Remarks on L^2-decay of small solutions to derivative nonlinear Schrödinger equations with weakly dissipative structure

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August 21, 2026

Dedicated to Professor Soichiro Katayama on the occasion of his sixtieth birthday

Abstract: We consider the initial value problem for cubic derivative nonlinear Schrödinger equations in one space dimension with small initial data of the form $\varepsilon\varphi(x)$. If the nonlinear term is weakly dissipative, it is known that the global solution decays like $O(\varepsilon^{1/2}(\log t)^{-1/4})$ in L^2 as $t \to +\infty$. In this paper, we show that this L^2-decay rate is optimal if the Fourier transform $\hat{\varphi}$ of φ does not vanish at the point ξ_0 where the dissipation is not effective; on the other hand, the L^2-decay is enhanced if $\hat{\varphi}(\xi_0) = 0$.

Key Words: Derivative nonlinear Schrödinger equation; Weakly dissipative structure; L^2-decay.

2010 Mathematics Subject Classification: 35Q55, 35B40.

1 Introduction and the results

This paper is a squel to the papers [3] and [4], which deal with the L^2-decay preperties of small solutions to one-dimensional cubic derivative nonlinear Schrödinger equations with

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weakly dissipative structure. We consider the initial value problem

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = N(u, \partial_x u), & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = \varepsilon\varphi(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $i = \sqrt{-1}$, $u = u(t, x)$ is a $\mathbb{C}$-valued unknown function on $[0, \infty) \times \mathbb{R}$, φ is a prescribed $\mathbb{C}$-valued function on $\mathbb{R}$ which belongs to $H^3 \cap H^{2,1}$, and $\varepsilon > 0$ is a small parameter which is responsible for the size of the initial data. Throughout this paper, for non-negative integers s and ℓ, H^s denotes the standard L^2-based Sobolev space of order s and the weighted Sobolev space $H^{s,\ell}$ is defined by $\{\phi \in L^2 \mid \langle \cdot \rangle^\ell \phi \in H^s\}$ equipped with the norm $\|\phi\|_{H^{s,\ell}} = \|\langle \cdot \rangle^\ell \phi\|_{H^s}$, where $\langle x \rangle = (1 + x^2)^{1/2}$. For $r \in (0, 1)$, C^r stands for the space of r-Hölder continuous functions on $\mathbb{R}$. The nonlinear term $N(u, \partial_x u)$ is a cubic homogeneous polynomial in $(u, \overline{u}, \partial_x u, \overline{\partial_x u})$ with complex coefficients. As in [3] and [4], let us introduce the following structural conditions on N.

Definition 1.1. We say that a cubic nonlinear term N is *weakly dissipative* if the following two conditions are satisfied:

(i) $N(e^{i\theta}, 0) = e^{i\theta} N(1, 0)$ for $\theta \in \mathbb{R}$.

(ii) There exist $c_0 > 0$ and $\xi_0 \in \mathbb{R}$ such that

$$\operatorname{Im} \nu(\xi) = -c_0(\xi - \xi_0)^2, \qquad \xi \in \mathbb{R},$$

where the function $\nu : \mathbb{R} \to \mathbb{C}$ is defined by

$$\nu(\xi) = \frac{1}{2\pi i} \oint_{|z|=1} N(z, i\xi z) \frac{dz}{z^2}. \tag{1.2}$$

First of all, let us briefly recall what have been done previously. It is well-known that cubic nonlinearity brings us to a borderline between short-range and long-range situation for the one-dimensional nonlinear Schrödinger equations. According to [9] (see also [2, 6, 7, 11, 12] etc.), typical results on the large-time behavior of global solutions to (1.1) under the condition (i) can be summarized in terms of ν as follows:

1. Small data global existence holds in $C([0, \infty); H^3 \cap H^{2,1})$ under the condition

$$\operatorname{Im} \nu(\xi) \le 0, \quad \xi \in \mathbb{R}. \tag{1.3}$$

2. The global solution has (at most) logarithmic phase correction in the asymptotic profile if

$$\operatorname{Im} \nu(\xi) = 0, \quad \xi \in \mathbb{R}. \tag{1.4}$$

In particular, L^2-decay does not occur under (1.4) for generic initial data of small amplitude.

