem.

Theorem 4. *The pseudo-Riemannian H-type nilmanifold $N_{3,4}(\mathfrak{v}_{\min})$ is a geodesic orbit manifold. Each geodesic is realized as an orbit of a one parameter subgroup which initial conditions form an eight dimensional affine space.*

It is known that the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{3,4})$ has two non-equivalent minimal admissible 8-dimensional modules $\mathfrak{v}^+_{\min}$ and $\mathfrak{v}^-_{\min}$. Moreover, the corresponding 15-dimensional pseudo H-type Lie algebras $N_{3,4}(\mathfrak{v}^+_{\min})$ and $N_{3,4}(\mathfrak{v}^-_{\min})$ are isomorphic and isometric, see [10, Theorem 12]. Therefore, it suffices to check only one pseudo H-type nilmanifold $N_{3,4}(\mathfrak{v}^-_{\min}) = N_{3,4}(\mathfrak{v}_{\min})$. The left invariant metric on $N_{3,4}(\mathfrak{v}_{\min})$ is generated by the scalar product $\langle\cdot\,,\cdot\rangle_{3,4} + \langle\cdot\,,\cdot\rangle_{4,4}$ on the Lie algebra $\mathfrak{n}_{3,4} \cong \mathbb{R}^{3,4} \oplus \mathbb{R}^{4,4}$. We write $(z_0, x_0) \in \mathfrak{z} \oplus \mathfrak{v}_{\min} \cong \mathbb{R}^{3,4} \oplus \mathbb{R}^{4,4}$ for an initial vector for a geodesic. For different type of the initial vectors we will use distinguished bases for the horizontal space $\mathfrak{v}_{\min}$ and for the centre $\mathfrak{z}$. We start from a basis for the centre.

3.1. Central component z_0 of initial conditions.

(C1) if $\|z_0\|^2 = \langle z_0, z_0\rangle_{\mathbb{R}^{3,4}} = 1$, then we set

$$\{Z_1 = z_0, Z_2, \dots, Z_7\},$$

as an orthonormal basis, satisfying

$$\|Z_1\|^2 = \|Z_2\|^2 = \|Z_3\|^2 = 1, \quad \|Z_4\|^2 = \|Z_5\|^2 = \|Z_6\|^2 = \|Z_7\|^2 = -1. \tag{6}$$

(C2) If $\|z_0\|^2 = \langle z_0, z_0\rangle_{\mathbb{R}^{3,4}} = -1$, then we set

$$\{Z_1, Z_2, Z_3, Z_4 = z_0, Z_5, Z_6, \dots, Z_7\},$$

such that (6) holds.

(C3) If $\|z_0\|^2 = \langle z_0, z_0\rangle_{\mathbb{R}^{3,4}} = 0$, then first we fix an arbitrary orthonormal basis $\{z_1^*, \dots, z_7^*\}$, satisfying (6) and write

$$z_0 = \sum_{i=1}^{3} \theta_i\, z_i^* + \sum_{i=4}^{7} \theta_i\, z_i^* := z_+^* + z_-^*.$$

Then, without loss of generality, we can assume $\sum_{i=1}^{3} {\theta_i}^2 = \sum_{i=4}^{7} {\theta_i}^2 = 1$. Since $\langle z_+^*,\ z_-^*\rangle_{\mathbb{R}^{3,4}} = 0$ we can select an orthonormal basis

$$\{Z_1 = z_+^*, Z_2, Z_3, Z_4 = z_-^*, Z_5, Z_6, Z_7\},$$

still satisfying (6). With these basis vectors we can denote $z_0 = Z_1 + Z_4$.

In what follows we use the construction of basis on the centre as above.

Remark 2. In any of the above constructions, the operators J_{Z_j}, $j = 1, \dots, 7$, span a 7-dimensional subspace in $J_{\mathfrak{z}} \subset \mathfrak{so}(4,4)$, nevertheless the vector space $\mathbf{V} = J_{\mathfrak{z}}$ is not a Lie subalgebra of $\mathfrak{so}(4,4)$. Hence, $N_{3,4}(\mathfrak{v}_{\min})$ is not a naturally reductive manifold, see Theorem 3.

