Revised note on surface-link of trivial components

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Abstract

Quasi-ribbon .surface-links are introduced as a generalization of the concept of ribbon surface -links: surface-links that are transformed into ribbon surface-links (without changing the components) by surgery along a locally standard self (1, 2)-handle pair system. For every disconnected closed oriented surface F with at most one aspheric component, every F-link of trivial components is a quasi-ribbon surface-link. For every disconnected closed oriented surface F, there are non-ribbon quasi-ribbon F-links of trivial components. For every closed oriented surface F with at least two aspheric components, there are non-quasi-ribbon F-links of trivial components.

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1. Introduction

Let $\mathbf{F}$ be a (possibly disconnected) closed oriented surface. An $\mathbf{F}$-link in the 4-sphere S^4 is the image of a smooth embedding $\mathbf{F} \to S^4$. A *surface-link* in S^4 is an $\mathbf{F}$-link for some $\mathbf{F}$. If $\mathbf{F}$ consists of some copies of the 2-sphere S^2, then it is also called an S^2-*link*. When $\mathbf{F}$ is connected, they are also called an $\mathbf{F}$-*knot*, *surface-knot* and S^2-*knot*, respectively. A *trivial surface-link* is a surface-link F in S^4 which bounds disjoint handlebodies smoothly embedded in S^4. A *1-handle* on a surface-link F in S^4 is a 1-handle h on F embedded smoothly in S^4, which is called a *self 1-handle* on F if the attaching part of h belongs to the same component of F. A *self-trivial 1-handle* on F is a self 1-handle h on F such that the core arc α of h is an interior push of an arc α_0 in the connected component F_0 of F containing the attaching disk system of h into $S^4 \setminus F_0$, where note that the disk bounded by the loop $\alpha \cup \alpha_0$ may meet the sublink $F \setminus F_0$. A *1-handle system* on a surface-link F in S^4 consists of *disjoint* 1-handles on F. A *self* or *self-trivial* 1-handle system on F is a 1-handle system consisting of self or self-trivial 1-handles on F, respectively. Let $F(h)$ be the surface-link obtained from F by surgery along a 1-handle system h. A *ribbon surface-link* is the surface-link $F = O(h^O)$ in S^4

obtained from a trivial S^2-link O by surgery along a 1-handle system h^O on O, [1, 2]. Let D_h be a transverse disk system of the 1-handle system h with one disk for each 1-handle. Let $D \times I$ be a 2-handle system on F, namely a disjoint 2-handle system smoothly embedded in S^4 with core disk system D. The surface-link obtained from F by surgery along a 2-handle system $D \times I$ on F is also denoted by $F(D \times I)$. For a 1-handle system h on F, let $D'_h \times I$ be a 2-handle system on $F(h)$ with $\partial D'_h = \partial D_h$. The pair $(h, D'_h \times I)$ is called a *$(1,2)$-handle pair system* on F, and a *self $(1,2)$-handle pair system* on F if h is a self 1-handle system. A self $(1,2)$-handle pair system $(h, D'_h \times I)$ on a *surface-knot* F in S^4 is *standard* if h is a self-trivial 1-handle system on F and the 2-handle core disk system D'_h is F-relatively isotopic to the 2-handle core disk system D_h. For a surface-link F of surface-knot components $F_i\,(i = 1, 2, \ldots, r)$ in S^4, a self $(1,2)$-handle pair system $(h, D'_h \times I)$ on F is *locally standard* if the subsystem $(h_i, D'_{h_i} \times I)$ on F_i is a standard $(1,2)$-handle pair system by forgetting the sublink $F \setminus F_i$ for every i. A *quasi-ribbon* surface-link is a surface-link F in S^4 such that the surface-link $F(h; D'_h)$ defined by $F(h; D'_h) = F(h)(D'_h \times I)$ for a locally standard self $(1,2)$-handle pair system $(h, D'_h \times I)$ is a ribbon surface-link in S^4. By taking $D'_h = D_h$, every ribbon surface-link is seen to be a quasi-ribbon surface-link. If F is a quasi-ribbon surface-link, then $F(h)$ is a ribbon surface-link, because $F(h)$ is obtained from the ribbon surface-link $F(h; D'_h)$ by surgery along the 1-handle given by $D'_h \times I$, and the components $F_i\,(i = 1, 2, \ldots, r)$ of F are ribbon surface-knots because $F_i(h_i)(D'_{h_i}) = F_i(h_i)(D_{h_i}) = F_i$ for every i. The following theorem, giving a characterization on quasi-ribbon surface-links corrects an earlier claimed characterization of when a surface-link of ribbon components is a ribbon surface-link, [3, Theorem 1.4].

