VECTOR-VALUED HARDY–LITTLEWOOD MAXIMAL INEQUALITIES ON BALL BANACH FUNCTION SPACES

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Abstract. The goal of this paper is to characterize the boundedness property of the uncentered Hardy–Littlewood maximal operator on ball Banach function spaces and their Köthe associate spaces in terms of the vector-valued maximal inequality. The main result of this paper supplements the recent results by Nieraeth.

1. Introduction

The goal of this paper is to investigate the boundedness property of operators acting on Banach lattices. Let $\mathcal{M}$ denote the uncentered Hardy–Littlewood maximal operator over axis-parallel cubes in $\mathbb{R}^n$. See (1.2) for the precise definition. For $1 < q < \infty$, we consider whether there is a constant $C_{q,X}$, independent of the positive integer N, such that

$$\left\| \left(\sum_{j=1}^{N} (\mathcal{M} f_j)^q \right)^{\frac{1}{q}} \right\|_X \le C_{q,X} \left\| \left(\sum_{j=1}^{N} |f_j|^q \right)^{\frac{1}{q}} \right\|_X \tag{1.1}$$

for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N$ of measurable functions for which the quantity on the right is finite. Here we use the following notation: Throughout the paper, for a measurable function f we allow $\int_Q |f(x)|\,dx$ to take the value $+\infty$, and we put

$$\mathcal{M} f(x) = \sup_Q \mathbf{1}_Q(x) \langle |f| \rangle_Q, \qquad \langle f \rangle_Q = \frac{1}{|Q|} \int_Q f, \tag{1.2}$$

where the supremum is over all axis-parallel cubes. Here it will be understood that $0 \cdot \infty = 0$. Thus $\mathcal{M} f$ is initially an extended nonnegative measurable function. If f is locally integrable, all the averages in (1.2) are finite.

The constant in (1.1) is required to be independent of N. This is the finite-sequence form of the classical vector-valued maximal inequality of Fefferman and Stein [3]. We also use its weighted form due to Andersen and John [1]. Throughout the paper, a constant in a vector-valued estimate is only required to be finite and independent of N.

Let X be a ball Banach function space (see Definition 2.1) and let X' be its Köthe associate space. Consider the two boundedness assumptions

$$\mathcal{M} : X \to X, \qquad \mathcal{M} : X' \to X'. \tag{1.3}$$

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It is known that these assumptions imply the vector-valued estimate (1.1) for every $q \in (1,\infty)$. This implication follows from [5, Theorem 1.1] by applying the extrapolation theorem there to the weighted vector-valued maximal inequality of Andersen and John [1]. As explained in [5], that extrapolation theorem is a special case of [12, Theorem 4.7 and Remark 4.8].

The relation between (1.3) and sparse averaging operators is also known. Lorist and Nieraeth [9, Lemma 3.4] and Nieraeth [13, Theorem 1.2] proved that (1.3) is equivalent to uniform boundedness of sparse averaging operators. We use this characterization in the proof of Theorem 1.1, after deriving the required uniform sparse bound from (1.1). Lerner obtained another criterion for the boundedness of $\mathcal{M}$ on X' in terms of local quantiles; see [6].

A related result concerns the linearized cube-average operators

$$T_Q f = \langle f\rangle_Q \, \mathbf{1}_Q.$$

Here and below $\mathbf{1}_E$ denotes the indicator function of a set E. Suppose that X is r_0-convex for some $r_0 > 1$. Let $\mathcal{Q}$ denote the countable collection of cubes with rational corners. Nieraeth considers the sequence-valued map

$$\{f_Q\}_{Q\in\mathcal{Q}} \longmapsto \{\langle f_Q\rangle_Q \, \mathbf{1}_Q\}_{Q\in\mathcal{Q}}.$$

Here the norm of a sequence $\{g_Q\}_{Q\in\mathcal{Q}}$ in $X[\ell^r(\mathcal{Q})]$ is

$$\|\{g_Q\}_{Q\in\mathcal{Q}}\|_{X[\ell^r(\mathcal{Q})]} = \left\| \left(\sum_{Q\in\mathcal{Q}} |g_Q|^r \right)^{1/r} \right\|_X .$$

By [13, Theorem 3.21], (1.3) is equivalent to the boundedness of this map on $X[\ell^r(\mathcal{Q})]$ for some $r \in (1, r_0]$. Restricting this bound to sequences supported on a finite set $\mathcal{F} \subset \mathcal{Q}$ gives

$$\left\| \left(\sum_{Q\in\mathcal{F}} |T_Q f_Q|^r \right)^{1/r} \right\|_X \le C \left\| \left(\sum_{Q\in\mathcal{F}} |f_Q|^r \right)^{1/r} \right\|_X \tag{A$_r$}$$

for every finite collection $\mathcal{F}$ of cubes with rational corners and every family $\{f_Q\}_{Q\in\mathcal{F}} \subset X$. The number r and the constant C are independent of $\mathcal{F}$ and of the family $\{f_Q\}_{Q\in\mathcal{F}}$.

Let $\mathcal{F} = \{Q_1, \dots, Q_N\}$ be an arbitrary finite collection of cubes, and let $\{f_j\}_{j=1}^N \subset X$ be an arbitrary family. For every j, choose cubes $Q_j^{(k)}$ with rational corners whose lower-left corners and side lengths converge to those of Q_j. The choices can be made so that the cubes $Q_1^{(k)}, \dots, Q_N^{(k)}$ are pairwise distinct for each k and all the cubes Q_j and $Q_j^{(k)}$ are contained in one fixed bounded cube. Then

$$|Q_j^{(k)} \triangle Q_j| \longrightarrow 0, \qquad |Q_j^{(k)}| \longrightarrow |Q_j|, \qquad \mathbf{1}_{Q_j^{(k)}} \longrightarrow \mathbf{1}_{Q_j} \quad \text{almost everywhere.}$$

Since $X \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$, each f_j is integrable on that fixed bounded cube. Hence the absolute continuity of the integral gives

$$\left| \int_{Q_j^{(k)}} f_j - \int_{Q_j} f_j \right| \le \int_{Q_j^{(k)} \triangle Q_j} |f_j| \longrightarrow 0.$$

Together with $|Q_j^{(k)}| \to |Q_j|$, this yields

$$\langle f_j\rangle_{Q_j^{(k)}} \longrightarrow \langle f_j\rangle_{Q_j} \qquad (1 \le j \le N).$$

Consequently,

$$G_k := \left(\sum_{j=1}^N |T_{Q_j^{(k)}} f_j|^r\right)^{1/r} \longrightarrow G := \left(\sum_{j=1}^N |T_{Q_j} f_j|^r\right)^{1/r} \qquad \text{almost everywhere.}$$

Apply (A_r) to $\{Q_1^{(k)}, \dots, Q_N^{(k)}\}$ and the corresponding functions $\{f_1, \dots, f_N\}$. For $m \in \mathbb{N}$, set

$$H_m = \inf_{k \ge m} G_k.$$

Then $0 \le H_m \uparrow G$ almost everywhere, and $H_m \le G_k$ for all $k \ge m$. By the lattice property of X,

$$\|H_m\|_X \le \inf_{k\ge m} \|G_k\|_X.$$

The Fatou property of X therefore gives

$$\|G\|_X = \sup_{m\in\mathbb{N}} \|H_m\|_X \le \liminf_{k\to\infty} \|G_k\|_X \le C \left\| \left(\sum_{j=1}^N |f_j|^r\right)^{1/r} \right\|_X.$$

Thus (A_r) also holds for arbitrary finite collections of cubes. Nieraeth's proof of the equivalence in [13, Theorem 3.21] between (1.3) and the boundedness of the sequence-valued cube-average map on $X[\ell^r(\mathcal{Q})]$ uses a theorem of Rutsky [16]. For $r = 2$, Nieraeth also proved that the implication from (A_r) to (1.3) holds for order-continuous Banach function spaces with the Fatou property, without a convexity assumption; see [13, Theorem A and Remark 3.22]. Since $|T_Q f_Q| \le \mathcal{M} f_Q$ pointwise, (1.1) with $q = 2$ implies (A_r) with $r = 2$. Therefore Nieraeth's result already gives the converse to (1.1) in the order-continuous case when $q = 2$.

To the best of our knowledge, the present paper removes the convexity and order-continuity assumptions from this converse implication. More precisely, we prove that if (1.1) holds for some $q \in (1,\infty)$ with a constant independent of N, then (1.3) holds without assuming r_0-convexity for any $r_0 > 1$ or order continuity. The main new step is to obtain a uniform scalar sparse bound from (1.1). First, the disjoint sets in a sparse family give a bound for a sparse ℓ^q operator. Next, a binomial iteration changes this ℓ^q bound into a bound for the ordinary sparse averaging operator. The finite dyadic sparse criterion in Proposition 4.2 below then gives (1.3), and in particular the boundedness of $\mathcal{M}$ on X'.

Theorem 1.1 (Vector-valued maximal characterization)**.** *Let X be a ball Banach function space on $\mathbb{R}^n$, and let X' be its Köthe associate space.*

- (*i*) *The following statements are equivalent.*
  - (*a*) *$\mathcal{M} : X \to X$ and $\mathcal{M} : X' \to X'$ are bounded.*
  - (*b*) *There exist $q \in (1,\infty)$ and a constant $C > 0$ such that, for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N$ of measurable functions, the*

finite vector-valued inequality

$$\left\| \left(\sum_{j=1}^{N} (\mathcal{M} f_j)^q \right)^{\frac{1}{q}} \right\|_X \le C \left\| \left(\sum_{j=1}^{N} |f_j|^q \right)^{\frac{1}{q}} \right\|_X \tag{1.4}$$

holds whenever the quantity on the right is finite. The constant C is independent of N and of the family $\{f_j\}_{j=1}^N$.

(c) *For every $q \in (1, \infty)$, there is a constant $C > 0$ such that (1.4) holds for every $N \in \mathbb{N}$, with C independent of N and of the family $\{f_j\}_{j=1}^N$.*

(ii) *The parameter $q \in (0, \infty)$ must satisfy $q > 1$ if there exists a finite constant C satisfying the inequality (1.4) for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N$ of measurable functions. More precisely, for $0 < q \le 1$ such an estimate fails on every ball quasi-Banach function space on $\mathbb{R}^n$.*

(iii) *Fix $1 < q < \infty$. Suppose that the inequality (1.4), with this value of q and constant C, holds for every $N \in \mathbb{N}$. Then, for every sequence $\{f_j\}_{j=1}^\infty$ satisfying*

$$\left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \in X,$$

one has

$$\left\| \left(\sum_{j=1}^{\infty} (\mathcal{M} f_j)^q \right)^{\frac{1}{q}} \right\|_X \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_X .$$

Conversely, this estimate for all sequences implies (1.4) by taking $f_j = 0$ for $j > N$, with the same constant C.

(iv) *The following statements are equivalent. The constant C may be chosen to be the same in (a)–(c).*

(a) *There is a constant $C > 0$ such that*

$$\|\mathcal{M} f\|_X \le C \|f\|_X \qquad (f \in X).$$

(b) *There is a constant $C > 0$ such that, for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N$ satisfying $\max\limits_{1 \le j \le N} |f_j| \in X$,*

$$\left\| \max_{1 \le j \le N} \mathcal{M} f_j \right\|_X \le C \left\| \max_{1 \le j \le N} |f_j| \right\|_X . \tag{1.5}$$

(c) *There is a constant $C > 0$ such that, for every sequence $\{f_j\}_{j=1}^\infty$ with $\sup\limits_{j \ge 1} |f_j| \in X$,*

$$\left\| \sup_{j \in \mathbb{N}} \mathcal{M} f_j \right\|_X \le C \left\| \sup_{j \in \mathbb{N}} |f_j| \right\|_X .$$

These equivalent conditions need not imply that $\mathcal{M}$ is bounded on X'.

The characterization in Theorem 1.1 is also useful for concrete ball Banach function spaces; see, for example, [19, Theorem 4.12].

Let $s \in (1,\infty)$ and let X be a Banach lattice. We say that X is *s-concave* if there exists a constant $C>0$ such that, for every finite family $\{x_j\}_{j=1}^N \subset X$,

$$\left(\sum_{j=1}^N \|x_j\|_X^s\right)^{1/s} \le C \left\| \left(\sum_{j=1}^N |x_j|^s\right)^{1/s} \right\|_X .$$

Lerner recently proved that, if X is an s-concave Banach function space for some $s \in (1,\infty)$ and $\mathcal{M}$ is bounded on X, then $\mathcal{M}$ is also bounded on X'; see [8, Corollary 1.4]. Our result is different in nature. Without assuming r_0-convexity for any $r_0 > 1$, s-concavity, or order continuity, Theorem 1.1 (i) characterizes the boundedness of $\mathcal{M}$ on X and X' by the finite vector-valued maximal inequality (1.4). The principal new contribution is the converse implication: the finite vector-valued inequality itself implies the boundedness of $\mathcal{M}$ on X', without any additional convexity, concavity, or order-continuity assumption.

We briefly describe the converse argument. Let $\mathscr{S}$ be a finite sparse subcollection of a dyadic grid and define

$$\mathcal{T}_{\mathscr{S},q} f = \left(\sum_{Q\in\mathscr{S}} \langle |f|\rangle_Q^q \,\mathbf{1}_Q\right)^{\frac{1}{q}}, \qquad \mathcal{A}_{\mathscr{S}} f = \sum_{Q\in\mathscr{S}} \langle |f|\rangle_Q \,\mathbf{1}_Q .$$

The estimate in (1.4) first gives a bound for $\mathcal{T}_{\mathscr{S},q}$ that is independent of $\mathscr{S}$. The binomial iteration in Theorem 3.2, combined with this bound, gives the uniform bound for $\mathcal{A}_{\mathscr{S}}$ stated in Corollary 3.3. The finite dyadic sparse criterion in Proposition 4.2 then yields the boundedness of $\mathcal{M}$ on X'.

To obtain the result in the quasi-Banach setting, let Y be a ball quasi-Banach function space and let $r>0$. Following the notation in [12], define

$$Y^r = \{h : \ |h|^{1/r} \in Y\}, \qquad \|h\|_{Y^r} = \left\| |h|^{1/r} \right\|_Y^r . \tag{1.6}$$

The following is a standard concavification fact; see [12, Definition 2.7 and the discussion following it].

Lemma 1.2 (Standard concavification facts)**.** *Let Y be an r-convex ball quasi-Banach function space with convexity constant one, and put $X = Y^r$. Then X is a Banach lattice with the Fatou property and contains the indicator of every ball. Moreover,*

$$\|H\|_Y^r = \|H^r\|_X \qquad (H \ge 0). \tag{1.7}$$

We assume that Y is r-convex with constant one (see Definition 5.3). Under this assumption, the quantity in (1.6) is a norm. Put $X = Y^r$. For a finite collection $\mathscr{S}$ of cubes and $f \in L^r_{\mathrm{loc}}(\mathbb{R}^n)$, define

$$\mathcal{M}_r f = \left[\mathcal{M}(|f|^r)\right]^{1/r}, \qquad \mathcal{A}_{\mathscr{S},r} f = \left[\mathcal{A}_{\mathscr{S}}(|f|^r)\right]^{1/r}. \tag{1.8}$$

The following theorem is a direct rescaling consequence of Theorem 1.1.

Theorem 1.3 (Powered vector-valued maximal characterization)**.** *Let Y be an r-convex ball quasi-Banach function space with convexity constant one for some $r>0$, and put $X = Y^r$ with the norm in* (1.6).

(i) *The following statements are equivalent.*

  (a) $\mathcal{M} : X \to X$ *and* $\mathcal{M} : X' \to X'$ *are bounded.*

(*b*) *There exist $q > r$ and a constant $C > 0$ such that, for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N \subset L^r_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\left\| \left(\sum_{j=1}^N (\mathcal{M}_r f_j)^q \right)^{\frac{1}{q}} \right\|_Y \le C \left\| \left(\sum_{j=1}^N |f_j|^q \right)^{\frac{1}{q}} \right\|_Y \tag{1.9}$$

whenever the quantity on the right is finite. The constant C is independent of N and of the family $\{f_j\}_{j=1}^N$.

(*c*) *For every $q > r$, there is a constant $C > 0$ such that* (1.9) *holds for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N \subset L^r_{\mathrm{loc}}(\mathbb{R}^n)$ for which the quantity on the right is finite, with C independent of N and of the family.*

(*d*) *There is a constant $C_{\mathrm{sp}} > 0$ such that*

$$\|\mathcal{A}_{\mathscr{S},r} f\|_Y \le C_{\mathrm{sp}} \|f\|_Y \tag{1.10}$$

for every finite $1/2$-sparse subcollection $\mathscr{S}$ of every dyadic grid and every $f \in Y \cap L^r_{\mathrm{loc}}(\mathbb{R}^n)$.