3.2. **Horizontal component x_0 of initial conditions and reduction by conjugate Clifford action.** Fix $z_0 \in \mathfrak{z}$ and choose a suitable basis as was described in Section 3.1. Now we have three cases.

$$(\mathbf{H1})\ \|x_0\|^2 = \langle x_0, x_0\rangle_{\mathbb{R}^{4,4}} = 1, \quad (\mathbf{H2})\ \|x_0\|^2 = -1, \quad (\mathbf{H3})\ \|x_0\|^2 = 0.$$

Before we consider these three options we let $v \in \mathfrak{v}_{\min}$ be such that $\|v\|^2 = 1$ and

$$J_{Z_1Z_2Z_4Z_5}(v) = v, \quad J_{Z_1Z_2Z_6Z_7}(v) = v, \quad J_{Z_1Z_3Z_5Z_7}(v) = v, \quad J_{Z_1Z_2Z_3}(v) = v. \tag{7}$$

The condition $J_{Z_1Z_2Z_3}(v) = v$ is equivalent to $J_{Z_1Z_2Z_3Z_4Z_5Z_6Z_7}(v) = -v$. This choice fixes the module $\mathfrak{v}_{\min} = \mathfrak{v}_{\min}^-$. Moreover, the choice of v as in (7) allows us to use the following orthonormal basis for the module $\mathfrak{v}_{\min}$:

$$\begin{array}{llll} X_1 = v, & X_2 = J_{Z_1}(v), & X_3 = J_{Z_2}(v), & X_4 = J_{Z_3}(v), \\ X_5 = J_{Z_4}(v), & X_6 = J_{Z_5}(v), & X_7 = J_{Z_6}(v), & X_8 = J_{Z_7}(v). \end{array} \tag{8}$$

Remark 3. Once we fix an orthonormal basis $\{z_i\}$ in $\mathbb{R}^{3,4}$ (or in general in $\mathbb{R}^{r,s}$), the number of a set of generating positive involutions $\{p_i\}$ of a maximal abelian subgroup S consisting of positive involutions, which is a subgroup in the finite group $\mathbb{G} = \mathbb{G}(\{z_i\})$ generated by $\{z_i\}$, is determined by the pair (r, s). In the case $(r, s) = (3, 4)$ it is 4, that is one set of them is listed in (7). It should be noted that in our case the correspondence $z \mapsto J_z$ is not one-to-one. This is common in all the cases of the Clifford algebra being not irreducible. In such case we have two irreducible modules and they are distinguished by the action of the volume form $J_{z_1} \cdots J_{z_{r+s}}$. As we can see easily that in our case $J_{z_1} \cdots J_{z_7}$ is in the center of the group $\mathbb{G}$, so that $J_{z_1}J_{z_2}J_{z_3}(v) = v$ is not only equivalent to $J_{z_1} \cdots J_{z_7}(v) = -v$, it is also equivalent to $J_{z_1} \cdots J_{z_7} \equiv -\mathrm{Id}$ as an operator acting on the space generated by $\{J_{z_i}(v)\}$. So we may distinguish two irreducible modules $\mathfrak{v}_{\min}^{\pm}$ by the action of the volume form $J_{z_1} \cdots J_{z_7} = \pm\mathrm{Id}$, see details in [10].

(H1) Let $x_0 \in \mathfrak{v}_{\min}$ be as in (**H1**). Let $T \in SO(4,4)$ be such that $x_0 = T(v)$ and consider the conjugate Clifford action

$$\tilde{J}_w := T^{-1} J_w T = T^{\tau} J_w T,$$

where T^{τ} is the transpose to T with respect to the scalar product $\langle \cdot\,, \cdot\rangle_{4,4}$. Then, a geodesic determined by the initial condition (z_0, x_0) is homogeneous if there is a pair $D = (C, A)$ such that

$$A(x_0) = J_{z_0}(x_0), \quad C(z_0) = 0, \quad [A, J_w] = J_{C(w)}, \quad \text{for all} \quad w \in \mathbb{R}^{3,4}. \tag{9}$$

Lemma 1. *The condition* (9) *is equivalent to*

$$\tilde{A}(v)=T^{-1}AT(v)=T^{-1}J_{z_0}T(v)=\tilde{J}_{z_0}(v),\quad C(z_0)=0,\quad [\tilde{A},\tilde{J}_w]=\tilde{J}_{C(w)},\tag{10}$$

for all $w\in\mathfrak{z}\cong\mathbb{R}^{3,4}$.

Proof. In fact, from (9)

$$\begin{aligned}&T^{-1}AT(v)=T^{-1}A(x_0)=T^{-1}J_{z_0}(x_0)=T^{-1}T\tilde{J}_{z_0}T^{-1}(x_0)=\tilde{J}_{z_0}(v),\\&[T^{-1}AT,T^{-1}J_wT]=T^{-1}[A,J_w]T=T^{-1}J_{C(w)}T=\tilde{J}_{C(w)},\end{aligned}$$

and vise versa. ■

We denote the H-type group $N_{3,4}(\mathfrak{v}_{\min})$ related to the Clifford action $J_z\colon\mathfrak{v}_{\min}\to\mathfrak{v}_{\min}$ and the H-type group $N^T_{3,4}(\mathfrak{v}_{\min})$ related to the Clifford action $\tilde{J}_z\colon\mathfrak{v}_{\min}\to\mathfrak{v}_{\min}$, $\tilde{J}_z=T^{-1}J_zT$. Then the Lie bracket $[x,y]^T$ defined by $\tilde{J}_z$ is given by

$$\langle z,[x,y]^T\rangle_{\mathbb{R}^{3,4}}=\langle\tilde{J}_z(x),y\rangle_{\mathfrak{v}_{\min}}.$$

