Haar-type one-hidden-layer neural-network approximation scheme in interval Banach function spaces

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Abstract We study a Haar-type approximation scheme for one-hidden-layer neural networks through its associated averaging operators. In the classical L^p setting, these operators correspond to piecewise constant approximations obtained by local averaging. Motivated by the study of neural-network approximation independently of a specific error norm, we extend the analysis to interval Banach function spaces, which form a special class of ball Banach function spaces on the real line. Our main result characterizes the convergence of the averaging operators on function spaces generated by a natural class of step functions. We prove that convergence is equivalent to the vanishing of the norms of characteristic functions of shrinking intervals centered at rational points. Under the assumption of uniform boundedness of the averaging operators, this characterization extends to the closure of the space of step functions. We also investigate the dyadic scheme, for which convergence holds without additional assumptions. These results demonstrate that convergence is governed at rational points by the local absolute continuity of the norm at rational points of the underlying function space rather than by the boundedness of the Hardy–Littlewood maximal operator. We also provide examples

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illustrating the restrictive nature of the assumptions required for this approximation procedure.

Keywords Haar approximation · Neural networks · Interval Banach function spaces · Averaging operators · Function space approximation

1 Introduction

In this paper, we study a simple approximation scheme based on piecewise constant functions generated by Haar-type characteristic functions, which can be interpreted as a one-hidden-layer neural-network model with step-function activation. Equivalently, the scheme can be described in terms of local averaging operators.

To make this more precise, we consider the following approximation scheme. Let $J \in \mathbb{N}$ and define

$$E_J f = J \sum_{j=1}^{J} \chi_{\left[\frac{j-1}{J}, \frac{j}{J}\right)} \int_{\frac{j-1}{J}}^{\frac{j}{J}} f(y)\, dy.$$

Here and throughout the paper, χ_E denotes the indicator function of a measurable set E.

A standard approach to studying the operators E_J is to compare them with the Hardy–Littlewood maximal operator, as in [2, 8]. Recall that the Hardy–Littlewood maximal operator is defined by

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)|\, dy,$$

where the supremum is taken over all intervals I such that $x \in I \subset [0, 1]$. Since

$$|E_J f(x)| \le Mf(x) \quad \text{for all } x,$$

one immediately obtains norm estimates for E_J whenever the maximal operator is bounded on X. However, as the example of $L^1([0, 1]$ shows, the boundedness of the Hardy–Littlewood maximal operator is a substantially stronger assumption than is actually needed for the uniform boundedness of E_J, $J \in \mathbb{N}$.

Let us now introduce interval Banach function spaces on $[0, 1]$ following [7]. This notion arises as the one-dimensional case of the ball Banach function spaces introduced in [7].

Let $\mathbb{M}([0, 1])$ denote the set of all (Lebesgue) measurable real-valued functions on $[0, 1]$, and let $\mathbb{M}^+([0, 1])$ be the subset of all non-negative measurable functions.

Let

$$\rho : \mathbb{M}^+([0, 1]) \to [0, \infty]$$

be a mapping. We say that ρ is an *interval Banach function norm* if, for all $f, g, \{f_j\}_{j=1}^{\infty} \subset \mathbb{M}^+([0, 1])$, all constants $a \ge 0$, and all intervals (balls) $I \subset [0, 1]$, the following properties hold:

(P1) **Norm properties.**

$$\rho(f) = 0 \iff f = 0 \text{ a.e.}, \qquad \rho(af) = a\,\rho(f), \qquad \rho(f+g) \le \rho(f) + \rho(g).$$

(P2) **Lattice property.** If $0 \le g \le f$ a.e., then $\rho(g) \le \rho(f)$.
(P3) **Fatou property.** If $0 \le f_j \uparrow f$ a.e., then $\rho(f_j) \uparrow \rho(f)$.
(P4) **Finite norm of characteristic functions of intervals.** For every interval $I \subset [0,1]$, we have $\rho(\chi_I) < \infty$.
(P5) **Local integrability.** For every interval $I \subset [0,1]$, there exists a constant $C > 0$, depending on I and ρ but is independent of f, such that

$$\|\chi_I f\|_{L^1([0,1])} \le C\rho(f).$$

Note that (P5) is a natural condition for us since the integral defining $E_J f$ makes sense due to this condition. Applying (P5) to $[0,1]$, we learn that the embedding $X \hookrightarrow L^1([0,1])$ is continuous, that is,

$$\|f\|_{L^1([0,1])} \le C\|f\|_X \tag{1}$$

for all $f \in X$. The collection

$$X([0,1]) := \{f \in \mathbb{M}([0,1]) : \rho(|f|) < \infty\},$$

equipped with the norm

$$\|f\|_X := \rho(|f|),$$

is called the *interval Banach function space over* $[0,1]$ generated by ρ.

We now turn to the main result of the paper, which characterizes the convergence of the averaging operators E_J in this setting. Unlike the classical L^p theory, where convergence follows from the boundedness of the Hardy–Littlewood maximal operator, such a reduction is no longer available in the present abstract framework.

Instead, convergence is governed by a local vanishing property of characteristic functions at rational points of the underlying space, expressed in terms of the behavior of characteristic functions of small intervals. Theorem 1 shows that convergence of $E_J f$ is essentially equivalent to a vanishing condition on the X-norm of shrinking neighborhoods of points, and it further clarifies how this condition interacts with density and uniform boundedness of the approximation scheme.

Theorem 1 *Let X be an interval Banach function space over $[0,1]$, and define the averaging operators*

$$E_J f = J \sum_{j=1}^{J} \chi_{\left[\frac{j-1}{J}, \frac{j}{J}\right)} \int_{\frac{j-1}{J}}^{\frac{j}{J}} f(y)\,dy, \qquad J \in \mathbb{N}.$$

Define

$$\mathcal{U} := \mathrm{Span}\left\{\chi_{[0,q)} : q \in \mathbb{Q} \cap (0,1)\right\}.$$

(1) *We have*

$$\lim_{J\to\infty} \|E_J f - f\|_X = 0 \tag{2}$$

for every $f \in \mathcal{U}$ if and only if

$$\lim_{r\downarrow 0} \|\chi_{(q-r,q+r)}\|_X = 0 \tag{3}$$

for every $q \in \mathbb{Q} \cap (0,1)$.

