

# Non-trivialization probability of arc system in three-dimensional space

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## ABSTRACT

The type-specific knotting probability of an arc diagram is earlier defined by using chord diagrams of ribbon surface-links in 4D space. By modifying this notion, Non-Trivialization probability (simply NT probability) for the arc diagram is introduced and generalized to an arc system diagram. Some properties of the NT probability are shown. The method of transforming a polygonal arc in 3D space into a unique arc diagram up to isomorphisms earlier developed is generalized to a polygonal arc system in 3D space to define the NT probability.

*Keywords:* Arc system, Ribbon surface-link, Chord diagram, Knotting probability, Non-trivialization probability.

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## 1. Introduction

An  $n$ -arc system in 3-dimensional space (simply 3D space)  $R^3$  is the system  $L$  of mutually disjoint polygonal oriented arcs  $a_i$  ( $i = 1, 2, \dots, n$ ) in  $R^3$  (Where unnecessary, the orientation of the arc  $a_i$  is omitted). An arc system  $D$  in an oriented plane  $P$  is an  $n$ -arc diagram if  $D$  has only transversely meeting crossing points with upper-lower relation (apart from the end points) in  $P$ . A standard arc system  $D$  in an oriented plane  $P$  is obtained as the image  $\lambda(L)$  of an  $n$ -arc system  $L$  in  $R^3$  under an orthogonal projection  $\lambda : R^3 \rightarrow P$  such that the singularity  $\mathcal{S}(L)$ , the set of a point  $x \in L$  with  $|\lambda^{-1}(\lambda(x))| \geq 2$ , consists of a non-vertex point  $x$  of an edge in  $L$  with  $|\lambda^{-1}(\lambda(x))| = 2$ . An arc system diagram is an  $n$ -arc diagram for some  $n$ . A chord graph is a trivalent graph  $(o; \alpha)$  in the 3D space  $R^3$  consisting of a trivial oriented link  $o$  (called a based loop system) and attaching simple arcs  $\alpha$  (called a chord system), and a chord diagram is a planar representation  $C(o; \alpha)$  of a chord graph  $(o; \alpha)$  in a

plane  $P$ , which represents a unique ribbon surface-link  $F(o; \alpha)$  in 4-dimensional space (simply 4D space)  $R^4$ , [1, 2]. Two chord diagrams  $C = C(o, \alpha)$  and  $C' = C(o', \alpha')$  in a plane  $P$  are *isomorphic* if there is an orientation-preserving self-homeomorphism  $f : P \rightarrow P$  sending  $C$  to  $C'$  which preserves the orientations of the based loops  $o$  and  $o'$ , and *equivalent* if the ribbon surface-link  $F(o; \alpha)$  is equivalent to the ribbon surface-link  $F(o'; \alpha')$ . Equivalence between  $C$  and  $C'$  in  $P$  is understood by a finite number of the moves  $M_i$  ( $i = 0, 1, 2$ ) on chord diagrams, [3]. This notion is also explained in Section 2. The chord diagram  $C = C(D) = C(o, \alpha)$  of an  $n$ -arc diagram  $D$  is *basically equivalent* to the chord diagram  $C' = C(D') = C(o', \alpha')$  of an  $n$ -arc diagram  $D'$  if, up to isomorphisms of  $C$ , the based loop  $o$  is identical to the based loop  $o'$  and the chord system  $\alpha'$  is obtained from the chord system  $\alpha$  by a finite number of moves consisting of a *chord-homotopy move* (which is a plane homotopy move of a chord which does not change around the starting and terminal based loop system  $o^s(a_i), o^t(a_i)$  ( $i = 1, 2, \dots, n$ )), a *chord slide move* and a *fusion-fission chord move*, illustrated in Fig. 1. These moves are cosequences of the moves  $M_i$  ( $i = 0, 1, 2$ ). An  $n$ -arc diagram  $D$  is *basically equivalent* to an  $n$ -arc diagram  $D'$  if  $C(D)$  is basically equivalent to  $C(D')$ .

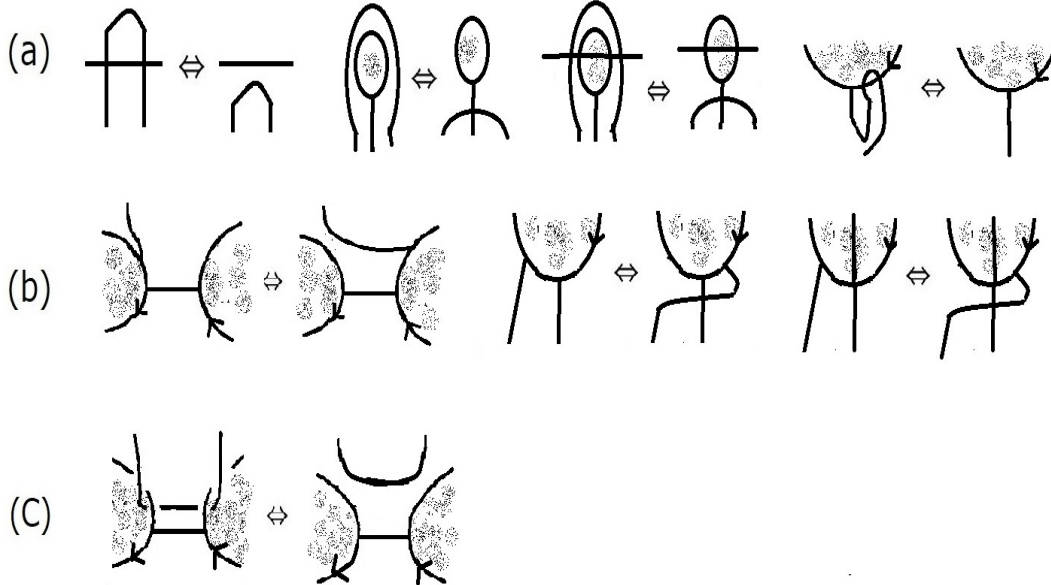


Figure 1: A chord-homotopy move (a), a chord slide move (b) and a fusion-fission chord move (c)

There is a method to transform every  $n$ -arc diagram  $D$  of  $c$  crossing number into a chord diagram  $C = C(D)$  with  $c + 2n$  based loops uniquely up to isomorphisms,

where the trivial based loops  $o^s(a_i)$  and  $o^t(a_i)$  are attached to the starting point  $s(a_i)$  and the terminal point  $t(a_i)$  of every arc  $a_i$  in  $D$ , respectively. This transformation was used in [4] and illustrated in Fig. 2. The original  $n$ -arc diagram  $D$  is recovered from the chord diagram  $C = C(D) = C(o, \alpha)$  by taking the upper arc part of every based loop system  $o$ . The first purpose of this paper is to show the following theorem.

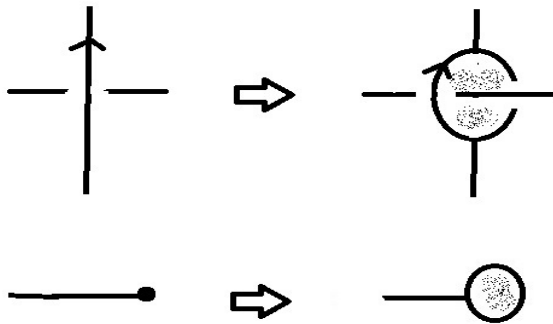


Figure 2: Transforming the areas around a crossing point and an endpoint into chord diagrams

**Theorem 1.1.** For any  $n$ -arc diagram  $D$  and any integer  $d$  satisfying  $1 \leq d \leq 4$ , there exists a basic equivalence invariant  $\kappa_d(D)$  on  $D$  that takes a rational value in the interval  $[0, 1]$  and is independent of choices of orientations of  $a_i$  ( $i = 1, 2, \dots, n$ ).

The proof of Theorem 1.1 is done in Section 3. This invariant  $\kappa_d(D)$  is called *Non-Trivialization probability* (simply *NT probability*) of *grade*  $d$  for the  $n$ -arc diagram  $D$ . The NT probabilities  $\kappa_1(D)$  and  $\kappa_4(D)$  of grades 1 and 4 are also called the *reduced NT probability* and the *full NT probability* for  $D$ , respectively. The *NT probability*  $\kappa(D)$  for an  $n$ -arc diagram  $D$  is the quadruplet

$$\kappa(D) = (\kappa_1(D), \kappa_2(D), \kappa_3(D), \kappa_4(D)).$$

Some properties of the NT probability  $\kappa(D)$  are shown in Theorem 3.1. The *mirror image*  $D^*$  of an  $n$ -arc diagram  $D$  is an  $n$ -arc diagram obtained from  $D$  by changing the upper-lower relation of every crossing point of  $D$ . The NT probability  $\kappa(D^*)$  for  $D^*$  is generally different from  $\kappa(D)$ . The NT probability  $\kappa(D)$  can be understood as the *NT probability of  $D$  toward the future*, and  $\kappa(D^*)$  as the *NT probability of  $D$  toward the past*. The second purpose of this paper is to show the following theorem.

**Theorem 1.2.** For every  $n$ -arc system  $L$  in  $R^3$  up to orientation-preserving Euclidean isometries of  $R^3$ , there exists a unique system of  $n$ -arc diagrams  $D(L)_{ij}$  ( $i, j = 1, 2, \dots, n$ ) up to isomorphisms.

The proof of Theorem 1.2 is done in Section 5. By Theorem 1.1, the NT probability

$$\kappa(D(L)_{ij}) = (\kappa_1(D(L)_{ij}), \kappa_2(D(L)_{ij}), \kappa_3(D(L)_{ij}), \kappa_4(D(L)_{ij}))$$

of the  $n$ -arc diagrams  $D(L)_{ij}$  is defined for every  $i, j$ . By taking the additive average

$$\kappa_d(L) = \sum_{i,j=1}^n \kappa(D(L)_{ij})/n^2,$$

the *NT probability*  $\kappa(L) = (\kappa_1(L), \kappa_2(L), \kappa_3(L), \kappa_4(L))$  of every  $n$ -arc system  $L$  up to orientation-preserving Euclidean isometries of  $R^3$  is defined.

In Section 2, basic concepts on the chord diagram of an  $n$ -arc diagram are discussed. In Section 3, the NT probability of an  $n$ -arc diagram is defined and some properties of the NT probability are shown. In Section 4, the unique system of  $n$ -arc diagrams for an  $n$ -arc system in 3D space  $R^3$  is shown. In Section 5, computation examples of 2-arc systems in 3D space are given.

At the end of the introduction, the author refers to several historical notes. Arcs in  $R^3$  from the random knotting viewpoint independent of the present viewpoint were studied, [5], “KnotProt” (<https://knotprot.cent.uw.edu.pl/>). This research come from the question: *How a linear scientific object such as a non-circular molecule (that is, a non-circular DNA, protein, linear polymer, etc.) is considered as a mathematical knot object?* Also, a knotting probability on classical knots was studied from the viewpoint of a random knotting, [6, 7]. A knotting probability of an arc in  $R^3$  was also studied by the author from a knotting structure of a graph in  $R^3$  but with the demerit that it depends on the heights of the crossing points of a diagram of the arc, [8, 9]. The knotting probability of an arc diagram that resolves this demerit has been studied in [4]. When attempting to generalize this knotting probability to the case of an  $n$ -arc diagram, we encountered a serious problem that does not manifest in the case of an arc, which motivated us to define the current NT probability. The study of an arc system in 3D space is a natural generalization of the study of an arc in 3D space, [10].

