ON A ČECH COHOMOLOGICAL EQUATION ALONG A COMPLEX CURVE WITH A NODE

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Abstract. Let C be a rational curve with a single node embedded in a complex surface V. We assume that the normal bundle $N_{C/V}$ is flat, corresponding to the complex number $\alpha \in \mathbb{C}^*$ with $|\alpha| = 1$. In this paper, we investigate how the cohomology of a certain finite Stein covering of V changes its behavior depending on the parameter α.

1. Introduction

Let C be a connected compact reduced complex curve holomorphically embedded in a non-singular complex surface V. The complex analytic structure of a neighborhood of C is related to the positivity or negativity of the normal line bundle $N_{C/V}$.

In the case where C is non-singular, several classical results characterize this relationship. We assume that C consists of irreducible components C_ν for $\nu \in \{1, 2, \ldots, N\}$. For the intersection number of irreducible components $a_{\nu\mu} := C_\nu \cdot C_\mu$, we denote the intersection matrix of C by $A_C = (a_{\nu\mu})$. According to the Grauert–Mumford criterion, the largest eigenvalue $\lambda(A_C)$ of A_C is negative if and only if C admits a fundamental system of strongly pseudoconvex neighborhoods (see [G] and [M]). Conversely, if $\lambda(A_C)$ is positive, C possesses a fundamental system of strongly pseudoconcave neighborhoods [KU].

However, in the critical case where $\lambda(A_C)$ is zero, the analytic properties of the neighborhood are not determined by topological data alone. For non-singular carves with a topologically trivial normal bundle, Ueda provided a classification into four types based on the analytic structure of their neighborhoods [U83].

In the present paper, we observe the complex analytic properties of neighborhoods of a curve C that is not a necessarily non-singular, specifically focusing on the case where $\lambda(A_C)$ is zero. Regarding the curve C as a divisor, we denote by $[C]$ the corresponding holomorphic line bundle on V, and define the normal bundle by $N_{C/V} := [C]|_C$. We suppose that C is a rational curve with a single node (an ordinary double point) and that its normal bundle $N_{C/V}$ is flat. Given the flatness of $N_{C/V}$, the bundle corresponds to the complex number $\alpha \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ (see §2.1). For such nodal curves, Ueda established the existence of an analytic neighborhood under the condition $|\alpha| \neq 1$ [U91]. Koike investigated the case where $|\alpha| = 1$, proving that a certain arithmetical condition on α determines the complex structure of a neighborhood of C [K]. Motivated by this result, we study the cohomology groups of neighborhoods of C with $|\alpha| = 1$ under such an arithmetical condition.

2020 *Mathematics Subject Classification.* 32J15; 32G05

Key words and phrases. Complex surface, Nodal curve, Singular holomorphic foliation.

In this paper, we investigate a Čech cohomological equation for a certain class of complex curves with a single node, constructed as follows.

Let S and $T = 1/S$ be the standard non-homogeneous coordinates of the complex projective line $\mathbb{P}^1$. For the line bundle $\pi\colon \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathbb{P}^1$, let x and y denote the fiber coordinates over the neighborhoods $\pi^{-1}(\{S=0\})$ and $\pi^{-1}(\{T=0\})$. We define the domains $\widetilde{U}_x$ and $\widetilde{U}_y$ by

$$\widetilde{U}_x := \{(S,x) \in \pi^{-1}(\mathbb{C}) \mid |S| < \varepsilon,\ |x| < \varepsilon\},$$
$$\widetilde{U}_y := \{(T,y) \in \pi^{-1}(\mathbb{P}^1 \setminus \{0\}) \mid |T| < \varepsilon,\ |y| < \varepsilon\},$$

where $\varepsilon \in (0,1)$ is a sufficiently small number. We then define a biholomorphism $r\colon \widetilde{U}_y \to \widetilde{U}_x$ by the relation

$$r^*S = y, \quad r^*(tx) = T.$$

Here, the parameter t satisfies $t \in \mathrm{U}(1) = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$ and let $\theta \in (0,1)$ be the value such that $t = e^{2\pi\sqrt{-1}\theta}$. By introducing a global coordinate $w = Sx = Ty$ on $\mathcal{O}_{\mathbb{P}^1}(-2)$, we define a neighborhood $\widetilde{V}$ of the zero-section $\widetilde{C}$ of $\mathcal{O}_{\mathbb{P}^1}(-2)$ as

$$\widetilde{V} = \{w \in \pi^{-1}(\mathbb{P}^1) \mid |w| < \varepsilon^2\}.$$

Then, we obtain a complex curve with a node, embedded in a non-singular complex surface, by taking the quotients

$$C := \widetilde{C}/\sim\ \subset V := \widetilde{V}/\sim,$$

where the equivalence relation $\sim$ on $\widetilde{V}$ is generated by $r(\eta) \sim \eta$. Furthermore, the family of hypersurfaces $\{w \in \mathcal{O}_{\mathbb{P}^1}(-2) \mid w = \text{constant}\}$ induces a singular foliation $\mathcal{F}$ on V. Let $\mathcal{C}_{\mathcal{F}}$ denote the sheaf of functions that are holomorphic on V and constant along each leaf of $\mathcal{F}$.