Lemma 2. *The transformation* $T\times\mathrm{Id}\colon N_{3,4}(\mathfrak{v}_{\min})\to N^T_{3,4}(\mathfrak{v}_{\min})$ *is an isometric isomorphism.*

Proof. We can see this by the equality:

$$\begin{aligned}\langle z,[T(x),T(y)]\rangle_{\mathfrak{v}_{\min}}&=\langle J_z(T(x)),T(y)\rangle_{\mathfrak{v}_{\min}}\\&=\langle T^{\tau}J_zT(x),y\rangle_{\mathfrak{v}_{\min}}=\langle\tilde{J}_z(x),y\rangle_{\mathfrak{v}_{\min}}=\langle z,[x,y]^T\rangle_{\mathbb{R}^{3,4}},\end{aligned}$$

that proves the lemma. ■

(H2) Let $x_0\in\mathfrak{v}_{\min}$ be as in (**H2**) and $v\in\mathfrak{v}_{\min}$ as in (7). We find $T\in SO(4,4)$, $T\colon\mathfrak{v}_{\min}\to\mathfrak{v}_{\min}$, such that $T(x_0)=J_{Z_4}(v)$. Then condition (9) is equivalent to

$$A(J_{Z_4}(v))=J_{z_0}(v),\quad C(z_0)=0,\quad [A,J_w]=J_{C(w)},\quad\text{for all}\quad w\in\mathbb{R}^{3,4}.$$

by changing the Clifford action from J_z to $\tilde{J}_z=T^{-1}J_zT$.

(H3) Let $x_0\in\mathfrak{v}_{\min}$ be as in (**H3**) and $v\in\mathfrak{v}_{\min}$ as in (7). We fix an orthonormal basis $\{Z_i\}_{i=1}^7$ as was described in (**C3**), and express x_0 in the basis (8) as follows

$$x_0=\sum_{i=0}^{3}\theta_iJ_{Z_i}(v)+\sum_{i=4}^{7}\theta_iJ_{Z_i}(v)=x_0^++x_0^-,\qquad\sum_{i=0}^{3}{\theta_i}^2=\sum_{i=4}^{7}{\theta_i}^2=1.$$

Since $\langle x_0^+,x_0^-\rangle_{\mathfrak{v}_{\min}}=0$, we can find an orthogonal transformation $T\in SO(4,0)\times SO(0,4)\subset SO(4,4)$ such that $T(x_0^+)=v$ and $T(x_0^-)=J_{Z_4}(v)$. Then for $\tilde{J}_z=T^{-1}J_zT$ we obtain

$$\begin{aligned}&T^{-1}AT(v+J_{Z_4}(v))=T^{-1}J_{Z_0}T(v+J_{z_4}(v)),\quad C(z_0)=0,\\&T^{-1}[A,J_w]T=[T^{-1}AT,T^{-}J_wT]=T^{-1}J_{C(w)}T\quad\text{for any}\quad w\in\mathbb{R}^{3,4}.\end{aligned}\tag{11}$$

which is equivalent to condition (9).

The discussions above can be summarized in the statement that an initial vector (z_0,x_0) for a geodesic can be restricted to the following nine cases.

$$\left.\begin{array}{l}Z_1\oplus v,\\Z_4\oplus v,\\(Z_1+Z_4)\oplus v\end{array}\right\}\in\mathbb{R}^{3,4}\oplus\mathfrak{v}_{\min},\tag{12}$$

$$\left.\begin{array}{l}Z_1\oplus J_{Z_4}(v),\\Z_4\oplus J_{Z_4}(v),\\(Z_1+Z_4)\oplus J_{Z_4}(v)\end{array}\right\}\in\mathbb{R}^{3,4}\oplus\mathfrak{v}_{\min},\tag{13}$$

$$\left.\begin{array}{l}Z_1\oplus(v+J_{Z_4}(v)),\\Z_4\oplus(v+J_{Z_4}(v)),\\(Z_1+Z_4)\oplus(v+J_{Z_4}(v))\end{array}\right\}\in\mathbb{R}^{3,4}\oplus\mathfrak{v}_{\min}.\tag{14}$$

We show not only the existence of a pair (C, A) for all the 9 cases (12), (13) and (14), but we also show that they form an affine space of a suitable dimension according to the initial conditions.

