Fusion of boundary surface-link and ribbonness

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ABSTRACT

A boundary surface-link in the 4-sphere is proved to be a ribbon surface-link if the surface-link obtained from it by surgery along a pairwise surgically nontrivial fusion 1-handle system is a ribbon surface-link.

Keywords: Boundary surface-link, Ribbon surface-link, fusion 1-handle.
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1. Introduction

A surface-link (of r components) is a closed oriented surface F of r connected components $F_i\,(i = 1, 2, \dots, r)$ smoothly embedded in the 4-sphere S^4. It is called a *surface-knot* when F is connected, and a *sphere-link* or an S^2*-link* when F consists of 2-spheres. A surface-link F of r components $F_i\,(i = 1, 2, \dots, r)$ is a *boundary surface-link* if there is a system V of disjoint compact connected oriented 3-manifolds $V_i\,(i = 1, 2, \dots, r)$ smoothly embedded in S^4 with $\partial V_i = F_i$, where V_i is called a *Seifert hypersurface* for F_i. The system V is called a *disjoint Seifert hypersurface system* for the boundary surface-link F. A *1-handle system* on a surface-link F is a system h of disjoint 1-handles $h_j\,(j = 1, 2, \dots, s)$ on F smoothly embedded in S^4. Let $F(h)$ be the surface-link obtained from F by surgery along a 1-handle system h. A *ribbon surface-link* is the surface-link $O(h)$ obtained from a trivial S^2-link O by surgery along a 1-handle system h, [1, 2]. A system h of 1-handles $h_j\,(j = 1, 2, \dots, s)$ on F is a *fusion 1-handle system* if $F(h)$ has just $r - s(\geq 1)$ connected components with h_j called a *fusion 1-handle*, and a *self 1-handle system* if h_j is a 1-handle on a connected component of F for all j. A special feature of a boundary surface-link F is the existence of a self 1-handle system β on F embedded in a disjoint Seifert hypersurface system V for F such that the closed complement $V(\beta) = \mathrm{cl}(V \setminus \beta)$ is a handlebody system and thus, the surface-link $F(\beta)$ is a trivial surface-link in S^4, [3]. The following definition is made for a fusion 1-handle on a boundary surface-link.

Definition. A fusion 1-handle h on a boundary surface-link F in S^4 is *surgically trivial*, or simply *s-trivial* if there is a self 1-handle system β on F embeded to a disjoint Seifert hypersurface system V of F and disjoint from h such that $V(\beta)$ is a handlebody system and $F(\beta)(h)$ is a trivial surface-link. Otherwise, h is *surgically nontrivial*, or simply *s-nontrivial*. Furhter, if $F(\beta)(h)$ is a split union of a nontrivial ribbon surface-knot and a trivial surface-link, then h is *pairwise surgically nontrivial*, or simply *pairwise s-nontrivial*.

The following lemma is basic to the definitions of s-trivial, s-nontrivial, and pairwise s-nontrivial fusion 1-handles on a boundary surface-link.

Lemma 1.1. Whether a fusion 1-handle h on a boundary surface-link F is s-trivial, s-nontrivial, or pairwise s-nontrivial does not depend on choices of a disjoint Seifert hypersurface system V of F and a self 1-handle system β in V with handlebody system $V(\beta)$.