Theorem 1.1. *Let C, V, and $\mathcal{F}$ be as above. Then there exists a finite Stein covering $\mathcal{V}$ of V which satisfies the following:*

(1) If θ is an asymptotically positive irrational number, i.e. there exist positive numbers c and $R < 1$ such that $\min_{m\in\mathbb{Z}} |n\theta - m| \geq cR^n$ holds for any positive integer n, then $\check{H}^1(\mathcal{V}, \mathcal{C}_{\mathcal{F}}) = 0$.

(2) If an irrational number θ is not asymptotically positive, then $\check{H}^1(\mathcal{V}, \mathcal{C}_{\mathcal{F}})$ is a non-Hausdorff topological space with respect to the topology of compact convergence.

We define the Čech cohomology group as the Hausdorff quotient of the standard Čech cohomology. Specifically, let $\mathcal{V}$ be an open covering of a neighborhood of C. We define

$$\check{H}^1_{\text{formal}}(\mathcal{V}, \mathcal{O}_V/C_{\mathcal{F}}) := \frac{\mathrm{Ker}(\delta\colon \check{C}^1(\mathcal{V}, \mathcal{O}_V/C_{\mathcal{F}}) \to \check{C}^2(\mathcal{V}, \mathcal{O}_V/C_{\mathcal{F}}))}{\overline{\mathrm{Img}}(\delta\colon \check{C}^0(\mathcal{V}, \mathcal{O}_V/C_{\mathcal{F}}) \to \check{C}^1(\mathcal{V}, \mathcal{O}_V/C_{\mathcal{F}}))},$$

where δ denotes the boundary operator and $\overline{\mathrm{Img}}$ represents the closure of the image of δ with respect to the topology of compact convergence. With this definition, the resulting cohomology group becomes a Hausdorff topological vector space. For the quotient sheaf $\mathcal{O}_V/C_{\mathcal{F}}$, we obtain the following result.

Theorem 1.2. *There exists a finite Stein covering $\mathcal{V}$ of V such that $\check{H}^1_{\mathrm{formal}}(\mathcal{V}, \mathcal{O}_V/C_{\mathcal{F}}) = 0$.*

The organization of this paper. §2 provides the setting of main theorems. In §2.1, we construct a curve with a single node as described above and introduce the finite Stein covering $\mathcal{V}$ of V. §2.2 is devoted to an explicit description of the Čech cohomological equations to be solved. Finally, we prove Theorems 1.1 and 1.2 in §3.

Acknowledgement. The authors would like to express his sincere gratitude to Professor Takayuki Koike for his valuable comments on the research direction. The authors also grateful to Professor Laurent Stolovitch for helpful suggestions regarding the convergence and divergence of the power series in the proof. This work was partly supported by Osaka Central Advanced Mathematical Institute: MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0723833165, and Osaka Metropolitan University Strategic Research Promotion Project (Development of International Research Hubs). The first author is supported by JST, the establishment of university fellowships towards the creation of science technology innovation, Grant Number JPMJFS2138. The second author is supported by JST SPRING Grant Number JPMJSP2139 and JSPS Bilateral Program Number JPJSBP120243210.

2. Čech cohomological equations

2.1. **Construction a finite Stein covering.** Let $\pi_r : \widetilde{V} \to V = \widetilde{V}/\sim$ be the quotient map associated with the biholomorphism $r : \widetilde{U}_y \to \widetilde{U}_x$ and the equivalence relation $\sim$ determined by r.

