HARDY OPERATORS FROM LOCAL TO GLOBAL WEIGHTED MORREY SPACES

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ABSTRACT. The goal of this paper is to study the boundedness of the n-dimensional Hardy operator from weighted local Morrey spaces of Komori–Shirai type to weighted Morrey spaces of Komori–Shirai type. Let

$$w_\alpha(x) = |x|^\alpha, \qquad x \in \mathbb{R}^n,$$

be a power weight. For $1 \le q < p < \infty$, we characterize the range of the parameter α for which the Hardy operator is bounded from the weighted local Morrey space $\mathrm{L}\mathcal{M}^p_1(w_\alpha)$ to the weighted Morrey space $\mathcal{M}^p_q(w_\alpha)$. More precisely, we prove that the Hardy operator is bounded if and only if

$$-n < \alpha < n(p-1).$$

This result provides a local-to-global boundedness theorem for the Hardy operator on weighted Morrey spaces and extends previous results for weighted Morrey spaces of Samko type.

1. Introduction

The goal of this paper is to study the boundedness of the classical n-dimensional Hardy operator

$$Hf(x) := \frac{1}{|x|^n} \int_{|y|<|x|} f(y)\, dy,$$

on local weighted Morrey spaces of Komori–Shirai type [2]. Since the kernel of H is positive, we may assume that f is non-negative if necessary.

Let

$$w_\alpha(x) := |x|^\alpha, \qquad x \in \mathbb{R}^n,$$

be a power weight with $\alpha \in \mathbb{R}$. Throughout the paper, we assume that w_α is locally integrable, that is,

$$\alpha > -n.$$

For $x \in \mathbb{R}^n$ and $r > 0$, let

$$B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$$

denote the open ball centered at x with radius r. When $x = 0$, we simply write $B(r) := B(0,r)$.

Let $1 \le q \le p < \infty$. The weighted local Morrey space $\mathrm{L}\mathcal{M}^p_q(w_\alpha)$ of Komori–Shirai type consists of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathrm{L}\mathcal{M}^p_q(w_\alpha)} := \sup_{r>0} w_\alpha(B(r))^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(r)} |f(y)|^q w_\alpha(y)\, dy \right)^{\frac{1}{q}} < \infty.$$

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$$\|f\|_{\mathcal{M}^p_q(w_\alpha)} := \sup_{\substack{x\in\mathbb{R}^n \\ r>0}} w_\alpha(B(x,r))^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q w_\alpha(y)\,dy\right)^{\frac{1}{q}} < \infty.$$

Here and below for a measurable set E, we write

$$w_\alpha(E) = \int_E w_\alpha(x)\,dx.$$

It follows from the definition that

$$\mathcal{M}^p_p(w_\alpha) = \mathrm{L}\mathcal{M}^p_p(w_\alpha) = L^p(w_\alpha).$$

See [3] for the Hardy operator acting on weighted Lebesgue spaces. So, we are interested in the case $q < p$ in this paper.

A direct consequence of the definitions is that

$$\mathcal{M}^p_q(w_\alpha) \hookrightarrow \mathrm{L}\mathcal{M}^p_q(w_\alpha) \tag{1.1}$$

continuously with embedding constant 1.

Let

$$1 \le \tilde{q} \le q \le p < \infty.$$

If we use Hölder's inequality for weighted measure $w_\alpha\,dx$, then we have

$$\mathcal{M}^p_q(w_\alpha) \hookrightarrow \mathcal{M}^p_{\tilde{q}}(w_\alpha)$$

and

$$\mathrm{L}\mathcal{M}^p_q(w_\alpha) \hookrightarrow \mathrm{L}\mathcal{M}^p_{\tilde{q}}(w_\alpha) \tag{1.2}$$

continuously, with embedding constant 1. Using the scaling properties of the power weight (see (1.5) below) and the corresponding Morrey norms, we prove the following theorem:

Theorem 1.1. *Let $1 \le q < p < \infty$. Then the following assertions are equivalent:*

(1) *The parameter α satisfies*

$$-n < \alpha < n(p-1). \tag{1.3}$$

(2) *H is bounded from $\mathrm{L}\mathcal{M}^p_1(w_\alpha)$ to $\mathcal{M}^p_q(w_\alpha)$, that is, there exists a constant $C > 0$ such that*

$$\|Hf\|_{\mathcal{M}^p_q(w_\alpha)} \le C\|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)} \tag{1.4}$$

for every $f \in \mathrm{L}\mathcal{M}^p_1(w_\alpha)$.

