A note on the Bergman metric of Siegel domains

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Abstract. In this article, we give an answer to the problem posed in [5] about the length of the gradient of a potential function of the Bergman metric of a bounded homogeneous domain. Indeed, our result is valid for any bounded domain biholomorphic to a Siegel domain.

§1. Main result.

Let Ω be an open convex cone in $\mathbb{R}^n$ containing no line, and $Q : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^n$ a Hermitian map satisfying $Q(u,u) \in \mathrm{Cl}(\Omega) \setminus \{0\} \quad (u \in \mathbb{C}^m \setminus \{0\})$, where $\mathrm{Cl}(\Omega)$ stands for the closure of Ω. We define a Siegel domain $\mathcal{D}$ by

$$\mathcal{D} := \{ (u,w) \in \mathbb{C}^m \times \mathbb{C}^n \, ; \, w + \bar{w} - Q(u,u) \in \Omega \} . \tag{1}$$

Our Siegel domain $\mathcal{D}$ is a complex domain in $\mathbb{C}^N$ with $N := m+n$, where the coordinates of $z = (u,w) \in \mathcal{D}$ is taken as

$$z_\alpha := \begin{cases} u_\alpha & (1 \le \alpha \le m), \\ w_{\alpha - m} & (m+1 \le \alpha \le N). \end{cases}$$

If $m = 0$, then $\mathcal{D}$ is the tube domain $\Omega + i\mathbb{R}^n \subset \mathbb{C}^n$.

Let $K_{\mathcal{D}} : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ be the Bergman kernel on $\mathcal{D}$, and $\phi : \mathcal{D} \to \mathbb{R}$ a function defined by $\phi(z) := \log K_{\mathcal{D}}(z,z) \quad (z \in \mathcal{D})$. Recall that the Bergman metric $ds^2_{\mathcal{D}}$ of $\mathcal{D}$ is a Kahler metric whose potential is ϕ:

$$ds^2_{\mathcal{D}} := \sum_{\alpha,\beta=1}^{N} \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha d\bar{z}_\beta.$$

Now we state our result.

Theorem 1. *Measured by the metric $ds^2_{\mathcal{D}}$, the length of the gradient of ϕ is equal to a constant $\sqrt{m+2n}$:*

$$\left| \sum_{\alpha=1}^{N} \frac{\partial \phi}{\partial z_\alpha} dz_\alpha \right|_{ds^2_{\mathcal{D}}} = \sqrt{m+2n}.$$

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It is known [6, Chapter 1, Sections 1,2] that any Siegel domain is holomorphically isomorphic to a bounded domain. Let $\sigma : \mathcal{U} \to \mathcal{D}$ be a biholomorphism from a bounded domain $\mathcal{U}$ onto the Siegel domain $\mathcal{D}$. We denote by $ds^2_{\mathcal{U}}$ the Bergman metric of $\mathcal{U}$, and let φ be the function $\phi \circ \sigma$ on $\mathcal{U}$. Then φ is a potential of $ds^2_{\mathcal{U}}$, and we have $\sigma^* ds^2_{\mathcal{D}} = ds^2_{\mathcal{U}}$. Therefore

Corollary 2. *On the bounded domain $\mathcal{U}$, one has*

$$\left| \sum_{\alpha=1}^{N} \frac{\partial \varphi}{\partial z_\alpha} dz_\alpha \right|_{ds^2_{\mathcal{U}}} = \sqrt{m+2n}.$$

Similarly to [5, Theorem 2], the result by Chen [1] tells us that

$$\dim H^{p,q}_{(2)}(\mathcal{U}) = \begin{cases} +\infty & (p+q=N), \\ 0 & (p+q \neq N), \end{cases}$$

where $H^{p,q}_{(2)}(\mathcal{U})$ denotes the L^2 $\bar{\partial}$-cohomology group of the bounded domain $\mathcal{D}$. Since every bounded homogeneous domain is biholomorphic to a Siegel domain ([8]), Corollary 2 gives the answer to the problem posed in [5].

§2. Proof of Theorem 1.