Let $p \in C$ denote the node of C. We define V_0 as a tubular neighborhood of $C \setminus \{p\}$ in V, and $V_1 := \pi_r(\widetilde{U}_x)$ as a neighborhood of p in V. Since V_0 is Stein by [Si], the sets form a finite Stein covering $\mathcal{V} = \{V_0, V_1\}$ of V. The intersection $V_0 \cap V_1$ consists of two connected components, denoted by V^+ and V^-. These components are chosen such that $\pi_r^{-1}(V^+)$ (resp. $\pi_r^{-1}(V^-)$) contains the axis $\{(x, S) \mid x = 0, 0 < |S| < \varepsilon\}$ (resp. $\{(y, T) \mid y = 0, 0 < |T| < \varepsilon\}$). Accordingly, we define the following four sets:

$$\widetilde{V}_x^+ := \widetilde{U}_x \cap \pi_r^{-1}(V^+), \quad \widetilde{V}_y^+ := \widetilde{U}_y \cap \pi_r^{-1}(V^+),$$
$$\widetilde{V}_x^- := \widetilde{U}_x \cap \pi_r^{-1}(V^-), \quad \widetilde{V}_y^- := \widetilde{U}_y \cap \pi_r^{-1}(V^-).$$

Finally, let $\widetilde{V}_0$ be the connected component of the union of $\pi_r^{-1}(V_0) \setminus (\widetilde{V}_x^- \cup \widetilde{V}_y^+)$ and two points $\{(x, S) = (0, 0)\} \cup \{(y, T) = (0, 0)\}$ which contains $\widetilde{C}$.

It is known that every topologically trivial line bundle $E \in \mathrm{Pic}^0(C)$ is uniquely represented by the non-zero complex number; that is, $H^1(C, \mathcal{O}_C^*) \cong \mathbb{C}^*$ [U91, Lemma 1]. Below, we provide an explicit description of this correspondence in our setting.

We consider an open covering $\{W_0, W_1\}$ of C, where $W_0 := C \setminus \{p\}$ and W_1 is a small neighborhood of the node in C. The intersection $W_0 \cap W_1$ consists of two disjoint connected components, which we identify with the previously defined $V^- \cap C$ and $V^+ \cap C$.

Suppose that $E = N_{C/V}$. Let ω_0 be a local defining function of $V_0 \cap C$ in V_0. As established in our construction, ω_0 satisfies the following transition relations:

$$\omega_0 = \begin{cases} tS \cdot x & (\text{on } V^-), \\ T \cdot y & (\text{on } V^+). \end{cases}$$

Then we have

$$N_{C/V} = [\{(V^- \cap C, t), (V^+ \cap C, 1)\}] \in \check{H}^1(\{W_i\}, \mathcal{O}_C^*) = H^1(C, \mathcal{O}_C^*).$$

Thus, the normal bundle $N_{C/V}$ is uniquely determined by the number $t \in U(1) \subset \mathbb{C}^*$.

2.2. **Čech cohomological equations.** Let $\mathcal{C}_\mathcal{F} \subset \mathcal{O}_V$ be the subsheaf induced by leafwise constant functions with respect to $\mathcal{F}$ as in §1. From the short exact sequence

$$0 \to \mathcal{C}_\mathcal{F} \to \mathcal{O}_V \to \mathcal{O}_V/\mathcal{C}_\mathcal{F} \to 0,$$

we obtain the corresponding long exact sequence in Čech cohomology $\check{H}^\bullet(\mathcal{V}, \bullet)$. Since the Stein covering $\mathcal{V}$ consists of only two sets V_0 and V_1, the Čech cohomology groups $\check{H}^k(\mathcal{V}, \bullet)$ vanish for all $k \geq 2$. Therefore, we focus on the following part of the long exact sequence:

$$\cdots \to \check{H}^1(\mathcal{V}, \mathcal{C}_\mathcal{F}) \to \check{H}^1(\mathcal{V}, \mathcal{O}_V) \to \check{H}^1(\mathcal{V}, \mathcal{O}_V/\mathcal{C}_\mathcal{F}) \to \cdots.$$

What we want to investigate is the first Čech cohomology $\check{H}^1(\mathcal{V}, \bullet)$. We denote $V_0 \cap V_1$ by V_{01} for instance. To compute $H^1(\mathcal{V}, \mathcal{O}_V)$, we consider the coboundary equation $F_0 - F_1 = f$, where $F_0 \in \mathcal{O}(V_0)$ and $F_1 \in \mathcal{O}(V_1)$ are unknown functions, and $\{(V_{01}, f)\} \in \check{C}^1(\mathcal{V}, \mathcal{O}_V)$ is a given 1-cocycle. Since the equation must be satisfied on each connected component of V_{01}, we express it as follows:

$$F_0 - F_1 = \begin{cases} f^+ & \text{on } V^+ \\ f^- & \text{on } V^- \end{cases}. \tag{1}$$

