

Quasi-ribbon but non-ribbon surface-link of trivial components

Akio Kawauchi

Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University

Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

kawauchi@omu.ac.jp

ABSTRACT

An operation on a surface-link preserving the surface-knot components called a quasi-equivalence is introduced, which induces an equivalence relation on the surface-links. A quasi-ribbon surface-link is a surface-link quasi-equivalent to a ribbon surface-link. It is shown that a surface-link of trivial components is a quasi-ribbon surface link if it has only at most one non-spherical component, whereas there exist examples that are not quasi-ribbon surface-links of trivial components when it has at least two aspherical components. For every closed disconnected, orientable surface, a quasi-ribbon but non-ribbon surface-link of trivial components is constructed. This construction is done by applying a positive solution to Cochran's conjecture, meaning that the sphere-knot obtained from the anti-parallel sphere-link of every non-ribbon sphere-knot by surgery along an s-nontrivial or s-trivial fusion 1-handle is a non-ribbon or trivial sphere-knot, respectively.

Keywords: Ribbon, Quasi-equivalence, Quasi-ribbon, Cochran's conjecture, s-trivial fusion 1-handle.

Mathematics Subject Classification 2020: 57K45; 57K40

1. Introduction

Let $\mathbf{F}$ be a (possibly disconnected) closed oriented surface. An $\mathbf{F}$ -link in the 4-sphere S^4 is the image of a smooth embedding $\mathbf{F} \rightarrow S^4$. A surface-link in S^4 is an $\mathbf{F}$ -link for some $\mathbf{F}$. If $\mathbf{F}$ consists of some copies of the 2-sphere S^2 , then it is also called an S^2 -link. When $\mathbf{F}$ is connected, they are also called an $\mathbf{F}$ -knot, surface-knot and S^2 -knot, respectively. A trivial surface-link is a surface-link F in S^4 which bounds

disjoint handlebodies smoothly embedded in S^4 . A 1-handle on a surface-link F in S^4 is a 1-handle h on F embedded smoothly in S^4 , which is called a *self 1-handle* on F if the attaching part of h belongs to the same component of F . Otherwise, the 1-handle h is called a *fusion 1-handle* on F . Let $F(h)$ be the surface-link obtained from F by surgery along a 1-handle system (i.e., a system of disjoint 1-handles) h . A ribbon surface-link is the surface-link $F = O(h^O)$ in S^4 obtained from a trivial S^2 -link O by surgery along a 1-handle system h^O on O , [1, 2]. An O2-handle pair on a surface-link F in S^4 is a pair $(D \times I, E \times I)$ of 2-handles $D \times I, E \times I$ on F in S^4 which intersect orthogonally only with the attaching parts $(\partial D) \times I, (\partial E) \times I$ to F so that the intersection $Q = (\partial D) \times I \cap (\partial E) \times I$ is a square, [3]. Let F_0 be a surface-knot in S^4 and h_0 a self-trivial 1-handle on F_0 . Then there is an O2-handle pair $(D \times I, E \times I)$ on the surface-knot $F_0(h_0)$. For any O2-handle pair $(D \times I, E \times I)$ on $F_0(h_0)$, the surface-knot $F_0(h_0)(D \times I, E \times I)$ is not only equivalent to the surface-knots $F_0(h_0)(D \times I)$ and $F_0(h_0)(E \times I)$ by the common 2-handle property, but also equivalent to the original surface-knot F_0 , [3, 4, 5]. More generally, let F be a surface-link with a component F_0 in S^4 , and h_0 a self 1-handle system on F in S^4 such that h_0 is a self-trivial 1-handle system on F_0 . Let $(D \times I, E \times I)$ be an O2-handle pair system on $F_0(h_0)$ such that the 2-handle system $E \times I$ possibly meets the complementary surface-link $F \setminus F_0$ in S^4 , which is referred to as a *quasi-O2-handle pair system* on F . Then the surface-link $F' = F(h_0)(D \times I)$ is a (possibly new) surface-link consisting of the components equivalent to the original components of F . This correspondence $F \rightarrow F'$ is called an *elementary quasi-equivalence*. A correspondence $F \rightarrow F^*$ for a surface-link F^* is called a *quasi-equivalence* if there is a sequence of surface-links $F^{(i)}$ ($i = 0, 1, 2, \dots, n$) with $F^{(0)} = F$ and $F^{(n)} = F^*$ such that the correspondence $F^{(i-1)} \rightarrow F^{(i)}$ is an elementary quasi-equivalence for all $i > 0$. *It would be interesting to observe that the inverse correspondence $F^* \rightarrow F$ is also a quasi-equivalence and thus, quasi-equivalence on a surface-link is an equivalence relation.* This is because the 2-handle system $D \times I$ on $F(h_0)$ is regarded as a self-trivial 1-handle system h'_0 on F' with $F'(h'_0) = F(h_0)$ and the self-trivial 1-handle system h_0 on F is regarded as a 2-handle system $D' \times I$ on $F'(h'_0)$ constituting a quasi-O2-handle pair system $(D' \times I, E' \times I)$ on $F'(h'_0)$ such that $F = F'(h'_0)(D' \times I)$. A *quasi-ribbon surface-link* is a surface-link quasi-equivalent to a ribbon surface-link. By definition, every ribbon surface-link is a quasi-ribbon surface-link and every quasi-ribbon surface-link consists of ribbon surface-knot components. In particular, a quasi-ribbon surface-knot is a ribbon surface-knot. The following theorem is shown.

