The crossing numbers of knots and links via 4D topology

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ABSTRACT

The diagrams of links (including knots) are characterized in terms of circular rigid disk-chord diagrams of their spun ribbon torus-links in the 4-sphere. As a result, the crossing numbers of links are equal to the chord indexes of their spun ribbon torus-links. By using this result, additivity on the crossing numbers of links under connected sums can be shown.

Keywords: Crossing number, Spun torus-link, Ribbon torus-link, Chord index.
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1. Introduction

A *diagram* D on the 2-sphere S^2 is an immersed oriented loop system on the 2-sphere S^2 whose singularities are only transverse double points with upper-lower relations. When the 2-sphere S^2 is fixed in the 3-sphere S^3, the diagram D represents a unique link k in S^3, obtained by pushing the interior of an upper arc of every double point of D to a positive direction of S^2 in S^3. Every link in S^3 up to equivalences correspond to a diagram on S^2 up to Reidemiester moves, [1]. A surface-link in the 4-sphere S^4 is a (possibly disconnected) closed oriented surface F smoothly embedded in S^4. Let $r = r(F)$ be the number of components of F. When $r = 1$, it is called a surface-knot. The surface-link F is an S^2-*link* or a *torus-link* if all the components of F are 2-spheres or tori, respectively. A *handled-sphere system* is a pair (O, h) where O is a trivial S^2-link and h is a 1-handle system on O, smoothly embedded in S^4. A *ribbon* surface-link F is a surface-link in S^4 constructed from a handled-sphere system (O, h) by surgery along h. The handled-sphere system (O, h) is uniquely constructed from a *chorded-sphere system* (O, α) in S^4 where α is a core arc system of the 1-handle h, called a *chord system*, [2]. The chorded-sphere system (O, α) is uniquely constructed from a *loop-chord graph* (o, α) located in a fixed standard 3-sphere S^3 in S^4 with o a trivial link, called a *based loop system*, by using the Horibe-Yanagawa's lemma, [3]. A regular diagram on S^2 of a chord graph (o, α) in S^3 is a *loop-chord diagram* $C(o, \alpha)$ on S^2, [4, 5]. By convention, the based loop system o on S^2 is oriented in a clockwise orientation and situated to bound a disjoint oriented disk system d in S^2, called the *based disk system*. Under this convention, the loop-chord diagram $C(o, \alpha)$ is also considered as a *disk-chord diagram* and denoted by $C(d, \alpha)$. Every ribbon surface-link up to equivalences is characterized by a loop-chord diagram $C(o, \alpha)$ on S^2 up to *equivalences*, namely up to finite numbers of the following moves: Trivalent graph Reidemeister move M_0, Fusion-fission move M_1 and Chord move M_2, illustrated in Fig. 1, [4, 5, 6, 7]. Two disk-chord diagrams $C(d, \alpha)$ and $C(d', \alpha')$ are *homotopic* if a neighborhood of d in $C(d, \alpha)$ is isotopic to a neighborhood of d' in $C(d', \alpha')$ and the remaining chord system of α is ∂-relatively homotopic to the remaining chord system of α' in S^2. Homotopic loop-chord diagrams are equivalent. The following definition is basic to this paper.

