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Quandles and Symmetric Spaces

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Abstract. The workshop "*Quandles and symmetric spaces*" has been held annually since 2018. This volume records the abstracts and some slides of talks presented in this workshop on 2019 and 2020.

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Preface

The workshop *Quandles and symmetric spaces* has been held annually since 2018 in order to encourage the cross-pollination among topology (knot theory), differential geometry (symmetric spaces), and other areas through *quandles*.

A quandle is an algebraic system, whose axioms are corresponding to the Reidemiseter moves of classical knots diagrams. The notion of quadles has been introduced by Joyce and Matveev independently in 1982, and nowadays it plays important roles in knot theory. For example, classical knots are classified in principle by their fundamental quandles. Quandle colorings, which means the numbers of homomorphisms from the fundamental quandles to a fixed finite quandle, are important examples of knot invariants, and have been widely studied.

The notion of quandle frequently appears in various fields other than knot theory. As Joyce has already pointed out, symmetric spaces are typical examples of (involutory) quandles. A symmetric space is a manifold that every point has a diffeomorphism called a point symmetry, whose conditions are corresponding to the axioms of quandles. Therefore, quandles can be considered as a generalization of symmetric spaces, and from this point of view it would be interesting and significant to research both of them interactively.

The series of workshops was organized by experts of knot theory (Kamada and Oshiro) and symmetric spaces (Kubo, Okuda, Tamaru, Tanaka and Tasaki), and the talks consisted of some instructive talks by experts and presentations by young researchers in addition to usual talks. There have been many presenters and participants from various fields, not only topology and differential geometry but also combinatorics, algebraic geometry, etc. During or after the talks, they exchanged their ideas and information, and discussed possible perspectives actively. The organizers are convinced that the workshops would be effective and successful to develop the theory of quandles.

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Quandle coloring quivers for links and quivers of quandles

Yuta Taniguchi

Abstract. In 2018, K.Cho and S.Nelson introduced the quandle coloring quiver which is an invariant for oriented links. In this paper, we will introduce the quandle quiver of a quandle and observe a relation between the quandle coloring quiver and the quandle quiver.

1 Quandle coloring quiver and quandle quiver

1.1 Quandle

A quandle is a non-empty set X with a binary operation $*$ satisfying the following conditions:

- For any $x \in X$, we have $x * x = x$.
- For any $y \in X$, the map $*y : X \to X; x \mapsto x * y$ is a bijection.
- For any $x, y, x \in X$, we have $(x * y) * z = (x * z) * (y * z)$.

Let M be a left $\mathbb{Z}[t^{\pm 1}]$-module. We define a binary operation $*$ on M by $x * y = tx + (1 - t)y$. Then, M is a quandle. We call it an *Alexander quandle*.

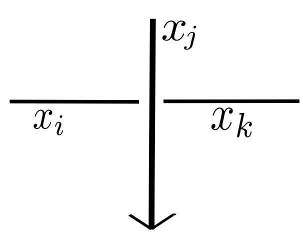
Let X be a quandle. A map $f : X \to X$ is a *quandle endomorphism* of X if $f(x*y) = f(x)*f(y)$ for any $x, y \in X$. We denote the set of all quandle endomorphisms of X by $\mathrm{End}(X)$.

A quandle X is an *abelian quandle* if $(x * y) * (z * w) = (x * z) * (y * w)$ for any $x, y, z, w \in X$. For example, an Alexander quandle is an abelian quandle. However, the converse is not true.

1.2 Quandle coloring

Let X be a finite quandle, D be a diagram of an oriented link L and $\mathrm{Arc}(D)$ be the set of all arcs of D. A map $c : \mathrm{Arc}(D) \to X$ is an *X-coloring of D* if for each crossing of D, c satisfies the following condition:

- Let x_i, x_j, x_k be arcs around a crossing as shown below. Then, we have $c(x_i) * c(x_j) = c(x_k)$.



We denote the set of all X-colorings of D by $\mathrm{Col}_X(D)$. Let D' be another diagram of L. It is known that the cardinalities of $\mathrm{Col}_X(D)$ and $\mathrm{Col}_X(D')$ are the same. Thus, the cardinality of $\mathrm{Col}_X(D)$ is an oriented link invariant, which is called the *coloring number* of D.

1.3 Quandle coloring quiver

In [1], Cho and Nelson introduced the quandle coloring quiver. Let X be a finite quandle and D be a diagram of an oriented link L. For any subset $S \subset \mathrm{End}(X)$, the *quandle coloring quiver* of D, which is denoted by $Q_X^S(D)$, the directed graph with a vertex for each X-coloring $c \in \mathrm{Col}_X(D)$ and an edge from v to w when $w = f(v)$ for an element $f \in S$.

Let D' be another diagram of L. Cho and Nelson showed that $Q_X^S(D)$ and $Q_X^S(D')$ are isomorphic as quivers for any subset $S \subset \mathrm{End}(X)$, which implies that the quandle coloring quiver $Q_X^S(D)$ is an oriented link invariant.

Note that the coloring number of D using a finite quandle X is the number of vertices of the quandle coloring quiver $Q_X^S(D)$ for any S. Thus, the quandle coloring quiver is in general a stronger invariant than the coloring number (see [1]).

1.4 Quandle quiver

In this section, we introduce the *quandle quiver*.

Definition 1. Let X be a finite quandle and S be a subset of $\mathrm{End}(X)$. The quandle quiver $Q(X, S)$ is the directed graph with a vertex for each element $x \in X$ and an edge from v to w when $w = f(v)$ for an element $f \in S$.

Lemma 1. *Suppose that finite quandles X and X' are quandle isomorphic. For any subset $S \subset \mathrm{End}(X)$, there exists a subset $S' \subset \mathrm{End}(X')$ such that $Q(X, S)$ and $Q(X', S')$ are isomorhic as quivers.*

In particular, when $S = \mathrm{End}(X)$, we have $S' = \mathrm{End}(X')$. Thus, the quandle quiver $Q(X, \mathrm{End}(X))$ is an invariant of a quandle.

2 Main result

In this section, we study a relation between the quandle coloring quiver and the quandle quiver.

Let X be a finite abelian quandle and D be an oriented link diagram. In [2], Crans and Nelson showed that when X is abelian quandle, $\mathrm{Col}_X(D)$ is a quandle with the operation $*$ defined by $(c * c')(x) = c(x) * c'(x)$ for any $x \in X$.

For any $f \in \mathrm{End}(X)$, we define the map $f_\# : \mathrm{Col}_X(D) \to \mathrm{Col}_X(D)$ by $f_\#(c) = f \circ c$. By the direct calculation, the map $f_\#$ is a quandle endomorphism. Thus, we can define the map $\psi_{X,D} : \mathrm{End}(X) \to \mathrm{End}(\mathrm{Col}_X(D))$ by $\psi_{X,D}(f) := f_\#$. For any $S \subset \mathrm{End}(X)$, we set $\tilde{S}_{X,D} := \psi_{X,D}(S)$.

Theorem 1. *Let D be an oriented link diagram and X be a finite abelian quandle. The quandle coloring quiver $Q_X^S(D)$ and the quandle quiver $Q(\mathrm{Col}_X(D), \tilde{S}_{X,D})$ are isomorphic for any subset $S \subset \mathrm{End}(X)$.*

By Theorem 1, we can interpret the quandle coloring quiver as the quandle quiver when a quandle is an abelian quandle.

References

[1] K. Cho and S. Nelson. Quandle coloring quivers. *J. Knot Theory Ramifications.* **28** (2019), no. 1, 1950001, 12 pp.

[2] A. S Crans and S. Nelson. Hom quandles. *J. Knot Theory Ramifications.* **23** (2014), no. 2, 1450010.

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Quandle coloring quivers for links and quivers of quandles

Yuta Taniguchi

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December 11, 2019

Quandle

Definition (Joyce, 1982)

X: a set, $* : X \times X \to X$: a binary operation.
$X = (X, *)$: a *quandle*.
$\Leftrightarrow$ $*$ satisfies the following three conditions:

1. $\forall x \in X$, $x * x = x$.
2. $\forall y \in X$, $*y : X \to X$; $x \mapsto x * y$: a bijection.
3. $\forall x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

ex.
$R_n = (\mathbb{Z}_n, *)$: *the dihedral quandle* of order n.
$(x * y = 2y - x)$

Abelian quandle

Definition (Joyce, 1982)

$X=(X,*)$: a quandle.
X: *an abelian quandle.*
$\Leftrightarrow \forall x,y,z,w\in X,\ (x*y)*(z*w)=(x*z)*(y*w)$.

ex1.
Let $M=(M,*)$ be an Alexander quandle.
(M: left $\mathbb{Z}[t^{\pm1}]$-mod; $x*y:=tx+(1-t)y(\forall x,y\in M)$.)
Then, M is an abelian quandle.
ex2.
Let X be $(\mathbb{Z}_2)^2$. $*$ is defined by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} * \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \left\{ (x_1\ x_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Then, $(X,*)$ is not an abelian quandle.

M-quandle

M : a monoid, X : a quandle.
End(X):=$\{f : X\to X \mid f$: a quandle homomorphism $\}$.
$\rho : M\to \mathrm{End}(X)$: a map.

Definition

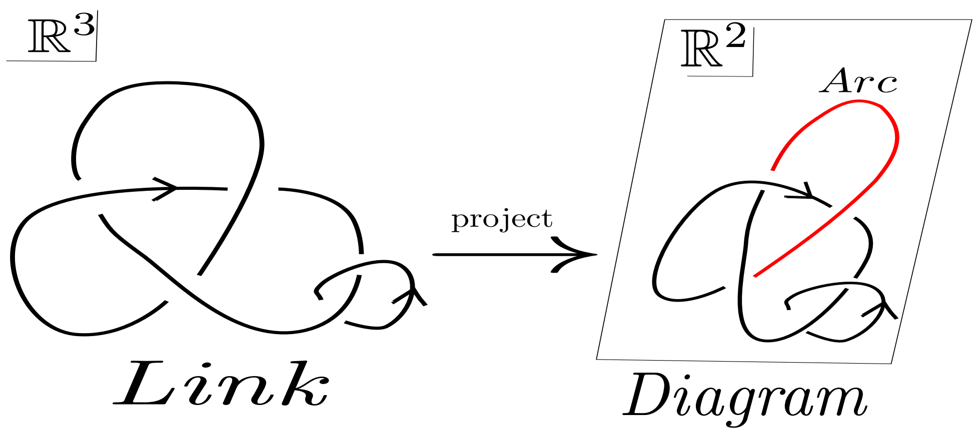
X : an (M,ρ)-*quandle* or M-*quandle.*
$\Leftrightarrow$ ρ: a monoid homomorphism, i.e.

1. $\forall m,m'\in M, \rho(m')\circ\rho(m)=\rho(m'm)$
2. $\rho(e)=id_X$

Definition

$X,\ X'$: M-quandles, $\varphi : X\to X'$: a quandle homomorphism.
φ: an *M-quandle homomorphism*
$\Leftrightarrow$ $\forall m\in M,\ \forall x\in X,\ \varphi(\rho(m)(x))=\rho'(m)(\varphi(x))$.
If φ is bijective, φ is called an *M-quandle isomorphism.*
Then, we write $X\cong X'$ as M-quandle.

Basics of knot theory



Theorem (Reidemeister, 1926)

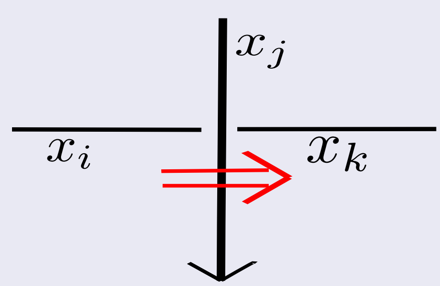
$\{$ links $\}/\sim \xleftrightarrow{1:1} \{$ diagrams $\}/$Reidemeister moves

Quandle coloring

X: a finite quandle, D: an oriented link diagram.
$\mathrm{Arc}(D) = \{x_1, \dots, x_n\}$: the set of arcs of D.

Definition (Joyce, 1982)

$c : \mathrm{Arc}(D) \to X$: an X-coloring of D.
$\Leftrightarrow$ For any crossing, $c(x_i) * c(x_j) = c(x_k)$.



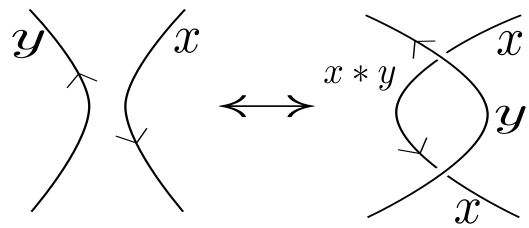
$\mathrm{Col}_X(D) := \{c : \ X\text{-colorings of } D\}$

Property of coloring

Proposition (Joyce, 1982)

D, D': diagrams of an oriented link.
$\Rightarrow \mathrm{Col}_X(D) \overset{1:1}{\longleftrightarrow} \mathrm{Col}_X(D')$.
Thus, $|\mathrm{Col}_X(D)|$ is a link invariant.

<u>skech.</u> In the case of Reidemeister move 2,



Theorem (Crans-Nelson, 2014)

X: an abelian quandle.
$\Rightarrow \mathrm{Col}_X(D)$: a quandle by $c * c'(x) := c(x) * c'(x)(\forall x \in \mathrm{Arc}(D))$

(Link) Quandle coloring quiver

<u>Remark.</u>
$\forall f \in \mathrm{End}(X)$.
$c \in \mathrm{Col}_X(D) \Rightarrow f \circ c \in \mathrm{Col}_X(D)$

Definition (Cho-Nelson, 2018)

X: a finite quandle, $S \subset \mathrm{End}(X)$
$Q_X^S(D)$: a *(link) quandle coloring quiver*
$\Leftrightarrow Q_X^S(D) = (V, E)$: an oriented graph

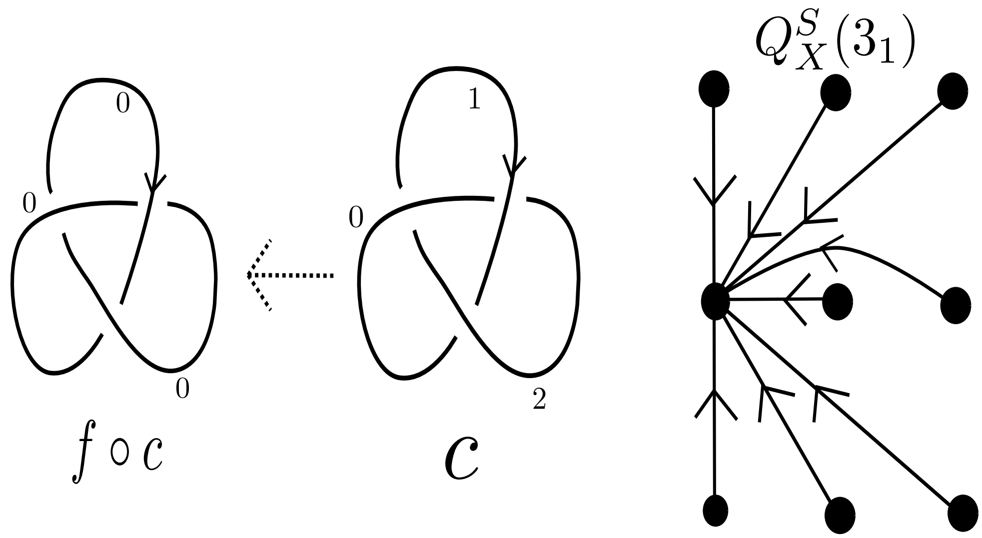
1. $V = \mathrm{Col}_X(D)$
2. $E = \{(v, w) \mid \exists f \in S \text{ s.t. } w = f \circ v\}$

Proposition (Cho-Nelson, 2018)

$D \sim D'$
$\Rightarrow \forall S \subset \mathrm{End}(X), Q_X^S(D) \cong Q_X^S(D')$.
Thus, $Q_X^S(D)$ is a link invariant.

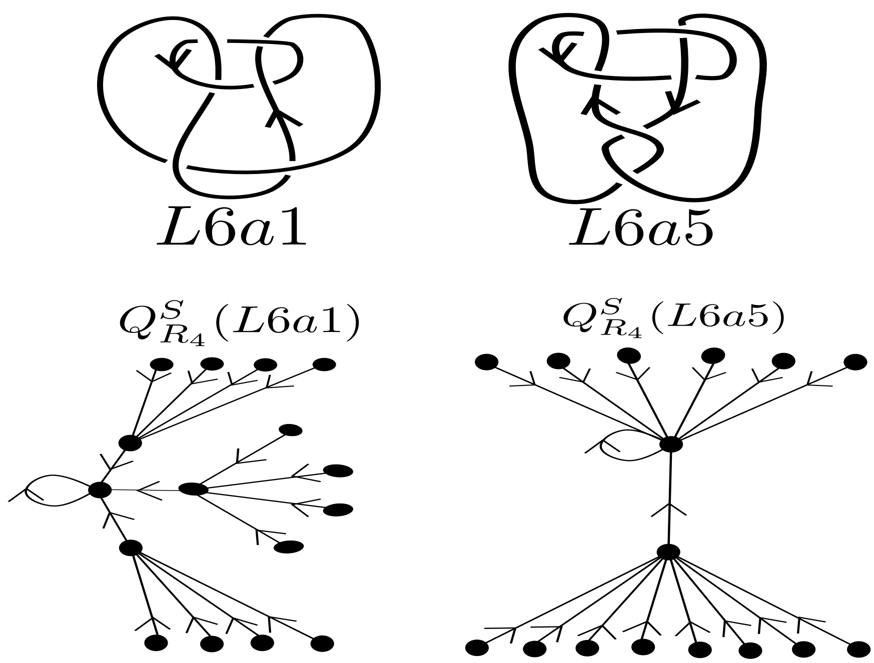
Example of quandle coloring quiver (1)

$f : R_3 \to R_3$ defined by $f(x) := 0$; $S = \{f\}$.



Example of quandle coloring quiver (2)

$f : R_4 \to R_4$ defined by $f(x) := 2x$; $S = \{f\}$



Quandle quiver

Definition

X: a finite quandle, $S \subset \mathrm{End}(X)$.
$Q(X,S)$: a *quandle quiver.*
$\Leftrightarrow Q(X,S) = (V,E)$: an oriented graph.

1. $V = X$.
2. $E = \{(v,w) \mid \exists f \in S \ s.t. \ w = f(v)\}$.

Proposition

$X \cong X'$.
$\Rightarrow \forall S \subset \mathrm{End}(X), \exists S' \subset \mathrm{End}(X')$ s.t. $Q(X,S) \cong Q(X',S')$.
In particular, $Q(X,\mathrm{End}(X)) \cong Q(X',\mathrm{End}(X'))$.

Property of quandle quiver

Remark.
X: a finite abelian quandle.

$$\forall f \in \mathrm{End}(X),\ f_\# : \begin{array}{ccc} \mathrm{Col}_X(D) & \to & \mathrm{Col}_X(D) \\ c & \longmapsto & f \circ c \end{array} \text{: a quandle homomorphism}$$

Then, we can define $\psi_{X,D} : \mathrm{End}(X) \to \mathrm{End}(\mathrm{Col}_X(D))$
by $\psi_{X,D}(f) := f_\#$.
$S \subset \mathrm{End}(X)$, $\tilde{S}_{X,D} := \psi_{X,D}(S)$.

Theorem

X: a finite abelian quandle.
$\Rightarrow \forall S \subset \mathrm{End}(X), Q_X^S(D) \cong Q(\mathrm{Col}_X(D), \tilde{S}_{X,D})$.

Application

By the definition of $\psi_{X,D}$, $\psi_{X,D}$ is a monoid homomorphism.
Then, $\mathrm{Col}_X(D)$ is an $\mathrm{End}(X)$-quandle.

Theorem

$\mathrm{Col}_X(D) \cong \mathrm{Col}_X(D')$ as $\mathrm{End}(X)$-quandle.
$\Rightarrow \forall S \subset \mathrm{End}(X), Q_X^S(D) \cong Q_X^S(D')$.

Theorem

p: a prime number.
X: a connected quandle of order p.
$|\mathrm{Col}_X(D)| = |\mathrm{Col}_X(D')|$.
$\Rightarrow \forall S \subset \mathrm{End}(X), Q_X^S(D) \cong Q_X^S(D')$.

Thank you for your attention.

Multiple conjugation quandle colorings for handlebody-knots

Tomo Murao

Abstract. A handlebody-knot is a handlebody embedded in the 3-sphere. The study of genus 1 handlebody-knots is exactly that of classical knots. In this paper, we introduce a coloring invariant for handlebody-knots by using a multiple conjugation quandles, where a multiple conjugation quandle is an algebra whose axioms are motivated from Reidemeister moves for handlebody-knots.

1 Multiple conjugation quandles

A *quandle* [3,4] is a non-empty set Q equipped with a binary operation $\triangleleft : Q \times Q \to Q$ satisfying the following axioms:

- For any $a \in Q$, $a \triangleleft a = a$.
- For any $a \in Q$, the map $\triangleleft a : Q \to Q$ defined by $\triangleleft a(x) = x \triangleleft a$ is bijective.
- For any $a, b, c \in Q$, $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$.

We denote $(\triangleleft a)^n : Q \to Q$ by $\triangleleft^n a$ for $n \in \mathbb{Z}$. We define the *type* of a quandle Q by

$$\operatorname{type} Q = \min\{n \in \mathbb{Z}_{>0} \mid x \triangleleft^n y = x \text{ (for any } x, y \in Q)\},$$

where we set $\min \emptyset := \infty$ for the empty set $\emptyset$. Any finite quandle is of finite type.

Let G be a group. We define a binary operation $\triangleleft$ on G by $a \triangleleft b = b^{-1}ab$. Then, $(G, \triangleleft)$ is a quandle. We call it the *conjugation quandle* of G and denote it by $\operatorname{Conj} G$. For a positive integer n, we denote by $\mathbb{Z}_n$ the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of order n. We define a binary operation $\triangleleft$ on $\mathbb{Z}_n$ by $a \triangleleft b = 2b - a$. Then, $(\mathbb{Z}_n, \triangleleft)$ is a quandle. We call it the *dihedral quandle* of order n and denote it by R_n.

Definition ([2])**.** A *multiple conjugation quandle* X is a disjoint union of groups $G_\lambda (\lambda \in \Lambda)$ with a binary operation $\triangleleft : X \times X \to X$ satisfying the following axioms:

- For any $a, b \in G_\lambda$, $a \triangleleft b = b^{-1}ab$.
- For any $x \in X$ and $a, b \in G_\lambda$, $x \triangleleft e_\lambda = x$ and $x \triangleleft (ab) = (x \triangleleft a) \triangleleft b$, where e_λ is the identity of G_λ.
- For any $x, y, z \in X$, $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.
- For any $x \in X$ and $a, b \in G_\lambda$, $(ab) \triangleleft x = (a \triangleleft x)(b \triangleleft x)$, where $a \triangleleft x, b \triangleleft x \in G_\mu$ for some $\mu \in \Lambda$.

Let $(Q, \triangleleft)$ be a quandle. Then $Q \times \mathbb{Z}_{\operatorname{type} Q} = \bigsqcup_{x \in Q} (\{x\} \times \mathbb{Z}_{\operatorname{type} Q})$ is an multiple conjugation quandle with

$$(x, a) \triangleleft (y, b) := (x \triangleleft^b y, a), \qquad (x, a)(x, b) := (x, a + b)$$

for any $x, y \in Q$ and $a, b \in \mathbb{Z}_{\operatorname{type} Q}$, where we put $\mathbb{Z}_\infty := \mathbb{Z}$. We call it the *associated multiple conjugation quandle* of Q.

2 Coloring invariants for handlebody-knots

A *handlebody-link* [1] is a disjoint union of handlebodies embedded in the 3-sphere S^3. A *handlebody-knot* is a one component handlebody-link. In this paper, we assume that every component of a handlebody-link is of genus at least 1. Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other. A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph whose regular neighborhood is the handlebody-link, where a spatial trivalent graph is a finite trivalent graph embedded in S^3. In this paper, a trivalent graph may contain circle components.

Let D be a diagram of a handlebody-link. A *Y-orientation* of D is a collection of orientations of all edges of D without sources and sinks with respect to the orientation as shown in Figure 1, where an edge of D is a piece of a curve each of whose endpoints is a vertex. In this paper, a circle component of D is also regarded as an edge of D. It is known that every diagram of a handlebody-link has a Y-orientation.

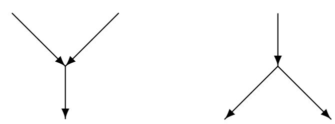


Figure 1: Y-orientations.

Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle and let D be a Y-oriented diagram of a handlebody-link. We denote by $\mathcal{A}(D)$ the set of arcs of D, where an arc is a piece of a curve each of whose endpoints is an undercrossing or a vertex. In this paper, we also regard a circle component of D as an arc of D. An X-*coloring* of D is a map from $\mathcal{A}(D)$ to X satisfying the conditions depicted in Figure 2 at each crossing and vertex of D. We denote by $\mathrm{Col}_X(D)$ the set of all X-colorings of D. Then we have the following theorem.

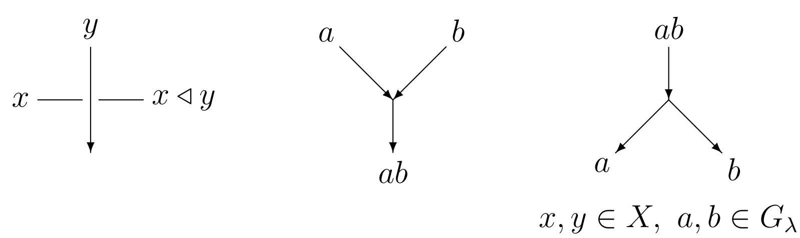


Figure 2: The coloring condition.

Theorem ([2])**.** *Let H be a handlebody-link represented by a Y-oriented diagram D. Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle. Then the cardinality $\#\mathrm{Col}_X(D)$ is an invariant of H.*

For a handlebody-link H represented by a Y-oriented diagram D and a multiple conjugation quandle X, we define the X-*coloring number* of H by $\mathrm{col}_X(H) = \#\mathrm{Col}_X(D)$.

Example. Let H_0 and H be handlebody-knots illustrated in Figure 3, respectively. Let $X = \bigsqcup_{x \in R_3} (\{x\} \times \mathbb{Z}_2)$ be the associated multiple conjugation quandle of the dihedral quandle R_3. Then we have $\mathrm{col}_X(H_0) = 12$ and $\mathrm{col}_X(H) = 18$. Therefore H_0 and H are not equivalent.

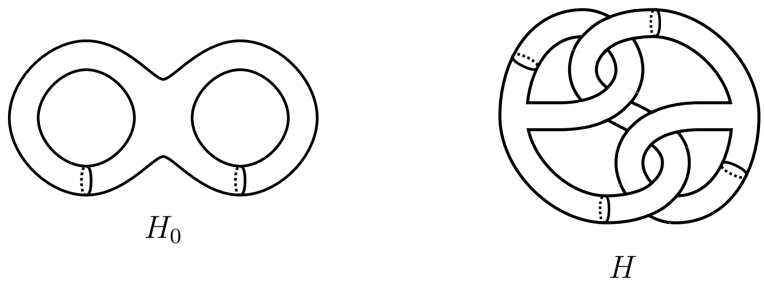


Figure 3: The handlebody-knots H_0 and H.

References

[1] A. Ishii, *Moves and invariants for knotted handlebodies*, Algebr. Geom. Topol. **8** (2008), 1403–1418.

[2] A. Ishii, *A multiple conjugation quandle and handlebody-knots*, Topology Appl. **196** (2015), 492–500.

[3] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982), no. 1, 37–65.

[4] S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) **119(161)** (1982), no. 1, 78–88, 160.

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Polars and antipodal sets

Hiroyuki Tasaki

The contents of this talk is based on a joint work [5] with Tanaka and Yasukura. Polars and antipodal sets are notions of compact Riemannian symmetric spaces, which are introduced by Chen-Nagano [1] and [2]. Their definitions are as follows. For a compact Riemannian symmetric space M with the geodesic symmetry s_x at each point x in M, a connected component of the fixed point set of the geodesic symmetry s_x is called a *polar* of M with respect to x. It is known that a polar is a totally geodesic submanifold. A subset A of M is called an *antipodal set*, if $s_x(y) = y$ for any points x, y in A. Polars and antipodal sets are closely related. Since each polar is an orbit of the isotropy group, we can describe it using canonical forms of the action of the isotropy group. When the number of polars is small, we can find some relationship between antipodal sets of M and those of polars. In the cases we treat in this talk a compact Riemannian symmetric space G/K is imbedded in G as a polar. We show a relationship between antipodal sets of G and those of G/K. We apply this method to the compact Lie group G_2 and the compact Riemannian symmetric space $G_2/SO(4)$ and give explicit descriptions of their maximal antipodal sets.

References

[1] B.-Y. Chen and T. Nagano, Totally geodesic submanifolds of symmetric spaces II, Duke Math. J., vol. 45, no. 2, (1978), 405–425.

[2] B.-Y. Chen and T. Nagano, A Riemannian geometric invariants and its applications to a problem of Borel and Serre, Trans. Amer. Math. Soc., vol. 308, no. 1, (1988), 273–297.