Theorem 1.1. Every surface-link F of trivial components with only at most one aspherical component in S^4 is a quasi-ribbon surface-link.

Theorem 1.1 corrects the author's earlier statement, [6, Theorem 1], [7, Theorem 1.1, Corollary 1.3]. The following theorem is a strengthened version of [6, Theorem 2] and is shown in Section 3.

Theorem 1.2. For every disconnected closed oriented surface $\mathbf{F}$ with at least two non-spherical components, there is a pair of $\mathbf{F}$ -links F and F' in S^4 both of trivial components with the same fundamental group up to meridian-preserving isomorphisms such that F is a ribbon surface-link but F' is not a quasi-ribbon surface-link.

For the proof of Theorem 1.2, a generalization of the null-homotopic Gauss sum invariant of a surface-knot to a surface-link is discussed, [8]. The following theorem is the main result of this paper.

Theorem 1.3. For every disconnected closed oriented surface $\mathbf{F}$, there is a quasi-ribbon but non-ribbon $\mathbf{F}$ -link of trivial components in S^4 .

Theorem 1.3 generalizes Ogasa's observation, [9]. The proof of Theorem 1.3 is done in Section 4 by applying a positive answer to Cochran's conjecture discussed there, [10].

2. Proof of Theorem 1.1

A *semi-unknotted multi-punctured handlebody system*, or simply a *SUPH system* for a surface-link F in S^4 is a compact oriented 3-manifold W smoothly embedded in S^4 such that W is a handlebody system with a finite number of open 3-balls removed and the boundary ∂W of W is given by $\partial W = F \cup O$ for a trivial S^2 -link O in S^4 , [4]. A typical SUPH system W is constructed from a ribbon surface-link F defined from a trivial S^2 -link O and a 1-handle system h^O on F as the union $O \times [0, 1] \cup h^O$ for a normal collar $O \times [0, 1]$ of O in S^4 with $O \times \{0\} = O$ where h^O does not meet $O \times [0, 1]$ except for the attaching part to O and $\partial W = F \cup O \times \{1\}$. For a SUPH system W with $\partial W = F \cup O$, there is a proper arc system α in W spanning O such that a regular neighborhood $N(O \cup \alpha)$ of the union $O \cup \alpha$ in W is diffeomorphic to the closed complement $\text{cl}(W \setminus c(F \times [0, 1]))$ of a boundary collar $c(F \times [0, 1])$ of F in W . This pair (O, α) is called a *chorded sphere system* of the SUPH system W . By replacing α with a 1-handle system h^O attaching to O with core arc system α , the surface-link F is a ribbon surface-link defined by O and h^O . In other words, giving a SUPH system W with $\partial W = F \cup O$ is the same as saying that the surface-link F is a ribbon surface-link with sphere system O . A *multi-fusion SUPH system* of a SUPH system W with $\partial W = F \cup O$ in S^4 is a SUPH system for F in S^4 obtained from W by deleting an open regular neighborhood of a disjoint simple proper arc system in W .

spanning O . A *multi-fission SUPH system* of a SUPH system W with $\partial W = F \cup O$ in S^4 is a SUPH system for F in S^4 obtained from W by adding a 2-handle system h^2 on O disjoint from W except for the attaching part in O where h^2 is taken in the complement $B \setminus D_W$ for a disjoint 3-ball system B bounded by O in S^4 with $(B \setminus O) \cap W = D_W$ a disjoint 2-disk system. For any two SUPH systems W and W' for a ribbon surface-link F in S^4 , there is an orientation-preserving diffeomorphism f of S^4 sending W to a multi-fusion SUPH system W^{**} of a multi-fission SUPH system W^* of the SUPH system W' , [10, Appendix]. The following theorem, giving a characterization of quasi-ribbon surface-links, corrects an earlier characterization of when a surface-link of ribbon components is a ribbon surface-link, [4, Theorem 1.4].

Theorem 2.1. A surface-link F in S^4 is a quasi-ribbon surface-link if and only if there is a self-trivial 1-handle system h on F such that the surface-link $F(h)$ is a ribbon surface-link in S^4 .