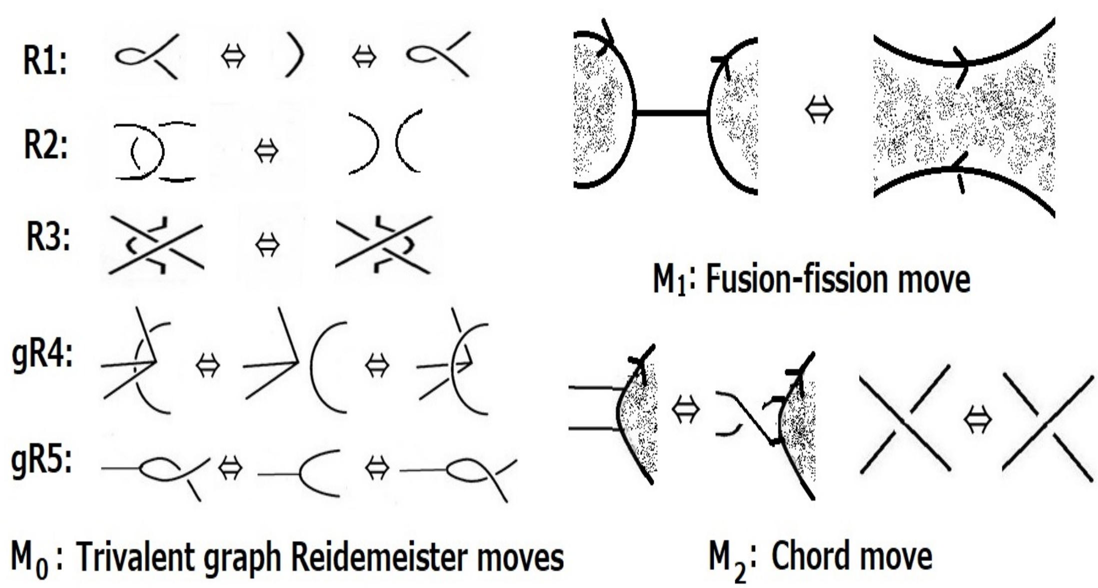

Figure 1: The moves for equivalence of loop-chord diagrams

Definition 1.1. A connected loop-chord system (o, α) in S^3 is *circular* if there are the same number of loops $o_i\,(i=1,2,\ldots,n)$ of o and chords $\alpha_i\,(i=1,2,\ldots,n)$ of α such that the chord α_i connects o_i to o_{i+1} for every i with $n+1\equiv 1$. A loop-chord system is *circular* if every connected component is a circular loop-chord system. A disk-chord system (d,α) is *circular* if the loop-chord system (o,α) is circular, and *circular primitive* or briefly *CP* if every chord of α intersects the interior of every disk in d transversely meets α at just one point.

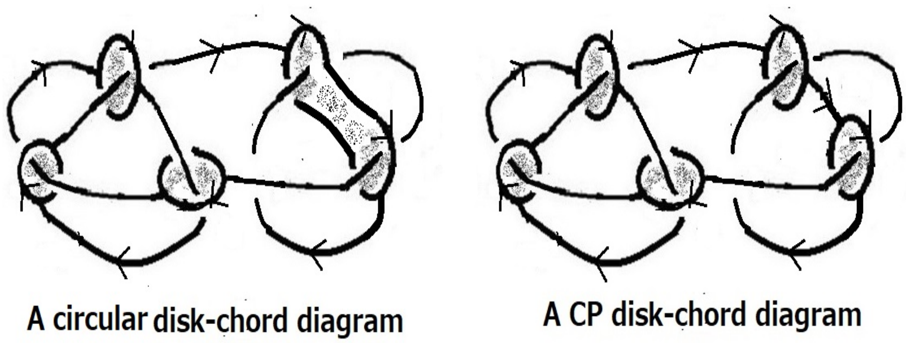

Figure 2: An example of a circular disk-chord diagram and a CP disk-chord diagram that are equivalent by the move M_1