Theorem 1.1. A surface-link F in S^4 is a quasi-ribbon surface-link if and only if the surface-link $F(h)$ for a self-trivial 1-handle system h on F is a ribbon surface-link in S^4.

The following lemma is implicitly used in the proofs of [3, Theorem 1.4] and [4, Theorem 1] although the full proof is given in this paper for convenience.

Lemma 1.2. For every surface-link F in S^4 with at most one aspheric component, there is a self 1-handle system h on F such that the surface-link $F(h)$ is a ribbon surface-link in S^4.

Note that every self 1-handle system on every surface-link F of trivial components is a self-trivial 1-handle system on F, [5]. Thus, the following corollary is a direct consequence of obtained from Theorem 1.1 and Lemma 1.2, which corrects the author's earlier statement, [4, Theorem 1].

Corollary 1.3. Every surface-link F of trivial components with at most one aspheric component is a quasi-ribbon surface-link.

This corollary leads to the following examples of quasi-ribbon surface-links that are not ribbon.

Example 1.4. For every disconnected closed oriented surface $\mathbf{F}$, there is a non-ribbon quasi-ribbon $\mathbf{F}$-link of trivial compthat are not ribbononents . In fact, for every disconnected closed oriented surface $\mathbf{F}$ with total genus 0 or with just one aspheric component of any even genus, non-ribbon $\mathbf{F}$-links of trivial components are constructed, [6, 7]. Let K be such a non-ribbon S^2-link of trivial components, which is quasi-ribbon by Corollary 1.3. For every disconnected closed oriented surface $\mathbf{F}$, a non-ribbon quasi-ribbon $\mathbf{F}$-link F is constructed by connected-summing trivial surface-knots with K and/or adding a trivial surface-link component to K as a splitting component. Non-ribbonness of F is seen from that if F is a ribbon surface-link, then K must be a ribbon S^2-link, [3].

The following theorem shows that there are surface-links of trivial components with at least two aspheric components that are not quasi-ribbon. This result is obtained by slightly strengthening an earlier result, [4, Theorem 2].

Theorem 1.5. Let $\mathbf{F}$ be any closed oriented disconnected surface with at least two aspheric components. Then there is a pair (F, F') of $\mathbf{F}$-links F, F' in S^4 both of trivial components with the same fundamental group up to meridian-preserving isomorphisms such that F is a ribbon surface-link and F' is not a quasi-ribbon surface-link.

