

Ribbonness on boundary surface-link, revised

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ABSTRACT

A revised proof of the author's earlier result is given. It is shown that a boundary surface-link in the 4-sphere is a ribbon surface-link if the surface-link obtained from it by surgery along a pairwise nontrivial fusion 1-handle system is a ribbon surface-link. As a corollary, the surface knot obtained from the anti-parallel surface-link of a non-ribbon surface-knot by surgery along a nontrivial or trivial fusion 1-handle is a non-ribbon or trivial surface-knot, respectively. This result answers Cochran's conjecture on non-ribbon sphere-knots in the affirmative.

Keywords: Boundary surface-link, Ribbon surface-link, Anti-parallel surface-link, Cochran's conjecture.

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1. Introduction

This paper is a revised version of the authors' previous paper (published missing in part from the manuscript), [1]. It also includes the affirmative answer to Cochran's conjecture on a non-ribbon sphere-knot and its generalization to a non-ribbon surface-knot and consequence as well as an appendix to improve the proof of one of the main results, Theorem 1.1. A surface-link (of r components) is a closed oriented surface F (of r connected components F_i ($i = 1, 2, \dots, r$)) smoothly embedded in the 4-sphere S^4 . When F is connected, F is called a *surface-knot*. If F consists of 2-spheres, then F is called a *sphere-link* or an S^2 -link. A *1-handle system* on a surface-link F is a system h of disjoint 1-handles h_j ($j = 1, 2, \dots, s$) on F smoothly

embedded in S^4 . Let $F(h)$ be the surface-link obtained from F by surgery along a 1-handle system h . A *ribbon surface-link* is the surface-link $O(h)$ obtained from a trivial S^2 -link O by surgery along a 1-handle system h , [2, 3]. A surface-link F of r components $F_i (i = 1, 2, \dots, r)$ is a *boundary surface-link* if there is a system V_* of disjoint compact connected oriented 3-manifolds $V_i (i = 1, 2, \dots, r)$ smoothly embedded in S^4 with $\partial V_i = F_i$, where V_i is called a *Seifert hypersurface* for F_i . The system V_* is called a *disjoint Seifert hypersurface system* for the boundary surface-link F . The 1-handle system h on F is a *fusion 1-handle system* if $F(h)$ has the $r - s (\geq 1)$ connected components, where the 1-handles $h_j (j = 1, 2, \dots, s)$ of h are called *fusion 1-handles*. Let F be a boundary surface-link in S^4 of components $F_i (i = 1, 2, \dots, r)$, and V_* a disjoint Seifert hypersurface system for F consisting of $V_i (i = 1, 2, \dots, r)$ with $\partial V_i = F_i (i = 1, 2, \dots, r)$. Let h be a fusion 1-handle system $h_j (j = 1, 2, \dots, s)$ on F . The Seifert hypersurface V_i for F_i is considered as a union $B_i \cup V_i^B$ pasting along a disk $d_i = B_i \cap V_i^B = (\partial B_i) \cap (\partial V_i^B)$. Let d_* , B_* and V_*^B be the systems of the disks $d_i (i = 1, 2, \dots, r)$, the 3-balls $B_i (i = 1, 2, \dots, r)$ and the 3-manifolds $V_i^B (i = 1, 2, \dots, r)$, respectively. Since the 3-manifold V_i^B is nothing but a 3-cell reduction of V_i through the 3-ball B_i , the surface-link F is canonically equivalent to the surface-link $\cup_{i=1}^r \partial V_i^B = \partial V_*^B$. The 3-ball system B_* and trivial S^2 -link $L = \partial B_*$ are called a *local 3-ball system* and a *local S^2 -link* of the boundary surface-link F , respectively. By sliding the transverse disk system of h meeting V_* into B_* through $B_* \cup V_*^B$, the fusion 1-handle system h on F is considered as a fusion 1-handle system h^L on the local S^2 -link L with an attaching disk system in the disk system $d_*^c = \text{cl}(L \setminus d_*)$. Thus, h^L does not intersect $V_* = B_* \cup V_*^B$ except if there is a transverse disk system of h^L inside B_* . This fusion 1-handle system h^L on L is called a *localized fusion 1-handle system* of h on F (with respect to the decomposition $B_* \cup V_*^B$ of V_*). The S^2 -link $L(h^L)$ is a ribbon S^2 -link. A fusion 1-handle h_j^L in h^L on L is *trivial* if the core arc α_j^L of h_j^L is made disjoint from the interior of the 3-ball system B_* by a ∂ -relative isotopy of α_j^L keeping L fixed. Otherwise, h_j^L is *nontrivial*. A fusion 1-handle h_j^L on L is *pairwise nontrivial* if the core arc α_j^L of h_j^L is nontrivial on L and made disjoint from the 3-balls of B_* excluding the two 3-balls attached from α_j^L . A fusion 1-handle system h on a boundary surface-link F is *trivial*, *nontrivial* or *pairwise nontrivial*, respectively, if the corresponding fusion 1-handle system h^L on L consists of trivial, nontrivial or pairwise nontrivial fusion 1-handles $h_j^L (j = 1, 2, \dots, s)$ for a decomposition $B_* \cup V_*^B$ of a disjoint Seifert hypersurface system V_* of F , respectively. The following theorem is a main result of this paper, correcting an incorrect result, [4, Theorem 1.4]. Some counterexamples, [5, 6].