Proof of Lemma 1.1. Let V and V' be disjoint Seifert hypersurface systems for F. Let $B \cup V^B$ be a decomposition of V into a 3-ball B and $V^B = \mathrm{cl}(V \setminus B)$ with $d^B = B \cap V^B$ a proper disk. Since a boundary collar of F in V is deformed to match a boundary collar of F in V', there is a decomposition $B \cup V'^B$ of V' with $d^B = B \cap V'^B$ by taking B in the boundary collar. Let β and β' be disjoint self 1-handle systems on F disjoint from B such that $V(\beta)$ and $V'(\beta')$ are handlebody systems. Let h be a fusion 1-handle on F which is disjoint from β and β' and attaches to F in the interior of the disk system $d'^B = \mathrm{cl}(\partial B \setminus d^B)$. Let α be a core arc of h meeting V and V' transversely. By a deformation of $V(\beta)$ keeping α and B fixed, it may be assumed that the handlebody system $V(\beta)$ transversely meets $V'(\beta')$ with a meridian disk system of the handlebody system $V(\beta)$ in the interior of $V'(\beta')$, and α meets $V(\beta)$ with $\alpha \cap V = \alpha \cap V(\beta)$. Note that $\alpha \cap V'(\beta') = \alpha \cap V'$ since $\alpha \cap \beta' = \emptyset$. Let $L = \partial B$ be a trivial S^2-link, and α^L a simple arc spanning L obtained from α by sliding every point of $\alpha \cap V^B$ into the interior of B through $V(\beta)$ and then sliding every point of $\alpha \cap V'^B$ into the interior of B through $V'(\beta')$ avoiding the meridian disk system $V(\beta) \cap V'(\beta')$ of $V(\beta)$. Let h^L be a fusion 1-handle on L with core arc α^L. The fundamental groups $\pi_1(S^4 \setminus L, b)$, $\pi_1(S^4 \setminus F(\beta), b)$ and $\pi_1(S^4 \setminus F(\beta'), b)$ are the same free group $< x_1, x_2, \dots, x_r >$ of rank r with basis $x_i\,(i = 1, 2, \dots, r)$ of meridian elements of L, where x_1 and x_2 are meridian elements of the components L_1 and L_2 of L attached by the fusion 1-handle h^L. The fundamental groups $\pi_1(S^4 \setminus L(h^L), b)$, $\pi_1(S^4 \setminus F(\beta)(h), b)$ and $\pi_1(S^4 \setminus F(\beta')(h), b)$ are mutually meridian-preservingly isomorphic groups and presented respectively by the following presentations. $< x_1, x_2, \dots, x_r |\, x_2 = w_L x_1 w_L^{-1} >$, $< x_1, x_2, \dots, x_r |\, x_2 = w x_1 w^{-1} >$

and $< x_1, x_2, \ldots, x_r | \, x_2 = w' x_1 w'^{-1} >$, where the words w_L, w, w' are taken reduced words in the free group $< x_1, x_2, \ldots, x_r >$. Then $w_L = w = w'$ by the conjugacy theorem for the one relator Wirtinger presentation, [4]. Thus, $w_L = 1$ if and if both $F(\beta)(h)$ and $F(\beta')(h)$ are trivial surface-links, $w_L \neq 1$ if and if both $F(\beta)(h)$ and $F(\beta')(h)$ are nontrivial surface-links, and $w_L \neq 1$ and the word w_L is written in x_1 and x_2 if and only if both $F(\beta)(h)$ and $F(\beta')(h)$ are equal to a split union of a nontrivial ribbon surface-knot and a trivial surface-link. This completes the proof of Lemma 1.1.

The following theorem is a main result, whose proof is done in Section 2 by using some properties of SUPH systems for the moves of a ribbon surface-link, explained in Appendix.

Theorem 1.2. Let F be a boundary surface-link of $r(\geq 2)$ components in S^4. If the surface-link $F(h)$ for a pairwise s-nontrivial fusion 1-handle system h on F is a ribbon surface-link, then the surface-link F is a ribbon surface-link.

2. A SUPH move of a 2-handle and the proof of Theorem 1.2

A ribbon surface-link F is a surface-link constructed from a pair (O, h) of a trivial S^2-link O and a 1-handle system h on O in S^4, called a *handled sphere system*, by surgery along h. This pair is uniquely constructed from a *chorded sphere system* (O, α) in S^4 with a core arc system α of h isotopically deformed into the standard 3-sphere S^3 of S^4, called a *chord system*, or from a *chord graph* (o, α) in S^3 with a trivial link o, called a *based loop system*, [1]. The chord graph (o, α) in S^3 is considered by a chord diagram $C = C(o, \alpha)$ in S^2. A *SUPH system* for a surface-link F in S^4 is a compact multi-punctured handlebody system W smoothly embedded in S^4 whose boundary ∂W is given by $\partial W = F \cup O$ for a trivial S^2-link O in S^4. A SUPH system W for a ribbon surface-link F given by a handled sphere system (O, h) is constructed by $W = O \times [0, 1] \cup h$ for a collar $O \times [0, 1]$ of O in S^4 with $O \times \{0\} = O$, where $O \times \{1\} = \partial W \setminus F$ is a trivial S^2-link. Conversely, if there is a SUPH system for a surface-link F, then F is a ribbon surface-link, [5]. The proof of Theorem 1.2 is done as follows.