A *CP disk-chord diagram* on S^2 is a diagram $C(d,\alpha)$ on S^2 of a CP disk-chord system (d,α) in S^3. Every ribbon torus-link T in S^4 admits a CP disk-chord diagram $C(d,\alpha)$, which is shown by an equivalent deformation of any given disk-chord diagram $C(d',\alpha')$ of T using a finite number of the moves M_0, M_1 and M_2. An example of a circular disk-chord diagram and a CP disk-chord diagram that are equivalent by the move M_1 is given in Fig. 2. A loop-chord system (o,α) is also *CP* if the disk-chord system (d,α) is CP. The chord systems α of a circular loop-chord diagram $C(o,\alpha)$ and a circular loop-chord diagram $C(d,\alpha)$ have the unique orientation induced from the clockwise orientation of o. From a disk-chord diagram $C(d,\alpha)$, the sphere-chord system (O,α) in S^4 is uniquely constructed by first realizing as a disk-chord system (d,α) in S^3 and then taking the trivial S^2-link $O=\partial(d\times[-1,1])$ for a collar $d\times[-1,1]$ of the disk system d in S^4 given by a collar $S^3\times[-1,1]$ of S^3 in S^4. From this viewpoint, the sphere-chord system (O,α) is *circular* if if the disk-chord system (d,α) is circular, and the statement "the chord system α meets transversely the interior of a disk of d at just n points"is expressed as "the corresponding sphere component of the based sphere system O is passed through n times by the chord system α". The spun torus-link $T(k)$ in S^4 for every link k in S^3 is equivalent to the ribbon torus-link $F(CD)$ of the unique CP disk-chord diagram $CD=C(d,\alpha)$ transformed from any given diagram D of the link k by the transformation given in Fig. 3. This is explained in [4], and for convenience in Section 2. Note that the diagram D is recovered from the CP disk-chord diagram CD by taking the upper arc system of the based loop system o, Fig. 4. The following theorem is a main result.

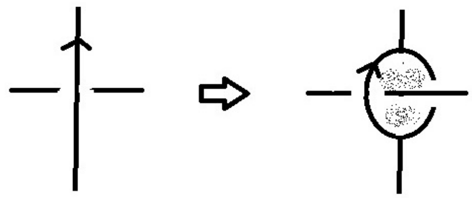

Figure 3: Changing the location from near a crossing point to near a loop with chords or a disk with chords

Theorem 1.2. Every CP disk-chord diagram $C(d, \alpha)$ on S^2 of the spun torus-link $T(k)$ in S^4 of a link k in S^3 is homotopic to the disk-chord diagram CD of a diagram D on S^2 of k.

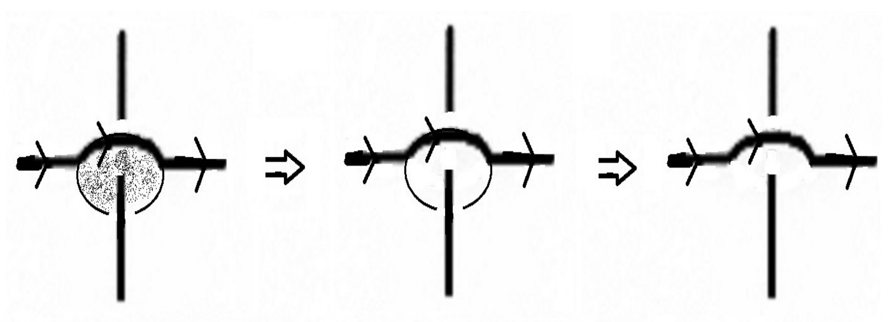

Figure 4: A link k with a disk system d

This proof of Theorem 1.2 is useful to clarify the method used in [8]. A double point of a diagram D is called a *crossing point* of D. The number of crossing points in D is called the *crossing number* of D and denoted by $c(D)$. The *crossing number* $c(k)$ of a link k is the minimum of the crossing numbers $c(D')$ of all diagrams D' of all links k' equivalent the link k. The *chord index* of a circular disk-chord diagram $C(d, \alpha)$ is the number $I(d, \alpha)$ of the geometric transverse intersection number of the interior of the disk system d and the chord system α of the disk-chord system (d, α) in S^3. The *chord index* of a ribbon torus-link F is the minimum $I(F)$ of the chord indexes $I(d', \alpha')$ of all circular disk-chord diagrams $C(d', \alpha')$ of ribbon torus-links F' equivalent to the ribbon torus-link F. If the chord index $I(F)$ of a ribbon torus-link F is n, then there is a CP disk-chord diagram (d, α) of chord index n of F because a circular disk-chord diagram of chord index n is equivalent to a CP disk-chord diagram of chord index n by Fusion-fission move M_1. Since the span torus-link $T(k)$ has the CP disk-chord diagram CD of any diagram D of k, the inequality $c(k) \geq I(T(k))$ holds. By Theorem 1.1, $c(k) \leq I(T(k))$. Thus, the following corollary is obtained.