Theorem 1.1. Let F be a boundary surface-link of $r (\geq 2)$ components in S^4 . If the surface-link $F(h)$ for a pairwise nontrivial fusion 1-handle system h on F is a ribbon

surface-link, then the surface-link F is a ribbon surface-link.

For a surface-knot F in S^4 , let $F \times [0, 1]$ be a normal $[0, 1]$ -bundle over F in S^4 such that the natural homomorphism $H_1(F \times \{1\}; Z) \rightarrow H_1(S^4 \setminus F \times \{0\}; Z)$ is the zero map. In other words, take $F \times [0, 1]$ a boundary collar of a Seifert hypersurface V of F in S^4 , [7]. The surface-link $P(F) = \partial(F \times [0, 1]) = F_0 \cup F_1$ with $F_i = F \times \{i\}$ is called the *anti-parallel surface-link* of F , where by convention F_0 and F_1 are identified with $-F$ (i.e., the orientation reversed F) and F , respectively. The anti-parallel surface-link $P(F)$ is a boundary surface-link, because $P(F)$ admits a disjoint Seifert hypersurface system $V_* = V_0 \cup V_1$ with $V_i = V \times \{i\}$ for a normal $[0, 1]$ -bundle $V \times [0, 1]$ of a Seifert hypersurface V for F in S^4 . The half part of the following theorem is a direct consequence of Theorem 1.1.

Theorem 1.2. Let $P(F)$ be the anti-parallel surface-link of a non-ribbon surface knot F in S^4 , and h a fusion 1-handle on $P(F)$. According to whether the 1-handle h is trivial or nontrivial, the surface-knot $P(F)(h)$ is a trivial surface-knot or a non-ribbon surface-knot, respectively.

This result solves Cochran's conjecture that if F is a non-ribbon S^2 -knot and h is any sufficiently complicated fusion 1-handle, then $P(F; h)$ must be a non-ribbon S^2 -knot, [8]. Theorem 1.2 leads to the following corollary, whose proof is done in Section 3.

Corollary 1.3. Let h and h_0 be a nontrivial and trivial fusion 1-handles on the anti-parallel surface-link $P(F)$ of a non-ribbon surface-knot F , respectively. Then the non-ribbon surface-knot $P(F)(h)$ is obtained from a surface-link of the trivial surface-knot component $P(F)(h_0)$ and a trivial S^2 -link component O by surgery along a fusion 1-handle. The trivial surface-knot $P(F)(h_0)$ is obtained from a surface-link of the non-ribbon surface-knot component $P(F)(h)$ and a trivial S^2 -link component O by surgery along a fusion 1-handle.

In this corollary, when F is taken a non-ribbon S^2 -knot, the case where the surface-knot $P(F)(h_0)$ is a trivial S^2 -knot is obtained since the genus of $P(F)(h_0)$ is twice the genus of F . When h is taken a simple nontrivial fusion 1-handle such as the 1-handle representing the spun trefoil knot, the case where O is a trivial S^2 -knot is obtained. Thus, this corollary is a generalization of Ogasa's observation, [9].

2. A SUPH move of a 2-handle and a change of a 1-handle on a surface-link

A ribbon surface-link F is a surface-link constructed from a pair (O, h) of a trivial