2.1: Proof of Theorem 1.2. Let F be a boundary surface-link, and h a pairwise s-nontrivial fusion 1-handle on F spanning the components $F_i \, (i = 1, 2)$ of F. If the surface-link $F(h)$ is a ribbon surface-link then F is a ribbon surface-link, then the proof of Theorem 1.2 is completed by inductive argument. For a disjoint Seifert hypersurface V of F, let $B \cup V^B$ be a decomposition , of V where B is a 3-ball system in V with just one 3-ball in each component of V and $V^B = \mathrm{cl}(V \setminus B)$ such that $d^B = B \cap V^B$ is a proper disk system in V. Let β be a self 1-handle system on F embedded in $V^B \setminus d^B$ and disjoint from h such that $V^B(\beta) = \mathrm{cl}(V^B \setminus \beta)$ is a handlebody system. The surface-link $F(\beta)$ is a trivial surface-link in S^4. The union

$V(\beta) = B \cup V^B(\beta)$ is a handlebody system. Let D_β be a transverse disk system of the 1-handle system β with one disk for every 1-handle of β, and k_β the boundary loop system of D_β in $F(\beta)$. The following two facts are used, [6, 7, 8].

(i) For a trivial surface-knot T in S^4, every *spin loop-pair system* (that is, system of disjoint oriented simple loop-pairs of geometric intersection number $+1$) on T extends to a *spin loop basis* of T (that is, spin loop-pair system on T representing a basis of $H_1(T; Z)$).

(ii) For any two spin loop bases $(k, \ell), (k', \ell')$ of a trivial surface-knot T in S^4, there is an orientation-preserving diffeomorphism of S^4 keeping T setwise fixed and sending (k, ℓ) to (k', ℓ'). For a handlebody system Vsmoothly embedded in S^4, the handlebody system V with any given spin loop basis (k, ℓ) such that the loop system k is a meridian loop system, namely the boundary of a meridian disk system of V is isotopic to a standard handlebody system V_0 in S^3 with a standard meridian-longitude loop basis (k_0, ℓ_0).

By the properties (i), (ii) above, for the spin loop-pair system (k_β, ℓ_β) on $F(\beta)$ in S^4, the handlebody system $V(\beta)$ is replaced by a handlebody system $V(\beta)' = B \cup V^B(\beta)'$ with the same trivial surface-link $F(\beta)$ such that the loop system k_β bounds a disjoint disk system D'_β in $V^B(\beta)'$. Since h is a pairwise s-nontrivial fusion 1-handle on F, the ribbon S^2-link $L(h^L)$ is a split union of a nontrivial ribbon S^2-knot $L_{12}(h^L)$ for the S^2-sublink L_{12} of L connected by h^L and the trivial S^2-sublink $L \setminus L_{12}$ of L, as seen in the proof of Lemma 1.1. Let B_{12} be the union of 3-balls in B connected by h^L, and $W(L_{12}; h^L) = B_{12}^{(0)} \cup h^L$ a SUPH system for the nontrivial ribbon S^2-knot $L_{12}(h^L)$ obtained from the boundary collar $B_{12}^{(0)}$ of L_{12} in B_{12}. Let $V(\beta)'' = \mathrm{cl}(V(\beta)' \setminus B_{12})$. From construction, the following claim is obtained.