3.3. **Relation between A and C in $[A, J_w] = J_{C(w)}$.** We can express A in the form

$$A = \sum_{i<k} a_{i,k} J_{Z_i} J_{Z_k}. \tag{15}$$

Indeed, we know that A belong to linear combination of $\{[J_{Z_i}, J_{Z_k}] \,|\, 1 \leq i < k \leq 7\}$, see details in [12, Proposition 4]. On the other hand, $[J_{Z_i}, J_{Z_k}] = J_{Z_i}J_{Z_k} - J_{Z_k}J_{Z_i} = 2J_{Z_i}J_{Z_k}$.

Then $[A, J_{Z_1}] = J_{C(Z_1)}$ implies

$$[A, J_{Z_1}] = AJ_{Z_1} - J_{Z_1}A = \sum_{i<k} a_{i,k}J_{Z_i}J_{Z_k}J_{Z_1} - J_{Z_1}\sum_{i<k} a_{i,k}J_{Z_i}J_{Z_k} = 2\sum_{k>1} a_{1,k}J_{Z_k} = J_{2\sum_{k=2}^{7} a_{1,k}Z_k}.$$

Thus $C(Z_1) = 2\sum_{k=2}^{7} a_{1,k}Z_k$. By similar calculations we obtain

$$C(Z_2) = -2a_{1,2}Z_1 + 2\sum_{k=3}^{7} a_{2,k}Z_k, \qquad C(Z_3) = -2\sum_{k=1}^{2} a_{k,3}Z_k + 2\sum_{k=4}^{7} a_{3,k}Z_k,$$
$$C(Z_4) = 2\sum_{k=1}^{3} a_{k,4}Z_k - 2\sum_{k=5}^{7} a_{4,k}Z_k, \qquad C(Z_5) = 2\sum_{k=1}^{4} a_{k,5}Z_k - 2\sum_{k=5}^{6} a_{5,k}Z_k,$$
$$C(Z_6) = 2\sum_{k=1}^{5} a_{k,6}Z_k - 2a_{6,7}Z_7, \qquad C(Z_7) = 2\sum_{k=1}^{6} a_{k,7}Z_k.$$

In the matrix form C is given by

$$C = 2 \times \begin{pmatrix} 0 & -a_{1,2} & -a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} \\ a_{1,2} & 0 & -a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} \\ a_{1,3} & a_{2,3} & 0 & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} \\ a_{1,4} & a_{2,4} & a_{3,4} & 0 & a_{4,5} & a_{4,6} & a_{4,7} \\ a_{1,5} & a_{2,5} & a_{3,5} & -a_{4,5} & 0 & a_{5,6} & a_{5,7} \\ a_{1,6} & a_{2,6} & a_{3,6} & -a_{4,6} & -a_{5,6}, & 0 & a_{6,7} \\ a_{1,7} & a_{2,7} & a_{3,7} & -a_{4,7} & -a_{5,7} & -a_{6,7} & 0. \end{pmatrix}. \tag{16}$$

3.4. **Writing $A(v)$ and $A(J_{Z_4}(v))$ in the basis (8).** Note that condition (7) implies

$$\begin{array}{rclclcl} J_{Z_1}(v) &=& -J_{Z_2}J_{Z_3}(v) &=& J_{Z_4}J_{Z_7}(v) &=& J_{Z_5}J_{Z_6}(v), \\ J_{Z_2}(v) &=& J_{Z_1}J_{Z_3}(v) &=& J_{Z_4}J_{Z_6}(v) &=& -J_{Z_5}J_{Z_7}(v), \\ J_{Z_3}(v) &=& -J_{Z_1}J_{Z_2}(v) &=& J_{Z_4}J_{Z_5}(v) &=& J_{Z_6}J_{Z_7}(v), \\ J_{Z_4}(v) &=& J_{Z_1}J_{Z_7}(v) &=& J_{Z_2}J_{Z_6}(v) &=& J_{Z_3}J_{Z_5}(v), \\ J_{Z_5}(v) &=& -J_{Z_2}J_{Z_7}(v) &=& -J_{Z_3}J_{Z_4}(v) &=& J_{Z_1}J_{Z_6}(v), \\ J_{Z_6}(v) &=& -J_{Z_1}J_{Z_5}(v) &=& -J_{Z_2}J_{Z_4}(v) &=& J_{Z_3}J_{Z_7}(v), \\ J_{Z_7}(v) &=& -J_{Z_1}J_{Z_4}(v) &=& J_{Z_2}J_{Z_5}(v) &=& -J_{Z_3}J_{Z_6}(v). \end{array} \tag{17}$$