First let us fix our notation used in what follows. We denote by $\langle \cdot , \cdot \rangle$ the standard inner product on $\mathbb{C}^p$: $\langle v, v' \rangle = \sum_{k=1}^{p} v_k v'_k$ $(v, v' \in \mathbb{C}^p)$. Let V_1 and V_2 be vector spaces, and f a V_2-valued function on an open set U in V_1. For $a \in U$ and $v \in V_1$, we denote by $D_v f(a)$ the derivative of f given by $D_v f(a) := (\frac{d}{dt})_{t=0} f(a+tv) \in V_2$. If V_1 and V_2 are complex vector spaces, the holomorphic and anti-holomorphic derivative of f are given as $\partial_v f := (D_v f - i D_{iv} f)/2$ and $\bar{\partial}_v f := (D_v f + i D_{iv} f)/2$ respectively. Making use of these conventions, we write

$$\sum_{\alpha=1}^{N} \zeta_\alpha \frac{\partial \phi}{\partial z_\alpha}(z) = \partial_\zeta \phi(z) \tag{2}$$

and

$$ds^2_{\mathcal{D}}(\zeta, \zeta')_z = \sum_{\alpha,\beta=1}^{N} \zeta_\alpha \bar{\zeta}'_\beta \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta}(z) = \partial_\zeta \bar{\partial}_{\zeta'} \phi(z) \tag{3}$$

for $z \in \mathcal{D}$ and $\zeta, \zeta' \in \mathbb{C}^N$.

By [4, Proposition 5.1], there exists a holomorphic function η on the tube domain $\Omega + i\mathbb{R}^n \subset \mathbb{C}^n$ such that

$$K_{\mathcal{D}}((u,w),(u',w')) = \eta(w + \bar{w}' - Q(u,u')) \qquad ((u,w),\ (u',w') \in \mathcal{D}). \tag{4}$$

Note that the right-hand side is well-defined because

$$\begin{aligned} &2\Re(w + \bar{w}' - Q(u,u')) \\ &\quad = w + \bar{w} - Q(u,u) + w' + \bar{w}' - Q(u',u') + Q(u-u', u-u') \in \Omega. \end{aligned}$$

Lemma 3. *For $c > 0$ and $x \in \Omega$, one has $\eta(cx) = c^{-(m+2n)}\eta(x)$.*

Proof. We denote by λ_c the map $\mathbb{C}^N \ni (u,w) \mapsto (\sqrt{c}u, cw) \in \mathbb{C}^N$. Then λ_c preserves the domain $\mathcal{D} \subset \mathbb{C}^N$ by (1). Therefore we see from (4) that

$$\begin{aligned} \eta(cx) &= K_{\mathcal{D}}(\lambda_c(0,x/2), \lambda_c(0,x/2)) = |\det \lambda_c|^{-2} K_{\mathcal{D}}((0,x/2),(0,x/2)) \\ &= |\det \lambda_c|^{-2}\eta(x), \end{aligned}$$

which together with $\det \lambda_c = c^{m/2+n}$ gives the formula. □

Define a map $\mathcal{I}_\eta : \Omega \ni x \mapsto \xi \in \mathbb{R}^n$ by $\xi_k := -\frac{\partial}{\partial x_k}\log \eta(x) \quad (k = 1, \ldots, n)$. Then

$$D_v \log \eta(x) = -\langle v, \mathcal{I}_\eta(x)\rangle \tag{5}$$

for $x \in \Omega$ and $v \in \mathbb{R}^n$. In particular, we see from Lemma 3 that

$$\begin{aligned} \langle x, \mathcal{I}_\eta(x)\rangle &= -D_x \log \eta(x) = -(\tfrac{d}{dt})_{t=0}\{\log \eta(x) - (m+2n)\log(1+t)\} \\ &= m + 2n. \end{aligned} \tag{6}$$

Since η is holomorphic on $\Omega + i\mathbb{R}^n$, we have $\frac{\partial}{\partial x_k}\log \eta(x) = \frac{\partial}{\partial z_k}\log \eta(x) \quad (k = 1, \ldots, n)$. Thus $\mathcal{I}_\eta$ is naturally extended to a holomorphic map, and (5) is still valid for $x \in \Omega + i\mathbb{R}^n$ and $v \in \mathbb{C}^n$.