Proof. A surface-link F is a quasi-ribbon surface-link if and only if there is a ribbon surface-link F' with the same components as F such that $F = F'(h')(D \times I)$ for a self 1-handle system h' on F' and a quasi-O2-handle pair system $(D \times I, E \times I)$ on $F'(h')$. Then the 2-handle system $D \times I$ on the ribbon surface-link $F'(h')$ is regarded as a self-trivial 1-handle system h on F such that $F(h) = F'(h')$. This shows the “only if part” of the theorem. Conversely, assume that $F(h)$ is a ribbon surface-link consisting of the ribbon surface-knots $F_i(h_i)$ ($i = 1, 2, \dots, r$) where F_i is a surface-knot component F_i of F and h_i is the self-trivial 1-handle system h_i on F_i with $h = \cup_{i=1}^r h_i$. By the ribbonness of $F_i(h_i)$ and the triviality of h_i on F_i , the surface-knots F_i ($i = 1, 2, \dots, r$) are ribbon surface-knots, [4]. Let W be a SUPH system for $F(h)$ which is a union of SUPH systems W_i for $F_i(h_i)$ ($i = 1, 2, \dots, r$). Let W_i^0 be a SUPH system for the ribbon surface-knot F_i . The trivial 1-handle system h_i on F_i is obtained so that the core arc system α_i of h_i is obtained from an arc system α_i^0 in F_i by pushing the interior of the arc system α_i^0 outside of W_i^0 . Then the union $W_i' = W_i^0 \cup h_i$ a SUPH system for $F_i(h_i)$ which admits an O2-handle pair system $(D_{h_i} \times I, E_i \times I)$ on $F_i(h_i)$ such that the core disk system D_{h_i} of the 2-handle system $D_{h_i} \times I$ is a transverse disk system of h_i and the core disk system E_i of the 2-handle system $E_i \times I$ does not meet W_i' . For the SUPH systems W_i and W_i' for $F_i(h_i)$, there is an orientation-preserving diffeomorphism f_i of S^4 sending W_i to a multi-fusion SUPH system W_i^{**} of a multi-fission SUPH system W_i^* of the SUPH system W_i' . Then the disk system D_{h_i} is in W_i' and hence in W_i^* , which is deformed into a disk system D_i^* in W_i^{**} with $\partial D_i^* = \partial D_{h_i}$. Since the core arc system α_i of h_i is an interior push of the arc system α_i^0 in F_i , the disk system E_i is made disjoint from a disjoint 3-ball system B_i bounded by the trivial S^2 -link $O_i = \partial W_i^0 \setminus F_i$ by sliding

the arc system α_i^0 along F_i . This means that the disk system E_i meets W_i^{**} only with the boundary loop system ∂E_i . The disk systems $D'_i = f_i^{-1}(D_i^*)$ and $E'_i = f_i^{-1}(E_i)$ constitute an O2-handle system $(D'_i \times I, E'_i \times I)$ on $F_i(h_i)$ such that D'_i is a proper disk system in W_i and E'_i meets W_i only with the boundary loop system $\partial E'_i$. Since the pair $(D'_i \times I, E'_i \times I)$ is a quasi-O2-handle pair system on $F(h_i)$, there is a quasi-O2-handle pair system $(D' \times I, E' \times I)$ on the ribbon surface-link $F(h)$ by taking $D' = \cup_{i=1}^r D'_i$ and $E' = \cup_{i=1}^r E'_i$ such that $F' = F(h)(D' \times I)$ is a ribbon surface-link with a SUPH system obtained from the SUPH system W for $F(h)$ by cutting along the proper disk system D' . Thus, F is quasi-equivalent to the ribbon surface-link F' . Since quasi-equivalence is an equivalence relation, F is a quasi-ribbon surface-link. This completes the proof of Theorem 2.1.

The following lemma is shown in [7, Lemma 1.2] (although similar ideas are found in the incorrect claims, [4, Theorem 1.4] and [6, Theorem 1]).

Lemma 2.2. For every surface-link F in S^4 with at most one aspherical component, there is a self 1-handle system h on F such that the surface-link $F(h)$ is a ribbon surface-link in S^4 .

The proof of Theorem 1.1 is done as follows.

2.3: Proof of Theorem 1.1. Every self 1-handle system on every surface-link F of trivial components is a self-trivial 1-handle system on F , [11]. Thus, Theorem 1.1 is obtained from Theorem 2.1 and Lemma 2.2, completing the proof of Theorem 1.1.

3. Proof of Theorem 1.2

The *quadratic function* $\eta : H_1(K; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ of a surface-knot K in S^4 is defined as follows. For a loop ℓ on K , let d be a compact (possibly non-orientable) surface in S^4 with $d \cap K = \partial d = \ell$. The value $\eta([\ell])$ is defined by the $\mathbb{Z}_2$ -self-intersection number $\text{Int}(d, d) \bmod 2$ with respect to the framing of the surface K which is independent of a choice of d by calculation. The function $\eta : H_1(K; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ is a $\mathbb{Z}_2$ -quadratic function with the identity

$$\eta(x + y) = \eta(x) + \eta(y) + x \cdot y,$$

where $x, y \in H_1(K; \mathbb{Z}_2)$ and $x \cdot y$ denotes the $\mathbb{Z}_2$ -intersection number of x and y in K . A loop ℓ on K is *spin* or *non-spin* according to whether $\eta([\ell])$ is 0 or 1, respectively. For a surface-link F in S^4 , the *quadratic function* $\eta : H_1(F; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ of F is defined to be the split sum of the quadratic functions $\eta_K : H_1(K; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ for all the components K of F . This quadratic function may be identified with