Here, $f^+ := f|_{V^+}$ and $f^- := f|_{V^-}$. For simplicity, we denote the pull-back $\pi_r^* F_0$, $\pi_r^* F_1$, and $\pi_r^* f^\pm$ by F_0, F_1, and $f^\pm$. Their Laurent series expansions are given as follows:

$$f^+(S, w) = \sum_{n \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}} a_{n,m} S^m w^n \quad \text{on } \widetilde{V}_x^+, \tag{2}$$

$$f^-(T, w) = \sum_{n \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}} b_{n,m} T^m w^n \quad \text{on } \widetilde{V}_y^-, \tag{3}$$

$$F_0(S, w) = \sum_{n \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}} p_{n,m} S^m w^n \quad \text{on } \widetilde{V}_0, \tag{4}$$

$$F_1(S, x) = \sum_{n, m \in \mathbb{Z}_{\geq 0}} q_{n,m} S^m x^n \quad \text{on } \widetilde{U}_x. \tag{5}$$

3. Proof of main theorems

3.1. **Proof of Theorem** 1.1. If F_0, F_1, and $f^{\pm}$ are leafwise constant, the expansions in (2)–(5) reduce to power series in w alone:

$$f^+(w) = \sum_{n\geq 0} A_n w^n, \quad f^-(w) = \sum_{n\geq 0} B_n w^n,$$

$$F_0(w) = \sum_{n\geq 0} \widehat{p}_n w^n, \quad F_1(w) = \sum_{n\geq 0} \widehat{q}_n w^n.$$

By substituting these series into the coboundary equation (1) on V^+,

$$F_0(w) - F_1(w) = \sum_{n\geq 0} \widehat{p}_n w^n - \sum_{n\geq 0} \widehat{q}_n w^n = \sum_{n\geq 0} A_n w^n.$$

Hence, we obtain the relation $\widehat{p}_n - \widehat{q}_n = A_n$ for each $n \in \mathbb{Z}_{\geq 0}$. On the other hand, the equation (1) on V^- yields the relation $t^{-n}\widehat{p}_n - \widehat{q}_n = B_n$. Note that the transformation $w \mapsto tw$ induced by r implies that the Laurent expansion of F_0 on V^- is given by $\sum_{n\geq 0} \widehat{p}_n (t^{-1}w)^n$, based on its expansion of F_0 on V^+. If $t = e^{2\pi\sqrt{-1}\theta}$ for an irrational number $\theta \in \mathbb{R}$, then $1 - t^n \neq 0$ holds for all $n \in \mathbb{Z}_{>}0$. Therefore, we obtain the coefficients by

$$\widehat{p}_n = \frac{B_n - A_n}{1 - t^{-n}}, \quad \widehat{q}_n = \frac{t^n B_n - A_n}{1 - t^n} \tag{6}$$

for any $n \in \mathbb{Z}_{>0}$ By the assumption of the first assertion, there exist $c > 0$ and $0 < R < 1$ such that

$$\frac{1}{|1 - t^n|} \leq \frac{1}{cR^n}$$

for any $n \in \mathbb{Z}_{>0}$. Therefore, F_0 and F_1 have a positive radius of convergence.

On the contrary, we show the second assertion. For any $d > 0$ and any positive number $r < 1$, there exists $N_{d,r} \in \mathbb{Z}_{>0}$ such that

$$\frac{1}{|1 - t^{N_{d,r}}|} \geq \frac{1}{dr^{N_{d,r}}}.$$

For example, for f^+ with $A_n = 0$ and f^- with $B_n = 1/\varepsilon^n$, neither F_0 nor F_1 has a positive radius of convergence. □

3.2. **Proof of Theorem** 1.2. In this subsection, we identify the coefficients $p_{n,m}$ and $q_{n,m}$ by substituting the expansions into the equation (1).

On the component $\widetilde{V}^+_x$, the equation $F_0 - F_1 = f^+$ is expressed in terms of the Laurent series as follows:

$$\sum_{n\in\mathbb{Z}_{\geq 0}, m\in\mathbb{Z}} p_{n,m} S^m w^n - \sum_{n,m\in\mathbb{Z}_{\geq 0}} q_{n,m} S^m x^n = \sum_{n\in\mathbb{Z}_{\geq 0}, m\in\mathbb{Z}} a_{n,m} S^m w^n.$$

By using the relation $w = Sx$ and comparing the coefficients of each monomial $S^m x^n$, we derive the following relation:

$$p_{n,m-n} - a_{n,m-n} = \begin{cases} q_{n,m} & (m \geq 0) \\ 0 & (m < 0) \end{cases}. \tag{7}$$