Lemma 4. *For $c > 0$ and $x \in \Omega$, one has $\mathcal{I}_\eta(cx) = c^{-1}\mathcal{I}_\eta(x)$.*

Proof. For $v \in \mathbb{R}^n$, we see from (5) and Lemma 3 that

$$\begin{aligned} \langle v, \mathcal{I}_\eta(cx)\rangle &= (\tfrac{d}{dt})_{t=0} \log \eta(cx + tv) \\ &= (\tfrac{d}{dt})_{t=0}\{\log \eta(x + tc^{-1}v) - (m+2n)\log c\} \\ &= \langle c^{-1}v, \mathcal{I}_\eta(x)\rangle, \end{aligned}$$

whence the formula follows. □

We set $\Phi(z) := w + \bar{w} - Q(u,u)$ for $z = (u,w) \in \mathcal{D}$. Then we have $\phi = \log \eta \circ \Phi$ by (4). Thanks to (2) and (5), we obtain

$$\begin{aligned}(7)\qquad \sum_{\alpha=1}^{N} \zeta_{0,\alpha} \frac{\partial \phi}{\partial z_\alpha}(z) = \partial_{\zeta_0}\phi(z) &= -\langle \partial_{\zeta_0}\Phi(z),\, \mathcal{I}_\eta(\Phi(z))\rangle \\ &= -\langle w_0 - Q(u_0,u),\, \mathcal{I}_\eta(\Phi(z))\rangle\end{aligned}$$

for $\zeta_0 = (u_0,w_0) \in \mathbb{C}^N$.

Let us define a map $H_\eta : \Omega \ni x \to (a_{kl})_{k,l=1}^n \in \mathrm{Sym}(n,\mathbb{R})$ by $a_{kl} := \frac{\partial^2}{\partial x_k \partial x_l} \log \eta(x)$. Namely, H_η is the Hessian of the function $\log\eta$. Recalling the definition of $\mathcal{I}_\eta$, we have

$$(8)\qquad D_v \mathcal{I}_\eta(x) = -H_\eta(x)v,$$

which is valid for $v \in \mathbb{C}^n$. We see from (3) and (7) that

$$\begin{aligned}(9)\qquad ds^2_{\mathcal{D}}(\zeta_0,\zeta_1)_z &= \bar\partial_{\zeta_1}\partial_{\zeta_0}\phi(z) = -\bar\partial_{\zeta_1}\langle \partial_{\zeta_0}\Phi(z),\, \mathcal{I}_\eta(\Phi(z))\rangle \\ &= -\langle \bar\partial_{\zeta_1}\partial_{\zeta_0}\Phi(z),\, \mathcal{I}_\eta(\Phi(z))\rangle - \langle \partial_{\zeta_0}\Phi(z), \bar\partial_{\zeta_1}\mathcal{I}_\eta(\Phi(z))\rangle\end{aligned}$$

for $\zeta_j = (u_j,w_j) \in \mathbb{C}^N$ $(j = 0,1)$. We observe $\bar\partial_{\zeta_1}\partial_{\zeta_0}\Phi(z) = -Q(u_0,u_1)$. On the other hand, since $\bar\partial_{\zeta_1}\Phi(z) = \bar{w}_1 - Q(u,u_1)$, we have by (8)

$$\bar\partial_{\zeta_1}\mathcal{I}_\eta(\Phi(z)) = -H_\eta(\Phi(z))(\bar{w}_1 - Q(u,u_1)).$$

Thus (9) tells us that

$$(10)\quad ds^2_{\mathcal{D}}(\zeta_0,\zeta_1)_z = \langle Q(u_0,u_1), \Phi(z)\rangle + \langle w_0 - Q(u_0,u), H_\eta(\Phi(z))(\bar{w}_1 - Q(u,u_1))\rangle.$$