Put $A(v) = \sum_{i<k} a_{i,k}J_{Z_i}J_{Z_k}(v) = \alpha_0 v + \sum_{j=1}^{7} \alpha_j J_{Z_j}(v)$. Then using (17) we find the relations between coefficients $a_{i,k}$ and α_j as follows

$$\begin{cases} \alpha_0 = 0, \\ \alpha_1 = -a_{2,3} + a_{4,7} + a_{5,6}, \\ \alpha_2 = a_{1,3} + a_{4,6} - a_{5,7}, \\ \alpha_3 = -a_{1,2} + a_{4,5} + a_{6,7}, \\ \alpha_4 = a_{1,7} + a_{2,6} + a_{3,5}, \\ \alpha_5 = a_{1,6} - a_{2,7} - a_{3,4}, \\ \alpha_6 = -a_{1,5} - a_{2,4} + a_{3,7}, \\ \alpha_7 = -a_{1,4} + a_{2,5} - a_{3,6}. \end{cases} \tag{18}$$

Write $A(J_{Z_4}(v))=\beta_0 v+\sum_{m=1}^{7}\beta_m J_{Z_m}(v)$, and note that

$$\begin{array}{rcrcrcr} J_{Z_1}(v) &=& J_{Z_1}J_{Z_4}J_{Z_4}(v) &=& J_{Z_2}J_{Z_5}J_{Z_4}(v) &=& -J_{Z_3}J_{Z_6}J_{Z_4}(v),\\ J_{Z_2}(v) &=& -J_{Z_1}J_{Z_5}J_{Z_4}(v) &=& J_{Z_2}J_{Z_4}J_{Z_4}(v) &=& J_{Z_3}J_{Z_7}J_{Z_4}(v),\\ J_{Z_3}(v) &=& J_{Z_1}J_{Z_6}J_{Z_4}(v) &=& -J_{Z_2}J_{Z_7}J_{Z_4}(v) &=& J_{Z_3}J_{Z_4}J_{Z_4}(v),\\ v &=& J_{Z_1}J_{Z_7}J_{Z_4}(v) &=& J_{Z_2}J_{Z_6}J_{Z_4}(v) &=& J_{Z_3}J_{Z_5}J_{Z_4}(v),\\ J_{Z_5}(v) &=& -J_{Z_1}J_{Z_2}J_{Z_4}(v) &=& -J_{Z_4}J_{Z_5}J_{Z_4}(v) &=& J_{Z_6}J_{Z_7}J_{Z_4}(v),\\ J_{Z_6}(v) &=& J_{Z_1}J_{Z_3}J_{Z_4}(v) &=& -J_{Z_4}J_{Z_6}J_{Z_4}(v) &=& -J_{Z_5}J_{Z_7}J_{Z_4}(v),\\ J_{Z_7}(v) &=& -J_{Z_2}J_{Z_3}J_{Z_4}(v) &=& -J_{Z_4}J_{Z_7}J_{Z_4}(v) &=& J_{Z_5}J_{Z_6}J_{Z_4}(v). \end{array} \tag{19}$$

By using (19) we find relations between $a_{i,k}$ and β_j as below:

$$\begin{cases} \beta_0 = a_{1,7}+a_{2,6}+a_{3,5},\\ \beta_1 = a_{1,4}+a_{2,5}-a_{3,6},\\ \beta_2 = -a_{1,5}+a_{2,4}+a_{3,7},\\ \beta_3 = a_{1,6}-a_{2,7}+a_{3,4},\\ \beta_4 = 0,\\ \beta_5 = -a_{1,2}-a_{4,5}+a_{6,7},\\ \beta_6 = a_{1,3}-a_{4,6}-a_{5,7},\\ \beta_7 = -a_{2,3}-a_{4,7}+a_{5,6}. \end{cases} \tag{20}$$

3.5. **Existence of $D=(C,A)$.** Based on data in the previous section, we discuss the existence of a pair (C,A) of condition (9) for each cases of initial conditions (12), (13), and (14).