the quadratic function $\eta_{\#} : H_1(F_{\#}; Z_2) \rightarrow Z_2$ of a surface-knot $F_{\#}$ in S^4 which is a fusion of F along a fusion 1-handle system on F under a canonical isomorphism $\iota : H_1(F; Z_2) \rightarrow H_1(F_{\#}; Z_2)$. To see this, let ℓ be a loop in a component K of F , d a compact surface in S^4 with $d \cap K = \partial d = \ell$, and $F_{\#}$ the connected surface-knot in S^4 obtained from F by surgery along a fusion 1-handle system h on F . Every transverse intersection point between d and $F \setminus K$ in S^4 can be moved into $K \setminus h \cap K$ through $F_{\#}$, so that the compact surface d is modified into a compact surface d_K in S^4 with $d_K \cap F = \partial d_K = \ell \cup o_K$ for a trivial loop system o_K in K . Since every loop of o_K is a spin loop in $F_{\#}$, the identity

$$\eta_{\#}(\iota([\ell])) = \eta_{\#}(\iota([\ell]) + \iota([o_K])) = \eta([\ell]),$$

holds, showing the identification of η to $\eta_{\#}$. Let $\Delta(F; Z_2)$ be the subgroup of $H_1(F; Z_2)$ consisting of an element represented by a loop ℓ in F which bounds an immersed disk d in S^4 with $d \cap F = \ell$. The restriction $\xi : \Delta = \Delta(F; Z_2) \rightarrow Z_2$ of the quadratic function η on $H_1(F; Z_2)$ is called the *null-homotopic quadratic function* of the surface-link F . The *null-homotopic Gauss sum* of F is the Gauss sum $GS_0(F)$ of ξ defined by

$$GS_0(F) = \sum_{x \in \Delta} \exp(\xi(x)\pi\sqrt{-1}).$$

This number $GS_0(F)$ is an invariant of a surface-link F , which is calculable as shown for the case of a surface-knot, [8]. The following lemma is used for the proof of Theorem 1.2.

Lemma 3.1. If F is a quasi-ribbon surface-link of total genus g , then $GS_0(F) = 2^g$.

Proof. If F is a ribbon surface-link of total genus g , then it is known that $GS_0(F) = 2^g$, [8]. For a self-trivial 1-handle h_0 on a surface-link F , the group $\Delta(F(h_0); Z_2)$ is the direct sum $\Delta(F; Z_2) \oplus Z_2 \oplus Z_2$ or $\Delta(F; Z_2) \oplus Z_2$ according to whether a simple loop ℓ in F induced from the core arc α_0 of h_0 bounds an immersed disk d in S^4 with $d \cap F = \ell$. In either case, it is calculated that $GS_0(F(h_0)) = GS_0(F) \cdot 2$. If F is a quasi-ribbon surface-link of total genus g , then there is a self-trivial 1-handle system h such that $F(h)$ is a ribbon surface-link of total genus $g + s$ so that $GS_0(F(h)) = GS_0(F) \cdot 2^s = 2^{g+s}$, for the number s of 1-handles in h . Thus, $GS_0(F) = 2^g$. This completes the proof of Lemma 3.1.

By using this invariant $GS_0(F)$, the proof of Theorem 1.2 is done as follows.

3.2: Proof of Theorem 1.2. Let $k \cup k'$ be a non-split link in the interior of a 3-ball B such that k and k' are trivial knots. For the boundary 2-sphere $S^B = \partial B$ and