On the other hand, applying (1) to the component V^-, we obtain similar relation. It should be noted that the representations of F_0 on $\widetilde{V}_x^+$ and on $\widetilde{V}_y^-$ are related via the coordinate transformations $ST = 1$ and $w = Sx = Ty$. Furthermore, the representations of F_1 on $\widetilde{V}_x^+$ and on $\widetilde{V}_y^-$ are identified through the gluing map r. Substituting these into the expansion on $\widetilde{V}_y^-$, we have

$$\sum_{n\in\mathbb{Z}_{\geq 0}, m\in\mathbb{Z}} p_{n,m} T^{n-m} y^n - \sum_{n,m\geq 0} q_{n,m} t^{-n} T^n y^m = \sum_{n\in\mathbb{Z}_{\geq 0}, m\in\mathbb{Z}} b_{n,m} T^m w^n.$$

By comparing the coefficients of each monomial $T^m y^n$,

$$p_{n,n-m} - b_{n,m-n} = \begin{cases} q_{m,n} t^{-m} & (m \geq 0) \\ 0 & (m < 0) \end{cases}. \tag{8}$$

The equations (7) and (8) give the relations for $p_{n,m}$ and $q_{n,m}$ as follows:

$$p_{n,m} = \begin{cases} b_{n,-m} & (m > n) \\ b_{n,-m} + q_{n-m,n} t^{m-n} & (m \leq n) \\ a_{n,m} + q_{n,m+n} & (m \geq -n) \\ a_{n,m} & (m < -n) \end{cases}.$$

By rearranging these relations, we obtain a system of equations for $p_{n,m}$ and $q_{n,m}$:

$$p_{n,m} = b_{n,-m} + q_{n-m,n} t^{m-n} \qquad (m \leq n), \tag{9}$$

$$p_{n,m} = a_{n,m} + q_{n,m+n} \qquad (m \geq -n), \tag{10}$$

$$q_{n,m} = p_{n,m-n} - a_{n,m-n} \qquad (n, m \geq 0), \tag{11}$$

$$q_{n,m} = t^n (p_{m,m-n} - b_{m,n-m}) \qquad (n, m \geq 0). \tag{12}$$

We determine the coefficients $p_{n,m}$ and $q_{n,m}$ by iteratively applying the relations (9) to (12). First, by combining the relation $p_{n,m} = b_{n,-m}$ for $m > n$ with (11), we find that

$$q_{n,m} = b_{n,n-m} - a_{n,m-n} \tag{13}$$

for pairs (n, m) satisfying $m > 2n$.

Next, by substituting (13) into (9), we obtain

$$p_{n,m} = b_{n,-m} + t^{m-n}(b_{n-m,-m} - a_{n-m,m}) \tag{14}$$

for (n, m) satisfying $n \geq m > \frac{n}{2}$. Similarly, applying (11) to (14) yields

$$q_{n,m} = b_{n,n-m} - a_{n,m-n} + t^{m-2n}(b_{2n-m,n-m} - a_{2n-m,m-n})$$

for (n, m) with $\frac{3}{2}n < m \leq 2n$.

By repeatedly alternating between (9) and (11), one can determine the coefficients $p_{n,m}$ and $q_{n,m}$ for (n, m) belonging to the specific regions. This inductive process leads to the following assertion.

Proposition 3.1. *For each integer $j \geq 1$, let P^j be the set defined by $P^j = \{(n,m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \mid \frac{n}{j+1} < m \leq \frac{n}{j}\}$. Then,*

$$p_{n,m} = b_{n,-m} + \sum_{\ell=1}^{j} t^{m \sum(0,\ell) - n\ell} (b_{n-m\ell,-m} - a_{n-m\ell,m})$$

holds for $(n,m) \in P^j$. Here, we denote the sum of integers from ℓ_1 to ℓ_2 by $\sum(\ell_1,\ell_2) = \sum_{\ell=\ell_1}^{\ell_2} \ell$.