Each of these conditions implies $Y \subset L^r_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, for $q > r$, a constant C works in (1.9) *if and only if C^r works in* (1.4) *on X with q/r in place of q.*

(*ii*) *Let $q > 0$. Suppose that there is a finite constant C such that* (1.9) *holds for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N \subset L^r_{\mathrm{loc}}(\mathbb{R}^n)$ for which the quantity on the right is finite, with C independent of N and of the family. Then necessarily $q > r$ and $Y \subset L^r_{\mathrm{loc}}(\mathbb{R}^n)$. In particular, no estimate in* (1.9) *can hold for every $N \in \mathbb{N}$ when $0 < q \le r$.*

(*iii*) *Fix $q > r$ and suppose that* (1.9)*, with constant C, holds for every $N \in \mathbb{N}$. Then, for every sequence $\{f_j\}_{j=1}^\infty$ satisfying $\left(\sum_{j=1}^\infty |f_j|^q\right)^{\frac{1}{q}} \in Y$, one has*

$$\left\| \left(\sum_{j=1}^\infty (\mathcal{M}_r f_j)^q \right)^{\frac{1}{q}} \right\|_Y \le C \left\| \left(\sum_{j=1}^\infty |f_j|^q \right)^{\frac{1}{q}} \right\|_Y .$$

Conversely, this estimate for all sequences implies (1.9) *by taking $f_j = 0$ for $j > N$, with the same constant C.*

(*iv*) *Suppose that $Y \subset L^r_{\mathrm{loc}}(\mathbb{R}^n)$. The following statements are equivalent. The constant C may be chosen to be the same in* (*a*)–(*c*).

(*a*) *There is a constant $C > 0$ such that*

$$\|\mathcal{M}_r f\|_Y \le C \|f\|_Y \qquad (f \in Y).$$

(*b*) *There is a constant $C > 0$ such that, for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N$ satisfying $\max\limits_{1 \le j \le N} |f_j| \in Y$,*

$$\left\| \max_{1 \le j \le N} \mathcal{M}_r f_j \right\|_Y \le C \left\| \max_{1 \le j \le N} |f_j| \right\|_Y .$$

(c) *There is a constant $C > 0$ such that, for every sequence $\{f_j\}_{j=1}^{\infty}$ with $\sup_{j\in\mathbb{N}} |f_j| \in Y$,*

$$\left\| \sup_{j\in\mathbb{N}} \mathcal{M}_r f_j \right\|_Y \le C \left\| \sup_{j\in\mathbb{N}} |f_j| \right\|_Y .$$

To see the relation between the two theorems, let $f_j \in L^r_{\mathrm{loc}}(\mathbb{R}^n)$, set $h_j = |f_j|^r$, and put $a = q/r$. Then

$$\left\| \left(\sum_{j=1}^N (\mathcal{M}_r f_j)^q \right)^{\frac{1}{q}} \right\|_Y^r = \left\| \left(\sum_{j=1}^N (\mathcal{M} h_j)^a \right)^{1/a} \right\|_X ,$$

$$\left\| \left(\sum_{j=1}^N |f_j|^q \right)^{\frac{1}{q}} \right\|_Y^r = \left\| \left(\sum_{j=1}^N |h_j|^a \right)^{1/a} \right\|_X .$$

Thus, after raising both sides to the power r, (1.9) becomes (1.4) on X, with $a = q/r$ in place of q. Similarly,

$$\|\mathcal{A}_{\mathscr{S},r} f\|_Y^r = \|\mathcal{A}_{\mathscr{S}}(|f|^r)\|_X .$$

These identities reduce the proof of Theorem 1.3 to Theorem 1.1, after the completeness and local-integrability properties of X have been verified.

The paper is organized as follows. Section 2 contains the definitions and known results used throughout the paper. Section 3 derives uniform sparse bounds from (1.4) under the assumption that this estimate holds for some $q \in (1, \infty)$. Section 4 recalls the finite dyadic sparse criterion needed here and then proves Theorem 1.1. Section 5 is devoted to the proof of Theorem 1.3.

Our results also extend to the local Hardy–Littlewood maximal operator by using Theorem 3.2. Finally, Section 6 establishes the local versions of Theorems 1.1 and 1.3 by applying the results of Section 3.

2. Ball function spaces, maximal operators, and sparse families

Section 2 collects preliminary facts.

2.1. **The function-space setting.** Fix $n \in \mathbb{N}$. We work on $\mathbb{R}^n$ with Lebesgue measure. For $1 < a < \infty$, write $a' = a/(a-1)$. The operators used below are positively homogeneous but need not be linear, and they may initially have the value $+\infty$. If T assigns an extended measurable function Tf to every element f of a quasi-normed function space Z, we set

$$\|T\|_{Z\to Z} := \sup_{0\ne f\in Z} \frac{\|Tf\|_Z}{\|f\|_Z} \in [0, \infty], \tag{2.1}$$

where the quotient is interpreted as $+\infty$ when $Tf \notin Z$. Thus $T : Z \to Z$ is bounded exactly when the quantity in (2.1) is finite.

We invoke the notion of ball Banach function spaces from [4].

Definition 2.1. Let X be a normed linear space of measurable functions on $\mathbb{R}^n$, where functions that agree almost everywhere are identified. We call X a *normed function lattice* if it satisfies property (i) below. We call X a *ball Banach function space* if it is complete and satisfies properties (i)–(iv).

(i) If $f \in X$ and $|g| \le |f|$ almost everywhere, then $g \in X$ and $\|g\|_X \le \|f\|_X$.

(ii) If $f_k \in X$, $0 \le f_k \uparrow f$ almost everywhere, and $\sup\limits_k \|f_k\|_X < \infty$, then $f \in X$ and

$$\|f\|_X = \sup_k \|f_k\|_X .$$

(iii) For every ball $B \subset \mathbb{R}^n$, one has $\mathbf{1}_B \in X$.

(iv) For every ball $B \subset \mathbb{R}^n$, there is a constant C_B such that

$$\int_B |f(x)|\,dx \le C_B \|f\|_X \qquad (f \in X).$$

We also present the definition of ball quasi-Banach function spaces.

Definition 2.2. A quasi-Banach space Y of measurable functions on $\mathbb{R}^n$ is called a *ball quasi-Banach function space* if the following properties hold.

(i) If $f \in Y$ and $|g| \le |f|$ almost everywhere, then $g \in Y$ and $\|g\|_Y \le \|f\|_Y$.

(ii) If $f_k \in Y$, $0 \le f_k \uparrow f$ almost everywhere, and $\sup\limits_k \|f_k\|_Y < \infty$, then $f \in Y$ and

$$\|f\|_Y = \sup_k \|f_k\|_Y .$$

(iii) For every ball $B \subset \mathbb{R}^n$, one has $\mathbf{1}_B \in Y$.

Let X be a normed function lattice that contains the indicator of every bounded measurable set. Its Köthe associate space is

$$X' = \left\{ g : \ \|g\|_{X'} := \sup_{\|f\|_X \le 1} \int_{\mathbb{R}^n} |f(x)g(x)|\,dx < \infty \right\}. \tag{2.2}$$

Since $\mathbf{1}_E \in X$ for every bounded measurable set E, each $g \in X'$ is locally integrable. More precisely,

$$\int_E |g(x)|\,dx \le \|\mathbf{1}_E\|_X \|g\|_{X'} < \infty.$$

The generalized Hölder inequality is

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le \|f\|_X \|g\|_{X'} . \tag{2.3}$$

Lemma 2.3. *Let X be a normed function lattice and let X' be defined by* (2.2)*. If $0 \le g_k \uparrow g$ almost everywhere and $\sup\limits_k \|g_k\|_{X'} < \infty$, then $g \in X'$ and*

$$\|g\|_{X'} = \sup_k \|g_k\|_{X'} .$$

In particular, X' has the Fatou property.

Proof. Put $C = \sup\limits_k \|g_k\|_{X'}$. For every $f \in X$, the monotone convergence theorem and (2.3) give

$$\int_{\mathbb{R}^n} |f(x)|g(x)\,dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} |f(x)|g_k(x)\,dx \le C \|f\|_X .$$

Thus $g \in X'$ and $\|g\|_{X'} \le C$. The reverse inequality follows from $g_k \le g$ and the lattice property of X'. □

We collect some fundamental properties of ball Banach function spaces.

Lemma 2.4. *Let X be a ball Banach function space.*

(i) *If E is a bounded measurable set, then $\mathbf{1}_E \in X$.*
(ii) *Both X and X' are continuously embedded into $L^1(E)$ for every bounded measurable set E.*
(iii) *If E has positive measure, then there is a measurable subset $F \subset E$ such that $|F| > 0$ and $\mathbf{1}_F \in X$.*

Proof. Let E be bounded and choose a ball B containing E. Since $\mathbf{1}_E \le \mathbf{1}_B$, part (i) follows from Definition 2.1(i),(iii). For $f \in X$,

$$\int_E |f(x)|\,dx \le \int_B |f(x)|\,dx \le C_B \|f\|_X ,$$

so the embedding for X in part (ii) follows from Definition 2.1(iv). Let us establish that X' is embedded into $L^1(E)$ in part (ii). If $g \in X'$, then

$$\int_E |g(x)|\,dx \le \|\mathbf{1}_E\|_X \|g\|_{X'} ,$$

which proves the estimate for X'. Finally, if $|E| > 0$, then $|E \cap B(0,R)| > 0$ for some $R > 0$. Part (i), applied to $F = E \cap B(0,R)$, proves part (iii). □

We recall [10, Theorem 3.6] and [20, Theorem 71.1].

Lemma 2.5. [10, Theorem 3.6] and [20, Theorem 71.1] *Let X be a ball Banach function space. Then X' is a ball Banach function space, both X and X' are continuously embedded into $L^1(E)$ for every bounded measurable set E, and $X'' = X$ with equality of norms.*

We do not recall the proof but we make some comments. Part (iii) of Lemma 2.4 is precisely the saturation property used in the standard theory of Banach function spaces. Together with the Fatou property in Definition 2.1(ii), the Lorentz–Luxemburg theorem gives $X'' = X$ with equality of norms.

2.2. **Maximal and vector-valued operators.** The extrapolation theorem in [5] is stated for a centered maximal operator over Euclidean balls $\mathcal{M}_{\text{center}}$. The operator $\mathcal{M}_{\text{center}}$ and the operator in (1.2) dominate each other pointwise up to constants depending only on n. Consequently, the boundedness on X and X', and the estimates obtained by extrapolation, are unchanged apart from dimensional constants.

A weight is a locally integrable function w such that $0 < w < \infty$ almost everywhere. For $1 < p < \infty$, the Muckenhoupt class A_p consists of the weights satisfying

$$[w]_{A_p} = \sup_Q \langle w \rangle_Q \left(\left\langle w^{-1/(p-1)} \right\rangle_Q \right)^{p-1} < \infty. \tag{2.4}$$

We write

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx \right)^{1/p} .$$

When $w \equiv 1$, we simply write $L^p = L^p(\mathbb{R}^n)$ and $\|f\|_{L^p}$ in place of $L^p(w)$ and $\|f\|_{L^p(w)}$, respectively.

The following weighted vector-valued estimate is due to Andersen and John [1].

Theorem 2.6. *Let $1 < p, q < \infty$. There is an increasing function $N_{n,p,q} : [1, \infty) \to (0, \infty)$ such that, for every $w \in A_p$, every $N \in \mathbb{N}$, and every finite family $\{f_j\}_{j=1}^N$ of measurable functions for which the quantity on the right-hand side of the following inequality is finite,*

$$\left\| \left(\sum_{j=1}^{N} (\mathcal{M} f_j)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \le N_{n,p,q}([w]_{A_p}) \left\| \left(\sum_{j=1}^{N} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} . \tag{2.5}$$

The function $N_{n,p,q}$ is independent of N.

We use the following known extrapolation theorem in the form stated in [5, Theorem 1.1]. That paper also explains that this statement is a special case of [12, Theorem 4.7 and Remark 4.8].

Theorem 2.7. *Let X be a ball Banach function space such that $\mathcal{M}$ is bounded on both X and X'. Fix $p_0 \in (1, \infty)$, and let $N : [1, \infty) \to (0, \infty)$ be increasing. Let $\mathcal{F}$ be a collection of pairs (f, g) of nonnegative measurable functions. Assume that, for every $w \in A_{p_0}$ and every $(f, g) \in \mathcal{F}$ with $g \in L^{p_0}(w)$, one has $f \in L^{p_0}(w)$ and*

$$\|f\|_{L^{p_0}(w)} \le N([w]_{A_{p_0}}) \|g\|_{L^{p_0}(w)} .$$

Then there is a constant $C = C(X, p_0, N)$ such that, for every $(f, g) \in \mathcal{F}$ with $g \in X$, one has $f \in X$ and

$$\|f\|_X \le C \|g\|_X .$$

The constant C is independent of $\mathcal{F}$ and of the particular pair (f, g).

The weighted spaces in [12] are written in the multiplier-weight convention $L^{p_0}_v = \{H : vH \in L^{p_0}\}$. Taking $v = w^{1/p_0}$ gives

$$\|H\|_{L^{p_0}_v} = \|H\|_{L^{p_0}(w)}, \qquad [v]_{p_0} = [w]_{A_{p_0}}^{1/p_0}.$$

Accordingly, [12, Theorem 4.7], applied with $\phi(t) = N(t^{p_0})$, yields the conclusion of Theorem 2.7 for each pair $(f, g) \in \mathcal{F}$. Remark 4.8 in [12] shows that the constant may be chosen independently of the particular pair. Hence a specific constant works uniformly for all $(f, g) \in \mathcal{F}$ with $g \in X$.

Let Z be a ball quasi-Banach function space and let $0 < q < \infty$. Given a constant $C > 0$, we say that the finite vector-valued estimate holds on Z with constant C if, for every $N \in \mathbb{N}$,

$$\left\| \left(\sum_{j=1}^{N} (\mathcal{M} f_j)^q \right)^{\frac{1}{q}} \right\|_Z \le C \left\| \left(\sum_{j=1}^{N} |f_j|^q \right)^{\frac{1}{q}} \right\|_Z \tag{2.6}$$

whenever the quantity on the right is finite. The constant C is independent of N and of the family. Here $\mathcal{M} f_j$ is the extended-valued function in (1.2). Whenever (2.6) holds with a finite constant C, taking $N = 1$ gives

$$\|\mathcal{M} f\|_Z \le C \|f\|_Z \qquad (f \in Z). \tag{2.7}$$

Hence $\mathcal{M} f$ is finite almost everywhere for every $f \in Z$. If $\|f\|_{L^1(Q)} = \infty$ for some cube Q, then $\mathcal{M} f = \infty$ on Q, which is impossible. Therefore every element of Z is locally integrable whenever (2.6) holds with a finite constant independent of N.

Lemma 2.8. *Let X be a ball quasi-Banach function space, and let $0 < q < \infty$. Assume that* (2.6)*, with constant C, holds for every $N \in \mathbb{N}$. Then $X \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and*

$$\left\| \left(\sum_{j=1}^{\infty} (\mathcal{M} f_j)^q \right)^{1/q} \right\|_X \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{1/q} \right\|_X \tag{2.8}$$

for every sequence for which the quantity on the right belongs to X.

Proof. If $\|f\|_{L^1(Q)} = \infty$ for some cube Q, then

$$\mathcal{M} f = \infty \tag{2.9}$$

on Q, which is impossible. Hence $\|f\|_{L^1(Q)} < \infty$ for every such cube. Since every bounded set E is contained in a finite union of these cubes, say $E \subset \bigcup_{\nu=1}^{m} Q_\nu$, one has

$$\int_E |f(x)|\, dx \le \sum_{\nu=1}^{m} \int_{Q_\nu} |f(x)|\, dx < \infty.$$

Thus $X \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Next, we justify the passage from inequality (2.6) for finitely many functions to inequality (2.8) for infinite sequences. Since $|f_j| \le F$, Definition 2.2(i) gives $f_j \in X$, and the assumed local embedding shows that each f_j is locally integrable. For $N \ge 1$, put

$$F_N = \left(\sum_{j=1}^{N} |f_j|^q \right)^{\frac{1}{q}}, \qquad G_N = \left(\sum_{j=1}^{N} (\mathcal{M} f_j)^q \right)^{\frac{1}{q}}.$$

Also set

$$F = \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}}, \qquad G = \left(\sum_{j=1}^{\infty} (\mathcal{M} f_j)^q \right)^{\frac{1}{q}}.$$

Then $F_N \le F$, and therefore

$$\|G_N\|_X \le C \|F_N\|_X \le C \|F\|_X .$$

Since $G_N \uparrow G$ pointwise, Definition 2.2(ii) gives $G \in X$ and

$$\|G\|_X = \sup_{N \in \mathbb{N}} \|G_N\|_X \le C \|F\|_X .$$

□

We follow [18, p. 75, 5.2] to prove the following proposition.

Proposition 2.9. *Let Y be a ball quasi-Banach function space on $\mathbb{R}^n$, and let $0 < q \le 1$. There is no constant C such that*

$$\left\| \left(\sum_{j=1}^{N} (\mathcal{M} f_j)^q \right)^{\frac{1}{q}} \right\|_Y \le C \left\| \left(\sum_{j=1}^{N} |f_j|^q \right)^{\frac{1}{q}} \right\|_Y \tag{2.10}$$

for every $N \in \mathbb{N}$ and all measurable functions for which the quantity on the right is finite.