(2) *Assume that the family $\{E_J\}_{J\in\mathbb{N}}$ is uniformly bounded on X. Then* (2) *holds for every $f \in \overline{\mathcal{U}}^X$, the closure of $\mathcal{U}$ in X, if and only if* (3) *holds for every $q \in \mathbb{Q} \cap (0,1)$.*

In the case of $X = L^p([0,1])$ with $1 \le p < \infty$, the approximation given by (2) for all $f \in X$ is proved by a straightforward calculation in [11].

The proof of Theorem 1 relies on the discontinuous nature of the kernels associated with the operators E_J, in contrast to the Kantorovich-type operators studied in [2, 8], whose kernel enjoys more continuity than that of E_J. The dyadic approximation scheme enjoys a stronger property because the interval $[0,1]$ is divided evenly at each scale. Let

$$\mathcal{D} := \left\{ \frac{m}{2^k} : k \in \mathbb{N},\ 1 \le m \le 2^k - 1 \right\} \subset \mathbb{Q} \cap (0,1)$$

be the set of dyadic rational points in $(0,1)$, and define

$$\mathcal{V} := \operatorname{Span}\left\{ \chi_{[0,q)} : q \in \mathcal{D} \right\}.$$

Then

$$\lim_{J\to\infty} \|E_{2^J} f - f\|_X = 0 \tag{4}$$

for every $f \in \mathcal{V}$. Indeed, for every $f \in \mathcal{V}$, there exists $J_0 \in \mathbb{N}$ such that

$$E_{2^J} f = f$$

for all $J \ge J_0$. This example of dyadic approximation illustrates that the Haar wavelet provides greater flexibility than the standard equal partition, as demonstrated in the recent papers [9, 10].

We emphasize that every function in $\mathcal{U}$ has only finitely many discontinuities, all of which are rational points in $(0,1)$. Consequently, the convergence criterion in Theorem 1 needs to be verified only at rational points; irrational points play no role in the approximation. Naturally, if an interval Banach function space $X([0,1])$ has an absolutely continuous norm, then it satisfies condition (3). Recall that $X([0,1])$ is said to have an *absolutely continuous norm* if, for every $f \in X([0,1])$,

$$\|\chi_{E_n} f\|_X \to 0 \tag{5}$$

whenever that $E_{n+1} \subset E_n \subset [0,1]$ for all n and

$$\left| \bigcap_{n=1}^{\infty} E_n \right| = 0.$$

This property is satisfied by many classical Banach function spaces, including Lorentz spaces, Orlicz spaces, Musielak–Orlicz spaces, and numerous Morrey-type spaces studied in [6]. Consequently, all of these spaces satisfy condition (3).

Theorem 1 shows, however, that the full strength of absolute continuity is not required for the convergence of the averaging operators. It suffices that condition (3) holds at rational points. On the other hand, $L^\infty([0,1])$ provides a typical example of a space on which the Hardy–Littlewood maximal operator M is bounded, while condition (3) fails at every point of $(0,1)$.

From a mathematical viewpoint, Theorem 1 provides a useful criterion for the convergence of E_J to the identity operator. From the viewpoint of applications, this approximation scheme has recently attracted increasing attention as a Haar-type one-hidden-layer neural-network model [11].

We may ask ourselves whether the convergence takes place for all $f \in X$ instead of $f \in \overline{\mathcal{U}}^X$. If we assume the converngece for all $f \in X$, we will have a topological restriction on X.

Theorem 2 *Let X be an interval Banach function space over $[0,1]$. The following assertions are equivalent.*

(1) *For every $f \in X$,*

$$\lim_{J\to\infty} \|E_J f - f\|_X = 0. \tag{6}$$

(2) *The norm of X is absolutely continuous and*

$$\sup_{J\in\mathbb{N}} \|E_J\|_{X\to X} < \infty. \tag{7}$$

In neural network approximation, the choice of the norm used to measure the approximation error is not merely a technical issue. It determines which features of the target function are emphasized and what types of discrepancies are regarded as significant. Although L^p-norms provide a classical and widely used scale of error criteria, many applications in data analysis and machine learning call for more flexible geometries, such as weighted function spaces, non-Hilbertian Banach spaces, or spaces with spatially inhomogeneous norms. From this perspective, it is natural to ask whether constructive approximation schemes originally developed in L^p-spaces remain valid in more general function spaces.

The main objective of this paper is to address this question. We show in Theorem 1 that the answer depends on the approximation scheme under consideration: different constructive schemes require different structural assumptions on the underlying function space.

Interval Banach function spaces form a broad class of function spaces. Their generality precludes the derivation of explicit approximation error orders in full generality. Nevertheless, they provide a natural abstract setting for the study of neural-network approximation. Indeed, they preserve the lattice structure of function spaces and the monotonicity of the norm with respect to absolute values, while encompassing numerous classical and non-classical spaces beyond the fixed L^p scale. Therefore, extending approximation results from L^p-spaces to ball Banach function spaces represents a step toward a norm-independent theory of neural-network approximation.

The present paper adopts this perspective. We revisit a simple one-hidden-layer neural-network approximation scheme based on Haar-type characteristic functions. In the classical L^p setting, this scheme reduces to a piecewise constant approximation obtained by local averaging. We investigate the same averaging mechanism in an interval Banach function space X and identify conditions under which the associated approximation operators converge in the norm of X. Our objective is not to propose a new training algorithm but rather to elucidate the functional-analytic structure underlying this basic constructive approximation scheme. In particular, we show that its convergence extends from the classical L^p framework to the more general setting of ball Banach function spaces in the sense that $E_J f$ converges back to $f \in \overline{\mathcal{U}}^X$ in X.