(2.1.1) The SUPH system $W' = W(L_{12}; h^L) \cup V(\beta)''$ for the surface-link $F(\beta)(h)$ in S^4 is obtained by disk-summing the SUPH system $W(L_{12}; h^L)$ for the nontrivial ribbon S^2-knot$L_{12}(h^L)$ and the handlebody system $V(\beta)''$.

By assumption, the surface-link $F(h)$ is a ribbon surface-link. Let $W(F(h))$ be a SUPH system for $F(h)$. If necessary, by replacing $W(F(h))$ with a compact multi-punctured manifold of $W(F(h))$, the union $W = W(F(h)) \cup \beta$ is a SUPH system for the surface-link $F(\beta)(h)$. As explained in Appendix, there is an orientation-preserving diffeomorphism $f : (S^4, W') \to (S^4, W_{**})$ for some multi-fusion SUPH system W_{**} of some multi-fission SUPH system W_* of the SUPH system W for $F(\beta)(h)$. Technically, every handlebody component in W' and W should be a once-punctured handlebody, but it is ultimately filled by the removed 3-ball. The SUPH system W_* for $F(\beta)(h)$ admits the disk system D_β in W, but the SUPH system W_* for $F(\beta)(h)$ admits a deformed disk system Δ_β of D_β keeping k_β fixed. By the fact (i), let $(k_\beta^+, \ell_\beta^+)$

be a spin loop basis on $F(\beta)(h)$ extending the spin loop-pair system (k_β, ℓ_β) such that the loop system k^+_β bounds a disjoint disk system Δ^+_β in W_{**} extending the disk system Δ_β. The preimage $(f^{-1}(k^+_\beta), f^{-1}(\ell^+_\beta))$ of the spin loop basis $(k^+_\beta, \ell^+_\beta)$ of $F(\beta)(h)$ by f is a spin loop basis of $F(\beta)(h)$ and the loop system $f^{-1}(k^+_\beta)$ bounds a disjoint disk system $f^{-1}(\Delta^+_\beta)$ in W'. By deforming the two disk-summands with $W(L_{12}; h^L)$ in W' into a single disk-summand and using the fact (ii), there is an orientation-preserving diffeomorphism σ of S^4 sending $(k^+_\beta, \ell^+_\beta)$ to $(f^{-1}(k^+_\beta), f^{-1}(\ell^+_\beta))$ and keeping W' setwise fixed and $W(L_{12}; h^L)$ fixed. The loop system $f^{-1}(k^+_\beta)$ bounds a disjoint disk system $f^{-1}(\Delta^+_\beta)$ in W', and bounds a disjoint disk system $f^{-1}(\Delta^+_\beta)^*$ in the handlebody system $V(\beta)''$ by the cut and paste operations on the transverse intersection loop system $f^{-1}(\Delta^+_\beta) \cap \partial V''$. The image $f(V(\beta)'')$ of the handlebody system $V(\beta)''$ in W_{**} is deformed into $V(\beta)''$ with the same spin basis $(k^+_\beta, \ell^+_\beta)$. Also, the image $f(W(L_{12}; h^L))$ of $W(L_{12}; h^L)$ in W_{**} is identified with $W(L_{12}; h^L)$. To see this, note that $W(L_{12}; h^L) = B^{(0)}_{12} \cup h^L$ is a compact 2-punctured 3-ball with $L_{12}(h^L)$ a nontrivial ribbon S^2-knot and $O^2 = \partial W(L_{12}; h^L) \setminus L_{12}(h^L)$ a trivial S^2-link of 2 components. As shown in Lemma 1.1, after identification $f(O^2) = O^2$, the 1-handle $f(h^L)$ on O^2 is unique up to isotopies of S^4 keeping $B^{(0)}_{12} cup V''$ fixed by regarding V'' as a spine graph. Thus, by the composite diffeomorphism $g = f\sigma$ of S^4, the following claim is obtained.

(2.1.2) There is an isotopic deformation of W' so that $W' = W_{**}$ and there is an orientation-preserving diffeomorphism g of S^4 sending W' to W_{**} whose restriction to the spin loop basis $(k^+_\beta, \ell^+_\beta)$ on $F(\beta)(h)$ is the identity map.