## 2. Basic concepts on chord diagram of arc system diagram

A chord diagram  $C = C(o; \alpha)$  is *regular* if the based loops  $o$  bound mutually disjoint disks in the plane  $P$  meeting only with the chords transversely. A regular chord diagram is *oriented* if every chord is attached to the based loops as in Fig. 3.

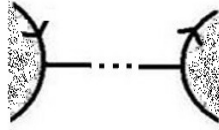


Figure 3: A chord in an oriented regular chord diagram

An *orientable chord diagram* is a chord diagram  $D$  which becomes an oriented regular chord diagram after applying a finite number of the move  $M_0$  to  $D$ . Unless otherwise mentioned, a chord diagram means an orientable regular chord diagram. There are three moves  $M_0$ ,  $M_1$  and  $M_2$  on chord diagrams, [1].

**Move  $M_0$ .** This move consists of the Reidemeister moves  $R_1$ ,  $R_2$ ,  $R_3$ ,  $gR_4$ ,  $gR_5$  as trivalent graphs in  $R^3$ , illustrated in Fig. 4, where note that two arcs in the three arcs together with a vertex or an arc can be a part of a based loop although the orientation and the shadow of the based loop are omitted.

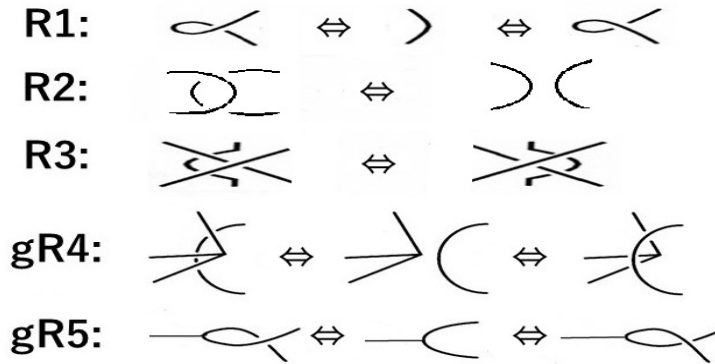


Figure 4: Move  $M_0$ : Reidemeister moves  $R_1$ ,  $R_2$ ,  $R_3$ ,  $gR_4$ ,  $gR_5$  for trivalent graph diagrams

**Move  $M_1$ .** This move is the *fusion-fission move*, illustrated in Fig. 5, where the fusion operation is done only for a chord between different based loops.

**Move  $M_2$ .** This move consists of moves on chords illustrated in Fig. 6.

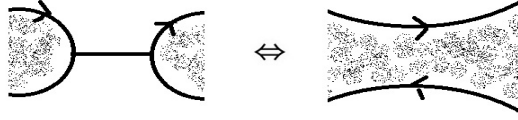


Figure 5: Fusion-fission move  $M_1$

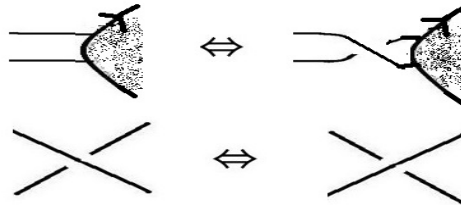


Figure 6: Chord moves  $M_2$

Two chord diagrams  $C = C(o, \alpha)$  and  $C' = C(o', \alpha')$  in a plane  $P$  are *equivalent* if one is obtained from the other by a finite number of the moves  $M_i$  ( $i = 0, 1, 2$ ) up to isomorphisms. The following result is known, [1, 3, 11].

**Lemma 2.1.** Two ribbon surface-links  $F(o; \alpha)$  and  $F(o'; \alpha')$  are equivalent (that is, sent by an orientation-preserving diffeomorphism of  $R^4$ ) if and only if the chord diagrams  $C(o'; \alpha')$  and  $C(o; \alpha)$  are equivalent.

The following lemma is basic to determine whether a chord diagram is trivial, [12].

**Lemma 2.2.** A surface-link  $F$  is a trivial surface-link in  $R^4$  if and only if the fundamental group  $\pi_1(R^4 \setminus F, x_0)$  is a free group with a meridian basis.

A useful invariant for determining whether a chord diagram  $C = C(o; \alpha)$  is trivial or not is the *finitely presented group*  $\pi(C)$ , which is identified with the fundamental group  $\pi_1(R^4 \setminus F(C), x_0)$  and calculated as in Fig. 7, where the generator system is represented by a meridian system of  $F(C)$  in  $R^4$ , [1, 13]. Since all the edges of  $C$  attaching to every based loop of  $o$  have the same color in the group presentation  $\pi(C)$ , the based loop system  $o$  is identified with the generator system of the finitely presented group  $\pi(C)$ . Note that the presented group  $\pi(C)$  is identical to the group of a virtual graph diagram interpreted from the chord diagram  $C$ , [1, 2]. For the group  $\pi(C)$  of a chord diagram  $C$ , let  $\gamma : \pi(C) \rightarrow \mathbf{Z}$  be the eimorphism sending the

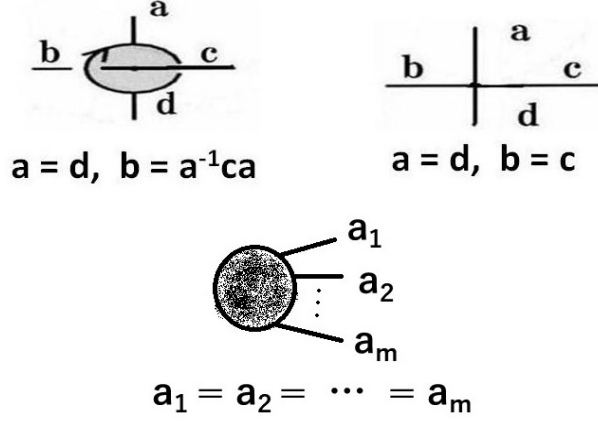


Figure 7: Group relations

generators of the presented group  $\pi(C(D))$  to the unit  $1 \in \mathbf{Z}$ , and  $K(\gamma)$  the kernel subgroup of  $\gamma$ . The quotient group  $K(\gamma)/K(\gamma)'$  for the commutator subgroup  $K(\gamma)'$  of  $K(\gamma)$  forms a finitely generated  $\Lambda$ -module  $M(C)$ , called the *module* of the chord diagram  $C$ , where  $\Lambda$  denotes the integral Laurent polynomial ring  $\mathbf{Z}[t, t^{-1}]$  identical to the integral group ring  $\mathbf{Z}[\mathbf{Z}]$ . The  $\Lambda$ -module  $M(C)$  can be obtained from the group presentation of  $\pi(C)$  by Fox's free differential calculus and is an invariant of an oriented chord diagram  $C$  up to equivalences, [14, 15, 16]. However, note that the  $\Lambda$ -module  $M(C)$  depends on choices of orientations of the chord diagram  $C$  in general (see Example 4.3 later).

### 3. Non-trivialization probability of arc system diagram

Let  $D$  be an  $n$ -arc diagram of an arc system  $L = \{a_i | i = 1, 2, \dots, r\}$ , and  $C(D)$  the chord diagram of  $D$ . For an arc  $a_i$  in  $L$ , let  $c_i$  be the upper crossing number (i.e., the number of the upper crossing points) of  $a_i$  in  $D$ . Let  $C(a_i^D)$  be the chord subdiagram of the chord diagram  $C(D)$  on the arc  $a_i$  which has  $c_i + 2$  based loops including the two trivial based loops  $\sigma^t(a_i)$  and  $\sigma^t(a_i)$  of the endpoints of  $a_i$ . Note that the chord diagram  $C(a_i^D)$  is obtained from the chord diagram of the 1-arc diagram of  $a_i$  in  $D$  by adding some extra trivial based loops. The proof of Theorem 1.1 is done as follows.

*3.1: Proof of Theorem 1.1.* An  $a_i$ -based adjoint chord diagram of the chord diagram  $C(D)$  is a chord diagram obtained from  $C(D)$  by connecting  $\sigma^t(a_i)$  and  $\sigma^t(a_i)$  to any different based loops of  $C(a_i^D)$  with a single chord each not passing the other based loops of  $C(D)$ . Let  $A(D, a_i)$  be the the system of  $a_i$ -based adjoint chord diagrams of

$C(D)$  consisting of  $(c_i + 2)^2$  chord diagrams. By removing some natural overlaps up to equivalences, the system  $A(D, a_i)$  is reduced to the system  $A^R(D, a_i)$  consisting of  $(c_i + 1)^2 + 1$  chord diagrams, which is classified into the four systems  $A^I(D, a_i)$  of type I,  $A^{II}(D, a_i)$  of type II,  $A^{III}(D, a_i)$  of type III, and  $A^{IV}(D, a_i)$  of type IV, which are illustrated in Fig. 8, [4].

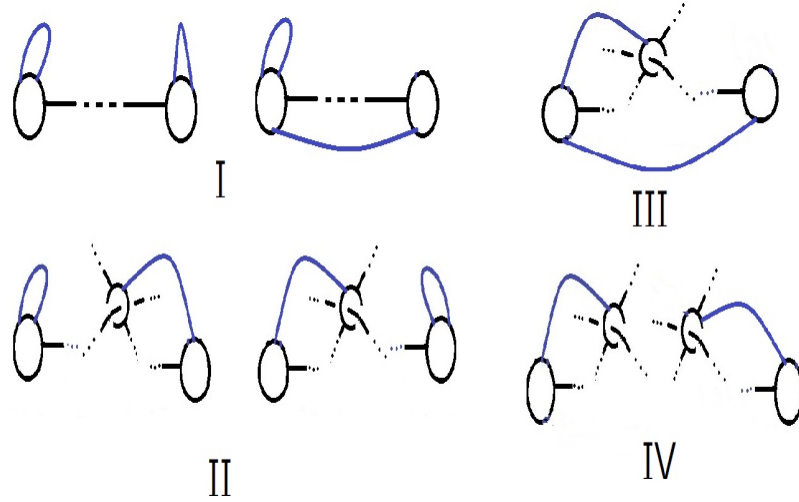


Figure 8: Chord diagrams of types I, II, III, IV

**Type I system**  $A^I(D, a_i)$ . Here are two members of  $a_i$ -based adjoint chord diagrams  $A_{ij} = A^{II}(D, a_i)_j$  ( $j = 1, 2$ ) of  $C(D)$ . One is the  $a_i$ -based adjoint chord diagram with two self-attaching additional chords. The other is the  $a_i$ -based adjoint chord diagram with a self-attaching additional chord on  $o^t(a_i)$  and an additional chord joining  $o^t(a_i)$  with  $o^t(a_i)$ . Let  $m_i^I$  be the number of non-trivial  $a_i$ -based adjoint chord diagrams in  $A^I(D, a_i)$ .