We organize the remainder of this paper as follows. Section 2 proves Theorem 1. Section 3 considers the convergence of $E_J f$ for all $f \in X$. Section 4 presents examples of spaces X satisfying the conditions in Theorem 1 illustrating both situations: (i) M is bounded on X but condition (3) fails at some $q \in (0,1) \setminus \mathbb{Q}$ (Example 1), and (ii) M is not bounded on X and condition (3) still fails at some $q \in (0,1) \setminus \mathbb{Q}$ (Example 2).

2 Proof of Theorem 1

2.1 Proof of Theorem 1(1)

We first prove the "if" part.

Let $q \in \mathbb{Q} \cap (0,1)$, and write

$$q = \frac{s}{d}$$

in lowest terms, where $s, d \in \mathbb{N}$ and $1 \le s < d$. Since the characteristic functions $\chi_{[0,q)}$ generate $\mathcal{U}$ from the definition of $\mathcal{U}$, it suffices to consider the case $f = \chi_{[0,q)}$.

For each $J \in \mathbb{N}$, choose $q(J) \in \mathbb{N}$ such that

$$\frac{q(J)-1}{J} \le q \le \frac{q(J)}{J}.$$

Then, by the definition of E_J,

$$|f - E_J f| \le \chi_{\left[\frac{q(J)-1}{J},\, \frac{q(J)}{J}\right]}.$$

Write

$$\delta_J = \max\left\{\frac{q(J)}{J} - q,\, q - \frac{q(J)-1}{J}\right\}.$$

Since

$$\left[\frac{q(J)-1}{J},\, \frac{q(J)}{J}\right] \subset [q-\delta_J,\, q+\delta_J] \subset (q-2\delta_J,\, q+2\delta_J),$$

it follows that

$$|f - E_J f| \le \chi_{[q-\delta_J,\, q+\delta_J]} \le \chi_{(q-2\delta_J,\, q+2\delta_J)}.$$

Since $(q-2\delta_J,\ q+2\delta_J)$ is an open interval centered at q for each $J \in \mathbb{N}$ and $2\delta_J \to 0$ as $J \to \infty$, condition (3) implies

$$\lim_{J\to\infty} \|E_J f - f\|_X = 0.$$

We next prove the "only if" part.

Assume that

$$\|E_J \chi_{[0,q)} - \chi_{[0,q)}\|_X \to 0 \qquad (J \to \infty)$$

for every $q \in \mathbb{Q} \cap (0,1)$.

Fix $q \in \mathbb{Q} \cap (0,1)$, and write

$$q = \frac{s}{d}$$

in lowest terms.

For each $n \in \mathbb{N}$, let

$$J_n = nd + 1.$$

Then

$$J_n q = (nd+1)\frac{s}{d} = ns + \frac{s}{d},$$

so the fractional part of $J_n q$ is independent of n and is equal to q. Let

$$m_n = \lfloor J_n q \rfloor = ns.$$

Then

$$m_n < J_n q < m_n + 1,$$

and hence the interval of the J_n-partition containing q is

$$I_n := \left[\frac{m_n}{J_n}, \frac{m_n+1}{J_n}\right).$$

A direct computation shows that

$$q - \frac{m_n}{J_n} = \frac{q}{J_n}, \qquad \frac{m_n+1}{J_n} - q = \frac{1-q}{J_n}.$$

Write $c_0 := \min\{q, 1-q\}$. Then

$$\left(q - \frac{c_0}{J_n},\ q + \frac{c_0}{J_n}\right) \subset I_n.$$

Moreover, for every $t \in I_n$,

$$E_{J_n}\chi_{[0,q)}(t) = J_n \int_{m_n/J_n}^{(m_n+1)/J_n} \chi_{[0,q)}(x)\,dx = J_n\left(q - \frac{m_n}{J_n}\right) = q.$$

Hence,

$$|E_{J_n}\chi_{[0,q)} - \chi_{[0,q)}| = 1-q \quad \text{on } I_n \cap [0,q), \tag{8}$$

and

$$|E_{J_n}\chi_{[0,q)} - \chi_{[0,q)}| = q \quad \text{on } I_n \cap (q,1).$$

Therefore,

$$|E_{J_n}\chi_{[0,q)} - \chi_{[0,q)}| \geq c_0\chi_{(q-c_0/J_n,\, q+c_0/J_n)}.$$

Since X is an interval Banach function space, the lattice property implies

$$\left\|\chi_{(q-c_0/J_n,\, q+c_0/J_n)}\right\|_X \leq \frac{1}{c_0}\left\|E_{J_n}\chi_{[0,q)} - \chi_{[0,q)}\right\|_X.$$

Hence,

$$\lim_{n\to\infty}\left\|\chi_{(q-c_0/J_n,\, q+c_0/J_n)}\right\|_X = 0.$$

Finally, let

$$\varepsilon_n = \frac{c_0}{J_n} = \frac{c_0}{nd+1}.$$

Then

$$\lim_{n\to\infty}\left\|\chi_{(q-\varepsilon_n,\, q+\varepsilon_n)}\right\|_X = 0.$$

Since $\{\varepsilon_n\}_{n=1}^{\infty}$ is a monotone sequence that converges to 0, for $r \in (0,\varepsilon_1)$, there exists $n \in \mathbb{N}$ such that

$$\varepsilon_{n+1} < r \leq \varepsilon_n.$$

This implies $(q-r, q+r) \subset (q-\varepsilon_n, q+\varepsilon_n)$. Hence, by the lattice property,

$$\|\chi_{(q-r,q+r)}\|_X \leq \|\chi_{(q-\varepsilon_n,q+\varepsilon_n)}\|_X.$$

Letting $r \downarrow 0$, we conclude that

$$\lim_{r\downarrow 0}\|\chi_{(q-r,q+r)}\|_X = 0.$$

This completes the proof.