CASE (z_0,x_0) WITH $\|x_0\|^2\neq 0$. Assume for the moment that $\|x_0\|^2=\|z_0\|^2=1$ and use Lemma 1. Then the conditions $A(v)=J_{Z_1}(v)$, $C(Z_1)=0$, and (18) give

$$a_{1,2}=a_{1,3}=a_{1,4}=a_{1,5}=a_{1,6}=a_{1,7}=0, \text{ and}$$

$$\begin{cases} \alpha_0 = 0,\\ \alpha_1 = -a_{2,3}+a_{4,7}+a_{5,6}=1,\\ \alpha_2 = a_{1,3}+a_{4,6}-a_{5,7}=0,\\ \alpha_3 = -a_{1,2}+a_{4,5}+a_{6,7}=0,\\ \alpha_4 = a_{1,7}+a_{2,6}+a_{3,5}=0,\\ \alpha_5 = a_{1,6}-a_{2,7}-a_{3,4}=0,\\ \alpha_6 = -a_{1,5}-a_{2,4}+a_{3,7}=0,\\ \alpha_7 = -a_{1,4}+a_{2,5}-a_{3,6}=0. \end{cases} \Longrightarrow \begin{cases} a_{5,6}=1+a_{2,3}-a_{4,7},\\ a_{5,7}=a_{4,6},\\ a_{6,7}=-a_{4,5},\\ a_{3,5}=-a_{2,6},\\ a_{3,4}=-a_{2,7},\\ a_{3,7}=a_{2,4},\\ a_{3,6}=a_{2,5}, \end{cases} \tag{21}$$

where $a_{2,3}, a_{2,4}, a_{2,5}, a_{2,6}, a_{2,7},\ a_{4,5}, a_{4,6}, a_{4,7}$ are free parameters.

Analogously to the above case one can obtain systems similar to (21) for any of the following initial conditions:

$$\begin{array}{c} Z_1\oplus v,\quad Z_4\oplus v,\quad Z_1\oplus J_{Z_4}(v),\quad Z_4\oplus J_{Z_4}(v),\\ (Z_1+Z_4)\oplus v,\qquad (Z_1+Z_4)\oplus J_{Z_4}(v). \end{array} \tag{22}$$

CASE (z_0,x_0) WITH $\|x_0\|=0$ OR $\|z_0\|=0$. The cases $Z_1\oplus(v+J_{Z_4}(v))$, $Z_4\oplus(v+J_{Z_4}(v))$, and $(Z_1+Z_4)\oplus(v+J_{Z_4}(v))$ are treated by a similar way, requiring more careful calculation. We consider only the $(Z_1+Z_4)\oplus(v+J_{Z_4}(v))$ in some details.

Conditions (9) are equivalent to

$$\begin{aligned} A(v+J_{Z_4}(v)) &= J_{Z_1+Z_4}(v+J_{Z_4}(v)) = J_{Z_1}(v)+J_{Z_1Z_4}(v)+J_{Z_4}(v)+v\\ &= \gamma_0 v+\sum_{i=1}^{7}\gamma_i J_{Z_i}(v). \end{aligned}$$

Noticing that $J_{Z_1}(v)+J_{Z_1Z_4}(v)+J_{Z_4}(v)+v=J_{Z_1}(v)-J_{Z_7}(v)+J_{Z_4}(v)+v$, we obtain the system

$$\begin{cases}\gamma_0=\alpha_0+\beta_0=a_{1,7}+a_{2,6}+a_{3,5}=1,\\ \gamma_1=\alpha_1+\beta_1=(-a_{2,3}+a_{5,6}-a_{3,6}+a_{2,5})+(a_{1,4}+a_{4,7})=1,\\ \gamma_2=\alpha_2+\beta_2=(a_{1,3}-a_{1,5}+a_{3,7}-a_{5,7})+(a_{4,6}+a_{2,4})=0,\\ \gamma_3=\alpha_3+\beta_3=(-a_{1,2}+a_{6,7}+a_{1,6}-a_{2,7})+(a_{4,5}+a_{3,4})=0\\ \gamma_4=\alpha_4+\beta_4=a_{1,7}+a_{2,6}+a_{3,5}=1,\\ \gamma_5=\alpha_5+\beta_5=(-a_{1,2}+a_{6,7}+a_{1,6}-a_{2,7})-(a_{4,5}+a_{3,4})=0\\ \gamma_6=\alpha_6+\beta_6=(a_{1,3}-a_{1,5}+a_{3,7}-a_{5,7})-(a_{4,6}+a_{2,4})=0,\\ \gamma_7=\alpha_7+\beta_7=(-a_{2,3}+a_{5,6}-a_{3,6}+a_{2,5})-(a_{1,4}+a_{4,7})=-1.\end{cases} \tag{23}$$

The condition $C(Z_1+Z_4)=0$ leads to

$$a_{1,4}=0,\ a_{1,2}+a_{2,4}=0,\ a_{1,3}+a_{3,4}=0,\ a_{1,5}-a_{4,5}=0,\ a_{1,6}-a_{4,6}=0,\ a_{1,7}-a_{4,7}=0,$$

From these expressions we obtain the solution

$$\begin{cases}a_{1,4}=0,\\ a_{4,6}=-a_{2,4}=a_{1,6}=a_{1,2},\\ a_{4,5}=-a_{3,4}=a_{1,5}=a_{1,3},\\ a_{4,7}=a_{1,7}=1,\\ a_{5,6}=a_{2,3}-a_{2,5}+a_{3,6},\\ a_{3,5}=-a_{2,6},\\ a_{6,7}=a_{2,7},\\ a_{5,7}=a_{3,7},\end{cases}$$

with eight free parameters $a_{1,2},a_{1,3},a_{2,3},a_{2,5},a_{2,6},a_{2,7},a_{3,6},a_{3,7}$.