3. L^2-decay of the global solution occurs under the condition

$$\sup_{\xi\in\mathbb{R}} \operatorname{Im}\nu(\xi) < 0. \tag{1.5}$$

It has been pointed out in [3] that (1.4) and (1.5) do not cover all cases of (1.3). For example, both (1.4) and (1.5) are violated by $N = -i|\partial_x u|^2 u$ since $\operatorname{Im}\nu(\xi) = -\xi^2 \, (\le 0)$. This observation leads us to the condition (ii). We also note that the condition (i) excludes just the worst terms u^3, $|u|^2\overline{u}$, $\overline{u}^3$, and it is known that these three terms make the situation much more complicated. For more details of the backgrounds on Definition 1.1, we refer the readers to [3], [4] and the survey article [5].

Now we turn our attentions to what we can say on the large time behavior of the L^2 norm of solution u to (1.1) under the weak dissipativity condition. We denote by $\hat{\varphi}$ or $\mathcal{F}\varphi$ the Fourier transform of φ, i.e.,

$$\hat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(y) e^{-iy\xi} \, dy.$$

The following two results have been shown in [4].

Proposition 1.1 ([4]). *Suppose that N is weakly dissipative. Then there exists a positive constant C, not depending on ε, such that the global solution u to* (1.1) *satisfies*

$$\limsup_{t\to+\infty} ((\log t)^{1/4} \|u(t)\|_{L^2_x}) \le C\varepsilon^{1/2}, \tag{1.6}$$

provided that ε is suitably small.

Proposition 1.2 ([4]). *Suppose that N is weakly dissipative, and that $\hat{\varphi}(\xi_0) \neq 0$, where ξ_0 comes from* (ii) *in Definition 1.1. Then the global solution u to* (1.1) *satisfies*

$$\liminf_{t\to+\infty} ((\log t)^{1/4} \|u(t)\|_{L^2_x}) > 0, \tag{1.7}$$

provided that ε is suitably small.

From these two results, we see that the global solution to (1.1) decays like $O((\log t)^{-1/4})$ in L^2 as $t \to +\infty$ if N is weakly dissipative, and this rate is best possible in general. However, when we compare these two results carefully, the following two questions arise:

Q1. There is still a gap with respect to the dependence of ε in the upper bound (1.6) and the lower bound (1.7). Is it possible to fill this gap?

Q2. What can we say about the lower L^2-decay bounds in the case of $\hat{\varphi}(\xi_0) = 0$?

Our aim in the present paper is to address these questions. In what follows, ξ_0 stands for the point appearing in (ii) in Definition 1.1. The main results are as follows.

Theorem 1.1. *Assume that N is weakly dissipative, and that $\hat{\varphi}(\xi_0) \neq 0$. Then there exist positive constants C_0 and ε_0 such that the global solution u to* (1.1) *satisfies*

$$\liminf_{t \to +\infty} ((\log t)^{1/4} \|u(t)\|_{L^2_x}) \geq C_0 \varepsilon^{1/2}$$

for $\varepsilon \in (0, \varepsilon_0]$.

Theorem 1.2. *Assume that N is weakly dissipative, and that $\hat{\varphi}(\xi) = O(|\xi - \xi_0|^\kappa)$ as $\xi \to \xi_0$ with some $\kappa \geq 0$. Then there exist positive constants C_1, C_2, ε_1 and M such that the global solution u to* (1.1) *satisfies*

$$\|u(t)\|_{L^2_x} \leq \frac{C_1 \varepsilon}{(\varepsilon^2 \log t)^{(1+2\kappa)/(4+4\kappa)}} + \frac{C_2 \varepsilon^{3/2}}{(\varepsilon^2 \log t)^{1/4}}$$

for $t \geq e^{M/\varepsilon^2}$, provided that $\varepsilon \in (0, \varepsilon_1]$.

