Orthogonal 2-sphere basis of stable 4-sphere

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ABSTRACT

It is known by Wall that any two orthogonal bases of every stable 4-sphere are transformed into each other by an orientation-preserving diffeomorphism of the stable 4-sphere. In this paper another proof of Wall's result is presented, strengthened in the sense that the diffeomorphism is taken as the lift of an equivalence of a trivial surface-knot space to the double branched covering space identified with the stable 4-sphere.

Keywords: Stable 4-sphere, Trivial surface-knot, O2-handle basis, Branched covering, Orthogonal basis, O2-sphere basis.

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1. Introduction

The *stable 4-sphere of genus* n is the connected sum 4-manifold $\Sigma = \Sigma(n)$ of n copies of $S^2 \times S^2$, which is canonically diffeomorphic to the double branched covering space $S^4(F)_2$ of the 4-sphere S^4 branched along a trivial surface-knot F of genus n in S^4, [1]. The pair (S^4, F) is called a *trivial surface-knot space* of genus n. It is known by Wall that any two orthogonal bases of every stable 4-sphere Σ are transformed into each other by an orientation-preserving diffeomorphism σ of the stable 4-sphere Σ, [2]. In this paper another proof of Wall's result is presented, strengthened in the sense that the diffeomorphism σ is taken as the lift f' of an *equivalence* f of a trivial surface-knot space (S^4, F), namely an orientation-preserving diffeomorphism f of (S^4, F) to the stable 4-sphere $S^4(F)_2 = \Sigma$. This paper is a revised version of an incorrect claim [3, Lemma 3.1] that was originally intended for use in the paper [1], which was written instead using Wall's result, [1, Conclusion]. An *orthogonal basis* of the stable 4-sphere Σ is a basis $(x_*, x'_*) = \{x_i, x'_i \,|\, i = 1, 2, \ldots, n\}$ of the second integral homology group $H_2(\Sigma; Z)$ which is a free abelian group of rank $2n$ with the intersection numbers

$$\mathrm{Int}(x_i, x_j) = \mathrm{Int}(x'_i, x'_j) = \mathrm{Int}(x'_i, x_j) = \mathrm{Int}(x_i, x'_j) = 0$$

for all i, j except for that $\mathrm{Int}(x_i, x'_i) = \mathrm{Int}(x'_i, x_i) = 1$ for all i. The following theorem is a main result of this paper.

Theorem 1.1. For any two orthogonal bases (x_*, x'_*) and (y_*, y'_*) of the stable 4-sphere Σ of genus n, there is an equivalence f of a trivial surface-knot space (S^4, F) of genus n in S^4 whose lift f' to $S^4(F)_2 = \Sigma$ induces $f'_*(x_i) = y_i$ and $f'_*(x'_i) = y'_i$ for all i.

A 2-*handle* on a trivial surface-knot F of genus n in S^4 is a 2-handle $D \times I$ on F embedded in S^4 such that