Proof. Suppose, to the contrary, that (2.10) holds with a constant C independent of N. Choose $L = 8M$ with $M \in \mathbb{N}$, and set

$$Q = [0,1)^n.$$

The L^n half-open cubes

$$Q_\nu = \prod_{i=1}^n \left[\frac{\nu_i - 1}{L}, \frac{\nu_i}{L}\right), \qquad \nu = (\nu_1, \dots, \nu_n) \in \{1, \dots, L\}^n,$$

are pairwise disjoint and their union is exactly Q. Put $f_\nu = \mathbf{1}_{Q_\nu}$. Thus we have a family of $N = L^n$ functions, and

$$\left(\sum_{\nu \in \{1,2,\dots,L\}^n} |f_\nu|^q\right)^{\frac{1}{q}} = \mathbf{1}_Q \quad \text{pointwise on } \mathbb{R}^n. \tag{2.11}$$

Let $E = (3/8, 5/8)^n$. For each $x \in E$, define

$$\kappa_i = \lfloor Lx_i \rfloor + 1 \qquad (1 \le i \le n), \qquad \kappa = (\kappa_1, \dots, \kappa_n).$$

Since $0 < x_i < 1$, one has $\kappa_i \in \{1, \dots, L\}$ and

$$\frac{\kappa_i - 1}{L} \le x_i < \frac{\kappa_i}{L}.$$

Hence $x \in Q_\kappa$. There is only one index κ such that $x \in Q_\kappa$ because the cubes Q_ν form a disjoint partition of Q. Moreover,

$$3M < Lx_i < 5M,$$

and therefore

$$3M + 1 \le \kappa_i \le 5M. \tag{2.12}$$

For each $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \{1, \dots, L\}^n$, define the lattice distance by

$$d(\nu, \kappa) = \max_{1 \le i \le n} |\nu_i - \kappa_i|.$$

We now construct explicitly a cube containing both x and Q_ν. For the i-th coordinate, write

$$I_{\nu_i} = \left[\frac{\nu_i - 1}{L}, \frac{\nu_i}{L}\right), \qquad x_i \in I_{\kappa_i},$$

and let

$$a_i = \min\left\{x_i, \frac{\nu_i - 1}{L}\right\}, \qquad b_i = \max\left\{x_i, \frac{\nu_i}{L}\right\}.$$

Then both the point x_i and the interval I_{ν_i} are contained in $[a_i, b_i]$. Moreover,

$$b_i - a_i = \begin{cases} \dfrac{\nu_i}{L} - x_i, & \nu_i > \kappa_i (\iff \nu_i \ge \kappa_i + 1), \\ \dfrac{1}{L}, & \nu_i = \kappa_i, \\ x_i - \dfrac{\nu_i - 1}{L}, & \nu_i < \kappa_i (\iff \nu_i \le \kappa_i - 1). \end{cases}$$

Recall that $\kappa_i = \lfloor Lx_i \rfloor + 1$. Therefore,

$$\frac{\kappa_i - 1}{L} \le x_i < \frac{\kappa_i}{L}.$$

Each of these three cases gives

$$b_i - a_i \le \frac{|\nu_i - \kappa_i| + 1}{L} \le \frac{d(\nu,\kappa) + 1}{L}. \tag{2.13}$$

Set

$$\delta = \frac{d(\nu,\kappa)+2}{L} \quad \text{and} \quad c_i = a_i - \frac{\delta - (b_i - a_i)}{2} = \frac{a_i + b_i - \delta}{2}.$$

Estimate (2.13) implies $b_i - a_i < \delta$ and $c_i < a_i$, and hence

$$[a_i, b_i] \subset (c_i, c_i + \delta).$$

Therefore the axis-parallel cube

$$R_{\nu,x} = \prod_{i=1}^{n} [c_i, c_i + \delta)$$

contains x and the whole cube Q_ν, and its side length is

$$\ell(R_{\nu,x}) = \delta = \frac{d(\nu,\kappa)+2}{L}.$$

Using this cube in the definition of $\mathcal{M}$, we obtain

$$\begin{aligned} \mathcal{M} f_\nu(x) &\ge \frac{1}{|R_{\nu,x}|} \int_{R_{\nu,x}} \mathbf{1}_{Q_\nu}(y)\, dy = \frac{|Q_\nu|}{|R_{\nu,x}|} \\ &\ge \frac{L^{-n}}{\big[(d(\nu,\kappa)+2)/L\big]^n} = \big(d(\nu,\kappa)+2\big)^{-n}. \end{aligned} \tag{2.14}$$

Let $s \in \mathbb{N}$. We now count the indices ν at a fixed lattice distance s from κ. Set

$$\Gamma_s(\kappa) = \{\nu \in \mathbb{Z}^n : \ d(\nu,\kappa) = s\}.$$

If $1 \le s \le M = L/8$, then since

$$2M + 1 \le \kappa_i - s \le \nu_i \le \kappa_i + s \le 6M < L$$

due to (2.12), every $\nu \in \Gamma_s(\kappa)$ belongs to $\{1,\dots,L\}^n$. The number of integer vectors satisfying $d(\nu,\kappa) \le s$ is $(2s+1)^n$, because each coordinate can take one of the $2s+1$ values

$$\kappa_i - s, \ \kappa_i - s + 1, \ \dots, \ \kappa_i + s.$$

Similarly, the number satisfying $d(\nu,\kappa) \le s-1$ is $(2s-1)^n$. Therefore

$$\#\Gamma_s(\kappa) = (2s+1)^n - (2s-1)^n.$$

The subtraction removes the vectors with $d(\nu,\kappa) \le s-1$ from those with $d(\nu,\kappa) \le s$, so the remaining vectors are exactly the elements of $\Gamma_s(\kappa)$. Moreover,

$$(2s+1)^n - (2s-1)^n = \int_{2s-1}^{2s+1} n t^{n-1}\, dt \ge 2n(2s-1)^{n-1} \ge 2n s^{n-1}.$$

Thus

$$\#\Gamma_s(\kappa) \ge 2n s^{n-1} \qquad (1 \le s \le M).$$

Remark that $s + 2 \le 3s$ for $s \in \mathbb{N}$. If $q = 1$, then (2.14) and the preceding count give

$$\sum_{\nu \in \{1,2,\dots,L\}^n} \mathcal{M} f_\nu(x) \ge \sum_{s=1}^{M} \sum_{\nu \in \Gamma_s(\kappa)} (s+2)^{-n} \ge \sum_{s=1}^{M} \frac{2n s^{n-1}}{(s+2)^n} \ge \sum_{s=1}^{M} \frac{2n}{3^n s}.$$

Furthermore, using $L = 8M$, we have

$$\sum_{s=1}^{M} \frac{2n}{3^n s} \geq \frac{2n}{3^n} \log(M+1) = \frac{2n}{4 \cdot 3^n} \log(M+1)^4 \geq \frac{2n}{4 \cdot 3^n} \log 8M = \frac{2n}{4 \cdot 3^n} \log L.$$

Hence

$$\sum_{\nu \in \{1,2,\ldots,L\}^n} \mathcal{M} f_\nu(x) \geq \frac{2n}{4 \cdot 3^n} \log L.$$

Let $0 < q < 1$. Again using (2.14), we have

$$\sum_{\nu \in \{1,2,\ldots,L\}^n} (\mathcal{M} f_\nu(x))^q \geq \sum_{s=1}^{M} \frac{\sharp \Gamma_s(\kappa)}{(s+2)^{nq}} \geq \sum_{s=1}^{M} \frac{2n s^{n-1}}{(s+2)^{nq}} \geq \sum_{s=1}^{M} \frac{2n}{3^{nq}} s^{n(1-q)-1}.$$

Hence

$$\sum_{\nu \in \{1,2,\ldots,L\}^n} (\mathcal{M} f_\nu(x))^q \geq \sum_{s=1}^{M} \frac{2n}{3^{nq}} s^{n(1-q)-1} \geq c_{n,q} L^{n(1-q)}.$$

The last inequality is the elementary power-sum estimate; for example, it follows by summing only over the integers s with $M/2 \leq s \leq M$.

Consequently, for every $x \in E$,

$$\left(\sum_{\nu \in \{1,2,\ldots,L\}^n} (\mathcal{M} f_\nu(x))^q \right)^{\frac{1}{q}} \geq \begin{cases} c_n \log L, & q = 1, \\ c_{n,q} L^{n(1/q-1)}, & 0 < q < 1. \end{cases}$$

The sets E and Q are bounded, so each is contained in a ball. Definition 2.2(i),(iii) therefore gives $\mathbf{1}_E, \mathbf{1}_Q \in Y$. Hence (2.11) and the lattice property give

$$\left\| \left(\sum_{\nu \in \{1,2,\ldots,L\}^n} (\mathcal{M} f_\nu)^q \right)^{\frac{1}{q}} \right\|_Y \geq \begin{cases} c_n (\log L) \left\| \mathbf{1}_E \right\|_Y, & q = 1, \\ c_{n,q} L^{n(1/q-1)} \left\| \mathbf{1}_E \right\|_Y, & 0 < q < 1, \end{cases}$$

whereas the right-hand side of (2.10), for this family, is $C \left\| \mathbf{1}_Q \right\|_Y$. The latter is independent of L, while either lower bound tends to infinity as $L \to \infty$. This contradiction proves the proposition. $\square$

Proposition 2.10. *Let Y be a ball quasi-Banach function space. The following statements are equivalent. The constant C may be chosen to be the same in (i)–(iii).*

(i) *There is a constant $C > 0$ such that*

$$\left\| \mathcal{M} f \right\|_Y \leq C \left\| f \right\|_Y \qquad (f \in Y).$$

(ii) *There is a constant $C > 0$ such that, for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N$ satisfying $\max_{1 \leq j \leq N} |f_j| \in Y$,*

$$\left\| \max_{1 \leq j \leq N} \mathcal{M} f_j \right\|_Y \leq C \left\| \max_{1 \leq j \leq N} |f_j| \right\|_Y.$$

(iii) *There is a constant $C > 0$ such that*

$$\left\| \sup_{j \in \mathbb{N}} \mathcal{M} f_j \right\|_Y \leq C \left\| \sup_{j \in \mathbb{N}} |f_j| \right\|_Y$$

whenever $\sup_{j\in\mathbb{N}} |f_j| \in Y$.

Moreover, even when Y is a ball Banach function space, these equivalent conditions need not imply that $\mathcal{M}$ is bounded on Y'.

Proof. For any finite or countable sequence,

$$\sup_j \mathcal{M} f_j \le \mathcal{M}\left(\sup_j |f_j|\right)$$

pointwise in $[0,\infty]$. Thus the scalar estimate in (i), with constant C, implies both (ii) and (iii) with the same constant. Conversely, applying either (ii) or (iii) to a family with one nonzero term gives (i), again with the same constant.

For the final assertion, take $Y = L^\infty(\mathbb{R}^n)$. Then $\mathcal{M}$ is bounded on Y, whereas $Y' = L^1(\mathbb{R}^n)$ and $\mathcal{M}$ is not bounded on $L^1(\mathbb{R}^n)$; see, for example, [17, Chapter I]. □

2.3. **Dyadic grids and sparse families.** A half-open cube is a set of the form

$$Q = \prod_{i=1}^n [a_i, a_i+\ell), \qquad a_i \in \mathbb{R}, \quad \ell > 0.$$

A collection $\mathscr{D}$ of half-open cubes is called a *dyadic grid* if it can be written as

$$\mathscr{D} = \bigcup_{k\in\mathbb{Z}} \mathscr{D}_k$$

and satisfies the following properties.

(i) For every $k \in \mathbb{Z}$, $\mathscr{D}_k$ is a partition of $\mathbb{R}^n$ into half-open cubes of side length 2^{-k}.

(ii) Every $Q \in \mathscr{D}_k$ is contained in a unique cube $Q^{(1)} \in \mathscr{D}_{k-1}$ and is the disjoint union of exactly 2^n cubes in $\mathscr{D}_{k+1}$.

(iii) If $P, Q \in \mathscr{D}$, then $P \cap Q = \varnothing$, $P \subset Q$, or $Q \subset P$.

A collection $\mathscr{S}$ of cubes is called η-sparse, where $0 < \eta < 1$, if there are measurable sets $E_Q \subset Q$, $Q \in \mathscr{S}$, such that

$$|E_Q| \ge \eta |Q| \quad \text{and} \quad E_Q \cap E_P = \varnothing \quad (Q \ne P). \tag{2.15}$$

If $\mathscr{S}$ is contained in a dyadic grid, it is called a dyadic sparse collection. For every finite collection $\mathscr{S}$ of cubes and every $0 < q < \infty$, define

$$\mathcal{T}_{\mathscr{S},q} f = \left(\sum_{Q\in\mathscr{S}} \langle |f| \rangle_Q^q \, \mathbf{1}_Q\right)^{\frac{1}{q}}, \tag{2.16}$$

Every sparse collection of cubes in $\mathbb{R}^n$ is at most countable. For an arbitrary sparse collection $\mathscr{S}$, define

$$\mathcal{A}_{\mathscr{S}} f(x) := \sup_{\substack{\mathscr{F}\subset\mathscr{S} \\ \mathscr{F} \text{ finite}}} \sum_{Q\in\mathscr{F}} \langle |f| \rangle_Q \, \mathbf{1}_Q(x) \in [0,\infty].$$

Since all the summands are nonnegative, this agrees with the pointwise sum

$$\mathcal{A}_{\mathscr{S}} f(x) = \sum_{Q\in\mathscr{S}} \langle |f| \rangle_Q \, \mathbf{1}_Q(x), \tag{2.17}$$

independently of the enumeration of $\mathscr{S}$. Note that the operator $\mathcal{T}_{\mathscr{S},q}$ will only be used below when $\mathscr{S}$ is finite.

For a dyadic grid $\mathscr{D}$ and a measurable function f, define the dyadic maximal operator associated with $\mathscr{D}$ by

$$\mathcal{M}_{\mathscr{D}}f(x) := \sup_{Q\in\mathscr{D}} \langle |f| \rangle_Q \mathbf{1}_Q(x) \in [0,\infty].$$

If $\mathscr{F} \subset \mathscr{D}$ is a nonempty finite subcollection, define the corresponding finite dyadic maximal operator by

$$\mathcal{M}_{\mathscr{F}}f(x) := \max_{Q\in\mathscr{F}} \langle |f| \rangle_Q \mathbf{1}_Q(x).$$

We use the convention $\mathcal{M}_{\varnothing}f = 0$.

Lemma 2.11 gives a finite sparse domination of the dyadic maximal operator associated with a finite family of cubes. Starting with the inclusion-maximal cubes in $\mathscr{F}$, which are pairwise disjoint, we recursively select proper subcubes whose averages are more than twice the average of the previously selected cube.

At each stage, we retain only those candidate subcubes that are not properly contained in another candidate. This standard recursive selection produces a finite 1/2-sparse collection and controls every average occurring in $\mathcal{M}_{\mathscr{F}}f$; compare [7]. We include the proof because Proposition 4.2 below requires precisely this finite form inside one fixed dyadic grid.

Lemma 2.11. *Let $\mathscr{D}$ be a dyadic grid and let $\mathscr{F} \subset \mathscr{D}$ be a finite collection. For every locally integrable f, there is a finite 1/2-sparse collection $\mathscr{S} \subset \mathscr{D}$ such that*

$$\mathcal{M}_{\mathscr{F}}f \le 2\mathcal{A}_{\mathscr{S}}f \qquad \textit{almost everywhere.} \tag{2.18}$$

Proof. If $\mathscr{F} = \varnothing$, take $\mathscr{S} = \varnothing$. Hence we may assume that $\mathscr{F} \neq \varnothing$.

Let $\mathcal{C} \subset \mathscr{F}$ be a subcollection. A cube $Q \in \mathcal{C}$ is called *inclusion-maximal in* $\mathcal{C}$ if there is no cube $R \in \mathcal{C}$ such that $Q \subsetneq R$.

We construct $\mathscr{S}$ recursively. First, select all cubes that are inclusion-maximal in $\mathscr{F}$. Suppose that Q has already been selected in the above procedure, and consider the finite collection

$$\mathcal{B}(Q) = \left\{ P \in \mathscr{F} :\ P \subsetneq Q,\ \langle |f| \rangle_P > 2\langle |f| \rangle_Q \right\}.$$

Assume that $\mathcal{B}(Q) \neq \emptyset$. Select all cubes that are inclusion-maximal in $\mathcal{B}(Q)$. We call them the *selected children* of Q. Here “children” refers only to this recursive selection: these cubes need not be the 2^n immediate dyadic children of Q. Apply the same rule to every newly selected cube. Each selected child is a proper subcube of its parent, and $\mathscr{F}$ is finite. Hence the procedure stops after finitely many steps. Let $\mathscr{S}$ be the collection of all selected cubes, and for $Q \in \mathscr{S}$ write $\mathrm{ch}_{\mathscr{S}}(Q)$ for the selected children of Q. Notice that a cube R belongs to $\mathscr{S}$ if and only if either R is inclusion-maximal in $\mathscr{F}$, or there exists a cube $Q \in \mathscr{S}$ such that R is inclusion-maximal in $\mathcal{B}(Q)$.