This finishes the proof of Theorem 4. We emphasize that this proof is constructive and it is easy to find a 1-parameter group (and even a multi-parameter family of such groups) whose orbit is an arbitrarily chosen geodesic.

4. Riemannian H-type nilmanifold $N_{7,0}=N_{7,0}(\mathfrak{v}_{\min})$

Here we consider another example of 15-dimensional manifold, related to Clifford algebra. It is the Riemannian H-type Lie group with 7-dimensional centre and 8-dimensional irreducible Clifford module. The metric on the centre and on the module is positive definite. It is known from the work of [25] that this Riemannian nilmanifold $N_{7,0}(\mathfrak{v}_{\min})$ is geodesic orbit. We give a proof which is similar to the proof in Section 3, but significantly simpler since all the vectors has positive square lengths. It also shows more complicate structure of pseudo-Riemannian nilmanifold, compare to Riemannian once.

Theorem 5. *The Riemannian H-type nilmanifold $N_{7,0}(\mathfrak{v}_{\min})$ is geodesic orbit. Each geodesic is realized as an orbit of a one parameter subgroup which initial conditions form an eight dimensional affine space.*

Proof. The Clifford algebra $\mathrm{Cl}(\mathbb{R}^{7,0})$ has two non-equivalent irreducible 8-dimensional modules $\mathfrak{v}^+_{\min}$ and $\mathfrak{v}^-_{\min}$. The corresponding 15-dimensional H-type Lie algebras $N_{7,0}(\mathfrak{v}^+_{\min})$ and $N_{7,0}(\mathfrak{v}^-_{\min})$ are isomorphic and isometric, see [Section 3.1.2][4]. The left invariant metric is generated by the positive definite scalar product $\langle\cdot\,,\cdot\rangle_{7,0}+\langle\cdot\,,\cdot\rangle_{8,0}$ on the Lie algebra $\mathfrak{n}_{7,0}\cong\mathbb{R}^{7,0}\oplus\mathbb{R}^{8,0}$. We write $(z_0,x_0)\in\mathfrak{z}\oplus\mathfrak{v}_{\min}\cong\mathbb{R}^{7,0}\oplus\mathbb{R}^{8,0}$ for an initial condition for a geodesic. As was shown in Section 3, without loss of generality we can assume that $z_0=Z_1$, where

$$\{Z_1,\dots,Z_7\},\quad \|Z_1\|^2_{\mathbb{R}^{7,0}}=\ldots=\|Z_7\|^2_{\mathbb{R}^{7,0}}=1$$

is an orthonormal basis for $\mathbb{R}^{7,0}$ and $x_0 = v \in \mathfrak{v}_{\min}$ is a vector satisfying (7) and $\|v\|^2_{\mathbb{R}^{8,0}} = 1$. We will also use basis (8).

Then, a geodesic determined by the initial condition $(z_0, x_0) = (Z_1, v)$ is homogeneous if there is a pair $D = (C, A)$ such that

$$A(v) = J_{Z_1}(v), \quad C(Z_1) = 0, \quad [A, J_w] = J_{C(w)}, \quad \text{for all} \quad w \in \mathbb{R}^{7,0}. \tag{24}$$

We write $A = \sum_{i<k} a_{i,k} J_{Z_i} J_{Z_k}$. By checking the condition $[A, J_w] = J_{C(w)}$ for $w = Z_k$, $k = 1, \dots, 7$ we come to the conclusion that

$$C = 2 \times \begin{pmatrix} 0 & -a_{1,2} & -a_{1,3} & -a_{1,4} & -a_{1,5} & -a_{1,6} & -a_{1,7} \\ a_{1,2} & 0 & -a_{2,3} & -a_{2,4} & -a_{2,5} & -a_{2,6} & -a_{2,7} \\ a_{1,3} & a_{2,3} & 0 & -a_{3,4} & -a_{3,5} & -a_{3,6} & -a_{3,7} \\ a_{1,4} & a_{2,4} & a_{3,4} & 0 & -a_{4,5} & -a_{4,6} & -a_{4,7} \\ a_{1,5} & a_{2,5} & a_{3,5} & a_{4,5} & 0 & -a_{5,6} & -a_{5,7} \\ a_{1,6} & a_{2,6} & a_{3,6} & a_{4,6} & a_{5,6}, & 0 & -a_{6,7} \\ a_{1,7} & a_{2,7} & a_{3,7} & a_{4,7} & a_{5,7} & a_{6,7} & 0 \end{pmatrix}. \tag{25}$$