Note that the case $\kappa = 0$ in Theorem 1.2 implies Proposition 1.1. In other words, Theorem 1.2 can be viewed as a refinement of Proposition 1.1.

As a simple consequence of Theorem 1.2, we have the following.

Corollary 1.1. *Assume that N is weakly dissipative, and that $\hat{\varphi}(\xi_0) = 0$. If there exist constants $C_* \geq 0$ and $\varepsilon_* > 0$ such that the global solution u to* (1.1) *satisfies*

$$\liminf_{t \to +\infty} ((\log t)^{1/4} \|u(t)\|_{L^2_x}) \geq C_* \varepsilon^{1/2} \tag{1.8}$$

for $\varepsilon \in (0, \varepsilon_*]$, then we have $C_* = 0$.

Theorem 1.1 and Corollary 1.1 tell us that the vanishing or non-vanishing of $\hat{\varphi}$ at the only one point ξ_0 actually affects the L^2-decay of the solution $u(t)$ as $t \to +\infty$. This should be remarkable because L^2-norm is not affected by information at a single point in general.

A next problem would be to specify the precise L^2-decay rate in the case of $\hat{\varphi}(\xi_0) = 0$. However, it turns out that this problem is much more delicate than it looks. To the best of authors' knowledge, there is no previous result on explicit lower L^2-decay bounds in this case. In view of the arguments in Section 3 below, it would be reasonable to expect that

$$\|u(t)\|_{L^2_x} \gtrsim \frac{\varepsilon}{(\varepsilon^2 \log t)^{(1+2k)/(4+4k)}}$$

holds for sufficiently large t if $\hat{\varphi}^{(k)}(\xi_0) \neq 0$ and $\hat{\varphi}^{(j)}(\xi_0) = 0$ for $0 \leq j < k$. However, the authors have been unable to prove it so far.

2 Preliminaries

In this section, we collect several lemmas which are needed in the proof of the main results. Since these results are essentially not new, we shall be brief. In what follows, several positive constants appearing in estimates will be denoted by the same letter C, which may vary from one line to another.

2.1 Reduction to ODE analysis

According to the previous works [9, 3, 4], we have already known the following lemmas. We set

$$\alpha(t,\xi) = \mathcal{F}[\mathcal{U}(-t)u(t,\cdot)](\xi) \tag{2.1}$$

for the solution $u(t,x)$ to (1.1), where $\mathcal{U}(t) = \exp(i\frac{t}{2}\partial_x^2)$ and $\mathcal{F}$ denotes the Fourier transformation. We call α given by (2.1) the *profile function* of u.

Lemma 2.1. *Assume that the condition* (i) *in Definition 1.1 and the condition* (1.3) *are fulfilled. If ε is sufficiently small, the global solution u to* (1.1) *satisfies*

$$\|u(t)\|_{H_x^3} + \|\mathcal{J}u(t)\|_{H_x^2} \le C\varepsilon(1+t)^\gamma, \quad t \ge 0,$$

where $\mathcal{J} = x + it\partial_x$ and $0 < \gamma < 1/12$. Moreover, for the profile function α of u, we have

$$|\alpha(t,\xi)| \le \frac{C\varepsilon}{\langle\xi\rangle^2}, \qquad t \ge 0,\ \xi \in \mathbb{R},$$

and

$$|\alpha(t,\xi) - \beta(t,\xi)| \le \frac{C\varepsilon^3}{t^{1/2}\langle\xi\rangle^4}, \qquad t \ge 1,\ \xi \in \mathbb{R}, \tag{2.2}$$

with some $\beta : [1,\infty) \times \mathbb{R} \to \mathbb{C}$ satisfying

$$\left| i\partial_t\beta(t,\xi) - \frac{\nu(\xi)}{t}|\beta(t,\xi)|^2\beta(t,\xi) \right| \le \frac{C\varepsilon^3}{t^{1+\delta}\langle\xi\rangle^2},$$

where ν is given by (1.2) *and $0 < \delta < 1/4$.*

The proof can be found in Section 2 of [3] or Section 4 of [9].