Corollary 1.3. $c(k) = I(T(k))$ for every link k and the spun torus-link $T(k)$.

The additivity of crossing numbers is known for large classes of knots, e.g., alternating knots, adequate knots, torus knots, and investigated by lots of researchers, [9, 10, 11, 12, 13, 14]. By further developing the argument on Theorem 1.1, the following general additivity of the crossing numbers can be shown.

Theorem 1.4. Let $k_1 \# k_2$ be a connected sum of any knots or links $k_i\,(i = 1, 2)$. Then the identity $c(k_1 \# k_2) = c(k_1) + c(k_2)$ holds.

2. Proof of Theorem 1.2.

Let B be a 3-ball in S^3 with boundary sphere $S = \partial B$. The spun torus-link of a link k in B is the torus-link $T(k) = k \times S^1$ in the 4-sphere $S^4 = B \times S^1 \cup S \times D^2$, where S^1 and D^2 denote the unit circle and the unit disk in the complex number plane, respectively. For a boundary collar $S \times [0, 1]$ of S in B with $S \times \{0\} = S$, assume that the link k is in $S \times [0, 1]$ and the image of k under the projection $S \times [0, 1] \to S$ onto the first factor S is a diagram D of k. Write every point of k as a pair (x, s) with $x \in S$ and $s \in [0, 1]$. Then the *projection trace* of k is the immersed annulus system (namely, the union of transversely meeting annuli) $P(k) = \cup_{(x,s)\in k}\{x\} \times [0, s]$ with $P(k) \cap S = D$. The union $V = P(k) \times S^1 \cup D \times D^2$ is an immersed solid torus system in S^4 with ribbon disk singularity whose boundary is $T(k)$. A multi-punctured solid torus system W is obtained from V by removing an open 3-ball neighborhood of the interior disk of every ribbon disk singularity, so that $\partial W = T(k) \cup O$ for a trivial S^2-link O in S^4. The multi-punctured solid torus system W is uniquely determined by a CP disk-chord system (d, α) in $S \times [0, 1]$ with the intersection arc system $k_d = (\partial d) \cap k$ and the complementary arc system $\alpha = \mathrm{cl}(k \setminus k_d)$. The multi-punctured solid torus system W is directly constructed from the CP handled sphere-chord system (O, h) with h a 1-handle system with core system α obtained from the disk-chord system $D(d, \alpha)$ of a diagram of k so that $W = O \times [0, 1] \cup h$ in S^4 with $\partial W = T(k) \cup O \times \{1\}$, where $O \times \{0\} = O$. From this viewpoint, the multi-punctured solid torus system W is called a *SUPH system* for the ribbon torus-link $T(k)$ in S^4, [15]. Let δ be a disk in S with $\delta^c = \mathrm{cl}(S \setminus \delta)$ the complementary disk of δ in S. Assume that the intersection $t = k \cap (\delta \times [0,1])$ is a *tangle* in the 3-ball $\delta \times [0,1]$, namely a proper arc system in $\delta \times [0,1]$ such that the boundary point set ∂t in the circle $(\partial\delta) \times \{0\} = \partial\delta$. Then the projection $\delta \times [0,1] \to \delta$ sends the tangle t to a tangle diagram t^D in δ with $\partial t^D = \partial t$, which is a subdiagram of the diagram D. The *spun S^2-link* of a tangle t in $\delta \times [0,1]$ is the S^2-linkcl$T(t)$ in S^4 given by the union $T(t) \cup (\partial t) \times D^2$. For the *projection trace* of the tange t, that is the union $P(t) = \cup_{(x,s)\in t}\{x\} \times [0,s]$, the union $V(t) = P(t) \times S^1 \cup D \times D^2$ is an immersed 3-ball system with ribbon disk singularity whose boundary is cl$T(t)$. A multi-punctured 3-ball system $W(t)$ of the CP disk-chord diagram $D(d_t, \alpha_t)$ of the diagram t^D of t is obtained from $V(k)$ by removing an open 3-ball neighborhood of the interior disk of every ribbon disk singularity. The *cross-index* of a tangle diagram t^D is the total number $\varepsilon(t^D)$ of distinct arc pairs in t^D such that each arc pair has the Z_2-intersection number 1 in δ, where define $\varepsilon(t^D) = 0$ for a single arc diagram t^D, [16]. The following lemma is used for the proof of Theorem 1.2.