Proof. The proof proceeds by induction on j. Let $j \geq 1$ be an integer and suppose that the following expression for $p_{n,m}$ holds for all $(n,m) \in P^j$:

$$p_{n,m} = b_{n,-m} + \sum_{\ell=1}^{j} t^{m \sum(0,\ell) - n\ell} (b_{n-m\ell,-m} - a_{n-m\ell,m}). \tag{15}$$

To advance the induction to P^{j+1}, we observe how the index shifts in the relation (11). Specifically, for a pair (n,m) to satisfy $(n, m-n) \in P^j$, the indices must fall within the range

$$\frac{j+2}{j+1} n < m \leq \frac{j+1}{j} n.$$

By substituting the inductive hypothesis (15) into the relation (11), it follows that

$$\begin{aligned}
q_{n,m} &= p_{n,m-n} - a_{n,m-n} \\
&= b_{n,-(m-n)} + \sum_{\ell=1}^{j} t^{(m-n)\sum(0,\ell) - n\ell} (b_{n-(m-n)\ell,-(m-n)} - a_{n-(m-n)\ell,m-n}) - a_{n,m-n} \\
&= b_{n,n-m} + \sum_{\ell=1}^{j} t^{(m-n)\sum(0,\ell) - n\ell} (b_{n-(m-n)\ell,n-m} - a_{n-(m-n)\ell,m-n}) - a_{n,m-n} \\
&= \sum_{\ell=0}^{j} t^{(m-n)\sum(0,\ell) - n\ell} (b_{n-(m-n)\ell,n-m} - a_{n-(m-n)\ell,m-n}) \\
&= \sum_{\ell=0}^{j} t^{m\sum(0,\ell) - n\sum(-1,\ell+1)} (b_{(\ell+1)n-\ell m,n-m} - a_{(\ell+1)n-\ell m,m-n}) \\
&= \sum_{\ell=1}^{j+1} t^{m\sum(0,\ell-1) - n\sum(-1,\ell)} (b_{\ell n-(\ell-1)m,n-m} - a_{\ell n-(\ell-1)m,m-n})
\end{aligned}$$

for indices (n,m) satisfying the condition

$$\frac{j+2}{j+1} n < m \leq \frac{j+1}{j} n. \tag{16}$$

Hence, we find that

$$q_{n,m} = \sum_{\ell=1}^{j+1} t^{m\sum(0,\ell-1) - n\sum(-1,\ell)} (b_{\ell n-(\ell-1)m,n-m} - a_{\ell n-(\ell-1)m,m-n}) \tag{17}$$

holds for (n,m) satisfying (16). Next, we substitute the expression (17) into (9). This substitution is valid when

$$(n-m,n) \in \left\{ (n,m) \in (\mathbb{Z}_{\geq 0})^2 \,\middle|\, \frac{j+2}{j+1}n < m \leq \frac{j+1}{j}n \right\}.$$

For $(n,m) \in P^{j+1}$, that is, indices satisfying

$$\frac{n}{j+2} < m \leq \frac{n}{j+1},$$

then we see that

$$\begin{aligned}
p_{n,m} &= b_{n,-m} + q_{n-m,n} t^{m-n} \\
&= b_{n,-m} + t^{m-n} \sum_{\ell=1}^{j+1} t^{n\sum(0,\ell-1)-(n-m)\sum(-1,\ell)} (b_{\ell(n-m)-(\ell-1)n,(n-m)-n} - a_{\ell(n-m)-(\ell-1)n,n-(n-m)}) \\
&= b_{n,-m} + t^{m-n} \sum_{\ell=1}^{j+1} t^{n\sum(0,\ell-1)-n\sum(-1,\ell)+m\sum(-1,\ell)} (b_{n-\ell m,-m} - a_{n-\ell m,m}) \\
&= b_{n,-m} + t^{m-n} \sum_{\ell=1}^{j+1} t^{m\sum(-1,\ell)-(\ell-1)n} (b_{n-\ell m,-m} - a_{n-\ell m,m}) \\
&= b_{n,-m} + \sum_{\ell=1}^{j+1} t^{m\sum(0,\ell)-n\ell} (b_{n-m\ell,-m} - a_{n-m\ell,m}).
\end{aligned}$$

This confirms that (15) holds for $(n,m) \in P^{j+1}$, thereby completing the induction. □

In the proof of Proposition 3.1, we also obtain the following.

Corollary 3.2. *For each integer $j \geq 1$, let Q^j be the set defined by $Q^j = \{(n,m) \in (\mathbb{Z}_{\geq 0})^2 \mid \frac{j+2}{j+1}n < m \leq \frac{j+1}{j}n\}$ for $j \in \mathbb{Z}$ with $j \geq 1$. Then,*

$$q_{n,m} = \sum_{\ell=1}^{j+1} t^{m\sum(0,\ell-1)-n\sum(-1,\ell)} (b_{\ell n-(\ell-1)m,n-m} - a_{\ell n-(\ell-1)m,m-n})$$

holds for $(n,m) \in Q^j$.

On the other hand, the relation $p_{n,m} = a_{n,m}$ holds for $m < -n$. By substituting this into (12) and applying the relations (10) and (12) alternately, as in the previous argument, we obtain the following proposition.