We first prove that $\mathscr{S}$ is 1/2-sparse. The cubes in $\mathrm{ch}_{\mathscr{S}}(Q)$ are pairwise disjoint due to maximality. If $\langle |f| \rangle_Q > 0$, the definition of the children and their pairwise disjointness give

$$2\langle |f| \rangle_Q \sum_{P\in\mathrm{ch}_{\mathscr{S}}(Q)} |P| < \sum_{P\in\mathrm{ch}_{\mathscr{S}}(Q)} \int_P |f| \le \int_Q |f| = \langle |f| \rangle_Q |Q|.$$

Consequently,

$$\left| \bigcup_{P\in\mathrm{ch}_{\mathscr{S}}(Q)} P \right| \le \frac{|Q|}{2}.$$

If $\langle |f| \rangle_Q = 0$, then $|f| = 0$ almost everywhere on Q. Hence every subcube $P \subset Q$ has $\langle |f| \rangle_P = 0$, and no child of Q is selected. In either case, the set

$$E_Q = Q \setminus \bigcup_{P \in \mathrm{ch}_{\mathscr{S}}(Q)} P$$

satisfies $|E_Q| \geq |Q|/2$.

It remains to check that the sets E_Q, $Q \in \mathscr{S}$, are pairwise disjoint. Let $Q, R \in \mathscr{S}$ and $Q \neq R$. If $Q \cap R = \varnothing$, then clearly $E_Q \cap E_R = \varnothing$. Otherwise, the dyadic-grid property implies that one cube contains the other. Suppose that $R \subsetneq Q$ by symmetry. By the recursive construction, there is a finite chain of selected cubes

$$Q = Q_0 \supsetneq Q_1 \supsetneq \cdots \supsetneq Q_m = R,$$

where Q_{i+1} is a child of Q_i for every i. In particular, $Q_1 \in \mathrm{ch}_{\mathscr{S}}(Q)$ and $R \subseteq Q_1$. Setting $P = Q_1$, we obtain

$$E_R \subset R \subseteq P, \qquad E_Q \subset Q \setminus P,$$

so $E_Q \cap E_R = \varnothing$. Thus $\mathscr{S}$ is 1/2-sparse.

We now prove the pointwise estimate (2.18). Fix $R \in \mathscr{F}$. Since $\mathscr{F}$ is finite and $\mathscr{D}$ is nesting, there is a unique inclusion-maximal cube Q_0 of $\mathscr{F}$ that contains R, and this cube was selected at the first stage.

Starting from Q_0, proceed as follows. If R is contained in a child of the current selected cube, replace the current cube by that child. Such a child is unique because the children are pairwise disjoint. Each replacement gives a proper subcube, and $\mathscr{S}$ is finite, so after finitely many steps we reach a selected cube Q such that

$$R \subseteq Q$$

and that

$$R \not\subseteq P \tag{2.19}$$

for every $P \in \mathrm{ch}_{\mathscr{S}}(Q)$. We show that this terminal cube Q satisfies

$$\langle |f| \rangle_R \leq 2 \langle |f| \rangle_Q .$$

If $R = Q$, this is immediate. Suppose that $R \subsetneq Q$ and that this inequality is false. Then

$$\mathcal{G} = \left\{ P \in \mathscr{F} : \ R \subseteq P \subsetneq Q, \ \langle |f| \rangle_P > 2 \langle |f| \rangle_Q \right\}$$

is nonempty because $R \in \mathcal{G}$. Choose an inclusion-maximal cube P in $\mathcal{G}$. There are two possibilities, and both lead to a contradiction.

- If P is not inclusion-maximal in $\mathcal{B}(Q)$, then there exists $P' \in \mathcal{B}(Q)$ such that $P \subsetneq P'$. Since $R \subseteq P \subsetneq P' \subsetneq Q$, we have $P' \in \mathcal{G}$, which contradicts the inclusion-maximality of P in $\mathcal{G}$.
- If P is inclusion-maximal in $\mathcal{B}(Q)$, then P is a selected child of Q. Since $R \subseteq P$, this contradicts (2.19).

Therefore

$$\langle |f| \rangle_R \leq 2 \langle |f| \rangle_Q .$$

For $x \in R$, the cube Q also contains x, and the term $\langle |f| \rangle_Q \mathbf{1}_Q(x)$ occurs in $\mathcal{A}_{\mathscr{S}} f(x)$. Hence

$$\langle |f| \rangle_R \mathbf{1}_R(x) \leq 2 \langle |f| \rangle_Q \mathbf{1}_Q(x) \leq 2 \mathcal{A}_{\mathscr{S}} f(x).$$

The same inequality is trivial when $x \notin R$ since the left-hand side is zero. Taking the maximum over $R \in \mathscr{F}$ proves (2.18). $\square$

We next explain how finitely many dyadic maximal operators control the ordinary maximal operator. A standard construction of adjacent dyadic grids provides $\mathscr{D}^1, \dots, \mathscr{D}^{3^n}$ with the following property: for every axis-parallel cube Q, there are an index $t \in \{1, \dots, 3^n\}$ and a cube $R \in \mathscr{D}^t$ such that

$$Q \subset R, \qquad |R| \le 6^n |Q|. \tag{2.20}$$

Thus an arbitrary cube can be enlarged to a dyadic cube from one of the fixed grids, while its measure increases by at most the factor 6^n. If $x \in Q$ and R is chosen as above, then

$$\langle |f| \rangle_Q \le \frac{|R|}{|Q|} \langle |f| \rangle_R \le 6^n \mathcal{M}_{\mathscr{D}^t} f(x).$$

Taking the supremum over all cubes Q containing x gives

$$\mathcal{M} f(x) \le 6^n \max_{1 \le t \le 3^n} \mathcal{M}_{\mathscr{D}^t} f(x) \le 6^n \sum_{t=1}^{3^n} \mathcal{M}_{\mathscr{D}^t} f(x), \tag{2.21}$$

See [7] for this construction.

3. Sparse bounds obtained from the vector-valued estimate

3.1. **A vector-valued sparse estimate.** The proof of the next proposition uses the sets E_Q from the definition of sparseness in two ways. For a fixed nonnegative function f and each $Q \in \mathscr{S}$, consider the auxiliary function $\langle f \rangle_Q \mathbf{1}_{E_Q}$. Since the sets E_Q are pairwise disjoint, the ℓ^q-sum

$$\left(\sum_{Q \in \mathscr{S}} | \langle f \rangle_Q \mathbf{1}_{E_Q} |^q \right)^{\frac{1}{q}}$$

of these auxiliary functions is controlled by $\mathcal{M} f$. On the other hand, the estimate $|E_Q| \ge \eta |Q|$ gives

$$\mathcal{M}\big(\langle f \rangle_Q \mathbf{1}_{E_Q} \big)(x) \ge \eta \langle f \rangle_Q \qquad (x \in Q).$$

These two observations give the estimate in the next proposition.

Proposition 3.1. *Let X be a ball Banach function space, let $1 < q < \infty$, and assume that* (2.6) *holds on X with a constant $C > 0$ independent of N. If $\mathscr{S}$ is a finite η-sparse collection, then*

$$\| \mathcal{T}_{\mathscr{S},q} f \|_X \le \eta^{-1} C^2 \| f \|_X \qquad (f \in X). \tag{3.1}$$

The constant is independent of $\mathscr{S}$ and of the measurable sets E_Q used in (2.15).

Proof. It is enough to consider the case $f \ge 0$. Choose measurable sets $E_Q \subset Q$, $Q \in \mathscr{S}$, satisfying (2.15), and define

$$h_Q = \langle f \rangle_Q \mathbf{1}_{E_Q}. \tag{3.2}$$

Each h_Q belongs to X by Lemma 2.4(i). Since the sets E_Q are pairwise disjoint,

$$\left(\sum_{Q \in \mathscr{S}} |h_Q|^q \right)^{\frac{1}{q}} = \sum_{Q \in \mathscr{S}} \langle f \rangle_Q \mathbf{1}_{E_Q}.$$

If $x \in E_Q$, then $x \in Q$, and hence $\langle f \rangle_Q \leq \mathcal{M} f(x)$. Therefore

$$\left(\sum_{Q \in \mathscr{S}} |h_Q|^q \right)^{\frac{1}{q}} \leq \mathcal{M} f. \tag{3.3}$$

For $x \in Q$,

$$\mathcal{M} h_Q(x) \geq \langle h_Q \rangle_Q = \langle f \rangle_Q \frac{|E_Q|}{|Q|} \geq \eta \langle f \rangle_Q .$$

Consequently,

$$\eta \mathcal{T}_{\mathscr{S},q} f = \eta \left(\sum_{Q \in \mathscr{S}} \langle |f| \rangle_Q^q \mathbf{1}_Q \right)^{\frac{1}{q}} \leq \left(\sum_{Q \in \mathscr{S}} (\mathcal{M} h_Q)^q \right)^{\frac{1}{q}} . \tag{3.4}$$

Recall that, by taking $N = 1$ in (2.6), we obtain

$$\|\mathcal{M} f\|_X \leq C \|f\|_X .$$

Therefore, by (3.4) and (3.3),

$$\eta \|\mathcal{T}_{\mathscr{S},q} f\|_X \leq C \left\| \left(\sum_{Q \in \mathscr{S}} |h_Q|^q \right)^{\frac{1}{q}} \right\|_X \leq C \|\mathcal{M} f\|_X \leq C^2 \|f\|_X .$$

This is the desired result. □

3.2. **Binomial iteration.** Let $\{a_j\}_{j=1}^N$ be a finite family of nonnegative numbers, and let $q > 1$. Then

$$\left(\sum_{j=1}^N a_j^q \right)^{\frac{1}{q}} \leq \sum_{j=1}^N a_j,$$

but the reverse inequality requires a constant dependent on N. Thus a bound for $\mathcal{T}_{\mathscr{S},q}$ does not immediately give a bound for $\mathcal{A}_{\mathscr{S}}$. The next theorem shows that repeated application of $\mathcal{T}_{\mathscr{S},q}$ produces binomial coefficients large enough to control the sum defining $\mathcal{A}_{\mathscr{S}}$. Since $\mathscr{S}$ is finite, if $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then every average occurring in $\mathcal{T}_{\mathscr{S},q} g$ is finite, and $\mathcal{T}_{\mathscr{S},q} g$ is a bounded measurable function supported on $\bigcup_{Q \in \mathscr{S}} Q$. Hence $\mathcal{T}_{\mathscr{S},q} g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, so every iteration below is well defined for locally integrable functions.

Theorem 3.2. *Let $1 < q < \infty$, let $m \in \mathbb{N}$ satisfy $m > q$, and let $\mathscr{S}$ be a finite subcollection of a dyadic grid. Put*

$$\Gamma_{q,m} = \left(\sum_{\ell=1}^{\infty} \binom{\ell + m - 2}{m - 1}^{-1/(q-1)} \right)^{1/q'} . \tag{3.5}$$

Then $\Gamma_{q,m} < \infty$, and

$$\mathcal{A}_{\mathscr{S}} f(x) \leq \Gamma_{q,m} \mathcal{T}_{\mathscr{S},q}^m f(x) \tag{3.6}$$

for every locally integrable f and almost every x. Here $\mathcal{T}_{\mathscr{S},q}^m$ denotes the m-fold composition of $\mathcal{T}_{\mathscr{S},q}$.

Proof. It is enough to consider $f \geq 0$. Put

$$\Omega_{\mathscr{S}} = \bigcup_{Q \in \mathscr{S}} Q.$$

If $x \notin \Omega_{\mathscr{S}}$, then $\mathbf{1}_Q(x) = 0$ for every $Q \in \mathscr{S}$. Hence, by (2.16) and (2.17),

$$\mathcal{A}_{\mathscr{S}} f(x) = 0 \qquad \text{and} \qquad \mathcal{T}_{\mathscr{S},q} g(x) = 0$$

for every function g for which $\mathcal{T}_{\mathscr{S},q} g$ is defined. It follows that $\mathcal{T}^m_{\mathscr{S},q} f(x) = 0$, so (3.6) holds at x. Therefore, it remains to prove (3.6) for $x \in \Omega_{\mathscr{S}}$.

Step 1. A lower bound for the iterations. For each $x \in \Omega_{\mathscr{S}}$, the cubes of $\mathscr{S}$ containing x form a nonempty finite chain, since $\mathscr{S}$ is a finite set. Write this chain as

$$Q_1 \supsetneq Q_2 \supsetneq \cdots \supsetneq Q_N, \qquad a_j = \langle f \rangle_{Q_j}.$$

Thus $N \geq 1$. We prove by induction on $k \geq 1$ that, for every $x \in \Omega_{\mathscr{S}}$,

$$\left(\mathcal{T}^k_{\mathscr{S},q} f(x)\right)^q \geq \sum_{j=1}^N \binom{N-j+k-1}{k-1} a_j^q, \tag{3.7}$$

where N, $Q_1, \ldots, Q_N$, and $a_1, \ldots, a_N$ are determined by x as above.

For $k = 1$, only the cubes $Q_1, \ldots, Q_N$ contribute to (2.16). Therefore,

$$\left(\mathcal{T}_{\mathscr{S},q} f(x)\right)^q = \sum_{j=1}^N a_j^q.$$

Since $\binom{N-j}{0} = 1$ for $1 \leq j \leq N$, this proves (3.7) when $k = 1$.

Assume that (3.7) holds for an integer $k \geq 1$ at every point of $\Omega_{\mathscr{S}}$. Let $1 \leq s \leq N$ and $y \in Q_s$. The cubes $Q_1, \ldots, Q_s$ all contain y. Write all cubes of $\mathscr{S}$ containing y in decreasing order as

$$R_1 \supsetneq R_2 \supsetneq \cdots \supsetneq R_L. \tag{3.8}$$

We claim that

$$R_j = Q_j \qquad (1 \leq j \leq s).$$

Indeed, let $R \in \mathscr{S}$ contain y and suppose that R is not strictly contained in Q_s. Since R and Q_s are dyadic cubes in the same grid and intersect at y, one of them must contain the other. By our assumption on R, we therefore have $Q_s \subset R$. Since $x \in Q_s$, it follows that R also contains x. Hence R is one of the cubes $Q_1, \ldots, Q_s$. Conversely, each of the cubes $Q_1, \ldots, Q_s$ contains Q_s and therefore contains y. Thus, $Q_1, \ldots, Q_s$ are the only cubes in $\mathscr{S}$ that contain y and are not strictly contained in Q_s. Since (3.8) holds, we have $Q_j = R_j$ for all $j = 1, 2, \ldots, s$. This proves the claim. In particular, $L \geq s$, and any further cubes $R_{s+1}, \ldots, R_L$, if they occur, are strictly contained in Q_s. Put $b_j = \langle f \rangle_{R_j}$ for $1 \leq j \leq L$. By the claim, $b_j = a_j$ for $1 \leq j \leq s$. Applying the induction hypothesis at the point y to the chain $R_1, \ldots, R_L$, and then keeping only its first s nonnegative terms, gives

$$\begin{aligned}
\left(\mathcal{T}^k_{\mathscr{S},q} f(y)\right)^q &\geq \sum_{j=1}^L \binom{L-j+k-1}{k-1} b_j^q \\
&\geq \sum_{j=1}^s \binom{L-j+k-1}{k-1} a_j^q
\end{aligned}$$

$$\geq \sum_{j=1}^{s} \binom{s-j+k-1}{k-1} a_j^q.$$

For the last inequality, we used $L \geq s$ and the fact that $u \mapsto \binom{u}{k-1}$ is nondecreasing for integers $u \geq k-1$.

Set

$$B_s = \sum_{j=1}^{s} \binom{s-j+k-1}{k-1} a_j^q.$$

The preceding estimate holds for every $y \in Q_s$, and hence

$$\mathcal{T}_{\mathscr{S},q}^k f(y) \geq B_s^{\frac{1}{q}} \qquad (y \in Q_s).$$

Taking the average over Q_s gives

$$\langle \mathcal{T}_{\mathscr{S},q}^k f \rangle_{Q_s} \geq B_s^{\frac{1}{q}}, \qquad \left(\langle \mathcal{T}_{\mathscr{S},q}^k f \rangle_{Q_s} \right)^q \geq B_s.$$

The cubes of $\mathscr{S}$ that contain x are exactly $Q_1, \dots, Q_N$. Therefore, by the definition of $\mathcal{T}_{\mathscr{S},q}$,

$$\begin{aligned}
(\mathcal{T}_{\mathscr{S},q}^{k+1} f(x))^q &= \sum_{s=1}^{N} \left(\langle \mathcal{T}_{\mathscr{S},q}^k f \rangle_{Q_s} \right)^q \\
&\geq \sum_{s=1}^{N} B_s \\
&= \sum_{s=1}^{N} \sum_{j=1}^{s} \binom{s-j+k-1}{k-1} a_j^q \\
&= \sum_{j=1}^{N} \left(\sum_{s=j}^{N} \binom{s-j+k-1}{k-1} \right) a_j^q \\
&= \sum_{j=1}^{N} \binom{N-j+k}{k} a_j^q.
\end{aligned}$$

In the last equality, we set $u = s-j$ and used the binomial identity

$$\sum_{s=j}^{N} \binom{s-j+k-1}{k-1} = \sum_{u=0}^{N-j} \binom{u+k-1}{k-1} = \binom{N-j+k}{k}.$$

The final expression is the right-hand side of (3.7) with $k+1$ in place of k. This proves (3.7) for $k+1$.