Lemma 2.2. *Let α be the profile function of the solution u to* (1.1) *with small ε. We have*

$$|\alpha(1,\xi) - \varepsilon\hat{\varphi}(\xi)| \le \frac{C\varepsilon^2}{\langle\xi\rangle^2}, \quad \xi \in \mathbb{R}.$$

For the proof, see Lemma 3.2 in [4].

Remark 2.1. Roughly speaking, these lemmas tell us that the solution u to (1.1) can be expressed as

$$u(t) = \mathcal{U}(t)\mathcal{F}^{-1}\alpha(t) = \mathcal{U}(t)\mathcal{F}^{-1}\beta(t) + \cdots$$

with the solution $\beta(t,\xi)$ to the ordinary differential equation

$$\begin{cases} i\partial_t\beta(t,\xi) = \dfrac{\nu(\xi)}{t}|\beta(t,\xi)|^2\beta(t,\xi) + \cdots, \\ \beta(1,\xi) = \varepsilon\hat{\varphi}(\xi) + \cdots, \end{cases} \tag{2.3}$$

where the terms "$+\cdots$" are expected to be harmless. The original idea of this reduction goes back to Hayashi–Naumkin [1].

2.2 A lemma on the reduced ODE

In view of Remark 2.1, it is natural to focus on an analysis of β satisfying (2.3) in order to study asymptotic behavior of the solution u to the initial problem (1.1). For this purpose, the following lemma due to [4] (see also [10, 2]) is useful.

Lemma 2.3. *Let $b_0(\xi)$, $\mu(\xi)$ be $\mathbb{C}$-valued continuous functions on $\mathbb{R}$ satisfying*

$$|b_0(\xi)| \le \frac{C\varepsilon}{\langle\xi\rangle^2}, \quad |\mu(\xi)| \le C\langle\xi\rangle^3, \quad \mathrm{Im}\,\mu(\xi) \le 0, \tag{2.4}$$

where $\varepsilon > 0$ is a small parameter. Let $R(t,\xi)$ be a $\mathbb{C}$-valued continuous function on $[1,\infty)\times\mathbb{R}$ satisfying

$$|R(t,\xi)| \le \frac{C\varepsilon^3}{t^{1+\delta}\langle\xi\rangle^2} \tag{2.5}$$

with some $\delta > 0$. If the function $B : [1,\infty)\times\mathbb{R} \to \mathbb{C}$ solves

$$i\partial_t B(t,\xi) = \frac{\mu(\xi)}{t}|B(t,\xi)|^2 B(t,\xi) + R(t,\xi), \quad B(1,\xi) = b_0(\xi) \tag{2.6}$$

and ε is suitably small, then we have

$$|B(t,\xi) - A(\log t,\xi)| \le \frac{C\varepsilon^3}{t^{\delta'}\langle\xi\rangle^2}$$

for $(t,\xi) \in [1,\infty)\times\mathbb{R}$, where $\delta' \in (0,\delta)$ can be taken arbitrarily close to δ, and the function $A : [0,\infty)\times\mathbb{R} \to \mathbb{C}$ solves

$$i\partial_\tau A(\tau,\xi) = \mu(\xi)|A(\tau,\xi)|^2 A(\tau,\xi), \quad A(0,\xi) = b_0(\xi) + b_1(\xi)$$

with some $b_1 : \mathbb{R} \to \mathbb{C}$ satisfying

$$|b_1(\xi)| \le \frac{C\varepsilon^3}{\langle\xi\rangle^2}.$$

For the proof, see Lemma 2.1 in [4].