[3] M. S. Tanaka, H. Tasaki and O. Yasukura, Maximal antipodal sets related to G_2, preprint.

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極地と対蹠集合

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定義 (Chen-長野)

M : コンパクト Riemann 対称空間

s_x : $x \in M$ における点対称

$S \subset M$: 対蹠集合

$\Leftrightarrow \forall x, y \in S \ s_x(y) = y$

$\#_2 M = \max\{|A| \mid A : 対蹠集合\}$

$F(s_o, M)$ の各連結成分 : 極地

極地を調べることが対蹠集合を調べることの手がかり

極地：全測地的部分多様体

特にコンパクト Riemann 対称空間

$x \in S^n(r)$ $F(s_x, S^n(r)) = \{\pm x\}$

$K = R, C, H$, $x \in P^n(K)$

$F(s_x, P^n(K)) = \{x\} \cup P^{n-1}(K)$

$F(s_{(x_1,x_2)}, M_1 \times M_2)$

$= F(s_{x_1}, M_1) \times F(s_{x_2}, M_2)$

M の極地：

$$F(s_o, M) = \{o\} \cup \bigcup_{i=1}^{a} \{o_i\} \cup M_1^+$$

が成り立つとき、$s_o = s_{o_i}$

M の極大対蹠集合 (MAS)

$\leftrightarrow$ M_1^+ の MAS

$\{o, o_1, \ldots, o_a\} \cup A \leftrightarrow A$

(合同類も対応する)

$\tilde{M} = S^{n_1}(r_1) \times S^{n_2}(r_2)$

$M = (S^{n_1}(r_1) \times S^{n_2}(r_2))/\{\pm 1\}$

$\tilde{M} \to M \; ; (x, y) \mapsto [x, y]$

被覆変換と点対称は可換

$s_{[x,y]}([x_1, y_1]) = [s_x(x_1), s_y(y_1)]$

$s_{[x,y]}([x_1, y_1]) = [x_1, y_1]$

$\Leftrightarrow$ 「$s_x(x_1) = x_1$ かつ $s_y(y_1) = y_1$」

または

「$s_x(x_1) = -x_1$ かつ $s_y(y_1) = -y_1$」

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$F(s_{[x,y]}, M)$

$= \{[x, \pm y]\} \cup (S^{n_1-1} \times S^{n_2-1})/\{\pm 1\}$

$(S^{n_1}(r_1) \times S^{n_2}(r_2))/\{\pm 1\}$ の MAS

$\{[x_1, \pm y_1], \ldots, [x_k, \pm y_k]\}$

$x_1, \ldots, x_{n_1+1}$: $\mathbb{R}^{n_1+1}$ の直交基底

$y_1, \ldots, y_{n_2+1}$: $\mathbb{R}^{n_2+1}$ の直交基底

$k = \min\{n_1, n_2\} + 1$

上の MAS は合同を除いて一意的

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コンパクト$\mathbf{Lie}$群
両側不変$\mathbf{Riemann}$計量
　$\rightarrow$ コンパクト$\mathbf{Riemann}$対称空間
　点対称 $s_x(y) = xy^{-1}x$
点対称を代数的に表現できる
コンパクト$\mathbf{Lie}$群を$\mathbf{Riemann}$対称空間
とみなすことにより、その代数構造を
幾何学的観点から調べることができる

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G : コンパクト$\mathbf{Lie}$群
A : 単位元を含む極大対蹠集合
$\forall x \in A\ x = s_e(x) = x^{-1},\ x^2 = e$
$\forall y \in A\ y = s_x(y) = xy^{-1}x,$
$xy = yx$
Aの元の積は可換　$\forall z \in A$

$$s_z(xy) = z(xy)^{-1}z = zy^{-1}zzx^{-1}z$$
$$= s_z(y)s_z(x) = xy$$

Aの極大性より$xy \in A$　Aは部分群

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Aの単位元以外の各元の位数は2

有限Abel群の基本定理より

$A \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$

$$F(s_e, G) = \{x \in G \mid x^2 = e\}$$
$$= \{e\} \cup \bigcup_{i=1}^{a} \{o_i\} \cup \bigcup_{j=1}^{b} M_j^+$$

$M_j^+ = \{g x_j g^{-1} \mid g \in G_0\}$：極地

ここでG_0はGの単位連結成分

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A：GのMASとすると

$s_e = s_{o_i}$これより$\{e, o_1, \ldots, o_a\} \subset A$

Aの極大性より$\exists j \ A \cap M_j^+ \neq \emptyset$

Aを共役なものに取り換える

$$x_j \in A \cap M_j^+ \quad A \subset Z_{x_j}(G)$$

Aは$Z_{x_j}(G)$のMAS

各$Z_{x_j}(G)$のMAS：GのMASの候補

上記一般論がうまくいく例：G_2

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$\boldsymbol{H}$：四元数体　$\boldsymbol{O} := \boldsymbol{H} \times \boldsymbol{H}$

$\boldsymbol{O}$：八元数　積の定義

$$(\boldsymbol{m},\boldsymbol{a})(\boldsymbol{n},\boldsymbol{b}) = (\boldsymbol{m}\boldsymbol{n} - \overline{\boldsymbol{b}}\boldsymbol{a},\ \boldsymbol{a}\overline{\boldsymbol{n}} + \boldsymbol{b}\boldsymbol{m})$$
$$((\boldsymbol{m},\boldsymbol{a}),(\boldsymbol{n},\boldsymbol{b}) \in \boldsymbol{O})$$

$$\mathbf{Aut}(\boldsymbol{O}) := \{\boldsymbol{\alpha} \in \boldsymbol{GL_R}(\boldsymbol{O}) \mid \boldsymbol{\alpha}(\boldsymbol{x}\boldsymbol{y}) = (\boldsymbol{\alpha}\boldsymbol{x})(\boldsymbol{\alpha}\boldsymbol{y})\ (\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{O})\}$$

$\mathbf{Aut}(\boldsymbol{O})$：$\boldsymbol{G}_2$型連結コンパクト**Lie**群

$\boldsymbol{G}_2 = \mathbf{Aut}(\boldsymbol{O})$で表す

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写像$\boldsymbol{\psi} : \boldsymbol{Sp}(1) \times \boldsymbol{Sp}(1) \to \boldsymbol{G}_2$を

$$\boldsymbol{\psi}(\boldsymbol{p},\boldsymbol{q})(\boldsymbol{m},\boldsymbol{a}) := (\boldsymbol{q}\boldsymbol{m}\overline{\boldsymbol{q}}, \boldsymbol{p}\boldsymbol{a}\overline{\boldsymbol{q}})$$
$$(\boldsymbol{p},\boldsymbol{q} \in \boldsymbol{Sp}(1),\ (\boldsymbol{m},\boldsymbol{a}) \in \boldsymbol{O})$$

と定める。$\boldsymbol{\psi}$は**Lie**群の準同型写像

$$\ker \boldsymbol{\psi} = \{\pm(1,1)\}$$
$$\boldsymbol{\psi}(\boldsymbol{Sp}(1)^2) = \boldsymbol{Z}_{\psi(1,-1)}(\boldsymbol{G}_2) \cong \boldsymbol{SO}(4)$$

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$G_2 \supset \psi(Sp(1)^2) \cong SO(4)$
ともに階数2
$T = \{\psi(e^{is}, e^{it}) \mid s, t \in \mathbb{R}\}$
は$\psi(Sp(1)^2)$とG_2の極大トーラス

$$G_2 = \bigcup_{g \in G_2} gTg^{-1}$$

$$F(s_e, G_2) = \bigcup_{g \in G_2} gF(s_e, T)g^{-1}$$

$$F(s_e, T) = \{\psi(1, \pm 1), \psi(i, \pm i)\}$$

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$\psi(1, -1), \psi(i, \pm i)$: 互いに共役
$F(s_e, G_2) \backslash \{e\}$
$= \{g\psi(1, -1)g^{-1} \mid g \in G_2\}$
e以外の極地は$\psi(1, -1)$の軌道のみ
A : G_2のMAS 共役なものに取り換え
$\psi(1, -1) \in A$とできる
$A \subset Z_{\psi(1,-1)}(G_2) = \psi(Sp(1)^2)$
Aは$\Psi = \{\psi(p, \pm p) \mid p = 1, i, j, k\}$
に共役　G_2のMASはΨのみ

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$\boldsymbol{Z_{\psi(1,-1)}(G_2)}$ の MAS も $\boldsymbol{\Psi}$ のみ

$\boldsymbol{\Psi = \{\psi(p, \pm p) \mid p = 1, i, j, k\}}$

$\quad \boldsymbol{\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2}$：階数 $\boldsymbol{3}$

$\boldsymbol{\#_2 G_2 = |\Psi| = 2^3}$

上記の階数 $\boldsymbol{> 2}$ ： $\boldsymbol{G_2}$ の階数

すなわち、$\boldsymbol{G_2}$ の MAS $\boldsymbol{\Psi}$ は $\boldsymbol{G_2}$ の極大トーラスに収まらない

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$$\boldsymbol{F(s_e, G_2) = \{e\} \cup M_1^+,}$$

$$\boldsymbol{M_1^+ = \{g\psi(1,-1)g^{-1} \mid g \in G_2\}}$$

$$\boldsymbol{\cong G_2/SO(4)}$$

$\boldsymbol{M_1^+}$ の極大対蹠集合 (MAS)

$$\boldsymbol{\Psi_1 = \{\psi(1,-1)\}}$$

$$\boldsymbol{\cup \{\psi(p, \pm p) \mid p = i, j, k\}}$$

$\boldsymbol{\#_2 M_1^+ = |\Psi_1| = 7}$

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$\tilde{G}_{\mathrm{ass}}$：結合的Grassmann多様体
$\tilde{G}_{\mathrm{ass}} \subset \tilde{G}_3(R^7)$ とみなせる
全測地的部分多様体になる
G_2 は $\tilde{G}_{\mathrm{ass}}$ に推移的に作用
$\tilde{G}_{\mathrm{ass}} \cong G_2/S_{\mathrm{Im}H\times\{0\}}(G_2)$
$S_{\mathrm{Im}H\times\{0\}}(G_2) = Z_{\psi(1,-1)}(G_2)$
$M_1^+ \cong G_2/Z_{\psi(1,-1)}(G_2)$
$\Psi_1 \subset M_1^+ \to \tilde{\Psi}_1 \subset \tilde{G}_{\mathrm{ass}} \subset \tilde{G}_3(R^7)$
$\tilde{\Psi}_1$ は $\tilde{G}_3(R^7)$ の対蹠集合、$|\tilde{\Psi}_1| = 7$

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$\tilde{G}_k(R^n)$ のMAS
$[n] = \{1, 2, 3, \ldots, n\}$
$\binom{[n]}{k}$：$[n]$ 内の濃度 k の部分集合全体
$A \subset \binom{[n]}{k}$：対蹠集合
$\Leftrightarrow \forall \alpha, \beta \in A \ |\alpha \backslash \beta|$：偶数
$\binom{[n]}{k}$ のMASの合同類と
$\tilde{G}_k(R^n)$ のMASの合同類
：一対一対応

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$e_1, \ldots, e_n$: R^n の正規直交基底

$\alpha \in \binom{[n]}{k}$ に対して $\alpha = \{\alpha_1, \ldots, \alpha_k\}$

$A \subset \binom{[n]}{k}$: 対蹠集合

$\mathcal{A}(A) = \{\pm\langle e_{\alpha_1}, \ldots, e_{\alpha_k}\rangle \mid \alpha \in A\}$

: $\tilde{G}_k(R^n)$ の対蹠集合

これにより

$\binom{[n]}{k}$ の MAS $\leftrightarrow$ $\tilde{G}_k(R^n)$ の MAS

$$A \leftrightarrow \mathcal{A}(A)$$

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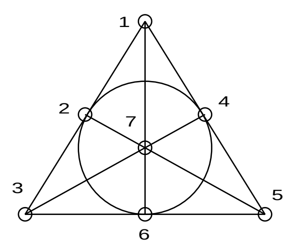
$\binom{[7]}{3}$ の MAS : Fano 平面 (7 点集合)



$\Psi_1 \subset M_1^+ \cong \tilde{G}_{\mathrm{ass}} \subset \tilde{G}_3(R^7)$

$\{\pm\xi \mid \xi \in \Psi_1\}$: $\tilde{G}_3(R^7)$ の MAS

Ψ_1 は Fano 平面の元の組合せによって

O の積の演算表を定める

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Quandle homology theory and quandle cocycle invariants of links

Kanako Oshiro

Abstract. A quandle is an algebraic system independently introduced by D. Joyce [9] and S. Matveev [4] in 1992. There are several studies using quandles in knot theory. In this paper, we review quandle homology theory and quandle cocycle invariants of oriented links (refer to [1, 2]).

1 Quandles

Definition 1. A *quandle* is a set Q equipped with a binary operation $* : Q \times Q \to Q$ satisfying the following conditions.

(Q1) For any $a \in Q$, $a * a = a$.

(Q2) For any $a, b \in Q$, there exists a unique $c \in Q$ such that $c * b = a$.

(Q3) For any $a, b, c \in Q$, $(a * b) * c = (a * c) * (b * c)$.

We write a quandle $(Q, *)$ simply Q when no confusion can arise.

Example 1. For an integer $n \geq 3$, we set a map $* : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n$ by $a * b = 2b - a$. Then $(\mathbb{Z}_n, *)$ is a quandle. We call it the *dihedral quandle* of order n and we denote it by R_n.

2 Quandle colorings

Let $(Q, *)$ be a quandle. Let D be a diagram of an oriented link.

Definition 2. A *Q-coloring* of D is an assignment of an element of Q to each arc of D satisfying the following crossing condition.

- For a crossing χ of D, let a, c and b denote the assigned elements to the under-arcs and the over-arc of χ as depicted in the left of Figure 4. Then $a * b = c$ holds.

A *shadow Q-coloring* of D is a Q-coloring of D with an assignment of an element of each region of D satisfying the following condition.

- For two regions of D adjacent by an arc, let x, y and a denote the assigned elements to the regions and the arc as depicted in the right of Figure 4. Then $x * a = y$ holds.

We call the assigned element of an arc (or a region) the *color* of the arc (or the region). We denote by $\mathrm{Col}_Q(D)$ the set of shadow Q-colorings of D.

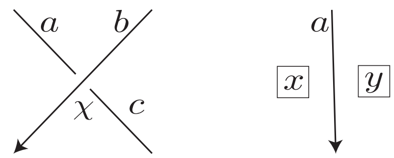

Figure 4: The crossing conditions

3 Quandle (co)homology groups

Let $(Q, *)$ be a quandle. Let $C_n(Q)$ be the free abelian group generated by n-tuples $(x_1, \ldots, x_n) \in Q^n$ when n is a positive integer, and let $C_n(Q) = 0$ otherwise. The boundary homomorphism $\partial_n : C_n(Q) \to C_{n-1}(Q)$ is defined by

$$\begin{aligned}\partial_n(x_1, \ldots, x_n) = & \sum_{i=1}^{n}(-1)^i\big\{(x_1, \ldots, \hat{x_i}, \ldots, x_n) \\ & -(x_1 * x_i, \ldots, x_{i-1} * x_i, \hat{x_i}, x_{i+1}, \ldots, x_n)\big\}\end{aligned}$$

for $n > 1$ and $\partial_n = 0$ otherwise, where $\hat{x_i}$ represents that x_i is removed. Then $C_*(Q) = \{C_n(Q), \partial_n\}_{n\in\mathbb{Z}}$ is a chain complex.

Let $D_n(Q)$ be the subgroup of $C_n(Q)$ generated by the elements of

$$\{(x_1, \ldots, x_n) \in Q^n \mid x_i = x_{i+1} \text{ for some } i \in \{1, \ldots, n-1\}\}.$$

Then $D_*(Q) = \{D_n(Q), \partial_n\}_{n\in\mathbb{Z}}$ is a subchain complex of $C_*(Q)$. Let $C_n^{\mathrm{Q}}(Q) = C_n(Q)/D_n(Q)$, and we denote by ∂_n^{Q} the induced boundary homomorphism $\partial_n^{\mathrm{Q}} : C_n^{\mathrm{Q}}(Q) \to C_{n-1}^{\mathrm{Q}}(Q)$. The quotient chain complex $C_*^{\mathrm{Q}}(Q) = \{C_n^{\mathrm{Q}}(Q), \partial_n^{\mathrm{Q}}\}_{n\in\mathbb{Z}}$ leads to the *quandle homology group* of Q by $H_n^{\mathrm{Q}}(Q) = \mathrm{Ker}\partial_n^{\mathrm{Q}}/\mathrm{Im}\partial_{n+1}^{\mathrm{Q}}$.

For an abelian group A, we define the cochain group $C_{\mathrm{Q}}^n(Q; A)$ and the coboundary homomorphism $\delta_{\mathrm{Q}}^n : C_{\mathrm{Q}}^n(Q; A) \to C_{\mathrm{Q}}^{n+1}(Q; A)$ by $C_{\mathrm{Q}}^n(Q; A) = \mathrm{Hom}(C_n^{\mathrm{Q}}(Q), A)$ and $\delta_{\mathrm{Q}}^n(\theta) = \theta \circ \partial_{n+1}^{\mathrm{Q}}$, respectively. The *quandle cohomology group with coefficients in* A is defined by $H_{\mathrm{Q}}^n(Q; A) = \mathrm{Ker}\delta_{\mathrm{Q}}^n/\mathrm{Im}\delta_{\mathrm{Q}}^{n-1}$.

4 Quandle cocycle invariants

Let $(Q, *)$ be a quandle, A an abelian group, and $\theta : C_3^{\mathrm{Q}}(Q) \to A$ a 3-cocycle. Let D be a diagram, and C a shadow Q-coloring of D.

For a crossing χ of D with C, the weigh of χ is defined by $W_\theta(\chi) := \varepsilon_\chi\theta(x, a, b)$, where $\varepsilon_\chi = +1$ (or -1) if χ is the positive (or negative) crossing, and x, a and b are the colors of the region, the under-arc and the over-arc, respectively, in the position depicted in Figure 5.

We define $\Phi_\theta(D)$ by the multiset

$$\Phi_\theta(D) = \{W_\theta(D, C) \mid C \in \mathrm{Col}_Q(D)\},$$

where $W_\theta(D, C) = \sum_{\chi\in\{\text{crossings}\}} W_\theta(\chi)$.

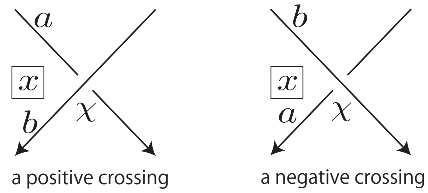

Figure 5: Crossings

Theorem 1. *The multiset $\Phi_\theta(D)$ is an invariant for oriented links.*

By Theorem 1, we may denote the invariant by $\Phi_\theta(L)$, where L is the oriented link which D represents.

Example 2. Let Q be the dihedral quandle R_3. Set $\theta : C_3^{\mathrm{Q}}(Q) \to \mathbb{Z}_3$ by

$$\theta(x,a,b) = (x-a)(a-b)^2 b.$$

We note that the homomorphism θ is a 3-cocycle.

Let K and K^* be the left- and right-handed trefoil knots, respectively (see Figure 6 for a diagram D of K). Then we have

$$\Phi_\theta(K) = \{\underbrace{0,\ldots,0}_{9},\underbrace{1,\ldots,1}_{18}\} \text{ and } \Phi_\theta(K^*) = \{\underbrace{0,\ldots,0}_{9},\underbrace{-1,\ldots,-1}_{18}\}.$$

Thus we can distinguish the left- and right-handed trefoil knots by using the quandle cocycle invariants.

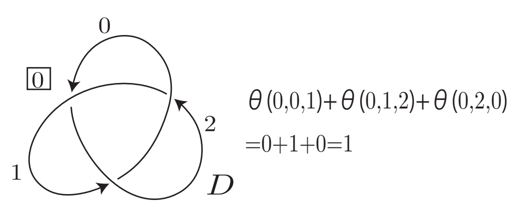

Figure 6: A diagram of the left-handed trefoil knot

References

[1] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. **355** (2003) 3947–3989.

[2] J. S. Carter, S. Kamada and M. Saito, *Surfaces in* 4*-space*, Encyclopaedia of Mathematical Sciences **142** (2004), Low-Dimensional Topology, III. Springer-Verlag, Berlin, xiv+213 pp.

[3] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Alg. **23** (1982) 37–65.

[4] S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) **119**(**161**) (1982) 78–88.

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Connectedness and homogeneity of antipodal sets

Yuuki Sasaki

1 Introduction

Let M be a compact Riemannian symmetric space and denote the geodesic symmetry at $x \in M$ by s_x. In this paper, we assume that M is connected. If $s_x(y) = y$ for two points $x, y \in M$, we say that x, y are antipodal. A subset S of M is an antipodal set, if any two points of S are antipodal. The 2-number $\#_2 M$ of M is the maximum of the cardinalities of antipodal sets of M. We call an antipodal set S in M great if $\# S = \#_2 M$. An antipodal set S is called maximal if there are no anitipodal sets including S properly. These notions were introduced by Chen-Nagano [1]. In general, any antipodal set of any Riemannian symmetric space of noncompact type is a one-point set, so we consider only compact symmetric spaces in this paper. We say that an antipodal set $A \subset M$ is homogeneous if there is a subgroup of the isometry group of M acting on A transitively.

It is known that any compact Lie group G is a Riemannian symmetric space with respect to a biinvariant metric and any maximal antipodal set including the unit element of G becomes a subgroup of G. Therefore, any maximal antipodal set of G is homogeneous. Moreover, Tanaka and Tasaki proved that any great antipodal set of any symmetric R-space is homogeneous [5]. Thus, we consider the following problem:

Problem 1. Is any maximal antipodal set of any compact Riemannian symmetric space homogeneous ?

We consider this problem introducing a concept of connectedness of antipodal sets.

2 connectedness of antipodal sets

Let M be a connected compact symmetric space. Then, there is a closed geodesic through any antipodal two points. Moreover, there is shortest closed geodesics in M. By using these properties, we introduce a concept of connectedness of antipodal sets as follows.

Definition 1. Let $p, q \in M (p \neq q)$ be antipodal two points. If there is a shortest closed geodesic through p and q, we say that p, q are connected.

Definition 2. Let S be an antipodal set.

(1) If a point series $\{p_i\}_{i=1}^{l}$ of S satisfies that p_i is connected to p_{i+1}, then we say this point series is a *connected point series*.

(2) If S satisfies the following condition, we say that S is *connected*: for any $p, q \in S$, there is a connected point series $\{p_i\}_{i=1}^{l}$ of S containing p and q.

(3) Let S be connected. If there are no connected anitpodal sets containing S properly, then we say that S is a *maximally connected* antipodal set.

(4) Let S be not necessarily connected and T be a connectd subset of S. If there are no connected antipodal subsets of S containing T properly, we say that T is a *connected component* of S.

For connected antipodal sets, we obtain the following proposition.

Proposition 1. *Let A be an antipodal set of M containing connected two points. Suppose that $p, q \in A$ are conncted points. Then, there is a shortest closed geodesic through p and q invariant under every $s_r (r \in A)$. Moreover, there is a totally geodesic sphere $M_{p,q}$ through p and q including any shortest closed geodesic through p and q.*

By using this property, we define the following sets. Let A be an antipodal set and $p \in A$.

Set $A_p = \{q \in A; q \text{ is connected to } p.\}$ and $q \in A_p$.

- $L(p, q, A)$: The set of all shortest closed geodesics through p and q invariant under every $s_r (r \in A)$.
- $CL(p, q, A)$: The set of all middle points of all geodesics of $L(p, q, A)$ between p and q.
- $L(A) := \bigcup_{p,q \in A} L(p, q, A)$, where p, q are connected.
- $CL(A) := \bigcup_{p,q \in A} CL(p, q, A)$, where p, q are connected.
- G_A : the group generated by $\{x; x \in CL(A)\}$.

Remark that G_A is not trivial if and only if A has connected two points.

3 Main results

By using connectedness of antipodal sets, we obtain the following propositions.

Proposition 2. *If A is an antipodal set containing connected two points, then $A \cup g(A)$ is an antipodal set for any $g \in G_A$.*

Corollary 1. *$G_A(A)$ is an antipodal set.*

Proof. It is sufficient to show that $g(A) \cup h(A)$ is an antipodal set for any $g, h \in G_W$. We see that $A \cup g^{-1}h(A)$ is an antipodal set by Proposition 2.

□

By the definition of maximal antipodal sets, we obtain the following corollary immediately.

Corollary 2. *If A is a maximal antipodal set containing connected two points, then $G_A(A) \subset A$. In other words, G_A acts on A.*

Next, we study how G_A acts on A in the case $G_A(A) \subset A$.

Proposition 3. *Let A be a connected antipodal set and $o \in A$. Then $G_A(o) = A$. In other words, G_A acts on A transitively.*

Summarizing these results we obtain the follwing theorem.

Theorem 1. *Let A be a maximal antipodal set. If A is connected, then A is homogeneous. In particular, G_A acts on A transitively.*

References

[1] B.Y.Chen, T.Nagano, A Riemannian geometric invariant and its applications to a problem of Borel and Serre, Trans. Amer. Math. Soc., **308**(1988), 273-297

[2] M.S.Tanaka, H.Tasaki, Antipodal sets of symmetric R-spaces, Osaka J. Math., **50**(2013), 161-169

[3] Y.Sasaki, Homogeneity of maximal antipodal sets, Osaka J. Math

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Connectedness and homogeneity of antipodal sets

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Abstract

We introduce a concept of "connectedness" of antipodal sets.

- We construct a method to make a bigger antipodal set from a given antipodal set.
- We construct a method to decide whether a given maximal antipodal set is homogeneous.

Contents

Symmetric space

Let M be a Riemannian manifold.

Definition 1.1

M is a symmetric space.
$\overset{\text{def}}{\Longleftrightarrow}$ For any $x \in M$, there is an isomerty s_x satisfying that

1. *x is an isolated fixed point of s_x, and*
2. *s_x is involutive ($s_x^2 = \mathrm{id}_M$).*

We call $s_x (x \in M)$ the symmetry at x.

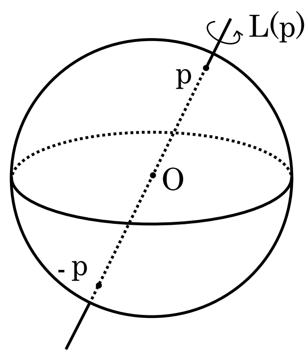
Antipodal set

Definition 1.2

- $x, y \in M$ are antipodal. $\overset{\text{def}}{\Longleftrightarrow} s_x(y) = y (\Leftrightarrow s_y(x) = x)$.
- A subset A of M is an antipodal set.
  $\overset{\text{def}}{\Longleftrightarrow}$ p, q are antipodal for any $p, q \in A$,.
- An antipodal set A is maximal.
  $\overset{\text{def}}{\Longleftrightarrow}$ There are no antipodal sets including A properly.
- The 2-number $\#_2 M$ of M is the maximum cardinality of maximal antipodal sets.
- An antipodal set A is great.
  $\overset{\text{def}}{\Longleftrightarrow}$ $\#A = \#_2 M$.

These notions were introduced by Chen-Nagano in 1988.

Example : S^2



- Let $p \in S^2$ and $L(p)$ be a line through o, p.
- The symmetry at $p \in S^2$ is the 180 degree rotation around $L(p)$.
- The fixed point set $F(s_p, S^2)$ of s_p is $\{p, -p\}$.
  Therefore, $\{p, -p\}$ is a great antipodal set of S^2 and $\#_2 S^2 = 2$.

Remark 1.3

- In a connected symmetric space, for any antipodal two points there is a closed geodesic through these points.
- If M is a symmetric space of non-compact type, any antipodal set of M is trivial.
  $\longrightarrow$ suppose that M is compact.
- Any antipodal set is a finite set.

Property of antipodal sets

Let M be a symmetric R space.
For example, the followings are symmetric R spaces

- spheres S^n
- real,complex,quaternion Grassmannians
- classical groups $SO(n), U(n), Sp(n)$

Theorem 1.4 (Takeuchi, 89)

$\#_2 M = \sum_i \dim(H_i(M, \mathbb{Z}_2))$

Motivation

Considering great antipodal sets and maximal antipodal sets is significant to study symmetric spaces.

We consider the following problem:

How every point of a maximal antipodal set is arranged in a symmetric spaces ?

$\longrightarrow$ Consider following two examples.

Ex1 : Compact Lie group

Let G be a compact Lie group.

- There is a biinvariant Riemannian metric on G and G becomes a compact symmetric space.
- The symmetry $s_g(g \in G)$ is $s_g : G \to G; h \mapsto gh^{-1}g$.
- It is known that a maximal antipodal set including the unit element of G is a subgroup.

In a compact Lie group G, for any maximal antipodal set A there exists a subgroup of the isometry group of G acting on A transitively.

This is a result of Chen-Nagano in 1988.

Ex2 : Symmetric R space

- Maximal antipodal sets in symmetric R spaces are studied well by Tanaka-Tasaki.