the disk D^2 with boundary circle S^1 , let K be the surface-link of torus-components $T = k \times S^1$ and $T' = k' \times S^1$ in the 4-sphere S^4 with $S^4 = B \times S^1 \cup S \times D^2$, which is a ribbon surface-link in S^4 , [2]. Then, $GS_0(K) = 2^2$. Since k and k' are trivial knots in B , the torus-knots T and T' are trivial torus-knots in S^4 by construction. Since $k \cup k'$ is non-split in B , there is a simple loop $t(k)$ in T coming from the longitude of k in B such that $t(k)$ does not bound any disk not meeting T' in S^4 , meaning that there is a simple loop c in T unique up to isotopies of T which bounds a disk d in S^4 not meeting T' , where c and d are given by $c = \{p\} \times S^1$ and $d = a \times S^1 \cup \{q\} \times D^2$ for a simple arc a in B joining a point p of k to a point q in S with $a \cap (k \cup k') = \{p\}$ and $a \cap S = \{q\}$. Regard the 3-ball B as the product $B = B_1 \times [0, 1]$ for a disk B_1 . Let τ_1 be a diffeomorphism of the solid torus $B_1 \times S^1$ given by one full-twist rounding the meridian disk B_1 one time along the S^1 -direction. Let $\tau = \tau_1 \times 1$ be the product diffeomorphism of $(B_1 \times S^1) \times [0, 1] = B \times S^1$ for the identity map 1 of $[0, 1]$. Let τ_∂ be the diffeomorphism of the boundary $S^B \times S^1$ of $B \times S^1$ obtained from τ by restricting to the boundary, and the 4-manifold M obtained from $B \times S^1$ and $S^B \times D^2$ by pasting the boundaries $\partial(B \times S^1) = S^B \times S^1$ and $\partial(S^B \times D^2) = S^B \times S^1$ by the diffeomorphism τ_∂ . Since the diffeomorphism τ_∂ of $S^B \times S^1$ extends to the diffeomorphism τ of $B \times S^1$, the 4-manifold M is diffeomorphic to S^4 . Let $K_M = T_M \cup T'_M$ be the surface-link of torus components T_M and T'_M in the 4-sphere M , arising from $K = T \cup T'$ in $B \times S^1$. There is a meridian-preserving isomorphism $\pi_1(S^4 \setminus K, x) \rightarrow \pi_1(M \setminus K_M, x)$ by van Kampen theorem. Let $(S^4, K') = (M, K_M)$. Let A be a 4-ball in S^4 such that $A \cap K = A \cap K'$ is a trivial disk system in A taking one disk from each component of K and from each component of K' to construct the pair (F, F') of a $\mathbf{F}$ -links F and F' constructed from the pair (K, K') by connected-summing a trivial surface-knot with both K and K' and/or adding a trivial surface-link component to both K and K' as a splitting component. By van Kampen theorem, it is shown that the fundamental groups $\pi_1(S^4 \setminus F, x)$ and $\pi_1(S^4 \setminus F', x)$ are the same group up to meridian-preserving isomorphisms. By construction, F is a ribbon surface-link. To show that F' is not a quasi-ribbon surface-link, note that the loop $t(k)$ in T_M does not bound any disk not meeting T'_M in M , so that the loop c in T_M is a unique simple loop up to isotopies of T_M which bounds a disk $d_M = a \times S^1 \cup D_M^2$ in M not meeting T'_M , where D_M^2 denotes a proper disk in $S^B \times D^2$ bounded by the loop $\tau_\partial(\{q\} \times S^1)$. An important observation is that the self-intersection number $\text{Int}(d_M, d_M)$ in M with respect to the surface-framing on K_M is ± 1 . This means that the loop c in T_M is a non-spin loop. Similarly, there is a unique non-spin loop c' in T'_M which bounds a disk d'_M with the self-intersection number $\text{Int}(d'_M, d'_M) = \pm 1$ with respect to the surface-framing on $K' = K_M$. Then it is calculated that $GS_0(K') = 0$, so that $GS_0(F') = 2^{g-2}$ for the total genus $g(\geq 2)$ of F' . Thus, F' is not a quasi-ribbon surface-link by Lemma 3.1. This completes the proof of Theorem 1.2.

In 3.2, the non-ribbon surface-link K' of two components starting from the Hopf link $k \cup k'$ in the interior of a 3-ball B has the free abelian fundamental group of rank 2. Then by van Kampen theorem, the surface-knot $K'(h')$ obtained from K' by surgery along any fusion 1-handle h' on K' has the infinite cyclic fundamental group, so that $K'(h')$ is a trivial surface-knot in S^4 by smooth unknotting result of a surface-knot, [3, 5]. This shows that there is a non-ribbon surface-link F' of ribbon components such that the surface-knot obtained from F' by surgery along any fusion 1-handle is a ribbon surface-knot, meaning that *non-ribbonness of a surface-link cannot detect in general by surgery along any fusion 1-handle system*, giving a strong counterexample to [4, Theorem 1.4]. The diffeomorphism τ_∂ of $S^B \times S^1$ in 3.2 coincides with Gluck's non-spin diffeomorphism of $S^2 \times S^1$, [12]. The surface-link (M, K_M) called a *turned torus-link* of a link $k \cup k'$ in B is an analogy of a *turned torus-knot* of a knot in B , [13].

4. Proof of Theorem 1.3

For a 1-handle h on a surface-link F in S^4 , a trivial S^2 -link *around* h is a trivial S^2 -link O^h in S^4 all of whose components are the boundary-spheres of fibers of a trivial normal 3-disk bundle of the core arc α of h in S^4 not meeting with $F \cup h$. The following lemma concerns a replacement of two spanning arcs of a surface-link.

Lemma 4.1. Let F be a surface-link in S^4 , and α_0 a simple arc spanning F in S^4 meeting F only with the endpoints. Then every simple arc α spanning F in S^4 meeting F only with the same endpoints as α_0 is isotopic to an arc obtained from the arc α_0 by band summing with a meridian loop system m of F by a smooth isotopy of S^4 keeping F fixed.

Proof. After a slight move of the interior of α , the union $\alpha_0 \cup \alpha$ bounds a disk D smoothly embedded in S^4 . The interior of the disk D transversely meets F in a finite point set $Q = \{q_1, q_2, \dots, q_n\}$, whose regular neighborhood in D is an unoriented meridian disk system $N(Q) = \{N(q_1), N(q_2), \dots, N(q_n)\}$ of F in S^4 . The arc α is ∂ -relatively isotopic in D to a band sum of the arc α_0 and the unoriented meridian loop system $m = \partial N(Q) = \{m_1, m_2, \dots, m_n\}$, $m_i = \partial N(q_i)$, along a band system $b = \{b_1, b_2, \dots, b_n\}$ in D , where b_i spans α and m_i . This completes the proof of Lemma 4.1.