Remark 2.2. It follows from Lemma 2.3 that

$$\|B(t) - A(\log t)\|_{L^2_\xi} \le \frac{C\varepsilon^3}{t^{\delta'}}$$

for $t \ge 1$, and that

$$|A(\tau,\xi)|^2 = \frac{|A(0,\xi)|^2}{1 - 2\mathrm{Im}\,\mu(\xi)|A(0,\xi)|^2\tau} \tag{2.7}$$

for $\tau \ge 0$. Therefore, it is natural to focus on the integral

$$\int_{\mathbb{R}} \frac{|A(0,\xi)|^2}{1 - 2\mathrm{Im}\,\mu(\xi)|A(0,\xi)|^2 \log t}\, d\xi$$

in order to investigate the large time behavior of the solution $B(t,\xi)$ to (2.6) in L^2.

3 Proof of the main results

This section is devoted to the proof of Theorems 1.1, 1.2 and Corollary 1.1. Before getting into the detail, we make some preparation. Let u be the solution to (1.1), and let α, β be as in Lemma 2.1. Taking $B(t,\xi) = \beta(t,\xi)$, we see that (2.4), (2.5), (2.6) are satisfied with $b_0(\xi) = \beta(1,\xi)$, $\mu(\xi) = \nu(\xi)$ and $\delta \in (0, 1/4)$. Therefore, by Lemma 2.3, (2.2) and (2.7), we can take a function $A(\tau,\xi)$ satisfying

$$|\alpha(1,\xi)| - \frac{C\varepsilon^3}{\langle\xi\rangle^2} \le |A(0,\xi)| \le |\alpha(1,\xi)| + \frac{C\varepsilon^3}{\langle\xi\rangle^2} \tag{3.1}$$

and

$$\|A(\log t)\|_{L^2_\xi}^2 = \int_{\mathbb{R}} \frac{|A(0,\xi)|^2}{1 + 2c_0(\xi-\xi_0)^2|A(0,\xi)|^2 \log t}\, d\xi \tag{3.2}$$

for $t \ge 1$, where c_0 and ξ_0 are from (ii) of Definition 1.1. Moreover, since $\mathcal{U}(t)$ and $\mathcal{F}$ are unitary in L^2, we have $\|u(t)\|_{L^2_x} = \|\alpha(t)\|_{L^2_\xi}$. This leads to

$$\begin{aligned}\big|\|u(t)\|_{L^2_x} - \|A(\log t)\|_{L^2_\xi}\big| \le& \|\alpha(t) - A(\log t)\|_{L^2_\xi}\\ \le& \|\alpha(t) - \beta(t)\|_{L^2_\xi} + \|\beta(t) - A(\log t)\|_{L^2_\xi}\\ \le& \frac{C\varepsilon^3}{t^{1/2}} + \frac{C\varepsilon^3}{t^{\delta'}},\end{aligned}$$

whence

$$\|A(\log t)\|_{L^2_\xi} - \frac{C\varepsilon^3}{t^{\delta'}} \le \|u(t)\|_{L^2_x} \le \|A(\log t)\|_{L^2_\xi} + \frac{C\varepsilon^3}{t^{\delta'}} \tag{3.3}$$

for $t \ge 1$. Here $\delta' \in (0, 1/4)$ can be taken arbitrarily close to $1/4$.