2. Proofs of Theorem 1.1, Lemma 1.2 and Theorem 1.5

A *semi-unknotted multi-punctured handlebody system* or simply a *SUPH system* for a surface-link F in S^4 is a compact oriented 3-manifold W smoothly embedded in S^4 such that W is a handlebody system with a finite number of open 3-balls removed and the boundary ∂W of W is given by $\partial W = F \cup O$ for a trivial S^2-link O in S^4, [3]. A typical SUPH system W is constructed from a ribbon surface-link F defined from a trivial S^2-link O and a 1-handle system h^O on F as the union $O \times [0,1] \cup h^O$ for a normal collar $O \times [0,1]$ of O in S^4 with $O \times \{0\} = O$ where h^O does not meet $O \times [0,1]$ except for the attaching part to O and $\partial W = F \cup O \times \{1\}$. For a SUPH system W with $\partial W = F \cup O$, there is a proper arc system α in W spanning O such that a regular neighborhood $N(O \cup \alpha)$ of the union $O \cup \alpha$ in W is diffeomorphic to the closed complement $\mathrm{cl}(W \setminus c(F \times [0,1]))$ of a boundary collar $c(F \times [0,1])$ of F in W. This pair (O, α) is called a *chorded sphere system* of the SUPH system W. By replacing α with a 1-handle system h^O attaching to O with core arc system α, the surface-link F is a ribbon surface-link defined by O and h^O. In other words, giving a SUPH system W with $\partial W = F \cup O$ is the same as saying that the surface-link F is a ribbon surface-link with sphere system O. A *multi-fusion SUPH system* of a SUPH system W with $\partial W = F \cup O$ in S^4 is a SUPH system for F in S^4 obtained from

W by deleting an open regular neighborhood of a disjoint simple proper arc system in W spanning O. A *multi-fission SUPH system* of a SUPH system W with $\partial W = F \cup O$ in S^4 is a SUPH system for F in S^4 obtained from W by adding a 2-handle system on O disjoint from W except for the attaching part in O where the 2-handle system can be taken in the complement $B \setminus D_W$ for a disjoint 3-ball system B bounded by O in S^4 whose interior meets W with a disjoint 2-disk system D_W, [7, Appendix]. Two ribbon structures of equivalent surface-links are moved into each other by a finite number of the moves M_0, M_1, M_2, [8]. This means that for any two SUPH systems W and W' for a ribbon surface-link F in S^4, there is an orientation-preserving diffeomorphism f of S^4 sending W to a multi-fusion SUPH system W^{**} of a multi-fission SUPH system W^* of the SUPH system W', [7, Appendix]. An *O2-handle pair* on a surface-link F in S^4 is a pair $(D \times I, E \times I)$ of 2-handles $D \times I$, $E \times I$ on F in S^4 which intersect orthogonally only with the attaching parts $(\partial D) \times I$, $(\partial E) \times I$ to F, so that the intersection $Q = (\partial D) \times I \cap (\partial E) \times I$ is a square, [9]. The proof of Theorem 1.1 is done as follows.