  From their results, we see the following fact.

In a symmetric R space M, for any maximal antipodal set A there exists a subgroup of the isometry group of M acting on A transitively.

Introduction

Definition 1.5

An antipodal set A of M is homogeneous.
$\overset{\text{def}}{\Longleftrightarrow}$ there is a subgroup of the isometry group of M acting on A transitively.

Problem

Are maximal antipodal sets of a symmetric space homogeneous ?

- the method to decide whether a given maximal antipodal set is homogeneous.
- the method to construct a bigger antipodal set from a given antipodal set.

Preparations

Connectedness of antipodal two points

Let M be a connected compact symmetric space and $I(M)$ be the isometry group of M.

Remark 2.1

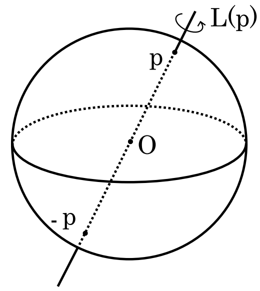
- The set of all lengths of closed geodesics of M is a discrete set in $\mathbb{R}$.
  $\longrightarrow$ There is a shortest closed geodesic on M.
- For any antipodal two points, there is a closed geodesic through these points.

Definition 2.2 (S)

Let $p, q \in M (p \neq q)$ be antipodal two points.
p, q are connected.
$\overset{\text{def}}{\Longleftrightarrow}$ There is a shortest closed geodesic through p and q.

Example 1 (S^2)

- $p \in S^2$ is antipodal to $-p$.
- Any geodesic on S^2 is a great circle on S^2.
- Therefore, p and $-p$ are connected.



Example 2 (compact rank 1 symmetric space M)

- Any geodesic on M is a closed geodesic and they have the common length.
- Therefore, any antipodal two points of M are connected.

Connectedness of antipodal sets

Definition 2.3 (S)

Let A be an antipodal set of M.

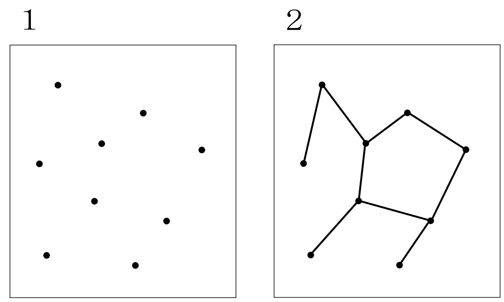
- A is connected
  $\overset{\text{def}}{\iff}$ for any $p, q \in A$, there is a point series $\{p_i\}_{i=1}^k \subset A$ s.t.
  (1) $p, q \in \{p_i\}_{i=1}^k$
  (2) $p_i = p_{i+1}$ or p_i is connected to $p_{i+1} (1 \leq i \leq k-1)$
- A connected subset $B \subset A$ is a connected component of A.
  $\overset{\text{def}}{\iff}$ There are no connected subsets of A including B properly.
- A conncetd antipodal set A is maximally connected.
  $\overset{\text{def}}{\iff}$ There are no connected antipodal sets including A properly.

Remark 2.4

Let A be an antipodal set.

1. Prot each point of A as a vertex.
2. Connect two connected points with a edge.

A is connected.
$\Longleftrightarrow$This graph is connected.



Example 3 (compact rank 1 symmetric space M)

Any antipodal two points are connected in M.

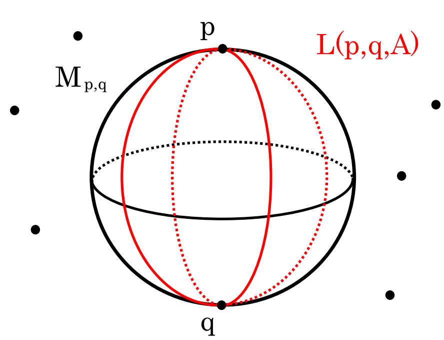
Hence, any antipodal set of M is connected.

Proposition 2.5 (S)

Let A be an antipodal set of M containing connected two points. Suppose that $p, q \in A$ are conncted points. It follows that:

(1) there is a shortest closed geodesic through p and q invariant under every $s_r (r \in A)$, and

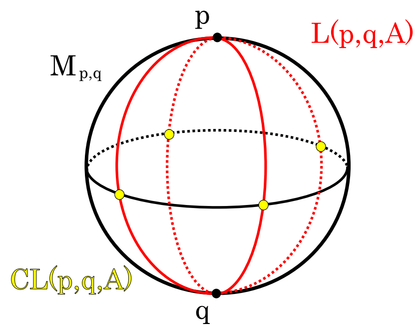
(2) there is a totally geodesic sphere $M_{p,q}$ through p and q including any shortest closed geodesic through p and q.



Some definitions

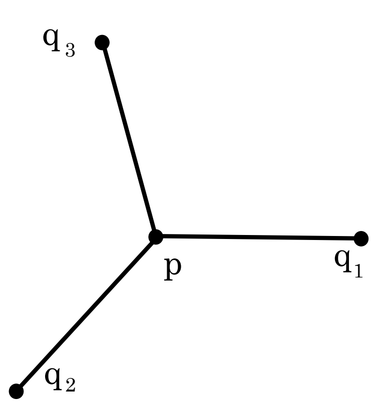
Let A be an antipodal set and $p \in A$.
Set $A_p = \{q \in A; q \text{ is connected to } p.\}$ and $q \in A_p$.

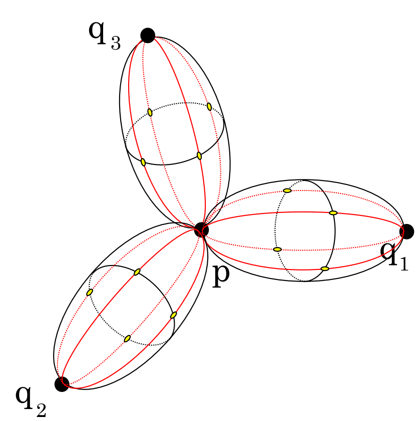
- $L(p, q, A) \overset{\text{def}}{\Longleftrightarrow}$ The set of all shortest closed geodesics through p and q invariant under every $s_r (r \in A)$.
- $CL(p, q, A) \overset{\text{def}}{\Longleftrightarrow}$ The set of all middle points of all geodesics of $L(p, q, A)$ between p and q.



- $L(p, A) := \bigcup_{q \in A_p} L(p, q, A), \quad CL(p, A) := \bigcup_{q \in A_p} CL(p, q, A)$

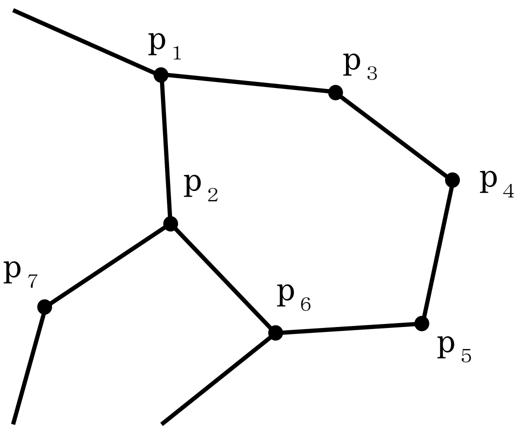
For example, let $A = \{p, q_1, q_2, q_3\}$. In this case, $A_p = \{q_1, q_2, q_3\}$.

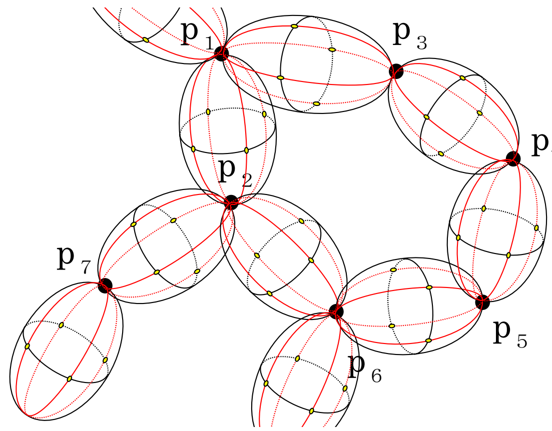




- $L(A) := \bigcup_{p,q\in A} L(p,q,A)$, where p, q are connected.
  $CL(A) := \bigcup_{p,q\in A} CL(p,q,A)$, where p, q are connected.

For example, let $A = \{p_1, p_2, p_3 \cdots\}$.





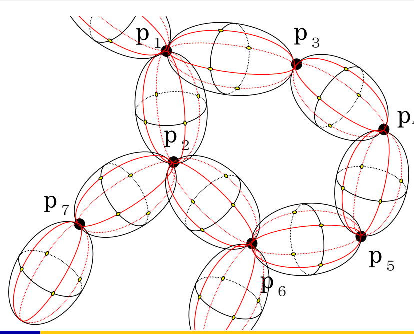
Let W be a subset of $CL(A)$.

- $G_W \overset{\text{def}}{\Longleftrightarrow}$ the group generated by $\{s_x; x \in W\}$.
- If $W = CL(A)$, denote G_W by G_A.
  If $W = CL(p, A)$, denote G_W by $G_{p,A}$.

G_A is a subgroup of the isometry group of M.
Remark that G_W is a subgroup of G_A.

Remark 2.6

$G_A \neq \phi \Longleftrightarrow A$ has connected two points.



Lemma 2.7 (S)

If A is an antipodal set containing connected two points, then it follows that:

(1) For any $p \in A$ and $x \in CL(q, A)(q \in A)$

$$s_p(x) = x \quad \text{or} \quad s_p(x) = s_q(x).$$

In particular,

$$s_p s_x = s_x s_p \quad \text{or} \quad s_p s_x = s_q s_x s_q s_p.$$

(2) If $m \in M$ be antipodal to every point of A and $x \in CL(q, A)$, then

$$s_q s_x s_q(m) = s_x(m)$$

Main results

Main results 1/3

Theorem 3.1 (S)

If A is an antipodal set containing connected two points, then $A \cup g(A)$ is an antipodal set for any $g \in G_A$.

(Proof)

- It is sufficient to show that p and $g(q)$ are antipodal for any $p, q \in A$.
- We can denote $g \in G_A$ by $g = s_{x_m} \cdots s_{x_1} (x_i \in CL(A))$, so we prove it by induction with respect to m.
  Let $x_i \in CL(p_i, q_i, A)(p_i, q_i \in A)$.
- In $m = 1$,

$$s_p(s_{x_1}(q)) = \begin{cases} s_{x_1} s_p(q) = s_{x_1}(q) \\ s_{p_1} s_{x_1} s_{p_1} s_p(q) = s_{p_1} s_{x_1} s_{p_1}(q) = s_{x_1}(q) \end{cases} .$$

Main results Main results 1/3

- Assume that it is true until $m - 1$. Then,

$$\begin{aligned} s_p\Big(s_{x_m} \cdots s_{x_1}(q)\Big) &= \big(\epsilon_m s_{x_m} \epsilon_m\big)\big(\epsilon_{m-1} s_{x_{m-1}} \epsilon_{m-1}\big) \cdots \big(\epsilon_1 s_{x_1} \epsilon_1\big) s_p(q) \\ &= \big(\epsilon_m s_{x_m} \epsilon_m\big)\big(\epsilon_{m-1} s_{x_{m-1}} \epsilon_{m-1}\big) \cdots \big(\epsilon_1 s_{x_1} \epsilon_1\big)(q) \\ &= \big(\epsilon_m s_{x_m} \epsilon_m\big)\Big(s_{x_{m-1}} \cdots s_{x_1}(q)\Big) \\ &= s_{x_m} s_{x_{m-1}} \cdots s_{x_1}(q), \end{aligned}$$

where $\epsilon_i = s_{p_i}$ or $\mathrm{id}|_M$.

Corollary 3.2

For any subset $W \subset CL(A)$, $G_W(A)$ is an antipodal set.

(Proof)

- $G_W(A) = \bigcup_{g \in G_W} g(A)$.
  It is sufficient to show that $g(A) \cup h(A)$ is an antipodal set for any $g, h \in G_W$.
- We see that $A \cup g^{-1}h(A)$ is an antipodal set.

We obtain a bigger antipodal set $G_W(A)$
from an antipodal set A and G_W.

$\longrightarrow$ This is the method to construct a bigger antipodal set.

Corollary 3.3

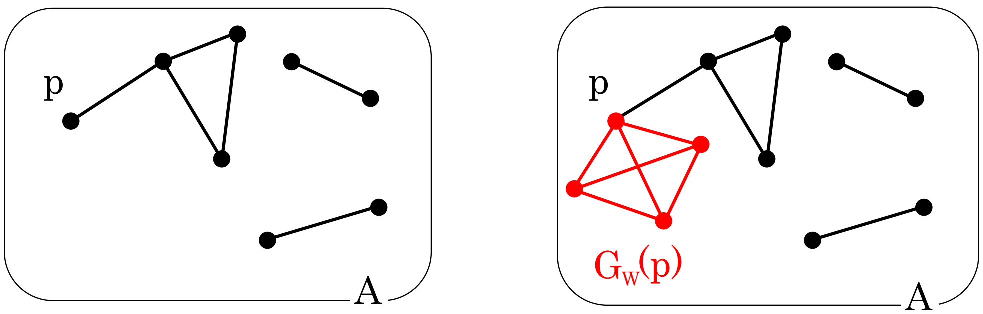
If A is a maximal antipodal set containing connected two points, then G_W acts on A.

Secondly, we study orbits of G_W in $G_W(A)$.

Main results 2/3

Theorem 3.4 (S)

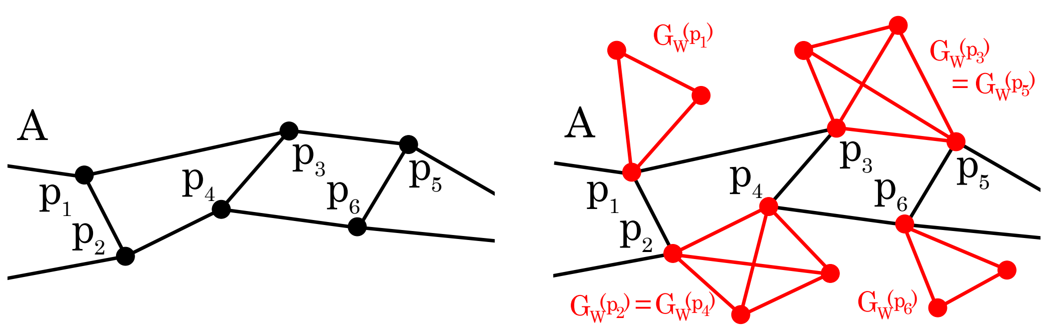
If A is an antipodal set containing connected two points, then $G_W(p)$ is a connected antipodal set for any $p \in A$.



Corollary 3.5

If A is a connected antipodal set, then $G_W(A)$ is a connected antipodal set for any $W \subset CL(A)$.

(Proof)



- Let $A = \{p_1, p_2, \cdots\}$.
  $G_W(A) = G_W(p_1) \cup G_W(p_2) \cup \cdots = \bigcup_{p \in A} G_W(p)$.
- Since A, $G_W(p_1), \cdots$ are connected, $G_W(A)$ is connected.

By the definition, we have the following corollary.

Corollary 3.6

If A is a connected maximal antipodal set, then G_W acts on A.

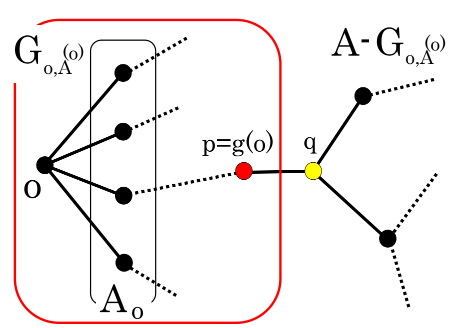
Main results 3/3

Theorem 3.7 (S)

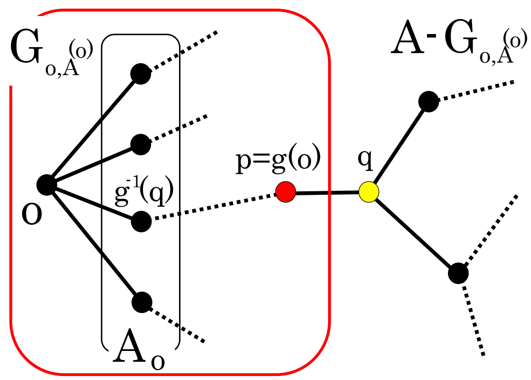
Let A be a connected antipodal set and $o \in A$.
If A is invariant under $G_{o,A}$, then $G_{o,A}(o) = A$.

(Proof)

- We see $A_o \cup \{o\} \subset G_{o,A}(o)$. We assume that $A - G_{o,A}(o) \neq \phi$.
- By the connectedness of A, there is $p \in G_{o,A}(o)$ and $q \in A - G_{o,A}(o)$ which are connected.
- Since $p \in G_{o,A}(o)$, we denote p by $p = g(o)(g \in G_{o,A})$.



- The connectedness is invariant under isometries. Hence, we have $g^{-1}(q) \in A_o$.
- Thus, we have $q \in G_{o,A}(o)$ because $A_o \subset G_{o,A}(o)$. However, this is a contradiction. We conclude that $A = G_{o,A}(o)$.



If A be a connected antipodal set and $G_{o,A}(A) \subset A$, then A is $G_{o,A}$-homogeneous.

For example, following antipodal sets satisfy the above condition.

- connected maximal antipodal sets.
- maximally connected antipodal sets

Is a maximal antipodal set A homogeneous ?

$\longrightarrow$ A is connected : Yes !

$\longrightarrow$ A is not connected : ?

Let B be a not connected maximal antipodal set and

$$B = A_1 \cup \cdots \cup A_n$$

be the decomposition of B by connected components $A_i (1 \leq i \leq n)$.

- Let H be a subgroup of isometry group of M.
  We assume that B is H-homogeneous.
- Each element of H permutates connected components of B.

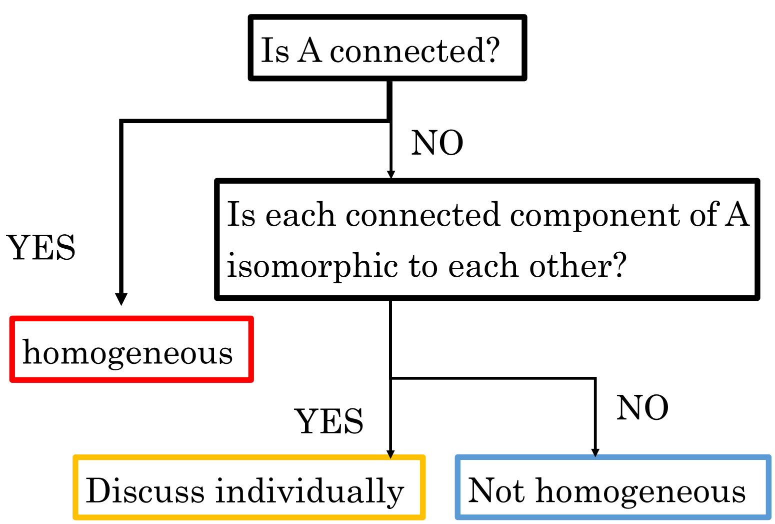
B is homogeneous.

$\Longrightarrow$ For any $i \neq j$, A_i is "isomorphic" to A_j with respect to the structure of the connectedness.

$\Longleftrightarrow$ The graph of A_i is graph isomorphic to the graph of A_j.

The method to decide the homogeneity.

Let A be a maximal antipodal set.



Examples

Ex1 : real oriented Grassmannian

Maximal antipodal sets of real oriented Grassmannians $\tilde{G}_k(\mathrm{R}^n) \cong SO(n)/S(O(k) \times O(n-k))$ are investigated by some mathematicians.

- In $k = 3, 4$, maximal antipodal sets are classified.
- In $k \geq 5$, the classification is not complete.
  However, many maximal antipodal sets are known.

We decide the homogeneity of these maximal antipodal sets.

Examples

$\tilde{G}_3(\mathbb{R}^n)$

n	$3,4$	5	6	$7,8$	$9 \leq n$	
	A(3,3)	A(3,5)	B(3,6)	B(3,7)	A(3,2l+1) $(l = [\frac{n-1}{2}])$	$B(3,7)$
connectedness	◯	◯	◯	◯	◯	◯
homogeneity	◯	◯	◯	◯	◯	◯

$\tilde{G}_4(\mathbb{R}^n)$

n	$4,5$	6	7	$8,9$	10	
	A(4,4)	A(4,6)	B(4,7)	B(4,8)	$A(4,10)$	B(4,8)
connectedness	◯	◯	◯	◯	◯	◯
homogeneity	◯	◯	◯	◯	◯	◯

n	$11 \leq n$			
	A(4,2l) $(l = [\frac{n}{2}])$	$B(4,7) \sqcup \cdots \sqcup B(4,7)$	$B(4,8) \sqcup \cdots \sqcup B(4,8)$	otherwise
connectedness	◯	×	×	×
homogeneity	◯	◯	◯	×

Examples

$\tilde{G}_{4m}(\mathbb{R}^n)$

n	$8m$	$8m+1$	$8m+2$	$8m+3$	$8m+4$	$8m+5$	$8m+6$	$8m+7$
	$E^+_{v_{8m}}$	$E^+_{v_{8m}}$	$E^+_{v_{8m}}$	$E^+_{v_{8m}}$				
connectedness	×	×	×	×				
homogeneity	×	×	×	×				

$\tilde{G}_{4m+1}(\mathbb{R}^n)$

n	$8m$	$8m+1$	$8m+2$	$8m+3$	$8m+4$	$8m+5$	$8m+6$	$8m+7$
			$E_{v_{8m+2}}$	$E_{v_{8m+2}}$	$E_{v_{8m+2}}$	$E^+_{v_{8m+2}}$		
connectedness			◯	◯	◯	×		
homogeneity			◯	◯	◯	×		

$\tilde{G}_{4m+2}(\mathbb{R}^n)$

n	$8m$	$8m+1$	$8m+2$	$8m+3$	$8m+4$	$8m+5$	$8m+6$	$8m+7$
					$E_{v_{8m+4}}$	$E_{v_{8m+4}}$	$E^+_{v_{8m+4}}$	
connectedness					◯	◯	×	
homogeneity					◯	◯	×	

$\tilde{G}_{4m+3}(\mathbb{R}^n)$

n	$8m$	$8m+1$	$8m+2$	$8m+3$	$8m+4$	$8m+5$	$8m+6$	$8m+7$
$k=3$							$E_{v_{8m+6}}$	$E^+_{v_{8m+6}}$
connectedness							◯	×
homogeneity							◯	×

$\tilde{G}_{2k}(\mathbb{R}^{2l})$, $\tilde{G}_{2k}(\mathbb{R}^{2l+1})$, $\tilde{G}_{2k+1}(\mathbb{R}^{2l+1})$, $\tilde{G}_{2k+1}(\mathbb{R}^{2l+2})$

	$\tilde{G}_{2k}(\mathbb{R}^{2l})$, $\tilde{G}_{2k}(\mathbb{R}^{2l+1})(l \geq 3k-1)$	$\tilde{G}_{2k+1}(\mathbb{R}^{2l+1})$, $\tilde{G}_{2k+1}(\mathbb{R}^{2l+2})(k \geq 2)$
	$A(2k, 2l)$	$A(2k+1, 2l+1)$
connectedness	◯	◯
homogeneity	◯	◯

In $2k, 2k+1 \geq 5$ and $n >> k$, these maximal antipodal sets are great.

Pay attention to great antipodal sets already known......

Great antipodal sets are connected.

Ex2 : Symmetric spaces having only one polar.

Definition 4.1

Let $o \in M$. Each connected component of the fixed point set $F(s_o, M)$ of s_o except for $\{o\}$ is called a polar.

- Let M be a symmetric space having only one polar.
- Then, it is known that for any antipodal two points of M there is a shortest closed geodesic through these points.

$\longrightarrow$ Any antipodal two points of M are connected.

Proposition 4.2

If M is a symmetric space having only one polar, then any antipodal set of M is connected.
In particular, any maximal antipodal set of M is homogeneous.

For example, following symmetric spaces have only one polar.

Example 4

$(E_6/F_4), (E_6/F_4)^*, F_4/\mathrm{Spin}(9), G_2/SO(4)$

In particular,

Great antipodal sets are connected.

Future works

- In some symmetric spaces, great antipodal sets are connected.
- Moreover, great antipodal sets of symmetric R spaces are connected.

Problem

In any compact symmetric space, is any great antipodal set connected?

Reference I

[1] B.Y. Chen, T.Nagano, A Riemannian geometric invariant and its applications to a problem of Borel and Serre, Trans. Amer. Math. Soc., **308**(1988), 273-297

[2] H.Tasaki, Antipodal sets in oriented real Grassmann manifolds, International Journal of Mathmatics, **24**(2013), no.8, 135006-1-28

[3] H.Tasaki, Sequences of Maximal Antipodal Sets of Oriented Real Grassmann Manifolds, Real and Complex Submanifolds. Springer Proceedings in Mathematics and Statistics, **106**(2014), 515-524

Reference II

[4] H.Tasaki, Sequences of maximal antipodal sets of oriented real Grassmann manifolds II, Springer Proceedings in Mathematics and Statistics **203**, Y.J. Suh et al. (eds.), "Hermitian-Grassmannian Submanifolds", (2017), 17-26

[5] H.Tasaki, Estimates of antipodal sets in oriented real Grassmann manifolds, "Global Analysis and Differential Geometry on Manifolds,", International Journal of Mathematics **26** no.5 (2015), 1541008-1-1

[6] B.Y. Chen, T.Nagano, Totally geodesic submanifolds of symmetric spaces II, Duke.math.J, **45**(1978), 405-425

[7] M.S.Tanaka, H.Tasaki, Antipodal sets of symmetric R-spaces, Osaka J. Math, **50**(2013), 161-169

Reference III

[8] M.Takeuchi, Two-number of symmetric R-spaces, Nagoya Math.J., **115**(1989), 43-46

[9] P.Frankl, N.Tokushige, Uniform eventown problems, Euro.J.Combi, **51**(2016), 280-286

[10] T.Nagano, The involutions of compact symmetric spaces, Tokyo.J.Math, **11**.No.1(1988), 57-79

Thank you for listening.

Homogeneous quandles arising from symmetric groups

Akihiro Higashitani

This report is based on the joint work [1] with Hirotake Kurihara.

Let X be a set equipping a map s_x for each $x \in X$. We say that the pair (X, s) is a *quandle* if the following three conditions are satisfied:

(Q1) $s_x(x) = x$ for any $x \in X$;

(Q2) for any $x \in X$, the map s_x is bijective;

(Q3) $s_x \circ s_y = s_{s_x(y)} \circ s_x$ for any $x, y \in X$.

(We simply denote it by X.) For two quandles (X, s) and (X', s'), we say that a map $f : X \to X'$ is a *quandle homomorphism* if it satisfies $f \circ s_x = s'_{f(x)} \circ s'_x$ for any $x \in X$. In particular, $s_x : X \to X$ is a quandle homomorphism for each $x \in X$. We call a bijective quandle homomorphism a *quandle automorphism*. We say that two quandles are isomorphic if there is a quandle automorphism between them. A quandle X is said to be *homogeneous* if for any $x, y \in X$, there is a quantle automorphism f such that $y = f(x)$. Homogeneous quandles can be regarded as a kind of discrete version of symmetric spaces.

For a group G, let $\mathrm{Aut}(G)$ denote the automorphism group of G. Given a group G and $\sigma \in \mathrm{Aut}(G)$, we can construct a homogeneous quandle as follows: Let $Q(G, \sigma)$ be G as a set equipping a map $s_g : G \to G$ for each $g \in G$ defined by

$$s_g(h) := g\sigma(g^{-1}h).$$

This is also known as the *generalized Alexander quandle* of G. Note that $Q(G, \sigma)$ is called an Alexander quandle if G is abelian.

What we would like to do is to determine the structure of

$$\mathcal{Q}(G) := \{Q(G, \sigma) : \sigma \in \mathrm{Aut}(G)\}/ \cong$$

for a given group G, where $\cong$ denotes "up to quandle isomorphism".

Let $\mathfrak{S}_n$ denote the symmetric group on $\{1, \ldots, n\}$. We discuss $\mathcal{Q}(\mathfrak{S}_n)$ for small n's. The main reuslt of this report is the following:

Theorem 1. *We have a one-to-one correspondence between $\mathcal{Q}(\mathfrak{S}_n)$ and the conjugacy classes of $\mathrm{Aut}(\mathfrak{S}_n)$ for any $n \in \{3, 4, \ldots, 30\} \setminus \{15\}$. In particular, we have a one-to-one correspondence between $\mathcal{Q}(\mathfrak{S}_n)$ and the conjugacy classes of $\mathfrak{S}_n$ for $n \in \{3, 4, \ldots, 30\} \setminus \{6, 15\}$.*

References

[1] Akihiro Higashitani and Hirotake Kurihara, Homogeneous quandles arising from automorphisms of symmetric groups, arXiv:2005.12057v1.