3.1 Proof of Theorem 1.1

We follow the approach of [8] (in which the wave equation case is treated) with minor modifications to fit the present purpose. It follows from Lemma 2.2 that

$$|\alpha(1,\xi_0)| \ge \varepsilon|\hat{\varphi}(\xi_0)| - C\varepsilon^2 \ge \frac{\varepsilon|\hat{\varphi}(\xi_0)|}{2},$$

if $\varepsilon > 0$ is suitably small and $\hat{\varphi}(\xi_0) \neq 0$. By the continuity of $\xi \mapsto \alpha(1,\xi)$, we can choose an open interval I with $I \ni \xi_0$ such that

$$\inf_{\xi\in I} |\alpha(1,\xi)| \ge \frac{\varepsilon|\hat{\varphi}(\xi_0)|}{3} > 0.$$

Then, it follows from (3.1) that

$$\begin{aligned}\inf_{\xi\in I}|A(0,\xi)| &\geq \inf_{\xi\in I}|\alpha(1,\xi)| - C\varepsilon^3\\ &\geq \frac{\varepsilon|\hat{\varphi}(\xi_0)|}{3} - C\varepsilon^3\\ &\geq \frac{\varepsilon|\hat{\varphi}(\xi_0)|}{4}.\end{aligned}$$

Since the function $[0,\infty) \ni X \mapsto X/(1+KX)$ is strictly increasing if $K \geq 0$, we have

$$\frac{|A(0,\xi)|^2}{1+2c_0(\xi-\xi_0)^2|A(0,\xi)|^2\log t} \geq \frac{\varepsilon^2|\hat{\varphi}(\xi_0)|^2/2}{8+c_0(\xi-\xi_0)^2\varepsilon^2|\hat{\varphi}(\xi_0)|^2\log t}$$

for $\xi \in I$. Now we take $m > 0$ so small that $[\xi_0 - m, \xi_0 + m] \subset I$. Then, for $t \geq e^{1/\varepsilon^2}$ (i.e., for $(\varepsilon^2 \log t)^{-1/2} \leq 1$), we deduce from (3.2) that

$$\begin{aligned}\|A(\log t)\|_{L^2_\xi}^2 \geq& \frac{1}{2}\int_I \frac{\varepsilon^2|\hat{\varphi}(\xi_0)|^2}{8+c_0(\xi-\xi_0)^2\varepsilon^2|\hat{\varphi}(\xi_0)|^2\log t}\,d\xi\\ \geq&\frac{1}{2}\int_{\xi_0-m(\varepsilon^2\log t)^{-1/2}}^{\xi_0+m(\varepsilon^2\log t)^{-1/2}} \frac{\varepsilon^2|\hat{\varphi}(\xi_0)|^2}{8+c_0(\xi-\xi_0)^2\varepsilon^2|\hat{\varphi}(\xi_0)|^2\log t}\,d\xi\\ =&\frac{C_0^2\varepsilon}{(\log t)^{1/2}}\end{aligned}$$

with

$$C_0 = \left(\frac{1}{2}\int_{-m}^{m}\frac{|\hat{\varphi}(\xi_0)|^2}{8+c_0|\hat{\varphi}(\xi_0)|^2\eta^2}\,d\eta\right)^{1/2},$$

where we have made the change of the variable $\eta = (\xi - \xi_0)(\varepsilon^2 \log t)^{1/2}$ in the last line. Therefore, by (3.3), we obtain

$$\|u(t)\|_{L^2_x} \geq \frac{C_0\varepsilon^{1/2}}{(\log t)^{1/4}} - \frac{C\varepsilon^3}{t^{\delta'}}$$

for $t \geq e^{1/\varepsilon^2}$, whence

$$\liminf_{t\to+\infty}\big((\log t)^{1/4}\|u(t)\|_{L^2_x}\big) \geq C_0\varepsilon^{1/2}.$$