Proof of Theorem 1.1. If there is a locally standard self $(1, 2)$-handle pair system $(h, D'_h \times I)$ on a surface-link F in S^4 such that $F(h; D'_h)$ is a ribbon surface-link, then h is a self-trivial 1-handle system on F and $F(h)$ is a ribbon surface-link. Conversely, assume that $F(h)$ is a ribbon surface-link for a self-trivial 1-handle system h on F. The ribbon surface-link $F(h)$ consisting of ribbon surface-knots $F_i(h_i)$ for self-trivial 1-handle system h_i on the components $F_i\,(i = 1, 2, \ldots, r)$ of F. By the ribbonness of $F_i(h_i)$ and the self-triviality of h_i, the surface-knot $F_i\,(i = 1, 2, \ldots, r)$ are ribbon surface-knots, [3]. Let W be a SUPH system for $F(h)$ which is the union of SUPH systems W_i for $F_i(h_i)\,(i = 1, 2, \ldots, r)$. Since F_i is a ribbon surface-knot and h_i is a self-trivial 1-handle system on F_i, let W_i^0 be a SUPH system for F_i, and $W'_i = W_i^0 \cup h_i$ a SUPH system for $F_i(h_i)$ such that the core arc α_i of h_i is obtained from an arc α_i^0 in $F_i(h_i)$ by pushing the interior of α_i^0 outside of W_i^0. Then there is an orientation-preserving diffeomorphism f_i of S^4 sending W_i to a multi-fusion SUPH system W_i^{**} of a multi-fission SUPH system W^* of the SUPH system W'_i. The disk system D_{h_i} is in W'_i and hence in W_i^*, which is modified into a disk system D'_{h_i} in W_i^{**} with $\partial D'_{h_i} = \partial D_{h_i}$. The disk system $D'_h = \cup_{i=1}^r f_i^{-1}(D'_{h_i})$ is a 2-handle core disk system on $F(h)$ in W with $\partial D'_h = \partial D_h$ and $(h, D'_h \times I)$ is a self$(1, 2)$-handle pair system on F. The surface-link $F(h; D'_h)$ is a ribbon surface-link because the 3-manifold obtained from W by splitting along the disk system D'_h is a SUPH system for $F(h; D'_h)$. To show that the self $(1, 2)$-handle pair system $(h, D'_h \times I)$ on F is locally standard, let h be a self-trivial 1-handle system on a *surface-knot* F in S^4. By definition, there is an O2-handle pair system $(D_h \times I, E \times I)$ on $F(h)$ where E is made from a 2-handle core disk system bounded by the simple loop system $\alpha_0 \cup \alpha$ for an arc system α_0 in F and the core arc system α of h. There is an O2-handle pair system $(D'_h \times I, E \times I)$ on $F(h)$ obtained by replacing the 2-handle system $D_h \times I$ with a 2-handle system $D'_h \times I$. Then the 2-handle system $D'_h \times I$ is equivalent to the 2-handle system $D_h \times I$ under 3-cell moves keeping $F(h)$ fixed by the common 2-handle property, [9, 10]. Thus, the self $(1, 2)$-handle pair system $(h, D'_h \times I)$ on F is locally standard. This completes the proof of Theorem 1.1.

The proof of Lemma 1.2 is done as follows.

Proof of Lemma 1.2. Let F be a surface-link in S^4 of a possibly non-sphere surface-knot component K and the remaining S^2-link $L = F \setminus K$. Since the second homology class $[K] = 0$ in $H_2(S^4 \setminus L; Z) = 0$, there is a compact connected oriented 3-manifold V_K smoothly embedded in S^4 with $\partial V_K = K$ and $V_K \cap L = \emptyset$. Let h_K be a 1-handle system on K in V_K such that the closed complement $H_K = \mathrm{cl}(V_K \setminus h_K)$ is a handlebody given by a decomposition into a 3-ball B_K and an attaching 1-handle system H^1_K. Let S be any S^2-knot component in L, which bounds a compact connected oriented 3-manifold V_S smoothly embedded in S^4 such that $V_S \cap (L \setminus S) = \emptyset$. The 3-ball B_K and the 1-handle system H^1_K are deformed in S^4 by shrinking B_K into a smaller 3-ball and the 1-handle system H^1_K into a thinner 1-handle system so that $V_S \cap B_K = \emptyset$ and the 1-handle system H^1_K transversely meets V_S with transversal disks in the interior of V_S. Then there is a 1-handle system h_S on S in V_S such that the closed complement $H_S = \mathrm{cl}(V_S \setminus h_S)$ is a handlebody given by a decomposition into a 3-ball B_S and an attaching 1-handle system H^1_S so that the transversal disks of H^1_S in the interior of V_S are in the interior of B_S. Then the surface-link $K(h_K) \cup S(h_S)$ is a ribbon surface-link given by the trivial S^2-link $\partial B_K \cup \partial B_S$ and the 1-handle system $H^1_K \cup H^1_S$. Because h_S is a 1-handle system on the surface-link $K(h_K) \cup S$ and the core arc system of h_S transversely meets the interior of h_S with finite points by general position, the 1-handle systems h_K and h_S are made disjoint by isotopic deformations of h_S keeping $K(h_K) \cup S$ fixed which are changing h_S into a thinner 1-handle system and then sliding h_S along h_K. Next, let T be any S^2-knot component in $L \setminus S$, which bounds a compact connected oriented 3-manifold V_T smoothly embedded in S^4 such that $V_T \cap (L \setminus (S \cup T)) = \emptyset$. The 3-balls B_K, B_S and the 1-handle systems H^1_K and H^1_S are deformed in S^4 so that $V_T \cap (B_K \cup B_S) = \emptyset$ and the 1-handle systems H^1_K and H^1_S transversely meet V_T with transversal disks in the interior of V_T. Then there is a 1-handle system h_T on T in V_T such that the closed complement $H(T) = \mathrm{cl}(V_T \setminus h_T)$ is a handlebody given by a decomposition into a 3-ball $B(T)$ and an attaching 1-handle system H^1_T such that the transversal disks of H^1_K and H^1_S in the interior of V_T are in the interior of B_T. Then the surface-link $K(h_K) \cup S(h_S) \cup T(h_T)$ is a ribbon surface-link given by the trivial S^2-link $\partial B_K \cup \partial B_S \cup \partial B_T$ and the 1-handle system $H^1_K \cup H^1_S \cup H^1_T$. Because h_K and h_S are disjoint and h_T is a 1-handle system on the surface-link $K(h_K) \cup S(h_S) \cup T$ and the core arc system of h_T transversely meets the interior of $h_K \cup h_S$ with finite points by general position, the 1-handle systems h_K, h_S and h_T are made disjoint by isotopic deformations of h_T keeping $K(h_K) \cup S(h_S) \cup T$ fixed. By continuing this process, it is shown that there is a self 1-handle system h on F such that the surface-link $F(h)$ is a ribbon .surface-link. This completes the proof of Lemma 1.2.