2.2 Proof of Theorem 1(2)

Assume that (2) and (3) hold as in Theorem 1(1) and that $\sup_{J\in\mathbb{N}}\|E_J\|_{X\to X} \leq C$.

Let $f \in \overline{\mathcal{U}}^X$. Then there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{U}$ such that

$$\lim_{n\to\infty}\|f_n - f\|_X = 0.$$

Fix $\varepsilon > 0$. Choose n sufficiently large so that

$$\|f - f_n\|_X < \frac{\varepsilon}{3(1+C)}.$$

Since $f_n \in \mathcal{U}$, by (1) there exists $J_0 \in \mathbb{N}$ such that for all $J \geq J_0$,

$$\|E_J f_n - f_n\|_X < \frac{\varepsilon}{3}.$$

Now we estimate:

$$\|E_J f - f\|_X \leq \|E_J(f - f_n)\|_X + \|E_J f_n - f_n\|_X + \|f_n - f\|_X.$$

For any $f \in X$, by assumption $\{E_J f\}_{J=1}^{\infty}$ converges to f as $J \to \infty$. Using the uniform boundedness of $\{E_J\}$, we obtain

$$\|E_J(f - f_n)\|_X \leq C\|f - f_n\|_X.$$

Hence, for all $J \geq J_0$,

$$\|E_J f - f\|_X \leq C\|f - f_n\|_X + \frac{\varepsilon}{3} + \|f_n - f\|_X = (1 + C)\|f - f_n\|_X + \frac{\varepsilon}{3}.$$

By the choice of n, we have

$$(1 + C)\|f - f_n\|_X < \frac{2\varepsilon}{3}.$$

Therefore,

$$\|E_J f - f\|_X < \varepsilon \quad \text{for all } J \geq J_0,$$

which implies

$$\lim_{J \to \infty} \|E_J f - f\|_X = 0.$$

Conversely, suppose that

$$\lim_{J \to \infty} \|E_J f - f\|_X = 0 \quad \text{for every } f \in \overline{\mathcal{U}}^X.$$

Since $\mathcal{U} \subset \overline{\mathcal{U}}^X$, the convergence also holds for every $f \in \mathcal{U}$. Therefore, Theorem 1(1) implies

$$\lim_{r \downarrow 0} \|\chi_{(q-r,q+r)}\|_X = 0$$

for every $q \in \mathbb{Q} \cap (0, 1)$. This completes the proof.

3 Convergence on the whole space

Here, we give a complete criterion for convergence of the averaging operators on the whole space X. We consider the following space of step functions:

$$\mathcal{W} := \operatorname{Span}\left\{\chi_{[0,q)} : q \in \mathbb{Q} \cap (0, 1]\right\}.$$

By (P4), every function in the spanning family defining $\mathcal{W}$ belongs to X, and hence $\mathcal{W} \subset X$. The endpoint $q = 1$ is included so that $\mathcal{W}$ contains $\chi_{[0,1)}$, which is equal almost everywhere on $[0, 1]$ to the constant function 1.

We obtain a crude estimate using the triangle inequality.

Lemma 1 *For every $J \in \mathbb{N}$, the operator E_J is a bounded linear operator on X, and*

$$E_J(X) \subset \mathcal{W}.$$

Proof. Property (P5) applied to $[0, 1]$ gives a constant $C_X > 0$ such that

$$\|f\|_{L^1([0,1])} \le C_X \|f\|_X \qquad (f \in X).$$

For

$$I_{J,j} := \left[\frac{j-1}{J}, \frac{j}{J}\right), \qquad 1 \le j \le J,$$

we have

$$|E_J f| \le J \sum_{j=1}^{J} \left| \int_{I_{J,j}} f(y)\,dy \right| \chi_{I_{J,j}}.$$

Therefore, by (P1), (P2), (P4), and (P5),

$$\|E_J f\|_X \le J \sum_{j=1}^{J} \left| \int_{I_{J,j}} f(y)\,dy \right| \|\chi_{I_{J,j}}\|_X \le J C_X \left(\sum_{j=1}^{J} \|\chi_{I_{J,j}}\|_X \right) \|f\|_X. \tag{9}$$

Thus E_J is bounded for every fixed J.

It remains to prove the inclusion $E_J(X) \subset \mathcal{W}$. For $1 \le j \le J$ and $f \in X$, set

$$a_{J,j} := J \int_{I_{J,j}} f(y)\,dy.$$

Then

$$E_J f = a_{J,J}\chi_{[0,1)} + \sum_{j=1}^{J-1} (a_{J,j} - a_{J,j+1})\chi_{[0,j/J)} \tag{10}$$

almost everywhere on $[0, 1]$. Indeed, if $x \in I_{J,k}$, then the right-hand side of (10) is

$$a_{J,J} + \sum_{j=k}^{J-1} (a_{J,j} - a_{J,j+1}) = a_{J,k}.$$

For $1 \le j \le J - 1$, we have

$$\frac{j}{J} \in \mathbb{Q} \cap (0, 1),$$

and hence $\chi_{[0,j/J)} \in \mathcal{W}$. Moreover, $\chi_{[0,1)} \in \mathcal{W}$ by the definition of $\mathcal{W}$. Therefore, (10) implies that $E_J f \in \mathcal{W}$. □

Note that estimate (9) depends on J.

The following lemma is an immediate consequence of absolute continuity:

Lemma 2 *Suppose that the norm of X is absolutely continuous. Then we have*

$$\lim_{\delta\downarrow 0} \sup_{A\subset[0,1],|A|\le\delta} \|\chi_A\|_X = 0, \tag{11}$$

where the supremum is taken over all Lebesgue measurable sets $A \subset [0,1]$ satisfying $|A| \le \delta$.