Now we set $\zeta_\phi(z) := (0, -\Phi(z)) \in \mathbb{C}^N$ for $z \in \mathcal{D}$. By (10) we have

$$(11)\qquad ds^2_{\mathcal{D}}(\zeta_0,\zeta_\phi(z))_z = -\langle w_0 - Q(u_0,u), H_\eta(\Phi(z))\Phi(z)\rangle.$$

Thanks to (8) and Lemma 4, we have

$$H_\eta(\Phi(z))\Phi(z) = -D_{\Phi(z)}\mathcal{I}_\eta(\Phi(z)) = \mathcal{I}_\eta(\Phi(z)).$$

Thus we see from (11) and (7) that

$$(12)\qquad ds^2_{\mathcal{D}}(\zeta_0,\zeta_\phi(z))_z = -\langle w_0 - Q(u_0,u),\, \mathcal{I}_\eta(\Phi(z))\rangle = \sum_{\alpha=1}^{N} \zeta_{0,\alpha}\frac{\partial\phi}{\partial z_\alpha}(z),$$

which implies that

$$\left| \sum_{\alpha=1}^{N} \frac{\partial \phi}{\partial z_\alpha} dz_\alpha \right|^2_{ds^2_{\mathcal{D}}} = ds^2_{\mathcal{D}}(\zeta_\phi(z), \zeta_\phi(z))_z.$$

By (12) and (6), the right-hand side is

$$\langle \Phi(z), \mathcal{I}_\eta(\Phi(z))\rangle = m + 2n.$$

Hence Theorem 1 is proved.

§3. Appendix.

The holomorphic function $\eta : \Omega + i\mathbb{R}^n \to \mathbb{C}$ in (4) equals the Fourier-Laplace transform of a measure on the dual cone of Ω ([4, Theorem 5.4]). Following the argument in [9, Chapter 3], we shall review this result briefly.

We denote by Ω^* the dual cone $\{\, \xi \in \mathbb{R}^n \,;\, \langle x, \xi \rangle > 0 \;\; (x \in \mathrm{Cl}(\Omega) \setminus \{0\}) \,\}$ of Ω. We define functions δ_{Ω^*} and δ_Q on Ω^* by

$$\delta_{\Omega^*}(\xi) := \int_\Omega e^{-\langle x, \xi \rangle} dx, \qquad \delta_Q(\xi) := \int_{\mathbb{C}^m} e^{-\langle Q(u,u), \xi \rangle} d\nu(u) \qquad (\xi \in \Omega^*)$$

respectively, where $d\nu$ stands for the Euclidean measure on $\mathbb{C}^m$. Let $d\mu$ be a measure on Ω^* given by $d\mu(\xi) := \pi^{-n} \delta_{\Omega^*}(\xi)^{-1} \delta_Q(\xi)^{-1} \, d\xi \;\; (\xi \in \Omega^*)$.

Lemma 5. *The Fourier-Laplace transform $\int_{\Omega^*} e^{-\langle w, \xi \rangle} d\mu(\xi)$ of the measure $d\mu$ converges absolutely for all $w \in \Omega + i\mathbb{R}^n$.*

Proof. It is enough to show the case that $w = x \in \Omega$. For $s > 0$, let $P_{x,s}$ be the hyperplane $\{\, \xi \in \mathbb{R}^n \,;\, \langle x, \xi \rangle = s \,\}$ in $\mathbb{R}^n$. Then we have

$$\int_{\Omega^*} e^{-\langle x, \xi \rangle} d\mu(\xi) = \int_0^\infty \left\{ \int_{P_{x,s} \cap \Omega^*} e^{-s} \cdot \pi^{-n} \delta_{\Omega^*}(\xi)^{-1} \delta_Q(\xi)^{-1} d\sigma_{x,s}(\xi) \right\} \frac{ds}{\|x\|}, \tag{13}$$

where $d\sigma_{x,s}$ denotes the Euclidean measure on $P_{x,s}$. It is easy to see that $d\sigma_{x,s}(s\xi) = s^{n-1} d\sigma_{x,1}(\xi)$ for $\xi \in P_{x,1}$. Moreover we have