(3) *H is bounded from $\mathrm{L}\mathcal{M}^p_q(w_\alpha)$ to $\mathcal{M}^p_q(w_\alpha)$.*

(4) *H is bounded on $\mathcal{M}^p_q(w_\alpha)$.*

When $p = q$, the weighted Morrey space coincides with the weighted Lebesgue space by the monotone convergence theorem [8]:

$$\mathcal{M}^q_q(w_\alpha) = \mathrm{L}\mathcal{M}^q_q(w_\alpha) = L^q(w_\alpha).$$

In view of the boundedness of the Hardy–Littlewood maximal operator, the boundedness of the Hardy operator on weighted Morrey spaces has been studied in [8]. The novelty of this paper lies in establishing a local-to-global boundedness theorem for the Hardy operator between weighted Morrey spaces.

We recall a useful formula for the weighted measure. Let σ_{n-1} denote the surface measure of the unit sphere

$$\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}.$$

For later use, we note that

$$w_\alpha(B(r)) = \int_{B(r)} |x|^\alpha \, dx = \int_0^r \sigma_{n-1} s^{n+\alpha-1} \, ds = \frac{\sigma_{n-1}}{n+\alpha} r^{n+\alpha}. \tag{1.5}$$

This formula shows that the quantity $n+\alpha$ plays the role of an effective dimension.

It is worth comparing our results with those for weighted Morrey spaces of Samko type. The weighted local Morrey space $\mathrm{L}\mathcal{M}^p_q(1, w_\alpha)$ of Samko type consists of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathrm{L}\mathcal{M}^p_q(1,w_\alpha)} := \sup_{r>0} |B(r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(r)} |f(y)|^q w_\alpha(y) \, dy \right)^{\frac{1}{q}} < \infty.$$

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$$\|f\|_{\mathcal{M}^p_q(1,w_\alpha)} := \sup_{\substack{x\in\mathbb{R}^n \\ r>0}} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q w_\alpha(y) \, dy \right)^{\frac{1}{q}} < \infty.$$

The counterpart of Theorem 1.1 for weighted Morrey spaces of Samko type can be found in [6, 7]. In fact, by carefully reexamining the proofs in [6, 7], we observe that the case $\alpha = 0$ is also covered by those results. The proof of Theorem 1.1 is based on the techniques developed in [6, 7]. We also refer to [1, 4, 8] for results concerning the boundedness of operators on weighted Morrey spaces, and to [9] for sharp constants in local Morrey spaces.

Here and below we use the following convention in this paper. Let $A, B \geq 0$. Then $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leq CB$, where C is usually independent of the functions we are considering. The symbol $A \sim B$ stands for the two-sided inequality $A \lesssim B \lesssim A$. Section 2 is devoted to the proof of Theorem 1.1 and the investigation of the sharpness of the parameter α. Section 3 calculates the exact value of the embedding constant.

2. Proof of Theorem 1.1

We first recall the following lemma. Notice that the condition $q < p$ is essential here. Indeed, when $p = q$, the conclusion generally fails.

Lemma 2.1. *Let $1 \leq q < p < \infty$ and assume* (1.3)*. Then*

$$|x|^{-\frac{n+\alpha}{p}} \in \mathcal{M}^p_q(w_\alpha).$$

See [8, Example 58 (2) (b)] for the proof.

Inspired by the arguments in [6, 7], we establish the following pointwise estimate.

Proposition 2.2. *Let $1 \le p < \infty$ and assume (1.3). Then, for every $f \in \mathrm{L}\mathcal{M}^p_1(w_\alpha)$,*

$$|Hf(x)| \lesssim |x|^{-\frac{n+\alpha}{p}} \|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)}, \qquad x \neq 0.$$

Proof. Let $R = |x|$. Since

$$B(R) = \bigcup_{k=0}^{\infty} A_k, \qquad A_k = \{y : 2^{-k-1}R < |y| \le 2^{-k}R\} \quad (k = 0, 1, \ldots),$$

we have

$$|Hf(x)| \le \frac{1}{R^n} \sum_{k=0}^{\infty} \int_{A_k} |f(y)|\, dy.$$

Since $|y| \sim 2^{-k}R$ on A_k,

$$w_\alpha(y) = |y|^\alpha \sim (2^{-k}R)^\alpha,$$

and therefore

$$\int_{A_k} |f(y)|\, dy \lesssim (2^{-k}R)^{-\alpha} \int_{A_k} |f(y)| w_\alpha(y)\, dy.$$