Note that condition (7) for $N_{7,0}(\mathfrak{v}_{\min})$ implies

$$\begin{array}{lllllll} J_{Z_1}(v) & = & -J_{Z_2}J_{Z_3}(v) & = & -J_{Z_4}J_{Z_7}(v) & = & -J_{Z_5}J_{Z_6}(v), \\ J_{Z_2}(v) & = & J_{Z_1}J_{Z_3}(v) & = & J_{Z_4}J_{Z_6}(v) & = & -J_{Z_5}J_{Z_7}(v), \\ J_{Z_3}(v) & = & -J_{Z_1}J_{Z_2}(v) & = & J_{Z_4}J_{Z_5}(v) & = & J_{Z_6}J_{Z_7}(v), \\ J_{Z_4}(v) & = & J_{Z_1}J_{Z_7}(v) & = & -J_{Z_2}J_{Z_6}(v) & = & -J_{Z_3}J_{Z_5}(v), \\ J_{Z_5}(v) & = & J_{Z_2}J_{Z_7}(v) & = & J_{Z_3}J_{Z_4}(v) & = & J_{Z_1}J_{Z_6}(v), \\ J_{Z_6}(v) & = & -J_{Z_1}J_{Z_5}(v) & = & J_{Z_2}J_{Z_4}(v) & = & -J_{Z_3}J_{Z_7}(v), \\ J_{Z_7}(v) & = & -J_{Z_1}J_{Z_4}(v) & = & -J_{Z_2}J_{Z_5}(v) & = & J_{Z_3}J_{Z_6}(v). \end{array} \tag{26}$$

Now we write the value $A(v)$ in the basis (8). Put $A(v) = \sum_{i<k} a_{i,k} J_{Z_i} J_{Z_k}(v) = \alpha_0 v + \sum_{j=1}^{7} \alpha_j J_{Z_j}(v)$ and solve the equation $A(v) = J_{Z_1}(v)$. We obtain the values of α_k in terms of $a_{j,k}$.

$$\begin{cases} \alpha_0 = 0, \\ \alpha_1 = -a_{2,3} - a_{4,7} - a_{5,6}, = 1, \\ \alpha_2 = a_{1,3} + a_{4,6} - a_{5,7} = 0, \\ \alpha_3 = -a_{1,2} + a_{4,5} + a_{6,7} = 0, \\ \alpha_4 = a_{1,7} - a_{2,6} - a_{3,5} = 0, \\ \alpha_5 = a_{1,6} + a_{2,7} + a_{3,4} = 0, \\ \alpha_6 = -a_{1,5} + a_{2,4} - a_{3,7} = 0, \\ \alpha_7 = -a_{1,4} - a_{2,5} + a_{3,6} = 0. \end{cases} \tag{27}$$

The condition $C(Z_1) = 0$ implies

$$a_{1,2} = a_{1,3} = a_{1,4} = a_{1,5} = a_{1,6} = a_{1,7} = 0.$$

The system (27) has the following solution:

$$a_{2,3}, a_{2,4}, a_{2,5}, a_{2,6}, a_{2,7}, a_{4,5}, a_{4,6}, a_{4,7} \quad \text{are free variables and}$$

$$\begin{cases} a_{3,4} & = -a_{2,7}, \\ a_{3,5} & = -a_{2,6}, \\ a_{3,6} & = a_{2,5}, \\ a_{3,7} & = a_{2,4}, \\ a_{5,6} & = 1 + a_{2,3} - a_{4,7}, \\ a_{5,7} & = a_{4,6}, \\ a_{6,7} & = -a_{4,5}. \end{cases}$$

This finishes the proof of Theorem 5. ■

The given examples of geodesic orbit pseudo-Riemannian metrics are obtained using important structural results for the corresponding pseudo H-type Lie algebras. We dare to suggest that similar ideas will be useful in constructing other examples of geodesic orbit pseudo-Riemannian manifolds.

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K. Furutani. Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University, Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan
Email address: kf46089@gmail.com

I. Markina. Department of Mathematics, University of Bergen, P.O. Box 7803, Bergen N-5020, Norway
Email address: irina.markina@uib.no

Yu.G. Nikonorov. Southern Mathematical Institute of VSC RAS, 53 Vatutina St., Vladikavkaz, 362025, Russia
Email address: nikonorov2006@mail.ru