**Type II system**  $A^{II}(D, a_i)$ . Here are  $2c_i$  members of  $a_i$ -based adjoint chord diagrams  $A_{ij} = A^{II}(D, a_i)_j$  ( $j = 1, 2, \dots, 2c_i$ ) of  $C(D)$ , which are given by the additional chord pairs consist of a self-attaching additional chord on  $o^t(a_i)$  (or  $o^t(a_i)$ , respectively) and an additional chord joining  $o^t(a_i)$  (or  $o^t(a_i)$ , respectively) with a based loop except for  $o^t(a_i)$  and  $o^t(a_i)$ . Let  $m_i^{II}$  be the number of non-trivial  $a_i$ -based adjoint chord diagrams in  $A^{II}(D, a_i)$ .

**Type III system**  $A^{III}(D, a_i)$ . Here are  $c_i$  members of  $a_i$ -based adjoint chord diagrams  $A_{ij} = A^{III}(D, a_i)_j$  ( $j = 1, 2, \dots, n_i$ ) of  $C(D)$  where the additional chord pairs

consist of an additional chord joining  $o^t(a_i)$  with  $o^t(a_i)$  and an additional chord joining  $o^t(a_i)$  with a based loop except for  $o^t(a_i)$  and  $o^t(a_i)$ . Let  $m_i^{\text{III}}$  be the number of non-trivial  $a_i$ -based adjoint chord diagrams in  $A^{\text{III}}(D, a_i)$ .

**Type IV system**  $A^{\text{IV}}(D, a_i)$ . Here are  $c_i(c_i - 1)$  members of  $a_i$ -based adjoint chord diagrams  $A_{ij} = A^{\text{IV}}(D, a_i)_j$  ( $j = 1, 2, \dots, c_i(c_i - 1)$ ) of  $C(D)$  where the additional chord pair joins the pair of  $o^t(a_i)$  and  $o^t(a_i)$  with a pair of distinct based loops except for  $o^t(a_i)$  and  $o^t(a_i)$ . Let  $m_i^{\text{IV}}$  be the number of non-trivial  $a_i$ -based adjoint chord diagrams in  $A^{\text{IV}}(D, a_i)$ .

Since  $2 + 2c_i + c_i + c_i(c_i - 1) = 1 + (c_i + 1)^2$ , the reduced system  $A^R(D, a_i)$  consists of just  $1 + (c_i + 1)^2$  members of  $a_i$ -based adjoint chord diagrams. A general idea of NT probability is to measure to what extent non-trivial chord diagrams are included within the reduced  $a_i$ -based adjoint chord system  $A^R(D, a_i)$ . By Lemma 2.2, an  $a_i$ -based adjoint chord diagram  $A_{ij}$  in  $A^R(D, a_i)$  is trivial if and only if the finitely presented group  $\pi(A_{ij})$  of  $A_{ij}$  is a free group with a basis in the generator system. For an  $n$ -arc diagram  $D$  and an integer  $d$  with  $1 \leq d \leq 4$ , the  $a_i$ -based NT probability  $\kappa_d(D, a_i)$  of grade  $d$  is defined as follows.

$$\begin{aligned}\kappa_1(D, a_i) &= m_i^{\text{I}}/2, \\ \kappa_2(D, a_i) &= (m_i^{\text{I}} + m_i^{\text{II}})/(2 + 2c_i), \\ \kappa_3(D, a_i) &= (m_i^{\text{I}} + m_i^{\text{II}} + m_i^{\text{III}})/(2 + 3c_i), \\ \kappa_4(D, a_i) &= (m_i^{\text{I}} + m_i^{\text{II}} + m_i^{\text{III}} + m_i^{\text{IV}})/(1 + (c_i + 1)^2).\end{aligned}$$

By Lemma 2.1 and definition, the value  $\kappa(D)$  is independent of choices of orientations of  $a_i$  ( $i = 1, 2, \dots, n$ ) and invariant under basic equivalences of an  $n$ -arc diagram  $D$ . This completes the proof of Theorem 1.1.

The NT probability of the  $n$ -arc diagram  $D$  of grade  $d$  is defined by

$$\kappa_d(D) = \sum_{i=1}^n \kappa_d(D, a_i)/n.$$

Here,  $\kappa_1(D)$  and  $\kappa_4(D)$  are also called the *reduced NT probability* and the *full NT probability* of the  $n$ -arc diagram  $D$ , respectively. The NT probability  $\kappa(D)$  of an  $n$ -arc diagram  $D$  is defined by the quadruplet

$$\kappa(D) = (\kappa_1(D), \kappa_2(D), \kappa_3(D), \kappa_4(D)).$$

The following notation is used.

**Notation.** The quadruplet  $\kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$  of numbers  $\kappa_i$  ( $i = 1, 2, 3, 4$ ) in the interval  $[0, 1]$  is written as  $\kappa = 1, \kappa > 0$ , or  $\kappa = 0$  according to whether  $\kappa_i = 1$  ( $i = 1, 2, 3, 4$ ),  $\kappa_i > 0$  ( $i = 1, 2, 3, 4$ ), or  $\kappa_i = 0$  ( $i = 1, 2, 3, 4$ ), respectively.

An  $n$ -arc diagram  $D$  of an  $n$ -arc system  $L = \{a_i | i = 1, 2, \dots, r\}$  is *inbound* if  $D$  is included in an  $n$ -component link diagram  $\text{cl}(D)$  without extra crossing points and the endpoints  $s(a_i), t(a_i)$  of  $a_i$  ( $i = 1, 2, \dots, n$ ) are in the same region in the regions of the plane  $P$  divided by  $D$ , whose region is called the *front region*. The link diagram  $\text{cl}(D)$  in  $P$  is called the *closed link diagram* of an inbound  $n$ -arc diagram  $D$ , which is the union of  $D$  and a disjoint simple arc system  $\gamma_i$  ( $i = 1, 2, \dots, n$ ) with  $\partial\gamma_i = \{s(a_i), t(a_i)\}$  ( $i = 1, 2, \dots, n$ ). The following theorem on some properties of NT probability is proved.

**Theorem 3.1.** The NT probability  $\kappa(D)$  of an  $n$ -arc diagram  $D$  of an arc system  $L = \{a_i | i = 1, 2, \dots, n\}$  has the following properties (1)-(3)

(1)  $\kappa(D) = 0$  if and only if the chord diagram  $C(a_i^D)$  is a trivial chord diagram for all  $i$  and  $C(D)$  is basically equivalent to a split sum of the trivial chord diagrams  $C(a_i^D)$  for all  $i$ .

(2)  $\kappa(D) = 1$  if and only if the quotient group obtained from the presented group  $\pi(C(D))$  by identifying each of the generators  $\sigma^t(a_i)$  and  $\sigma^t(a_i)$  with a single generator in the based loop system of  $C(a_i^D)$  is not a free group with a basis in the generator system for every  $i$  ( $i = 1, 2, \dots, n$ ).

(3) If  $D$  is an inbound  $n$ -arc diagram and  $D^*$  is the mirror image of  $D$ , then  $\kappa(D) = \kappa(D^*)$  and  $\kappa_1(D) = m/2n$  for an integer  $m$  satisfying  $0 \leq m \leq n$  excluding  $m = 1, 3$  when  $n = 2$ , and excluding  $m = 1$  when  $n \neq 2$ .

A realization of the reduced NT probability  $\kappa_1(D)$  of an inbound  $n$ -arc diagram  $D$  excluding the exceptional values in Theorem 3.1(3) is achieved in Example 4.4.

**Proof of Theorem 3.1.** For (1), if  $\kappa(D) = 0$ , then the chord diagram  $C(D)$  is basically equivalent to a split sum of the chord diagrams  $C(a_i^D)$  ( $i = 1, 2, \dots, n$ ) and  $\kappa(C(a_i^D)) = 0$  for every  $i$ . In particular,  $\text{cl}_u(D)$  is a trivial link diagram since  $\kappa(C(a_i^D)) = 0$  implies that the knot diagram  $\text{cl}_u(a_i^D)$  is a trivial knot diagram for every  $i$ , [4]. The converse is obvious. This shows (1). For (2), let  $A_{ij}$  be any  $a_i$ -based adjoint chord diagram of  $C(D)$  in  $A^\Gamma(D, a_i)$ . The presented group  $\pi(A_{ij})$  is

obtained from the presented group  $\pi(C(D))$  by identifying the generator  $o^t(a_i)$  with a based loop generator of  $C(a_i^D)$  and the generator  $o^t(a_i)$  with a based loop generator of  $C(a_i^D)$ . By Lemma 2.2,  $\kappa(D) = 1$  if and only if  $\pi(A_{ij})$  is not a free group with based loop basis for all  $i$  and  $j$ , completing the proof of (2). For (3), by Artin's spinning construction of an inbound arc diagram  $D$ , the ribbon  $S^2$ -link  $F(C)$  of the chord diagram  $C = C(D)$  is sent to the ribbon  $S^2$ -link  $F(C^*)$  of the mirror image  $C^* = C(D^*)$  by an orientation-reversing diffeomorphism of  $R^4$ , [4]. This means that  $\kappa(D) = \kappa(D^*)$ . To research the value of the reduced NT probability  $\kappa_1(D)$  of an inbound arc diagram  $D$ , let  $C = C(D)$  be the chord diagram of an inbound  $n$ -arc diagram  $D$ . Let  $\text{cl}(D) * v$  be a diagram of an  $n$ -leafed bouquet  $L_v^+$  with vertex  $v$  in the 3-sphere  $R^3$  obtained from the link diagram  $\text{cl}(D)$  by adding  $n$  simple arcs  $\gamma'_i (i = 1, 2, \dots, n)$  disjoint except for the vertex  $v$  such that  $\gamma'_i$  joins a point of  $\gamma_i$  to the vertex  $v$  and  $\text{cl}(D) * v$  has no crossing points other than the crossing points of  $\text{cl}(D)$ . Let  $\text{cl}(D) * v \setminus \text{Int}\beta'_i$  is the subdiagram of  $\text{cl}(D) * v$  which is a diagram of the subgraph  $L_v^+ \setminus \text{Int}\beta'_i$  in  $R^3$  obtained from  $L_v^+$  by removing the interior  $\text{Int}\beta'_i$  of the arc  $\beta'_i$ . As explained in [15, p.204] and [17], by Artin's spinning construction, the fundamental group  $G$  of  $L_v^+ \setminus \text{Int}\beta'_i$  in  $R^3$  is identical to the finitely presented group  $\pi(C)$  of the chord diagram  $C$  such that the generator set of  $\pi(C)$  is a meridian system of  $L_v^+ \setminus \{v_i\}$ , and the fundamental group  $G_i$  of  $L_v^+ \setminus \text{Int}\beta'_i$  in  $R^3$  is identical to the finitely presented group  $\pi(C \cup \beta_i)$  of the chord diagram  $C(D) \cup \beta_i$  such that the generator set of  $\pi(C \cup \beta_i)$  is a meridian system of  $G_i$ . If  $\kappa_1(C) = 1/2n$ , then  $G$  and  $G_i$  for every  $i$  except for one index  $i$ , say  $i = 1$  are free groups with basis in the specified generator systems. By applying Dehn's lemma to the compact exterior of  $L_v^+ \setminus \text{Int}\beta'_i$  in  $R^3$ , the loop  $a_i \cup \beta_i$  bounds a disk  $d_i$  in  $R^3$  disjoint from the other component of  $L_v^+ \setminus \text{Int}\beta'_i$ . The disks  $d_i (i = 2, 3, \dots, n)$  are made disjoint. Then the loop  $a_i \cup \beta_i$  also bounds a disk  $d_n$  in  $R^3$  disjoint from the other component of  $L_v^+ \setminus \text{Int}\beta'_n$  and the disks  $d_i (i = 2, 3, \dots, n)$ . This means that the surface-links  $F(C)$  and  $F(C \cup \beta_i)$  are trivial surface-links for all  $i$ , so that  $\kappa_1(C) = 0$ , a contradiction. Hence  $\kappa_1(C) \neq 1/2n$ . Let  $n = 2$  and suppose that  $\kappa_1(C) = 3/4$ . Then the group  $G_1$  or  $G_2$ , say  $G_1$  must be a free group with basis in the specified generator system. By Dehn's lemma, the group  $G_2$  is also a free group with basis in the specified generator system. Then  $\kappa_1(C) = 0$ , a contradiction. Hence  $\kappa_1(C) \neq 3/4$ , showing (3). This completes the proof of Theorem 3.1.