Moreover, by the definition of the local Morrey norm,

$$\int_{A_k} |f(y)| w_\alpha(y)\, dy \le \int_{B(2^{-k}R)} |f(y)| w_\alpha(y)\, dy \lesssim w_\alpha(B(2^{-k}R))^{1-\frac{1}{p}} \|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)}.$$

Hence, by the definition of the weighted Morrey norm of Komori–Shirai type,

$$\int_{A_k} |f(y)|\, dy \lesssim (2^{-k}R)^{-\alpha} w_\alpha(B(2^{-k}R))^{1-\frac{1}{p}} \|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)}.$$

It follows from (1.5) that

$$\int_{A_k} |f(y)|\, dy \lesssim (2^{-k}R)^{-\alpha} (2^{-k}R)^{(n+\alpha)(1-\frac{1}{p})} \|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)} = (2^{-k}R)^{n-\frac{n+\alpha}{p}} \|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)}.$$

Therefore,

$$|Hf(x)| \lesssim R^{-n} \sum_{k=0}^{\infty} (2^{-k}R)^{n-\frac{n+\alpha}{p}} \|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)} = R^{-\frac{n+\alpha}{p}} \sum_{k=0}^{\infty} 2^{-k\left(n-\frac{n+\alpha}{p}\right)} \|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)}.$$

Since $\alpha < n(p-1)$, the geometric series converges. Consequently, using $|x| = R$, we obtain

$$|Hf(x)| \lesssim |x|^{-\frac{n+\alpha}{p}} \|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)},$$

which completes the proof. □

Now we prove Theorem 1.1.
(1) $\Longrightarrow$ (2). By Lemma 2.1 and Proposition 2.2, we obtain

$$\|Hf\|_{\mathcal{M}^p_q(w_\alpha)} \lesssim \left\| |\cdot|^{-\frac{n+\alpha}{p}} \right\|_{\mathcal{M}^p_q(w_\alpha)} \|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)} \lesssim \|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)}.$$

This proves (1.4).

(2) $\Longrightarrow$ (3). This is an immediate consequence of the continuous embedding

$$\mathrm{L}\mathcal{M}^p_q(w_\alpha) \hookrightarrow \mathrm{L}\mathcal{M}^p_1(w_\alpha):$$

See (1.2) with $\tilde{q} = 1$. Indeed,

$$\|Hf\|_{\mathcal{M}^p_q(w_\alpha)} \lesssim \|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)} \leq \|f\|_{\mathrm{L}\mathcal{M}^p_q(w_\alpha)}$$

for all $f \in \mathrm{L}\mathcal{M}^p_q(w_\alpha)$.
(3) $\Longrightarrow$ (4). This follows immediately from the continuous embedding

$$\mathcal{M}^p_q(w_\alpha) \hookrightarrow \mathrm{L}\mathcal{M}^p_q(w_\alpha):$$

See (1.1). Indeed,

$$\|Hf\|_{\mathcal{M}^p_q(w_\alpha)} \lesssim \|f\|_{\mathrm{L}\mathcal{M}^p_q(w_\alpha)} \leq \|f\|_{\mathcal{M}^p_q(w_\alpha)}$$

for all $f \in \mathcal{M}^p_q(w_\alpha)$.
(4) $\Longrightarrow$ (1). The necessity of the condition on α follows from a suitable test function. Let

$$f(x) = |x|^{-n}\chi_{B(1)}(x).$$

Then $Hf(x) = \infty$ for every $x \neq 0$, since

$$Hf(x) = \frac{1}{|x|^n}\int_{|y|<|x|} |y|^{-n}\, dy = \frac{\sigma_{n-1}}{|x|^n}\int_0^{|x|} r^{-1}\, dr = \infty.$$

Hence, if H is bounded from $\mathcal{M}^p_q(w_\alpha)$ to $\mathcal{M}^p_q(w_\alpha)$, then $f \notin \mathcal{M}^p_q(w_\alpha)$. By [8, Example 58 (3) (b)], which asserts that

$$f \in \mathcal{M}^p_q(w_\alpha) \quad \Longleftrightarrow \quad \alpha \geq n(p-1),$$

we obtain $\alpha < n(p-1)$. Together with the standing assumption $\alpha > -n$, this yields the desired range.

This completes the proof of Theorem 1.1.

3. Exact value of the pointwise inequality

In connection with [9], we find the best constant in Proposition 2.2.