The following lemma is obtained from Lemma 4.1 by replacing the spanning arcs α_0 and α with the 1-handles h_0 and h on F with core arcs α_0 and α , respectively.

Lemma 4.2. Let F be a surface-link in S^4 , and h_0 a 1-handle on F . Then for every 1-handle h on F with the same attaching part as h_0 , there is a trivial S^2 -link O around h_0 in S^4 with an equivalence $F(h) = (F(h_0) \cup O)(\xi^O)$ for a fusion 1-handle system ξ^O connecting the trivial surface-knot $F(h_0)$ and the trivial S^2 -link O .

Proof. In Lemma 4.1, let $B = \{B_1, B_2, \dots, B_n\}$ be a disjoint 3-ball system in S^4 attaching to F with a disk system $d = \{d_1, d_2, \dots, d_n\}$ and meeting a meridian loop system m of F transversely so that $m \cap B = \{t_1, t_2, \dots, t_n\}$ for a point $t_i = m_i \cap (B_i \setminus \partial B_i)$ ($i = 1, 2, \dots, n$). The thickening 1-handle h of α meets B_i with a transverse disk centered at t_i . For the complementary disk system $d^c = \text{cl}(\partial B \setminus d)$ of d in the boundary ∂B of B , regard the surface-link $F' = \text{cl}(F \setminus d) \cup d^c$ as F . Deform the arc α into the arc α_0 by taking together the 3-ball system B along the band system b . Then the 1-handle h is deformed into the 1-handle h_0 thickening the arc α_0 by taking together the 3-ball system B . This means that there is an equivalence $F(h) = (F(h_0) \cup O)(\xi^O)$ for the trivial S^2 -link $O = \partial B$ around h_0 and a fusion 1-handle system ξ^O connecting $F(h_0)$ and O which is constructed by thickening and stretching the complementary disk system d^c along the band system b . This completes the proof of Lemma 4.2.

For a surface-knot F in S^4 , let V be a connected Seifert hypersurface V of F in S^4 , and $(V, F) \times [0, 1]$ a normal $[0, 1]$ -bundle over (V, F) in S^4 . Let $(V_i, F_i) = (V, F) \times \{i\}$ ($i = 0, 1$). Then the natural homomorphism $H_1(F_1; \mathbb{Z}) \rightarrow H_1(S^4 \setminus F_0; \mathbb{Z})$ is the zero. The surface-link $P(F) = F_0 \cup F_1$ is called the *anti-parallel surface-link* of F , where by convention F_0 and F_1 are identified with $-F$ (i.e., the orientation reversed F) and F , respectively. Cochran's conjecture says that when F is a non-ribbon S^2 -knot, the anti-parallel S^2 -knot $P(F; h)$ must be a non-ribbon S^2 -knot for a sufficiently complicated fusion 1-handle h , [10]. Let h be a fusion 1-handle on $P(F)$, and β be a 1-handle system on F embedded in V with $V(\beta) = \text{cl}(V \setminus \beta)$ a handlebody such that $\beta_i = \beta \times \{i\}$ is embedded in V_i ($i = 0, 1$) and disjoint from h . A fusion 1-handle h on $P(F)$ is *surgically trivial*, or simply *s-trivial* if $P(\beta_0 \cup \beta_1)(h)$ is a trivial surface-knot. Otherwise, h is *surgically nontrivial*, or simply *s-nontrivial*. Since $P(F)$ is a boundary surface-link, this definition does not depend on choices of V and β , [14, Lemma 1.1]. Cochran's conjecture says that for the anti-parallel S^2 -link $P(K)$ of a non-ribbon S^2 -knot K in S^4 , if h is a "sufficiently complicated" fusion 1-handle h on $P(K)$ must be a non-ribbon S^2 -knot, [10]. By replacing "sufficiently complicated" with "s-nontrivial", the following theorem not only confirms Cochran's conjecture affirmatively but also shows that the result holds even when the non-ribbon S^2 -knot K is generalized to any non-ribbon surface-knot F , which was announced in [15] with incomplete proof.

Theorem 4.3. Let $P(F)$ be the anti-parallel surface-link of a non-ribbon surface-

knot F in S^4 . According to whether a fusion 1-handle h on $P(F)$ is s-trivial or s-nontrivial, the surface-knot $P(F)(h)$ is a trivial surface-knot or a non-ribbon surface-knot, respectively.