S^2 -link O and a 1-handle system h on O in S^4 , called a *handled sphere system*, by surgery along h , which is uniquely constructed from a *chorded sphere system* (O, α) in S^4 with the core arc system α of h in the standard 3-sphere S^3 of S^4 , called a *chord system*, or from a *chord graph* (o, α) in S^3 with a trivial link o , called a *based loop system*, [2]. The chord graph (o, α) in S^3 is considered by a chord diagram $C = C(o, \alpha)$ in S^2 . A *SUPH system* for a surface-link F in S^4 is a compact multi-punctured handlebody system W smoothly embedded in S^4 whose boundary ∂W is given by $\partial W = F \cup O$ for a trivial S^2 -link O in S^4 . If F is a ribbon surface-link given by a handled sphere system (O, h) , a SUPH system W for F is constructed by $W = O \times [0, 1] \cup h$ for a collar $O \times [0, 1]$ of O in S^4 with $O \times \{0\} = O$, where $O \times \{1\} = \partial W \setminus F$ is a trivial S^2 -link. Conversely, it is observed that if there is a SUPH system for a surface-link F , then F is a ribbon surface-link, [4]. Let D be a *2-handle core disk* on a surface-link F in S^4 , namely the 2-handle core of a 2-handle $D \times I$ on a surface-link F in S^4 . Let U be a SUPH system for a ribbon S^2 -knot K in S^4 such that $\delta = K \cap D = U \cap D$ is a disk and $K \cap F = \emptyset$. The 2-handle core disk D' on F given by $D' = \text{cl}(D \setminus \delta) \cup \text{cl}(K \setminus \delta)$ is called a 2-handle core disk obtained from D by an *elementary SUPH-move*. A 2-handle core disk D^* on F is obtained from D on F by a *SUPH-move* if D^* is obtained from D on F by a finite sequence of elementary SUPH moves. Note that if D^* is obtained from D on F by a SUPH-move, then D is obtained from D^* on F by a SUPH-move. For a 2-handle $D \times I$ on a surface-link F , let $F(D \times I)$ denote the surface-link obtained from F by surgery along $D \times I$. To prove Theorem 1,1, the following lemma is used.

Lemma 2.1. For a 2-handle $D \times I$ on a surface-link F in S^4 , assume that the surface-link $F(D \times I)$ is a ribbon surface-link. Let D' be a disk obtained from the 2-handle core disk D on F by an elementary SUPH-move. Then the surface-link $F(D' \times I)$ in S^4 is a ribbon surface-link.

Proof of Lemma 2.1. Let (O, h) be a handled sphere system for the ribbon surface-link $F(D \times I)$ in S^4 . Let $W = O \times [0, 1] \cup h$ be a SUPH system for $F(D \times I)$ with $O \times \{0\} = O$. The 1-handle system h on $O \times [0, 1]$ attached to O is made disjoint from the 2-handle $D \times I$, where let $I = [-\varepsilon, \varepsilon]$ for small positive number ε . Let $D_{\varepsilon t} = D \times \{\varepsilon t\}$ for every t in $[-1, 1]$ where D_0 is identified with the core disk D of $D \times I$. The disks $D_{-\varepsilon}$ and D_{ε} are located on $O \setminus h \cap O$. Let U be a SUPH system for a ribbon S^2 -knot K in S^4 which is used for the elementary SUPH move from D to D' , where $\delta = K \cap D = U \cap D$ is a disk and $K \cap F(D \times I) = \emptyset$. The intersection $(D \times I) \cap U$ is assumed to be a collar $\delta \times [0, \varepsilon]$ of δ in U , so that the union $W^+ = W \cup D \times I \cup U$ is a compact oriented 3-manifold in S^4 . Let $W^+ \times [-1, 1]$ be a bi-collar of W^+ in S^4

with $W^+ \times \{0\} = W^+$. In $W^+ \times [-1, 1]$, let

$$W^* = W \cup_{0 \leq t \leq 1} D_{\varepsilon(t-1)} \times \{-t\} \cup U \times \{-1\} \cup_{0 \leq t \leq 1} D_{\varepsilon(1-t)} \times \{t\} \cup U \times \{1\},$$

which is a SUPH system for a surface-link equivalent to the surface-link obtained from $F(D' \times I)$ by pushing the interior of the disk $D'_{-\varepsilon}$ into $W^+ \times [-1, 0)$ and the interior of the disk D'_ε into $W^+ \times (0, 1]$, which is equivalent to $F(D' \times I)$ in S^4 . Thus, $F(D' \times I)$ is a ribbon surface-link. This completes the proof of Lemma 2.1.

The following lemma concerns a replacement of a spanning arc of a surface-link.

Lemma 2.2. Let F be a surface-link in S^4 , and α_0 a simple arc spanning F in S^4 meeting F only with the endpoints. Then every simple arc α spanning F in S^4 meeting F only with the same endpoints as α_0 is F -relatively isotopic to an arc obtained from the arc α_0 by band summing with a meridian loop system m of F in S^4 .