Proof. For $\delta > 0$, set

$$\Phi(\delta) := \sup_{A\subset[0,1],|A|\le\delta} \|\chi_A\|_X.$$

By the lattice property,

$$0 \le \Phi(\delta) \le \|\chi_{[0,1]}\|_X < \infty.$$

Moreover, Φ is nondecreasing as a function of δ. Consequently, the limit

$$L := \lim_{\delta\downarrow 0} \Phi(\delta) = \inf_{\delta>0} \Phi(\delta)$$

exists.

Suppose that (11) does not hold. Then $L > 0$. Put $\varepsilon_0 := L/2$. For every $n \in \mathbb{N}$, we have

$$\Phi(2^{-n}) \ge L > \varepsilon_0.$$

Hence, by the definition of the supremum, there exists a measurable set $A_n \subset [0,1]$ such that

$$|A_n| \le 2^{-n} \qquad \text{and} \qquad \|\chi_{A_n}\|_X > \varepsilon_0.$$

Define

$$B_n := \bigcup_{k=n}^{\infty} A_k.$$

Then $B_{n+1} \subset B_n$ and

$$|B_n| \le \sum_{k=n}^{\infty} 2^{-k} = 2^{-n+1}.$$

It thus follows that $\bigcap_{n=1}^{\infty} B_n$ differs from the empty set by a set of measure zero. Since $\chi_{[0,1]} \in X$ by (P4), absolute continuity gives

$$\lim_{n\to\infty} \|\chi_{B_n}\|_X = 0.$$

This contradicts $A_n \subset B_n$ and (P2). Therefore, (11) holds.

□

Proposition 1 *Let X be an interval Banach function space over $[0,1]$. The following conditions are equivalent.*

(1) *The norm of X is absolutely continuous.*
(2) *The space $\mathcal{W}$ is dense in X.*

(3) *The space X is separable.*

We know that (1) and (3) are equivalent; see [1, Chapter 1 Theorem 5.5].

Proof. We first prove $(1) \Rightarrow (2)$. We approximate measurable sets by finite unions of intervals with rational endpoints. Let $A \subset [0,1]$ be measurable. By (P2) and (P4), we have $\chi_A \in X$, since

$$0 \le \chi_A \le \chi_{[0,1]}$$

almost everywhere. For every $n \in \mathbb{N}$, the inner and outer regularity of Lebesgue measure give a compact set $K_n \subset A$ and an open set $O_n \subset \mathbb{R}$ such that

$$A \subset O_n \qquad \text{and} \qquad |O_n \setminus K_n| < \frac{1}{n}.$$

For each $x \in K_n$, choose an open interval I_x with rational endpoints such that

$$x \in I_x \qquad \text{and} \qquad \overline{I_x} \cap [0,1] \subset O_n.$$

A finite number of these intervals cover the compact set K_n. Let V_n be their union. Then

$$K_n \subset V_n \cap [0,1] \subset O_n.$$

Since V_n is a finite union of open intervals with rational endpoints, the set $V_n \cap [0,1]$ differs only by finitely many boundary points from a finite union R_n of pairwise disjoint half-open intervals $[a,b)$ with

$$a, b \in \mathbb{Q} \cap [0,1], \qquad a < b.$$

As a result,

$$|(V_n \cap [0,1]) \triangle R_n| = 0.$$

Moreover, since K_n is contained in both A and $V_n \cap [0,1]$, while both of these sets are contained in O_n, we have

$$A \triangle (V_n \cap [0,1]) \subset O_n \setminus K_n.$$

Therefore,

$$|A \triangle R_n| \le |A \triangle (V_n \cap [0,1])| + |(V_n \cap [0,1]) \triangle R_n| \le |O_n \setminus K_n| < \frac{1}{n}. \tag{12}$$

For rational numbers $0 \le a < b \le 1$, we have

$$\chi_{[a,b)} = \chi_{[0,b)} - \chi_{[0,a)} \in \mathcal{W}.$$

Hence $\chi_{R_n} \in \mathcal{W}$. Since

$$|\chi_A - \chi_{R_n}| = \chi_{A \triangle R_n}$$

on $[0,1]$ and $|A \triangle R_n| < 1/n$, we deduce from (11) and (12) that

$$\|\chi_A - \chi_{R_n}\|_X = \|\chi_{A\triangle R_n}\|_X \le \sup_{\substack{B\subset[0,1] \\ |B|\le 1/n}} \|\chi_B\|_X \to 0 \tag{13}$$

as $n \to \infty$. Thus the characteristic function of every measurable set belongs to the closure of $\mathcal{W}$. Consequently, every measurable simple function belongs to the closure of $\mathcal{W}$.

Let $f \in X$. For $N \in \mathbb{N}$, define

$$f_N := \max\{-N, \min\{f, N\}\}.$$

The sets

$$F_N := \{x \in [0,1] : |f(x)| > N\}$$

decrease to a null set because f is finite almost everywhere. Since

$$|f - f_N| \le |f|\chi_{F_N},$$

absolute continuity and (P2) give

$$\lim_{N\to\infty} \|f - f_N\|_X = 0. \tag{14}$$

For each fixed N, there are measurable simple functions s_N such that

$$\|f_N - s_N\|_{L^\infty([0,1])} \le \frac{1}{N}.$$

Since

$$\|h\|_X \le \|h\|_{L^\infty([0,1])}\|\chi_{[0,1]}\|_X$$

for every bounded measurable h, we have $s_N - f_N \to 0$ in X as $N \to \infty$. To complete the proof of density, let $\varepsilon > 0$. First choose $N > \dfrac{3}{\varepsilon}$ so that $\|f - f_N\|_X < \varepsilon/3$, so that $\|f_N - s_N\|_X < \varepsilon/3$. Since s_N belongs to the closure of $\mathcal{W}$, choose $g \in \mathcal{W}$ such that $\|s_N - g\|_X < \varepsilon/3$. Then $\|f - g\|_X < \varepsilon$. Hence $\mathcal{W}$ is dense in X.