$$(D \times I) \cap F = (\partial D) \times I,$$

where I denotes a closed interval with 0 as the center and $D \times 0$ is called the *core disk* of the 2-handle $D \times I$ and identified with D. An *orthogonal 2-handle pair* or simply an *O2-handle pair* on F in S^4 is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I$ and $D' \times I$ on F which *meet orthogonally* on F, in other words, which meet F only with the attaching annuli $(\partial D) \times I$ and $(\partial D') \times I$ so that the loops ∂D and $\partial D'$ meet transversely at just one point q and the intersection $(\partial D) \times I \cap (\partial D') \times I$ is diffeomorphic to the square $Q = \{q\} \times I \times I$, [4]. For a trivial surface-knot F of genus n in S^4, an O2-*handle basis* of F is a system $(D_* \times I, D'_* \times I)$ of mutually disjoint O2-handle pairs $(D_i \times I, D'_i \times I)$ $(i = 1, 2, \ldots, n)$ on F in S^4. A *loop basis* of F is a pair system (e_*, e'_*) of oriented simple loop pairs (e_i, e'_i) $(i = 1, 2, \ldots, n)$ on F representing a basis for $H_1(F; Z)$ such that $e_i \cap e_j = e'_i \cap e'_j = e_i \cap e'_j = \emptyset$ for all distinct i, j and $e_i \cap e'_i$ is one point with the intersection number $\mathrm{Int}(e_i, e'_i) = +1$ in F for all i. A loop basis (e_*, e'_*) of F is *spin* if the Z_2-quadratic function $q : H_1(F; Z) \to Z_2$ associated with the surface-knot F in S^4 has $q(e_i) = q(e'_i) = 0$ for all i. The loop pair system $(\partial D_*, \partial D'_*)$ given by $\{(\partial D_i, \partial D'_i) | \, i = 1, 2, \ldots, n\}$ is a spin loop basis of F. Every trivial surface-knot F of genus n in S^4 is moved into the boundary $F_H$of a standard handlebody H of genus n in the equatorial 3-sphere S^3 of S^4, where a *standard* O2-handle basis and a *standard* loop basis of F_H are taken. Any two spin loop bases on F are transformed into each other by an equivalence of (S^4, F), [4, (2.5.1), (2.5.2)], [5]. Hence every spin loop basis on F bounds the core disk pair system of an O2-handle basis on F. Further, the core disk pair systems of any two O2-handle bases on F bounded by the same loop basis are transformed into each other by an equivalence of (S^4, F), [6]. For the double branched covering projection $p : S^4(F)_2 \to S^4$ branched along F, the non-trivial covering involution of $S^4(F)_2$ is denoted by α. The *standard O2-sphere basis* of the stable 4-sphere Σ of genus n is the 2-sphere pair system $(S^2 \times 1_*, 1 \times S^2_*)$ of $(S^2 \times 1_i, 1 \times S^2_i)\,(i = 1, 2, \ldots, n)$ in Σ. An *O2-sphere basis* of Σ is a 2-sphere pair system (S_*, S'_*) in Σ sent to the standard O2-sphere basis $(S^2 \times 1_*, 1 \times S^2_*)$ by an orientation-preserving diffeomorphism of Σ. The preimage $p^{-1}(F)$ in Σ of F is the fixed point set of α, diffeomorphic to F and written by the same notation as F. For every O2-handle basis $(D_* \times I, D'_* \times I)$ on F in S^4, it is shown that there is an orientation-preserving diffeomorphism $\iota : S^4(F)_2 \to \Sigma$ sending the 2-sphere pair system $(S(D_*), S(D'_*))$ with $S(D_i) = D_i \cup \alpha D_i$ and $S(D'_i) = D'_i \cup \alpha D'_i\,(i = 1, 2, \ldots, n)$ to $(S^2 \times 1_*, 1 \times S^2_*)$, [1]. Thus, $(S(D_*), S(D'_*))$ is an O2-sphere basis of Σ. A key to show Theorem 1.1 is the following lemma.

Lemma 1.2 (Key Lemma). For every orthogonal basis (x_*, x'_*) of Σ, there is an O2-handle basis $(E_* \times I, E'_* \times I)$ on a trivial surface-knot F of genus n in S^4 such that $x_i = [S(E_i)]$ and $x'_i = [S(E'_i)]$ for all i.

The proof of Theorem 1.1 assuming Lemma 1.2 is done as follows.

Proof of Theorem 1.1 assuming Lemma 1.2. By Lemma 1.2, let $x_i = [S(D_i)], x'_i = [S(D'_i)]$ and $y_i = [S(E_i)], x'_i = [S(E'_i)]$ f or O2-handle bases $(D_* \times I, D'_* \times I)$ and $(E_* \times I, E'_* \times I)$ on F. Then there is an equivalence g of (S^4, F) sending the spin loop basis $(\partial D_*, \partial D'_*)$ of F to the spin loop basis $(\partial E_*, \partial E'_*)$ of F, [4], [5]. On the O2-handle bases $(gD_* \times I, gD'_* \times I)$ and $(E_* \times I, E'_* \times I)$ on F with the same attaching part in F, there is an equivalence g' of (S^4, F) such that $(g'gD_* \times I, g'gD'_* \times I) = (E_* \times I, E'_* \times I)$, [6]. The composite equivalence $f = g'g$ of (S^4, F) lifts to a diffeomorphism f' of $\Sigma = S^4(F)_2$ sending $(x_i, x'_i) = ([S(D_i)], [S(D'_i)])$ to $(y_i, y'_i) = ([S(E_i)], [S(E'_i)])$ for all i. This completes the proof of Theorem 1.1 assuming Lemma 1.2.