Proof of Lemma 2.2. After a slight move of the interior of α , the union $\alpha_0 \cup \alpha$ bounds a disk D in S^4 . The interior of the disk D transversely meets F in a finite point set $Q = \{q_1, q_2, \dots, q_n\}$, whose regular neighborhood in D is an unoriented meridian disk system $N(Q) = \{N(q_1), N(q_2), \dots, N(q_n)\}$ of F in S^4 . The arc α is ∂ -relatively isotopic in D to a band sum of the arc α_0 and the unoriented meridian loop system $m = \partial N(Q) = \{m_1, m_2, \dots, m_n\}$, $m_i = \partial N(q_i)$, along a band system $b = \{b_1, b_2, \dots, b_n\}$ in D , where b_i spans α and m_i . This completes the proof of Lemma 2.2.

The following corollary is obtained from Lemma 2.2 by replacing the spanning arcs α_0 and α with the 1-handles h_0 and h on F with core arcs α_0 and α , respectively.

Corollary 2.3. Let F be a surface-link in S^4 , and h_0 a 1-handle on F . Then the surface-link $F(h)$ for every 1-handle h on F with the same attaching part as h_0 is isotopic to the surface-link obtained from a surface-link of the surface-link component $F(h_0)$ and a trivial S^2 -link component O in S^4 by surgery along a fusion 1-handle system h^O joining $F(h_0)$ with O .

Proof of Corollary 2.3. In Lemma 2.2, let $B = \{B_1, B_2, \dots, B_n\}$ be a disjoint 3-ball system in S^4 attaching to F with a disk system $d = \{d_1, d_2, \dots, d_n\}$ and meeting the meridian loop system m transversely such that $m \cap B = \{t_1, t_2, \dots, t_n\}$ with $t_i = m_i \cap (B_i \setminus \partial B_i)$. The thickening 1-handle h of α meets B_i with a transverse disk centered at t_i . Deform the arc α into the arc α_0 by taking the 3-ball system B with

it along the band system b . Then the 1-handle h is deformed into the 1-handle h_0 thickening the arc α_0 by taking the 3-ball system B with it. This means that the surface-link $F(h)$ is equivalent to a surface-link obtained from a surface-link of the surface-link component $F(h_0)$ and a trivial S^2 -link component $O = \partial B$ by surgery along a 1-handle system h^O consisting of a 1-handle h_i^O made by thickening and stretching the disk d_i along the band b_i , which joins the sphere component $O_i = \partial B_i$ to F ($i = 1, 2, \dots, n$). This completes the proof of Corollary 2.3.

3. Proof of Theorem 1.1

Throughout this section, the proof of Theorem 1.1 is done. Let $F = F_1 \cup F_2 \cup F_{3*}$ be a boundary surface-link of surface-knot components F_i ($i = 1, 2$) and a surface-link component F_{3*} , and h a pairwise nontrivial fusion 1-handle on F spanning F_i ($i = 1, 2$). If it can be shown that if the surface-link $F(h)$ is a ribbon surface-link then F is a ribbon surface-link, then the proof of Theorem 1.1 is complete by inductive argument. For a disjoint Seifert hypersurface V_* of F with $V_* = V_{1,2} \cup V_{3*}$ where $V_{1,2} = V_1 \cup V_2$, $\partial V_i = F_i$ and V_{3*} is a disjoint Seifert hypersurface system of F_{3*} , let β be a 1-handle system on F of 1-handle systems β_i on F_i embedded in V_i ($i = 1, 2, 3*$) such that $V_i(\beta) = \text{cl}(V_i \setminus \beta_i)$ is a handlebody for $i = 1, 2$ and a handlebody system for $i = 3*$, [10]. The surface-link $F(\beta) = F_{1,2}(\beta) \cup F_{3*}(\beta)$, where $F_i(\beta_i)$ is denoted by $F_i(\beta)$ and $F_{1,2}(\beta) = F_1(\beta) \cup F_2(\beta)$, is a trivial surface-link in S^4 since it bounds the handlebody system $V_*(\beta) = V_{1,2}(\beta) \cup V_{3*}(\beta)$ with $V_{1,2}(\beta) = V_1(\beta) \cup V_2(\beta)$. Let D_β be a transverse disk system of the 1-handle system β with one disk for every 1-handle of β , and k_β the boundary loop system of D_β in $F(\beta)$. Let ℓ_β be a spin loop system on $F(\beta)$ in S^4 such that the pair (k_β, ℓ_β) is a spin loop pair system, namely a system of disjoint oriented simple loop pairs of intersection number $+1$, on $F(\beta)$ in S^4 . It is known that for a trivial surface-knot T in S^4 , every spin loop pair system on T extends to a spin loop basis of T , namely a spin loop pair system of T representing a basis of $H_1(T; \mathbb{Z})$, and for any two spin loop bases $(k, \ell), (k', \ell')$ of T in S^4 , there is an orientation-preserving diffeomorphism of S^4 keeping T setwise fixed and sending (k, ℓ) to (k', ℓ') , [11, 12]. Further, for a handlebody system V smoothly embedded in S^4 , it is also known that the handlebody system V with any given spin loop basis (k, ℓ) such that the loop system k is a meridian loop system, namely the boundary of a meridian disk system of V is isotopic to a standard handlebody system V_0 in S^3 with a standard meridian-longitude loop basis (k_0, ℓ_0) , cite11,12, 13. By these properties, the handlebody system $V_*(\beta)$ is replaced by a handlebody system $V_*(\beta)' = V_{1,2}(\beta)' \cup V_{3*}(\beta)'$, where $V_{1,2}(\beta)' = V_1(\beta)' \cup V_2(\beta)'$, with the same surface-link $F(\beta)$ which admits a loop pair system (k_β, ℓ_β) such that the loop system k_β bounds a disjoint disk system D'_β in $V_*(\beta)'$. For $i = 1, 2$, let B_i be the 3-ball of a decomposition $B_i \cup V_i(\beta)''$ of $V_i(\beta)'$ along a proper disk d_i . The 3-ball system $B_* = \{B_1, B_2\}$ is