Lemma 2.1. For every tangle t in $\delta \times [0,1]$ with tangle diagram t^D such that the cross-index $\varepsilon(t^D)$ is $n > 0$, then every CP disk-chord diagram $t^D(d_t, \alpha)$ of the spun S^2-link cl$T(t)$ must have at least n based spheres.

Proof. Let (t_1, t^2) be a pair of distinct strings of the tangle t such that the diagram pair (t^D_{1p}, t^D_{2p}) in δ has the Z_2-intersection number 1. For every transverse intersection point p between t^D_1 and t^D_2, take the point (p, t_{ip}) in t_i $(i = 1, 2)$. Say $t_{1p} < t_{2p}$. Then t_{1p} belongs to the half-open interval $(t_{2p}, 0]$. Then the arc t_1 intersects the immersed 3-ball $V(t_2)$ transversely at the point (p, t_{1p}). By counting the case $t_{1p} > t_{2p}$, it is concluded that the Z_2-intersection number of (t^D_{1p}, t^D_{2p}) is 1 in δ if and only if the sum of the Z_2-intersection numbers of $(t_1, V(t_2))$ and $(t_2, V(t_1))$ is 1. This claim holds regardless of choices of immersed 3-balls $V(t_i)$ bounded by cl$T(t_i)$ $(i = 1, 2)$ since the Z_2-linking numbers of $(t_1, \mathrm{cl}T(t_2))$ and $(t_2, \mathrm{cl}T(t_1))$ are well-defined. Thus, for $\varepsilon(t^D) = n$, every CP disk-chord diagram Ct^D of the spun S^2-link cl$T(t)$ must have at least n based spheres. This completes the proof of Lemma 2.1.

The proof of Theorem 1.2 is done as follows.