The SUPH system W_* for $F(\beta)(h)$ is obtained from W_{**} by adding 2-handles attaching to the trivial S^2-link $O^2 = \partial W_{**} \setminus F(\beta)(h)$ and the disk system D_β is in W_*. Let D_h be a transverse disk of the 1-handle h, and $k_h = \partial D_h$. Then $g(k_h) = k_h$ is assumed and $g(D_h)$ is ∂-relatively isotopic to D_h in $W' = W_{**}$ and hence in W_*. The disk system $g(D_h)$ meets the disk system D_β in W_*. Let $g(D_h)'$ be a proper disk with $\partial g(D_h)' = k_h$ and $g(D_h)' \cap D_\beta = \emptyset$ in W_* obtained from $g(D_h)$ by the cut and paste operations on the transverse intersection loop system $g(D_h) \cap D_\beta$. The compact 3-manifold W'_* obtained from W_* by splitting along the disjoint disk system $g(D_h)' \cup D_\beta$ is a SUPH system for the surface-link $F(h)(g(D_h)' \times I)$ for a 2-handle $g(D_h)' \times I$ on $F(h)$ with core disk $g(D_h)'$, so that $F(g(D_h)' \times I)$ is a ribbon surface-link. The disk D_h is obtained from the 2-handle core disk $g(D_h)'$ on $F(h)$ by a SUPH-move as explained in Appendix. By Proposition A.3, the surface-link $F = F(h)(D_h \times I)$ is a ribbon surface-link. This completes the proof of Theorem 1.2.

Appendix: Moves on the SUPH systems of a ribbon surface-link

There are three kinds of moves M_0, M_1, M_2 on a chord diagram $C = C(o, \alpha)$ for equivalence of a ribbon surface-link F, which are called the *Reidemeister move*, the *fusion-fission move*, and the *chord move*, respectively, [1, 9]. The fusion-fission move M_1 consists of the fusion move $M_1(fusion)$ decreasing the number of based loops and the fission move $M_1(fission)$ increasing the number of based loops. For a SUPH system W' obtained from a SUPH system W by applying the Reidemeister move M_0 or the chord move M_2, there is an orientation-preserving diffeomorphism of S^4 sending W to W'. If the fusion move $M_1(fusion)$ or its iteration gives a SUPH system W' from a SUPH system W, then W' is obtained from W by removing a 1-handle or a 1-handle system on O in W and called a *fusion SUPH system* or a *multi-fusion SUPH system* of W, respectively. If the fission move $M_1(fission)$ or its iteration produces a SUPH system W' from a SUPH system W, then W' is obtained from W by attaching a 2-handle or a 2-handle system on O in W and called a *fission SUPH system* or a *multi-fission SUPH system* of W, respectively. If a 2-handle h_2 on a component O_1 of O is used for a fission SUPH system W' of the SUPH system W, then there is a 3-ball A_1 with $\partial A_1 = O_1$ in S^4 such that W meets the interior of A_1 only with a disjoint proper 2-disk system d_1 of W and the 2-handle h_2 is embedded in $A_1 \setminus d_1$. This property is obtained from the following proposition.

Proposition A.1. If a 1-handle h_1 on a trivial S^2-link O of r components in S^4 produces a trivial S^2-link O' of $r-1$ components by surgery, then the 1-handle h_1 is a trivial 1-handle on O, that is, a 1-handle not meeting the interior of a 3-ball system bounded by O.

A similar argument to this proposition is used in the proof of Lemma 1.1 and obtained by the conjugacy theorem for the one relator Wirtinger presentation, [4]. Note that this method detects classification of 1-handles but does not classify 1-fusion ribbon S^2-links. By this proposition, the fusion move $M_1(fusion)$ and the fission move $M_1(fission)$ are dual concepts on the SUPH systems for a ribbon surface-link. For any two SUPH systems W and W' for a ribbon surface-link F, it is known that W' is obtained from W by a finite number of the moves $M_i\,(i = 0, 1, 2)$, [9]. By identifying two SUPH systems sent by an orientation-preserving diffeomorphism of S^4, it may be considered that W' is obtained from W by a finite number of the fusion move $M_1(fusion)$ and/or the fission move $M_1(fission)$. The ordering of these moves is arranged so that a finite number of the fusion move $M_1(fusion)$ are performed after a finite number of the fission move $M_1(fission)$ are performed. This means the following proposition.