Step 2. Comparison with $\mathcal{A}_{\mathscr{S}}$. Apply (3.7) with $k = m$ and set

$$w_j = \binom{N-j+m-1}{m-1}.$$

Hölder's inequality gives

$$\sum_{j=1}^{N} a_j \leq \left(\sum_{j=1}^{N} w_j a_j^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^{N} w_j^{-1/(q-1)} \right)^{1/q'} \leq \Gamma_{q,m} \, \mathcal{T}_{\mathscr{S},q}^m f(x).$$

The left-hand side is $\mathcal{A}_{\mathscr{S}} f(x)$. To finish the proof, it remains to verify the convergence in (3.5). Since $m > q > 1$, one has $m \geq 2$. For every integer $\ell \geq 1$,

$$\binom{\ell+m-2}{m-1} = \frac{1}{(m-1)!} \prod_{i=0}^{m-2} (\ell+i).$$

Each factor in the product is at least ℓ. Also, $\ell + i \leq \ell + m - 2 \leq (m-1)\ell$ for $0 \leq i \leq m-2$. Therefore

$$\frac{\ell^{m-1}}{(m-1)!} \leq \binom{\ell+m-2}{m-1} \leq \frac{(m-1)^{m-1}}{(m-1)!} \ell^{m-1} \qquad (\ell \geq 1). \tag{3.9}$$

Thus the binomial coefficient in (3.9) is comparable to ℓ^{m-1} for every $\ell \geq 1$, with constants depending only on m. In particular, the summand in (3.5) is bounded above by a constant depending only on m and q times

$$\ell^{-(m-1)/(q-1)}.$$

Since $m > q$, one has $(m-1)/(q-1) > 1$. The corresponding p-series converges, and hence $\Gamma_{q,m} < \infty$. □

Corollary 3.3. *Let X be a ball Banach function space, let $1 < q < \infty$, and assume that* (2.6) *holds on X with a constant $C > 0$ independent of N. Let $m \in \mathbb{N}$ satisfy $m > q$. If $\mathscr{S}$ is a finite η-sparse subcollection of a dyadic grid, then*

$$\|\mathcal{A}_{\mathscr{S}} f\|_X \leq \Gamma_{q,m} \big(\eta^{-1} C^2\big)^m \|f\|_X \qquad (f \in X). \tag{3.10}$$

In particular, the constant is independent of the dyadic grid and of $\mathscr{S}$.

Proof. By Proposition 3.1,

$$\|\mathcal{T}_{\mathscr{S},q} f\|_X \leq K \|f\|_X, \qquad K = \eta^{-1} C^2.$$

Applying this estimate m times yields

$$\left\|\mathcal{T}^m_{\mathscr{S},q} f\right\|_X \leq K^m \|f\|_X.$$

Now use (3.6). □

4. A finite dyadic sparse criterion and the proof of Theorem 1.1

4.1. **The finite dyadic sparse criterion.** Lorist and Nieraeth proved the following observation [9, Lemma 3.4].

Lemma 4.1. *For a Banach function space X, the two bounds*

$$\mathcal{M} : X \to X \qquad \text{and} \qquad \mathcal{M} : X' \to X'$$

are equivalent to the uniform estimate

$$\sup_{\mathscr{S} \text{ is } 1/2\text{-sparse}} \|\mathcal{A}_{\mathscr{S}}\|_{X \to X} < \infty.$$

The same equivalence is also stated in [13, Theorem 1.2, conditions (iii) and (iv)]. The spaces considered here satisfy the hypotheses of these results: the Fatou property is part of Definition 2.1, and the saturation property follows from Lemma 2.4(iii).

Section 3 gives a uniform estimate only for finite 1/2-sparse subcollections of dyadic grids. For the reader's convenience, we explain why this restricted class is sufficient. The implication from the two maximal-operator bounds to the finite dyadic sparse estimate follows from [9, Lemma 3.4]. The converse follows directly from Lemma 2.11, the Fatou properties of X and X', and the finite family of adjacent dyadic grids in (2.20)–(2.21). We record the precise form needed below.

Proposition 4.2. *Let X be a ball Banach function space. The following statements are equivalent.*

(*i*) *The operators $\mathcal{M} : X \to X$ and $\mathcal{M} : X' \to X'$ are bounded.*

(*ii*) *There is a constant C_S such that*

$$\|\mathcal{A}_{\mathscr{S}} f\|_X \le C_S \|f\|_X \qquad (f \in X)$$

for every finite $1/2$-sparse subcollection $\mathscr{S}$ of every dyadic grid.

Proof. Assume (i). By the sparse characterization in [9, Lemma 3.4], the operators $\mathcal{A}_{\mathscr{S}}$ are uniformly bounded on X for all sparse collections. In particular, the estimate in (ii) holds for every finite $1/2$-sparse subcollection of every dyadic grid.

Conversely, assume (ii). We first show that the same sparse estimate holds on X'. Let $\mathscr{S}$ be a finite $1/2$-sparse subcollection of a dyadic grid and let $g \in X'$. For every $f \in X$, the finiteness of $\mathscr{S}$ gives

$$\int_{\mathbb{R}^n} |f| \, \mathcal{A}_{\mathscr{S}} g = \sum_{Q \in \mathscr{S}} \langle |g| \rangle_Q \int_Q |f| = \sum_{Q \in \mathscr{S}} \langle |f| \rangle_Q \int_Q |g| = \int_{\mathbb{R}^n} \mathcal{A}_{\mathscr{S}} f \, |g|.$$

Hence, by (2.3) and (ii),

$$\int_{\mathbb{R}^n} |f| \, \mathcal{A}_{\mathscr{S}} g \le \|\mathcal{A}_{\mathscr{S}} f\|_X \|g\|_{X'} \le C_S \|f\|_X \|g\|_{X'} .$$

Taking the supremum over all $f \in X$ with $\|f\|_X \le 1$ yields

$$\|\mathcal{A}_{\mathscr{S}} g\|_{X'} \le C_S \|g\|_{X'} . \tag{4.1}$$

Fix a dyadic grid $\mathscr{D}$ and a finite subcollection $\mathscr{F} \subset \mathscr{D}$. Let Z denote either X or X', and let $f \in Z$. By Lemma 2.4(ii), the function f is locally integrable. Lemma 2.11 gives a finite $1/2$-sparse collection $\mathscr{S} \subset \mathscr{D}$ such that

$$\mathcal{M}_{\mathscr{F}} f \le 2\mathcal{A}_{\mathscr{S}} f \qquad \text{almost everywhere.}$$

Using (ii) when $Z = X$ and (4.1) when $Z = X'$, we obtain

$$\|\mathcal{M}_{\mathscr{F}} f\|_Z \le 2C_S \|f\|_Z . \tag{4.2}$$

Each generation $\mathscr{D}_m$ is countable: its cubes have pairwise disjoint interiors, and each such interior contains a rational point. Hence $\mathscr{D} = \bigcup_{m \in \mathbb{Z}} \mathscr{D}_m$ is countable. Choose increasing finite subcollections $\mathscr{F}_k \subset \mathscr{D}$ such that $\bigcup_{k=1}^{\infty} \mathscr{F}_k = \mathscr{D}$. Then $\mathcal{M}_{\mathscr{F}_k} f \uparrow \mathcal{M}_{\mathscr{D}} f$ pointwise. The Fatou property of X and Lemma 2.3 for X' therefore give, from (4.2),

$$\|\mathcal{M}_{\mathscr{D}} f\|_Z \le 2C_S \|f\|_Z \qquad (Z = X \text{ or } X').$$

Finally, let $\mathscr{D}^1, \ldots, \mathscr{D}^{3^n}$ be the adjacent dyadic grids in (2.20). By (2.21),

$$\|\mathcal{M} f\|_Z \le 6^n \sum_{t=1}^{3^n} \|\mathcal{M}_{\mathscr{D}^t} f\|_Z \le 2 \cdot 18^n C_S \|f\|_Z \qquad (Z = X \text{ or } X').$$

Thus $\mathcal{M}$ is bounded on both X and X', which proves (i). □

4.2. **Proof of Theorem 1.1.** We first prove parts (ii), (iii), and (iv). Part (ii) is Proposition 2.9. For part (iii), fix $1<q<\infty$. Lemma 2.8 shows that an estimate valid for every $N\in\mathbb{N}$ extends, with the same constant, to every sequence $\{f_j\}_{j=1}^{\infty}$ for which $\left(\sum_{j=1}^{\infty}|f_j|^q\right)^{\frac{1}{q}}$ belongs to X. The converse follows by adding zero terms. This proves part (iii). Part (iv) is Proposition 2.10.

It remains to prove part (i).

Step 1. $(b)\implies(a)$. Assume (b). Then there are $q\in(1,\infty)$ and a finite constant C such that (1.4) holds for every N, with C independent of N. If we take $N=1$, then we see that $\mathcal{M}$ is bounded on X. By Corollary 3.3, the operators $\mathcal{A}_{\mathscr{S}}$ are uniformly bounded on X over all finite 1/2-sparse subcollections of all dyadic grids. Proposition 4.2 then shows $\mathcal{M}$ is bounded on X'. Thus (a) holds.

Step 2. $(a)\implies(c)$. Assume (a), and fix $q\in(1,\infty)$. Choose $p_0\in(1,\infty)$. For every $N\in\mathbb{N}$ and every finite family $\{f_j\}_{j=1}^{N}$ of measurable functions, put

$$G=\left(\sum_{j=1}^{N}(\mathcal{M}f_j)^q\right)^{\frac{1}{q}},\qquad F=\left(\sum_{j=1}^{N}|f_j|^q\right)^{\frac{1}{q}}.$$

The weighted inequality in Theorem 2.6 verifies the hypothesis of Theorem 2.7, with a bound independent of N. Applying that theorem to the collection of all pairs (G,F) gives (1.4) on X. Since $q\in(1,\infty)$ was arbitrary, (c) follows.

Step 3. $(c)\implies(b)$. This implication is immediate.

This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.3

Theorem 1.3 is proved by passing from the ball quasi-Banach function space Y to the rescaled space $X=Y^r$. We first recall the properties of this rescaling that are needed in the proof.

5.1. **Power rescaling.** We recall the r-concavification of a ball quasi-Banach function space Y.

Definition 5.1. Let Y be a ball quasi-Banach function space and let $r>0$. Define

$$Y^r=\{h:\ |h|^{1/r}\in Y\},\qquad \|h\|_{Y^r}=\left\||h|^{1/r}\right\|_Y^r. \tag{5.1}$$

The space Y^r, equipped with the quasi-norm in (5.1), is called the r-concavification of Y.

This is the power notation used in [12].

The next lemma records the elementary properties of the power space Y^r that are used below.

Lemma 5.2. *Let Y be a ball quasi-Banach function space and let $r>0$. Then Y^r, equipped with the quasi-norm in* (5.1)*, is a ball quasi-Banach function space. For every nonnegative $H\in Y$,*

$$\|H\|_Y^r=\|H^r\|_{Y^r}. \tag{5.2}$$

Proof. By Definition 2.2(i),(iii), Y has the saturation property. Hence the standard concavification result in [12, p. 12] applies and shows that Y^r, equipped with the quasi-norm in (5.1), is a quasi-Banach function space. Since $\mathbf{1}_B^{1/r}=\mathbf{1}_B$ for every ball B, the ball-indicator property is inherited from Y. Finally, (5.2) follows directly from (5.1). $\square$

We also recall the r-convexity of constant one.

Definition 5.3. Let Y be a ball quasi-Banach function space and let $r > 0$. Following [12, Definition 2.7], we say that Y is *r-convex with constant one* if

$$\left\| \left(|f|^r + |g|^r \right)^{1/r} \right\|_Y \leq \left(\|f\|_Y^r + \|g\|_Y^r \right)^{1/r} \qquad (f, g \in Y). \tag{5.3}$$

By the standard concavification criterion, Y is r-convex with constant one if and only if the quasi-norm in (5.1) is a norm on Y^r; see [12, p. 12]. Combining this criterion with Lemma 5.2, we obtain Lemma 5.4 below. At this stage, the continuous embedding of Y^r into $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ has not yet been established. This will follow from Lemma 5.5 below whenever the maximal operator is bounded on Y^r.

Lemma 5.4. *Let Y be an r-convex ball quasi-Banach function space with constant one, and put $X = Y^r$ with the quasi-norm in (5.1). Then this quasi-norm is actually a norm, and X is a Banach lattice of measurable functions with the Fatou property and containing the indicator of every ball.*

Proof. By the standard concavification criterion [12, p. 12], the quasi-norm in (5.1) is a norm on Y^r because Y is r-convex. The remaining assertions follow from Lemma 5.2. $\square$

5.2. **Local integrability and powered operators.** Comparing Definitions 2.1 and 2.2, we see that, for a ball quasi-Banach function space Z to be a ball Banach function space, it is necessary, in addition to the triangle inequality for the norm, that every element of Z belong to L^1_{loc}. The following lemma provides a sufficient condition for this property.

Lemma 5.5. *Let Z be a ball quasi-Banach function space whose given quasi-norm is a norm. If $\mathcal{M}$ is bounded on Z, then Z is continuously embedded into $L^1(B)$ for every ball B. Consequently, Z is a ball Banach function space.*

Remark that Z is a normed function space (see Definition 2.1). Therefore, its Köthe dual is defined by (2.2).

Proof. By Definition 2.2(i),(iii), Z has the saturation property. We may therefore apply [12, Lemma 2.26] to the basis of axis-parallel cubes. It follows that $\mathbf{1}_Q \in Z'$ for every axis-parallel cube Q. Let B be a ball and choose such a cube Q_B containing B. Then the generalized Hölder inequality gives

$$\int_B |f(x)|\, dx \leq \int_{Q_B} |f(x)|\, dx \leq \|f\|_Z \, \|\mathbf{1}_{Q_B}\|_{Z'} \qquad (f \in Z).$$

Thus Z is continuously embedded into $L^1(B)$. Since the given quasi-norm is a norm, all the remaining axioms in Definition 2.1 are already part of the assumptions on Z. $\square$

It is useful to compare the preceding lemma with an earlier related result. In [11, Lemma 2.4], we assumed that $\mathcal{M}$ is bounded on Z'. In both cases, we use the existence of a non-trivial element in Z'.

Let $r > 0$, let $\mathscr{S}$ be a finite collection of cubes. We recall again that we defined $\mathcal{M}_r$ and $\mathcal{A}_{\mathscr{S},r}$ in (1.8).

The assumption $f \in L^r_{\mathrm{loc}}(\mathbb{R}^n)$ means precisely that $|f|^r \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, so every cube average in (1.8) is finite. The function $\mathcal{M}_r f$ may nevertheless take the value $+\infty$, because the supremum defining $\mathcal{M}$ is taken over all cubes. In the proof below, any one of the conditions in part (i) of Theorem 1.3 will imply $Y \subset L^r_{\mathrm{loc}}(\mathbb{R}^n)$. For conditions (b) and (d) in that part, this is proved by applying the assumed estimate to bounded compactly supported truncations.

5.3. **Proof of Theorem 1.3.** Put $X = Y^r$. By Lemma 5.4, X is a Banach lattice with the Fatou property and contains the indicator of every ball.

Step 1. The case $0 < q < \infty$. Let $0 < q < \infty$, put $a = q/r$, and let $f_j \in L^r_{\mathrm{loc}}(\mathbb{R}^n)$. Set $h_j = |f_j|^r$. From (5.1) and (1.8),

$$\left\| \left(\sum_{j=1}^N (\mathcal{M}_r f_j)^q \right)^{\frac{1}{q}} \right\|_Y^r = \left\| \left(\sum_{j=1}^N (\mathcal{M} h_j)^a \right)^{1/a} \right\|_X, \tag{5.4}$$

$$\left\| \left(\sum_{j=1}^N |f_j|^q \right)^{\frac{1}{q}} \right\|_Y^r = \left\| \left(\sum_{j=1}^N |h_j|^a \right)^{1/a} \right\|_X. \tag{5.5}$$

Thus an estimate on X with exponent a and constant D gives (1.9) with constant $D^{1/r}$. Conversely, (1.9) with constant C gives the corresponding estimate on X with constant C^r for finite families of nonnegative locally integrable functions for which the quantity on the right is finite.