assumed to be disjoint from the loop pair system (k_β, ℓ_β) . Since h is a pairwise nontrivial fusion 1-handle on F , the 1-handle h on F is taken as a nontrivial fusion 1-handle on the trivial S^2 -link $L = L_1 \cup L_2$ with $L_i = \partial B_i$ such that L is disjoint from the handlebody system $V_{1,2}(\beta)'' \cup V_{3*}(\beta)'$, where $V_{1,2}(\beta)'' = V_1(\beta)'' \cup V_2(\beta)''$. Let $W(L; h) = B_*(0) \cup h$ be a SUPH system for the nontrivial ribbon S^2 -knot $L(h)$, where $B_*(0)$ denotes the system of compact once-punctured 3-balls $B_i(0)$ of B_i ($i = 1, 2$). Then there is a SUPH system $W' = W'_{1,2} \cup V_{3*}(\beta)'$ in S^4 for the surface-link $F(\beta)(h)$ such that $W'_{1,2} = W(L; h) \cup V_{1,2}(\beta)''$ is a SUPH system for the surface-knot $F_{1,2}(\beta)(h) = (F_1 \cup F_2)(h)$. By regarding $\partial V_i(\beta)''$ as $F_i(\beta)$ ($i = 1, 2$) and by considering the handlebody system $V_{1,2}(\beta)'' \cup V_{3*}(\beta)'$ as its spine graph system, the following claim is shown.

(3.2.1) The SUPH system W' is the split union of $W'_{1,2}$ and $V_{3*}(\beta)'$ in S^4 with $W'_{1,2}$ the disk sum of $W(L; h)$ and $V_i(\beta)''$ ($i = 1, 2$) obtained from their split union pasting along a 1-handle system thickening and stretching the disk system $d = \{d_1, d_2\}$ in $L(h)$ disjoint from h .

By assumption, the surface-link $F(h)$ is a ribbon surface-link. Let $W(F(h))$ be a SUPH system for $F(h)$. If necessary, by replacing $W(F(h))$ with a compact multi-punctured manifold of $W(F(h))$, the union $W = W(F(h)) \cup \beta$ is a SUPH system for the surface-link $F(\beta)(h)$. As it is explained in Appendix, there is an orientation-preserving diffeomorphism

$$f : (S^4, W') \rightarrow (S^4, W_{**})$$

for some multi-fusion SUPH system W_{**} of some multi-fission SUPH system W_* of the SUPH system W . Technically, the handlebody system component $V_{3*}(\beta)'$ of W' in (3.2.1) should be considered as a system of their compact punctured handlebodies by a 3-ball system, but it is ultimately filled by the removed 3-ball system. The SUPH system $W'_{1,2}$ in (3.2.1) is isotopically deformed in S^4 into a 3-manifold $W'_{1\#2}$ obtained by a disk sum of $W(L; h)$ and a handlebody $V_{1\#2}(\beta)''$ disk summing the handlebody system $V_{1,2}(\beta)''$. Let $F_{1\#2}(\beta) = \partial V_{1\#2}(\beta)''$ which is a connected sum of the trivial surface-knots $F_i(\beta)$ ($i = 1, 2$). Let $W'_{\#}$ be the split union of $W'_{1\#2}$ and $V_{3*}(\beta)'$, which is isotopically deformed from W' in S^4 . The preimage $(f^{-1}(k_\beta), f^{-1}(\ell_\beta))$ of the spin loop pair system (k_β, ℓ_β) on $F(\beta)(h)$ by f is a spin loop spin loop pair system on $F(\beta)(h)$, which is deformed to meet the 1-handle h only in simple lines parallel to the core arc α of h in W' . Thus, the spin loop pair system $(f^{-1}(k_\beta), f^{-1}(\ell_\beta))$ is assumed to belong to the split union $F_{1\#2}(\beta) \cup F_{3*}(\beta)$. By using that the loop system $f^{-1}(k_\beta)$ bounds a disk system in the handlebody system $V_{1\#2}(\beta)'' \cup V_{3*}(\beta)'$, an orientation-preserving diffeomorphism σ of S^4 preserving $V_{1\#2}(\beta)'' \cup V_{3*}(\beta)'$ setwise and sending (k_β, ℓ_β) to