Proof. For any s-nontrivial fusion 1-handle h on $P(F)$, the surface-knot $P(F)(h)$ is a non-ribbon surface-knot, [14]. Let h be an s-trivial fusion 1-handle on $P(F)$. Let V be a connected Seifert hypersurface for F , β a 1-handle system on F embedded in V such that $\text{cl}(V \setminus \beta)$ is a handlebody. Let $(V, F) \times [0, 1]$ be a normal $[0, 1]$ -bundle bundle over (V, F) in S^4 . For a subset X in V , let X_i denote $X \times \{i\}$ for $i = 0$ or 1 . Let $P(F) = F_0 \cup F_1$. For a disk d in F , let $h_0 = d \times [0, 1]$ be the standard trivial 1-handle on $P(F)$, and assume that the attaching part of h to $P(F)$ coincides with the disk union $d_0 \cup d_1$. Note that the surface-knot $P(F)(h_0)$ is a trivial surface-knot since it bounds a handlebody $\text{cl}(F \setminus d) \times [0, 1]$. Let α and α_0 be the core arcs of the 1-handles h and h_0 on $P(F)$, respectively, which are disjoint except for the same endpoints p_i in d_i ($i = 0, 1$). The simple loop $\alpha_0 \cup \alpha$ meets $V_0 \cup V_1$ only with the points p_i ($i = 0, 1$), and bounds a smoothly embedded disk Δ in S^4 , [11]. Let D_i^β be a transverse disk system of the 1-handle system β_i with just one disk for every 1-handle ($i = 0, 1$). By regarding the handlebody systems $V_i(\beta_i)$ ($i = 0, 1$) as spine graphs attaching to the points p_i ($i = 0, 1$), the disk Δ can avoid intersecting the handlebody system $V_0(\beta_0) \cup V_1(\beta_1)$ except for the points p_i ($i = 0, 1$) by general position. In particular, the disk Δ does not meet the trivial surface-link $P(F)(\beta_0 \cup \beta_1)$ except for the points p_i ($i = 0, 1$). Thus, the disk Δ meets transversely the disk system $D_0^\beta \cup D_1^\beta$ with a finite number of interior points. By Lemma 4.1 applied to the disk Δ , the arc α is obtained from the arc α_0 by band-summing a meridian loop system m of the disk system $D_0^\beta \cup D_1^\beta$ in S^4 . By Lemma 4.2, there is a disk system E_i^β which is obtained from D_i^β and a trivial S^2 -link O_i around the 1-handle h_0 by surgery along a fusion 1-handle system ξ_i such that the surface-knot $P(F)(h)(\beta_0 \cup \beta_1)(D_0^\beta \times I \cup D_1^\beta \times I)$ is equivalent to the surface-knot $P(F)(h_0)(\beta_0 \cup \beta_1)(E_0^\beta \times I \cup E_1^\beta \times I)$. By choosing a disjoint arc system γ in $\text{cl}(F \setminus d)$ such that each arc of γ meets transversely the loop system ∂D^γ with just one point, the disk system $\Lambda = \gamma \times [0, 1]$ is considered as a core disk system of a 2-handle $\Lambda \times I$ on $P(F)(h_0)(\beta_0 \cup \beta_1)$ such that the pairs $(E_i^\beta \times I, \Lambda \times I)$ ($i = 0, 1$) and $(D_i^\beta \times I, \Lambda \times I)$ are O2-handle pair systems on $P(F)(h_0)\beta_0 \cup \beta_1$ ($i = 0, 1$) with the same attaching parts for each i , where the 1-handle systems ξ_i ($i = 0, 1$) are chosen to be disjoint from the disk Λ . By common 2-handle property, the O2-handle pair system $(E_i^\beta \times I, \Lambda \times I)$ is sent to the O2-handle pair system $(D_i^\beta \times I, \Lambda \times I)$ for each i by an orientation-preserving diffeomorphism f of S^4 keeping $P(F)(h_0)(\beta_0 \cup \beta_1)$ fixed, [3, 5]. In particular, $P(F)(h) = P(F)(h_0)(\beta_0 \cup \beta_1)(E_0^\beta \times I \cup E_1^\beta \times I)$ is equivalent to $P(F)(h_0)(\beta_0 \cup \beta_1)(D_0^\beta \times I \cup D_1^\beta \times I) = P(F)(h_0)$, which is a trivial surface-link. This completes the proof of Theorem 1.2.

The proof of Theorem 1.3 is done as follows.