Assume that (1.9) holds with a constant C. We now remove the local-integrability restriction. Let $h_1, \dots, h_N$ be nonnegative measurable functions such that

$$H = \left(\sum_{j=1}^N h_j^a \right)^{1/a} \in X.$$

For $k \in \mathbb{N}$, define

$$h_{j,k} = \min\{h_j, k\} \mathbf{1}_{B(0,k)}, \qquad f_{j,k} = h_{j,k}^{1/r}.$$

Then $f_{j,k} \in Y \cap L^r_{\mathrm{loc}}(\mathbb{R}^n)$ and $\left(\sum_{j=1}^N h_{j,k}^a\right)^{1/a} \le H$. Hence (1.9) and (5.4)–(5.5) give

$$\left\| \left(\sum_{j=1}^N (\mathcal{M} h_{j,k})^a \right)^{1/a} \right\|_X \le C^r \|H\|_X.$$

As $k \to \infty$, one has $h_{j,k} \uparrow h_j$ and $\mathcal{M} h_{j,k} \uparrow \mathcal{M} h_j$. The Fatou property of X therefore gives

$$\left\| \left(\sum_{j=1}^N (\mathcal{M} h_j)^a \right)^{1/a} \right\|_X \le C^r \left\| \left(\sum_{j=1}^N h_j^a \right)^{1/a} \right\|_X.$$

Taking $N = 1$ shows that $\mathcal{M}$ is bounded on X. By Lemma 5.5, X is a ball Banach function space, and hence $Y \subset L^r_{\mathrm{loc}}(\mathbb{R}^n)$. If $q \le r$, then $a \le 1$, and the last vector-valued estimate contradicts Proposition 2.9. Thus $q > r$. Theorem 1.1 now gives $\mathcal{M}$ is bounded on X', so condition (b) in part (i) implies condition (a). The same argument proves part (ii).

Conversely, assume condition (a) in part (i). Lemma 5.5 shows that X is a ball Banach function space and therefore that $Y \subset L^r_{\mathrm{loc}}(\mathbb{R}^n)$. For every $q > r$, one has $a = q/r > 1$. Theorem 1.1 gives (1.4) on X with exponent a and some finite constant D. The identities (5.4)–(5.5) then give (1.9) with constant $D^{1/r}$. Hence, within part (i), (a) implies (c), while (c) implies (b) immediately.

Under the hypothesis of part (iii), part (ii) gives $Y \subset L^r_{\mathrm{loc}}(\mathbb{R}^n)$. Since $q > r$ by assumption, one has $a = q/r > 1$. For every sequence considered in part (iii),

$$|f_j| \leq \left(\sum_{k=1}^{\infty} |f_k|^q\right)^{\frac{1}{q}} \in Y,$$

so each f_j belongs to $L^r_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathcal{M}_r f_j$ is well defined. The argument above gives the finite vector-valued estimate on X with exponent $a = q/r$ and constant C^r. Theorem 1.1(iii), together with (5.4)–(5.5) and increasing partial sums, therefore gives the estimate in part (iii) with constant C.

Step 2. The sparse condition. For every finite sparse collection $\mathscr{S}$ and every $f \in Y \cap L^r_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\|\mathcal{A}_{\mathscr{S},r} f\|_Y^r = \|\mathcal{A}_{\mathscr{S}}(|f|^r)\|_X . \tag{5.6}$$

Assume condition (a) in part (i). Lemma 5.5 shows that X is a ball Banach function space, so $Y \subset L^r_{\mathrm{loc}}(\mathbb{R}^n)$. Proposition 4.2 gives a constant $D > 0$, independent of $\mathscr{S}$, such that

$$\|\mathcal{A}_{\mathscr{S}} h\|_X \leq D \|h\|_X \qquad (h \in X).$$

By (5.6),

$$\|\mathcal{A}_{\mathscr{S},r} f\|_Y \leq D^{1/r} \|f\|_Y \qquad (f \in Y).$$

Hence (a) implies (d) in part (i).

Conversely, assume condition (d) in part (i), and let C_{sp} be the constant in that condition. We first verify local integrability of the elements of X. Let $h \in X$, and for $k \in \mathbb{N}$ define

$$h_k = \min\{|h|, k\}\mathbf{1}_{B(0,k)}, \qquad f_k = h_k^{1/r}.$$

Then $f_k \in Y \cap L^r_{\mathrm{loc}}(\mathbb{R}^n)$. Let B be a ball, choose an axis-parallel cube Q_0 containing B, and use (2.20) to choose a dyadic cube Q such that $Q_0 \subset Q$. The one-cube collection $\{Q\}$ is 1/2-sparse. By condition (d) and (5.6),

$$\langle h_k \rangle_Q \|\mathbf{1}_Q\|_X = \left\|\mathcal{A}_{\{Q\}} h_k\right\|_X = \left\|\mathcal{A}_{\{Q\},r} f_k\right\|_Y^r \leq C_{\mathrm{sp}}^r \|f_k\|_Y^r \leq C_{\mathrm{sp}}^r \|h\|_X .$$

Letting $k \to \infty$ and using the monotone convergence theorem gives

$$\int_B |h(x)|\, dx \leq \frac{|Q| C_{\mathrm{sp}}^r}{\|\mathbf{1}_Q\|_X} \|h\|_X .$$

Thus X is a ball Banach function space and $Y \subset L^r_{\mathrm{loc}}(\mathbb{R}^n)$. For $h \in X$, apply condition (d) to $f = |h|^{1/r}$. Equation (5.6) gives

$$\|\mathcal{A}_{\mathscr{S}} h\|_X \leq C_{\mathrm{sp}}^r \|h\|_X$$

for every finite 1/2-sparse subcollection $\mathscr{S}$ of every dyadic grid. Proposition 4.2 now gives $\mathcal{M}$ is bounded on both X and X'. Hence (d) implies (a) in part (i).

Step 3. The supremum estimate. Assume that $Y \subset L^r_{\mathrm{loc}}(\mathbb{R}^n)$, as in part (iv) of the theorem. Let $\{f_j\}$ be a finite or countable family such that

$$F := \sup_j |f_j| \in Y.$$

For every j, one has $|f_j|^r \le F^r$. Hence the monotonicity of $\mathcal{M}$ gives

$$\mathcal{M}_r f_j = \left[\mathcal{M}(|f_j|^r)\right]^{1/r} \le \left[\mathcal{M}(F^r)\right]^{1/r} = \mathcal{M}_r F.$$

Therefore

$$\sup_j \mathcal{M}_r f_j \le \mathcal{M}_r F \qquad \text{pointwise.} \tag{5.7}$$

If $\mathcal{M}_r : Y \to Y$ is bounded and $\|\mathcal{M}_r f\|_Y \le C\,\|f\|_Y$ for every $f \in Y$, then (5.7) and the lattice property give

$$\left\|\sup_j \mathcal{M}_r f_j\right\|_Y \le \|\mathcal{M}_r F\|_Y \le C\,\|F\|_Y = C\left\|\sup_j |f_j|\right\|_Y.$$

Conversely, if the supremum estimate holds with a constant C, then a family with one nonzero term gives

$$\|\mathcal{M}_r f\|_Y \le C\,\|f\|_Y \qquad (f \in Y).$$

Thus the same constant works in the scalar estimate and in the supremum estimate. This proves part (iv).

Taking $Y = L^\infty(\mathbb{R}^n)$ shows that boundedness of $\mathcal{M}_r$ on Y need not imply boundedness of $\mathcal{M}$ on $X' = L^1(\mathbb{R}^n)$.

This completes the proof of Theorem 1.3.

6. Applications to local maximal operators

We consider the local version of Theorems 1.1 and 1.3. Let f be a measurable function on $\mathbb{R}^n$. For $\rho \in (0,\infty]$, define

$$\mathcal{M}^{\mathrm{loc}}_\rho f(x) = \sup_{\substack{Q \ni x \\ \ell(Q) \le \rho}} \langle |f| \rangle_Q,$$

where the supremum is taken over all axis-parallel cubes in $\mathbb{R}^n$. We call ρ the truncation parameter; the case $\rho = \infty$ corresponds to the untruncated operator. When $\rho = \infty$, that is, when the restriction on $\ell(Q)$ is removed, we have

$$\mathcal{M}^{\mathrm{loc}}_\infty = \mathcal{M}.$$

Furthermore, $\mathcal{M}^{\mathrm{loc}}_1$ is the usual normalized local maximal operator [15].

We organize Section 6 as follows. Section 6.1 states the main results of Section 6. Section 6.2 compares various finite truncations with different parameters. Section 6.3 collects inequalities related to $\mathcal{M}^{\mathrm{loc}}_a$. Section 6.4 deals with the endpoint cases. Section 6.5 establishes the sparse criterion for a finite truncation parameter. Section 6.6 derives a sparse estimate from the vector-valued inequality, and Section 6.7 proves Theorem 6.1.

6.1. **The characterization.** Let X be a ball Banach function space and let X' be its Köthe associate space. For $0 < q < \infty$ and $\rho \in (0,\infty)$ consider

$$\left\|\left(\sum_{j=1}^N (\mathcal{M}^{\mathrm{loc}}_\rho f_j)^q\right)^{1/q}\right\|_X \le C\left\|\left(\sum_{j=1}^N |f_j|^q\right)^{1/q}\right\|_X, \tag{6.1}$$

where C is independent of $N \in \mathbb{N}$ and of the finite sequence $\{f_j\}_{j=1}^N$.

We have the corresponding statement to Theorem 1.1.

Theorem 6.1. *Let $\rho \in (0,\infty)$. Let X be a ball Banach function space on $\mathbb{R}^n$, and let X' be its Köthe associate space.*

(*i*) *The following statements are equivalent.*

(*a*) $\mathcal{M}_\rho^{\rm loc} : X \to X$ *and* $\mathcal{M}_\rho^{\rm loc} : X' \to X'$ *are bounded.*

(*b*) *There exist $q \in (1,\infty)$ and a constant $C > 0$ such that, for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N$ of measurable functions, the finite vector-valued inequality*

$$\left\| \left(\sum_{j=1}^N (\mathcal{M}_\rho^{\rm loc} f_j)^q \right)^{\frac{1}{q}} \right\|_X \le C \left\| \left(\sum_{j=1}^N |f_j|^q \right)^{\frac{1}{q}} \right\|_X \tag{6.2}$$

holds whenever the quantity on the right is finite. The constant C is independent of N and of the family $\{f_j\}_{j=1}^N$.

(*c*) *For every $q \in (1,\infty)$, there is a constant $C > 0$ such that (6.2) holds for every $N \in \mathbb{N}$, with C independent of N and of the family $\{f_j\}_{j=1}^N$.*

(*ii*) *The parameter $q \in (0,\infty)$ must satisfy $q > 1$ if there exists a finite constant C satisfying the inequality (6.2) for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N$ of measurable functions. More precisely, for $0 < q \le 1$ such an estimate fails on every ball quasi-Banach function space on $\mathbb{R}^n$.*

(*iii*) *Fix $1 < q < \infty$. Suppose that the inequality (6.2), with this value of q and constant C, holds for every $N \in \mathbb{N}$. Then, for every sequence $\{f_j\}_{j=1}^\infty$ satisfying*

$$\left(\sum_{j=1}^\infty |f_j|^q \right)^{\frac{1}{q}} \in X,$$

one has

$$\left\| \left(\sum_{j=1}^\infty (\mathcal{M}_\rho^{\rm loc} f_j)^q \right)^{\frac{1}{q}} \right\|_X \le C \left\| \left(\sum_{j=1}^\infty |f_j|^q \right)^{\frac{1}{q}} \right\|_X . \tag{6.3}$$

Conversely, this estimate for all sequences implies (6.2) by taking $f_j = 0$ for $j > N$, with the same constant C.

(*iv*) *The following statements are equivalent. The constant C may be chosen to be the same in (a)–(c).*

(*a*) *There is a constant $C > 0$ such that*

$$\|\mathcal{M}_\rho^{\rm loc} f\|_X \le C \|f\|_X \qquad (f \in X).$$

(*b*) *There is a constant $C > 0$ such that, for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N$ satisfying $\max\limits_{1\le j\le N} |f_j| \in X$,*

$$\left\| \max_{1\le j\le N} \mathcal{M}_\rho^{\rm loc} f_j \right\|_X \le C \left\| \max_{1\le j\le N} |f_j| \right\|_X .$$

(*c*) *There is a constant $C > 0$ such that, for every sequence $\{f_j\}_{j=1}^\infty$ with $\sup\limits_{j\in\mathbb{N}} |f_j| \in X$,*

$$\left\| \sup_{j\in\mathbb{N}} \mathcal{M}_\rho^{\rm loc} f_j \right\|_X \le C \left\| \sup_{j\in\mathbb{N}} |f_j| \right\|_X .$$

These equivalent conditions need not imply that $\mathcal{M}_\rho^{\mathrm{loc}}$ is bounded on X'.

We have the corresponding statement to Theorem 1.3. To formulate the powered result, let Y be a ball quasi-Banach function space and let $r > 0$. Recall that Y^r is defined by (1.6) and that Y is r-convex with constant one if (5.3) holds. Under this assumption, $X = Y^r$ is a Banach lattice with the Fatou property.

For $\rho \in (0, \infty]$, define

$$\mathcal{M}_{\rho,r}^{\mathrm{loc}} f = \left[\mathcal{M}_\rho^{\mathrm{loc}}(|f|^r)\right]^{1/r}. \tag{6.4}$$

Theorem 6.2. *Let $\rho \in (0, \infty)$. Let Y be an r-convex ball quasi-Banach function space with convexity constant one for some $r > 0$, and put $X = Y^r$ with the norm in* (1.6).

(*i*) *The following statements are equivalent.*
  (*a*) $\mathcal{M}_\rho^{\mathrm{loc}} : X \to X$ *and* $\mathcal{M}_\rho^{\mathrm{loc}} : X' \to X'$ *are bounded.*
  (*b*) *There exist $q > r$ and a constant $C > 0$ such that, for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N \subset L_{\mathrm{loc}}^r(\mathbb{R}^n)$,*

$$\left\| \left(\sum_{j=1}^N (\mathcal{M}_{\rho,r}^{\mathrm{loc}} f_j)^q \right)^{\frac{1}{q}} \right\|_Y \le C \left\| \left(\sum_{j=1}^N |f_j|^q \right)^{\frac{1}{q}} \right\|_Y \tag{6.5}$$

  whenever the quantity on the right is finite. The constant C is independent of N and of the family $\{f_j\}_{j=1}^N$.
  (*c*) *For every $q > r$, there is a constant $C > 0$ such that* (6.5) *holds for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N \subset L_{\mathrm{loc}}^r(\mathbb{R}^n)$ for which the quantity on the right is finite, with C independent of N and of the family.*

Each of these conditions implies $Y \subset L_{\mathrm{loc}}^r(\mathbb{R}^n)$. Moreover, for $q > r$, a constant C works in (6.5) *if and only if C^r works in* (6.2) *on X with q/r in place of q.*

(*ii*) *Let $q > 0$. Suppose that there is a finite constant C such that* (6.5) *holds for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N \subset L_{\mathrm{loc}}^r(\mathbb{R}^n)$ for which the quantity on the right is finite, with C independent of N and of the family. Then necessarily $q > r$ and $Y \subset L_{\mathrm{loc}}^r(\mathbb{R}^n)$. In particular, no estimate in* (6.5) *can hold for every $N \in \mathbb{N}$ when $0 < q \le r$.*

(*iii*) *Fix $q > r$ and suppose that* (6.5)*, with constant C, holds for every $N \in \mathbb{N}$. Then, for every sequence $\{f_j\}_{j=1}^\infty$ satisfying $\left(\sum_{j=1}^\infty |f_j|^q\right)^{\frac{1}{q}} \in Y$, one has*

$$\left\| \left(\sum_{j=1}^\infty (\mathcal{M}_{\rho,r}^{\mathrm{loc}} f_j)^q \right)^{\frac{1}{q}} \right\|_Y \le C \left\| \left(\sum_{j=1}^\infty |f_j|^q \right)^{\frac{1}{q}} \right\|_Y. \tag{6.6}$$

Conversely, this estimate for all sequences implies (6.5) *by taking $f_j = 0$ for $j > N$, with the same constant C.*

(*iv*) *Suppose that $Y \subset L_{\mathrm{loc}}^r(\mathbb{R}^n)$. The following statements are equivalent. The constant C may be chosen to be the same in (a)–(c).*
  (*a*) *There is a constant $C > 0$ such that*

$$\|\mathcal{M}_{\rho,r}^{\mathrm{loc}} f\|_Y \le C \|f\|_Y \qquad (f \in Y).$$

(*b*) *There is a constant $C > 0$ such that, for every $N \in \mathbb{N}$ and every finite family $\{f_j\}_{j=1}^N$ satisfying $\max_{1\le j\le N} |f_j| \in Y$,*

$$\left\| \max_{1\le j\le N} \mathcal{M}^{\mathrm{loc}}_{\rho,r} f_j \right\|_Y \le C \left\| \max_{1\le j\le N} |f_j| \right\|_Y .$$

(*c*) *There is a constant $C > 0$ such that, for every sequence $\{f_j\}_{j=1}^\infty$ with $\sup_{j\in\mathbb{N}} |f_j| \in Y$,*

$$\left\| \sup_{j\in\mathbb{N}} \mathcal{M}^{\mathrm{loc}}_{\rho,r} f_j \right\|_Y \le C \left\| \sup_{j\in\mathbb{N}} |f_j| \right\|_Y .$$

Remark 6.3. Let $a, b \in (0,\infty)$. Replacing the truncation parameter a by b does not change any of the boundedness or vector-valued properties in Theorems 6.1 and 6.2. More precisely, each statement involving $\mathcal{M}^{\mathrm{loc}}_a$ holds if and only if the corresponding statement involving $\mathcal{M}^{\mathrm{loc}}_b$ holds; in the powered case, the same assertion holds with $\mathcal{M}^{\mathrm{loc}}_{a,r}$ and $\mathcal{M}^{\mathrm{loc}}_{b,r}$. The constants may depend on a and b. This equivalence concerns only finite truncation parameters and does not compare these operators with the global Hardy–Littlewood maximal operator $\mathcal{M} = \mathcal{M}^{\mathrm{loc}}_\infty$. Consequently, boundedness for one, and hence every, finite truncation parameter does not by itself imply boundedness of $\mathcal{M}$.