The following corollary gives a criterion for an  $n$ -arc diagram  $D$  with  $\kappa(D) = 1$

**Corollary 3.2.** Let  $M(C)$  be the module of the chord diagram  $C = C(D)$  of an  $n$ -arc diagram  $D$  of an arc system  $L = \{a_i | i = 1, 2, \dots, n\}$ . Unless the module  $M(C)$  is  $\Lambda$ -isomorphic to the direct sum  $\Lambda^{n-1} \oplus TM(C)$  of a free  $\Lambda$ -module  $\Lambda^{n-1}$  of rank  $n - 1$

and a  $\Lambda$ -torsion module  $TM(C)$  generated by two  $\Lambda$ -elements, the NT probability  $\kappa(D) = 1$ .

**Proof of Corollary 3.2.** By Theorem 3.1 (2),  $\kappa(D) = 1$  if the  $\Lambda$ -module  $M(A_{ij})$  is not isomorphic to the free  $\Lambda$ -module  $\Lambda^{n-1}$  of rank  $n - 1$  for all  $i, j$ . Note that the module  $M(C)$  has  $\Lambda$ -rank  $n - 1$  and the  $\Lambda$ -module  $M(A_{ij})$  is a quotient  $\Lambda$ -module of  $M(C)$  by two  $\Lambda$ -elements. If the  $\Lambda$ -module  $M(A_{ij})$  for some  $i, j$  is a free module of rank  $n - 1$ , then the module  $M(C)$  must be  $\Lambda$ -isomorphic to the direct sum  $\Lambda^{n-1} \oplus TM(C)$  for a  $\Lambda$ -torsion module  $TM(C)$  generated by two  $\Lambda$ -elements, which contradicts the assumption of the module  $M(C)$ . Thus,  $\kappa(D) = 1$ . This completes the proof of Corollary 3.2.

For a 1-arc diagram  $D$ , the *knotted probabilities of types I, II, III, and IV* denoted by  $p^I(D)$ ,  $p^{II}(D)$ ,  $p^{III}(D)$ , and  $p^{IV}(D)$ , respectively were defined in [4]:

$$p^I(D) = m_1^I/2, \quad p^{II}(D) = m_1^{II}/2c_1, \quad p^{III}(D) = m_1^{III}/c_1, \quad p^{IV}(D) = m_1^{IV}/c_1(c_1 - 1).$$

The *type-specific knotted probability*  $p(D)$  of a 1-arc diagram  $D$  is defined to be the quadruplet

$$p(D) = (p^I(D), p^{II}(D), p^{III}(D), p^{IV}(D)).$$

The inconvenience of this definition is that the knotted probabilities  $p^{II}(D)$ ,  $p^{III}(D)$ ,  $p^{IV}(D)$  are not defined when  $c_1 = 0$ , and the knotted probability  $p^{IV}(D)$  is not defined when  $c_1 = 1$ , although these values were defined by 0, respectively. This inconvenience is serious to consider a general  $n$ -arc diagram  $D$  because  $c_i = 0$  happens often. From this reason, the NT probability  $\kappa(D)$  of an  $n$ -arc diagram  $D$  is introduced.

#### 4. Computation example on $n$ -arc diagram

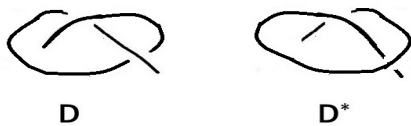


Figure 9: A 1-arc diagram  $D$  with 2 crossings and its mirror image  $D^*$

This section presents several examples of how to calculate the NT probability for  $n$ -arc diagrams. In the examples, the notation T or NT attached to each chord diagram in the system  $A^R(D, a_i)$  means a trivial chord diagram or a nontrivial chord diagram, respectively. If the upper crossing number  $c_1$  of a 1-arc diagram  $D$  is a previous

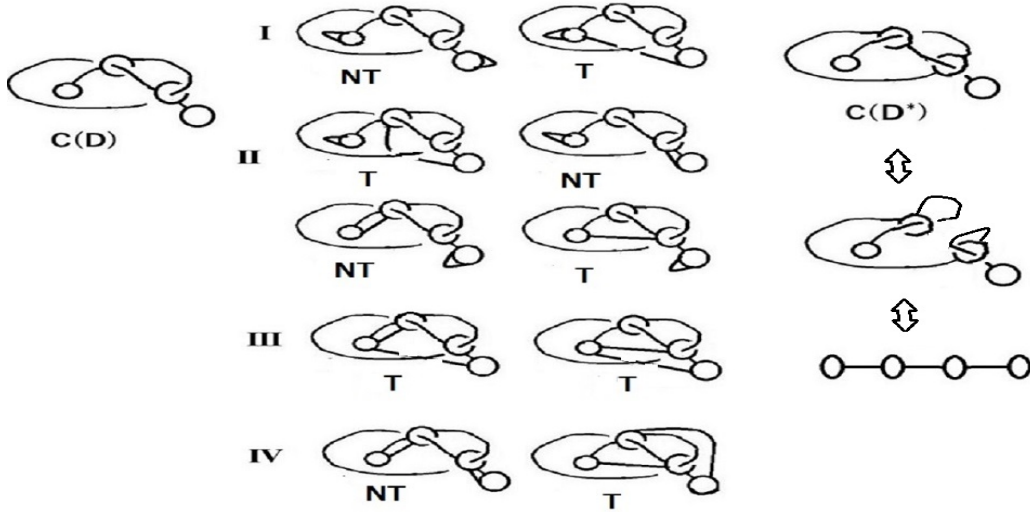


Figure 10: The system  $A^R(D, a_1)$  of  $a_1$ -based adjoint chord diagrams of the chord diagram  $C(D)$  of the diagram  $D$  in Fig. 9 and a basic equivalence of the mirror image  $D^*$  of  $D$  to a chord diagram without crossing

known integer greater than or equal to 2, then the type-specific knotting probability  $p(D)$  and the NT probability  $\kappa(D)$  are computable from each other. Some examples of the knotting probability  $p(D)$  for the 1-arc diagram  $D$  are calculated, [4]. So, the NT probabilities  $\kappa(D)$  of these examples are obtained by rewriting the values of these examples of  $p(D)$ . In the following example, the knotting probability was calculated in [4], but here the NT probability is directly calculated.

**Example 4.1.** For the 1-arc diagram  $D$  with  $c_1 = 2$  in Fig. 9, the system  $A^R(D, a_1)$  of  $a_1$ -based adjoint chord diagrams on the chord system  $C = C(D)$  of  $D$  is listed in Fig. 10. The knotting probability  $p(D)$  of  $D$  is calculated as  $p^I(D) = 1/2$ ,  $p^{II}(D) = 2/4 = 1/2$ ,  $p^{III}(D) = 0/2 = 0$ , and  $p^{IV}(D) = 1/2$ . Hence  $p(D) = (1/2, 1/2, 0, 1/2)$ . The NT probability  $\kappa(D) = \kappa(D, a_1)$  of  $D$  is calculated as  $\kappa_1(D) = 1/2$ ,  $\kappa_2(D) = (1 + 2)/(2 + 4) = 1/2$ ,  $\kappa_3(D) = (1 + 2 + 0)/(2 + 4 + 2) = 3/8$ , and  $\kappa_4(D) = (1 + 2 + 0 + 1)/(2 + 4 + 2 + 2) = 2/5$ . Hence  $\kappa(D) = (1/2, 1/2, 3/8, 2/5)$ . The mirror image  $D^*$  of  $D$  in Fig. 9 is basically equivalent to a chord without crossing, so that  $\kappa(D^*) = 0$  by Theorem 1.1. This shows that the NT probability  $\kappa(D^*)$  of the mirror image  $D^*$  of an  $n$ -arc diagram  $D$  is generally different from the NT probability  $\kappa(D)$  of  $D$ .

**Example 4.2.** The 2-arc diagram  $D$  with  $c_1 = 2$  and  $c_2 = 1$  in Fig. 11 in-

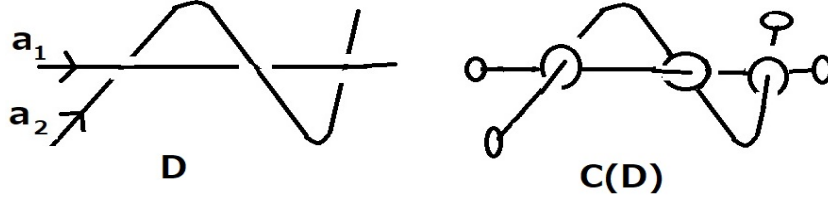


Figure 11: A 2-arc diagram  $D$  with 3 crossings and the chord diagram  $C(D)$

duces the system  $A^R(D, a_i)$  of  $a_i$ -based adjoint chord diagrams ( $i = 1, 2$ ) in Fig. 12. Then  $\kappa_1(D, a_1) = 1/2$ ,  $\kappa_2(D, a_1) = (1 + 2)/(2 + 4) = 1/2$ ,  $\kappa_3(D, a_1) = (1 + 2 + 2)/(2 + 4 + 2) = 5/8$ ,  $\kappa_4(D, a_1) = (1 + 2 + 2 + 1)/(2 + 4 + 2 + 2) = 3/5$ . Hence,  $\kappa(D, a_1) = (1/2, 1/2, 5/8, 3/5)$ . Also,  $\kappa_1(D, a_2) = 1/2$ ,  $\kappa_2(D, a_2) = (1 + 2)/(2 + 2) = 3/4$ ,  $\kappa_3(D, a_2) = (1 + 2 + 1)/(2 + 2 + 1) = 4/5 = \kappa_4(D, a_2)$ . Hence,  $\kappa(D, a_2) = (1/2, 3/4, 4/5, 4/5)$ . Thus,  $\kappa(D) = (1/2, 5/8, 57/80, 7/10)$ . Although  $D$  is not inbound, the mirror image  $D^*$  of the 2-arc diagram  $D$  is basically equivalent to  $D$ , so that by Theorem 1.1,

$$\kappa(D^*) = \kappa(D) = (1/2, 5/8, 57/80, 7/10).$$

The following example is an example on Theorem 3.1 (2) and Corollary 3.2.