$(f^{-1}(k_\beta), f^{-1}(\ell_\beta))$. Then this diffeomorphism σ of S^4 is deformed into an orientation-preserving diffeomorphism σ' of S^4 preserving W' setwise, keeping $W(L; h)$ fixed and sending (k_β, ℓ_β) to $(f^{-1}(k_\beta), f^{-1}(\ell_\beta))$. By replacing the composite $f\sigma'$ of S^4 with f , the following claim is shown.

(3.2.2) There is a diffeomorphism f' of S^4 sending W' to W_{**} whose restriction to the spin loop pair system (k_β, ℓ_β) is the identity map.

By (3.2.2), the handlebody systems $f(V_i(\beta)'')$ ($i = 1, 2$) and $f(V_{3*}(\beta)')$ in W_{**} are identified with $V_i(\beta)''$ ($i = 1, 2$) and $V_{3*}(\beta)'$ with the same spin loop pair system (k_β, ℓ_β) , respectively. For the image $f(W(L; h))$, note that $W(L; h)$ is a compact 2-punctured 3-ball. Then every chord α in $W(L; h)$ spanning the 2-component trivial S^2 -link $O^2 = \partial W(L; h) \setminus L(h)$ is unique up to isotopies of $W(L; h)$. Further, the compact complement $\text{cl}(W(L; h) \setminus N)$ for a regular neighborhood N of the union $O^2 \cup \alpha$ in $W(L; h)$ is diffeomorphic to $L(h) \times [0, 1]$. This means that $W(L; h)$ is uniquely obtained by the union $O^2 \cup \alpha$, and hence, by taking f invariant for O^2 , the image $f(W(L; h))$ is also determined by the union $O^2 \cup \alpha$. Thus, $f(W(L; h))$ is identified with $W(L; h)$. Let D_h be a transverse disk of the 1-handle h , and $k_h = \partial D_h$. Then the disk D_h is contained in W_{**} . Since the SUPH system W_* for $F(\beta)(h)$ is obtained from a multi-punctured 3-manifold $W_{**}^{(0)}$ of W_{**} by adding 2-handles attaching to the trivial S^2 -link $\partial W_{**}^{(0)} \setminus F(\beta)(h)$, the following claim is shown.

(3.2.3) The 2-handle core disk D_h and the 2-handle core disk system D_β are contained in the SUPH system W_* for the surface-link $F(\beta)(h)$.

Since k_h and k_β are disjoint and in the boundary of W_* , the transverse intersection $D_h \cap D_\beta$ consists of simple loops in W_* . A 2-handle core disk D_h^* in W_* is obtained from D_h with $D_h^* \cap D_\beta = \emptyset$ by cut and paste operations. The compact 3-manifold obtained from W_* by splitting along the disk system $D_h^* \cup D_\beta$ is a SUPH system of the surface-link G obtained from $F(\beta)(h)$ by surgery along the 2-handle system $D_h^* \times I \cup D_\beta \times I$, so that G is a ribbon surface-link. Let $W_* \times [0, 1]$ be a collar of W_* in S^4 with $W_* \times \{0\} = W_*$. Let $D_\beta^+ = k_\beta \times [0, 1] \cup D_\beta \times \{1\}$ be a disk system in $W_* \times [0, 1]$ which is the core disk of a 2-handle system $D_\beta^+ \times I$ on $F(\beta)(h)$. Since the 2-handle system $D_h^* \times I \cup D_\beta \times I$ on $F(\beta)(h)$ is equivalent to the 2-handle system $D_h^* \times I \cup D_\beta^+ \times I$ on $F(\beta)(h)$, the surface-link G^+ obtained from $F(\beta)(h)$ by surgery along the 2-handle system $D_h^* \times I \cup D_\beta^+ \times I$ is equivalent to the ribbon surface-link G . The 2-handle core disk D_h in W_* is obtained from the 2-handle core disk D_h^* in W_* by a SUPH move. By Lemma 2.1, the surface-link F^+ obtained from $F(\beta)(h)$ by surgery along the 2-handle system $D_h \times I \cup D_\beta^+ \times I$ is a ribbon surface-link. The

2-handle system $D_h \times I \cup D_\beta^+ \times I$ on $F(\beta)(h)$ is equivalent to the 2-handle system $D_h \times I \cup D_\beta \times I$ on $F(\beta)(h)$, the surface-link F is a ribbon surface-link since it is equivalent to the ribbon surface-link F^+ . This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

The proof of Theorem 1.2 is done as follows.