Proposition 3.3. *For each integer $j \geq 1$, let P_j be the set defined by $P_j = \{(n,m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \mid -\frac{n}{j} \leq m < -\frac{n}{j+1}\}$. Then the following holds for $(n,m) \in P_j$:*

$$p_{n,m} = a_{n,m} + \sum_{\ell=0}^{j-1} t^{m\sum(0,\ell)+n(\ell+1)} (a_{m(\ell+1)+n,m} - b_{m(\ell+1)+n,-m}).$$

□

Proof. This proof also proceeds by induction on j. Let j be a positive integer, and suppose that the expression

$$p_{n,m} = a_{n,m} + \sum_{\ell=0}^{j-1} t^{m\sum(0,\ell)+n(\ell+1)}(a_{m(\ell+1)+n,m} - b_{m(\ell+1)+n,-m}) \tag{18}$$

holds for $(n,m) \in P_j$. To establish the induction step, we consider the relation (12), which involves the shifted index $(m, m-n)$. From the definition of P_j, the condition $(m, m-n) \in P_j$ is equivalent to the following inequality for m:

$$\frac{j}{j+1}n \leq m < \frac{j+1}{j+2}n.$$

By substituting the hypothesis (18) into the relation (12), it follows that

$$\begin{aligned}
q_{n,m} &= t^n(p_{m,m-n} - b_{m,n-m}) \\
&= t^n\left(a_{m,m-n} + \sum_{\ell=0}^{j-1} t^{(m-n)\sum(0,\ell)+m(\ell+1)}(a_{(m-n)(\ell+1)+m,m-n} - b_{(m-n)(\ell+1)+m,-(m-n)}) - b_{m,n-m}\right) \\
&= t^n\left(a_{m,m-n} - b_{m,n-m} + \sum_{\ell=0}^{j-1} t^{m\sum(0,\ell+1)-n\sum(0,\ell)}(a_{(m-n)(\ell+1)+m,m-n} - b_{(m-n)(\ell+1)+m,-(m-n)})\right) \\
&= \sum_{\ell=-1}^{j-1} t^{m\sum(0,\ell+1)-n\sum(-1,\ell)}(a_{(m-n)(\ell+1)+m,m-n} - b_{(m-n)(\ell+1)+m,-(m-n)}) \\
&= \sum_{\ell=0}^{j} t^{m\sum(0,\ell)-n\sum(-1,\ell-1)}(a_{(m-n)\ell+m,m-n} - b_{(m-n)\ell+m,-(m-n)})
\end{aligned}$$

for indices (n,m) satisfying the condition $(m, m-n) \in P_j$. Hence, we find that

$$q_{n,m} = \sum_{\ell=0}^{j} t^{m\sum(0,\ell)-n\sum(-1,\ell-1)}(a_{(m-n)\ell+m,m-n} - b_{(m-n)\ell+m,-(m-n)}) \tag{19}$$

holds for (n,m) satisfying the condition

$$\frac{j}{j+1}n \leq m < \frac{j+1}{j+2}n.$$

Subsequently, we substitute (19) into (10) when the condition

$$(n, m+n) \in \left\{(n,m) \in (\mathbb{Z}_{\geq 0})^2 \,\middle|\, \frac{j}{j+1}n \leq m < \frac{j+1}{j+2}n\right\}.$$

Then we obtain the equation

$$\begin{aligned}
p_{n,m} &= a_{n,m} + q_{n,m+n} \\
&= a_{n,m} + \sum_{\ell=0}^{j} t^{(m+n)\sum(0,\ell)-n\sum(-1,\ell-1)}(a_{((m+n)-n)\ell+(m+n),(m+n)-n} - b_{((m+n)-n)\ell+(m+n),-((m+n)-n)}) \\
&= a_{n,m} + \sum_{\ell=0}^{j} t^{m\sum(0,\ell)+n(\ell+1)}(a_{(m+1)\ell+n,m} - b_{(m+1)\ell+n,-m})
\end{aligned}$$

for (n, m) satisfying the condition

$$-\frac{n}{j+1} < m \leq -\frac{n}{j+2}.$$

Therefore, the equation (18) holds if $(n, m) \in P_{j+1}$. $\square$

In the proof of Proposition 3.3, we obtain the following.