2. Proof of Key Lemma (Lemma 1.2)

For a disk D, let $D^o = D \setminus \partial D$. The following lemma gives a basic information on the intersection numbers of the lifting O2-sphere bases of two O2-handle bases of F in S^4, which corrects a computation error of [3, Lemma 3.1].

Lemma 2.1. Let $(k_*, k'_*) = (\partial D_*, \partial D'_*)$ be the spin loop basis of F. For a 2-handle $E \times I$ on F in S^4, assume that the homology class $[e]$ of the simple loop $e = \partial E$ in F is given by the intersection numbers $\mathrm{Int}([e], [k'_j]) = s_j$ and $\mathrm{Int}([e], [k_j]) = s'_j$ in F for some integers s_j, s'_j $(j = 1, 2, \ldots, n)$. Then the homology class $[S(E)]$ in Σ is given by

$$[S(E)] = \sum_{j=1}^{n}(s_j + 2m_j)[S(D_j)] + \sum_{j=1}^{n}(s'_j + 2m'_j)[S(D'_j)],$$

where m_j and m'_j are integers given by the intersection numbers $\mathrm{Int}(E^o, (D'_j)^o)$ and $\mathrm{Int}(E^o, D^o_j)$, respectively.

Proof of Lemma 2.1. Let $N(F)^c = \mathrm{cl}(S^4 \setminus N(F))$ for a regular neighborhood $N(F)$ of F in S^4. Consider that the disk $E^e = E \cap N(F)^c$ transversely meets the disks

$$D^e_j = D_j \cap N(F)^c, \quad D'^e_j = D'_j \cap N(F)^c \quad (j = 1, 2, \ldots, n)$$

with the intersection points $E^o \cap {D_j}^o$ and $E^o \cap (D'_j)^o$ for all i, respectively. The intersection number of the lift of E^e and the lift of D^e_j to $S^4(F)_2 = \Sigma$ is equal to

$$\mathrm{Int}(E^e, D^e_j) + \mathrm{Int}(\alpha E^e, \alpha D^e_j)) = 2\mathrm{Int}(E^e, D^e_j) = 2\mathrm{Int}(E^o, (D_j)^o) = 2m'_j.$$

Similarly, the intersection number of the lift of E^e and the lift of D'^e_j to $S^4(F)_2 = \Sigma$ is equal to

$$\mathrm{Int}(E^e, D'^e_j j) + \mathrm{Int}(\alpha E^e, \alpha D'^e_j)) = 2\mathrm{Int}(E^e, D'^e_j) = 2\mathrm{Int}(E^o, (D')^o) = 2m_j.$$

By using the intersection numbers $\mathrm{Int}([e], [k_j]) = s'_j$ and $\mathrm{Int}([e], [k'_j]) = s_j$ in F and examining the geometric intersections between the lift of $E \cap N(F)$ and the lifts of $D_j \cap N(F)$, $D'_j \cap N(F)$ to $S^4(F)_2 = \Sigma$, it is seen that

$$\mathrm{Int}([S(E)], [S(D_j)]) = s'_j + 2m'_j \quad \text{and} \quad \mathrm{Int}([S(E)], [S(D'_j)]) = s_j + 2m_j.$$

Since $([S(D_*)], [S(D'_*)])$ is an orthogonal basis of Σ, the desired identity is obtained. This completes the proof of Lemma 2.1.

By using Lemma 2.1, the following lemma is obtained.

Lemma 2.2. For every orthogonal basis (x_*, x'_*) of Σ, there is an O2-handle basis $(E_* \times I, E'_* \times I)$ on the trivial surface-knot F of genus n in S^4 such that $x_i = u_i[S(E_i)] + 2A_i + 2A'_i$ and $x'_i = u'_i[S(E'_i)] + 2B_i + 2B'_i$ for all i, where u_i, u'_i are odd integers and

$$A_i = \sum_{j=1}^{n} a_{ij}[S(E_j)], \;\; A'_i = \sum_{j=1}^{n} a'_{ij}[S(E'_j)], \;\; B_i = \sum_{j=1}^{n} b_{ij}[S(E_j)], \;\; B'_i = \sum_{j=1}^{n} b'_{ij}[S(E'_j)],$$

for some integers a_{ij}, a'_{ij}, b_{ij}, b'_{ij} with $a_{ii} = b'_{ii} = 0$ for all i.