2.2: Proof of Theorem 1.2. Let $C(d, \alpha)$ be the CP disk-chord diagram on the sphere $S = \partial B$ of the spun torus-link $T(k)$ in S^4, and (d, α) a CP disk-chord system in the interior of a boundary-collar $S \times [0,1]$ of S in B which is sent to the CP disk-chord diagram$C(d,\alpha)$ on S under the projection $S\times[0,1]\to S$. Assume that the union $d\cup\alpha$ is connected since otherwise the task can be done for each connected component. Further, by assuming that the spun torus-link $T(k)$ is constructed from a link k in the interior of $S\times[0,1]$ and then deforming the union $d\cup\alpha$ in S^4 into B by an ambient isotopy of S^4 keeping the link $k=k\times\{1\}\subset T(k)$ fixed, the union $d\cup\alpha$ is written as the union $d\cup k$ in $S\times[0,1]$ by counting the relationship between $d\cup\alpha$ and $d\cup k$ around the based disk system d, illustrated in Fig. 4 in $S\times[0,1]$. Let Γ be a connected graph in $S\times[0,1]$ obtained from $d\cup k$ by shrinking every disk of d into a vertex. Let (Δ,τ^+) be a regular neighborhood of τ in $(S\times[0,1],\tau)$ with Δ a 3-ball. Let $e=\mathrm{cl}(\Gamma\setminus\tau^+)$ and $t=\mathrm{cl}(k\setminus e)$ be be arc systems in k. Let $\tau^+(d,t)$ be the disk-arc system obtained by replacing the vertex system of Γ with the disk system d. Deform the 3-ball Δ into the 3-ball $\delta\times[0,1]$ in $S\times[0,1]$ for a disk δ in S, so that the projection $\delta\times[0,1]to\delta$ sends the disk-tangle system $\tau^+(d,t)$ into the disk-tangle diagram $D\tau^+(d,t)$ in δ and the boundary point system ∂t of t is in the boundary circle $\partial\delta$ of δ. Note that the disk-tangle system $D\tau^+(d,t)$ is a disk-chord diagram of a tangle diagram t^D in δ. The arc system e is a tangle in the 3-ball $\delta^c\times[0,1]$ for the disk $\delta^c=\mathrm{cl}(S\setminus\delta)$ with the boundary point system ∂e in the boundary circle $\partial\delta^c$. By construction, the spun S^2-link $\mathrm{cl}T(t)$ bounds an immersed 3-ball system $V(t)$ with ribbon disk singularity given by the disk-tangle diagram $D\tau^+(d,t)$ and the spun S^2-link $\mathrm{cl}T(e)$ bounds a solid torus system $V(e)$ such that the union $V(t)\cup V(e)$ is an immersed handlebody system constructed from the CP disk-chord diagram $C(d,\alpha)$ on the sphere S. By Lemma 2.1, the cross-index $\varepsilon(e^D)$ of the diagram e^D of e in the disk δ^c is 0. Since the spun S^2-link $\mathrm{cl}T(e)$ of the tangle e in the 3-ball $\delta^c\times[0,1]$ is a trivial S^2-link in S^4, the tangle e is a trivial tangle in $\delta^c\times[0,1]$. In fact, the compact exterior E of e in $\delta^c\times[0,1]$ has a meridian-based free fundamental group $\pi_1(E,b_0)$ and E is a handlebody by Dehn's lemma, [1]. Thus, there is a disjoint disk system d_e in $\delta^c\times[0,1]$ with $\partial d_e=e\cup e^c$ for a disjoint arc system e^c in the annulus $(\partial\delta^c)\times[0,1]$. The cross-index $\varepsilon(e^D)=0$ implies that the arc system e is isotopic into an arc system in δ^c by an isotopy of $\delta^c\times[0,1]$ keeping the boundary point system ∂e fixed. Thus, there is a link diagram D of k on S such that the CP disk-chord diagram $C(d,\alpha)$ is the disk-chord diagram CD of a link diagram D. This completes the proof of Theorem 1.2.

3. Proof of Theorem 1.4.

If k_1 is a trivial knot, then the identity $c(k_1\#k_2)=c(k_1)+c(k_2)$ holds. If k_1 is a split link of sublinks k_1' and k_1'' and a connected sum $k_1\#k_2$ is made as $k_1'\#k_2\cup k_1''$ and the identity $c(k_1'\#k_2)=c(k_1')+c(k_2)$ is known, then $c(k_1\#k_2)=c(k_1)+c(k_2)$ also holds. Thus, for the proof of Theorem 1.4, it may be assumed that the links $k_i\,(i=1,2)$ are non-trivial and non-split links. Then every component of every connected sum