The Köthe dual of $\mathrm{L}\mathcal{M}^p_1(w_\alpha)$ is defined by

$$(\mathrm{L}\mathcal{M}^p_1(w_\alpha))' = \left\{ g : \|g\|_{(\mathrm{L}\mathcal{M}^p_1(w_\alpha))'} := \sup_{f \in \mathrm{L}\mathcal{M}^p_1(w_\alpha)\setminus\{0\}} \frac{1}{\|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)}} \int_{\mathbb{R}^n} |f(x)g(x)|\, dx < \infty \right\}.$$

Proposition 2.2 states that

$$\int_{B(1)} |f(x)|\, dx \lesssim \|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)}.$$

Therefore,

$$\chi_{B(1)} \in (\mathrm{L}\mathcal{M}^p_1(w_\alpha))'$$

under the assumptions of Proposition 2.2.

Moreover, the best constant in the inequality

$$\int_{B(1)} |f(x)|\, dx \leq C\|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)}$$

is given by
$$C = \left\|\chi_{B(1)}\right\|_{\left(\mathrm{L}\mathcal{M}^p_1(w_\alpha)\right)'}.$$
It is interesting to determine its exact value. Set
$$\theta = 1 - \frac{1}{p}, \quad d = n + \alpha, \quad \kappa_\alpha = \frac{\sigma_{n-1}}{d}.$$

Lemma 3.1. *Let*
$$A_f(R) := \int_{B(0,R)} |f(x)| w_\alpha(x)\, dx.$$
Then
$$\|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)} \le 1$$
if and only if
$$A_f(R) \le \kappa_\alpha^\theta R^{d\theta} \tag{3.1}$$
for all $R > 0$.

Proof. By the definition of the weighted Morrey norm, we have
$$\|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)} = \sup_{R>0} w_\alpha(B(0,R))^{\frac{1}{p}-1} A_f(R).$$
It follows from (1.5) that
$$\sup_{R>0} w_\alpha(B(R))^{\frac{1}{p}-1} A_f(R) = \sup_{R>0} (\kappa_\alpha R^d)^{-\theta} A_f(R). \tag{3.2}$$
Therefore,
$$\|f\|_{\mathrm{L}\mathcal{M}^p_1(w_\alpha)} \le 1$$
is equivalent to
$$(\kappa_\alpha R^d)^{-\theta} A_f(R) \le 1$$
for every $R > 0$, which is precisely
$$A_f(R) \le \kappa_\alpha^\theta R^{d\theta}.$$
Hence the assertion follows. □

Proposition 3.2. *Let $-n < \alpha \le 0$. Then*
$$\left\|\chi_{B(0,1)}\right\|_{\left(\mathrm{L}\mathcal{M}^p_1(w_\alpha)\right)'} = \kappa_\alpha^\theta. \tag{3.3}$$

Proof. Set
$$C_\alpha := \left\|\chi_{B(0,1)}\right\|_{\left(\mathrm{L}\mathcal{M}^p_1(w_\alpha)\right)'}.$$
Let f satisfy (3.1). Since
$$|x|^{-\alpha} \le 1 \qquad (x \in B(0,1)),$$
we have
$$\int_{B(0,1)} |f(x)|\, dx = \int_{B(0,1)} |x|^{-\alpha} |f(x)| w_\alpha(x)\, dx \le A_f(1) \le \kappa_\alpha^\theta.$$
Therefore,
$$C_\alpha \le \kappa_\alpha^\theta. \tag{3.4}$$

We prove the converse inequality. Let $0 < \rho < 1$ and define

$$E_\rho := \{x \in \mathbb{R}^n : \rho < |x| < 1\}.$$

Set

$$f_\rho(x) := \frac{\kappa_\alpha^\theta \rho^{d\theta}}{w_\alpha(E_\rho)} \chi_{E_\rho}(x).$$

Then

$$A_{f_\rho}(R) = 0$$

for $R \le \rho$. On the other hand, for $R > \rho$,

$$A_{f_\rho}(R) \le \int_{E_\rho \cap B(0,R)} |f_\rho(x)| w_\alpha(x)\, dx \le \kappa_\alpha^\theta \rho^{d\theta} \le \kappa_\alpha^\theta R^{d\theta}.$$

Hence, by Lemma 3.1,

$$\|f_\rho\|_{\mathrm{L}\mathcal{M}_1^p(w_\alpha)} \le 1.$$

Consequently,

$$C_\alpha \ge \int_{B(0,1)} |f_\rho(x)|\, dx.$$

Since f_ρ is supported in E_ρ and E_ρ is a subset of $B(1)$, we obtain

$$|E_\rho| \ge \rho^{-\alpha} w_\alpha(E_\rho), \quad \int_{B(0,1)} |f_\rho(x)|\, dx = \frac{\kappa_\alpha^\theta \rho^{d\theta}}{w_\alpha(E_\rho)} |E_\rho| \ge \kappa_\alpha^\theta \rho^{d\theta - \alpha}.$$