Proof of Lemma 2.2. Represent x_i and x'_i as integral linear combinations of the homology classes on $[S(D_*)]$ and $[S(D'_*)]$ such that

$$x_i = \sum_{j=1}^{n} c_{ij}[S(D_j)] + \sum_{j=1}^{n} c'_{ij}[S(D'_j)], \quad x'_i = \sum_{j=1}^{n} d_{ij}[S(D_j)] + \sum_{j=1}^{n} d'_{ij}[S(D'_j)]$$

for all i. Then note that

$$\sum_{j=1}^{n} c_{ij}c'_{ij} = \sum_{j=1}^{n} d_{ij}d'_{ij} = 0, \quad \sum_{j=1}^{n} (c_{ij}d'_{ij} + c'_{ij}d_{ij}) = 1$$

for all i. Let $(k_*, k'_*) = (\partial D_*, \partial D'_*)$ be a spin loop basis of F. Then there is a spin loop basis (ℓ_*, ℓ'_*) on F such that

$$[\ell_i] = \sum_{j=1}^{n} c_{ij}[k_j] + \sum_{j=1}^{n} c'_{ij}[k'_j], \quad [\ell'_i] = -\sum_{j=1}^{n} d_{ij}[k_j] + \sum_{j=1}^{n} d'_{ij}[k'_j]$$

in $H_1(F; Z)$. In fact, a loop basis (ℓ_*, ℓ'_*) on F with the identities above are constructed by a simple loop realization of a symplectic basis of F. Then the Z_2-quadratic function $q : H_1(F; Z) \to Z_2$ gives

$$q([\ell_i]) = \sum_{j=1}^{n} c_{ij}c'_{ij} = 0, \quad q([\ell'_i]) = \sum_{j=1}^{n} d_{ij}d'_{ij} = 0$$

for all i, showing that the loop basis (ℓ_*, ℓ') is a spin loop basis on F. Let $(E_* \times I, E'_* \times I)$ be an O2-handle basis of F with $(\partial E_*, \partial E'_*) = (\ell_*.\ell'_*)$. By Lemma 2.1,

$$[S(E_i)] = \sum_{j=1}^{n} (c_{ij} + 2m_{ij})[S(D_j)] + \sum_{j=1}^{n} (c'_{ij} + 2m'_{ij})[S(D'_j)],$$

$$[S(E'_i)] = \sum_{j=1}^{n} (d_{ij} + 2n_{ij})[S(D_j)] + \sum_{j=1}^{n} (d'_{ij} + 2n'_{ij})[S(D'_j)]$$

with some integers $m_{ij}, m'_{ij}, n_{ij}, n'_{ij}$ for all i, j, Thus,

$$x_i = [S(E_i)] - 2\sum_{j=1}^{n} m_{ij}[S(D_j)] - 2\sum_{j=1}^{n} m'_{ij}[S(D'_j)],$$

$$x'_i = [S(E'_i)] - 2\sum_{j=1}^{n} n_{ij}[S(D_j)] - 2\sum_{j=1}^{n} n'_{ij}[S(D'_j)]$$

for all i. The homology classes $[S(D_j)]$, $[S(D'_j)]$ are Z-linear combinations of the basis $([S(E_*)], [S(E'_*)])$ of $H_2(\Sigma; Z)$, the desired identities with some $u_i, u'_i, A_i, A'_i, B_i, B'_i$ are obtained. This completes the proof of Lemma 2.2.