We next prove (2) $\Rightarrow$ (3). Consider all finite linear combinations of the functions $\chi_{[0,q)}$ ($q \in \mathbb{Q} \cap (0,1]$) with rational coefficients. This set is countable. We show that it is dense in $\mathcal{W}$. Let

$$g = \sum_{k=1}^m c_k \chi_{[0,q_k)} \in \mathcal{W},$$

where $c_k \in \mathbb{R}$ and $q_k \in \mathbb{Q} \cap (0,1]$. For each k, choose rational numbers $c_{k,n}$ such that $c_{k,n} \to c_k$ as $n \to \infty$. Then

$$\left\| g - \sum_{k=1}^m c_{k,n}\chi_{[0,q_k)} \right\|_X \le \sum_{k=1}^m |c_k - c_{k,n}| \|\chi_{[0,q_k)}\|_X \to 0$$

as $n \to \infty$. Therefore, the finite linear combinations with rational coefficients form a countable dense subset of $\mathcal{W}$. Thus $\mathcal{W}$ is separable. If $\mathcal{W}$ is dense in X, then X is separable. $\square$

We now compare assertion (2) with the strong convergence of E_J back to the identity operator. We prove Theorem 2.

Suppose first that (1) holds. Lemma 1 shows that every E_J is a bounded linear operator on X. For every fixed $f \in X$, condition (6) implies that $\{E_J f : J \in \mathbb{N}\}$ is bounded in X. The uniform boundedness principle gives (7). Lemma 1 also gives $E_J f \in \mathcal{W}$ for every $f \in X$ and every J. Since $E_J f \to f$ in X, every $f \in X$ belongs to the closure of $\mathcal{W}$. Thus $\mathcal{W}$ is dense in X. Proposition 1 shows that the norm of X is absolutely continuous. Hence (2) holds.

Conversely, assume (2). By Proposition 1, the space $\mathcal{W}$ is dense in X. We first consider the functions used in the definition of $\mathcal{W}$. We prove that

$$\lim_{J\to\infty} \|E_J \chi_{[0,q)} - \chi_{[0,q)}\|_X = 0$$

for every $q \in \mathbb{Q} \cap (0,1]$. Let $q \in \mathbb{Q} \cap (0,1)$. If $Jq \in \mathbb{Z}$, then

$$E_J \chi_{[0,q)} = \chi_{[0,q)} \quad \text{almost everywhere.}$$

Suppose that $Jq \notin \mathbb{Z}$ and put $m_J := \lfloor Jq \rfloor$. The functions $E_J \chi_{[0,q)}$ and $\chi_{[0,q)}$ agree outside

$$I_J := \left[\frac{m_J}{J}, \frac{m_J+1}{J}\right).$$

On I_J, both functions take values in $[0,1]$. Since

$$I_J \subset \left(q - \frac{1}{J}, q + \frac{1}{J}\right),$$

we obtain

$$\left|E_J \chi_{[0,q)} - \chi_{[0,q)}\right| \le \chi_{(q-1/J,q+1/J)\cap[0,1]}. \tag{15}$$

We have

$$\left(q - \frac{1}{J}, q + \frac{1}{J}\right) \cap [0,1] \downarrow \{q\}$$

as $J \to \infty$. Since $\{q\}$ has measure zero, absolute continuity applied to $\chi_{[0,1]}$ gives

$$\lim_{J\to\infty} \left\|\chi_{(q-1/J,q+1/J)\cap[0,1]}\right\|_X = 0.$$

It follows from (15) that

$$\lim_{J\to\infty} \|E_J \chi_{[0,q)} - \chi_{[0,q)}\|_X = 0. \tag{16}$$

for all $q \in [0,1)$. Note that (16) remains valid for $q = 1$, since we deduce from (8) $E_J \chi_{[0,1)} = \chi_{[0,1)}$ almost everywhere for every $J \in \mathbb{N}$. Linearity allows us to extend (16) for any function $g \in \mathcal{W}$:

$$\lim_{J\to\infty} \|E_J g - g\|_X = 0 \qquad (g \in \mathcal{W}). \tag{17}$$

Set

$$C := \sup_{J\in\mathbb{N}} \|E_J\|_{X\to X}.$$

Let $f \in X$ and $\varepsilon > 0$. Since $\mathcal{W}$ is dense in X, choose $g \in \mathcal{W}$ such that

$$\|f - g\|_X < \frac{\varepsilon}{2(C+1)}.$$

By (17), there is $J_0 \in \mathbb{N}$ such that

$$\|E_J g - g\|_X < \frac{\varepsilon}{2} \qquad (J \geq J_0).$$

For $J \geq J_0$,

$$\begin{aligned}\|E_J f - f\|_X &\leq \|E_J(f-g)\|_X + \|E_J g - g\|_X + \|g - f\|_X \\ &\leq (C+1)\|f - g\|_X + \|E_J g - g\|_X < \varepsilon.\end{aligned}$$

This proves (6).

Theorem 2 is therefore proved.

Proposition 2 below shows that, for any prescribed sequence of positive numbers $\{a_J\}_{J\in\mathbb{N}}$ satisfying $a_J \to 0$, there exists a function $f \in X$ such that no constant $C > 0$ satisfies

$$\|E_J f - f\|_X \leq C a_J$$

for all sufficiently large J. The proposition itself does not assume that

$$\lim_{J\to\infty} \|E_J f - f\|_X = 0$$

for every $f \in X$. If the equivalent conditions of Theorem 2 hold, then E_J converges to the identity operator in the strong operator topology, and the proposition shows that this convergence can be arbitrarily slow.