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Combinatorics and Fourier analysis on compact symmetric spaces

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1 Introduction

Codes and designs on association schemes are important research themes in combinatorics. Delsarte [7, Philips Res. Rep. Suppl. (1973)] gave linear programming bounds for cardinalities of codes and designs on commutative association schemes in terms of eigen-matrices. On spheres, by Delsarte–Goethals–Seidel [8, Geometriae Dedicata (1977)], definitions of codes and designs, and Delsarte's bounds for them were established in terms of spherical Fourier transforms. See Bannai–Bannai [3, European J. Combin. (2009)] for a survey on studies of codes and designs on spheres.

For other compact symmetric spaces, codes, designs and Delsarte's bounds have been studied by many resarchers. For example, by Bannai–Hoggar [4, Proc. Japan Acad. Ser. A Math. Sci. (1985)] on rank one compact symmetric spaces, by Bachoc–Coulangeon–Nebe [2, J. Algebraic Combin. (2002)] and Bachoc–Bannai–Couangeon [1, Discrete Math. (2004)] on real Grassmannians, by Roy [12, J. Algebraic Combin. (2010)] on complex Grassmannians, and by Roy–Scott [13, Des. Codes Cryptogr. (2009)] on Unitary groups. However, general definitions of codes and designs, and general formulations of Delsarte's bounds could not be found in existence literatures.

In this paper, we give a general definition of codes and designs, and a general formulation of Delsare's bounds for them on compact symmetric spaces. As an application of it, we also introduce a result in Kurihara–Okuda [10, J. Algebra (2020)] for design theoretic properties of great antipodal sets on complex Grassnaniann s.

The paper is organized as follows: In Sections 2 and 3, we fix our terminologies for compact symmetric spaces and spherical Fourier transforms on them. Codes and designs on compact symmetric spaces are defined in Section 4. We give a general formulation of Delsarte's linear programming bounds for codes and designs on compact symmetric spaces in Section 5. Finally, as an application of our Delsarte's linear programming bounds, we introduce results in [10] for design theoretic properties of great antipodal sets on complex Grassmannians.

2 Compact symmetric spaces

In this section, we fix our terminologies for compact symmetric spaces as follows: A compact connected C^∞-manifold M equipped with a family of diffeomorphisms $\{s_p : M \to M\}_{p \in M}$ indexed by M satisfying the following four conditions is said to be a compact symmetric space [11]:

*the presenter

1. The map $M \times M \to M,\ (p,q) \mapsto s_p(q)$ is C^∞.

2. For each $p \in M$, the point p is an isolated fixed point of $s_p : M \to M$.

3. $s_p^2 = \mathrm{id}_M$ for each $p \in M$.

4. $s_p \circ s_q = s_q \circ s_{s_q(p)}$ for each $p, q \in M$.

We call $s_p : M \to M$ the point symmetry on M at the point $p \in M$. Note that $(M, \{s_p\}_{p \in M})$ is an involutive quandle (see Joyce [9]).

Any compact symmetric space can be obtained as a homogeneous space of a symmetric pair of compact Lie groups as follows: Let G be a compact connected Lie group and $\sigma : G \to G$ is an involutive (i.e. $\sigma^2 = \mathrm{id}_G$) Lie group automorphism on G. We take an open and closed subgroup K of $G^\sigma := \{g \in G \mid \sigma(g) = g\}$. Then the coset manifold $M = G/K$ can be considered as a compact symmetric space by defining the point symmetries $s_{gK} : G/K \to G/K$ for each $gK \in G/K$ by

$$s_{gK} : G/K \to G/K,\ hK \mapsto g\sigma(g^{-1}h)K.$$

Conversely, let $(M, \{s_x\}_{x \in M})$ be a compact symmetric space. Then

$$\mathrm{Aut}(M) := \{g : M \to M \mid g \text{ is a diffeomorphism preserving point symmetries}\}$$

can be considered as a compact Lie group. We denote by G the identity component of the Lie group $\mathrm{Aut}(M)$, and then the G-action on M is known to be transitive. Let us fix a point $x_0 \in M$, and define

$$\sigma : G \to G,\ g \mapsto s_{x_0} \circ g \circ s_{x_0}.$$

Then $\sigma : G \to G$ is well-defined and involutive Lie group automorphism on G. Let us denote by the isotropy group

$$K := G^{x_0} = \{g \in G \mid gx_0 = x_0\}.$$

Then M can be considered as a coset manifolds G/K. It is known that K is an open and closed subgroup of G^σ, i.e. (G, K) is a compact symmetric pair with respect to σ.

3 Spherical Fourier transforms on compact symmetric spaces

Let M be a compact symmetric space and fix an open and closed subgroup G of the compact Lie group $\mathrm{Aut}(M)$. Then the G-action on M is transitive. In this section, we define spherical Fourier transform on M with respect to the transitive G-action.

3.1 Invariant binary relations

Throughout this paper, we define the subgroup $\operatorname{diag} G$ of the direct product group $G \times G$ by

$$\operatorname{diag} G := \{(g, g) \mid g \in G\} \subset G \times G.$$

Then $\operatorname{diag} G$ acts on $M \times M$ in the natural sense. We denote by $\mathcal{I} := \mathcal{I}_M$ the quotient space of $M \times M$ for the $(\operatorname{diag} G)$-action. Let us also put

$$R : M \times M \to \mathcal{I}$$

the quotient map. With respect to the quotient topology, the space $\mathcal{I}$ is compact Hausdorff topological space (but not necessary to be a manifold), and the map R is open and surjective. The $(\operatorname{diag} G)$-orbit $\{(p, p) \mid p \in M\}$ is denoted by $\alpha_0 \in \mathcal{I}$.

We say that a binary relation $\mathcal{A} \subset M \times M$ on M is G-invariant if $\mathcal{A}$ is stable by the action of $\operatorname{diag} G$ on $M \times M$. One can easily observe that each subset of $\mathcal{I}$ corresponds to each G-invariant binary relations on M in the following sense:

Observation 1. For each G-invariant binary relation $\mathcal{A}$ on M, the image $R(\mathcal{A})$ of $\mathcal{A}$ by R is a subset of $\mathcal{I}$. Conversely, for each subset S of $\mathcal{I}$, the inverse image $R^{-1}(S)$ of S by R is a G-invariant binary relation on M. Furthermore, the correspondences above gives a bijection between the family of all subsets in $\mathcal{I}$ and the family of all G-invariant binary relations on M. Note that a G-invariant binary relation on M is reflexive if and only if $\alpha_0 := \{(p, p) \mid p \in M\}$ belongs to the corresponding subset of $\mathcal{I}$.

3.2 Harmonic analysis

Throughout this paper, we denote by μ the probability G-invariant Radon measure on M. It is well-known that such μ exists uniquely. We write $C(M)$ for the vector space of all $\mathbb{C}$-valued continuous functions on M. The L^2-innerproduct with respect to μ on $C(M)$ is denoted by $(\cdot, \cdot)_\mu$, that is, for each $f_1, f_2 \in C(M)$,

$$(f_1, f_2)_\mu := \int_M (f_1 \cdot \overline{f_2})\, d\mu \in \mathbb{C}.$$

For each $g \in G$ and $f \in C(M)$, we put $g \cdot f \in C(M)$ by

$$g \cdot f : M \to \mathbb{C},\ p \mapsto f(g^{-1} \cdot p).$$

This defines a linear G-representation on $C(M)$ which preserves the L^2-innerproduct. The completion of the G-representation on $C(M)$ with respect to the L^2-innerproduct $(\cdot, \cdot)_\mu$ is written as $L^2(M)$. Then $L^2(M)$ can be considered as a unitary G-representation, and called the regular G-representation on M.

Throughout this paper, we define $\mathcal{J} = \mathcal{J}_M$ by the family of all irreducible unitary G-subrepresentations of $L^2(M)$. We consider $\mathcal{J}$ as a topological space equipped with the discrete topology. We also put

$$V_0 := \{\text{ constants on } M\} \subset C(M).$$

Then V_0 is one-dimensional irreducible G-subrepresentation and hence $V_0 \in \mathcal{J}$.

For the irreducible decomposition of the regular G-representation, the theorem below is well-known as a kind of the Peter–Weyl theorem (see [14]).

Theorem 1 (A Peter–Weyl type theorem for compact symmetric spaces)**.** *In the setting above, the following holds:*

1. *For any $V \in \mathcal{J}$, V is finite-dimensional and $V \subset C(M)$.*
2. *Let $V, W \in \mathcal{J}$ with $V \neq W$. Then V and W are not equivalent as unitary G-representations and $V \perp W$ with respect to the L^2-innerproduct. In particular, the regular G-representation $L^2(M)$ on M is multiplicity-free.*
3. *The orthogonal direct sum*
$$\bigoplus_{V \in \mathcal{J}} V$$
is dense in $C(M)$ with respect to the supremum norm.

3.3 Reproducing kernels and spherical functions

Recall that $\mathcal{J} = \mathcal{J}_M$ denotes the family of all irreducible unitary G-subrepresentations of $L^2(M)$ (see Section 3.2). Fix $V \in \mathcal{J}$. In this subsection, we recall definition and some properties of reproducing kernel and spherical function of V.

Since V is finite-dimensional, V itself is a Hilbert space with respect to the L^2-innerproduct $(\cdot, \cdot)_\mu$ defined in the previous subsection. Let us define the reproducing kernel of the finite-dimensional Hilbert space V as follows: For each $p \in M$,

$$V \to \mathbb{C}, \ f \mapsto f(p)$$

is a linear functional on V. In particular, by the Riesz representation theorem, there uniquely exists

$$\delta_V^p \in V$$

satisfying that

$$(f, \delta_V^p) = f(p).$$

Let us define the reproducing kernel $\mathcal{K}^V$ of V by

$$\mathcal{K}_V : M \times M \to \mathbb{C}, \ (p, q) \mapsto (\delta_V^p, \delta_V^q)_\mu.$$

Then the following proposition holds:

Proposition 1. *1. We put $N := \dim V$. Let $\{e_1, \ldots, e_N\}$ be any orthonormal basis of V. Then for each $p \in M$,*

$$\delta_V^p = \sum_{i=1}^{N} \overline{e_i(p)} e_i.$$

Furthermore, for each $(p, q) \in M \times M$,

$$\mathcal{K}_V(p, q) = \sum_{i=1}^{N} \overline{e_i(p)} e_i(q).$$

2. *The reproducing kernel $\mathcal{K}_V$ is a continuous function on $M \times M$, and diagonal G-invariant, that is, the equality*

$$\mathcal{K}_V(g \cdot p, g \cdot q) = \mathcal{K}_V(p, q)$$

holds for any $g \in G$ and any $p, q \in M$.

Recall that $\mathcal{I} = \mathcal{I}_M$ is defined as a quotient space of the diagonal G-action on $M \times M$, and $R : M \times M \to \mathcal{I}$ denotes the quotient map (see Section 3.1). Let us denote by $C(\mathcal{I})$ the vector space of all continuous $\mathbb{C}$-valued functions on $\mathcal{I}$. Since $\mathcal{K}_V : M \times M \to \mathbb{C}$ is diagonal G-invariant continuous function, there uniquely exists $Q_V \in C(\mathcal{I})$ satisfying that

$$Q_V \circ R = \mathcal{K}_V.$$

In this paper, we call Q_V the spherical function for $V \in \mathcal{J}$.

3.4 Spherical Fourier transforms and its dual

In this subsection, we define one of the variation of the spherical Fourier transforms and its dual on M in which we need.

Let us define the vector space $C_c(\mathcal{J})$ by

$$C_c(\mathcal{J}) := \{\eta : \mathcal{J} \to \mathbb{C} \mid \eta(V) = 0 \text{ without finitely many } V \in \mathcal{J}\}.$$

For each $\eta \in C_c(\mathcal{J})$, we define $\widehat{\eta} \in C(\mathcal{I})$ by

$$\widehat{\eta} := \sum_{V \in \mathcal{J}} \eta(V) \cdot Q_V.$$

We call $\widehat{\eta} \in C(\mathcal{I})$ the spherical Fourier transform of $\eta \in C_c(\mathcal{J})$. Let us also put

$$\mathcal{SF} : C_c(\mathcal{J}) \to C(\mathcal{I}),\ \eta \mapsto \widehat{\eta}.$$

Then $\mathcal{SF} : C_c(\mathcal{J}) \to C(\mathcal{I})$ is a linear map.

Remark 1. Note that the map $\mathcal{SF} : C_c(\mathcal{J}) \to C(\mathcal{I})$ is known to be injective but not surjective. It is well-known as a kind of a Parseval's theorem that for suitable measures on $\mathcal{I}$ and $\mathcal{J}$, by taking the completion, the linear map $\mathcal{SF}$ induces an isometry between $L^2(\mathcal{J})$ and $L^2(\mathcal{I})$ (see [14]).

Let us denote by $C^\vee(\mathcal{I})$ the vector space of all bounded linear functionals on $C(\mathcal{I})$ with respect to the supremum norm, and by $C_c^\vee(\mathcal{J})$ the vector space of all linear functionals on $C_c(\mathcal{J})$. For each $a \in C^\vee(\mathcal{I})$, we define the linear functional $\widehat{a}$ on $C_c(\mathcal{J})$ by

$$\widehat{a} : C_c(\mathcal{J}) \to \mathbb{C},\ \eta \mapsto a(\widehat{\eta}).$$

We also call $\widehat{a} \in C_c^\vee(\mathcal{J})$ the spherical Fourier transform of $a \in C^\vee(\mathcal{I})$. Let us put

$$\mathcal{SF}^\vee : C^\vee(\mathcal{I}) \to C_c^\vee(\mathcal{J}),\ a \mapsto \widehat{a}.$$

Then $\mathcal{SF}^\vee : C^\vee(\mathcal{I}) \to C_c^\vee(\mathcal{J})$ is a linear map.

Remark 2. It can be proved that the linear map

$$\mathcal{SF}^\vee : C^\vee(\mathcal{I}) \to C^\vee_c(\mathcal{J}),\ a \mapsto \widehat{a}$$

is injective but not necessary to be surjective. We tried but not succeeded to determine the image of the map $\mathcal{SF}^\vee$.

4 Codes and designs on compact symmetric spaces

Let us consider the setting in Section 3. In this section, for each non-empty finte subset X of M, we give definitions of the $\mathcal{I}$-distribution a^X of X and the $\mathcal{J}$-distribution b^X of X. Furthermore, we also give definitions of codes and designs on M, and characterize them in terms of distributions a^X and b^X of X.

4.1 $\mathcal{I}$-distributions

As in Section 3.1, we denote by $\mathcal{I}$ the quotient space of $M \times M$ with respect to the natural $(\operatorname{diag} G)$-action on $M \times M$, and by $R : M \times M \to \mathcal{I}$ the quotient map. The $(\operatorname{diag} G)$-orbit $\{(p,p) \mid p \in M\}$ is denoted by $\alpha_0 \in \mathcal{I}$.

Recall that $C(\mathcal{I})$ denotes the vector space of all $\mathbb{C}$-valued continuous functions on the compact Hausdorff space $\mathcal{I}$, and $C^\vee(\mathcal{I})$ the vector space of all bounded (or equivalently continuous) linear functionals on $C(\mathcal{I})$ with respect to the supremum norm. Note that for each $\alpha \in \mathcal{I}$,

$$\delta^\alpha_\mathcal{I} : C(\mathcal{I}) \to \mathbb{C},\ \xi \mapsto \xi(\alpha)$$

defines an element of $C^\vee(\mathcal{I})$, and $\{\delta^\alpha_\mathcal{I} \mid \alpha \in \mathcal{I}\}$ is linearly independent in $C^\vee(\mathcal{I})$.

Let us define the $\mathcal{I}$-distribution a^X of each non-empty finite subset X of M as an element of $C^\vee(\mathcal{I})$ below:

Definition 1. For each non-empty finite subset X of M, we define the $\mathcal{I}$-distribution

$$a^X := \sum_{\alpha \in \mathcal{I}} a^X_\alpha \delta^\alpha_\mathcal{I} \in C^\vee(\mathcal{I})$$

of X by putting

$$a^X_\alpha := \frac{\#\{(x,y) \in X \times X \mid R(x,y) = \alpha\}}{(\#X)^2}$$

for each $\alpha \in \mathcal{I}$.

Note that for each non-empty finite subset X of M, $a^X_\alpha = 0$ without finitely many $\alpha \in \mathcal{I}$, and hence $a^X \in C^\vee(\mathcal{I})$ is well-defined.

One can easily observe that the concept of $\mathcal{I}$-distribution is G-invariant, that is, $a^X = a^{gX}$ in $C^\vee(\mathcal{I})$ for any non-empty subset X of M and any $g \in G$.

4.2 Codes

Let $\mathcal{A} \subset M \times M$ be a G-invariant reflexive binary relation on M. Let us give definitions of $\mathcal{A}$-codes on M as follows:

Definition 2 ($\mathcal{A}$-codes)**.** Let X be a non-empty finite subset of M. We say that X is an $\mathcal{A}$-code on M if

$$(x, y) \in \mathcal{A} \text{ for any } x, y \in X.$$

One can easily observe that the concept of $\mathcal{A}$-codes on M is G-invariant, that is, X is a $\mathcal{A}$-code if and only if gX is an $\mathcal{A}$-code on M for any non-empty finite subset X of M and $g \in G$.

The following easy observation gives a characterization of $\mathcal{A}$-codes in terms of $\mathcal{I}$-distribution a^X of X defined in the previous subsection:

Observation 2. Let X be a non-empty finite subset of M. Then X is an $\mathcal{A}$-code if and only if

$$a^X_\alpha = 0 \text{ for any } \alpha \notin R(\mathcal{A}).$$

4.3 $\mathcal{J}$-distributions

As in Section 3.2, we denote by $\mathcal{J}$ the discrete space of all irreducible unitary G-subrepresentations in $L^2(M)$. The purpose of this subsection is to give a definition of the $\mathcal{J}$-distribution b^X of each non-empty finite subset X of M. Recall that we denote by

$$C_c(\mathcal{J}) := \{\eta : \mathcal{J} \to \mathbb{C} \mid \eta(V) = 0 \text{ without finitely many } V \in \mathcal{J}\},$$

and by $C_c^\vee(\mathcal{J})$ the vector space of all linear functionals on $C_c(\mathcal{J})$. Note that for each $V \in \mathcal{J}$,

$$\delta^V_{\mathcal{J}} : C_c(\mathcal{J}) \to \mathbb{C}, \ \eta \mapsto \eta(V)$$

defines an element of $C_c^\vee(\mathcal{J})$. Furthermore, for any function $\phi : \mathcal{J} \to \mathbb{C}$, the map

$$\sum_{V \in \mathcal{J}} \phi(V)\delta^V_{\mathcal{J}} : C_c(\mathcal{J}) \to \mathbb{C}, \ \eta \mapsto \sum_{V \in \mathcal{J}} \phi(V) \cdot \eta(V)$$

is well-defined and

$$\sum_{V \in \mathcal{J}} \phi(V)\delta^V_{\mathcal{J}} \in C_c^\vee(\mathcal{J}).$$

Note that for functions $\phi_1, \phi_2 : \mathcal{J} \to \mathbb{C}$, the equality

$$\sum_{V \in \mathcal{J}} \phi_1(V)\delta^V_{\mathcal{J}} = \sum_{V \in \mathcal{J}} \phi_2(V)\delta^V_{\mathcal{J}}$$

imples $\phi_1 = \phi_2$.

In order to define $\mathcal{J}$-distributions, we introduce the following notations: For each $V \in \mathcal{J}$ and each non-empty finite subset X of M, we define the linear functional avg^V_X on V by

$$\mathrm{avg}^V_X : V \to \mathbb{C}, \ f \mapsto \frac{1}{\# X} \sum_{x \in X} f(x).$$

We denote by $\| \operatorname{avg}_X^V \|_{\mathrm{op}} \in \mathbb{R}_{\geq 0}$ the operator norm of the linear functional avg_X^V on V with respect to the L^2-innerproduct, that is,

$$\| \operatorname{avg}_X^V \|_{\mathrm{op}} := \sup_{f \in V \setminus \{0\}} \frac{|\operatorname{avg}_X^V(f)|}{\sqrt{(f,f)_\mu}}.$$

Let us define the $\mathcal{J}$-distribution b^X of each non-empty finite subset X of M as an element of $C_c^\vee(\mathcal{J})$ below:

Definition 3. For each non-empty finite subset X of M, we define the $\mathcal{J}$-distribution

$$b^X := \sum_{V \in \mathcal{J}} b_V^X \cdot \delta_{\mathcal{J}}^V \in C_c^\vee(\mathcal{J})$$

of X by putting

$$b_V^X := \| \operatorname{avg}_X^V \|_{\mathrm{op}}^2$$

for each $V \in \mathcal{J}$.

One can easily observe that the concept of $\mathcal{J}$-distribution is G-invariant, that is, $b^X = b^{gX}$ in $C_c^\vee(\mathcal{J})$ for any non-empty subset X of M and any $g \in G$.

4.4 Designs

Let us fix a finite-dimensional G-stable subspace $\mathcal{H}$ of $C(M)$. In this subsection, we give the definition of $\mathcal{H}$-design on M and their characerization in terms of $\mathcal{J}$-distributions.

Definition 4 ($\mathcal{H}$-designs)**.** We say that a non-empty finite subset X of M is an $\mathcal{H}$-design if the equality below holds:

$$\int_{x \in M} f(x) d\mu(x) = \frac{1}{\# X} \sum_{x \in X} f(x) \quad \text{for any } f \in \mathcal{H}.$$

One can easily observe that the concept of $\mathcal{H}$-design is G-invariant, that is, X is an $\mathcal{H}$-design if and only if gX is an $\mathcal{H}$-design on M for any non-empty finite subset X of M and any $g \in G$.

Recall that $V_0 := \{$ constants on $M\} \in \mathcal{J}$. We also give a characterization of $\mathcal{H}$-designs on M in terms of $\mathcal{J}$-enumerator b^X of X as follows:

Theorem 2. *Let X be a non-empty finite subset of M and $\mathcal{H}$ a G-stable finite-dimensional subspace of $C(M)$. Then X is an $\mathcal{H}$-design if and only if*

$$b_V^X = 0 \textit{ for any } V \in \mathcal{J} \setminus \{V_0\} \textit{ with } V \subset \mathcal{H}.$$

5 Delsarte theory on compact symmetric spaces

Let us consider the setting in Section 3 and fix a non-empty finite subset X of M. In this section, as in Theorem 3, we give a relationship between the $\mathcal{I}$-distribution a^X and the $\mathcal{J}$-distribution b^X in terms spherical Fourier transforms, and furthermore, as an application of such the relationship, Delsarte's bounds for the cardinalities of codes and designs are given as in Theorem 4.

First, we give a relationship between a^X and b^X as follows:

Theorem 3. *The equality below holds:*

$$\widehat{a^X} = b^X,$$

where $\widehat{a^X} \in C_c^\vee(\mathcal{J})$ denotes the spherical Fourier transform of $a^X \in C^\vee(\mathcal{I})$ defined in Section 3.4.

Fix a G-invariant reflexive binary relation $\mathcal{A}$ on M and a finite-dimensional G-stable subspace $\mathcal{H}$ in $C(M)$. In the rest of this section, we study estimations of cardinality of a non-empty finite subset X satisfying that X is an $\mathcal{A}$-code and an $\mathcal{H}$-design on M simultaneously.

Let us define a subset $\mathrm{Cone}(\mathcal{A},\mathcal{H})$ in $C^\vee(\mathcal{I})$ by

$$\mathrm{Cone}(\mathcal{A},\mathcal{H}) := \left\{ a = \sum_{\alpha\in\mathcal{I}} a_\alpha \delta_{\mathcal{I}}^\alpha \in C^\vee(\mathcal{I}) \;\middle|\; a \in C^\vee(\mathcal{I}) \text{ satisfies the following five conditions} \right\}.$$

Condition (1): $a_\alpha \in \mathbb{R}_{\geq 0}$ for any $\alpha \in \mathcal{I}$, and $a_\alpha = 0$ without finitely many $\alpha \in \mathcal{I}$.

Condition (2): $a_{\alpha_0} > 0$.

Condition (3): $a^\alpha = 0$ for any $\alpha \notin R(\mathcal{A})$.

Condition (4): $(\widehat{a})_V \in \mathbb{R}_{\geq 0}$ for any $V \in \mathcal{J}$.

Condition (5): For any $V \in \mathcal{J} \setminus \{V_0\}$ with $V \subset \mathcal{H}$,

$$(\widehat{a})_V = 0.$$

Then one can easily see that $\mathrm{Cone}(\mathcal{A},\mathcal{H})$ is a convex cone in the vector space $C^\vee(\mathcal{I})$, that is, $\mathrm{Cone}(\mathcal{A},\mathcal{H})$ is closed under the addition and positive scalar multiplications in $C^\vee(\mathcal{I})$.

For each $a = \sum_{\alpha\in\mathcal{I}} a_\alpha \delta_\alpha \in \mathrm{Cone}(\mathcal{A},\mathcal{H})$, we define

$$\Xi(a) := \frac{\sum_{\alpha\in\mathcal{I}} a^\alpha}{a_{i_0}} \in \mathbb{R}_{>0},$$

and put

$$\Xi(\mathcal{A},\mathcal{H}) := \{\Xi(a) \mid a \in \mathrm{Cone}(\mathcal{A},\mathcal{H})\} \subset \mathbb{R}_{>0}.$$

The subset $\Xi(\mathcal{A},\mathcal{H})$ of $\mathbb{R}_{>0}$ gives an estimate of cardinalities of $\mathcal{A}$-codes and $\mathcal{H}$-designs as below:

Theorem 4 (Delsarte's bounds for codes and designs)**.** *Let X be a non-empty finite subset of M which is an $\mathcal{A}$-code and an $\mathcal{H}$-design on M simultenuously. Then the $\mathcal{I}$-enumerator a^X of X is an element of* $\mathrm{Cone}(\mathcal{A},\mathcal{H})$*, and*

$$\#X \in \Xi(\mathcal{A},\mathcal{H}).$$

Theorem 4 follows from Theorem 3.

Remark 3. If $\mathcal{A} = M \times M$, then any non-empty finite set in M is an $\mathcal{A}$-code. Therefore, by Theorem 4,

$$\#X \in \Xi(M \times M, \mathcal{H}).$$

holds for any $\mathcal{H}$-design X on M. Simalarly, if $\mathcal{H} = 0$, then any non-empty finite set in M is an $\mathcal{H}$-design, and hence

$$\#X \in \Xi(\mathcal{A}, 0)$$

holds for any $\mathcal{A}$-codes on M.

Remark 4. Studying the supremum and the infimum of $\Xi(\mathcal{A},\mathcal{H}) \subset \mathbb{R}_{>0}$ is considered as a linear programming problem (with infinitely constraints). Therefore, one can evaluate them by studying the dual problem. We omit the details here.

6 Applications for antipodal sets on compact symmetric spaces

Let us consider the setting in Section 3. In this section, we see that antipodal sets on compact symmetric space M can be considered as codes on M. Furthermore, as an application of Delsarte's bounds in Section 5, we introduce a result in Kurihara–Okuda [10] for design theory on great antipodal sets on complex Grassmannians.

6.1 Antipodal sets as codes

Let us recall the definition of antipodal sets on compact symmetric space M below:

Definition 5 (Antipodal sets (Chen–Nagano [6]))**.** A subset X of M is called *antipodal* if $s_x(y) = y$ for any $x, y \in X$.

It is well-known that any antipodal set should be of finite, and furthermore, there exists $\#_2 M \in \mathbb{Z}_{\geq 0}$ such that

$$\#_2 M = \max\{\#X \mid X \text{ is an antipodal subset of } M\}.$$

Such the number $\#_2 M$ is called the two-number of the compact symmetric space M. An antipodal set X on M is said to be *great* if $\#X = \#_2 M$. See Chen [5] for a survey on studies of two-numbers and great antipodal sets on compact symmetric spaces.

Let us put

$$\mathcal{A}_{\text{antipodal}} := \{(p,q) \in M \times M \mid s_p(q) = q\} \subset M \times M$$

Then $\mathcal{A}_{\text{antipodal}}$ is a G-invariant (furthermore, $\mathrm{Aut}(M)$-invariant) reflexive and symmetric binary relation on M. For a non-empty finite set X of M, one can easily observe that X is antipodal if and only if X is an $\mathcal{A}_{\text{antipodal}}$-code on M.

6.2 A design theoritic characterization of great antipodal sets on complex Grassmannian manifolds

In this subsection, let us fix $n, k \in \mathbb{Z}_{\geq 1}$ with $k \leq n/2$, and denotes

$$M := \mathrm{Gr}_k(\mathbb{C}^n) := \{k\text{-dimensional linear subspaces in } \mathbb{C}^n\}$$

the rank k complex Grassmannian for $\mathbb{C}^n$. For each $p \in \mathrm{Gr}_k(\mathbb{C}^n)$, take a orthogonal complement $p^\perp$ in $\mathbb{C}^n$, define the linear involutive isomorphism

$$r_p : \mathbb{C}^n \to \mathbb{C}^n,\ v + w \mapsto v - w$$

for $v \in p$ and $w \in p^\perp$, and put

$$s_p : \mathrm{Gr}_k(\mathbb{C}^n) \to \mathrm{Gr}_k(\mathbb{C}^n),\ q \mapsto r_p(q).$$

Then $M = \mathrm{Gr}_k(\mathbb{C}^n)$ can be considered as a compact symmetric space with respect to the point symmetries $\{s_p\}_{p \in \mathrm{Gr}_k(\mathbb{C}^n)}$. We also consider $G := U(n)$ the unitary group $U(n)$ of size n, then (M, G) is in the setting in Section 3.