**Example 4.3.** The finite presented group  $\pi(C)$  of the chord diagram  $C = C(D)$  of the 2-arc diagram  $D$ , illustrated in Fig. 13 is given by

$$\pi(C) = \langle x_0, x_1, y \mid x_0^{-1}x_1^{-1}yx_1y^{-1}x_1x_0y^{-1}x_0^{-1}y \rangle .$$

The module  $M(C)$  is calculated to be  $\Lambda$ -isomorphic to the  $\Lambda$ -ideal  $(t - 2, 2t - 1)$  of  $\Lambda$ , which is a torsion-free  $\Lambda$ -module of rank one, but not a free  $\Lambda$ -module. Thus,  $\kappa(D) = 1$  by Corollary 3.2. Let  $D'$  be the 2-arc diagram obtained from  $D$  by changing the orientation of only the arc  $a_2$  into the opposite orientation, and  $C' = C(D')$  the chord diagram of  $D'$ . Then the group  $\pi(C')$  has the presentation

$$\pi(C') = \langle x_0, x_1, y \mid x_0^{-1}x_1^{-1}y^{-1}x_1yx_1x_0yx_0^{-1}y^{-1} \rangle .$$

The module  $M(D')$  is calculated to be  $\Lambda$ -isomorphic to the direct sum  $\Lambda \oplus \Lambda / (t^2 - t + 1)$ , which cannot be used to determine that  $\kappa(D') = \kappa(D) = 1$ .

The following example shows that the value  $\kappa_1(D)$  of an inbound  $n$ -arc diagram  $D$  excluding the exceptional values in Theorem 3.1(3) is complete.

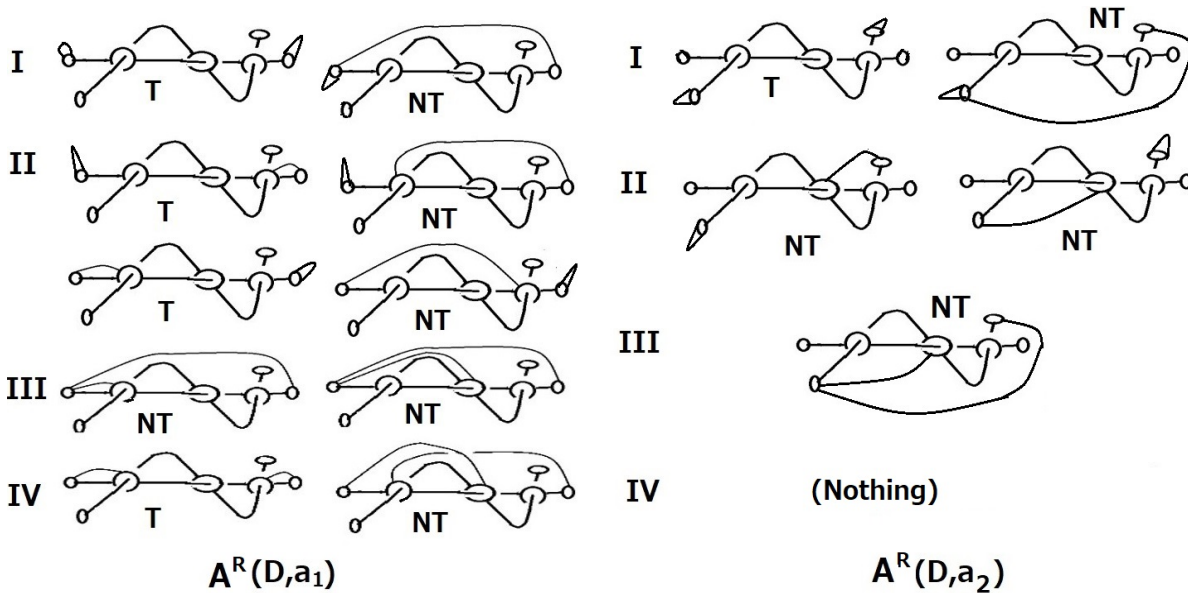


Figure 12: The system  $A^R(D, a_i)$  ( $i = 1, 2$ ) of  $a_i$ -based adjoint chord diagrams of the chord diagram  $C(D)$  of the diagram  $D$  in Fig. 11

**Example 4.4.** By Theorem 3.1 (3), the reduced NT probability  $\kappa_1(D)$  of an inbound  $n$ -arc diagram  $D$  of  $L = \{a_i | i = 1, 2, \dots, n\}$  takes a value  $m/2n$  with  $0 \leq m \leq 2n$  and  $m \neq 1, 2n - 1$ . Any such value  $m/2n$  is realized by an inbound  $n$ -arc diagram  $D$ . The split union diagram  $D$  of  $n$  copies of diagram  $D_1$  of Fig. 14 (a) has  $\kappa_1(D) = 0$  by definition. The split union diagram  $D$  of  $n$  copies of the diagram  $D_2$  of Fig. 14 (a) has  $\kappa_1(D) = 1$  since  $\kappa_1(D_2) = 1$ , [4]. So, assume that  $2 \leq m \leq 2n - 2$ . If  $m = 2p$  for an integer  $p$ , then  $1 \leq p \leq n - 1$ . Then take as  $D$  a split diagram of  $p$

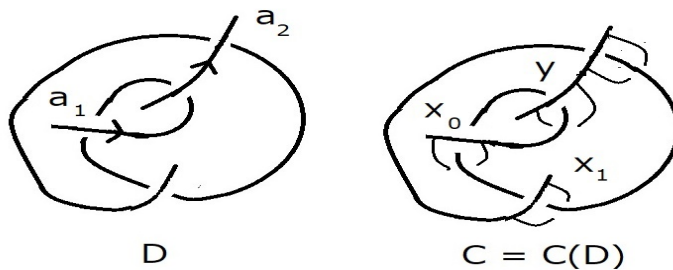


Figure 13: A 2-arc diagram  $D$  and the chord diagram  $C = C(D)$

copies of the diagram  $D_2$  of Fig. 14 (b) and  $n - p$  copies of the diagram  $D_1$ . Then  $\kappa_1(D) = p/n = m/2n$ . If  $m = 2p + 3$  for an integer  $p$ , then  $0 \leq p \leq n - 3$  and  $n \geq 3$ . If  $n = 3$ , then  $p = 0$  and take as  $D$  the diagram  $D_3$  of Fig. 14 (c) where  $\kappa_1(D) = 1/2 = m/2n$ , computed as  $\kappa_1(D, a_i) = 1/2$  ( $i = 1, 2, 3$ ). If  $n \geq 4$ , then take as  $D$  the split diagram of  $p$  copies of the diagram  $D_2$ , one copy of  $D_3$ , and  $n - 3 - p$  copies of the diagram  $D_1$ . Then  $\kappa_1(D) = (1 \times p + (1/2) \times 3)/n = m/2n$ . If a non-split diagram is wanted, then  $D$  can be transformed into a non-split inbound diagram by using R2-moves in Reidemeister move (cf. Fig. 4), where the resulting diagram is basically equivalent to the original diagram and thus, the reduced NT probability is unchanged. For  $n \geq 3$ , it is shown that there is an inbound non-split  $n$ -arc diagram  $D$  with  $\kappa_1(D) = (2n - 1)/n$ . Let  $o_1 \cup o_2$  be a split union of two trivial loops  $o_1, o'_1$  in  $R^3$ . Let  $\beta'_1$  be a simple arc in  $R^3$  joining a point of  $o_1$  and a point of  $o'_1$  such that the union  $o_1 \cup o'_1 \cup \beta'_1$  is a non-trivial (= non-planar) graph in  $R^3$ . Construct a trivial  $(n - 1)$ -leafed bouquet  $(\cup_{i=2}^n o_i) \cup (\cup_{i=2}^n \beta'_i)$  with vertex  $v$  in a solid torus neighborhood  $N(o'_1)$  of  $o'_1$  in  $R^3$  such that  $o_2 = o'_1$  and the leaf  $o_i$  ( $i \geq 3$ ) is parallel to  $o'_1$  in  $N(o'_1)$ , where  $\beta'_i$  denotes a leg (i.e., a simple arc joining a point of  $o_i$  to the vertex  $v$  with  $\beta'_i \setminus \{v\}$  ( $i = 2, 3, \dots, n$ ) disjoint). By sliding the endpoint of the arc  $\beta'_1$  in  $o'_1$  to the vertex  $v$  along  $o_2 \cup \beta'_2$ , regard  $\Gamma = (\cup_{i=1}^n o_i) \cup (\cup_{i=1}^n \beta'_i)$  as an  $n$ -leafed bouquet in  $R^3$ . Let  $B'$  be a regular neighborhood 3-ball of  $\cup_{i=1}^n \beta'_i$  in  $R^3$ , and  $D$  an inbound  $n$ -arc diagram in a plane  $P$  of a proper  $n$ -arc system  $L = \text{cl}(\Gamma \setminus \Gamma \cap B') = \cap_{i=1}^n a_i$  in a 3-ball  $B \subset R^3$  with  $a_i = \text{cl}(o_i \setminus o_i \cap B_i)$ . The  $n$ -leafed bouquet  $\Gamma$  is a non-trivial graph in  $R^3$ , by Artin's spinning construction, the chord diagram  $C = C(D)$  of the inbound  $n$ -arc diagram  $D$  is non-trivial. Since  $\Gamma$  is a non-trivial graph and  $(\cup_{i=1}^n o_i) \cup \beta'_1 \cup \beta'_j$  is a non-splittable non-trivial graph for  $j \geq 2$ , and  $(\cup_{i=1}^n o_i) \cup (\cup_{i=2}^n \beta'_i)$  is a trivial graph in  $R^3$ , it is computed via Artin's spinning construction that  $\kappa_1(D, a_1) = 1/2$  and  $\kappa_1(D, a_i) = 1$  for all  $i \geq 2$ , so that  $\kappa_1(D) = (1/2 + 1 \times (n - 1))/n = (2n - 1)/2n$ , as desired.