The power rescaling underlying Theorem 6.2 is particularly simple. If $h_j = |f_j|^r$ and $a = q/r$, then

$$\left\| \left(\sum_{j=1}^N (\mathcal{M}^{\mathrm{loc}}_{\rho,r} f_j)^q \right)^{1/q} \right\|_Y^r = \left\| \left(\sum_{j=1}^N (\mathcal{M}^{\mathrm{loc}}_{\rho} h_j)^a \right)^{1/a} \right\|_X , \tag{6.7}$$

$$\left\| \left(\sum_{j=1}^N |f_j|^q \right)^{1/q} \right\|_Y^r = \left\| \left(\sum_{j=1}^N h_j^a \right)^{1/a} \right\|_X . \tag{6.8}$$

Thus the threshold $q > r$ is exactly the threshold $a > 1$ in Theorem 6.1.

6.2. Comparison of finite truncation parameters.

Lemma 6.4 (Pointwise comparison of finite truncation parameters)**.** *Let $0 < a < b < \infty$. There exist $m \in \mathbb{N}$ and $C = C(n, b/a) > 0$ such that*

$$\mathcal{M}^{\mathrm{loc}}_b f(x) \le C (\mathcal{M}^{\mathrm{loc}}_a)^m f(x) \tag{6.9}$$

for every measurable function f and every $x \in \mathbb{R}^n$, with both sides understood as extended nonnegative quantities.

Proof. Set $R = b/a$. In [14, proof of Proposition 3.3], the following pointwise estimate is established: there exist $m = m(R) \in \mathbb{N}$ and $C_{n,R} > 0$ such that

$$\mathcal{M}^{\mathrm{loc}}_R h(z) \le C_{n,R} (\mathcal{M}^{\mathrm{loc}}_1)^m h(z)$$

for every measurable function h and every $z \in \mathbb{R}^n$.

Let $g(y) = f(ay)$. A change of variables in the defining averages gives

$$\mathcal{M}^{\mathrm{loc}}_R g(x/a) = \mathcal{M}^{\mathrm{loc}}_b f(x), \qquad \mathcal{M}^{\mathrm{loc}}_1 g(y) = \mathcal{M}^{\mathrm{loc}}_a f(ay).$$

Iterating the second identity yields

$$(\mathcal{M}_1^{\mathrm{loc}})^m g(x/a) = (\mathcal{M}_a^{\mathrm{loc}})^m f(x).$$

Therefore,

$$\mathcal{M}_b^{\mathrm{loc}} f(x) = \mathcal{M}_R^{\mathrm{loc}} g(x/a) \le C_{n,R}(\mathcal{M}_1^{\mathrm{loc}})^m g(x/a) = C_{n,R}(\mathcal{M}_a^{\mathrm{loc}})^m f(x),$$

which proves (6.9). □

If we apply (6.9), we see that the value a in $\mathcal{M}_a^{\mathrm{loc}}$ is not so important as long as $a > 0$.

Corollary 6.5. *Let Z be a ball quasi-Banach function space continuously embedded into $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and let $a, b \in (0,\infty)$.*

(*i*) *The operator $\mathcal{M}_a^{\mathrm{loc}}$ is bounded on Z if and only if $\mathcal{M}_b^{\mathrm{loc}}$ is bounded on Z.*

(*ii*) *Fix $0 < q < \infty$. The following statements are equivalent.*

(*a*) *There is $C_a > 0$ such that*

$$\left\| \left(\sum_{j=1}^N (\mathcal{M}_a^{\mathrm{loc}} f_j)^q \right)^{1/q} \right\|_Z \le C_a \left\| \left(\sum_{j=1}^N |f_j|^q \right)^{1/q} \right\|_Z$$

for every $N \in \mathbb{N}$ and every sequence $\{f_j\}_{j=1}^N$ for which the quantity on the right is finite, with C_a independent of N and of the sequence.

(*b*) *There is $C_b > 0$ such that*

$$\left\| \left(\sum_{j=1}^N (\mathcal{M}_b^{\mathrm{loc}} f_j)^q \right)^{1/q} \right\|_Z \le C_b \left\| \left(\sum_{j=1}^N |f_j|^q \right)^{1/q} \right\|_Z$$

for every $N \in \mathbb{N}$ and every sequence $\{f_j\}_{j=1}^N$ for which the quantity on the right is finite, with C_b independent of N and of the sequence.

Proof. It is enough to consider $a < b$. If $\mathcal{M}_a^{\mathrm{loc}}$ is bounded on Z, then (6.9) and iteration give boundedness of $\mathcal{M}_b^{\mathrm{loc}}$. Likewise, repeated application of the corresponding vector-valued estimate for $\mathcal{M}_a^{\mathrm{loc}}$ gives

$$\left\| \left(\sum_{j=1}^N [(\mathcal{M}_a^{\mathrm{loc}})^m f_j]^q \right)^{1/q} \right\|_Z \le C_a^m \left\| \left(\sum_{j=1}^N |f_j|^q \right)^{1/q} \right\|_Z .$$

Combining this with (6.9) proves the estimate for $\mathcal{M}_b^{\mathrm{loc}}$. The reverse implications follow from $\mathcal{M}_a^{\mathrm{loc}} f \le \mathcal{M}_b^{\mathrm{loc}} f$. □

We slightly refine Lemma 5.5.

Lemma 6.6. *Let Z be a ball quasi-Banach function space whose given quasi-norm is a norm. If $\mathcal{M}_\rho^{\mathrm{loc}}$ is bounded on Z for some $\rho \in (0,\infty)$, then Z is continuously embedded into $L^1(B)$ for every ball B. Consequently, Z is a ball Banach function space.*

Proof. This follows by covering B with cubes of volume ρ^n and applying the argument in the proof of Lemma 5.5. □

6.3. **Weighted inequalities and extrapolation at a fixed truncation parameter.** A weight is a locally integrable function $w : \mathbb{R}^n \to (0,\infty)$ that is positive and finite almost everywhere. For $\rho \in (0,\infty]$ and $1 < p < \infty$, define

$$[w]_{A_p(\rho)} = \sup_{\ell(Q)\le\rho} \langle w\rangle_Q \left(\left\langle w^{-1/(p-1)}\right\rangle_Q\right)^{p-1}. \tag{6.10}$$

We write $w \in A_p(\rho)$ when this quantity is finite. Thus $A_p(1) = A_p^{\mathrm{loc}}$ and $A_p(\infty) = A_p$. For $p = 1$, set

$$[w]_{A_1(\rho)} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \frac{\mathcal{M}_\rho^{\mathrm{loc}} w(x)}{w(x)}. \tag{6.11}$$

Proposition 6.7 (Weighted vector-valued inequality with truncation parameter ρ)**.** *Let $\rho \in (0,\infty)$ and $1 < p, q < \infty$. There is an increasing function $\mathcal{N}_{n,p,q} : [1,\infty) \to (0,\infty)$ such that, for every $w \in A_p(\rho)$, every $N \in \mathbb{N}$, and every finite family for which the quantity on the right is finite,*

$$\left\| \left(\sum_{j=1}^N (\mathcal{M}_\rho^{\mathrm{loc}} f_j)^q \right)^{1/q} \right\|_{L^p(w)} \le \mathcal{N}_{n,p,q}([w]_{A_p(\rho)}) \left\| \left(\sum_{j=1}^N |f_j|^q \right)^{1/q} \right\|_{L^p(w)}. \tag{6.12}$$

The function $\mathcal{N}_{n,p,q}$ is independent of N.

For $\rho = \infty$, (6.12) is the weighted vector-valued maximal inequality of Andersen and John [1], with the same increasing-envelope convention; see Theorem 2.6.

Proof. We remark that, when $\rho = 1$, (6.12) is precisely [15, Lemma 2.11]. We now explain how to extend this inequality to arbitrary values of $\rho < \infty$. For $0 < \rho < \infty$, set

$$\widetilde{f}_j(y) = f_j(\rho y), \qquad \widetilde{w}(y) = w(\rho y).$$

A change of variables gives

$$\mathcal{M}_1^{\mathrm{loc}} \widetilde{f}_j(y) = \mathcal{M}_\rho^{\mathrm{loc}} f_j(\rho y), \qquad [\widetilde{w}]_{A_p(1)} = [w]_{A_p(\rho)},$$

and, for every measurable H,

$$\|H(\rho\,\cdot)\|_{L^p(\widetilde{w})} = \rho^{-n/p} \|H\|_{L^p(w)}.$$

Hence Rychkov's estimate for $\mathcal{M}_1^{\mathrm{loc}}$ [15, Lemma 2.11] yields (6.12) with truncation parameter ρ. The same local result also appears in [14, Proposition 1.9]. The constants in these arguments are uniform when the corresponding A_p characteristic is bounded; taking the increasing envelope of this dependence gives $\mathcal{N}_{n,p,q}$. $\square$

We point out that the local counterpart to Theorem 2.7 is obtained in [2, Theorem 3.1]. Proposition 6.8 is known as [2, Theorem 3.1]. To keep track of the constant in (6.13), we supply the proof.

Proposition 6.8 (Extrapolation at a fixed truncation parameter)**.** *Let $\rho \in (0,\infty)$. Let X be a ball Banach function space such that $\mathcal{M}_\rho^{\mathrm{loc}}$ is bounded on both X and X'. Fix $p_0 \in (1,\infty)$ and let $\mathcal{N} : [1,\infty) \to (0,\infty)$ be increasing. Let $\mathcal{F}$ be a family of pairs (F,G) of nonnegative measurable functions. Assume that, for every $w \in A_{p_0}(\rho)$ and every $(F,G) \in \mathcal{F}$ with $G \in L^{p_0}(w)$, one has $F \in L^{p_0}(w)$ and*

$$\|F\|_{L^{p_0}(w)} \le \mathcal{N}([w]_{A_{p_0}(\rho)}) \|G\|_{L^{p_0}(w)}. \tag{6.13}$$

Then there is $C = C(X, \rho, p_0, \mathcal{N}) > 0$ such that
$$\|F\|_X \le C\|G\|_X \tag{6.14}$$
for every $(F, G) \in \mathcal{F}$ with $G \in X$.

When $\rho = \infty$, the assertion follows from the extrapolation theorem of Nieraeth [12, Theorem 4.7 and Remark 4.8]. See Theorem 2.7.

Proof. Assume $0 < \rho < \infty$ and set
$$A_X = \max\{1, \|\mathcal{M}_\rho^{\rm loc}\|_{X\to X}\}, \qquad A_{X'} = \max\{1, \|\mathcal{M}_\rho^{\rm loc}\|_{X'\to X'}\}.$$
The half-open unit cubes $k + [0,1)^n$, $k \in \mathbb{Z}^n$, are pairwise disjoint and their union is $\mathbb{R}^n$. Since this collection is countable, choose a sequence $\{Q_\nu\}_{\nu=1}^\infty$ in which each of these cubes occurs exactly once, and put
$$u_0 = \sum_{\nu=1}^\infty \frac{2^{-\nu}}{1 + \|\mathbf{1}_{Q_\nu}\|_X}\mathbf{1}_{Q_\nu}, \qquad v_0 = \sum_{\nu=1}^\infty \frac{2^{-\nu}}{1 + \|\mathbf{1}_{Q_\nu}\|_{X'}}\mathbf{1}_{Q_\nu}.$$
The lattice and Fatou properties imply $u_0 \in X$, $v_0 \in X'$, $u_0, v_0 > 0$ almost everywhere, and $\|u_0\|_X, \|v_0\|_{X'} \le 1$.

For $Z \in \{X, X'\}$, let A_Z denote A_X or $A_{X'}$, respectively. For every nonnegative $h \in Z$, define
$$\mathcal{R}_Z h = \sum_{k=0}^\infty \frac{(\mathcal{M}_\rho^{\rm loc})^k h}{(2A_Z)^k}. \tag{6.15}$$
The Fatou property and the boundedness of $\mathcal{M}_\rho^{\rm loc}$ give
$$h \le \mathcal{R}_Z h, \qquad \|\mathcal{R}_Z h\|_Z \le 2\|h\|_Z, \qquad \mathcal{M}_\rho^{\rm loc}(\mathcal{R}_Z h) \le 2A_Z \mathcal{R}_Z h. \tag{6.16}$$
Consequently, if $h > 0$ almost everywhere, then $\mathcal{R}_Z h \in A_1(\rho)$ and $[\mathcal{R}_Z h]_{A_1(\rho)} \le 2A_Z$.

Fix $(F, G) \in \mathcal{F}$ with $G \in X$. Let $0 \le h \in X'$ satisfy $\|h\|_{X'} \le 1$, and fix $\varepsilon > 0$. Set
$$U = \mathcal{R}_X(G + \varepsilon u_0), \qquad V = \mathcal{R}_{X'}(h + \varepsilon v_0).$$
Then U, V are strictly positive $A_1(\rho)$ weights and
$$\|U\|_X \le 2(\|G\|_X + \varepsilon), \qquad \|V\|_{X'} \le 2(1 + \varepsilon). \tag{6.17}$$
Write $p = p_0$ and define $W = U^{1-p}V$. For every cube Q with $\ell(Q) \le \rho$, the $A_1(\rho)$ inequalities yield
$$\langle W\rangle_Q \le [U]_{A_1(\rho)}^{p-1} \langle U\rangle_Q^{1-p} \langle V\rangle_Q,$$
$$\left\langle W^{-1/(p-1)}\right\rangle_Q \le [V]_{A_1(\rho)}^{1/(p-1)} \langle V\rangle_Q^{-1/(p-1)} \langle U\rangle_Q.$$
Thus
$$[W]_{A_p(\rho)} \le (2A_X)^{p-1}(2A_{X'}) =: K. \tag{6.18}$$
The same calculation shows the required local integrability of the two functions in the $A_p(\rho)$ condition.

Let $I = \int_{\mathbb{R}^n} U(x)V(x)\,dx$. By (2.3) and (6.17),
$$I \le 4(\|G\|_X + \varepsilon)(1 + \varepsilon). \tag{6.19}$$
Since $G \le U$ and $h \le V$,
$$\int_{\mathbb{R}^n} G^p W\,dx \le I, \qquad \int_{\mathbb{R}^n} h^{p'} W^{-1/(p-1)}\,dx \le I.$$

Weighted Hölder's inequality, (6.13), and (6.18) give

$$\int_{\mathbb{R}^n} Fh\,dx \le \mathcal{N}(K)I \le 4\mathcal{N}(K)(\|G\|_X+\varepsilon)(1+\varepsilon).$$

Taking the supremum over all such h, using $X''=X$ from Lemma 2.5, and then letting $\varepsilon \downarrow 0$ proves (6.14). □

6.4. **Infinite-sequence estimates and the cases $0<q\le 1$ and $q=\infty$.** Corresponding to Lemma 2.8, we have the following result.

Proposition 6.9. *Let $\rho\in(0,\infty)$, let X be a ball quasi-Banach function space, and let $0<q<\infty$. Assume that (6.1) holds with constant C for every $N\in\mathbb{N}$. Then $X\subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and*

$$\left\|\left(\sum_{j=1}^{\infty}(\mathcal{M}^{\mathrm{loc}}_{\rho}f_j)^q\right)^{1/q}\right\|_X \le C\left\|\left(\sum_{j=1}^{\infty}|f_j|^q\right)^{1/q}\right\|_X \tag{6.20}$$

for every sequence $\{f_j\}_{j=1}^{\infty}$ such that

$$\left(\sum_{j=1}^{\infty}|f_j|^q\right)^{1/q}\in X.$$

Proof. Taking $N=1$ in (6.1), we obtain

$$\|\mathcal{M}^{\mathrm{loc}}_{\rho}f\|_X \le C\|f\|_X \qquad (f\in X).$$

Therefore, the argument in the proof of Lemma 2.8 works to show that $X\subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For example, instead of (2.9) we can show that

$$\mathcal{M}^{\mathrm{loc}}_{\rho}f=\infty \tag{6.21}$$

on Q if $\|f\|_{L^1(Q)}=\infty$ for some cube Q with $\ell(Q)\le\rho$. Having established $X\subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define F_N, G_N, F, G exactly as in the proof of Lemma 2.8, with $\mathcal{M}$ replaced by $\mathcal{M}^{\mathrm{loc}}_{\rho}$. Then

$$\|G_N\|_X\le C\|F_N\|_X\le C\|F\|_X.$$

Since $G_N\uparrow G$, the Fatou property gives (6.20). □

Corresponding to Proposition 2.9, we have the following negative result.