In the setting above, in terms of principal angles on complex Grassmannians, one can identified $\mathcal{I}$ with

$$\mathrm{Range}(\mathrm{Gr}_k(\mathbb{C}^n)) := \{(\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k \mid 1 \geq \alpha_1 \geq \cdots \geq \alpha_k \geq 0\}$$

and then

$$R(\mathcal{A}_{\mathrm{antipodal}}) \simeq \{(\underbrace{1, \dots, 1}_{l}, \underbrace{0, \dots, 0}_{k-l}) \mid l = 0, \dots, k\}$$

(see [10, Section 4] for the details).

For the great antipodal sets (that is, $\mathcal{A}_{\mathrm{antipodal}}$-codes with the largest cardinality) on $\mathrm{Gr}_k(\mathbb{C}^n)$, the following theorem is well-known:

Theorem 5 ((see Chen–Nagano [6])). *The two-number of* $\mathrm{Gr}_k(\mathbb{C}^n)$ *is given by*

$$\#_2\mathrm{Gr}_k(\mathbb{C}^n) = \binom{n}{k}.$$

Furthermore, for any great antipodal set X *of* $\mathrm{Gr}_k(\mathbb{C}^n)$*, there exists an orthonormal basis* $\{e_1, \dots, e_n\}$ *of* $\mathbb{C}^n$ *such that*

$$X = \left\{ \mathbb{C}\text{-span } \{e_{i_1}, \dots, e_{i_k}\} \;\middle|\; \{i_1, \dots, i_k\} \in \binom{\{1, \dots, n\}}{k} \right\}$$

where $\binom{\{1,\dots,n\}}{k}$ *denotes the k-points subset of* $\{1, \dots, n\}$*.In particular, great antipodal sets on* $\mathrm{Gr}_k(\mathbb{C}^n)$ *are unique up to* $U(n)$*-congruent.*

For $\mathcal{J}$ of $\mathrm{Gr}_k(\mathbb{C}^n)$, in terms of "the left half part" of the highest weights, one can identified $\mathcal{J}$ with

$$\mathcal{P}_k := \{(\nu_1, \dots, \nu_k) \in \mathbb{Z}^k \mid \nu_1 \geq \cdots \geq \nu_k \geq 0\}$$

(see [10, Section 3.1] for the details). For each $\nu \in \mathcal{P}_k$, we denote by $V_\nu \in \mathcal{J}$ the corresponding irreducible representation in $L^2(\mathrm{Gr}_k(\mathbb{C}^n))$.

As an application of Theorem 4, we obtain the following characterization of great antipodal sets on $\mathrm{Gr}_k(\mathbb{C}^n)$:

Theorem 6 (Kurihara–Okuda [10]). *Let us define the subsets $\mathcal{E}$ and $\mathcal{F}$ of $\mathcal{P}_k$ by*

$$\mathcal{E} := \{(\underbrace{1,1,\dots,1}_{l},\underbrace{0,0,\dots,0}_{k-l}) \mid l = 0,1,\dots,k\},$$
$$\mathcal{F} := \{(2,\underbrace{1,1,\dots,1}_{l-1},\underbrace{0,0,\dots,0}_{k-l}) \mid l = 2,\dots,k\}.$$

We put the finite-dimensional $U(n)$-stable subspaces $\mathcal{H}_\mathcal{E}$ and $\mathcal{H}_\mathcal{F}$ in $C(\mathrm{Gr}_k(\mathbb{C}^n))$ by

$$\mathcal{H}_\mathcal{E} := \bigoplus_{\nu\in\mathcal{E}} V_\nu,$$
$$\mathcal{H}_\mathcal{F} := \bigoplus_{\nu\in\mathcal{F}} V_\nu.$$

Then the following holds:

1. *Any great antipodal set on $\mathrm{Gr}_k(\mathbb{C}^n)$ is an $\mathcal{H}_\mathcal{E}$-design on $\mathrm{Gr}_k(\mathbb{C}^n)$ with the smallest cardinality.*

2. *For a non-empty finite subset X of $\mathrm{Gr}_k(\mathbb{C}^n)$, the following two conditions are equivalent:*

   (a) X is a great antipodal set on $\mathrm{Gr}_k(\mathbb{C}^n)$, i.e. an $\mathcal{A}_{\mathrm{antipodal}}$-code on $\mathrm{Gr}_k(\mathbb{C}^n)$ with the largest cardinality.

   (b) X is an $\mathcal{H}_\mathcal{E} \oplus \mathcal{H}_\mathcal{F}$-design on $\mathrm{Gr}_k(\mathbb{C}^n)$ with the smallest cardinality.

Remark 5. There exists an example of an $\mathcal{H}_\mathcal{E}$-design with the smallest cardinality but not a great antipodal set on $\mathrm{Gr}_k(\mathbb{C}^n)$ (see [10, Appendix B]).

References

[1] C. Bachoc, E. Bannai and R. Coulangeon, *Codes and designs in Grassmannian spaces*, Discrete Math., **277**:15–28, (2004).

[2] C. Bachoc, R. Coulangeon and G. Nebe, *Designs in Grassmannian spaces and lattices*, J. Algebraic Combin., **16**:5–19, (2002).

[3] Ei. Bannai and Et Bannai, *A survey on spherical designs and algebraic combinatorics on spheres*, European J. Combin., **30**:1392–1425, (2009).

[4] E. Bannai and S.G. Hoggar, *On tight t-designs in compact symmetric spaces of rank one*, Proc. Japan Acad. Ser. A Math. Sci., **61**:78–82, (1985).

[5] B.-Y. Chen, *Two-numbers and their applications—a survey.* Bull. Belg. Math. Soc. Simon Stevin, **25**:565–596, (2018).

[6] B.-Y. Chen and T. Nagano, *A Riemannian Geometric Invariant and its Applications to a Problem of Borel and Serre.* Trans. Amer. Math. Soc., **308**:273–297, (1988).

[7] P. Delsarte, *An algebraic approach to the association schemes of coding theory*, Philips Res. Rep. Suppl. 97 pages, (1973).

[8] P. Delsarte, J.M. Goethals and J.J. Seidel, *Spherical codes and designs*, Geometriae Dedicata, **6**:363–388, (1977).

[9] D. Joyce, *A classifying invariant of knots, the knot quandle.* J. Pure Appl. Algebra, **23**:37–65, (1982).

[10] H. Kurihara and T. Okuda, *Great antipodal sets on complex Grassmannian manifolds as designs with the smallest cardinalities*, J. Algebra **559**:432–466, (2020)

[11] O. Loos, *Symmetric spaces I, II.* W. A. Benjamin, Inc., New York-Amsterdam (1969)

[12] A. Roy, *Bounds for codes and designs in complex subspaces*, J. Algebraic Combin., **31**:1–32, (2010).

[13] A. Roy and A.J. Scott, *Unitary designs and codes*, Des. Codes Cryptogr., **53**:13–31, (2009).

[14] M. Takeuchi, *Modern spherical functions*, American Mathematical Society, Providence, RI., Translations of Mathematical Monographs **135** (1994). Translated from the 1975 Japanese original by Toshinobu Nagura.

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Morse functions and maximal antipodal sets of $G_2/SO(4)$

Yuuki Sasaki

1 Introduction

Let M be a compact Riemannian symmetric space and denote the geodesic symmetry at $x \in M$ by s_x. In this paper, we assume that M is connected. If $s_x(y) = y$ for two points $x, y \in M$, we say that x, y are antipodal. A subset S of M is an antipodal set, if any two points of S are antipodal. The 2-number $\#_2 M$ of M is the maximum of the cardinalities of antipodal sets of M. We call an antipodal set S in M great if $\#S = \#_2 M$. An antipodal set S is called maximal if there are no anitipodal sets including S properly. These notions were introduced by Chen-Nagano [1]. In general, any antipodal set of any Riemannian symmetric space of noncompact type is a one-point set, so we consider only compact symmetric spaces in this paper.

It is known that there is some relation between great antipodal sets and the topology of compact symmetric spaces. For example, the following theorem is known.

Theorem 1 ([4])**.** *Let M be a symmetric R-space. Then,*

$$\#_2 M = \sum_i \dim_{\mathbb{Z}_2} H_i(M; \mathbb{Z}_2).$$

As the background of this theorem, there are $\mathbb{Z}_2$-perfect Morse functions of symmetric R spaces whose set of all critical points is a great antipodal set In non symmetric R spaces, it is known that there are some symmetric spaces which have $\mathbb{Z}_2$-perfect Morse functions whose set of all critical points is a great antipodal set. The special unitary group $SU(n)$ is one of the examples [2]. Moreover, the exceptional compact Lie group G_2 is one of the examples.

In $G_2/SO(4)$, it is known that $\#_2(G_2/SO(4)) = \dim_{\mathbb{Z}_2} H_*(G_2/SO(4); \mathbb{Z}_2)$. Hence, it is expected that there is a $\mathbb{Z}_2$-perfect Morse function of $G_2/SO(4)$ whose set of all critical points is a great antipodal set. In this paper, we construct Morse functions of $G_2/SO(4)$ satisfying such properties.

2 Preparation

For any $x = {}^t(x_0, \cdots, x_7)$ and $y = {}^t(y_0, \cdots, y_7) \in \mathbb{R}^7$ satisfying $x \perp y$, we set $xy \in \mathbb{R}^7$ as follows.

$$xy = \begin{pmatrix} (x_0y_1 + x_1y_0) + (x_2y_3 - x_3y_2) + (x_4y_5 - x_5y_4) + (x_6y_7 - x_7y_6) \\ (x_0y_2 + x_2y_0) + (x_3y_1 - x_1y_3) + (x_6y_4 - x_4y_6) + (x_5y_7 - x_7y_5) \\ (x_0y_3 + x_3y_0) + (x_1y_2 - x_2y_1) + (x_4y_7 - x_7y_4) + (x_5y_6 - x_6y_5) \\ (x_0y_4 + x_4y_0) + (x_5y_1 - x_1y_5) + (x_2y_6 - x_6y_2) + (x_7y_3 - x_3y_7) \\ (x_0y_5 + x_5y_0) + (x_1y_4 - x_4y_1) + (x_7y_2 - x_2y_7) + (x_6y_3 - x_3y_6) \\ (x_0y_6 + x_6y_0) + (x_7y_1 - x_1y_7) + (x_4y_2 - x_2y_4) + (x_3y_5 - x_5y_3) \\ (x_0y_7 + x_7y_0) + (x_1y_6 - x_6y_1) + (x_2y_5 - x_5y_2) + (x_3y_4 - x_4y_3) \end{pmatrix}.$$

We denote the set of all 7×7 real matrices by $M(\mathbb{R}, 7)$. For any matrix g, we denote the i-th law by g_i. It is known that G_2 is given by followings.

Proposition 1 ([2]). *Let $SO(7) = \{A \in M(7, \mathbb{R})\ ;\ {}^tA = A^{-1}, \det A = 1\}$. Then,*

$$G_2 = \left\{ (a_1, \cdots, a_7) \in SO(7)\ ;\ \begin{array}{ll} a_3 = a_1 a_2, & a_5 = a_1 a_4 \\ a_6 = a_4 a_2, & a_7 = a_1 a_6 = a_1(a_4 a_2) \end{array} \right\}.$$

Denote the unit element of G_2 by 1_7. Set $M \subset G_2$ as follows.

$$M = \{a \in G_2\ ;\ a^{-1} = a\} - \{1_7\} = \{a \in G_2\ ;\ {}^ta = a\} - \{1_7\}.$$

Then, it is known that G_2 acts on M transitively and $M \cong G_2/SO(4)$. In particular, M is a realization of $G_2/SO(4)$. We denote

$$\begin{pmatrix} x_1 & & & & \\ & x_2 & & & \\ & & \ddots & & \\ & & & x_6 & \\ & & & & x_7 \end{pmatrix} \in M(7, \mathbb{R})$$

by $d(x_1, x_2, \cdots, x_6, x_7)$.

Proposition 2 ([5]). *Set $p_1, \cdots, p_7 \in M$ as follows.*

$$\begin{aligned}
&p_1 = d(\ \ 1,\ \ 1,\ \ 1, -1, -1, -1, -1), p_2 = d(\ \ 1, -1, -1,\ \ 1,\ \ 1, -1, -1), \\
&p_3 = d(-1,\ \ 1, -1,\ \ 1, -1,\ \ 1, -1), p_4 = d(-1, -1,\ \ 1, -1,\ \ 1,\ \ 1, -1), \\
&p_5 = d(\ \ 1, -1, -1, -1, -1,\ \ 1,\ \ 1), p_6 = d(-1,\ \ 1, -1, -1,\ \ 1, -1,\ \ 1), \\
&p_7 = d(-1, -1,\ \ 1,\ \ 1, -1, -1,\ \ 1).
\end{aligned}$$

Then, $S = \{p_1, \cdots, p_7\}$ is a maximal antipodal set of M and any maximal antipodal set of M is congruent to S.

3 Main result

Set a inner product $(X, Y) = \mathrm{tr}({}^tXY)$ for any $X, Y \in M(7, \mathbb{R})$. For each $A \in M(7, \mathbb{R})$, we set the function $h_A : M \to \mathbb{R}; B \mapsto (A, B)$. We call h_A the height function of M with respect to A. We consider the following lemma which is proved in [2].

Lemma 1. *([2]) There are positive number $0 < c_1 < c_2 < c_4$ satisfying the following conditions:*

(1) $2c_1 < c_2, 2c_2 < c_4$

(2) *For any $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$,*

$$\begin{cases} \theta_1 \pm \theta_2 \pm \theta_3 \equiv 0 \mod 2\pi \\ c_1 \sin\theta_1 = c_2 \sin\theta_2 = c_4 \sin\theta_3 \end{cases} \implies \quad \theta_1 \equiv \theta_2 \equiv \theta_3 \equiv 0 \mod \pi$$

Let $X = d(c_1, c_2, 0, c_4, 0, 0, 0)$.

Theorem 2. *The function h_X is a Morse function of $M \cong G_2/SO(4)$ and the set of all critical points is the great antipodal set $\{p_1, \cdots, p_7\}$. The index $n(p_i)$ of h_X at each p_i is as follows:*

	p_1	p_2	p_3	p_4	p_5	p_6	p_7
$n(p_i)$	5	6	8	0	2	3	4

Therefore, h_X is a $\mathbb{Z}_2$-perfect Morse function of M.

References

[1] B.Y.Chen, T.Nagano, A Riemannian geometric invariant and its applications to a problem of Borel and Serre, Trans.Amer.Math.Soc, 308(1988), 273-297

[2] H.Kamiya, Weighted tarce functions as examples of Morse functions, J.Fac.Sci.Shinshu Univ, vol.7(1971), 85-96

[3] I.Yokota, Manifold and Morse function, Gendaisuugakusha, 2016 (in Japanese)

[4] M.Takeuchi, Two-number of symmetric R-spaces, Nagoya Math J, 115(1989), 43-46

[5] M.S.Tanaka, H.Tasaki, O.Yasukura, Maximal antipodal sets related to G_2, preprint

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$G_2/SO(4)$ の Morse 関数と極大対蹠集合

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今日の内容

- G 型コンパクト対称空間 $G_2/SO(4)$ の Morse 関数で，臨界点集合が極大対蹠集合となるようなものを構成できたので，これを紹介する.

1. 背景
2. 準備
3. 主結果
4. 今後の課題

1, 背景

定義 1.1

リーマン多様体 M について，各点 $x \in M$ に対して次を満たす等長変換 s_x が存在するとき，M を対称空間という．

1. x は s_x の孤立固定点である．
2. s_x は対合的である $(s_x^2 = \mathrm{id}_M)$．

s_x を x における点対称と呼ぶ．

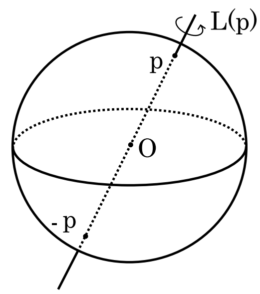
定義 1.2

- M の 2 点 $x, y \in M$ が対蹠的 $\overset{\text{def}}{\Longleftrightarrow}$ $s_x(y) = y (\Leftrightarrow s_y(x) = x)$．
- M の部分集合 S が対蹠集合 $\overset{\text{def}}{\Longleftrightarrow}$ S の任意の 2 点が対蹠的．
- M の対蹠集合 S が大対蹠集合
  $\overset{\text{def}}{\Longleftrightarrow}$ S は濃度が最大になる対蹠集合．
  大対蹠集合の濃度を M の 2-number といい，$\#_2 M$ とかく．

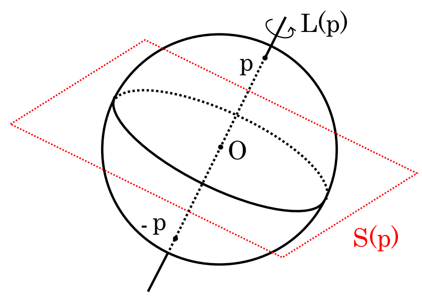
- 対蹠集合は，1988 年に Chen-Nagano により導入された．
- 非コンパクト型対称空間では，対蹠集合は常に 1 点集合である．
  $\longrightarrow$ 対蹠集合を考える上では，コンパクト対称空間を考える．

背景

Morse 関数と対蹠集合 (球面 S^2)



- $p \in S^2$ とし，$L(p)$ を中心 o と p を通る直線とする．
- p における点対称 s_p は $L(p)$ を回転軸とした 180 度回転となる．
- s_p の不動点集合は $\{p, -p\}$ となる．
  したがって，$\{p, -p\}$ は S^2 の大対蹠集合となり，$\#_2 S^2 = 2$.



- $S(p)$: o を通る $L(p)$ に直交する超平面
  $f : S^2 \to \mathbb{R}$: S^2 における $S(p)$ からの高さを測る関数
- このとき，f の臨界点集合は $\{p, -p\}$．とくに f は $\mathbb{Z}_2$-perfect Morse 関数となる．
- とくに，S^2 では大対蹠集合を臨界点集合とするような $\mathbb{Z}_2$-perfect Morse 関数が存在する．よって，

$$\#_2 S^2 = \dim H_*(S^2; \mathbb{Z}_2) = 2.$$

Morse 関数と対蹠集合 (対称 R 空間)

- コンパクト対称空間の特別なクラスとして，対称 R 空間が存在する．例えば，以下は対称 R 空間である．
  - 球面 S^n
  - 実・複素・四元数グラスマン多様体
  - 特殊直交群 $SO(n)$，ユニタリ群 $U(n)$，シンプレクティック群 $Sp(n)$
- 対称 R 空間には，標準埋め込みというユークリッド空間への埋め込みが存在する．
  - 球面 S^n では，$\mathbb{R}^{n+1}$ の round sphere としての実現
  - 特殊直交群 $SO(n)$，ユニタリ群 $U(n)$，シンプレクティック群 $Sp(n)$ では，通常の行列としての実現

- 対称 R 空間をユークリッド空間へ標準埋め込みにより埋め込んだとき，高さ関数では，臨界点集合が大対蹠集合となるような $\mathbb{Z}_2$-perfect Morse 関数が存在する．

定理 1.3 (Takeuchi, 89)

M を対称 R 空間とする．このとき，次が成り立つ．

$$\#_2 M = \dim H_*(M; \mathbb{Z}_2)$$

Morse 関数と対蹠集合

問題

対称 R 空間でないコンパクト対称空間 M について，

$$\#_2 M = \dim H_*(M;\mathbb{Z}_2)$$

であるなら，臨界点集合が大対蹠集合であるような $\mathbb{Z}_2$-perfect Morse 関数は存在するか？

例 1.4 (特殊ユニタリ群 $SU(n)$(Kamiya, 71))

- $\#_2 SU(n) = \dim H_*(SU(n);\mathbb{Z}_2) = 2^{n-1}$
- 行列としての実現 $SU(n) \subset M(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$ について，高さ関数の中で，そのような $\mathbb{Z}_2$-perfect Morse 関数が存在する．

例 1.5 (例外型コンパクトリー群 G_2(Kamiya,71))

- $\#_2 G_2 = \dim H_*(G_2;\mathbb{Z}_2) = 8$
- G_2 は $SO(8)$ の部分群として実現される．
- $SO(8)$ の行列としての実現を用いて埋め込み $G_2 \subset M(8,\mathbb{R})$ を考えると，高さ関数の中で，そのような $\mathbb{Z}_2$-perfect Morse 関数が存在する．

- G 型コンパクト対称空間 $G_2/SO(4)$ について，

$$\#_2 G_2/SO(4) = \dim H_*(G_2/SO(4); \mathbb{Z}_2) = 7$$

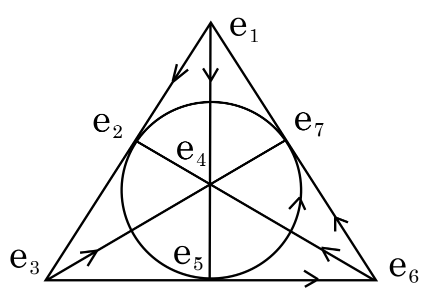
  である．したがって，$G_2/SO(4)$ においても臨界点集合が大対蹠集合であるような $\mathbb{Z}_2$-perfect Morse 関数の存在が期待される．

- 本講演では，そのような Morse 関数を具体的に構成することができたため，これを紹介する．

2, 準備

$\mathbb{O} = \sum_{1=0}^{7} \mathbb{R}e_i$ を八元数とする．積は次のように定義される．

- e_0 は積の単位元とする．単に 1 と書く．
- 各 $1 \leq i, j \leq 7$ について，$e_i^2 = -1, e_i e_j = -e_j e_i$．
- 積は分配法則を満たしている．
- 以下の図により積を定める．（例：$e_1 e_2 = e_3, e_1 e_4 = e_5$）



$\mathbb{O}$ の積においては，結合法則が成り立っていない．

- 線形同型 $\eta : \mathbb{O} \to \mathbb{R}^8$; $\sum_{i=0}^{7} x_i e_i \mapsto {}^t(x_0, \cdots, x_7)$ を考え，$\mathbb{R}^8$ における積を次のように定める：

$$xy := \eta(\ \eta^{-1}(x)\eta^{-1}(y)\ ).$$

すなわち，$x = {}^t(x_0, \cdot, x_7), {}^t y = (y_0, \cdots, y_7)$ について

$$xy = \begin{pmatrix} x_0y_0 - (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5 + x_6y_6 + x_7y_7) \\ (x_0y_1 + x_1y_0) + (x_2y_3 - x_3y_2) + (x_4y_5 - x_5y_4) + (x_6y_7 - x_7y_6) \\ (x_0y_2 + x_2y_0) + (x_3y_1 - x_1y_3) + (x_6y_4 - x_4y_6) + (x_5y_7 - x_7y_5) \\ (x_0y_3 + x_3y_0) + (x_1y_2 - x_2y_1) + (x_4y_7 - x_7y_4) + (x_5y_6 - x_6y_5) \\ (x_0y_4 + x_4y_0) + (x_5y_1 - x_1y_5) + (x_2y_6 - x_6y_2) + (x_7y_3 - x_3y_7) \\ (x_0y_5 + x_5y_0) + (x_1y_4 - x_4y_1) + (x_7y_2 - x_2y_7) + (x_6y_3 - x_3y_6) \\ (x_0y_6 + x_6y_0) + (x_7y_1 - x_1y_7) + (x_4y_2 - x_2y_4) + (x_3y_5 - x_5y_3) \\ (x_0y_7 + x_7y_0) + (x_1y_6 - x_6y_1) + (x_2y_5 - x_5y_2) + (x_3y_4 - x_4y_3) \end{pmatrix} \in \mathbb{R}^8.$$

- $x, y \in V := \{{}^t(0, x_1, \cdots, x_7) \in \mathbb{R}^8\}$ について，$x \perp y$ ならば $xy \in V$ となる．
- $\mathbb{R}^7$ においても $x \perp y$ なる $x, y \in \mathbb{R}^7$ について積を定める．

- $a \in M(7, \mathbb{R})$ の縦ベクトルを $a_1, \cdots, a_7$ と書き，$a = (a_1, \cdots, a_7)$ と記す．

命題 2.1

$SO(7) = \{A \in M(7, \mathbb{R})\ ;\ {}^tA = A^{-1}, \det A = 1\}$ とする．このとき，

$$G_2 = \left\{ (a_1, \cdots, a_7) \in SO(7)\ ;\ \begin{matrix} a_3 = a_1a_2, & a_5 = a_1a_4 \\ a_6 = a_4a_2, & a_7 = a_1a_6 = a_1(a_4a_2) \end{matrix} \right\}.$$

- G_2 は $\mathbb{O}$ の自己同型群として実現されるが，$\mathbb{O}$ の基底 $e_0, \cdots, e_7$ に関して行列表示すると，G_2 は上のように実現される．

$G_2/SO(4)$ に関して

- $M \subset G_2$ を次で定める：

$$M = \{a \in G_2 \ ; \ a^{-1} = a\} - \{1_7\} = \{a \in G_2 \ ; \ {}^t a = a\} - \{1_7\}.$$

このとき，M は連結であり，$\dim M = 8$.

- G_2 は M へ共役により作用している．すなわち，$g \in G_2, a \in M$ について，

$$g(a) = gag^{-1}.$$

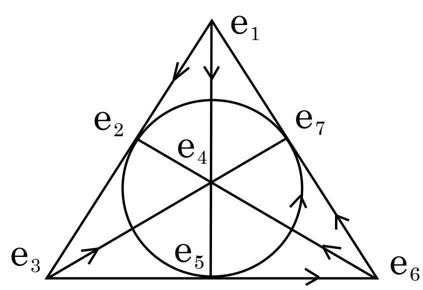
この作用は推移的になることが知られており，等質空間として $M = G_2/SO(4)$ となる．

- $x_1, \cdots, x_7 \in \mathbb{R}$ を対角成分に持つ対角行列を $\mathrm{d}(x_1, \cdots, x_7)$ とかく．

定理 2.2 (Tanaka-Tasaki-Yasukura)

$p_1, \cdots, p_7 \in M$ を以下で定める．

$$\begin{aligned}
p_1 &= \mathrm{d}(\ \ 1,\ \ 1,\ \ 1, -1, -1, -1, -1),\\
p_2 &= \mathrm{d}(\ \ 1, -1, -1,\ \ 1,\ \ 1, -1, -1),\\
p_3 &= \mathrm{d}(-1,\ \ 1, -1,\ \ 1, -1,\ \ 1, -1),\\
p_4 &= \mathrm{d}(-1, -1,\ \ 1, -1,\ \ 1,\ \ 1, -1),\\
p_5 &= \mathrm{d}(\ \ 1, -1, -1, -1, -1,\ \ 1,\ \ 1),\\
p_6 &= \mathrm{d}(-1,\ \ 1, -1, -1,\ \ 1, -1,\ \ 1),\\
p_7 &= \mathrm{d}(-1, -1,\ \ 1,\ \ 1, -1, -1,\ \ 1).
\end{aligned}$$



このとき，$S := \{p_1, \cdots, p_7\}$ は M の大対蹠集合．また，M の任意の極大対蹠集合は S と G_2 作用により移り合う．

埋め込み $G_2/SO(4) \cong M \subset G_2 \subset M(7, \mathbb{R})$ に関して，高さ関数で，S が臨界点集合となるような $\mathbb{Z}_2$-perfect Morse 関数となるものを探す．

3. 主結果

- $M(7,\mathbb{R})$ 上の内積 $(,)$ を次で定める．$X, Y \in M(7,\mathbb{R})$ について

$$(X, Y) = \mathrm{tr}({}^tXY).$$

定義 3.1

各 $X \in M(7,\mathbb{R})$ について，関数 $h_X : M(7,\mathbb{R}) \to \mathbb{R}; A \mapsto (X, A)$ を M へ制限して得られる関数

$$h_X|_M : M \to \mathbb{R}$$

を M における X-方向の高さ関数という．

補題 3.2

正数 $0 < c_1 < c_2 < c_4$ で次の (1),(2) を満たすものが存在する．

(1) $2c_1 < c_2,\quad 2c_2 < c_4.$

(2) $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ について

$$\begin{cases} \theta_1 + \theta_2 + \theta_3 \equiv 0 \pmod{2\pi} \\ c_1 \sin\theta_1 = c_2 \sin\theta_2 = c_4 \sin\theta_3 \end{cases} \implies \theta_1 \equiv \theta_2 \equiv \theta_3 \equiv 0 \pmod{\pi}$$

- 正数 c_1, c_2, c_4 を補題を満たすものとし，$X \in M(7, \mathbb{R})$ を次のように定める．

$$X = \begin{pmatrix} c_1 & & & & & \\ & c_2 & & & & \\ & & 0 & & & \\ & & & c_4 & & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix}.$$

定理 3.3 (S)

M における X-方向の高さ関数 $h_X|_M$ は，$\mathbb{Z}_2$-perfect Morse 関数となり，その臨界点集合は S となる．

証明のスケッチ 1(臨界点集合の決定)

$f := h_X|_M$ とする．まず f の臨界点集合 $C(f)$ を決定する．

- $\mathfrak{g}_2$ を G_2 の Lie 環とする．このとき，$\dim \mathfrak{g}_2 = 14$.
  $V_1, \cdots, V_{14}$ を $\mathfrak{g}_2$ の基底であるとする．
- $\Delta \subset M$ を以下で定める．

$$\begin{aligned} \Delta &= \left\{ A \in M \,;\, \frac{d}{dt} f(\exp(tX) A \exp(-tX))|_{t=0} = 0 \text{ for any } X \in \mathfrak{g}_2 \right\} \\ &= \left\{ A \in M \,;\, \frac{d}{dt} f(\exp(tV_i) A \exp(-tV_i))|_{t=0} = 0 \ (1 \leq i \leq 14) \right\} \end{aligned}$$

  このとき，G_2 の M への作用は推移的なので $C(f) = \Delta$.
- とくに Δ は次のようになる．

$$\Delta = \{ A \in M \,; (X, [V_i, A]) = 0 \ (1 \leq i \leq 14) \}$$

- E_{ij} で (i,j) 成分 1，その他の成分が 0 の 7 次正方行列とする．$G_{ij} := E_{ij} - E_{ji}$ $(i \neq j)$.
- $V_1, \cdots, V_{14}$ を以下のように定める．

$$\begin{array}{llll} V_1 = G_{15} - G_{37}, & V_2 = G_{26} + G_{37}, & V_3 = G_{14} + G_{36}, & V_4 = G_{27} - G_{36}, \\ V_5 = G_{17} + G_{35}, & V_6 = G_{24} + G_{35}, & V_7 = G_{16} - G_{34}, & V_8 = G_{25} - G_{34}, \\ V_9 = G_{23} - G_{67}, & V_{10} = G_{45} - G_{67}, & V_{11} = G_{13} + G_{57}, & V_{12} = G_{46} + G_{57}, \\ V_{13} = G_{12} - G_{56}, & V_{14} = G_{47} - G_{56}. & & \end{array}$$

- このとき，$\Delta \subset \{p_1, \cdots, p_7\} = S$ となる．とくに，$C(f) \subset S$.
- 一方で，S の各点における f の勾配ベクトル場を計算すれば，S の各点が f の臨界点だとわかる．とくに，$S \subset C(f)$.