Proposition 2 *Let X be an interval Banach function space over $[0,1]$. For every sequence*

$$\{a_J\}_{J\in\mathbb{N}} \subset (0,\infty), \qquad \lim_{J\to\infty} a_J = 0,$$

there exists a function $f \in X$ with

$$\|f\|_X = 1$$

such that

$$\limsup_{J\to\infty} \frac{\|E_J f - f\|_X}{a_J} = \infty.$$

Proof. Let I_X denote the identity operator on X. For each $J \in \mathbb{N}$, define

$$h_J := \chi_{[0,1/(2J))} - \chi_{[1/(2J),1/J)}.$$

By property (P4), we have $h_J \in X$. Moreover, h_J is not the zero function, and hence

$$\|h_J\|_X > 0.$$

The function h_J is supported in the first interval

$$I_{J,1} = \left[0, \frac{1}{J}\right)$$

of the J-partition. Its integral over this interval is zero:

$$\int_{I_{J,1}} h_J(y)\, dy = \frac{1}{2J} - \frac{1}{2J} = 0.$$

It also vanishes on every other interval of the J-partition. Therefore, $E_J h_J = 0$ and hence $(I_X - E_J)h_J = h_J$.

Set

$$u_J := \frac{h_J}{\|h_J\|_X}.$$

Then

$$\|u_J\|_X = 1$$

and

$$(I_X - E_J)u_J = u_J.$$

It follows that

$$\|I_X - E_J\|_{X\to X} \geq \|(I_X - E_J)u_J\|_X = 1 \qquad (J \in \mathbb{N}).$$

Now let $\{a_J\}_{J\in\mathbb{N}}$ be a sequence of positive numbers such that $a_J \to 0$, and define

$$T_J := \frac{I_X - E_J}{a_J}.$$

By Lemma 1, every E_J is a bounded linear operator on X. Hence every T_J is also a bounded linear operator on X. Moreover,

$$\|T_J\|_{X\to X} = \frac{\|I_X - E_J\|_{X\to X}}{a_J} \geq \frac{1}{a_J}.$$

Since $a_J \to 0$, we obtain

$$\sup_{J\in\mathbb{N}} \|T_J\|_{X\to X} = \infty.$$

Since X is a Banach space, the contrapositive of the Banach–Steinhaus theorem yields a function $g \in X$ such that

$$\sup_{J\in\mathbb{N}} \|T_J g\|_X = \infty.$$

In particular, $g \neq 0$. Replacing g by

$$f := \frac{g}{\|g\|_X},$$

we may assume that $\|f\|_X = 1$. For each fixed J, the quantity $\|T_J f\|_X$ is finite. Therefore, the unboundedness of the sequence $\{\|T_J f\|_X\}_{J\in\mathbb{N}}$ cannot be caused by finitely many terms. Hence, for every $N \in \mathbb{N}$,

$$\sup_{J\geq N} \|T_J f\|_X = \infty,$$

which is equivalent to

$$\limsup_{J\to\infty} \frac{\|E_J f - f\|_X}{a_J} = \infty.$$

□

Corollary 1 *Assume that one (and hence both) of the equivalent assertions in Theorem 2 holds. Then there exists a function $f \in X$ with*

$$\|f\|_X = 1$$

such that

$$\lim_{J\to\infty} \|E_J f - f\|_X = 0,$$

while, for every $\alpha > 0$,

$$\limsup_{J\to\infty} J^{\alpha}\|E_J f - f\|_X = \infty.$$

Equivalently, for this same function f, there is no $\alpha > 0$ such that

$$\|E_J f - f\|_X = O(J^{-\alpha}) \qquad (J \to \infty).$$

Proof. By Theorem 2, we have

$$\lim_{J\to\infty} \|E_J g - g\|_X = 0$$

for every $g \in X$.

Apply Proposition 2 to the positive sequence

$$a_J := \frac{1}{\log(J+1)}.$$

Since $a_J \to 0$, there exists $f \in X$ with $\|f\|_X = 1$ such that

$$\limsup_{J\to\infty} \log(J+1)\|E_J f - f\|_X = \infty.$$

In particular, the convergence

$$\lim_{J\to\infty} \|E_J f - f\|_X = 0$$

follows from Theorem 2.

Fix $\alpha > 0$. There exists a strictly increasing sequence $\{J_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \log(J_n+1)\|E_{J_n}f - f\|_X = \infty.$$

On the other hand,

$$\lim_{n\to\infty} \frac{J_n^\alpha}{\log(J_n+1)} = \infty.$$

As a consequence,

$$J_n^\alpha \|E_{J_n}f - f\|_X = \frac{J_n^\alpha}{\log(J_n+1)} \left(\log(J_n+1)\|E_{J_n}f - f\|_X\right) \to \infty$$

as $n \to \infty$. Therefore,

$$\limsup_{J\to\infty} J^\alpha \|E_J f - f\|_X = \infty.$$

For each fixed $\alpha > 0$, the estimate

$$\|E_J f - f\|_X = O(J^{-\alpha}) \qquad (J \to \infty) \tag{18}$$

holds if and only if the sequence

$$\{J^\alpha \|E_J f - f\|_X\}_{J\in\mathbb{N}}$$

is bounded. Thus the final assertion of Corollary 1 implies the failure of (18). □

4 Examples

4.1 Variable Lebesgue exponent spaces

Let $p(\cdot) : [0,1] \to [1,\infty)$ be a measurable function, which we assume to be finite-valued almost everywhere. We define

$$p_- := \operatorname*{ess\,inf}_{x\in[0,1]} p(x), \qquad p_+ := \operatorname*{ess\,sup}_{x\in[0,1]} p(x),$$

and assume that

$$1 \le p_- \le p_+ \le \infty.$$

For a measurable function f on $[0,1]$, we define the modular

$$\rho_{p(\cdot)}(f) = \int_0^1 |f(x)|^{p(x)}\, dx.$$

The variable exponent Lebesgue space $L^{p(\cdot)}([0,1])$ is defined by

$$L^{p(\cdot)}([0,1]) := \left\{ f \in \mathbb{M}([0,1]) : \rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) < \infty \text{ for some } \lambda > 0 \right\}.$$

The Luxemburg norm on $L^{p(\cdot)}([0,1])$ is given by

$$\|f\|_{L^{p(\cdot)}([0,1])} = \inf\left\{ \lambda > 0 : \rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1 \right\}.$$

Equipped with this norm, $L^{p(\cdot)}([0,1])$ becomes an interval Banach function space over $[0,1]$. We conclude this paper with two examples $p(\cdot)$ and $q(\cdot)$ of functions related to Theorem 1.