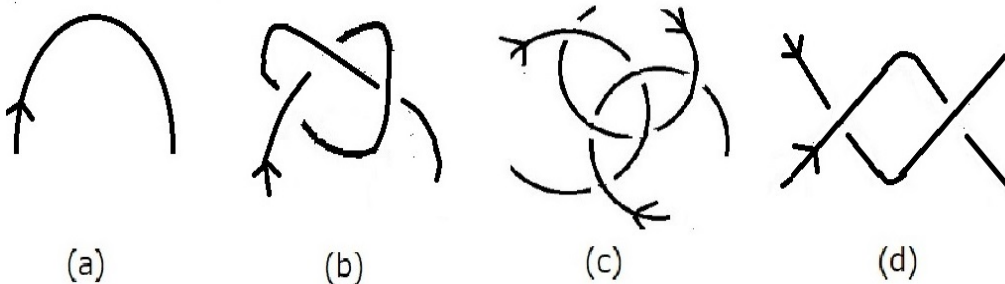


Figure 14: Diagrams (a), (b), (c), (d)

## 5. Unique $n$ -arc diagram system for $n$ -arc system in 3D space

Let  $L = \{a_i | i = 1, 2, \dots, n\}$  be an oriented polygonal  $n$ -arc system in 3D space  $R^3$ . In this section, first it will be observed how the projection image of  $L$  to an oriented plane  $P$  is approximated by an  $n$ -arc diagram. Next, it will be shown how a unique system of finitely many  $n$ -arc chord diagrams up to isomorphisms is constructed from any given  $n$ -arc system  $L$  in  $R^3$ . Let

$$S^2 = \{u \in R^3 | \|u\| = 1\}$$

be the unit sphere, where  $\| \cdot \|$  denotes the norm on  $R^3$ . Every element  $u \in S^2$  is called a *unit vector from the origin*  $(0, 0, 0)$ . For a unit vector  $u \in R^3$ , let  $P(u)$  be the oriented plane containing the origin  $(0, 0, 0)$  such that the unit vector  $u$  is a positively normal vector to  $P(u)$ . For a unit vector  $u = (a, b, c)$ , the plane  $P(u)$  denotes the plane  $ax + by + cz = 0$  in  $R^3$  with  $xyz$ -coordinate system. The orthogonal projection from  $R^3$  to the plane  $P(u)$  is called the *projection along* the unit vector  $u \in S^2$  and denoted by

$$\lambda(u) : R^3 \rightarrow P(u).$$

For a small positive number  $\delta$ , a  $\delta$ -*approximation* of the projection  $\lambda(u) : R^3 \rightarrow P(u)$  along  $u \in S^2$  is the projection  $\lambda(u') : R^3 \rightarrow P(u')$  along a unit vector  $u' \in S^2$  with  $\|u' - u\| < \delta$ , which is denoted by

$$\lambda(u)^\delta : R^3 \rightarrow P(u).$$

The following theorem is obtained.

**Theorem 5.1.** Let  $L$  be an oriented polygonal  $n$ -arc system in the 3-space  $R^3$  with  $xyz$  coordinate system, and  $\lambda = \lambda(u_z) : R^3 \rightarrow P$  the projection along the  $z$ -axis unit vector  $u_z = (0, 0, 1) \in S^2$  to the plane  $P = P(u_z)$ . For any sufficiently small positive number  $\delta$ , the projection  $\lambda$  has a  $\delta$ -approximation

$$\lambda^\delta : R^3 \rightarrow P$$

such that the projection image  $\lambda^\delta(L) = \lambda(u_z)^\delta(L)$  is an  $n$ -arc diagram uniquely determined up to isomorphisms from  $L$  and the projection  $\lambda$ .

This proof is done by refining an argument transforming a classical knot in  $R^3$  into a regular knot diagram and was done for a 1-arc in  $R^3$ , [10, 14]. Let  $L = \{a_i | i = 1, 2, \dots, n\}$  be an  $n$ -arc system in  $R^3$ , and  $s(a_i)$  and  $t(a_i)$  the starting point and the terminal point of the arcs  $a_i$  ( $i = 1, 2, \dots, n$ ), respectively. An *edge line* of  $L$  is the oriented straight line in  $R^3$  extending an edge in  $L$  with orientation induced from the oriented arc  $a_i$  of  $L$  containing the edge. A *front line* of  $L$  is the oriented straight line

in  $R^3$  containing the points  $s(a_i)$  and  $t(a_i)$  with orientation from  $s(a_i)$  to  $t(a_i)$  for an arc  $a_i$  of  $L$ . For any plane  $P$  in  $R^3$ , the *great circle*  $C$  of  $P$  in  $R^3$  is the great circle obtained by the intersection of  $S^2$  and the plane  $P(u)$  (containing the origin 0) which is parallel to  $P$ . The *trace set*  $T$  of an  $n$ -arc system  $L$  is the subset of  $S^2$  consisting of the great circles and the unit vectors obtained from  $L$  arising in the following cases (i) and (ii):

- (i) The great circle  $C$  of  $S^2$  of the plane  $P$  in  $R^3$  determined from an edge or front line  $\ell$  of  $L$  and a vertex  $v$  of  $L$  disjoint from  $\ell$ .
- (ii) The unit vectors  $\pm u_\ell$  of an edge line or a front line  $\ell$  of  $L$  and the unit vectors  $\pm u_\eta \in S^2$  of a straight line  $\eta$  that meets three lines  $\ell_i$  ( $i = 1, 2, 3$ ) at three points, where  $\ell_i$  ( $i = 1, 2, 3$ ) consist of edge or front lines such that no two lines lie on the same plane.

In (i), note that the trace set  $T$  contains the following great circles: the great circle of the plane determined by two parallel distinct front or edge lines, and the great circle of the plane determined by two distinct edge or front lines meeting at a point. In particular, the trace set  $T$  contains the great circle of any plane containing any three distinct front or edge lines. This means that the unit vectors  $\pm u(\eta) \in S^2$  of a straight line  $\eta$  in  $R^3$  that meets three lines  $\ell_i$  ( $i = 1, 2, 3$ ) at three points, where  $\ell_i$  ( $i = 1, 2, 3$ ) consist of edge or front lines such that some two lines lie on the same plane. In (ii), note that a line  $\eta$  meeting  $\ell_i$  ( $i = 1, 2, 3$ ) at three points is unique by uniqueness of the solution to a system of three linear equations with regular matrix coefficients. Also, note that if a unit vector  $u \in S^2$  is in the trace set  $T$ , then the unit vector  $-u$  is also in  $T$ . The following lemma is used for the proof of Theorem 5.1.

**Lemma 5.2.** For every unit vector  $u \in S^2 \setminus T$ , the projection image  $\lambda(u)(L)$  is an  $n$ -arc diagram in the plane  $P(u)$ . Further, the  $n$ -arc diagram  $\lambda(u)(L)$  up to isomorphisms is independent of any choice of a unit vector  $u'$  in the connected region  $R(u)$  of  $S^2 \setminus T$  containing  $u$ .

**Proof of Lemma 5.2.** If a unit vector  $u \in S^2$  is not in (i), then every edge of  $L$  and every front line  $\ell_\gamma$  are embedded into the plane  $P(u)$  by the projection  $\lambda(u)$ . If a unit vector  $u \in S^2$  is in neither (i) nor (ii), then the set of vertices of  $L$  is embedded into  $P(u)$  by the projection  $\lambda(u)$  whose image is disjoint from the image of any open edge of  $L$ . In particular, any two distinct parallel edge lines are disjointedly embedded into  $P(u)$ . Further, the images of the edges of  $L$  meet only in the images of the open edges of  $L$ . Thus, if a unit vector  $u \in S^2$  is in neither (i) nor (ii), namely if  $u \in S^2 \setminus T$ , then the meeting points among the edges of  $L$  consisting of double points between

two open edges of  $L$  and hence the projection image  $\lambda(u)(L)$  is an  $n$ -arc diagram in the plane  $P(u)$ . The  $n$ -arc diagram  $\lambda(u')(L)$  is unchanged up to isomorphisms for any unit vector  $u'$  in a connected open neighborhood of  $u$  in  $S^2 \setminus T$ , so that the  $n$ -arc diagram  $\lambda(u')(L)$  is unchanged up to isomorphisms for any unit vector  $u'$  in the connected region  $R(u)$ . This completes the proof of Lemma 5.2

For a number  $x$  with  $0 \leq x \leq 1$ , let  $x^c = \sqrt{1 - x^2}$ . Then the proof of Theorem 5.1 is done as follows:

*5.3: Proof of Theorem 5.1.* The idea of the proof is to specify a unique connected region  $R(u)$  of  $S^2 \setminus T$  adjacent to the unit vector  $u_z = (0, 0, 1) \in S^2$ . By taking a positive number  $r$  sufficiently small, take a unit vector  $u(r) = (r, 0, r^c) \in S^2$  so that the unit vector  $u(x) = (x, 0, x^c)$  with  $0 < x \leq r$  does not meet  $T$  except for the great circle  $x^2 + z^2 = 1$  in  $T$  (if it is in  $T$ ). Then the connected region  $R(u)$  is taken to be the unique connected region of  $S^2 \setminus T$  which is adjacent to the unit vector  $u(r)$  and contains the unit vector  $u(r, \theta) = (r \cos \theta, r \sin \theta, r^c)$  of  $S^2$  for a sufficiently small positive number  $\theta$ , which is orthogonal to the plane  $P(r, \theta) : (r \cos \theta)x + (r \sin \theta)y + r^c z = 0$ . The connected region  $R(u)$  is uniquely specified. By Lemma 5.2, the image  $\lambda(u)(L)$  of  $L$  under the orthogonal projection  $\lambda(u) : R^3 \rightarrow P(u)$  is an  $n$ -arc diagram in the plane  $P(u)$  which is a unique  $n$ -arc diagram up to isomorphisms independent of choices of a unit vector  $u'$  in the connected region  $R(u)$ . Since the connected region  $R(u)$  is adjacent to the normal vector  $u_z \in S^2$ , for every  $\delta > 0$  there is a unit vectors  $u' \in R(u)$  with  $\|u' - u\| < \delta$ , so that the orthogonal projection  $\lambda(u') : R^3 \rightarrow P(u')$  is a desired  $\delta$ -approximation  $\lambda^\delta : R^3 \rightarrow P$ . This completes the proof of Theorem 5.1.