$$\delta_{\Omega^*}(s\xi)^{-1} = s^n \delta_{\Omega^*}(\xi), \qquad \delta_Q(s\xi)^{-1} = s^m \delta_Q(\xi)$$

by definition. Thus the right-hand side of (13) is rewritten as

$$\begin{aligned} &\frac{1}{\pi^n \|x\|} \int_0^{+\infty} e^{-s} s^{2n+m-1} ds \int_{P_{x,1} \cap \Omega^*} \delta_{\Omega^*}(\xi)^{-1} \delta_Q(\xi)^{-1} \, d\sigma_{x,1}(\xi) \\ &\quad = \frac{\Gamma(2n+m)}{\pi^n \|x\|} \int_{P_{x,1} \cap \Omega^*} \delta_{\Omega^*}(\xi)^{-1} \delta_Q(\xi)^{-1} \, d\sigma_{x,1}(\xi). \end{aligned} \tag{14}$$

Now we introduce a map $R : \Omega^* \to \mathrm{Herm}(m, \mathbb{C})$ defined in such a way that $\langle R(\xi)u, \bar{u}' \rangle := \langle Q(u, u'), \xi \rangle \;\; (\xi \in \Omega^*,\, u, u' \in \mathbb{C}^m)$. Then we have

$$\delta_Q(\xi) = \pi^m \det R(\xi)^{-1}.$$

In particular, $\delta_Q(\xi)^{-1}$ is a polynomial function of degree m. On the other hand, for any point ξ_0 on the boundary of the cone Ω^*, we have $\lim_{\xi \in \Omega^*, \xi \to \xi_0} \delta_{\Omega^*}(\xi)^{-1} = 0$ thanks to [10, Chapter 1, Proposition 3]. Therefore $\delta_{\Omega^*}(\xi)^{-1} \delta_Q(\xi)^{-1}$ is bounded on the relatively compact set $P_{x,1} \cap \Omega^*$, so that the integral in (14) converges. $\square$

For $\xi \in \Omega^*$, we write $\mathcal{F}_\xi$ for the Fock-Bargmann space on $\mathbb{C}^m$ whose reproducing kernel is $e^{\xi \circ Q}$. Namely, $\mathcal{F}_\xi$ is the Hilbert space consisting of holomorphic square-integrable functions on $\mathbb{C}^m$ with respect to the Gaussian measure $\delta_Q(\xi)^{-1} e^{-\langle Q(u,u), \xi \rangle} d\nu(u)$ $(u \in \mathbb{C}^m)$.

Let $\mathcal{L}(\Omega^* \times \mathbb{C}^m; d\mu)$ be the space of equivalence classes of measurable functions f on $\Omega^* \times \mathbb{C}^m$ with respect to $d\mu \otimes d\nu$ such that
(L1) $f(\xi, \cdot) \in \mathcal{F}_\xi$ for almost all $\xi \in \Omega^*$,
(L2) $\|f\|^2_{\mathcal{L}} := \int_{\Omega^*} \|f(\xi, \cdot)\|^2_{\mathcal{F}_\xi} d\mu(\xi) < +\infty$.
Then $\mathcal{L}(\Omega^* \times \mathbb{C}^m; d\mu)$ forms a Hilbert space (see [9, Lemma 3.2.7]). For an element (u_0, w_0) of the Siegel domain $\mathcal{D}$, the function $k_{(u_0,w_0)}$ on $\Omega^* \times \mathbb{C}^m$ given by

$$k_{(u_0,w_0)}(\xi, u) := e^{-\langle \bar{w}_0, \xi \rangle} e^{\langle Q(u,u_0), \xi \rangle} \quad (\xi \in \Omega^*,\, u \in \mathbb{C}^m)$$

belongs to $\mathcal{L}(\Omega^* \times \mathbb{C}^m; d\mu)$. Indeed, we have $\|k_{(u_0,w_0)}(\xi, \cdot)\|^2_{\mathcal{F}_\xi} = e^{-\langle w_0 + \bar{w}_0 - Q(u_0,u_0), \xi \rangle}$, so that $\|k_{(u_0,w_0)}\|^2_{\mathcal{L}}$ is finite by Lemma 5 (see (19) below).