$k_1\#k_2$ does not bound any singular disk with interior disjoint from $k_1\#k_2$, by Dehn's lemma, [1, 17]. Under this assumption, the ribbon torus-link $T(k_1\#k_2)$ admits a *unique meridian system* in S^4, that is, there is a unique simple loop up to homotopy in every component of the ribbon torus-link $T(k_1\#k_2)$ which bounds a disk in S^4 with the interior disjoint from $T(k_1\#k_2)$, because there is a natural isomorphism $\pi_1(S^3 \setminus k_1\#k_2, x_0) \to \pi_1(S^4 \setminus T(k_1\#k_2), x_0)$. Let D_i be a diagram of a link k_i with $c(D_i) = c(k_i)\,(i = 1, 2)$, and D_{12} a diagram of a connected sum $k_1\#k_2$ with $c(D_{12}) = c(k_1\#k_2)$. Let $A_i\,(i = 1, 2)$ be 4-balls in S^4 with $A_1 \cap A_2 = B_0$ a 3-ball in the boundary 3-spheres $\partial A_i\,(i = 1, 2)$. Take a handled-sphere system (O_i, h_i) for the torus-link $T(k_i)$ in the 4-ball A_i constructed from the disk-chord system $CD_i = C(d_i, \alpha_i)$ of the diagram D_i, where h_i denotes a 1-handle system with core system α_i an arc system in k_i. Let (O_{12}, h_{12}) be a handled-sphere system for the torus-link $T(k_1\#k_2)$ in the 4-ball $A_1 \cup A_2$ constructed from the CP disk-chord system CD_{12} of the diagram D_{12}, where h_{12} denotes a 1-handle system with core system α_{12} an arc system in $k_1\#k_2$. By deforming α_i in $A_i\,(i = 1, 2)$, assume that the chord systems $\alpha_i\,(i = 1, 2)$ meets the 3-ball B_0 with an arc $\|a_0b_0\|$ with endpoint pair (a_0, b_0). Let $O'_{12} = O_1 \cup O_2$ and $\alpha'_{12} = \mathrm{cl}(\alpha_1 \setminus \|a_0b_0\|) \cup \mathrm{cl}(\alpha_2 \setminus \|a_0b_0\|)$. Then there is a handled-sphere system (O'_{12}, h'_{12}) for $T(k_1\#k_2)$ where h'_{12} denotes a 1-handle system with core system α'_{12}. meeting the 3-ball B_0 with two transverse disks μ, ν of a 1-handle in h'_{12}. Under these preliminaries, the proof of Theorem 1.4 is done as follows.