補題 3.4

f の臨界点集合 $C(f)$ は S となる．

証明のスケッチ 2(臨界点における指数)

f の各臨界点 p における Hessian の固有値を調べる．

まず，$p_1 \in S$ における Hessian を計算する．

- K_{p_1} を p_1 における G_2 のイソトロピー群とし，$\mathfrak{k}_{p_1}$ を K_{p_1} の Lie 環とし，$\mathfrak{p}_{p_1} = \mathfrak{k}_{p_1}^{\perp}$ とする．このとき，

$$\mathfrak{p}_{p_1} = \sum_{i=1}^{8} \mathbb{R} V_i, \qquad \mathfrak{k}_{p_1} = \sum_{i=9}^{14} \mathbb{R} V_i.$$

- このとき，次の ϕ は各 $|t_k| (1 \leq k \leq 8)$ が十分に小さいとき，p_1 の近傍の局所座標を与える．

$$\begin{aligned} \phi : \mathbb{R}^8 \ni (t_1, \cdots, t_8) \\ \mapsto \exp(t_1 V_1) \cdots \exp(t_8 V_8) p_1 \exp(-t_8 V_8) \cdots \exp(-t_1 V_1) \in M \end{aligned}$$

- この局所座標に関する f の Hess 行列の固有値は，

$$-4c_1, -4c_1, -4c_2, -4c_2, 4(c_4 - c_1), 4(c_4 - c_2),$$
$$2((c_4 - c_1) + (c_4 - c_2) \pm \sqrt{4c_4^2 + (c_2 - c_1)^2}).$$

とくに，0 の固有値はなく，指数 (負の固有値の数) は 5．

- $C(f)$ のその他の点 p について，ある $g \in G_2$ で $gp_1g^{-1} = p$ となる．このとき，

$$g\phi g^{-1} : \mathbb{R}^8 \xrightarrow{\phi} M \xrightarrow{g(\cdot)g^{-1}} M$$

は p の近傍に局所座標を与える．

定理 3.5 (S)

f の臨界点の各点において，そのヘッシアンは固有値として 0 を持たない．したがって，f は M の Morse 関数である．f の各臨界点における指数は次のよう．

	p_1	p_2	p_3	p_4	p_5	p_6	p_7
指数	5	6	8	0	2	3	4

とくに，f は臨界点集合が大対蹠集合であるような $\mathbb{Z}_2$-perfect Morse 関数である．

今後の課題

問題

対称 R 空間でないコンパクト対称空間 M について，

$$\#_2 M = \dim H_*(M;\mathbb{Z}_2)$$

であるなら，臨界点集合が大対蹠集合であるような $\mathbb{Z}_2$-perfect Morse 関数は存在するか？

- 特殊ユニタリ群 $SU(n)$，例外型コンパクトリー群 G_2，G 型コンパクト対称空間 $G_2/SO(4)$ はそのような Morse 関数を持っていた．
- そもそも，$\#_2 M = \dim H_*(M;\mathbb{Z}_2)$ となる M は他にどんなものがあるのか？

例 4.1 (射影古典群 $PU(n) = SU(n)/\mathbb{Z}_n (n = 2$ または n は奇数))

$$\#_2 PU(n) = \dim H_*(PU(n)\ ;\ \mathbb{Z}_2) = \begin{cases} 4\ (n=2) \\ 2^{n-1}(n{:}\text{奇数}) \end{cases}$$

例 4.2 (射影古典群 $PO(n) = SO(n)/\mathbb{Z}_2$ $(n = 2, 4)$)

$$\#_2 PO(n) = \dim H_*(PO(n)\ ;\ \mathbb{Z}_2) = \begin{cases} 2\ (n=2) \\ 16\ (n=4) \end{cases}$$

例 4.3 (射影古典群 $PSp(n) = Sp(n)/\mathbb{Z}_2$ $(n = 2, 4$ または n は奇数))

$$\#_2 PSp(n) = \dim H_*(PSp(n)\ ;\ \mathbb{Z}_2) = \begin{cases} 16\ (n=2) \\ 64\ (n=4) \\ 2^{n+1}(n{:}\text{奇数}) \end{cases}$$

例 4.4 (例外型コンパクトリー群 F_4)

$$\#_2 F_4 = \dim H_*(F_4\ ;\ \mathbb{Z}_2) = 32$$

一方で，$\#_2 M \neq \dim H_*(M;\mathbb{Z}_2)$ である対称空間も存在する．
以下，2^k を n を割り切る最大の 2 のべき乗とする．

例 4.5 (射影古典群 $PU(n) = SU(n)/\mathbb{Z}_n (n$ は 2 以外の偶数$)$)

$$\#_2 PU(n) = 2^n \text{ or } 2^{n-1} < \dim H_*(PU(n)\ ;\ \mathbb{Z}_2) = 2^{n+k-1}$$

例 4.6 (射影古典群 $PO(n) = SO(n)/\mathbb{Z}_2$ (n は 6 以上の偶数))

$$\#_2 PO(n) = 2^{n-2} < \dim H_*(PO(n)\ ;\ \mathbb{Z}_2) = 2^{n+k-2}$$

例 4.7 (射影古典群 $PSp(n) = Sp(n)/\mathbb{Z}_2$ (n は 6 以上の偶数))

$$\#_2 PSp(n) = 2^{n+1} < \dim H_*(PSp(n)\ ;\ \mathbb{Z}_2) = 2^{n+k+1}$$

例 4.8 (例外型コンパクトリー群 E_6, E_7, E_8)

$$\#_2 E_6 = 2^6 < \dim H_*(E_6;\mathbb{Z}_2) = 2^7$$
$$\#_2 E_7 = 2^7 < \dim H_*(E_7;\mathbb{Z}_2) = 2^{10}$$
$$\#_2 E_8 = 2^9 < \dim H_*(E_8;\mathbb{Z}_2) = 2^{15}$$

- これらの例では，$\#_2 M < \dim H_*(M;\mathbb{Z}_2)$ であるが，これは一般的なことか？

注意 4.9 (Chen-Nagano.88)

M のオイラー数を $\chi(M)$ とすれば，$\chi(M) \leq \#_2 M$ となることも知られている (Chen-Nagano,88)．

まとめ

- 対称 R 空間や $SU(n)$, G_2 では
$$\#_2 M = \dim H_*(M; \mathbb{Z}_2) \qquad (*)$$
が成り立つ．その背景には，臨界点集合が大対蹠集合となるような $\mathbb{Z}_2$-perfect Morse 関数の存在がある．
- G 型コンパクト対称空間 $G_2/SO(4)$ でも，$(*)$ が成り立つ．この場合でも，臨界点集合が大対蹠集合となるような $\mathbb{Z}_2$-perfect Morse 関数を構成することができた．
- その他にも，$(*)$ をみたすコンパクト対称空間は存在する．
    - その場合，今回調べたような Morse 関数は存在するのか？
    - $\#_2 M \leq \dim H_*(M; \mathbb{Z}_2)$ が多くのコンパクト対称空間で成り立つが，これは一般的なことか？

Quandle coloring quivers for spatial graphs

Hiroki Ito

Abstract. A quandle coloring quiver is a link invariant defined by S. Nelson and K. Cho. In this paper, we define a quandle coloring for spatial graphs. Using this quandle coloring, we extend quandle coloring quivers to spatial graphs. And we show that the quandle coloring quiver is a stronger invariant than the quandle coloring number.

1 Quandle coloring quiver

1.1 Quandle

Definition 1.1. A *quandle* is a set X with a binary operation $* : X \times X \to X$ satisfying the following three axioms:
(1) $\forall x \in X, x * x = x$.
(2) $\forall x \in X$, a map $S_x : X \to X; a \mapsto a * x$ is bijective.
(3) $\forall x, y, z \in X, (x * y) * z = (x * z) * (y * z)$.

These three axioms correspond to Reidemeister moves. Here, we give some examples of quandles.

Example 1.2. Let n be a natural number and $X := \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Then, X is a quandle if $* : X \times X \to X$ is a binary operation defined by $x * y := 2y - x$ $(x, y \in X)$. We call this quandle a *dihedral quandle*.

Example 1.3. Let n be a natural number and $X_n := \mathbb{Z}_n \sqcup \{\omega\}$. Then, X_n is a quandle if $* : X_n \times X_n \to X_n$ is a binary operation defined by

$$x * y := \begin{cases} 2y - x & (x, y \in \mathbb{Z}_n) \\ x & (otherwise) \end{cases}.$$

We call this quandle a *semi dihedral quandle*.

1.2 Quandle coloring

First, we introduce a proper map $\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ and θ-admissibility to define a quandle coloring for spatial graphs.

Definition 1.4. A map $\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ is a *proper map* if the following two conditions are satisfied.
(1) $\forall m \in \mathbb{N}, \theta(m) \leq m$.
(2) $\forall m \in \mathbb{N}, \theta(m) \equiv m \pmod 2$.

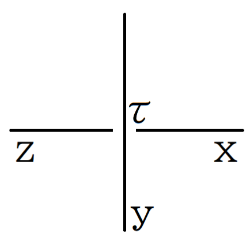
Definition 1.5. Let X_n be a semi dihedral quandle, $\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ be a proper map. Then, $(a_1, \ldots, a_m) \in {X_n}^m$ is *θ-admissible* if the following two conditions are satisfied.

(1) $S_{a_m} \circ \cdots \circ S_{a_1} = id_{X_n}$.

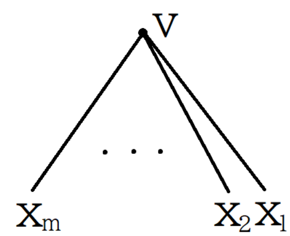
(2) $\#\{i \mid a_i = \omega\} = \theta(m)$.

Next, we define a quandle coloring for a spatial graph diagram.

Definition 1.6. Let g be a spatial graph diagram, X_n be a semi dihedral quandle, $\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ be a proper map and $\mathcal{A}(g) := \{$ arcs of g $\}$. Then, a map $C : \mathcal{A}(g) \to X_n$ is a *(X_n, θ)-coloring* of g if the values assigned arcs around each crossing τ and vertex v of g satisfy the conditions in the figure.



$C(x) * C(y) = C(z)$



$(C(x_1), \ldots, C(x_m))$: θ-admissible

We denote by $\mathrm{Col}_{(X_n,\theta)}(g)$ the set of (X_n, θ)-colorings of g, and by $\mathrm{col}_{(X_n,\theta)}(g)$ the cardinal number of $\mathrm{Col}_{(X_n,\theta)}(g)$. We call $\mathrm{col}_{(X_n,\theta)}(g)$ a *quandle coloring number* of g.

Proposition 1.7. If g and g' are diagrams of a spatial graph, then $\mathrm{col}_{(X_n,\theta)}(g) = \mathrm{col}_{(X_n,\theta)}(g')$.

By Proposition 1.7, the quandle coloring number $\mathrm{col}_{(X_n,\theta)}(g)$ is an invariant of spatial graphs.

1.3 Quandle coloring quiver

Let X be a quandle. We denote by $\mathrm{End}(X)$ the set of quandle endomorphisms of X.

Let X_n be a semi dihedral quandle, and let $\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ be a proper map. Then, we consider a subset $\mathrm{End}(X_n, \theta)$ of $\mathrm{End}(X_n)$ defined by

$$\mathrm{End}(X_n, \theta) := \{ f \in \mathrm{End}(X_n) \mid \begin{array}{c} \forall (a_1, \ldots, a_m) : \theta\text{-admissible}, \\ (f(a_1), \ldots, f(a_m)) : \theta\text{-admissible}. \end{array} \}.$$

The following definition gives an extention of quandle coloring quivers defined by S. Nelson and K. Cho to spatial graphs.

Definition 1.8. Let g be a spatial graph diagram, X_n be a semi dihedral quandle, $\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ be a proper map. Then, for any subset $S \subset \mathrm{End}(X_n, \theta)$, a *quandle coloring quiver* $Q^S_{(X_n,\theta)}(g) = (V, E)$ is defined as the following.

- $V = \mathrm{Col}_{(X_n,\theta)}(g)$.
- $E = \{(v, w, f) \in V \times V \times S \mid w = f \circ v\}$,
  where an edge (v, w, f) is directed from v to w.

2 Main result

We have the following theorem.

Theorem 2.1. If g and g' are diagrams of a spatial graph, then for any subset $S \subset \mathrm{End}(X_n, \theta)$, the quandle coloring quivers $Q^S_{(X_n,\theta)}(g)$ and $Q^S_{(X_n,\theta)}(g')$ are isomorphic as a quiver.

By Theorem 2.1, for any subset $S \subset \mathrm{End}(X_n, \theta)$, the quandle coloring quiver $Q^S_{(X_n,\theta)}(g)$ is an invariant of spatial graphs. Moreover, we obtain the following proposition.

Proposition 2.2. There exists spatial graph diagrams g, g', a proper map $\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ and a subset $S \subset \mathrm{End}(X_n, \theta)$ such that the following two conditions are satisfied.
(1) $\mathrm{col}_{(X_n,\theta)}(g) = \mathrm{col}_{(X_n,\theta)}(g')$.
(2) $Q^S_{(X_n,\theta)}(g)$ and $Q^S_{(X_n,\theta)}(g')$ are not isomorphic as a quiver.

By Proposition 2.2, the quandle coloring quiver $Q^S_{(X_n,\theta)}(g)$ is a stronger invariant than the quandle coloring number $\mathrm{col}_{(X_n,\theta)}(g)$.

References

[1] L. H. Kauffman, Invariants of graphs in three-space, *Trans. Amer. Math. Soc.* **311** (1989) 697–710.

[2] S. Yamada, An invariant of spatial graphs, *J. Graph Theory.* **13** (1989), 537–551.

[3] D. Joyce, A classifying invariants of knots, the knot quandle, *J. Pure Appl. Algebra.* **23** (1982), 37–65.

[4] K. Oshiro, On pallets for Fox colorings of spatial graphs, *Topology and its Applications.* **159** (2012), 1092–1105.

[5] K. Cho and S. Nelson. Quandle coloring quivers. *J. Knot T heory Ramications.* **28**(1) (2019), 1950001, 12.

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Quandle coloring quivers for spatial graphs

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目次

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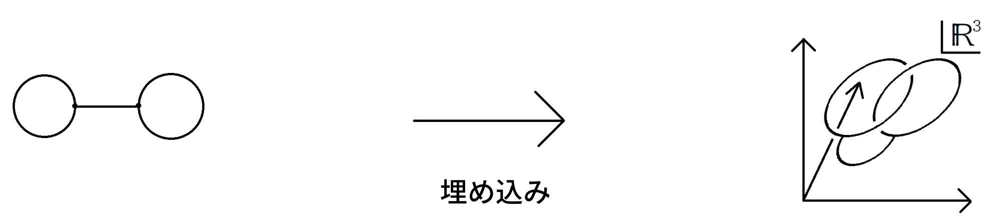
空間グラフ

定義

抽象グラフ $G = (V(G), E(G))$ を, $\mathbb{R}^3$ へ埋め込んだものを G の空間グラフといい, $\widetilde{G}$ と表記する.
また, 空間グラフ $\widetilde{G}, \widetilde{G}'$ に対して, 向きを保つ同相写像 $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ が存在して, $\phi(\widetilde{G}) = \widetilde{G}'$, $\phi(V(\widetilde{G})) = V(\widetilde{G}')$ となるとき, $\widetilde{G}, \widetilde{G}'$ はアンビエント・イソトピックであるという.

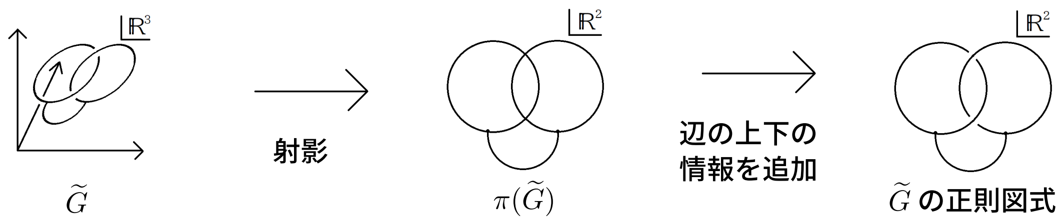


$G = (V(G), E(G))$ $\widetilde{G}$

空間グラフ

定義

$\widetilde{G}$ を空間グラフとする．$\pi : \mathbb{R}^3 \to \mathbb{R}^2$ を平面への射影であって，$\pi(\widetilde{G})$ の多重点は有限個の 2 重点のみであるとする．その時，$\pi(\widetilde{G})$ の 2 重点に辺の上下の情報を加えたものを，$\widetilde{G}$ の正則図式という．

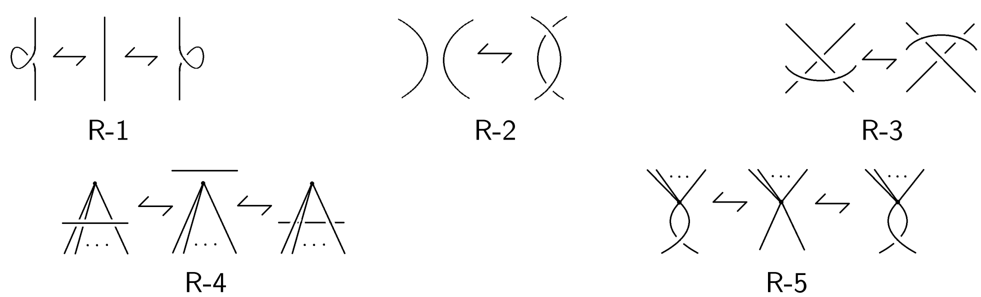


空間グラフ

g, g' をそれぞれ空間グラフ $\widetilde{G}, \widetilde{G}'$ の正則図式とする．

定義

$g \sim g' \Leftrightarrow g, g'$ が有限回の R-1 から R-5 の局所変形で移りあう．



命題

$\widetilde{G}, \widetilde{G}'$ がアンビエント・イソトピックである $\Leftrightarrow g \sim g'$

目次

Quandle

定義

X を集合とする．その時，X と 2 項演算 $*: X \times X \to X$ の組 $(X, *)$ が quandle である．
$\Leftrightarrow$ 以下の 3 つの条件を満たす．
(1) $\forall x \in X$, $x * x = x$
(2) $\forall y \in X$, 写像 $S_y : X \to X; x \mapsto x * y$ が全単射
(3) $\forall x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$

以後，$(X, *)$ を単に X と書く．

Quandle

例

$n \in \mathbb{N}$, $X := \mathbb{Z}_n$, $x * y := 2y - x$ $(x, y \in X)$ と定めると, X は quandle になる. この quandle を 2 面体 quandle という.

例

$n \in \mathbb{N}$, $X_n := \mathbb{Z}_n \sqcup \{\omega\}$, $* : X_n \times X_n \to X_n$ を以下のように定めると, X_n は quandle になる. この quandle を 準 2 面体 quandle という.

$$x * y := \begin{cases} 2y - x & (x, y \in \mathbb{Z}_n) \\ x & (otherwise) \end{cases}$$

	ω	0	1	2
ω	ω	ω	ω	ω
0	0	0	2	1
1	1	2	1	0
2	2	1	0	2

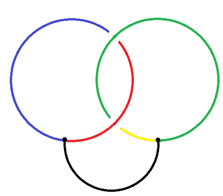
X_3

Quandle coloring

定義

g を空間グラフの正則図式とする. その時, $g \setminus V(g)$ の各連結成分を g の arc といい, $\mathcal{A}(g) := \{g \text{ の arc}\}$ と定める.

<u>例</u>　g を下図の正則図式とすると, g の arc は色を付けた 5 つの部分



g

Quandle coloring

定義

写像 $\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ が良い写像である. $\Leftrightarrow$ 以下の 2 つの条件を満たす.
(1) $\forall m \in \mathbb{N}, \theta(m) \leq m$
(2) $\forall m \in \mathbb{N}, \theta(m) \equiv m \pmod 2$

X_n を準 2 面体 quandle, $\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ を良い写像とする.

定義

$(a_1, \ldots, a_m) \in {X_n}^m$ が θ-admissible である. $\Leftrightarrow$ 以下の 2 つの条件を満たす.
(1) $S_{a_m} \circ \cdots \circ S_{a_1} = id_{X_n}$
(2) $\#\{i \mid a_i = \omega\} = \theta(m)$

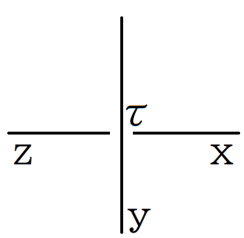
$a \in X_n$ に対して, 写像 S_a は, $S_a : X_n \to X_n; x \mapsto x * a$ である.

Quandle coloring

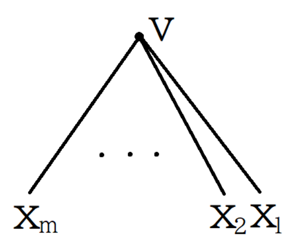
g を空間グラフの正則図式, X_n を準 2 面体 quandle, $\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ を良い写像とする.

定義

$C : \mathcal{A}(g) \to X_n$ が, g の (X_n, θ)-coloring である.
$\Leftrightarrow$ g の各交点 τ と各頂点 v の周りの arc に対して, 以下の関係式が成り立つ.
($\deg(v) = m$ とする.)



$C(x) * C(y) = C(z)$



$(C(x_1), \ldots, C(x_m))$: θ-admissible

Quandle coloring

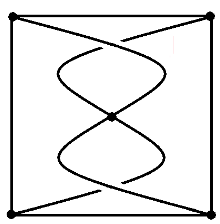
例　g を左下図の正則図式とし，$X_3 = \mathbb{Z}_3 \sqcup \{\omega\}$ を用いた g の coloring を考える．

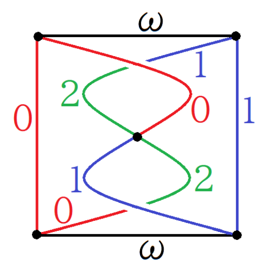
$\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ を $\theta(m) := \begin{cases} 1 & (m=3) \\ 0 & (m=4) \\ m & (otherwise) \end{cases}$ と定める．この時，右下図の coloring は

$S_0 \circ S_0 \circ S_\omega = id_{X_3}$, $S_1 \circ S_1 \circ S_\omega = id_{X_3}$, $S_2 \circ S_1 \circ S_2 \circ S_0 = id_{X_3}$

を満たすので，$(\omega, 0, 0), (\omega, 1, 1), (0, 2, 1, 2)$ は θ-admissible

故に，右下図の coloring は g の (X_3, θ)-coloring である．





Quandle coloring

g, g' を空間グラフの正則図式とする．

定義

$\mathrm{Col}_{(X_n,\theta)}(g) := \{g$ の (X_n, θ)-coloring$\}$, $\mathrm{col}_{(X_n,\theta)}(g) := \#\mathrm{Col}_{(X_n,\theta)}(g)$
と定める．

命題

$\mathrm{col}_{(X_n,\theta)}(g)$ は空間グラフの不変量である．即ち，
$g \sim g' \Rightarrow \mathrm{col}_{(X_n,\theta)}(g) = \mathrm{col}_{(X_n,\theta)}(g')$

Quandle coloring

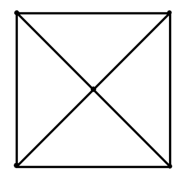
注意

$\mathrm{col}_{(X_n,\theta)}(g)$ は, θ の取り方に依る.

実際, g を右図の正則図式とし, θ_1, θ_2 を以下のように定める.

$$\theta_1(m) := \begin{cases} 1 & (m=3) \\ 2 & (m=4) \\ m & (otherwise) \end{cases} \qquad \theta_2(m) := \begin{cases} 1 & (m=3) \\ m & (otherwise) \end{cases}$$

その時, $\mathrm{col}_{(X_3,\theta_1)}(g) = 12$, $\mathrm{col}_{(X_3,\theta_2)}(g) = 3$ となる.



g

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Quandle coloring quiver

定義

X を quandle とする. その時, $f : X \to X$ が X の quandle 自己準同型
$\Leftrightarrow \forall x, y \in X, f(x * y) = f(x) * f(y)$
また, $\mathrm{End}(X) := \{ f : X$ の quandle 自己準同型 $\}$
と定める.

X_n を準 2 面体 quandle, $\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ を良い写像とする.

定義

$$\mathrm{End}(X_n, \theta) := \{ f \in \mathrm{End}(X_n) \mid \begin{array}{c} \forall (a_1, \ldots, a_m) : \theta\text{-admissible}, \\ (f(a_1), \ldots, f(a_m)) : \theta\text{-admissible} \end{array} \}$$

と定める.

Quandle coloring quiver

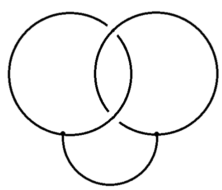
X_n を準 2 面体 quandle, $\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ を良い写像とする.

定義

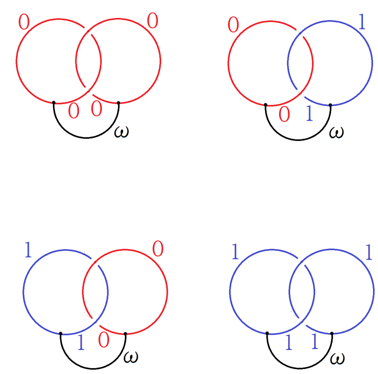
$S \subset \mathrm{End}(X_n, \theta)$ を任意の部分集合とする. その時,
g の Quandle coloring quiver $Q^S_{(X_n,\theta)}(g) = (V, E)$ を次で定義する.
(1) $V = \mathrm{Col}_{(X_n,\theta)}(g)$
(2) $E = \{(v, w, f) \in V \times V \times S \mid w = f \circ v\}$
但し, 辺は第 1 成分を始点, 第 2 成分を終点として向きを定める.

Quandle coloring quiver

<u>例</u>　g を左下図の正則図式とし，$\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ を $\theta(m) := \begin{cases} 1 & (m = 3) \\ m & (otherwise) \end{cases}$ と定めると，$\mathrm{Col}_{(X_2,\theta)}(g)$ は右下図の 4 通りとなる．

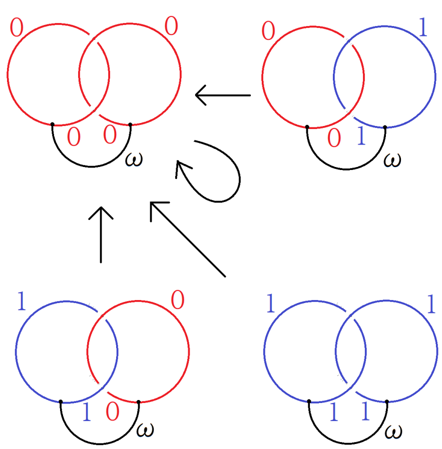


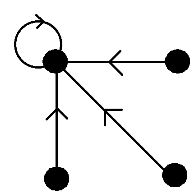
g



Quandle coloring quiver

$f(x) := \begin{cases} 0 & (x = 0, 1) \\ \omega & (x = \omega) \end{cases}$ とし，$S = \{f\}$ とすると，$Q^S_{(X_2,\theta)}(g)$ は以下のようになる．





$Q^S_{(X_2,\theta)}(g)$

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主結果

g, g' を空間グラフの正則図式とする.