Theorem 6 (cf. [9, Theorem 3.2.8]). *One has a unitary isomorphism Ψ from $\mathcal{L}(\Omega^* \times \mathbb{C}^m; d\mu)$ onto the Bergman space $A^2(\mathcal{D})$ on the Siegel domain $\mathcal{D}$ defined by*

$$\Psi f(u,w) := \int_\Omega e^{-\langle w, \xi \rangle} f(\xi, u)\, d\mu(\xi), \qquad ((u,w) \in \mathcal{D}) \tag{15}$$

where the integral converges absolutely.

Proof. Thanks to (L1) and the reproducing property of $e^{\xi \circ Q}$, we have $f(\xi, u) = (f(\xi, \cdot) | e^{\langle Q(\cdot, u), \xi \rangle})_{\mathcal{F}_\xi}$ for almost all $\xi \in \Omega^*$, so that (15) is rewritten as

$$\Psi f(u,w) = (f | k_{(u,w)})_{\mathcal{L}}, \tag{16}$$

which is always finite. Now we put $F := \Psi f$. Applying Morera's criterion and Fubini's theorem, we see that F is holomorphic. If $w = x + iy$ with $x, y \in \mathbb{R}^n$, we have

$$F(u, x+iy) = (2\pi)^{-n} \int_{\Omega^*} e^{-i\langle y, \xi \rangle} e^{-\langle x, \xi \rangle} f(\xi, u)\, \delta_{\Omega^*}(2\xi)^{-1} \delta_Q(\xi)^{-1} d\xi \tag{17}$$

because $\delta_{\Omega^*}(2\xi)^{-1} = 2^n \delta_{\Omega^*}(\xi)^{-1}$. By the Plancherel formula, we get

$$\int_{\mathbb{R}^n} |F(u, x+iy)|^2 dy = (2\pi)^{-n} \int_{\Omega^*} e^{-\langle x, 2\xi \rangle} |f(\xi, u)|^2 \delta_{\Omega^*}(2\xi)^{-2} \delta_Q(\xi)^{-2} d\xi,$$

so that

$$\int_{\mathbb{C}^m}\int_{\mathbb{R}^n}|F(u,x+iy)|^2dyd\nu(u)$$
$$= (2\pi)^{-n}\int_{\Omega^*} e^{-\langle x-Q(u,u)/2,2\xi\rangle}\left(\delta_Q(\xi)^{-1}\int_{\mathbb{C}^m}|f(\xi,u)|^2e^{-\langle Q(u,u),\xi\rangle}d\nu(u)\right)\delta_{\Omega^*}(2\xi)^{-2}\delta_Q(\xi)^{-1}d\xi$$
$$= (2\pi)^{-n}\int_{\Omega^*} e^{-\langle x-Q(u,u)/2,2\xi\rangle}\|f(\xi,\cdot)\|^2_{\mathcal{F}_\xi}\delta_{\Omega^*}(2\xi)^{-2}\delta_Q(\xi)^{-1}d\xi$$

by Fubini's theorem. Since $(u,w)\in\mathcal{D}$ if and only if $t := x - Q(u,u)/2 \in \Omega$, we obtain

$$\begin{aligned}
&\int_{\mathcal{D}}|F(u,x+iy)|^2d\nu(u)dxdy\\
&\quad=\int_{\Omega}\int_{\mathbb{R}^n}\int_{\mathbb{C}^m}|F(u,t+Q(u,u)/2+iy)|^2\,d\nu(u)\,dy\,dt\\
&\quad=(2\pi)^{-n}\int_{\Omega^*}\left(\int_{\Omega}e^{-\langle t,2\xi\rangle}dt\right)\|f(\xi,\cdot)\|^2_{\mathcal{F}_\xi}\delta_{\Omega^*}(2\xi)^{-2}\delta_Q(\xi)^{-1}d\xi\\
&\quad=(2\pi)^{-n}\int_{\Omega^*}\|f(\xi,\cdot)\|^2_{\mathcal{F}_\xi}\delta_{\Omega^*}(2\xi)^{-1}\delta_Q(\xi)^{-1}d\xi\\
&\quad=\int_{\Omega^*}\|f(\xi,u)\|^2_{\mathcal{F}_\xi}\,d\mu(\xi).
\end{aligned}$$