Let S^4 be the one-point compactification of the 4-space R^4. For the 3-space R^3 and an interval $J \subset R$, the notation $R^3J = \{(x,t) \in R^4 | x \in R^3, t \in J\}$ is used. Consider the trivial surface-knot F as a standard surface in R^3 and the O2-handle basis $(D_* \times I, D'_* \times I)$ on F is embedded in R^3. Let D_i^+ and $(D'_i)^+$ be slightly extended disks of D_i and D'_i so that the boundary loops ∂D_i^+ and $\partial(D'_i)^+$ are disjoint from F and meet transversely D'_i, D_i at single points, respectively. A *2-sphere surrounding* D_i is a 2-sphere S_i in R^4 which is the boundary of the 3-ball $D_i^+[-2,1]$ in R^4, and a *2-sphere surrounding* D'_i is a 2-sphere S'_i in R^4 which is the boundary of the 3-ball $(D'_i)^+[-1,2]$ in S^4. The 2-spheres $S_i, S'_i\,(i = 1, 2, \ldots, n)$ are disjoint for distinct indexes i and meet transversely with just two points of opposite signs in $R^3[\pm 1]$ for the same index i. The 2-sphere pair (S_i, S'_i) in S^4 is called *Montesinos's twin*, [7]. The connected sum $D_i \# S_i$ is made along an arc β' in S'_i joining the intersection point $D_i \cap S'_i$ in $R^3[0]$ with the intersection point of $S_i \cap S'_i$ in $R^3[1]$. The 2-sphere S_i is oriented so that the disk $D_i \# S_i$ is oriented with the orientation inherited from D_i. Similarly, the connected sum $D'_i \# S'_i$ is made along an arc β in S_i joining the intersection point $D'_i \cap S_i$ in $R^3[0]$ with the intersection point of $S_i \cap S'_i$ in $R^3[-1]$. The 2-sphere S'_i is oriented so that the disk $D'_i \# S'_i$ is oriented with the orientation inherited from D'_i. To show Lemma 1.2, the following lemma is used.

Lemma 2.3. The homology classes $[S_i]$ and $[S'_i]$ in Σ are given by $[S_i] = -[S(D_i)]$, $[S'_i] = -[S(D'_i)]$ for all i.

Proof of Lemma 2.3. From construction, the self-intersection numbers of $[S_i]$ and $[S'_i]$ are zero and the intersection numbers $\mathrm{Int}([S(D'_i)], [S_i])$ and $\mathrm{Int}([S(D_i)], [S'_i])$ are ± 1. Since $[\alpha S(D_i)] = -[S(D_i)]$ and $\mathrm{Int}([S(D_i)], [S'_i]) = \mathrm{Int}([\alpha S(D_i)], [\alpha S'_i])$, the identity $[\alpha S'_i] = -[S'_i]$ is obtained. Similarly, since $[\alpha S(D'_i)] = -[S(D'_i)]$ and $\mathrm{Int}([S(D'_i)], [S_i]) = \mathrm{Int}([\alpha S(D'_i)], [\alpha S_i])$, the identity $[\alpha S_i] = -[S_i]$ is obtained. Let $(E_* \times I, E'_* \times I)$ be the O2-handle basis on F given by $(E_i, E'_i) = (D_i \# S_i, D'_i \# S'_i)$ and $(E_j, E'_j) = (D_j, D'_j)$ for $j \neq i$. Then the following identities are obtained in Σ.

$$[S(E_i)] = [S(D_i] + [S_i] - [\alpha S_i] = [S(D_i)] + 2[S_i],$$

$$[S(E'_i)] = [S(D'_i)] + [S'_i] - [\alpha S'_i] = [S(D'_i)] + 2[S'_i].$$

Then the identities $[S_i] = -[S(D_i)]$, $[S'_i] = -[S(D'_i)]$ in Σ are obtained by the identities $[S_i] = \pm[S(D_i)]$, $[S'_i] = \pm[S(D'_i)]$ and $\mathrm{Int}([S(E_i], [S(E'_i)]) = \mathrm{Int}([S(D_i], [S(D'_i)]) = 1$. This completes the proof of Lemma 2.3.

As a by-product of the proof of Lemma 2.3, an elementary transformation on an O2-handle basis on a trivial surface-knot in S^4 was found. In fact, the following elementary transformation on the O2-handle basis $(D_* \times I, D'_* \times I)$ on F is presented.

Operation I. An O2-handle basis $(D_* \times I, D'_* \times I)$ on F is changed into an O2-handle basis $(E_* \times I, E'_* \times I)$ on F with the same attachment as $(D_* \times I, D'_* \times I)$ such that

$$[S(E_i)] = -[S(D_i)], [S(E'_i)] = -[S(D'_i)] \text{ in } \Sigma \text{ and } (E_j, E'_j) = (D_j, D'_j), j \neq i,$$

for any given i.