定理 A

任意の $S \subset \mathrm{End}(X_n, \theta)$ に対して, $Q^S_{(X_n,\theta)}(g)$ は空間グラフの不変量である. 即ち,
$g \sim g' \Rightarrow Q^S_{(X_n,\theta)}(g) \cong Q^S_{(X_n,\theta)}(g')$ が成り立つ.

定理 B

以下の 2 つの条件を満たす正則図式 g, g', 良い写像 $\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$, $S \subset \mathrm{End}(X_n, \theta)$ が存在する.
(1) $\mathrm{col}_{(X_n,\theta)}(g) = \mathrm{col}_{(X_n,\theta)}(g')$
(2) $Q^S_{(X_n,\theta)}(g) \not\cong Q^S_{(X_n,\theta)}(g')$

注意

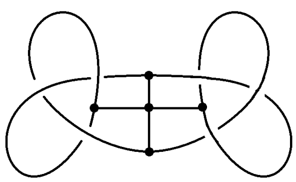
定理 B より, $Q^S_{(X_n,\theta)}(g)$ は $\mathrm{col}_{(X_n,\theta)}(g)$ より真に強い不変量である.

主結果

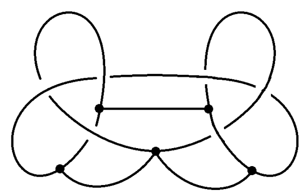
定理 B の証明

g, g' を下図の正則図式とし，$\theta : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ を $\theta(m) := \begin{cases} 1 & (m=3) \\ 2 & (m=4) \\ m & (otherwise) \end{cases}$ と定めると，

$\mathrm{col}_{(X_3,\theta)}(g) = \mathrm{col}_{(X_3,\theta)}(g) = 36$ となる．



g

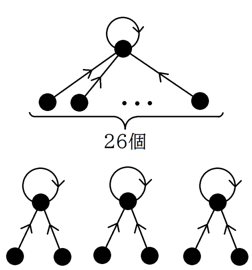


g'

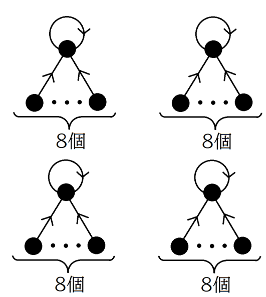
主結果

ここで，$f(x) := \begin{cases} 0 & (x = 0,1,2) \\ \omega & (x = \omega) \end{cases}$ とし，$S = \{f\}$ とすると，

$Q^S_{(X_3,\theta)}(g), Q^S_{(X_3,\theta)}(g')$ はそれぞれ以下のようになる．



$Q^S_{(X_3,\theta)}(g)$



$Q^S_{(X_3,\theta)}(g')$

故に，$Q^S_{(X_3,\theta)}(g) \ncong Q^S_{(X_3,\theta)}(g')$ $\square$

Thank you for your attention.

Quandle coloring quivers for vitrual links using a quandle endowed with an automorphism

Ryotaro Ueda

Abstract. S.Nelson and K.Cho introduced the notion of a quandle coloring quiver, which is a quiver-valued classical link invariant. In this talk, we extend Nelson-Cho's invariant to virtual links using virtual quandle colorings.

1 Virtual quandle colorings

Definition 1. A *virtual quandle* is a pair (X, f) of a quandle X and $f \in \mathrm{Aut}(X)$.

Definition 2. Let (X, f) and (X, g) be virtual quandles. (X, f) and (X, g) are *equivalent* if there exists an automorphism $\phi \in \mathrm{Aut}(X)$ which satisfies $f = \phi^{-1} g \phi$.

Let D be a virtual link diagram. A *virtual arc* of D is an oriented interval in D divided by under and virtual crossings. We denote the set of the virtual arcs of D by $\mathcal{VA}(D)$.

Definition 3. Let (X, f) be a virtual quandle and D be a virtual link diagram. A map $c : \mathcal{VA}(D) \to X$ is an (X, f)-*coloring* if c satisfies the condition at every crossing of D shown in Figure 7.

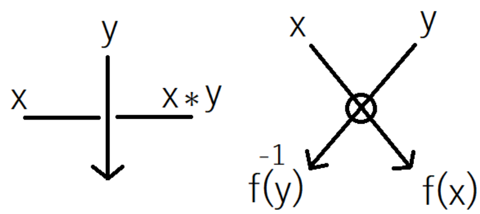


Figure 7: The coloring condition.

We denote by $\mathrm{Col}_{(X,f)}(D)$ the set of (X, f)-colorings of D.

2 Virtual quandle coloring quivers

K. Cho and S. Nelson [2] introduced the notion of a quandle coloring quiver. We extend Nelson-Cho's invariant to virtual links using virtual quandle colorings.

Definition 4. Let (X, f) be a virtual quandle and D be a virtual link diagram. For any subset $S \subset \mathrm{Com}_f(X) := \{g \in \mathrm{End}(X) \mid f \circ g = g \circ f\}$, the *virtual quandle coloring quiver* of D, which is denoted by $VQ^S_{(X,f)}(D)$, is the quiver with a vertex for each (X, f)-coloring $c \in \mathrm{Col}_{(X,f)}(D)$ and an edge directed from v to w when $w = g \circ v$ for an element $g \in S$. When $S = \mathrm{Com}_f(X)$, we denote the quandle coloring quiver by $VQ_{(X,f)}(D)$, which we call the *full virtual quandle coloring quiver*.

Example 5. Let D be a virtual Hopf link diagram in Figure 8. Let (R_3, f) be a virtual quandle defined by the dihedral quandle R_3 and $f \in \mathrm{Aut}(R_3)$ defined by $f(x) = 2x$. We put $g \in \mathrm{Com}_f(R_3)$ defined by $g(x) = 0$. Let S be a subset of $\mathrm{Com}_f(R_3)$ consisting of the element g. Then a virtual quandle coloring quiver $VQ^S_{(R_3,f)}(D)$ is the oriented graph shown in Figure 8.

D $\qquad$ $VQ^S_{(R_3,f)}(D)$

Figure 8: An example of virtual quandle coloring quiver.

3 Main results

Theorem 6. *If two virtual link diagrams D and D' are related by generalized Reidemeister moves, then virtual quandle coloring quivers $VQ^S_{(X,f)}(D)$ and $VQ^S_{(X,f)}(D')$ are isomorphic for any finite virtual quandle (X, f) and $S \subset Com_f(X)$.*

By Theorem 6, the virtual quandle coloring quiver is a virtual link invariant.

Theorem 7. *Let D be a virtual link diagram. Let (X, f) and (X, g) be finite virtual quandles. If (X, f) and (X, g) are equivalent, there exists an automorphism $\phi \in \mathrm{Aut}(X)$ which satisfies $f = \phi^{-1}g\phi$. we define a map $\Phi_\phi : \mathrm{Com}_f(X) \to \mathrm{Com}_g(X); h \mapsto \phi h \phi^{-1}$. Then, virtual quandle coloring quivers $VQ^S_{(X,f)}(D)$ and $VQ^{\Phi_\phi(S)}_{(X,g)}(D)$ are isomorphic for any $S \subset \mathrm{Com}_f(X)$. In particular, the full virtual quandle coloring quivers $VQ_{(X,f)}(D)$ and $VQ_{(X,g)}(D)$ are isomorphic.*

By Theorem 7, when we fix a virtual link diagram, the full virtual quandle coloring quiver is an invariant of equivalence classes of virtual quandles.

References

[1] D. Joyce, A classifying invariant of knots, the knot quandle. *J. Pure Appl. Algebra.* **23** (1982), no.1, 37-65.

[2] K. Cho and S. Nelson, Quandle coloring quivers. *J. Knot Theory Ramications.* **28** (2019), no. 1, 1950001, 12 pp.

[3] Louis H. Kauffman, Virtual knot theory, *European J. Combin.* **20** (1999), no. 7, 663-690, DOI 10.1006/eujc.1999.0314.

[4] S. Matveev, Distributive groupoids in knot theory. *Math. USSR. Sbornik.* **47** (1884), 73-83.

[5] V. O . Manturov, On Invariants of Virtual Links, *Acta Applicandae Mathematicae* **72**, 295-309 (2002). https://doi.org/10.1023/A:1016258728022

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Quandle coloring quivers for vitrual links using a quandle endowed with an automorphism

Ryotaro Ueda

Osaka University

研究集会「カンドルと対称空間」
December 17, 2020

Today's contents

1. Virtual links and virtual quandle colorings
2. Definition of virtual quandle coloring quivers
3. Main results

Today's contents

Virtual link diagrams

Definition (Kauffman, 1999)

$Virtual\ link\ diagram$ ⇔ 平面へはめ込まれたいくつかの向き付けられた円周であり, その多重点は有限個の横断的に交わる 2 重点のみで, そこには $classical\ crossing$ または $virtual\ crossing$ の情報が与えられているもの.

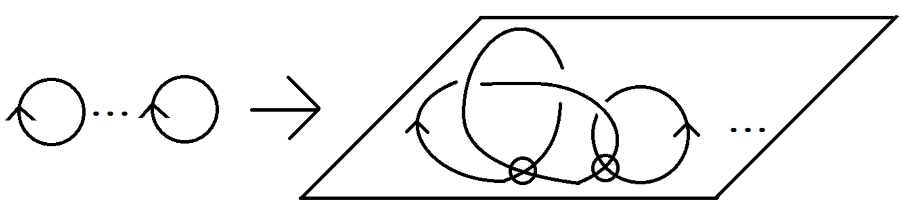


Figure: Virtual link diagram.

Virtual links

Definition (Kauffman, 1999)

D, D' : virtual link diagrams.
$D \sim D' \Leftrightarrow$ 有限回の *Generalized Reidemeister moves* で移り合う.
virtual link $\Leftrightarrow$ virtual link diagram の同値類.

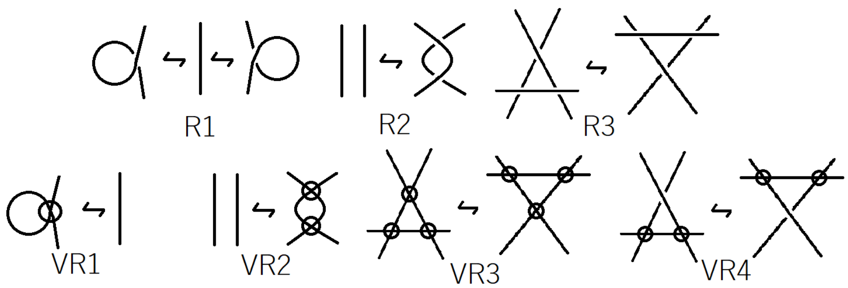


Figure: Generalized Reidemeister moves.

Virtual links and virtual quandle colorings

Quandles

Definition (Joyce, 1982)

次の 3 条件を満たす集合 X と 2 項演算 $*$ の組 $(X,*)$ を *quandle* という.

1. ${}^{\forall}x \in X$, $x * x = x$.
2. ${}^{\forall}x, y \in X$, ${}^{\exists!}z \in X$ s.t. $z * y = x$.
3. ${}^{\forall}x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

以後 quandle $(X, *)$ を X と書き, 有限なものとする.

Example

X を剰余群 $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, 演算 $*$ を $x * y = 2y - x$ とすれば quandle になる,
この quandle を位数 n の *dihedral quandle* と呼び, R_n と書く.

Quandle automorphisms and endomorphisms

Notation

X : quandle
End$(X) := \{\ f : X \to X \mid\ f$ は X の endomorphism $\}$.
Aut$(X) := \{\ f : X \to X \mid\ f$ は X の automorphism $\}$.

Fact

$n \in \mathbb{N}$, ${}^{\forall} f : R_n \to R_n$; endomorphism , ${}^{\exists} a, b \in \mathbb{Z}_n$ s.t. $f(x) = ax + b$.
特に f が automorphism の時は $(a, n) = 1$.

Virtual links and virtual quandle colorings

Virtual quandles

Definition (Manturov, 2002)

X : quandle , $f \in \mathrm{Aut}(X)$. このとき 組 (X, f) を *virtual quandle* という.

注意として (X, f) は quandle ではないが, Manturov に従ってこのように呼ぶ.

Definition (Ceniceros-Nelson, 2009)

$(X, f), (X, g)$: virtual quandles.
$(X, f) \sim (X, g) \Leftrightarrow {}^{\exists}\ \phi \in$ Aut(X) s.t. $f = \phi^{-1} \circ g \circ \phi$.

$$
\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "g"] \\
X \arrow[r, "\phi"'] & X
\end{tikzcd}
$$

Example of virtual quandles

Example

R_3 : 位数 3 の dihedral quandle ,
$f, g \in \mathrm{Aut}(R_3)$, $f(x) = x + 1,\ g(x) = x + 2$.
このとき $(R_3, f) \sim (R_3, g)$ である.
実際, $\phi(x) = 2x$ とすると $f = \phi^{-1} \circ g \circ \phi$ となる.

Fact

$\mathrm{Aut}(R_3) \cong S_3$: symmetric group である.
このとき S_3 の共役類は $(1), (12), (123)$ なので, R_3 上の virtual quandle の同値類は 3 種類ある.

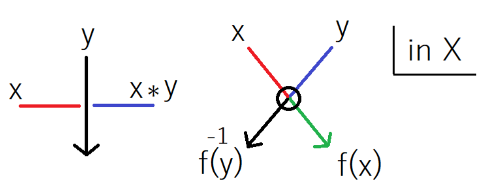
Virtual quandle colorings

Definition

D : virtual link diagram.
D の *virtual arc* $\Leftrightarrow$ D を classical under と virtual crossing で分割したときの各連結成分.
$\mathcal{VA}(D) := \{\ D$ の virtual arc $\}$.

Definition

(X, f) : virtual quandle, D : virtual link diagram.
$c : \mathcal{VA}(D) \to X$ が (X, f)*-coloring* $\Leftrightarrow$ D の各交点で以下の図の条件を満たす.
$\mathrm{Col}_{(X,f)}(D) := \{\ D$ の (X, f)-coloring $\}$.



Example of virtual quandle colorings

D : virtual Hopf link diagram, (R_3, f) : virtual quandle, $f(x) = 2x$.
このとき virtual quandle coloring は次の 3 通りである.

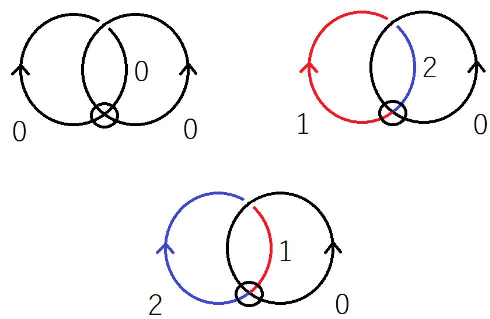


Figure: Virtual Hopf link diagram の virtual quandle coloring.

Property of virtual quandle colorings

Proposition (Manturov, 2002)

(X, f) : virtual quandle.
$D \sim D' \Rightarrow |\mathrm{Col}_{(X,f)}(D)| = |\mathrm{Col}_{(X,f)}(D')|$.
つまり $|\mathrm{Col}_{(X,f)}(D)|$ は virtual link の不変量である.

Proposition (U.)

D : virtual link diagram.
$(X, f) \sim (X, g) \Rightarrow |\mathrm{Col}_{(X,f)}(D)| = |\mathrm{Col}_{(X,g)}(D)|$.
つまり $|\mathrm{Col}_{(X,f)}(D)|$ は virtual quandle の同値類の不変量である.

Today's contents

Virtual quandle coloring quivers

Definiton (U.) (cf. Cho-Nelson, 2018)

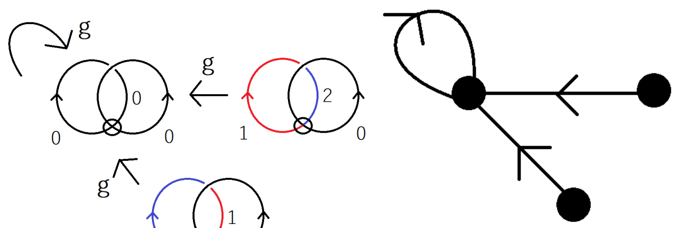
D : virtual link diagram , (X, f) : virtual quandle.
$S \subset \mathrm{Com}_f(X) := \{\ g \in \mathrm{End}(X) \mid f \circ g = g \circ f\ \}$,
$VQ^S_{(X,f)}(D)$: *virtual quandle coloring quiver*
$\Leftrightarrow VQ^S_{(X,f)}(D) = (V, E)$: an oriented graph.

1. $V = \mathrm{Col}_{(X,f)}(D)$.
2. $E = \{\ (v, w, g) \in V \times V \times S \mid w = g \circ v\ \}$.

ここで辺集合 E には第 1 成分を始点, 第 2 成分を終点として向きを入れる.

Example of virtual quandle coloring quivers (1)

D : virtual Hopf link diagram,
(R_3, f) : virtual quandle, $f(x) = 2x$, $g(x) = 0$, $S = \{g\} \subset \mathrm{Com}_f(R_3)$



$VQ^S_{(R_3,f)}(D)$

Today's contents

1. Virtual links and virtual quandle colorings
2. Definition of virtual quandle coloring quivers
3. Main results

Main results

Main theorem (1) (U.)

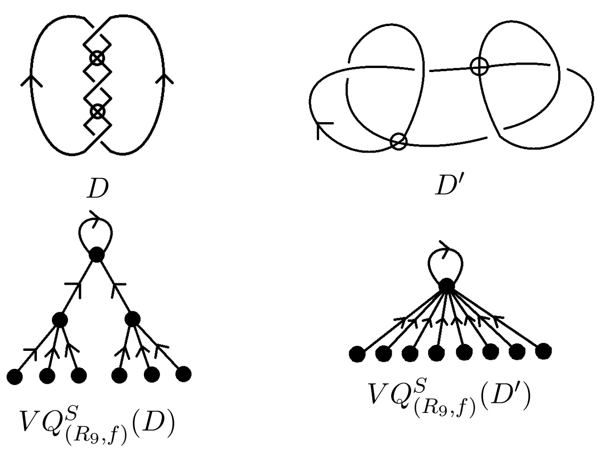
$D \sim D' \Rightarrow {}^{\forall}S \subset \mathrm{Com}_f(X)$, $VQ^S_{(X,f)}(D) \cong VQ^S_{(X,f)}(D')$.
つまり $VQ^S_{(X,f)}(D)$ は virtual link の不変量である.

Remark

$VQ^S_{(X,f)}(D)$ は $|\mathrm{Col}_{(X,f)}(D)|$ より真に強い virtual link の不変量である.

Example of virtual quandle coloring quivers (2)

(R_9, f) : virtual quandle, $f(x) = 2x$, $g(x) = 3x$, $S = \{g\} \subset \mathrm{Com}_f(R_9)$,
$|\mathrm{Col}_{(R_9,f)}(D)| = |\mathrm{Col}_{(R_9,f)}(D')| = 9$.



つまり $D \not\sim D'$ $\square$

Main results

Main theorem (2) (U.)

D : virtual link diagram,
$(X,f) \sim (X,g)$, つまり $^{\exists}\phi \in \mathrm{Aut}(X)$ s.t. $f = \phi^{-1} \circ g \circ \phi$.
このとき $\Phi_\phi :\ \mathrm{Com}_f(X) \to \mathrm{Com}_g(X)$; $h \mapsto \phi h \phi^{-1}$ とすると,
$^{\forall}S \subset \mathrm{Com}_f(X)$, $VQ^{S}_{(X,f)}(D) \cong VQ^{\Phi_\phi(S)}_{(X,g)}(D)$.

Remark

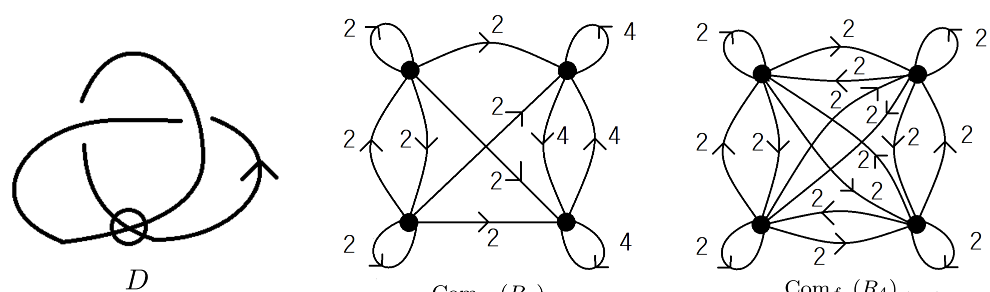
特に Φ_ϕ は全単射なので, $VQ^{\mathrm{Com}_f(X)}_{(X,f)}(D) \cong VQ^{\mathrm{Com}_g(X)}_{(X,g)}(D)$.
つまり $VQ^{\mathrm{Com}_f(X)}_{(X,f)}(D)$ は virtual quandle の同値類の不変量である.

Remark

$VQ^{\mathrm{Com}_f(X)}_{(X,f)}(D)$ は $|\mathrm{Col}_{(X,f)}(D)|$ より真に強い virtual quandle の同値類の不変量である.

Example of virtual quandle coloring quivers (3)

D : virtual trefoil diagram.
$(R_4, f_1), (R_4, f_2)$: virutual quandles, $f_1(x) = 3x + 2$, $f_2(x) = x + 2$,
$|\mathrm{Col}_{(R_4,f_1)}(D)| = |\mathrm{Col}_{(R_4,f_2)}(D)| = 4$.



D $\qquad VQ^{\mathrm{Com}_{f_1}(R_4)}_{(R_4,f_1)}(D)$ $\qquad VQ^{\mathrm{Com}_{f_2}(R_4)}_{(R_4,f_2)}(D)$

つまり $(R_4, f_1) \nsim (R_4, f_2)$ $\square$

Thank you for your attention.

Modules over geometric quandles and representations of Lie-Yamaguti algebras

Nobuyoshi Takahashi

Abstract. We study quandle modules over geometric quandles Q, i.e. quandles endowed with geometric structures. In the case Q is a regular s-manifold, we exhibit how modules over Q are related with representations of Lie-Yamaguti algebras.
As an application, we classify rank 1 linear modules over a general conjugacy class in $\mathrm{SL}(2, \mathbb{C})$.

1 Introduction

We consider quandles endowed with a geometric structure and call them *geometric quandles*. More specifically, topological quandles, smooth quandles and quandle varieties are defined as quandles with the structure of a topological space, smooth manifold, and algebraic variety, respectively. We will mainly be concerned with an especially nice class of smooth quandles, called *regular s-manifolds*, which has been studied as a generalization of the notion of a symmetric space ([K80], [F77]).

From the algebraic point of view, it is expected that the notion of a module over a quandle plays an important role, as in the group and ring theory. It would also provide a natural framework for the cohomology theory. The definition of a general quandle module was given by Jackson ([Ja05]; see also [AG03]). For a topological quandle Q, the notion of a module over Q was defined in [EM16], and it readily generalizes to other geometric quandles.

In this note, we explain how a quandle module over a regular s-manifold can be described by using infinitesimal data.

2 Definitions

Definition 1. A *smooth quandle* is a smooth manifold Q equipped with a quandle operation $\rhd$ such that $Q \times Q \to Q \times Q; (q, r) \mapsto (q, q \rhd r)$ is a diffeomorphism.

A regular s-manifold is a smooth quandle for which $1 - d_x s_x \in \mathrm{End}(T_x Q)$ is invertible for any $x \in Q$.

Among the fundamental results on a regular s-manifold Q is the fact that Q can be described as a homogeneous space G/H. Furthermore, we can endow $T = T_q Q$ with the structure of an "infinitesimal s-manifold," which determines Q locally, just as in the correspondence of Lie groups and Lie algebras.

Definition 2. ([Y69, §1], [K80, Definition III.20]) (1) A *Lie-Yamaguti algebra* over a field k is a triplet $(T, *, [\,])$, where T is a finite dimensional k-vector space, $* : T \times T \to T$ is a bilinear operation, and $[\,] : T \times T \times T \to T$ is a trilinear operation, such that the following hold:

(LY1) $x * x = 0$.

(LY2) $[x, x, y] = 0$.

(LY3) $[x, y, z] + [y, z, x] + [z, x, y] + (x * y) * z + (y * z) * x + (z * x) * y = 0$.

(LY4) $[x * y, z, w] + [y * z, x, w] + [z * x, y, w] = 0$.

(LY5) $[x, y, z * w] = [x, y, z] * w + z * [x, y, w]$.

(LY6) $[x, y, [z, v, w]] = [[x, y, z], v, w] + [z, [x, y, v], w] + [z, v, [x, y, w]]$.

(2) An *infinitesimal s-manifold* is a pair (T, σ) of a Lie-Yamaguti algebra T and a linear map $\sigma : T \to T$ satisfying the following.

(ISM0) Both σ and $id_T - \sigma$ are invertible.

(ISM1) $\sigma(x * y) = \sigma(x) * \sigma(y)$.

(ISM2) $\sigma([x, y, z]) = [\sigma(x), \sigma(y), \sigma(z)]$.

(ISM3) $\sigma([x, y, z]) = [x, y, \sigma(z)]$.

As for quandle modules, we give the definition of a *linear* quandle module over a smooth quandle Q, for simplicity. Let $\pi_i : Q^2 \to Q$ and $p_i : Q^3 \to Q$ denote the i-th projection maps, $p_{ij} = (p_i, p_j) : Q^3 \to Q^2$, $\mu : Q^2 \to Q$ the quandle operation and $\mu_{ij} := \mu \circ p_{ij} : Q^3 \to Q$.

Definition 3. (1) A *linear quandle module* over Q is a triplet $(\mathcal{A}, \eta, \tau)$, where $\mathcal{A}$ is a vector bundle, $\eta : \pi_2^*\mathcal{A} \to \mu^*\mathcal{A}$ is an isomorphism over $Q \times Q$ and $\tau : \pi_1^*\mathcal{A} \to \mu^*\mathcal{A}$ is a homomorphism over $Q \times Q$, satisfying the following conditions:

(a) $(p_1, \mu_{23})^*\eta \circ p_{23}^*\eta = (\mu_{12}, \mu_{13})^*\eta \circ p_{13}^*\eta$,

(b) $(p_1, \mu_{23})^*\eta \circ p_{23}^*\tau = (\mu_{12}, \mu_{13})^*\tau \circ p_{12}^*\eta$,

(c) $(p_1, \mu_{23})^*\tau = (\mu_{12}, \mu_{13})^*\eta \circ p_{13}^*\tau + (\mu_{12}, \mu_{13})^*\tau \circ p_{12}^*\tau$,

(d) $\Delta^*\eta + \Delta^*\tau = id_{\mathcal{A}}$, where $\Delta : Q \to Q \times Q$ is the diagonal map.

(2) Let Q be a connected regular s-manifold and $(\mathcal{A}, \eta, \tau)$ a linear quandle module over Q. We say $\mathcal{A}$ is *regular* if $id_{\mathcal{A}_q} - \eta_{qq} = \tau_{qq} : \mathcal{A}_q \to \mathcal{A}_q$ is invertible for any (or, equivalently, some) $q \in Q$.

On the infinitesimal side, we can consider representations of an infinitesimal s-manifold.

Definition 4 ([Y69])**.** A *representation of a Lie-Yamaguti algebra T* is a quadruplet $(V, \rho, \delta, \theta)$, where V is a vector space, $\rho : T \to \mathrm{End}(V)$ is a linear map and $\delta, \theta : T \times T \to \mathrm{End}(V)$ are bilinear maps, such that the following hold for any $x, y, z, w \in T$.

(RLY1) $\delta(x,y)+\theta(x,y)-\theta(y,x)=[\rho(x),\rho(y)]-\rho(x*y)$.

(RLY2) $\theta(x,y*z)-\rho(y)\theta(x,z)+\rho(z)\theta(x,y)=0$.

(RLY3) $\theta(x*y,z)-\theta(x,z)\rho(y)+\theta(y,z)\rho(x)=0$.

(RLY4) $\theta(z,w)\theta(x,y)-\theta(y,w)\theta(x,z)-\theta(x,[y,z,w])+\delta(y,z)\theta(x,w)=0$.

(RLY5) $[\delta(x,y),\rho(z)]=\rho([x,y,z])$.

(RLY6) $[\delta(x,y),\theta(z,w)]=\theta([x,y,z],w)+\theta(z,[x,y,w])$.

Definition 5. A *representation of an infinitesimal s-manifold* (T,σ) is given by data $(V,\rho,\delta,\theta,\psi)$ where (V,ρ,δ,θ) is a representation of T and $\psi\in\mathrm{End}(V)$ is an invertible linear transformation satisfying the following for any $x,y\in T$.

(RISM1) $\rho(\sigma(x))=\psi\circ\rho(x)\circ\psi^{-1}$.