Therefore F belongs to $A^2(\mathcal{D})$ and Ψ is an isometry. Owing to the Fourier inverse formula, we see from (17) that

$$f(\xi,u) = e^{\langle x,\xi\rangle}\,\delta_{\Omega^*}(2\xi)\,\delta_Q(\xi)\int_{\mathbb{R}^n}e^{i\langle y,\xi\rangle}F(u,x+iy)\,dy. \tag{18}$$

Thus, in order to show the surjectivity of Ψ, we have to check that the right-hand side of (18) is independent of $x\in\Omega$, and defines an element of $\mathcal{L}(\Omega^*\times\mathbb{C}^m;d\mu)$ for any $F\in A^2(\mathcal{D})$. We merely refer to [7, Proposition 2.5] and [7, Lemma 2.29] for details. □

Since Ψ is unitary, we see from (16) that

$$F(u,w) = (f|k_{(u,w)}) = (F|\Psi k_{(u,w)}) \qquad ((u,w)\in\mathcal{D})$$

for any $F = \Psi f \in A^2(\mathcal{D})$ $(f\in\mathcal{L}(\Omega^*\times\mathbb{C}^m;d\mu))$. This means that the Bergman kernel $K_{\mathcal{D}}$ is given by

$$K_{\mathcal{D}}((u,w),(u',w')) = \Psi k_{(u',w')}(u,w) = (k_{(u',w')}|k_{(u,w)})_{\mathcal{L}} \qquad ((u,w),\,(u',w')\in\mathcal{D}),$$

where the last term equals

$$\int_{\Omega^*}e^{-\langle\bar{w}',\xi\rangle}e^{-\langle w,\xi\rangle}(e^{\langle Q(\cdot,u'),\xi\rangle}|e^{\langle Q(\cdot,u),\xi\rangle})_{\mathcal{F}_\xi}\,d\mu(\xi) = \int_{\Omega^*}e^{-\langle w+\bar{w}'-Q(u,u'),\xi\rangle}d\mu(\xi). \tag{19}$$

Therefore, putting

$$\eta(w) := \int_{\Omega^*} e^{-\langle w,\xi\rangle} d\mu(\xi) \quad (w \in \Omega + i\mathbb{R}^n),$$

we have (4).

References

[1] B.-Y. Chen, *Infinite dimensionality of the middle L^2-cohomology on non-compact*, to appear in Publ. RIMS.

[2] H. Donnelly, *L_2-cohomology of the Bergman metric for weakly pseudoconvex domains*, Illinois J. Math. **41** (1997), 151–160.

[3] H. Donnelly and C. Fefferman, *L_2-cohomology and index theorem for the Bergman metric*, Ann. Math. **118** (1983), 593–618.

[4] S. G. Gindikin, *Analysis in homogeneous domains*, Russian Math. Surveys **19** (1964), 1–89.

[5] C. Kai and T. Ohsawa, *A note on the Bergman metric of bounded homogeneous domains*, Nagoya Math. J. **186** (2007), 157–163.

[6] I. I. Piatetskii-Shapiro, Automorphic functions and the geometry of classical domains, Gordon and Breach, New York, 1969.

[7] H. Rossi and M. Vergne, *Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions and the application to the holomorphic discrete series of a semisimple Lie group*, J. Funct. Anal. **13** (1973), 324–389.

[8] E. B. Vinberg, S. G. Gindikin and I. I. Piatetskii-Shapiro, *Classification and canonical realization of complex bounded homogeneous domains*, Trans. Moscow Math. Soc. **12** (1963), 404–437.

[9] M. Vergne and H. Rossi, *Analytic continuation of the holomorphic discrete series of a semi-simple Lie group*, Acta Math. **136** (1976), 1–59.

[10] E. B. Vinberg, *The theory of convex homogeneous cones*, Trans. Moscow Math. Soc., **12** (1963), 340–403.