Here are further elementary transformation s on the O2-handle basis $(D_* \times I, D'_* \times I)$ on F.

Operation II. Let $i \neq j$ and join the intersection point $v_i = D_i \cap S'_i$ with the intersection point $v'_j = D'_j \cap S_j$ by an arc γ in S^4 whose interior is disjoint from F and $(D_* \times I, D'_* \times I)$, where the 2-spheres S'_i and S_j are taken to be unoriented. Construct the connected sum pairs $(E_i, E'_i) = (D_i \# S_j, D'_i)$ and $(E_j, E'_j) = (D_j, D'_j \# S'_i)$ along the arc γ. For $h \neq i, j$, let $(E_h, E'_h) = (D_h, D'_h)$. Then $(E_* \times I, E'_* \times I)$ is an O2-handle basis on F with the same attachment as $(D_* \times I, D'_* \times I)$ such that

$$[S(E_i)] = [S(D_i] \pm 2[S(D_j)], [S(E'_j)] = [S(D'_j)] \mp 2[S(D'_i)] \text{ in } \Sigma \text{ and}$$

$$E'_i = D'_i, E_j = D_j, (E_h, E'_h) = (D_h, D'_h), h \neq i, j,$$

for any given distinct indexes i and j. In fact, since by Lemma 2.3 $[S(E_i)] = [S(D_i] + 2\varepsilon[S_j] = [S(D_i)] - 2\varepsilon[S(D_j)]$ and $[S(E'_j)]) = [S(D'_j)] + 2\varepsilon'[S'_i] = [S(D'_j)] - 2\varepsilon'[S(D'_i)]$ for signs $\varepsilon, \varepsilon'$ being ±1, the intersection number $\mathrm{Int}([S(E_i)], [S(E'_j)]) = 0$ implies $\varepsilon + \varepsilon' = 0$.

Operation III. Let $i \neq j$ and join the intersection point $v_i = D_i \cap S'_i$ with the intersection point $v'_j = D_j \cap S'_j$ by an arc γ in S^4 whose interior is disjoint from F and $(D_* \times I, D'_* \times I)$, where the 2-spheres S'_i and S'_j are taken to be unoriented. Construct the connected sum pairs $(E_i, E'_i) = (D_i \# S'_j, D'_i)$ and $(E_j, E'_j) = (D_j \# S'_i, D'_j)$ along the arc γ. For $h \neq i, j$, let $(E_h, E'_h) = (D_h, D'_h)$. Then $(E_* \times I, E'_* \times I)$ is an O2-handle basis on F with the same attachment as $(D_* \times I, D'_* \times I)$ such that

$$[S(E_i)] = [S(D_i)] \pm 2[S(D'_j)], [S(E_j)] = [S(D_j)] \mp 2[S(D'_i)] \text{ in } \Sigma \text{ and}$$

$$E'_i = D'_i, E'_j = D'_j, (E_h, E'_h) = (D_h, D'_h), h \neq i, j,$$

for any given distinct indexes i and j. In fact, since by Lemma 2.3 $[S(E_i)] = [S(D_i)] + 2\varepsilon[S'_j] = [S(D_i)] - 2\varepsilon[S(D'_j)]$ and $[S(E_j)]) = [S(D_j)] + 2\varepsilon'[S'_i] = [S(D_j)] - 2\varepsilon'[S(D'_i)]$ for signs $\varepsilon, \varepsilon'$ being ± 1, the intersection number $\mathrm{Int}([S(E_i)], [S(E_j)]) = 0$ implies $\varepsilon + \varepsilon' = 0$.