Example 1 We consider the variable exponent

$$p(t) := \frac{1}{\left|t - \frac{\sqrt{3}}{2}\right|} \qquad \left(t \in [0,1] \setminus \left\{\frac{\sqrt{3}}{2}\right\}\right),$$

and define

$$q(t) := \frac{p(t)}{p_-}, \quad (t \in [0,1]).$$

Then the space $L^{p(\cdot)}([0,1])$ satisfies the condition appearing in Theorem 1. However, although $p(\cdot)$ is finitely valued a.e., we claim that its norm is not absolutely continuous. Indeed, let

$$0 < r < 1 - \frac{\sqrt{3}}{2}.$$

Then, by the definition of the Luxemburg norm and the fact that

$$\int_{\frac{\sqrt{3}}{2}-r}^{\frac{\sqrt{3}}{2}+r} \left(\frac{1}{\lambda}\right)^{\frac{1}{\left|t-\frac{\sqrt{3}}{2}\right|}} dt = \infty$$

whenever $0 < \lambda < 1$, we deduce

$$\begin{aligned}\left\|\chi_{\left(\frac{\sqrt{3}}{2}-r,\frac{\sqrt{3}}{2}+r\right)}\right\|_{L^{p(\cdot)}} &= \inf\left\{\lambda > 0 : \int_{\frac{\sqrt{3}}{2}-r}^{\frac{\sqrt{3}}{2}+r} \left(\frac{1}{\lambda}\right)^{\frac{1}{\left|t-\frac{\sqrt{3}}{2}\right|}} dt \le 1\right\} \\ &= \inf\left\{\lambda \ge 1 : \int_{\frac{\sqrt{3}}{2}-r}^{\frac{\sqrt{3}}{2}+r} \left(\frac{1}{\lambda}\right)^{\frac{1}{\left|t-\frac{\sqrt{3}}{2}\right|}} dt \le 1\right\}.\end{aligned}$$

Meanwhile, when $\lambda = 1$,

$$\int_{\frac{\sqrt{3}}{2}-r}^{\frac{\sqrt{3}}{2}+r} 1\, dt = 2r \le 1.$$

This implies

$$1 \in \left\{ \lambda \geq 1 : \int_{\frac{\sqrt{3}}{2}-r}^{\frac{\sqrt{3}}{2}+r} \left(\frac{1}{\lambda}\right)^{\frac{1}{\left|t-\frac{\sqrt{3}}{2}\right|}} dt \leq 1 \right\}$$

and hence

$$\left\| \chi_{\left(\frac{\sqrt{3}}{2}-r, \frac{\sqrt{3}}{2}+r\right)} \right\|_{L^{p(\cdot)}} = 1.$$

Hence,

$$\lim_{r \downarrow 0} \left\| \chi_{\left(\frac{\sqrt{3}}{2}-r, \frac{\sqrt{3}}{2}+r\right)} \right\|_{L^{p(\cdot)}} = 1.$$

Since this violates (5) with $f = 1$, the space $L^{p(\cdot)}([0,1])$ does not have absolutely continuous norm.

On the other hand, if $x \in (0,1) \setminus \{\frac{\sqrt{3}}{2}\}$, then $p(\cdot)$ is bounded in a neighborhood of x, and therefore

$$\lim_{r \downarrow 0} \left\| \chi_{(x-r,x+r)} \right\|_{L^{p(\cdot)}} = 0.$$

This is an example of an interval Banach function space showing that the conclusion of Theorem 1 holds but X does not have an absolutely continuous norm.

Finally, we claim that the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}([0,1])$ since $p_- > 1$ and $1/p(\cdot)$ is log-Hölder continuous. Although the result in [3, Theorem 1.2] is stated for exponents defined on the whole real line, this causes no difficulty in our setting.

Indeed, we can extend $p(\cdot)$ from $[0,1]$ to $\mathbb{R}$ so that p_- remains unchanged, that the extension $\frac{1}{p(\cdot)}$ is Lipschitz continuous outside $(0,1)$, and that

$$\lim_{x \to \infty} p(x) =: p_\infty$$

exists.

Hence, the above results apply, and therefore M is bounded on $L^{p(\cdot)}([0,1])$.

The space $X = L^{p(\cdot)}([0,1])$ is the first example. We now turn to the second example.

Example 2 In Example 1, if we set $q(\cdot) = \frac{p(\cdot)}{p_-}$, then we see that $q_- = 1$ and that M is not bounded on $L^{q(\cdot)}([0,1])$. See [5, Proposition 21.2] and [3, Theorem 6.3]. However, by the same argument as above, we have

$$\liminf_{r \downarrow 0} \left\| \chi_{\left(\frac{\sqrt{3}}{2}-r, \frac{\sqrt{3}}{2}+r\right)} \right\|_{L^{q(\cdot)}} \geq 1,$$

and

$$\lim_{r \downarrow 0} \left\| \chi_{(x-r,x+r)} \right\|_{L^{q(\cdot)}} = 0 \qquad \text{for } x \neq \frac{\sqrt{3}}{2}.$$

Remark 1 When $p_+ < \infty$, the variable exponent Lebesgue space $L^{p(\cdot)}([0,1])$ is separable. See [4, Theorem 1.13]. To the best of the authors' knowledge, much less is known about the case where $p_+ = \infty$.

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