4.4: Proof of Theorem 1.3. Let K be a non-ribbon S^2 -knot in S^4 , and V a connected Seifert hypersurface for K . Let $B \cup V^B$ be a decomposition of V with B a 3-ball and $d^B = B \cap V^B$ a proper disk. For a collar $V \times [0, 1]$ of V in S^4 , Let $h_0 = d \times [0, 1]$ be the trivial fusion 1-handle in the anti-parallel S^2 -link $P(K) = K_0 \cup K_1$ of K for a disk d in d^B , and h a non-trivial fusion 1-handle on $P(K)$ with the same attachment as h_0 such that h is in a regular neighborhood N of the 3-ball $B_0 \cup h_0 \cup B_1$ in S^4 and meets the interior of the 3-ball B_0 with n transversal disks of h , for any given positive integer n . Let h' be a fusion 1-handle on $P(K)$ in N with the same attachment as h such that the core arc α' of h' does not meet B_0 except for the attaching point and the core arc α of h is a band sum of α' and a meridian loop system on K_0 by an argument of Lemma 4.1. Since α' is ∂ -relatively isotopic to α_0 in N , the S^2 -knot component $(P(K))(h')$ is equivalent to the trivial S^2 -knot $(P(K))(h_0)$. By Lemma 4.2, there is an equivalence $P(K)(h) = (P(K)(h_0) \cup O)(\xi^O)$ for a trivial S^2 -link O with n -components around h_0 and a fusion 1-handle system ξ^O connecting the surface-knot $P(K)(h_0)$ and O . By Theorem 4.3, the S^2 -knot $P(K)(h)$ is non-ribbon, so that the S^2 -link $F_K = P(K)(h_0) \cup O$ is a non-ribbon S^2 -link of $n + 1$ components. By Theorem 1.1, F_K is quasi-ribbon. For every disconnected closed oriented surface $\mathbf{F}$, a quasi-ribbon but non-ribbon $\mathbf{F}$ -link F with trivial components is constructed from F_K for some n by connected-summing trivial surface-knots and/or adding a trivial surface-link component as a splitting component. Non-ribbonness of F is seen from that if F is a ribbon surface-link, then F_K must be a ribbon S^2 -link, [4]. To see that the $\mathbf{F}$ -link F is quasi-ribbon, note that F_K is quasi-ribbon and by Theorem 2.1 there is a self-trivial 1-handle system h^* on F_K such that $F_K(h^*)$ is a ribbon surface-link. The connected-summand of trivial surface-knots and/or addition of a trivial surface-link component is taken locally apart from h^* , and $F(h^*)$ is taken as a ribbon surface-link. By Theorem 2.1, F is a quasi-ribbon but non-ribbon $\mathbf{F}$ -link of trivial components. This completes the proof of Theorem 1.3.

Conclusion

The original question is when a surface-link F of ribbon surface-knot components becomes a ribbon surface-link, [4]. Theorem 1.1 shows that a boundary surface-link F is a ribbon surface-link if there is a pairwise nontrivial fusion 1-handle system h on F with $F(h)$ a ribbon surface-link. For surface-links with at least two aspheric components, there are examples of non-ribbon surface-links F of ribbon components. In particular, there are examples of non-ribbon surface-links of trivial components which cannot be detected by the fundamental groups, as shown in Theorem 1.2 and

[6]. According to Theorem 1.1, a surface-link of trivial components with at most one aspherical component is a quasi-ribbon surface-link, which shares properties analogous to those of ribbon surface-links. However, as Theorem 1.3 indicates, additional conditions are required for it to be a ribbon surface-link.

Acknowledgements

This work was partly supported by JSPS KAKENHI Grant Numbers JP21H00978 and JP26K06456, and MEXT Promotion of Distinctive Joint Research Center Program JPMXP0723833165 and Osaka Metropolitan University Strategic Research Promotion Project (Development of International Research Hubs).

References

- [1] Kawauchi, A., Shibuya, T., Suzuki, S. (1982). Descriptions on surfaces in four-space, II: Singularities and cross-sectional links, Math Sem Notes Kobe Univ, 11: 31-69.
- [2] Kawauchi, A. (2015). A chord diagram of a ribbon surface-link, J Knot Theory Ramifications, 24: 1540002 (24 pages).
- [3] Kawauchi, A. (2021). Ribbonness of a stable-ribbon surface-link, I. A stably trivial surface-link, Topology and its Applications, 301: 107522 (16pages).
- [4] Kawauchi, A. (2025). Ribbonness of a stable-ribbon surface-link, II: General case, (MDPI) Mathematics, 13 (3): 402 (1-11).
- [5] Kawauchi, A. (2023). Uniqueness of an orthogonal 2-handle pair on a surface-link, Contemporary Mathematics (UWP), 4: 182-188.
- [6] Kawauchi, A. (2024). Note on surface-link of trivial components, Journal of Comprehensive Pure and Applied Mathematics, 2 (1) : 1 - 5.
- [7] Kawauchi, A. (2025). Revised note on surface-link of trivial components. J Comp Pure Appl Math, 3(2) (2025), 1-10.
- [8] Kawauchi, A. (2002). On pseudo-ribbon surface-links, J Knot Theory Ramifications, 11: 1043-1062.
- [9] Ogasa, E. (2001). Nonribbon 2-links all of whose components are trivial knots and some of whose band-sums are nonribbon knots, J. Knot Theory Ramifications 10, 913-922.
- [10] Cochran, T. (1983). Ribbon knots in S^4 , J London Math Soc (2) 28, 563-576.

- [11] Hosokawa, F. and Kawauchi, A. (1979). Proposals for unknotted surfaces in four-space, Osaka J. Math, 16: 233-248.
- [12] Gluck, H. (1962). The embedding of two-spheres in the four-sphere, Trans Amer Math Soc, 104: 308-333.
- [13] Boyle, J. (1993). The turned torus knot in S^4 , J Knot Theory Ramifications, 2: 239-249.
- [14] Kawauchi, A. (2026). Fusion of boundary surface-link and ribbonness. <https://sites.google.com/view/kawauchiwriting>
- [15] Kawauchi, A. (2025). Ribbonness on boundary surface-link. Open Access J. Phys. Math. 1(2), 1-4.