□

3.2 Proof of Theorem 1.2

By the assumption on φ, we can take $c_1 > 0$ and $\delta_1 > 0$ such that

$$|\hat{\varphi}(\xi)| \leq c_1|\xi-\xi_0|^\kappa$$

for $|\xi-\xi_0|<\delta_1$. Then, it follows from (3.1) and Lemma 2.2 that

$$|A(0,\xi)|^2 \le \left(\varepsilon|\hat{\varphi}(\xi)|+C\varepsilon^2\right)^2 \le 2c_1^2\varepsilon^2|\xi-\xi_0|^{2\kappa}+C\varepsilon^4$$

and

$$\frac{|A(0,\xi)|^2}{1+2c_0(\xi-\xi_0)^2|A(0,\xi)|^2\log t} \le \frac{2c_1^2\varepsilon^2|\xi-\xi_0|^{2\kappa}}{1+4c_0c_1^2|\xi-\xi_0|^{2+2\kappa}\varepsilon^2\log t}+\frac{C\varepsilon^4}{1+2c_0C(\xi-\xi_0)^2\varepsilon^4\log t}$$

for $|\xi-\xi_0|<\delta_1$. Here we have used the fact that $X\mapsto X/(1+KX)$ is strictly increasing, together with the inequality

$$\frac{X+Y}{1+K(X+Y)} \le \frac{X}{1+KX}+\frac{Y}{1+KY}$$

for $X\ge 0$, $Y\ge 0$, $K\ge 0$. Integrating the both sides in $\xi\in(\xi_0-\delta_1,\xi_0+\delta_1)$, we obtain

$$\begin{aligned}
&\int_{|\xi-\xi_0|<\delta_1}\frac{|A(0,\xi)|^2}{1+2c_0(\xi-\xi_0)^2|A(0,\xi)|^2\log t}\,d\xi\\
&\le \int_{|\xi-\xi_0|<\delta_1}\frac{2c_1^2\varepsilon^2|\xi-\xi_0|^{2\kappa}}{1+4c_0c_1^2|\xi-\xi_0|^{2+2\kappa}\varepsilon^2\log t}\,d\xi+\int_{|\xi-\xi_0|<\delta_1}\frac{C\varepsilon^4}{1+2c_0C(\xi-\xi_0)^2\varepsilon^4\log t}\,d\xi\\
&=: J_1+J_2
\end{aligned}$$

for $t\ge 1$. By the change of the variable $\eta=(\xi-\xi_0)(\varepsilon^2\log t)^{1/(2+2\kappa)}$ in J_1, we have

$$J_1 \le \frac{1}{2}\frac{C_1^2\varepsilon^2}{(\varepsilon^2\log t)^{(1+2\kappa)/(2+2\kappa)}}$$

with

$$C_1=\left(\int_{\mathbb{R}}\frac{4c_1^2|\eta|^{2\kappa}}{1+4c_0c_1^2|\eta|^{2+2\kappa}}\,d\eta\right)^{1/2}.$$

As for J_2, the change of the variable $\eta=(\xi-\xi_0)(\varepsilon^4\log t)^{1/2}$ yields

$$J_2 \le \frac{C\varepsilon^2}{(\log t)^{1/2}}\int_{\mathbb{R}}\frac{d\eta}{1+2c_0C\eta^2}.$$

On the other hand, it holds that

$$\int_{|\xi-\xi_0|\ge\delta_1}\frac{|A(0,\xi)|^2}{1+2c_0(\xi-\xi_0)^2|A(0,\xi)|^2\log t}\,d\xi \le \frac{1}{2c_0\log t}\int_{|\xi-\xi_0|\ge\delta_1}\frac{d\xi}{(\xi-\xi_0)^2}$$

regardless of the structure of $A(0,\xi)$. Piecing them together, we have

$$\begin{aligned}
\|A(\log t)\|_{L^2_\xi}^2 \le&\frac{1}{2}\frac{C_1^2\varepsilon^2}{(\varepsilon^2\log t)^{(1+2\kappa)/(2+2\kappa)}}+\frac{C\varepsilon^2}{(\log t)^{1/2}}+\frac{C}{\log t}\\
\le&\frac{1}{2}\frac{C_1^2\varepsilon^2}{(\varepsilon^2\log t)^{(1+2\kappa)/(2+2\kappa)}}+\frac{C\varepsilon^3}{(\varepsilon^2\log t)^{1/2}}+\frac{C\varepsilon^2}{M^{1/(2+2\kappa)}(\varepsilon^2\log t)^{(1+2\kappa)/(2+2\kappa)}}\\
\le&\frac{3}{4}\frac{C_1^2\varepsilon^2}{(\varepsilon^2\log t)^{(1+2\kappa)/(2+2\kappa)}}+\frac{C\varepsilon^3}{(\varepsilon^2\log t)^{1/2}}
\end{aligned}$$