Before proving Theorem 1.5, a generalization of the null-homotopic Gauss sum invariant of a surface-knot to a surface-link is discussed, [11]. The *quadratic function* $\eta : H_1(K; Z_2) \to Z_2$ of a surface-knot K in S^4 is defined as follows. For a loop ℓ on K, let d be a compact (possibly non-orientable) surface in S^4 with $d \cap K = \partial d = \ell$. The value $\eta([\ell])$ is defined by the Z_2-self-intersection number $\mathrm{Int}(d, d) \mod 2$ with respect to the framing of the surface K which is independent of a choice of d by calculation. The function $\eta : H_1(K; Z_2) \to Z_2$ is a Z_2-quadratic function with the identity

$$\eta(x + y) = \eta(x) + \eta(y) + x \cdot y,$$

where $x, y \in H_1(K; Z_2)$ and $x \cdot y$ denotes the Z_2-intersection number of x and y in K. A loop ℓ on K is *spin* or *non-spin* according to whether $\eta([\ell])$ is 0 or 1, respectively. For a surface-link F in S^4, the *quadratic function* $\eta : H_1(F; Z_2) \to Z_2$ of F is defined to be the split sum of the quadratic functions $\eta_K : H_1(K; Z_2) \to Z_2$ for all the components K of F. This quadratic function may be identified with the quadratic function $\eta_\# : H_1(F_\#; Z_2) \to Z_2$ of a surface-knot $F_\#$ in S^4 which is a fusion of F along a fusion 1-handle system on F under a canonical isomorphism $\iota : H_1(F; Z_2) \to H_1(F_\#; Z_2)$. To see this, let ℓ be a loop in a component K of F, d a compact surface in S^4 with $d \cap K = \partial d = \ell$, and $F_\#$ the surface obtained from F by surgery along a fusion 1-handle system h on F. Every transverse intersection point between d and $F \setminus K$ in S^4 can be moved into $K \setminus h \cap K$ through $F_\#$, so that the compact surface d is modified into a compact surface d_K in S^4 with $d_K \cap F = \partial d_K = \ell \cup o_K$ for a trivial loop system o_K in K. Since every loop of o_K is a spin loop in $F_\#$, the identity