Note that the vectors  $u_X = (r^c \cos \theta, r^c \sin \theta, -r)$ ,  $u_Y = (-\sin \theta, \cos \theta, 0)$  and  $u_Z = u(r, \theta)$  are mutually orthogonal unit vectors such that the plane  $P(r, \theta)$  is spanned by  $u_X, u_Y$ . The following matrix  $O(r, \theta)$  and the transpose matrix  $U^t(r, \theta) = (u_X^t \ u_Y^t \ u_Z^t)$  (which are mutually inverse matrices of determinant one) are important for calculations.

$$O(r, \theta) = \begin{pmatrix} u_X \\ u_Y \\ u_Z \end{pmatrix} = \begin{pmatrix} r^c \cos \theta & r^c \sin \theta & -r \\ -\sin \theta & \cos \theta & 0 \\ r \cos \theta & r \sin \theta & r^c \end{pmatrix},$$

$$O(r, \theta)^t = (u_X^t \ u_Y^t \ u_Z^t) = \begin{pmatrix} r^c \cos \theta & -\sin \theta & r \cos \theta \\ r^c \sin \theta & \cos \theta & r \sin \theta \\ -r & 0 & r^c \end{pmatrix}.$$

By the mutually orthogonal unit vectors  $u_X, u_Y$  and  $u_Z$ , the new  $XYZ$ -coordinate system is considered. Since  $u_X O(r, \theta)^t = (1 \ 0 \ 0)$ ,  $u_Y O(r, \theta)^t = (0 \ 1 \ 0)$  and  $u_Z O(r, \theta)^t =$

(001), the point  $p = (a, b, c)$  in the  $xyz$ -coordinate system is transformed into the point  $p = (a', b', c')$  in the  $XYZ$ -coordinate system by the matrix equation

$$(a' b' c') = (a b c)O(r, \theta)^t.$$

From now, the proof of Theorem 1.2 is done as follows.

*5.4: Proof of Theorem 1.2.* It is shown how a unique system of finitely many  $n$ -arc chord diagrams up to isomorphisms is constructed from any given  $n$ -arc system  $L$  in  $R^3$ . For every arc  $a_i$  of  $L$ , let  $\ell_i$  be the front line of  $a_i$ . For every arc  $a_j$  of  $L$  not belonging to  $\ell_i$ , let  $e_j$  be the edge of  $a_j$  which pops for the first time from  $\ell_i$  when a point is going on  $a_j$  along the orientation of  $a_j$ , and  $\ell_{ij}$  be the edge line of  $e_j$ . Let  $P_{ij}$  be the plane in  $R^3$  containing the origin  $(0, 0, 0)$  which is parallel to the plane  $P(\ell_i, \ell_{ij})$  spanned by the straight lines  $\ell_i$  and  $\ell_{ij}$  in  $R^3$ . Then a new unique  $XYZ$ -coordinate system of  $R^3$  is defined as follows. Take the  $X$ -axis given by the normal vector of  $\ell_i$ . The  $Y$ -axis is defined by the normal vector of the oriented line  $\ell_{ij}^*$  in  $P(\ell_i, \ell_{ij})$  meeting  $\ell_i$  orthogonally at  $s(a_i)$  such that the inner product of the normal vectors of  $\ell_{ij}^*$  and  $\ell_{ij}$  is positive. The  $Z$ -axis is constructed in  $R^3$  by taking a positive unit vector  $u_{ij}$  orthogonal to the oriented plane  $P_{ij}$  with the origin  $(0, 0, 0)$ . By applying Theorem 5.1 to the image  $\lambda(L)$  of  $L$  under the projection  $\lambda = \lambda(u_{ij}) : R^3 \rightarrow P_{ij}$ , a unique  $n$ -arc diagram  $D(L)_{ij}$  of  $L$  up to isomorphisms is obtained. For every arc  $a_j$  of  $L$  belonging to  $\ell_i$ , let  $P_{ij}$  be a plane in  $R^3$  containing the origin  $(0, 0, 0)$  and the straight line  $\ell_i$ , and  $D(L)_{ij}$  the union of disjoint intervals  $I_{ij}$  ( $j = 1, 2, \dots, n_i$ ) in  $P_{ij}$  such that the interval  $I_{ij}$  is obtained from the image interval  $\hat{\lambda}_{ij}(a_j)$  of the arc  $a_j$  in  $\ell_i$  under the orthogonal projection  $\hat{\lambda}_{ij} : R^3 \rightarrow \ell_i$  by pushing it into  $P_{ij} \setminus \ell_i$ . This completes the proof of Theorem 1.2.

Note that for every arc  $a_j$  of  $L$  belonging to  $\ell_i$ , the  $n$ -arc diagram  $D(L)_{ij}$  has no crossing point, so that the NT probability  $\kappa(D(L)_{ij}) = 0$ . An  $n$ -arc system  $L$  in  $R^3$  is *even* if the front pop planes  $P_{ij}$  ( $j = 1, 2, \dots, n_i$ ) for every arc  $a_i$  of  $L$  are taken as the same plane  $P$  in  $R^3$  (although the orientations of  $P_{ij}$  ( $j = 1, 2, \dots, n_i$ ) do not necessarily coincide due to the definition of  $Y$ -axis.) Then the plane  $P$  is called the *front pop plane* of  $L$ . By the definition of an even  $n$ -arc system, note that for every  $n$ -arc system  $L$  and a plane  $P$  in  $R^3$  such that the intersection  $P \cap L$  is a finite point set containing the boundary point set  $\partial L$  of  $L$ , there is an even  $n$ -arc system  $L^+$  in  $R^3$  obtained from  $L$  by adding some small edges in  $P$  to  $\partial L$  such that all of the front pop planes of  $L^+$  coincide with the plane  $P$ . The following corollary is direct from the definition of NT probability of  $L$ .

**Corollary 5.5.** Assume that the orthogonal projection image of an even  $n$ -arc system  $L$  in  $R^3$  into the front pop plane  $P$  is an  $n$ -arc diagram  $D(L)$ . Then each of the  $n$ -arc

diagrams  $D(L)_{ij}$  ( $i, j = 1, 2, \dots, n$ ) of  $L$  is isomorphic to  $D(L)$  or the mirror image  $D(L)^*$  of  $D(L)$ . If the  $n$ -arc diagram  $D(L)$  of an even  $n$ -arc system  $L$  in  $R^3$  is inbound, then the NT probability  $\kappa(L)$  of  $L$  is equal to the NT probability  $\kappa(D(L))$ .

**Proof of Corollary 5.5.** The first assertion is obtained from Theorem 1.2 and definition. The second assertion is obtained from Theorem 3.1 (3), completing the proof of Corollary 5.5.

In particular, for every  $n$ -arc system  $L$  embedded in a plane  $P$  of  $R^3$ , the NT probability  $\kappa(L)$  of  $L$  is 0.

## 6. Computation example of arc system in 3D space

In this section, two elementary examples on a 2-arc system  $L$  in  $R^3$  are computed to show how to perform the calculation, where a point  $(x, y, z)$  of  $R^3$  is represented as  $(x, y)^z$  in the plane  $P$  defined by  $z = 0$  and  $z$  is called the *height* of the point  $(x, y)$ .

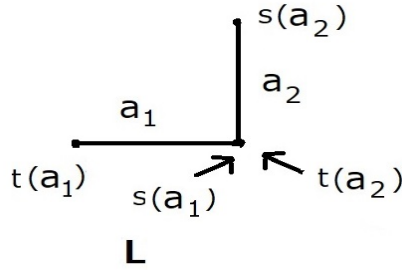


Figure 15: The non-even 2-arc system  $L = \{a_1, a_2\}$  represented in the  $xyz$ -coordinate system of  $R^3$  with the following ordered vertices  $(x, y)^z$ :  $a_1$ :  $s(a_1) = (0, 0)^0$ ,  $t(a_1) = (-1, 0)^0$ ,  $a_2$ :  $s(a_2) = (0, 1)^0$ ,  $t(a_2) = (0, 0)^1$

**Example 6.1.** Let  $L = \{a_1, a_2\}$  be a non-even 2-arc system in  $R^3$  illustrated in Fig. 15. By definition,  $\kappa(D(L)_{11}) = \kappa(D(L)_{22}) = 0$ . To compute  $\kappa(D(L)_{12})$ , let  $P_{12}$  be the front pop plane induced from  $\partial a_1$  and  $s(a_2)$ , which coincides with the plane  $P$  with standard orientation. The 2-arc system  $L$  in the  $XYZ$ -coordinate system by the matrix  $O(r, \theta)^t = (u_X^t \ u_Y^t \ u_Z^t)$  is given as follows.

$$\begin{aligned} a_1 : \quad & s(a_1) = (0, 0)^0, \quad t(a_1) = (-r^c \cos \theta, \sin \theta)^{-r \cos \theta}, \\ a_2 : \quad & s(a_2) = (r^c \sin \theta, \cos \theta)^{r \sin \theta}, \quad t(a_2) = (-r, 0)^{r^c}. \end{aligned}$$

By taking  $r$  and  $\theta$  to be sufficiently small positive numbers, the numbers  $r' = r/r^c$  and  $\theta' = \tan \theta$  are taken as sufficiently small positive numbers. To obtain a 2-arc diagram

$D(L)_{12}$  up to isomorphisms, one can use the  $XYZ$ -coordinate system obtained by applying the following matrix

$$V(r, \theta) = ((1/r^c \cos \theta)u_1^t (1/\cos \theta)u_2^t (1/r^c)u_3^t) = \begin{pmatrix} 1 & -\theta' & r' \\ \theta' & 1 & r'\theta' \\ -r' & 0 & 1 \end{pmatrix}$$

instead of  $O(r, \theta)^t = (u_1^t u_2^t u_3^t)$ . Then the 2-arc system  $L$  is given as follows.

$$\begin{aligned} a_1 & : s(a_1) = (0, 0)^0, t(a_1) = (-1, \theta')^{-r'}, \\ a_2 & : s(a_2) = (\theta', 1)^{r'\theta'}, t(a_2) = (-r', 0)^1. \end{aligned}$$

Then the 2-arc diagram  $D(L)_{12}$  is illustrated in Fig. 16 and the computation results  $\kappa(D(L)_{12}, a_1) = (1/2, 1/2, 1/2, 1/2)$  and  $\kappa(D(L)_{12}, a_2) = 0$  are obtained. Hence,  $\kappa(D(L)_{12}) = (1/4, 1/4, 1/4, 1/4)$ . Next the front pop plane  $P_{21}$  is induced from  $\partial a_2$  and  $s(a_1)$ , which is spanned by the unit vectors  $u_X = (0, -1/\sqrt{2}, 1/\sqrt{2})$ ,  $u_Y = (0, -1/\sqrt{2}, -1/\sqrt{2})$  for the new  $XY$ -coordinate system of the plane  $P_{21}$ . The  $Z$ -axis is defined by the unit vector  $u_Z = (1, 0, 0)$ . The plane  $P_{21}$  is defined by  $x = 0$ . Let  $U$  be the  $(2, 2)$ -matrix given by

$$U = (u_X^t u_Y^t u_Z^t) = \begin{pmatrix} 0 & 0 & 1 \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}.$$

Since  $u_X U = (100)$ ,  $u_Y U = (010)$  and  $u_Z U = (001)$ , the 2-arc system  $L$  in the  $XYZ$ -coordinate system of  $R^3$  is obtained by applying the matrix  $U$ . However, since it is sufficient to obtain a 2-arc diagram up to isomorphisms, one can use the 2-arc diagram in the  $XYZ$ -coordinate system of  $L$  obtained by using the following matrix

$$V = (-\sqrt{2}u_X^t - \sqrt{2}u_Y^t u_Z^t) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

instead of  $U$ . By  $V$ , the 2-arc system  $L$  is given by

$$\begin{aligned} a_1 & : s(a_1) = (0, 0)^0, t(a_1) = (0, 0)^{-1}, \\ a_2 & : s(a_2) = (1, 1)^0, t(a_2) = (-1, 1)^0. \end{aligned}$$