Proposition 6.10. *Let $\rho\in(0,\infty)$, let X be a ball quasi-Banach function space on $\mathbb{R}^n$, and let $0<q\le 1$. There is no constant $C>0$ such that*

$$\left\|\left(\sum_{j=1}^{N}(\mathcal{M}^{\mathrm{loc}}_{\rho}f_j)^q\right)^{1/q}\right\|_X \le C\left\|\left(\sum_{j=1}^{N}|f_j|^q\right)^{1/q}\right\|_X \tag{6.22}$$

holds for every $N\in\mathbb{N}$ and every finite family of measurable functions for which the quantity on the right is finite.

Proof. Since the proof of Proposition 2.9 used compactly supported functions, the same argument as 2.9 works. In fact, choose $\tau>0$ such that $2\tau\le\rho$, and reexamine the proof of Proposition 2.9 with $[0,1)^n$ replaced by $[0,\tau)^n$. □

The next proposition is a counterpart to Theorem 1.1(iv).

Proposition 6.11 (ℓ^∞-valued estimates)**.** *Let $\rho \in (0,\infty)$ and let X be a ball quasi-Banach function space continuously embedded into $L^1_{\rm loc}(\mathbb{R}^n)$. The following statements are equivalent. The constant C may be chosen to be the same in (i)–(iii).*

(i) *There is $C>0$ such that*

$$\|\mathcal{M}^{\rm loc}_\rho f\|_X \le C\|f\|_X \qquad (f \in X).$$

(ii) *There is $C>0$ such that, for every $N \in \mathbb{N}$ and every sequence $\{f_j\}_{j=1}^N$ satisfying $\max\limits_{1\le j\le N}|f_j| \in X$,*

$$\left\|\max_{1\le j\le N}\mathcal{M}^{\rm loc}_\rho f_j\right\|_X \le C\left\|\max_{1\le j\le N}|f_j|\right\|_X.$$

(iii) *There is $C>0$ such that, for every sequence $\{f_j\}_{j=1}^\infty$ satisfying $\sup\limits_{j\in\mathbb{N}}|f_j| \in X$,*

$$\left\|\sup_{j\in\mathbb{N}}\mathcal{M}^{\rm loc}_\rho f_j\right\|_X \le C\left\|\sup_{j\in\mathbb{N}}|f_j|\right\|_X.$$

Moreover, even when X is a ball Banach function space, these equivalent conditions need not imply that $\mathcal{M}^{\rm loc}_\rho$ is bounded on X'.

Proof. Go through the same argument as Theorem 1.1 (*iv*) works.

For the final assertion, take $X = L^\infty(\mathbb{R}^n)$. Then $\mathcal{M}^{\rm loc}_\rho$ is bounded on X, whereas $X' = L^1(\mathbb{R}^n)$ and $\mathcal{M}^{\rm loc}_\rho$ is not bounded on $L^1(\mathbb{R}^n)$. □

6.5. **Dyadic grids and the sparse criterion at a fixed truncation parameter.** We use the same system $\mathcal{D}$ as Section 2.3. This adjacent-grid fact is standard; see [7]. Consequently, for $\rho \in (0,\infty)$,

$$\mathcal{M}^{\rm loc}_\rho f \le 6^n \sum_{t=1}^{3^n} \mathcal{M}_{\mathscr{D}^t,(6\rho)} f, \tag{6.23}$$

where

$$\mathcal{M}_{\mathscr{D},(R)} f(x) = \sup_{Q\in\mathscr{D}(R)} \mathbf{1}_Q(x)\,\langle |f|\rangle_Q\,.$$

Here, for a dyadic grid $\mathscr{D}$ and $R \in (0,\infty]$, write

$$\mathscr{D}(R) = \{Q \in \mathscr{D} : \ell(Q) \le R\}, \qquad \mathscr{D}(\infty) = \mathscr{D}. \tag{6.24}$$

When $\mathscr{S} \subset \mathscr{D}(R)$, we also write $\mathcal{T}_{\mathscr{S},q,(R)} := \mathcal{T}_{\mathscr{S},q}$ and $\mathcal{A}_{\mathscr{S},(R)} := \mathcal{A}_{\mathscr{S}}$ to make the upper bound R on the side lengths of the cubes in $\mathscr{S}$ explicit.

Lemma 6.12 (Finite dyadic sparse domination)**.** *Let $R \in (0,\infty)$, let $\mathscr{D}$ be a dyadic grid, and let $\mathscr{F} \subset \mathscr{D}(R)$ be a finite collection. Set*

$$\mathcal{M}_{\mathscr{F},(R)} f(x) = \max_{Q\in\mathscr{F}} \langle |f|\rangle_Q\, \mathbf{1}_Q(x)$$

when $\mathscr{F} \ne \varnothing$, and use the convention $\mathcal{M}_{\varnothing,(R)} f = 0$. For every locally integrable f, there is a finite $1/2$-sparse collection $\mathscr{S} \subset \mathscr{F}$ such that

$$\mathcal{M}_{\mathscr{F},(R)} f \le 2\mathcal{A}_{\mathscr{S},(R)} f \quad \textit{almost everywhere.} \tag{6.25}$$

Proof. Apply the recursive selection in Lemma 2.11 to $\mathscr{F}$. Since every selected cube belongs to $\mathscr{F}$, one has

$$\mathscr{S} \subset \mathscr{F} \subset \mathscr{D}(R).$$

The pointwise estimate follows from (2.18). □

The proof of the next proposition follows the corresponding argument in Proposition 4.2 with $\mathscr{D}$ replaced by $\mathscr{D}(6\rho)$ and the local maximal operators kept explicit throughout.

Proposition 6.13 (Sparse criterion with truncation parameter ρ)**.** *Let $\rho \in (0, \infty)$ and let X be a ball Banach function space. The following statements are equivalent.*

(*i*) *The operator $\mathcal{M}^{\mathrm{loc}}_{\rho}$ is bounded on both X and X'.*

(*ii*) *There is a constant $C_{\mathrm{sp}} > 0$ such that*

$$\|\mathcal{A}_{\mathscr{S},(6\rho)} f\|_X \leq C_{\mathrm{sp}} \|f\|_X \tag{6.26}$$

for every finite $1/2$-sparse subcollection $\mathscr{S}$ of $\mathscr{D}(6\rho)$, where $\mathscr{D}$ is an arbitrary dyadic grid, and every $f \in X$, with a constant independent of $\mathscr{D}$ and $\mathscr{S}$.

Proof. The implication (i) ⇒ (ii) follows from Corollary 6.5 and the sparse sets E_Q by duality if we argue similarly to Lemma 4.1.

Conversely, assume (ii), namely, (6.26). Then the boundedness of $\mathcal{M}^{\mathrm{loc}}$ on X follows by repeating the corresponding part of Proposition 4.2 with $\mathcal{D}$ replaced by $\mathcal{D}(6\rho)$, using Lemma 6.12, and the Fatou property of X, and (6.23). Since (6.26) is symmetric with respect to X, that is, we can replace X with X' assuming (6.26), we see that $\mathcal{M}^{\mathrm{loc}}_{\rho}$ is bounded also on X'. □

6.6. **From a vector-valued bound to sparse averaging.** For a finite truncation parameter, the proof of the next proposition follows the argument of Proposition 3.1. It applies the vector-valued maximal inequality to the functions $h_Q = \langle |f| \rangle_Q \mathbf{1}_{E_Q}$, where the pairwise disjoint sets $E_Q \subset Q$ witness sparseness. By Corollary 6.5, the required vector-valued estimate is available for $\mathcal{M}^{\mathrm{loc}}_{6\rho}$.

Proposition 6.14 (A sparse ℓ^q estimate)**.** *Let $\rho \in (0, \infty)$, let X be a ball Banach function space, and let $1 < q < \infty$. Suppose that* (6.1) *holds with constant C. Then there is $K > 0$ such that*

$$\|\mathcal{T}_{\mathscr{S},q,(6\rho)} f\|_X \leq K \|f\|_X \tag{6.27}$$

for every finite $1/2$-sparse subcollection $\mathscr{S}$ of $\mathscr{D}(6\rho)$, where $\mathscr{D}$ is an arbitrary dyadic grid, and every $f \in X$. The constant is independent of $\mathscr{D}$ and $\mathscr{S}$.

Proof. We follow the idea of Proposition 3.1. We use $\mathcal{M}^{\mathrm{loc}}$ instead of $\mathcal{M}$. Assume $0 < \rho < \infty$. By Corollary 6.5, the vector-valued estimate for $\mathcal{M}^{\mathrm{loc}}_{6\rho}$ over finite families holds with some constant $C_{6\rho}$. Choose measurable sets $E_Q \subset Q$, $Q \in \mathscr{S}$, such that $|E_Q| \geq |Q|/2$ for every $Q \in \mathscr{S}$ and the sets E_Q are pairwise disjoint, and put

$$h_Q = \langle |f| \rangle_Q \mathbf{1}_{E_Q}.$$

Since the sets E_Q are disjoint and every $Q \in \mathscr{S}$ satisfies $\ell(Q) \leq 6\rho$,

$$\left(\sum_{Q \in \mathscr{S}} |h_Q|^q \right)^{1/q} \leq \mathcal{M}^{\mathrm{loc}}_{6\rho} f.$$

Moreover, for $x \in Q$,

$$\mathcal{M}^{\mathrm{loc}}_{6\rho} h_Q(x) \geq \langle h_Q \rangle_Q = \langle |f| \rangle_Q \frac{|E_Q|}{|Q|} \geq \frac{1}{2} \langle |f| \rangle_Q .$$

Therefore

$$\begin{aligned} \frac{1}{2} \|\mathcal{T}_{\mathscr{S},q,(6\rho)} f\|_X &\leq \left\| \left(\sum_{Q \in \mathscr{S}} (\mathcal{M}^{\mathrm{loc}}_{6\rho} h_Q)^q \right)^{1/q} \right\|_X \\ &\leq C_{6\rho} \left\| \left(\sum_{Q \in \mathscr{S}} |h_Q|^q \right)^{1/q} \right\|_X \\ &\leq C_{6\rho} \|\mathcal{M}^{\mathrm{loc}}_{6\rho} f\|_X \\ &\leq C_{6\rho}^2 \|f\|_X . \end{aligned}$$

This proves (6.27). □

As we did in Theorem 1.1, if we assume that $\mathcal{M}^{\mathrm{loc}}_{\rho}$ satisfies the vector-valued inequality (6.1), we have uniform sparse averaging bound of the operators $\mathcal{A}_{\mathscr{S},(6\rho)}$ are uniformly bounded on X over all dyadic grids and all finite $1/2$-sparse collections $\mathscr{S} \subset \mathscr{D}(6\rho)$.

Corollary 6.15 (Uniform sparse averaging bound)**.** *Let $1 < q < \infty$. Let $\rho \in (0, \infty)$. Suppose that* (6.1) *holds with constant C. The operators $\mathcal{A}_{\mathscr{S},(6\rho)}$ are uniformly bounded on X over all dyadic grids and all finite $1/2$-sparse collections $\mathscr{S} \subset \mathscr{D}(6\rho)$.*

Proof. Let $m \in \mathbb{N}$ satisfy $m > q$. We use Theorem 3.2 to have

$$\mathcal{A}_{\mathscr{S},(6\rho)} f \leq \Gamma_{q,m} (\mathcal{T}_{\mathscr{S},q})^m f = \Gamma_{q,m} (\mathcal{T}_{\mathscr{S},q,(6\rho)})^m f \quad \text{almost everywhere.} \tag{6.28}$$

Let K be the uniform bound in (6.27). Pointwise estimate (6.28) gives

$$\|\mathcal{A}_{\mathscr{S},(6\rho)} f\|_X \leq \Gamma_{q,m} \|(\mathcal{T}_{\mathscr{S},q,(6\rho)})^m f\|_X \leq \Gamma_{q,m} K^m \|f\|_X .$$

Thus, the operators $\mathcal{A}_{\mathscr{S},(6\rho)}$ are uniformly bounded on X over all dyadic grids and all finite $1/2$-sparse collections $\mathscr{S} \subset \mathscr{D}(6\rho)$. □

6.7. **Proof of Theorem 6.1.** We remark that the proof of Theorem 6.2 is similar to that of Theorem 1.3. We concentrate on the proof of Theorem 6.1.

Proof of Theorem 6.1. Part (ii) is Proposition 6.10, part (iii) is Proposition 6.9, and part (iv) is Proposition 6.11. It remains to prove part (i).

Assume (b). Proposition 6.14 and Corollary 6.15 give a uniform sparse bound. Proposition 6.13 then yields (a). Thus $(b) \Rightarrow (a)$.

Assume (a), and fix $q \in (1, \infty)$. Choose any $p_0 \in (1, \infty)$. For a finite family $\{f_j\}_{j=1}^N$, put

$$F = \left(\sum_{j=1}^N (\mathcal{M}^{\mathrm{loc}}_{\rho} f_j)^q \right)^{1/q}, \qquad G = \left(\sum_{j=1}^N |f_j|^q \right)^{1/q}.$$

Proposition 6.7 verifies the weighted hypothesis of Proposition 6.8, with a bound independent of N. Applying extrapolation at a fixed truncation parameter to the family of all such pairs (F, G) gives (6.1) on X. Since q was arbitrary, (c) holds. The implication $(c) \Rightarrow (b)$ is immediate. Hence all three conditions are equivalent. $\square$

Data availability

Not applicable.

Author contributions

The three authors contributed equally to the correctness of this paper.

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References

[1] K. F. Andersen and R. T. John, *Weighted inequalities for vector-valued maximal functions and singular integrals*, Studia Math. **69** (1981), no. 1, 19–31, DOI: 10.4064/sm-69-1-19-31.

[2] M. M. Cao, J. J. Marín, and J. M. Martell, Extrapolation on function and modular spaces, and applications, *Adv. Math.*, **406** (2022), 108520. https://doi.org/10.1016/j.aim.2022.108520.

[3] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. **93** (1971), no. 1, 107–115, DOI: 10.2307/2373450.

[4] Y. Sawano, K.-P. Ho, D. Yang, and S. Yang, *Hardy spaces for ball quasi-Banach function spaces*, Dissertationes Math. (Rozprawy Mat.), **525** (2017), 102 pp.

[5] M. Izuki, T. Noi, and Y. Sawano, *Applications of extrapolations to wavelet characterization of various function spaces and extension operators*, Bull. Korean Math. Soc. **63** (2026), no. 2, 477–499, DOI: 10.4134/BKMS.b250020.

[6] A. K. Lerner, *A note on the maximal operator on Banach function spaces*, J. Geom. Anal. **35** (2025), article 177, DOI: 10.1007/s12220-025-02015-2.

[7] A. K. Lerner, *A simple proof of the A_2 conjecture*, Int. Math. Res. Not. IMRN 2013, no. 14, 3159–3170.

[8] A. K. Lerner, *Dual boundedness of the maximal operator on concave Banach function spaces*, arXiv:2608.01140.

[9] E. Lorist and Z. Nieraeth, *Extrapolation of compactness on Banach function spaces*, J. Fourier Anal. Appl. **30** (2024), no. 3, article 30, DOI: 10.1007/s00041-024-10087-x.

[10] E. Lorist and Z. Nieraeth, *Banach function spaces done right*, Indag. Math. (N.S.) **35** (2024), no. 2, 247–268, DOI: 10.1016/j.indag.2023.11.004.

[11] M. Mastyło and Y. Sawano, The Navier–Stokes equation in some general Banach spaces, Analysis and Applications, Vol. 24, No. 4 (2026) 1055–1121

[12] Z. Nieraeth, *Extrapolation in general quasi-Banach function spaces*, J. Funct. Anal. **285** (2023), no. 10, article 110130, DOI: 10.1016/j.jfa.2023.110130.

[13] Z. Nieraeth, *The Muckenhoupt condition*, J. Funct. Anal. **290** (2026), no. 1, article 111209, DOI: 10.1016/j.jfa.2025.111209.

[14] T. Nogayama and Y. Sawano, *Local Muckenhoupt class for variable exponents*, J. Inequal. Appl. 2021, article 70, DOI: 10.1186/s13660-021-02601-2.

[15] V. S. Rychkov, *Littlewood–Paley theory and function spaces with A_p^{loc} weights*, Math. Nachr. **224** (2001), no. 1, 145–180.

[16] D. V. Rutsky, *Remarks on A_p-regular lattices of measurable functions*, St. Petersburg Math. J. **27** (2016), no. 5, 813–823, DOI: 10.1090/spmj/1418.
[17] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, NJ, 1970.
[18] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, Vol. 43, Princeton University Press, Princeton, NJ, 1993.
[19] Y. Wan, D. Yang, and Y. Zhao, *Grand Triebel–Lizorkin–Bourgain–Morrey spaces: nontriviality, embeddings, and boundedness of certain operators*, Bull. Math. Sci. **16** (2026), no. 1, article 2550031, 59 pp., DOI: 10.1142/S1664360725500316.
[20] A.C. Zaanen, Integration, North-Holland Publishing Co. Amsterdam; Interscience Publishers John Wiley & Sons, Inc. New York, 1967.

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