4.1: Proof of Theorem 1.2. Assume that h is a nontrivial fusion 1-handle on the anti-parallel surface-link $P(F)$. Since $P(F)$ is a boundary surface-link, if $P(F)$ is a ribbon surface-knot, then $P(F)$ is a ribbon surface-link by Theorem 1.1, contradicting the assumption that F is a non-ribbon surface-knot. Thus, $P(F)(h)$ is a non-ribbon surface-knot. Next, assume that h is a trivial fusion 1-handle on $P(F)$. Let $P(F) = \partial(F \times [0, 1]) = F_0 \cup F_1$ with $F_i = F \times \{i\}$, and $d_* = \{d_0, d_1\}$ with $d_i = d \times \{i\}$ for a disk d in F the attaching disk system of h to $P(F)$. Let $h_0 = d \times [0, 1]$ be the trivial 1-handle on $P(F)$. For a Seifert hypersurface V for F , let β be a 1-handle system on F embedded in V such that $V(\beta) = \text{cl}(V \setminus \beta)$ is a handlebody. Let $V_* = V_0 \cup V_1$ with $V_i = V \times \{i\}$. Let $V_*(\beta) = V_0(\beta) \cup V_1(\beta)$ with $V_i(\beta) = V(\beta) \times \{i\}$ be a handlebody system for the trivial surface-link $P(F)(\beta_*)$ for the 1-handle system $\beta_* = \beta_0 \cup \beta_1$ with $\beta_i = \beta \times \{i\}$ on $P(F)$. Let D_{β_*} be a transverse disk system of the 1-handle system β_* with just one disk for every 1-handle. The disjoint Seifert hypersurface V_* for $P(F)$ is obtained from the handlebody system $V_*(\beta)$ by adding the 2-handle system $D_{\beta_*} \times I$. Let α and α_0 be the core arcs with the same endpoints of the 1-handles h and h_0 on $P(F)$, respectively. Since h is a trivial fusion 1-handle, the interior of the arc α is disjoint from $V_*(\beta)$ by regarding it as the union of a 3-ball system B_* and an attaching graph, and disjoint from the 2-handle system $D_{\beta_*} \times I$. Only the difference between the arc α and the arc α_0 which are $V_*(\beta)$ -relatively isotopic in S^4 is that α is obtained from α_0 by band summing a meridian loop system m of D_{β_*} in S^4 . Deform α into α_0 by taking the trivial S^2 -link around α_0 connected to D_{β_*} via a fusion 1-handle system, as in Lemma 2.2 and Corollary 2.3. The 2-handle system $D_{\beta_*} \times I$ on the surface-knot $P(F)(\beta_*)(h_0)$ changes into a 2-handle system $D'_{\beta_*} \times I$ on $P(F)(\beta_*)(h_0)$ with the same attached part by surgery along this fusion 1-handle system. For every 2-handle $D \times I$ of $D_{\beta_*} \times I$ on $P(F)(\beta_*)(h_0)$ and the corresponding 2-handle $D' \times I$ of $D'_{\beta_*} \times I$, there is a common 2-handle $E \times I$ on $P(F)(\beta_*)(h_0)$ constructed in the handlebody $\text{cl}(F \setminus d) \times [0, 1]$ bounded by the surface-knot $P(F)(\beta_*)(h_0)$ such that $(D \times I, E \times I)$ and $(D' \times I, E \times I)$ are O2-handle pairs on $P(F)(\beta_*)(h_0)$ with the same attached parts. By uniqueness of O2-handle pair, the surface-knot obtained from $P(F)(\beta_*)(h_0)$ by surgery along the 2-handle $D \times I$ to equivalent to the surface-knot obtained from $P(F)(\beta_*)(h_0)$ by surgery along the 2-handle $D' \times I$, [11, 13]. By continuing this procedure, the surface-knot $P(F)(h)$

which is obtained from the surface-knot $P(F)(\beta_*)(h_0)$ by surgery along the 2-handle $D'_{\beta_*} \times I$ is equivalent to the surface-knot obtained from the surface-knot $P(F)(\beta_*)(h_0)$ by surgery along the 2-handle $D_{\beta_*} \times I$, which is the trivial surface-knot $P(F)(h_0)$. Thus, the surface-knot $P(F)(h)$ is a trivial surface-knot. This completes the proof of Theorem 1.2.