$$\eta_\#(\iota([\ell])) = \eta_\#(\iota([\ell]) + \iota([o_K])) = \eta([\ell]),$$

holds, showing the identification of η to $\eta_\#$. Let $\Delta(F; Z_2)$ be the subgroup of $H_1(F; Z_2)$ consisting of an element represented by a loop ℓ in F which bounds an immersed disk d in S^4 with $d \cap F = \ell$. The restriction $\xi : \Delta = \Delta(F; Z_2) \to Z_2$ of the quadratic function η on $H_1(F; Z_2)$ is called the *null-homotopic quadratic function* of the surface-link F. The *null-homotopic Gauss sum* of F is the Gauss sum $GS_0(F)$ of ξ defined by

$$GS_0(F) = \sum_{x \in \Delta} \exp(\xi(x)\pi\sqrt{-1}),$$

where $\xi : \Delta = \Delta(F; Z_2)$. This number $GS_0(F)$ is an invariant of a surface-link F, which is calculable as shown for the case of a surface-knot, [11]. Then it is known that *if F is a ribbon surface-link of total genus g, then $GS_0(F) = 2^g$.* By using this invariant $GS_0(F)$, the proof of Theorem 1.5 strengthening an earlier result of [4, Theorem 2] is obtained as shown below.

Proof of Theorem 1.5. Let $k \cup k'$ be a non-split link in the interior of a 3-ball B such that k and k' are trivial knots. For the boundary 2-sphere $S^B = \partial B$ and the disk D^2 with boundary circle S^1, let K be the surface-link of torus-components $T = k \times S^1$ and $T' = k' \times S^1$ in the