Operation IV. Let $i \neq j$ and join the intersection point $v_i = D'_i \cap S_i$ with the intersection point $v'_j = D'_j \cap S_j$ by an arc γ in S^4 whose interior is disjoint from F and $(D_* \times I, D'_* \times I)$, where the 2-spheres S_i and S_j are taken to be unoriented. Construct the connected sum pairs $(E_i, E'_i) = (D_i, D'_i \# S_j)$ and $(E_j, E'_j) = (D_j, D'_j \# S_i)$ along the arc γ. For $h \neq i, j$, let $(E_h, E'_h) = (D_h, D'_h)$. Then $(E_* \times I, E'_* \times I)$ is an O2-handle basis on F with the same attachment as $(D_* \times I, D'_* \times I)$ such that

$$[S(E'_i)] = [S(D'_i)] \pm 2[S(D_j)], [S(E'_j)] = [S(D'_j)] \mp 2[S(D_i)] \text{ in } \Sigma \text{ and}$$

$$E_i = D_i, E_j = D_j, (E_h, E'_h) = (D_h, D'_h), h \neq i, j,$$

for any given distinct indexes i and j. In fact, since by Lemma 2.3 $[S(E'_i)] = [S(D'_i)] + 2\varepsilon[S_j] = [S(D'_i)] - 2\varepsilon[S(D_j)]$ and $[S(E'_j)]) = [S(D'_j)] + 2\varepsilon'[S_i] = [S(D'_j)] - 2\varepsilon'[S(D_i)]$ for signs $\varepsilon, \varepsilon'$ being ± 1, the intersection number $\mathrm{Int}([S(E'_i)], [S(E'_j)]) = 0$ implies $\varepsilon + \varepsilon' = 0$.

In Operations II, III, IV, the sign ε can take both $+1$ and -1, because a normal 3-disk bundle of γ in S^4 admits the Hopf link bundle of γ used for the connected sums and the Hopf link has a component-preserving inversion.

The following lemma is easily obtained, where the sign $\varepsilon = \pm 1$ is taken to be the sign of the product integer ua.

Lemma 2.4. Let u be an odd integer, and a a non-zero integer. If $|a| \geq |u|$, then there is a sign $\varepsilon = \pm 1$ such that $0 \leq |2a - 2\varepsilon u| < 2|a|$. If $|u| > |a|$, then there is a sign $\varepsilon = \pm 1$ such that $u - 2\varepsilon a$ is an odd integer with $1 \leq |u - 2\varepsilon a| < |u|$.

The proof of Lemma 1.2 is done as follows.

Proof of Lemma 1.2. Fix the orthogonal basis (x_*, x'_*) of Σ. By Lemma 2.2, there is an O2-handle basis $(E_* \times I, E'_* \times I)$ on F such that $x_i = u_i[S(E_i)] + 2A_i + 2A'_i$ and $x'_i = u'_i[S(E'_i)] + 2B_i + 2B'_i$ for all i, where u_i, u'_i are odd integers and

$$A_i = \sum_{j=1}^{n} a_{ij}[S(E_j)], \quad A'_i = \sum_{j=1}^{n} a'_{ij}[S(E'_j)], \quad B_i = \sum_{j=1}^{n} b_{ij}[S(E_j)], \quad B'_i = \sum_{j=1}^{n} b'_{ij}[S(E'_j)],$$

for some integers a_{ij}, a'_{ij}, b_{ij}, b'_{ij} with $a_{ii} = b'_{ii} = 0$ for all i.

Let $n = 1$. Since (x_1, x'_1) and $([S(E_1)], [S(E'_1)])$ are orthogonal basis of Σ and $a_{11} = b'_{11} = 0$, the identities $x_1 = u_1[S(E_1)]$ and $x'_1 = u'_1[S(E'_1)]$ and $u_1u'_1 = 1$ are obtained. By Operation I, there is an O2-handle basis $(E_* \times I, E'_* \times I)$ on F with $u_1 = u'_1 = 1$. Then $x_1 = [S(E_1)]$ and $x'_1 = [S(E'_1)]$.