By applying  $O(r, \theta)^t$  to this 2-arc system  $L$ , the 2-arc diagram  $D(L)_{21}$  is a 2-arc diagram without crossing, illustrated in Fig. 16. Thus,  $\kappa(D(L)_{21}) = 0$ , and

$$\kappa(L) = \kappa(D(L)_{12})/4 = (1/16, 1/16, 1/16, 1/16).$$

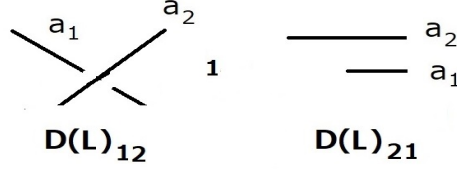


Figure 16: The 2-arc diagrams  $D(L)_{12}$  and  $D(L)_{21}$  of the 2-arc system  $L$  in Fig. 15

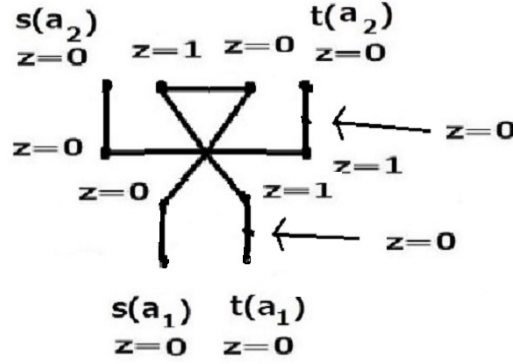


Figure 17: The even 2-arc system  $L = \{a_1, a_2\}$  represented in the  $xyz$ -coordinate system of  $R^3$  with the following ordered vertices  $(x, y)^z$ :  $a_1$ :  $s(a_1) = (1, 0)^0, (1, 1)^0, (2, 3)^0, (1, 3)^1, (2, 1)^1, (2, 0.5)^0, t(a_1) = (2, 0)^0, a_2$ :  $s(a_2) = (0, 3)^0, (0, 2)^0, (3, 2)^1, (3, 2.5)^0, t(a_2) = (3, 3)^0$ .

**Example 6.2.** Let  $L$  be an even 2-arc system in  $R^3$  illustrated in Fig. 17. The front pop planes  $P_{1j}$  ( $j = 1, 2$ ) are the plane  $z = 0$  with ordinal  $xy$ -orientation since  $s(a_1) = (1, 0, 0)$ ,  $u_X = (100)$ ,  $u_Y = (010)$ ,  $u_Z = (001)$ . The new coordinate system is given by changing the point  $(1, 0, 0)$  into the origin  $(0, 0, 0)$ , namely by the matrix equation  $(XYZ) = ((xyz) - (100))E$  with the identity matrix  $E$ . The 2-arc system  $L = \{a_1, a_2\}$  is given as follows.

$$\begin{aligned}
 a_1 : s(a_1) &= (0, 0)^0, (0, 1)^0, (1, 3)^0, (0, 3)^1, (1, 1)^1, (1, 0.5)^0, t(a_1) = (1, 0)^0, \\
 a_2 : s(a_2) &= (-1, 3)^0, (-1, 2)^0, (2, 2)^1, (2, 2.5)^0, t(a_2) = (2, 3)^0.
 \end{aligned}$$

Let  $a_{11}$ ,  $a_{12}$  and  $a_{21}$  be oriented edges of  $L$  with  $a_{11}$ ,  $a_{12}$  in the arc  $a_1$  and  $a_{21}$  in the arc  $a_2$  such that

$$\begin{aligned}
 a_{11} &: s(a_{11}) = (0, 1)^0, t(a_{11}) = (1, 3)^0, \\
 a_{12} &: s(a_{12}) = (0, 3)^1, t(a_{12}) = (1, 1)^1, \\
 a_{21} &: s(a_{21}) = (-1, 2)^0, t(a_{21}) = (2, 2)^1.
 \end{aligned}$$

Let  $V(r, 0)$  be the matrix obtained by setting  $\theta = 0$  to the matrix  $V(r, \theta)$  used in Example 6.1.

$$V(r, 0) = \begin{pmatrix} 1 & 0 & r' \\ 0 & 1 & 0 \\ -r' & 0 & 1 \end{pmatrix}.$$

In the  $XYZ$  coordinate system given by  $(X Y Z) = (x y z)V(r, 0)$ , the oriented edges  $a_{11}$ ,  $a_{12}$  and  $a_{21}$  are written as follows:

$$\begin{aligned} a_{11} & : s(a_{11}) = (0, 1)^0, t(a_{11}) = (1, 3)^{r'}, \\ a_{12} & : s(a_{12}) = (-r', 3)^1, t(a_{12}) = (1 - r', 1)^{1+r'}, \\ a_{21} & : s(a_{21}) = (-1, 2)^{-r'}, t(a_{21}) = (3 - r', 2)^{1+3r'}. \end{aligned}$$

The 2-arc diagram  $D(L)_{1j}$  ( $j = 1, 2$ ) and the chord diagrams  $C(D(L)_{1j})$  ( $j = 1, 2$ ) up to basic equivalences are illustrated in Fig. 18. The NT probabilities  $\kappa(D(L)_{1j}, a_1)$  and  $\kappa(D(L)_{1j}, a_2)$  for  $j = 1, 2$  are calculated to be

$$\kappa(D(L)_{1j}, a_1) = (1/2, 1/2, 5/8, 7/10), \quad \kappa(D(L)_{1j}, a_2) = (1/2, 1/2, 3/5, 3/5).$$

Thus,

$$\kappa(D(L)_{1j}) = (1/2, 1/2, 49/80, 13/20) \quad (j = 1, 2).$$

The front pop planes  $P_{2j}$  ( $j = 1, 2$ ) are the plane  $z = 0$  with opposite  $xy$ -orientation since  $s(a_1) = (0, 3, 0)$ ,  $u_X = (100)$ ,  $u_Y = (0 - 10)$ ,  $y_Z = (00 - 1)$ . The new coordinate systems for  $P_{2j}$  ( $j = 1, 2$ ) are given by the matrix equation  $(X Y Z) = ((x y z) - (030))U$  with the following matrix  $U$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The 2-arc system  $L = \{a_1, a_2\}$  is given as follows.

$$\begin{aligned} a_1 & : s(a_1) = (2, 3)^0, (1, 2)^0, (2, 0)^0, (1, 0)^{-1}, (2, 2)^{-1}, (2, 2.5)^0, t(a_1) = (2, 3)^0, \\ a_2 & : s(a_2) = (0, 0)^0, (0, 1)^0, (3, 1)^{-1}, (3, 0.5)^0, t(a_2) = (3, 0)^0. \end{aligned}$$

Let  $a_{11}$ ,  $a_{12}$  and  $a_{21}$  be oriented edges of  $L$  with  $a_{12}$ ,  $a_{12}$  in the arc  $a_1$  and  $a_{21}$  in the arc  $a_2$  such that

$$\begin{aligned} a_{11} & : s(a_{11}) = (1, 2)^0, t(a_{11}) = (2, 0)^0, \\ a_{12} & : s(a_{12}) = (1, 0)^{-1}, t(a_{12}) = (2, 2)^{-1}, \\ a_{21} & : s(a_{21}) = (0, 1)^0, t(a_{21}) = (3, 1)^{-1}. \end{aligned}$$

In the  $XYZ$  coordinate system given by  $(X, Y, Z) = (x, y, z)V(r, 0)$ , the oriented edges  $a_{11}$ ,  $a_{12}$  and  $a_{21}$  are written as follows:

$$\begin{aligned} a_{11} & : s(a_{11}) = (1, 2)^{r'}, t(a_{11}) = (2, 0)^{2r'}, \\ a_{12} & : s(a_{12}) = (1 + r', 0)^{-1+r'}, t(a_{12}) = (2 + r', 2)^{-1+2r'}, \\ a_{21} & : s(a_{21}) = (0, 1)^0, t(a_{21}) = (3 + r', 1)^{-1+3r'}. \end{aligned}$$

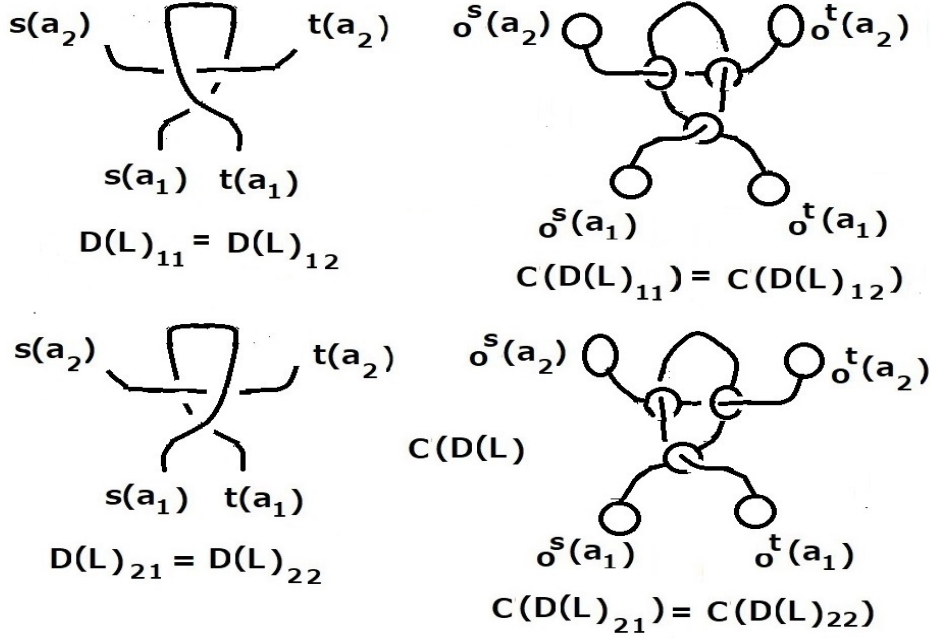


Figure 18: 2-arc diagrams  $D(L)_{ij}$  ( $i, j = 1, 2$ ) and the chord diagrams  $C(D(L)_{ij})$  ( $i, j = 1, 2$ )

The 2-arc diagrams  $D(L)_{2j}$  ( $j = 1, 2$ ) are the same inbound 2-arc diagram which is the mirror image of the same 2-arc diagram  $D(L)_{1j}$  ( $j = 1, 2$ ), as illustrated in Fig. 18. By Corollary 5.5, the NT probabilities  $\kappa(D(L)_{2j})$  ( $j = 1, 2$ ) are equal to  $\kappa(D(L)_{1j})$  ( $j = 1, 2$ ). Thus,

$$\kappa(L) = (1/2, 1/2, 49/80, 13/20).$$

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