4-sphere S^4 with $S^4 = B \times S^1 \cup S \times D^2$, which is a ribbon surface-link in S^4, [2]. Then, $GS_0(K) = 2^2$. Since k and k' are trivial knots in B, the torus-knots T and T' are trivial torus-knots in S^4 by construction. Since $k \cup k'$ is non-split in B, there is a simple loop $t(k)$ in T coming from the longitude of k in B such that $t(k)$ does not bound any disk not meeting T' in S^4, meaning that there is a simple loop c in T unique up to isotopies of T which bounds a disk d in S^4 not meeting T', where c and d are given by $c = \{p\} \times S^1$ and $d = a \times S^1 \cup \{q\} \times D^2$ for a simple arc a in B joining a point p of k to a point q in S with $a \cap (k \cup k') = \{p\}$ and $a \cap S = \{q\}$. Regard the 3-ball B as the product $B = B_1 \times [0, 1]$ for a disk B_1. Let τ_1 be a diffeomorphism of the solid torus $B_1 \times S^1$ given by one full-twist rounding the meridian disk B_1 one time along the S^1-direction. Let $\tau = \tau_1 \times 1$ be the product diffeomorphism of $(B_1 \times S^1) \times [0, 1] = B \times S^1$ for the identity map 1 of $[0, 1]$. Let τ_∂ be the diffeomorphism of the boundary $S^B \times S^1$ of $B \times S^1$ obtained from τ by restricting to the boundary, and the 4-manifold M obtained from $B \times S^1$ and $S^B \times D^2$ by pasting the boundaries $\partial(B \times S^1) = S^B \times S^1$ and $\partial(S^B \times D^2) = S^B \times S^1$ by the diffeomorphism τ_∂. Since the diffeomorphism τ_∂ of $S^B \times S^1$ extends to the diffeomorphism τ of $B \times S^1$, the 4-manifold M is diffeomorphic to S^4. Let $K_M = T_M \cup T'_M$ be the surface-link of torus components T_M and T'_M in the 4-sphere M, arising from $K = T \cup T'$ in $B \times S^1$. There is a meridian-preserving isomorphism $\pi_1(S^4 \setminus K, x) \to \pi_1(M \setminus K_M, x)$ by van Kampen theorem. The loop $t(k)$ in T_M does not bound any disk not meeting T'_M in M, so that the loop c in T_M is a unique simple loop up to isotopies of T_M which bounds a disk $d_M = a \times S^1 \cup D^2_M$ in M not meeting T'_M, where D^2_M denotes a proper disk in $S^B \times D^2$ bounded by the loop $\tau_\partial(\{q\} \times S^1)$. An important observation is that the self-intersection number $\mathrm{Int}(d_M, d_M)$ in M with respect to the surface-framing on K_M is ± 1. This means that the loop c in T_M is a non-spin loop. Similarly, there is a unique non-spin loop c' in T'_M which bounds a disk d'_M with the self-intersection number $\mathrm{Int}(d'_M, d'_M) = \pm 1$ with respect to the surface-framing on K_M. Then it is calculated that $GS_0(K_M) = 0$ and the surface-link K_M in M is a non-ribbon torus-link, [11]. Let $(S^4, K') = (M, K_M)$. For every closed oriented disconnected surface $\mathbf{F}$ with at least two aspheric components, the pair (F, F') of a ribbon $\mathbf{F}$-link F and a non-ribbon $\mathbf{F}$-link F' is constructed from the pair (K, K') by connected-summing a trivial surface-knot with both K and K' and/or adding a trivial surface-link component to both K and K' as a splitting component, because if F' is a ribbon surface-link then K' must be a ribbon S^2-link, [3]. For the total genus $g(\geq 2)$ of F, it is calculated that $GS_0(F) = 2^g$ and $GS_0(F') = 2^{g-2}$. Let A be a 4-ball in S^4 such that $A \cap K = A \cap K'$ is a trivial disk system in A taking one disk from each component of K and from each component of K'. Assume that the connected sum and the addition operations used to construct (F, F') from (K, K') are done in the 4-ball A. By van Kampen theorem, it is shown that the fundamental groups $\pi_1(S^4 \setminus F, x)$ and $\pi_1(S^4 \setminus F', x)$ are the same group up to meridian-preserving isomorphisms. Since the boundary loop of a transverse disk of a 1-handle is a spin loop, the null-homotopic Gauss sum invariant is shown to be independent of choices of a self 1-handle by a calculation of the Z_2-quadratic function identity, [11]. Thus, if the self-trivial 1-handle system h of s

members are done for F and F', then $F(h)$ is a ribbon surface-link of total genus $g+s$ with $GS_0(F(h)) = 2^{g+s}$ and $F'(h)$ is a non-ribbon surface-link with $GS_0(F'(h)) = 2^{g-2+s}$, so that F' is not a quasi-ribbon surface-link. This completes the proof of Theorem 1.5.

In the proof of Theorem 1.5, note that the non-ribbon surface-linkK' of two components starting from the Hopf link $k \cup k'$ in the interior of a 3-ball B has the free abelian fundamental group of rank 2. Then by van Kampen theorem, the surface-knot $K'(h)$ obtained from K' by surgery along any fusion 1-handle h on K' has the infinite cyclic fundamental group, so that $K'(h)$ is a trivial surface-knot in S^4 by smooth unknotting result of a surface-knot, [9, 10]. This example shows that there is a non-ribbon surface-link F' of ribbon components such that the surface-knot obtained from F' by surgery along any fusion 1-handle is a ribbon surface-knot meaning that non-ribbonness of a surface-link cannot detect by surgery along any fusion 1-handle, giving a strong counterexample to [3, Theorem 1.4]. A positive result for a boundary surface-link, [7]. The diffeomorphism τ_∂ of $S^B \times S^1$ in the proof of Theorem 1.4 coincides with Gluck's non-spin diffeomorphism of $S^2 \times S^1$, [12]. The surface-link (M, K_M) called a *turned torus-link* of a link $k \cup k'$ in B is an analogy of a *turned torus-knot* of a knot in B, [13].

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