Therefore,

$$\int_{B(0,1)} |f_\rho| \ge \kappa_\alpha^\theta \rho^{d\theta-\alpha} \to \kappa_\alpha^\theta.$$

Letting $\rho \to 1^-$ gives

$$C_\alpha \ge \kappa_\alpha^\theta.$$

Together with (3.4), we conclude that (3.3) holds. □

Proposition 3.3. *Let $0 < \alpha < n(p-1)$. Then*

$$\left\|\chi_{B(0,1)}\right\|_{\left(\mathrm{L}\mathcal{M}_1^p(w_\alpha)\right)'} = \kappa_\alpha^\theta \frac{(n+\alpha)(p-1)}{n(p-1)-\alpha}.$$

Proof. Set

$$C_\alpha := \left\|\chi_{B(0,1)}\right\|_{\left(\mathrm{L}\mathcal{M}_1^p(w_\alpha)\right)'}.$$

First, we prove the upper estimate. Let f satisfy (3.1). Then by Tonelli's theorem, it follows that

$$\begin{aligned}
\int_{B(0,1)} |f(x)|\, dx &= \int_{B(0,1)} |x|^{-\alpha} |f(x)| w_\alpha(x)\, dx \\
&= A_f(1) + \alpha \int_{B(0,1)} \left(\int_{|x|}^1 r^{-\alpha-1}\, dr \right) |f(x)| w_\alpha(x)\, dx \\
&= A_f(1) + \alpha \int_0^1 r^{-\alpha-1} \left(\int_{B(0,r)} |f(x)| w_\alpha(x)\, dx \right) dr
\end{aligned}$$

$$= A_f(1) + \alpha \int_0^1 r^{-\alpha-1} A_f(r)\, dr. \tag{3.5}$$

Using (3.1), we obtain

$$\int_{B(0,1)} |f(x)|\, dx \le \kappa_\alpha^\theta \frac{d\theta}{d\theta - \alpha}.$$

Consequently,

$$C_\alpha \le \kappa_\alpha^\theta \frac{d\theta}{d\theta - \alpha}. \tag{3.6}$$

Since

$$d = n + \alpha \qquad \text{and} \qquad \theta = 1 - \frac{1}{p},$$

we have

$$\frac{d\theta}{d\theta - \alpha} = \frac{(n+\alpha)(p-1)}{n(p-1) - \alpha}.$$

Hence,

$$C_\alpha \le \kappa_\alpha^\theta \frac{(n+\alpha)(p-1)}{n(p-1) - \alpha}.$$

We next prove the reverse inequality. Consider

$$f(x) := |x|^{-\frac{d}{p}}. \tag{3.7}$$

A direct computation gives

$$\int_{B(0,1)} f(x)\, dx = \frac{\sigma_{n-1}}{n - \frac{d}{p}}.$$

Moreover, since $w_\alpha(B(r)) = \kappa_\alpha r^d$,

$$\begin{aligned}
|w_\alpha(B(r))|^{\frac{1}{p}-1} \int_{B(r)} |f(x)| w_\alpha(x)\, dx &= (\kappa_\alpha r^d)^{\frac{1}{p}-1} \int_{B(r)} |x|^{-\frac{d}{p}} w_\alpha(x)\, dx \\
&= (\kappa_\alpha r^d)^{\frac{1}{p}-1} \int_{B(r)} |x|^{\alpha - \frac{d}{p}}\, dx \\
&= (\kappa_\alpha r^d)^{\frac{1}{p}-1} \sigma_{n-1} \int_0^r r^{d-1-\frac{d}{p}}\, dr \\
&= (\kappa_\alpha r^d)^{\frac{1}{p}-1} \frac{p\sigma_{n-1}}{d(p-1)} r^{d(1-\frac{1}{p})} \\
&= \kappa_\alpha^{\frac{1}{p}-1} \frac{p\sigma_{n-1}}{d(p-1)}.
\end{aligned}$$

Thus

$$C_\alpha \ge \frac{\sigma_{n-1}}{n - \frac{d}{p}} \div \kappa_\alpha^{\frac{1}{p}-1} \frac{p\sigma_{n-1}}{d(p-1)} = \kappa_\alpha^\theta \frac{(n+\alpha)(p-1)}{n(p-1) - \alpha}.$$

Combining this with the upper estimate completes the proof. $\square$

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