if $\varepsilon^2 \log t$ is greater than a sufficiently large constant M. With the aid of (3.3), we arrive at

$$\begin{aligned}
\|u(t)\|_{L^2_x} \leq & \|A(\log t)\|_{L^2_\xi} + \frac{C\varepsilon^3}{t^{\delta'}} \\
\leq & \left(\frac{\sqrt{3}}{2} \frac{C_1 \varepsilon}{(\varepsilon^2 \log t)^{(1+2\kappa)/(4+4\kappa)}} + \frac{C\varepsilon^{3/2}}{(\varepsilon^2 \log t)^{1/4}} \right) \\
& + \frac{\varepsilon}{(\varepsilon^2 \log t)^{(1+2\kappa)/(4+4\kappa)}} \cdot C\varepsilon^{(3+2\kappa)/(2+2\kappa)} \frac{(\log t)^{(1+2\kappa)/(4+4\kappa)}}{t^{\delta'}} \\
\leq & \frac{C_1 \varepsilon}{(\varepsilon^2 \log t)^{(1+2\kappa)/(4+4\kappa)}} + \frac{C\varepsilon^{3/2}}{(\varepsilon^2 \log t)^{1/4}}
\end{aligned}$$

for $t \geq e^{M/\varepsilon^2}$, provided that ε is small enough. □

3.3 Proof of Corollary 1.1

We first note that $\hat{\varphi} \in H^1$ because $\varphi \in H^{2,1} \hookrightarrow H^{0,1}$. Then, by the Sobolev embedding $H^1 \hookrightarrow C^{1/2}$ in $\mathbb{R}^1$ and the assumption $\hat{\varphi}(\xi_0)$=0, we have

$$|\hat{\varphi}(\xi)| = |\hat{\varphi}(\xi) - \hat{\varphi}(\xi_0)| \leq C|\xi - \xi_0|^{1/2}.$$

Therefore we can apply Theorem 1.2 with $\kappa = 1/2$. Since

$$\left. \frac{1+2\kappa}{4+4\kappa} \right|_{\kappa=1/2} = \frac{1}{3},$$

we obtain

$$\|u(t)\|_{L^2_x} \leq \frac{C_1 \varepsilon}{(\varepsilon^2 \log t)^{1/3}} + \frac{C_2 \varepsilon^{3/2}}{(\varepsilon^2 \log t)^{1/4}}$$

for $t \geq e^{M/\varepsilon^2}$. Connecting this to (1.8) yields

$$C_* \varepsilon^{1/2} \leq \liminf_{t \to +\infty} ((\log t)^{1/4} \|u(t)\|_{L^2_x}) \leq \limsup_{t \to +\infty} ((\log t)^{1/4} \|u(t)\|_{L^2_x}) \leq C_2 \varepsilon.$$

Dividing the both sides by $\varepsilon^{1/2}$ and letting $\varepsilon \to +0$, we have $C_* = 0$. This completes the proof. □

Acknowledgments

The work of C. L. is supported by the National Natural Science Foundation of China (NSFC) under Grant No. 12361051, and partly by the Jilin Provincial International Joint Research Center for Differential Equations and Their Innovative Applications under Grant No. YDZJ202602CXJD091. The work of H. S. is supported by JSPS KAKENHI 21K03314, 24H00024 and 26K06883. This work is partly supported by MEXT Promotion of Distinctive Joint Research Center Program JPMXP0723833165 and Osaka Metropolitan University Strategic Research Promotion Project (Development of International Research Hubs).

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