Corollary 3.4. *For each integer $j \geq 1$, let Q_j be the set defined by $\{(n, m) \in (\mathbb{Z}_{\geq 0})^2 \mid \frac{j}{j+1} n \leq m < \frac{j+1}{j+2} n\}$. Then we find that*

$$q_{n,m} = \sum_{\ell=0}^{j} t^{m \sum(0,\ell) - n \sum(-1,\ell-1)} (a_{(\ell+1)m - \ell n, m-n} - b_{(\ell+1)m - \ell n, n-m})$$

holds for $(n, m) \in Q_j$. $\square$

In §3.1, we observed that the coefficients $\widehat{p}_n$ of $F_0 \in \mathcal{C}_{\mathcal{F}}$ correspond to $p_{n,m}$ for indices (n, m) lying on the line $\{(n, m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \mid m = 0\}$. Similarly, the coefficients $\widehat{q}_n$ of $F_1 \in \mathcal{C}_{\mathcal{F}}$ correspond to $q_{n,m}$ when (n, m) lies on the diagonal $\{(n, m) \in (\mathbb{Z}_{\geq 0})^2 \mid n = m\}$. In the preceding sections, we identified $p_{n,m}$ (resp. $q_{n,m}$) except for the line $\{m = 0\}$ (resp. $\{n = m\}$). Consequently, we have formally solved the equation (1) in the quotient space $\mathcal{O}_V / \mathcal{C}_{\mathcal{F}}$. Specifically, we have obtained the N-th partial sum of the solution to (1) for any positive integer N. We define the N-th partial sum as follows:

$$f^+(S, w)_N := \sum_{k=0}^{N} \left(\sum_{(n,m) \in P_k \cup P^k} a_{n,m} S^m w^n \right),$$

$$f^-(T, w)_N := \sum_{k=0}^{N} \left(\sum_{(n,m) \in P_k \cup P^k} b_{n,m} T^m w^n \right),$$

$$F_0(S, w)_N := \sum_{k=0}^{N} \left(\sum_{(n,m) \in P_k \cup P^k} p_{n,m} S^m w^n \right),$$

$$F_1(S, x)_N := \sum_{k=0}^{N} \left(\sum_{(n,m) \in Q_k \cup Q^k} q_{n,m} S^m x^n \right).$$

Here, we set

$$P_0 := \{(n, m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \mid m < -n\},$$
$$P^0 := \{(n, m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \mid n < m\},$$
$$Q_0 := \{(n, m) \in (\mathbb{Z}_{\geq 0})^2 \mid m < \frac{n}{2}\},$$
$$Q^0 := \{(n, m) \in (\mathbb{Z}_{\geq 0})^2 \mid 2n < m\}.$$

For example, on the domain $\widetilde{V}_x^+$, the following series of equalities holds:

$$\begin{aligned} F_0(S,w)_N - F_1(S,x)_N &= \sum_{k=0}^{N} \left(\sum_{(n,m)\in P_k\cup P^k} p_{n,m} S^m w^n - \sum_{(n,m)\in Q_k\cup Q^k} q_{n,m} S^m x^n \right) \\ &= \sum_{k=0}^{N} \left(\sum_{(n,m)\in P_k\cup P^k} p_{n,m} S^m w^n - \sum_{(n,m)\in P_k\cup P^k} q_{n,m+n} S^m w^n \right) \\ &= \sum_{k=0}^{N} \sum_{(n,m)\in P_k\cup P^k} a_{n,m} S^m w^n = f^+(S,w)_N. \end{aligned}$$

Therefore, we obtain Theorem 1.2.

References

[G] H. Grauert, Über modifikationen und exzeptionelle analytische mengen, Math. Ann., **146**, 331–368 (1962).

[GR] H. Grauert and R. Remmert, Theory of Stein spaces, Classics in Mathematics. Springer, Berlin (2004).

[K] T. Koike, Arnol'd's type theorem on a neighborhood of a cycle of rational curves, JAMA(2022), **147** 41–67.

[KU] T. Koike and T. Ueda, Intersection matrix and neighborhood structure of complex curves, preprint.

[M] D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, Publications Mathématiques de l'Institut des Hautes Études Scientifiques, **9** (1961), 5–22.

[Su] O. Suzuki, Neighborhoods of a compact non-singular algebraic curve imbedded in a 2-dimensional complex manifold, Publ. Res. Inst. Math. Sci. Kyoto Univ. **11** (1975), 185–199.

[Si] Y. Siu, Every Stein subvariety admits a Stein neighborhood, Invent Math **38**, 89–100 (1976).

[U83] T. Ueda, On the neighborhood of a compact Riemann surface with topologically trivial normal bundle, Math. Kyoto Univ., **22** (1983), 583–607.

[U91] T. Ueda, Neighborhood of a rational curve with a node, Publ. RIMS, Kyoto Univ. **27** (1991), 681–693.

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