Let $n \geq 2$. By Lemma 2.4, $a_{nn} = 0$ and a finite number of applications of Operation II using the transformations $[S(E_n)] \to [S(E_n)] \pm 2[S(E_j)]$ and $[S(E_j)] \to [S(E_i)] \pm 2[S(E_n)]$ for $j < n$, there is an O2-handle basis $(E_* \times I, E'_* \times I)$ on F such that $x_n = u_n[S(E_n)] + 2A'_n$ and $x'_n = u'_n[S(E'_n)] + 2B_n + 2B'_n$ with some odd integers u_n, u'_n and some A'_n, B_n, B'_n. By Lemma 2.4, a finite number of applications of Operation III using the transformations $[S(E_n)] \to [S(E_n)] \pm 2[S(E'_j)]$ for $j < n$ and a finite number of applications of Operation IV using $[S(E'_j)] \to [S(E'_j)] \pm 2[S(E_n)]$ for $j < n$, there is an O2-handle basis $(E_* \times I, E'_* \times I)$ on F such that $x_n = u_n[S(E_n)]$ and $x'_n = u'_n[S(E'_n)] + 2B_n + 2B'_n$ with some odd integers un, u'_n and some B_n, B'_n. Since $b'_{nn} = 0$, the identity $u_nu'_n = 1$ holds. By Operation I, there is an O2-handle basis $(E_* \times I, E'_* \times I)$ on F with $u_n = u'_n = 1$. Then $x_n = [S(E_n)]$ and $x'_n = [S(E'_n)] + 2B_n + 2B'_n$. By using a finite number of applications of Operation IV using the transformation $[S(E'_n)] \to [S(E'_n)] \pm 2[S(E_j)]$ for $j < n$, $x_n = [S(E_n)]$ and $x'_n = [S(E'_n)] + 2B'_n$. Then $B'_n = 0$ since (x_*, x'_*) and $(S(E_*), S(E'_*))$ are orthogonal bases of Σ. Thus, the identities $x_n = [S(E_n)]$ and $x'_n = [S(E'_n)]$ are obtained for an O2-handle basis $(E_* \times I, E'_* \times I)$ on F.

Let F_{n-1} be the trivial surface-knot of genus $n - 1$ in S^4 obtained from F by surgery along the O2-handle pair $(E_n \times I, E'_n \times I)$, [4]. Then the system $(x_i, x'_i)\,(i = 1, 2, \ldots, n-1)$ are regarded as an orthogonal basis for the stable 4-sphere $\Sigma(n-1) = S^4(F_{n-1})_2$ of genus $n-1$. By inductive assumption on n, there is a replacement of the O2-handle basis $(E_i \times I, E'_i \times I)\,(i = 1, 2, \ldots, n-1)$ on F_{n-1} keeping the attaching part fixed so that $x_i = [S(E_i)]$ and $x'_i = [S(E'_i)]$ for all $i\,(i = 1, 2, \ldots, n-1)$ in $\Sigma(n-1)$. Thus, there is an O2-handle basis $(E_* \times I, E'_* \times I)$ on F such that $x_i = [S(E_i)]$ and $x'_i = [S(E'_i)]$ for all $i\,(i = 1, 2, \ldots, n)$ in $\Sigma = \Sigma(n)$. This completes the proof of Lemma 1.2.

This completes the proof of Theorem 1.1.

Conclusion. For every orthogonal basis (x_*, x'_*) of the stable 4-sphere Σ of genus n, it is shown in Lemma 2.2 that the Z_2-reduction of the orthogonal basis (x_*, x'_*) is realized by the Z_2-deduction of the orthogonal basis $([S(E_*)], [S(E'_*)])$ for an O2-handle basis $(E_* \times I, E'_* \times I)$ on a trivial surface-knot F of genus n in S^4. By the proof of Lemma 1.2, it is shown that there is a replacement of the O2-handle basis $(E_* \times I, E'_* \times I)$ on F keeping the attaching part fixed so that $x_i = [S(E_i)]$ and $x'_i = [S(E'_i)]$ for all $i\,(i = 1, 2, \ldots, n)$ in Σ. Since any two O2-handle bases on F are transformable into each other by an equivalence of the surface-knot space, any two orthogonal bases of Σ are transformable into each other by an orientation-preserving diffeomorphism of Σ. It is shown that *any two homotopic diffeomorphisms of* Σ *are isotopic if one diffeomorphism is replaced with a composition by an identity-shift*, [1, Theorem 4.1]. For the proof, Gabai's 4D light bulb theorem is needed, [8]. This means that every O2-sphere basis (S_*, S'_*) of Σ with $x_i = [S_i], x'_i = [S'_i]\,(i = 1, 2, \ldots, n)$ is piecewise-linearly isotopic to the O2-sphere basis $(S(E_*), S(E'_*))$ for an O2-handle basis $(E_* \times I, E'_* \times I)$ on a trivial surface-knot F in S^4. To establish the same claim in smooth category, the unsolved problem on an identity-shift composition remains.

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