The proof of Corollary 1.3 is done as follows.

4.2 Proof of Corollary 1.3. By taking $P(F)$ for F in Corollary 2.3, the desired result is obtained from Theorem 1.2 and Corollary 2.3, completing the proof of Corollary 1.3.

5. Appendix: Moves on the SUPH systems of a ribbon surface-link

There are three kinds of moves M_0, M_1, M_2 on a chord diagram $C = C(o, \alpha)$ for equivalence of a ribbon surface-link F , which are called the *Reidemeister move*, the *fusion-fission move*, and the *chord move*, respectively, [2, 14]. The fusion-fission move M_1 consists of the fusion move $M_1(\text{fusion})$ decreasing the number of based loops and the fission move $M_1(\text{fission})$ increasing the number of based loops. For a SUPH system W' obtained from a SUPH system W by applying the move M_0 or M_2 , there is an orientation-preserving diffeomorphism of S^4 sending W to W' . If the fusion move $M_1(\text{fusion})$ or its iteration gives a SUPH system W' from a SUPH system W , then W' is obtained from W by removing a 1-handle or a 1-handle system on O in W and called a *fusion SUPH system* or a *multi-fusion SUPH system* of W , respectively. If the fission move $M_1(\text{fission})$ or its iteration produces a SUPH system W' from a SUPH system W , then W' is obtained from W by attaching a 2-handle or a 2-handle system on O in W and called a *fission SUPH system* or a *multi-fission SUPH system* of W , respectively. If a 2-handle h_2 on a component O_1 of O is used for a fission SUPH system W' of the SUPH system W , then there is a 3-ball A_1 with $\partial A_1 = O_1$ in S^4 such that W meets the interior of A_1 only with a disjoint proper 2-disk system d_1 of W and the 2-handle h_2 is embedded in $A_1 \setminus d_1$. This property is obtained from the following proposition.

Proposition 5.1. If a 1-handle h_1 on a trivial S^2 -link O of r components in S^4 produces a trivial S^2 -link O' of $r - 1$ components by surgery, then the 1-handle h_1 is a trivial 1-handle on O , that is, a 1-handle not meeting the interior of a 3-ball system bounded by O .

This proposition is shown by the conjugacy theorem for the one relator Wirtinger presentation, [15]. Note that this method detects classification of 1-handles, but does

not classify 1-fusion ribbon S^2 -links. By this proposition, the fusion move $M_1(\text{fusion})$ and the fission move $M_1(\text{fission})$ are dual concepts on the SUPH systems for a ribbon surface-link. For any two SUPH systems W and W' for a ribbon surface-link F , it is known that W' is obtained from W by a finite number of the moves M_i ($i = 0, 1, 2$), [14]. By identifying two SUPH systems sent by an orientation-preserving diffeomorphism of S^4 , it may be considered that W' is obtained from W by a finite number of the fusion move $M_1(\text{fusion})$ and/or the fission move $M_1(\text{fission})$. The ordering of these moves is arranged so that a finite number of the fusion move $M_1(\text{fusion})$ are performed after a finite number of the fission move $M_1(\text{fission})$ are performed. This means the following proposition.

Proposition 5.2. For any two SUPH systems W and W' of a ribbon surface-link F with $\partial W \setminus F \neq \emptyset$ and $\partial W' \setminus F \neq \emptyset$, a multi-fusion SUPH system W_{**} of a multi-fission SUPH system W_* of W is sent to the SUPH system W' by an orientation-preserving diffeomorphism of S^4 . In other words, a multi-fission SUPH system W_* of W is sent to a multi-fusion SUPH system W'_* of W' by an orientation-preserving diffeomorphism of S^4 .

Conclusion

The original question is when a surface-link F of ribbon surface-knot components becomes a ribbon surface-link, [4]. In the case of the surface-links with at least two aspheric components, there are examples of non-ribbon surface-links F of ribbon components (containing non-ribbon surface-links of trivial components), [5, 6]. Theorem 1.1 shows that if F is a boundary surface-link of ribbon components and there is a pairwise nontrivial fusion 1-handle system h on F with $F(h)$ a ribbon surface-link, then F is a ribbon surface-link. By Corollary 1.3, even such surface-link of trivial components containing at most one aspheric component requires further conditions to be a ribbon surface-link. A revision paper, [6].

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