Proposition A.2. For any SUPH systems W and W' of a ribbon surface-link F with $\partial W \setminus F \neq \emptyset$ and $\partial W' \setminus F \neq \emptyset$, a multi-fusion SUPH system W_{**} of a multi-fission SUPH system W_* of W is sent to the SUPH system W' by an orientation-preserving diffeomorphism of S^4. In other words, a multi-fission SUPH system W_* of W is sent to a multi-fission SUPH system W'_* of W' by an orientation-preserving diffeomorphism of S^4.

Let D be a 2-handle core disk on a surface-link F in S^4, and U a SUPH system for a ribbon S^2-knot K in S^4 with $K \cap F = \emptyset$ such that $\delta = D \cap K = D \cap U$ is a disk. The 2-handle core disk D' on F given by $D' = \mathrm{cl}(D \setminus \delta) \cup \mathrm{cl}(K \setminus \delta)$ is called a 2-handle core disk obtained from D by an *elementary SUPH-move*. A 2-handle core disk D^* on F is obtained from D on F by a *SUPH-move* if D^* is obtained from D on F by a finite sequence of elementary SUPH moves. Note that if D^* is obtained from D on F by a SUPH-move, then D is obtained from D^* on F by a SUPH-move. The following proposition is used for the proof of Theorem 1.2.

Proposition A.3. Assume that the surface-link $F(D \times I)$ obtained from a surface-link F by surgery along a 2-handle $D \times I$ is a ribbon surface-link. Let D' be a disk obtained from the 2-handle core disk D on F by a SUPH-move. Then the surface-link $F(D' \times I)$ is a ribbon surface-link.

Proof of Proposition A.3. Let (O, h) be a handled sphere system for the ribbon surface-link $F(D \times I)$ in S^4. Let $W = O \times [0,1] \cup h$ be a SUPH system for $F(D \times I)$ with $O \times \{0\} = O$. The 1-handle system h on $O \times [0,1]$ attached to O is made disjoint from the 2-handle $D \times I$, where let $I = [-\varepsilon, \varepsilon]$ for small positive number ε. Let $D_{\varepsilon t} = D \times \{\varepsilon t\}$ for every t in $[-1,1]$ where D_0 is identified with the core disk D of $D \times I$. The disks $D_{-\varepsilon}$ and D_ε are located on $O \setminus h \cap O$. Let U be a SUPH system for a ribbon S^2-knot K in S^4 which is used for the elementary SUPH move from D to D', where $\delta = K \cap D = U \cap D$ is a disk and $K \cap F(D \times I) = \emptyset$. The intersection $(D \times I) \cap U$ is assumed to be a collar $\delta \times [0, \varepsilon]$ of δ in U, so that the union $W^+ = W \cup D \times I \cup U$ is a compact oriented 3-manifold in S^4. Let $W^+ \times [-1,1]$ be a bi-collar of W^+ in S^4 with $W^+ \times \{0\} = W^+$. In $W^+ \times [-1,1]$, let

$$W^* = W \cup_{0 \le t \le 1} D_{\varepsilon(t-1)} \times \{-t\} \cup U \times \{-1\} \cup_{0 \le t \le 1} D_{\varepsilon(1-t)} \times \{t\} \cup U \times \{1\},$$

which is a SUPH system for a surface-link equivalent to the surface-link obtained from $F(D' \times I)$ by pushing the interior of the disk $D'_{-\varepsilon}$ into $W^+ \times [-1, 0)$ and the interior of the disk D'_ε into $W^+ \times (0,1]$, which is equivalent to $F(D' \times I)$ in S^4. Thus, $F(D' \times I)$ is a ribbon surface-link. This completes the proof of Proposition A.3.

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