Proof of Theorem 1.4. Consider that the connected sum $k_1\#k_2$ is done between a component k_1^0 of k_1 and a component k_2^0 of k_2 and let $k'_i = k_i \setminus k_i^0\,(i = 1, 2)$. Let $W_{12} = O_{12} \times [0, 1] \cup h_{12}$ and $W'_{12} = O'_{12} \times [0, 1] \cup h'_{12}$ be the SUPH systems for $T(k_1\#k_2)$. After an isotopic deformation of the chord system h'_{12} in $A_1 \cup A_2$ fixing $\|a_0b_0\| \cup O'_{12}$, there is a multi-punctured solid torus system W'_* in $A_1 \cup A_2$ obtained from W'_{12} by adding a standard 2-handle system on $O'_{12} \times \{1\}$ such that there is an orientation-preserving diffeomorphism of $A_1 \cup A_2$ sending W_{12} is sent to the manifold W'_{**} obtained from W'_* by removing an embedded 1-handle system h'' on the trivial S^2-link $O'_* = \partial W'_* \setminus T(k_1\#k_2)$, [15, Appendix]. Let δ'' be the transverse disk system of the 1-handle system h'' cut by the disk union $\mu \cup \nu$. Assume $W'_{**} = W_{12}$. Let $(W_{12})_0$ be the SUPH system component of W_{12} containing the disk union $\mu \cup \nu$, and $(O_{12}, \alpha_{12})_0 = ((O_{12})_0, (\alpha_{12})_0)$ the circular spherical chord system component of the CP spherical chord system (O_{12}, α_{12}) in $(W_{12})_0$. In terms of $(O_{12}, \alpha_{12})_0$, the disk system δ'' is a proper disk system in a 3-ball system B_{12} in the 4-ball $A_1 \cup A_2$ bounded by the trivial S^2-link $(O_{12})_0$ after the chord system $(\alpha_{12})_0$ is deformed by a ∂-relative isotopy in $A_1 \cup A_2$, [15, Proposition A.1]. Since the circular disk-chord system CD_{12} is primitive, the intersection $\delta'' \cap B_{12}$ is changed into the intersection $\delta'' \cap (h_{12})_0$ for the 1-handle system $(h_{12})_0$ with $(\alpha_{12})_0$ the core arc system. Thus, the intersection of the handled sphere system $(O_{12}, h_{12})_0$ and the disk union $\mu \cup \nu$ can be assumed to be the intersection of $(h_{12})_0$ and $\mu \cup \nu$. Then, since the compact exterior E_{12} of $O_{12} \cup \alpha_{12}$ in W_{12} is diffeomorphic to $T(k_1 \# k_2) \times [0,1]$, $T(k_1 \# k_2)$ has a unique meridian system in S^4, and $(O_{12}, \alpha_{12})_0$ is a circular spherical chord system of $(W_{12})_0$, the oriented chord system $(\alpha_{12})_0$ meets μ and ν with intersection numbers $+1$ and -1 in $(W_{12})_0$, respectively. There is an band system with a core arc system in $\mu \cup \nu$ in $(W_{12})_0$ which spans chords in $(\alpha_{12})_0$ and splits the connected circular sphere-chord system $(O_{12}, \alpha_{12})_0$ into a connected circular sphere-chord system C_0 meeting μ and ν at one point each and some circular sphere-chord systems $C_j\,(j = 1, 2, \ldots, s)$ not meeting $\mu \cup \nu$, where C_0 is slided on μ and ν to set $C_0 \cap \mu = a_0$ and $C_0 \cap \nu = b_0$. Let C'_0 be a circular sphere-chord system in $(W_{12})_0$ by joining C_0 and $C_j\,(j = 1, 2, \ldots, s)$ by a band not meeting $\mu \cup \nu$. Let $(O^C_{12}, \alpha^C_{12})$ be a CP sphere-chord system in W_{12} obtained from (O_{12}, α_{12}) by replacing $(O_{12}, \alpha_{12})_0$ with C'_0, which is a circular sphere-chord system of W_{12}, because the 3-manifolds obtained from $(W_{12})_0$ by splitting along the disk system are simply connected. Let $W'_i\,(i = 1, 2)$ be the 3-manifolds obtained from W_{12} by splitting along $\mu \cup \nu$, where a boundary collar of $\mu \cup \nu$ in W'_i belongs to A_i. Since $W'_i\,(i = 1, 2)$ are disjoint in $A_1 \cup A_2$ but in general not split in S^4, move W'_i into A_i by disregarding the other 3-manifold $W'_{i'}\,(i' \neq i)$ and keeping $\mu \cup \nu$ fixed . After this move, while W_{12} remains a SUPH system for $T(k_1 \# k_2)$ and $(O^C_{12}, \alpha^C_{12})$ is a circular sphere-chord system, but $(O^C_{12}, \alpha^C_{12})$ is no longer any CP sphere-chord system. Let W'_i be a SUPH system for $T(k_i)$ obtained from W'_i by adding a 1-handle h_0 on $\mu \cup \nu$ with core arc $\|a_0 b_0\|\,(i = 1, 2)$, which is a SUPH system of a circular sphere-chord system (O^C_i, α^C_i) obtained from $(O^C_{12}, \alpha^C_{12})$ by splitting along a band with core arc $\|a_0 b_0\|\,(i = 1, 2)$.

$$c(k_1 \# k_2) = r(O_{12}) = r(O^C_{12}) = r(O^C_1) + r(O^C_2).$$

Every sphere of $O^C_i\,(i = 1, 2)$ is passed through at most once time by the chord system α^C_i by construction. By Corollary 1.3, $r(O^C_i) \geq I(T(k_i)) = c(k_i)\,(i = 1, 2)$. so that $c(k_1 \# k_2) \geq c(k_1) + c(k_2)$. Since $c(k_1 \# k_2) \leq c(k_1) + c(k_2)$ by definition, the identity $c(k_1 \# k_2) = c(k_1) + c(k_2)$ holds. This completes the proof of Theorem 1.4.

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