(RISM2) $\theta(x,\sigma(y))=\psi\circ\theta(x,y),\ \ \theta(\sigma(x),y)=\theta(x,y)\circ\psi^{-1}$.

(RISM3) $\delta(x,y)=\psi\circ\delta(x,y)\circ\psi^{-1}$.

3 Main theorem

Let us state our main theorem. Similar statements hold also in the complex analytic case.

Theorem 1 ([Ta21])**.** *Let Q be a connected regular s-manifold and $q\in Q$ a point.*

Given a regular quandle module $\mathcal{A}$ over Q, there is a natural structure of a regular representation of (T_qQ,d_qs_q) on $\mathcal{A}_q$.

This gives a faithful functor between the following categories:

- *The category $\mathbf{Mod}^r_{\mathbb{R}}(Q)$ of regular quandle modules $(\mathcal{A},\eta,\tau)$ over $(Q,\triangleright)$.*
- *The category $\mathbf{Rep}^r(T_qQ,d_qs_q)$ of regular representations $(V,\rho,\delta,\theta,\psi)$ of the infinitesimal s-manifold $(T_qQ,*,[\ ],d_qs_q)$.*

If Q is simply-connected, it is an equivalence.

For the proof, we look at the following correspondences:

- Regular quandle modules over Q and "vector bundle objects over Q".
- Regular representations of T_qQ and split extensions by abelian ideals.

Then we use the fact that the assignment $\mathcal{T}:Q\mapsto T_qQ$ extends to a functor, i.e. a quandle homomorphism induces a homomorphism of infinitesimal s-manifolds, although the proof is not so straightforward.

As an application, we can give a classification of regular quandle modules of rank 1 over a general conjugacy class Q in $\mathrm{SL}(2,\mathbb{C})$.

References

[AG03] Andruskiewitsch, N., and M. Graña, *From racks to pointed Hopf algebras*, Adv. Math. **178** (2003), 177–243.

[EM16] Elhamdadi, M., and E. M. Moutuou, *Foundations of topological racks and quandles*, J. Knot Theory Ramifications **25** (2016), 1640002, 17 pp.

[F77] Fedenko, A. S., "Prostranstva s simmetriyami"(Russian) ("Spaces with symmetries"), Izdat. Belorussk. Gos. Univ., Minsk, 1977.

[Ja05] Jackson, N., *Extensions of racks and quandles*, Homology Homotopy Appl. **7** (2005), 151–167.

[Jo82] Joyce, D., *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982), 37–65.

[K80] Kowalski, O., "Generalized Symmetric Spaces," Lecture Notes in Mathematics, 805, Springer-Verlag, 1980.

[M82] Matveev, S. V., *Distributive groupoids in knot theory* (Russian), Mat. Sb. **119** (1982), 78–88. (English translation: Math. USSR-Sb. **47** (1984), 73–83).

[Ta21] Takahashi, N., *Modules over Geometric Quandles and Representations ofLie-Yamaguti Algebras*, to appear in Journal of Lie Theory.

[Y69] Yamaguti, K., *On cohomology groups of general Lie triple systems*, Kumamoto J. Sci. Ser. A **8** (1967/1969), 135–146.

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カンドル空間上の加群と Lie-山口代数の表現

高橋 宣能

2020.12.17

Overview

1. カンドル上の加群
   - 加群と拡大、加法群対象
2. カンドル空間
   - カンドル演算付きの位相空間、可微分多様体、複素多様体、代数多様体...
   - 正則 s 多様体とLie-山口代数
3. カンドル空間上の加群
   - 正則 s 多様体上の正則な加群と Lie-山口代数の表現
   - 例: $\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\text{の共役類}\right)$ 上の線形な 1 階の加群

プレプリント arXiv:2010.05564

記法

以下、「左から作用する」記法で書く。

Definition

カンドル (quandle) とは、以下を満たす二項演算 $\triangleright$ の与えられた集合 Q.

(1) $q \triangleright q = q$

(2) $q \triangleright (-)$ は全単射

(3) $q \triangleright (r \triangleright s) = (q \triangleright r) \triangleright (q \triangleright s)$

Examples: Conjugation quandle, φ-space

群 G を共役操作

$$g \triangleright h := ghg^{-1}$$

によりカンドルと見たものを $\mathrm{Conj}(G)$ と書く。

共役類の和集合などもカンドル。

G を群、$\varphi \in \mathrm{Aut}(G)$, H を $G^{\varphi} := \{g \in G \mid \varphi(g) = g\}$ の部分群とする。
$\Rightarrow$ G/H は $xH \triangleright_{\varphi} yH := x\varphi(x^{-1}y)H$ によりカンドル

G が Lie 群で $H \supseteq (G^{\varphi})_0$ (G^{φ} の e での連結成分)
のとき G/H を φ 空間と呼ぶ。

Transitive quandles

Definition

- 自己同型群 $\mathrm{Aut}_{\triangleright}(Q)$
- 内部自己同型群　$\mathrm{Inn}(Q) := \langle s_q \rangle$
- Q が推移的(あるいは「代数的に連結」)とは $\mathrm{Inn}(Q)$ の Q への作用が推移的であること

Proposition

Q が推移的ならば、$q \in Q$ に対して

$$Q \cong (\mathrm{Inn}(Q)/\mathrm{Inn}(Q)_q, \triangleright_\varphi),$$

φ は $\mathrm{Inn}(Q)$ の自己同型 $g \mapsto s_q \circ g \circ s_q^{-1}$.
$(\overline{x} \triangleright_\varphi \overline{y} = \overline{x\varphi(x^{-1}y)} = \overline{xs_qx^{-1}ys_q^{-1}}.)$

Quandle modules

Definition (Andruskiewitsch-Graña, Jackson)

Q 上のカンドル加群とは、以下のような組:

$$((A_x)_{x\in Q}, (\eta_{x,y})_{x,y\in Q}, (\tau_{x,y})_{x,y\in Q}).$$

各 $x \in Q$ に対し A_x は 加法群
$\eta_{x,y} : A_y \to A_{x\triangleright y}$ は同型 $(\forall x, y \in Q)$,
$\tau_{x,y} : A_x \to A_{x\triangleright y}$ は準同型 $(\forall x, y \in Q)$, ただし

(1) $\eta_{x,y\triangleright z}\eta_{y,z} = \eta_{x\triangleright y,x\triangleright z}\eta_{x,z}$,

(2) $\eta_{x,y\triangleright z}\tau_{y,z} = \tau_{x\triangleright y,x\triangleright z}\eta_{x,y}$,

(3) $\tau_{x,y\triangleright z} = \eta_{x\triangleright y,x\triangleright z}\tau_{x,z} + \tau_{x\triangleright y,x\triangleright z}\tau_{x,y}$,

(4) $\eta_{x,x} + \tau_{x,x} = id_{A_x}$.

Quandle modules(cont.)

$\eta_{x,y} : A_y \to A_{x \triangleright y}$, $\tau_{x,y} : A_x \to A_{x \triangleright y}$

(1) $\underline{\eta_{x,y \triangleright z}\eta_{y,z} = \eta_{x \triangleright y, x \triangleright z}\eta_{x,z}}$,

(2) $\eta_{x,y \triangleright z}\tau_{y,z} = \tau_{x \triangleright y, x \triangleright z}\eta_{x,y}$,

(3) $\tau_{x,y \triangleright z} = \eta_{x \triangleright y, x \triangleright z}\tau_{x,z} + \tau_{x \triangleright y, x \triangleright z}\tau_{x,y}$,

(4) $\eta_{x,x} + \tau_{x,x} = id_{A_x}$.

たとえば (1) は以下の可換性:

$$\begin{tikzcd} A_z \arrow[r, "\eta_{y,z}"] \arrow[d, "\eta_{x,z}"] & A_{y \triangleright z} \arrow[d, "\eta_{x,y \triangleright z}"] \\ A_{x \triangleright z} \arrow[r, "\eta_{x \triangleright y, x \triangleright z}"] & A_{x \triangleright (y \triangleright z)} = A_{(x \triangleright y) \triangleright (x \triangleright z)} \end{tikzcd}$$

(2), (3) の項も $A_y \to A_{x \triangleright (y \triangleright z)}$, $A_x \to A_{x \triangleright (y \triangleright z)}$ として考えられるものを尽くしており、それなりに自然な条件。

Quandle modules: Example 1

素朴には次のようなものが考えられる:

- 加法群 A, および
- 「作用」 $\triangleright_A : Q \times A \to A$: Q の元ごとに全単射

で、加法性と以下を満たすもの:

$$x \triangleright_A (y \triangleright_A a) = (x \triangleright y) \triangleright_A (x \triangleright_A a) \qquad x, y \in Q, a \in A.$$

言い換えると、$\mathrm{As}(Q) := \langle g_x \ (x \in Q) \mid g_{x \triangleright y} = g_x g_y g_x^{-1} \rangle$ 上の加群。

($\because$ $t_x := x \triangleright_A (-)$ として $t_x t_y = t_{x \triangleright y} t_x$, すなわち $t_{x \triangleright y} = t_x t_y t_x^{-1}$)

これは、上の定義の特別な場合と考えることができる:

$$\eta_{x,y}(a) := x \triangleright_A a, \quad \tau_{x,y}(a) := a - (x \triangleright y) \triangleright_A a.$$

Quandle modules: Example 1(cont.)

$$x \rhd_A (y \rhd_A a) = (x \rhd y) \rhd_A (x \rhd_A a) \qquad x, y \in Q, a \in A,$$

$$\eta_{x,y}(a) := x \rhd_A a, \quad \tau_{x,y}(a) := a - (x \rhd y) \rhd_A a.$$

たとえば、条件 (2):

$$\begin{aligned}
\eta_{x,y \rhd z}\tau_{y,z}(a) &= x \rhd (a - (y \rhd z) \rhd_A a) \\
&= x \rhd_A a - x \rhd_A ((y \rhd z) \rhd_A a) \\
&= x \rhd_A a - (x \rhd (y \rhd z)) \rhd_A (x \rhd_A a), \\
\tau_{x \rhd y, x \rhd z}\eta_{x,y}(a) &= \tau_{x \rhd y, x \rhd z}(x \rhd_A a) \\
&= x \rhd_A a - ((x \rhd y) \rhd (x \rhd z)) \rhd_A (x \rhd_A a)
\end{aligned}$$

一般に、群 G 上の加群 A とカンドル準同型 $f : Q \to \mathrm{Conj}(G)$ ($\Leftrightarrow$ 群準同型 $\mathrm{As}(Q) \to G$) について

$$\eta_{x,y} := f(x), \quad \tau_{x,y} := 1 - f(x \rhd y)$$

で Q 上の加群が定まる。(上は $G = \mathrm{As}(Q)$ の場合)

Quandle modules(cont. 2)

- (ある種の) 代数 T 上の加群 (T の表現) は、$T \oplus V$ 上の代数としての構造で、T を部分代数、V をイデアルとし、V 上で演算が 0 であるものと対応すべき (Eilenberg)
- 代数系 T 上の加群は T 上の「加法群対象」と対応すべき (Beck 加群)

例: 可換環 R 上の加群 M に対し、$\tilde{R} := R \oplus M$ 上で

$$(r_1, m_1)(r_2, m_2) := (r_1 r_2, r_1 m_2 + r_2 m_1)$$

と定めると $\tilde{R}$ は可換環。

R は $\tilde{R}$ の部分環、M は $\tilde{R}$ のイデアル。

Quandle modules(cont. 3)

群 G 上の加群 M に対し、$\tilde{G} := G \times M$ 上で

$$(g_1, m_1)(g_2, m_2) := (g_1 g_2, m_1 + g_1 m_2)$$

と定めると $\tilde{G}$ は群。

射影 $\Pi : \tilde{G} \to G$ は群準同型: $\tilde{G}$ を「G 上の群 (G への準同型が与えられた群)」と見る。

「ファイバー毎の加法」$A : \tilde{G} \times_G \tilde{G} \to \tilde{G}$ も群準同型。ただし、

$$\tilde{G} \times_G \tilde{G} = \{(\tilde{g}_1, \tilde{g}_2) \in \tilde{G} \times \tilde{G} \mid \Pi(\tilde{g}_1) = \Pi(\tilde{g}_2)\}.$$

これは $G \times M \times M$ と同一視でき、群演算は

$$(g_1, m_1, m_1')(g_2, m_2, m_2') = (g_1 g_2, m_1 + g_1 m_2, m_1' + g_1 m_2'),$$

また $A(g, m, m') = (g, m + m')$ と定義される。

「零元」$Z : G \to \tilde{G}$, 「ファイバー毎の逆元を取る写像」$I : \tilde{G} \to \tilde{G}$ をあわせて、$(\tilde{G}, \Pi, Z, A, I)$ は

「G 上の群の圏における加法群対象」

Quandle modules(cont. 4)

カンドル Q 上の加群 $((A_x)_{x\in Q}, (\eta_{x,y})_{x,y\in Q}, (\tau_{x,y})_{x,y\in Q})$ に対し、$\mathcal{A} := \coprod_{x\in Q} A_x$ とおき、$a \in A_x$ を (x, a) と書く。

$$(x, a) \triangleright (y, b) := (x \triangleright y, \eta_{x,y}(b) + \tau_{x,y}(a)).$$

と定めると $\mathcal{A}$ はカンドル、Q への射影はカンドル準同型。
またファイバー毎の和 (等) もカンドル準同型。

この対応は、

- 「カンドル Q 上の加群の圏」と
- 「Q 上のカンドルの圏における加法群対象の圏」

の同値を与える。

前に与えた「群上の加群 $\mapsto$ カンドル上の加群」はこのような対応を通して得られるもの。

Quandle spaces

Definition

位相カンドル (resp. smooth quandle, quandle variety) とは、
位相空間 (resp. C^∞ 多様体、代数多様体) Q と
演算 $\triangleright : Q \times Q \to Q$ の組で、以下を満たすもの。

(1) $q \triangleright q = q$,

(2) $Q \times Q \to Q \times Q; (q, r) \mapsto (q, q \triangleright r)$ は同相 (resp. 微分同相、双正則),

(3) $q \triangleright (r \triangleright s) = (q \triangleright r) \triangleright (q \triangleright s)$.

Examples

- 位相群/Lie 群/代数群 G に対し、$\mathrm{Conj}(G)$.
- G, H, φ を位相群 etc. として、$(G/H, \triangleright_\varphi)$.

Regular s-manifolds

正則 s 多様体とは

- (C^∞ 級、複素解析的、代数) 多様体 Q
- Q 上の C^∞ 級の (resp. 複素解析的、正則) 演算 $\triangleright$

の組で、カンドルの公理と以下を満たすもの:

$$\forall q \in Q,\ \ 1 - d_q s_q \in \mathrm{End}(T_q Q) \text{ は可逆 } (s_x(y) := x \triangleright y).$$

- 対称空間は正則 s 多様体
- 正則 s 多様体は「簡約等質空間」

Example

$\mathrm{GL}(2,\mathbb{C})$ において、行列 A の共役類 Q_A は、共役演算

$$X \triangleright Y := XYX^{-1}$$

によりカンドル多様体。特に、$\mathrm{diag}(\alpha,\beta) := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$
に対し $Q_{\mathrm{diag}(\alpha,\beta)} = Q_{\alpha,\beta}$ と書くと、$\alpha \neq \beta$ のとき
正則 s 多様体: $\mathrm{diag}(\alpha,\beta)$ での接空間の基底として
$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ が取れ、

$$d_A s_A(E) = AEA^{-1} = \alpha\beta^{-1}E, \quad d_A s_A(F) = \beta\alpha^{-1}F$$

$Q_{\alpha,\beta} \xrightarrow{\sim} Q_{\lambda\alpha,\lambda\beta}; X \mapsto \lambda X$ がわかる
$\rightsquigarrow \quad Q_\alpha := Q_{\alpha,\alpha^{-1}}$ を考える。

Regular s-manifolds are homogeneous

Theorem (Fedenko, Kowalski (1970's))

Q が連結な正則 s 多様体のとき、

1. Q は推移的である。
2. Q は φ 空間である。すなわち、ある *Lie* 群 G, $\varphi \in \mathrm{Aut}(G)$, $H \subseteq G^\varphi$ に対して $Q \cong (G/H, \triangleright_\varphi)$.
3. $x \in Q$ に対し接空間 $T = T_xQ$ は*Lie-*山口代数の構造を持ち、また自己同型 $\sigma = d_x s_x$*(*で、ある条件を満たすもの*)* がある。
Q は局所的に (T,σ) で定まる。

より一般に、連結で推移的な smooth quandle や quandle variety (標数 0) はほぼ φ 空間 (smooth: Katsumi Ishikawa, algebraic: T.)

Lie-Yamaguti algebra

Definition

Lie-山口代数とは、ベクトル空間 T と双線形・三重線形な演算

$$(x,y) \mapsto x * y, \qquad (x,y,z) \mapsto [x,y,z]$$

の組で、以下を満たすもの。

- $x * x = 0$, $[x,x,y] = 0$.
- $\mathcal{C}([x,y,z] + (x*y)*z) = 0$ (1st Bianchi identity), $\mathcal{C}$ は x,y,z についての巡回和。
- $\mathcal{C}[x*y,z,w] = 0$ (2nd Bianchi identity).
- $z \mapsto [x,y,z]$ は $*$, $[\,]$ に関する derivation.

Lie-Yamaguti algebra

Definition

無限小 s 多様体とは、Lie-山口代数 $(T,*,[\,])$ とその自己同型 σ で

- $[\sigma(x),\sigma(y),z] = [x,y,z]$
- $1-\sigma$ は可逆

を満たすもの。

(前の定理の「ある条件」とはこれ。)

Example

Q_α(= (diag(α, α^{-1}) の共役類)) に対する無限小 s 多様体 T は:

- $\dim_{\mathbb{C}} T = 2, \quad T = \langle E, F \rangle.$
- $E * E = E * F = F * E = F * F = 0$
- $[E, E, \bullet] = [F, F, \bullet] = 0.$
- $[E, F, E] = 2E, \quad [E, F, F] = -2F, \quad [F, E, \bullet] = -[E, F, \bullet].$
- $\sigma(E) = \alpha^2 E, \quad \sigma(F) = \alpha^{-2} F.$

これは、$Q_\alpha = \mathrm{SL}(2, \mathbb{C})/\{\det = 1$ の対角行列 $\}$ と書けることから、$sl(2, \mathbb{C}) = T \oplus \mathfrak{h}$, $T = \langle E, F \rangle$, $\mathfrak{h} = \langle H \rangle$ と分解し、$X, Y, Z \in T$ に対して

$$X * Y = [X, Y]_T, \quad [X, Y, Z] = [[X, Y]_{\mathfrak{h}}, Z]$$

などとしたもの。

Modules on a quandle space

Q: 位相カンドル/smooth quandle/quandle variety

Definition

Q 上の (線形)カンドル加群とは、次のような組 $(\mathcal{A}, \eta, \tau)$:

$\mathcal{A}$ $\quad$ Q 上のベクトル束や連接層
$\eta : p_2^* \mathcal{A} \to \mu^* \mathcal{A}$ $\quad$ $Q \times Q$ 上の同型
$\tau : p_1^* \mathcal{A} \to \mu^* \mathcal{A}$ $\quad$ $Q \times Q$ 上の準同型、ただし以下を満たす:

(1) $(p_1, \mu_{23})^* \eta \circ p_{23}^* \eta = (\mu_{12}, \mu_{13})^* \eta \circ p_{13}^* \eta$

(2) $(p_1, \mu_{23})^* \eta \circ p_{23}^* \tau = (\mu_{12}, \mu_{13})^* \tau \circ p_{12}^* \eta$

(3) $(p_1, \mu_{23})^* \tau = (\mu_{12}, \mu_{13})^* \eta \circ p_{13}^* \tau + (\mu_{12}, \mu_{13})^* \tau \circ p_{12}^* \tau$

(4) $\Delta^* \eta + \Delta^* \tau = id_{\mathcal{A}}$ $\quad$ (Δ: 対角写像)

(位相カンドル上: Elhamdadi-Moutuou, コホモロジー: Elhamdadi-Saito-Zappala)

Example

$\mathrm{GL}(2,\mathbb{C})$ の標準的表現 $\mathbb{C}^2$ から $Q := \mathrm{Conj}(\mathrm{GL}(2,\mathbb{C}))$ 上の加群 $V = (Q \times \mathbb{C}^2, \eta, \tau)$ が

$$\eta_{XY}(\boldsymbol{v}) = X\boldsymbol{v}, \quad \tau_{XY}(\boldsymbol{v}) = (1 - XYX^{-1})(\boldsymbol{v})$$

で定まる。これから定まる $Q \times \mathbb{C}^2$ 上のカンドル演算は $\left\{ \begin{pmatrix} X & \boldsymbol{v} \\ 0 & 1 \end{pmatrix} \right\}$ 上の conjugation と同じ:

$$\begin{aligned}
\begin{pmatrix} X & \boldsymbol{v} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} Y & \boldsymbol{w} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X & \boldsymbol{v} \\ 0 & 1 \end{pmatrix}^{-1} &= \begin{pmatrix} X & \boldsymbol{v} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} Y & \boldsymbol{w} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X^{-1} & -X^{-1}\boldsymbol{v} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} X & \boldsymbol{v} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} YX^{-1} & \boldsymbol{w} - YX^{-1}\boldsymbol{v} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} XYX^{-1} & \boldsymbol{v} + X(\boldsymbol{w} - YX^{-1}\boldsymbol{v}) \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} XYX^{-1} & \eta_{XY}(\boldsymbol{w}) + \tau_{XY}(\boldsymbol{v}) \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Example(cont.)

$Q_\alpha = \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \text{の共役類} \right)$ 上の加群 $\mathcal{S}_\lambda$ が

$$Q_\alpha \cong Q_{\lambda\alpha, \lambda\alpha^{-1}} \subseteq \mathrm{Conj}(\mathrm{GL}(2,\mathbb{C}))$$

で引き戻すことにより得られる:

$$\begin{aligned}
&(\mathcal{S}_\lambda)_X = \mathbb{C}^2 \\
&\eta_{XY}(\boldsymbol{v}) = \lambda X\boldsymbol{v}, \quad \tau_{XY}(\boldsymbol{v}) = (1 - \lambda XYX^{-1})(\boldsymbol{v})
\end{aligned}$$

ここまでのまとめ:

カンドル上の加群: $((A_x), (\eta_{xy}), (\tau_{xy}))$. $\mathcal{A} := \coprod A_x$ 上

$$(x, a) \triangleright (y, b) := (x \triangleright y, \tau_{xy}(a) + \eta_{xy}(b))$$

として、カンドルのある種の拡大に対応

正則 s 多様体: 特に性質の良いカンドル空間

- 正則 s 多様体は「無限小 s 多様体」と対応 (Lie-山口代数 + ある条件を満たす自己同型)

正則 s 多様体上の加群は?

Representations of Lie-Yamaguti algebra

Definition (山口, 1969)

Lie-山口代数 T の表現とは、ベクトル空間 V と線形/双線形写像 $\rho : T \to \mathrm{End}(V)$, $\theta : T \times T \to \mathrm{End}(V)$ の組で以下を満たすもの
ただし $\delta(x, y) := [\rho(x), \rho(y)] - \rho(x * y) - (\theta(x, y) - \theta(y, x))$.

- $\theta(x, y * z) = \rho(y)\theta(x, z) - \rho(z)\theta(x, y)$,
  $\theta(x * y, z) = \theta(x, z)\rho(y) - \theta(y, z)\rho(x)$,
- $\theta(z, w)\theta(x, y) - \theta(y, w)\theta(x, z) - \theta(x, [y, z, w]) + \delta(y, z)\theta(x, w) = 0$,
- $[\delta(x, y), \rho(z)] = \rho([x, y, z])$,
- $[\delta(x, y), \theta(z, w)] = \theta([x, y, z], w) + \theta(z, [x, y, w])$.

T の表現は、T の abelian ideal による split extension に対応。

Quandle modules on regular s-manifolds

Definition

無限小 s 多様体 $(T, *, [\ ], \sigma)$ の表現とは、T の表現 V と $\psi \in \mathrm{GL}(V)$ の組 (V, ψ) で

- $\rho(\sigma(x)) = \psi \circ \rho(x) \circ \psi^{-1}$
- $\theta(x, \sigma(y)) = \psi \circ \theta(x, y),\ \ \theta(\sigma(x), y) = \theta(x, y) \circ \psi^{-1}$

を満たすもの (σ との「可換性」$+\alpha$)。

(有限次元の) 表現が正則: $1 - \psi$ が可逆

Definition

正則 s 多様体上の加群 $(\mathcal{A}, \eta, \tau)$ が正則とは、$id_{\mathcal{A}_x} - \eta_{x,x} : \mathcal{A}_x \to \mathcal{A}_x$ が可逆であること。

Quandle modules on regular s-manifolds(2)

Theorem

Q を正則 s 多様体、$q \in Q$ とするとき、
Q 上の正則なカンドル加群 $(\mathcal{A}, \eta, \tau)$ から
(T_qQ, d_qs_q) の正則な表現 $(\mathcal{A}_q, \eta_{qq})$ が定まる。
この対応は忠実な関手。

Q が連結かつ単連結な (*smooth* または複素解析的) 正則 s 多様体ならば、同値関手。

証明:

- 正則な加群と「Q 上の正則 s 多様体の圏におけるベクトル空間対象」の対応
- 正則な表現と「abelian ideal による split extension」の対応

および次の対応 (+ もう少し議論) を用いる。

Quandle modules on regular s-mfds(4)

Proposition (Fedenko?)

Q, Q' を正則 s 多様体、Q は単連結、$x \in Q$, $x' \in Q'$ とするとき、以下の集合に一対一対応がある:

- $\{$ 準同型 $f : Q \to Q'$ で $f(x) = x'$ となるもの $\}$
- $\{$ 無限小 s 多様体の準同型 $(T_x Q, d_x s_x) \to (T_{x'} Q', d_{x'} s_{x'})\}$.

$Q \to Q'$ が全射ならば $\mathrm{Inn}(Q) \to \mathrm{Inn}(Q')$ があるが、一般にはもう少し複雑。

Examples

$Q_\alpha(= (\mathrm{diag}(\alpha, \alpha^{-1})$ の共役類$))$ 上の線形な rank 1 加群を考える。ただし $\alpha^4 \neq 1$ とする。

対応する無限小 s 多様体 (T, σ) の 1 次元表現 $(\mathbb{C}, \rho, \theta, \psi)$ は$\cdots$

- $(\mathbb{C}, \rho, \theta)$ は T の表現、$\psi : \mathbb{C} \to \mathbb{C}$ は線形同型
- $\rho(\sigma(x)) = \psi \circ \rho(x) \circ \psi^{-1}$
- $\theta(x, \sigma(y)) = \psi \circ \theta(x, y)$, $\theta(\sigma(x), y) = \theta(x, y) \circ \psi^{-1}$

$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ として $T = \langle E, F \rangle$, $\sigma(E) = \alpha^2 E, \sigma(F) = \alpha^{-2} F$

ψ, $\rho(E)$ などはスカラーと考えられるから

$$\rho(\sigma(E)) = \psi \circ \rho(E) \circ \psi^{-1} = \rho(E),$$

左辺は $\alpha^2 \rho(E)$ だから $\rho(E) = 0$. 同様に考えて $\rho \equiv 0$.
同様にして、$\theta(E, E) = \theta(F, F) = 0$.

Examples

$\alpha^4 \neq 1$, $(\mathbb{C}, \rho, \theta)$ は T の表現、$\psi : \mathbb{C} \to \mathbb{C}$ は線形同型、

- $\rho(\sigma(x)) = \psi \circ \rho(x) \circ \psi^{-1}$
- $\theta(x, \sigma(y)) = \psi \circ \theta(x, y)$, $\theta(\sigma(x), y) = \theta(x, y) \circ \psi^{-1}$

$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ として $T = \langle E, F \rangle$, $\sigma(E) = \alpha^2 E, \sigma(F) = \alpha^{-2} E$

$$\theta(E, F) = \theta(\sigma(E), F) \circ \psi = \alpha^2 \psi \cdot \theta(E, F)$$

よって $\theta(E, F) = 0$ または $\psi = \alpha^{-2}$.
同様に $\theta(F, E) = 0$ または $\psi = \alpha^2$.

Examples

Lie-山口代数の表現という条件も使ってゆくと、次のいずれかに同型:

1. $\rho \equiv 0$, $\theta \equiv 0$, $\psi \in \mathbb{C} \setminus \{0\}$.
2. $\rho \equiv 0$, $\theta(E, E) = \theta(F, F) = \theta(E, F) = 0$, $\theta(F, E) = 1$, $\psi = \alpha^2$.
3. $\rho \equiv 0$, $\theta(E, E) = \theta(F, F) = \theta(F, E) = 0$, $\theta(E, F) = -1$, $\psi = \alpha^{-2}$.

(1) は $Q_\alpha \times \mathbb{C}$ 上で $\eta \equiv \psi$, $\tau \equiv 1 - \psi$ としたものに対応。
$(Q_\alpha \to \mathrm{Conj}(\mathrm{GL}(1, \mathbb{C})); x \mapsto \psi$ に対応、「Alexander 加群」)

Examples

(2) $\rho \equiv 0$, $\theta(E,E) = \theta(F,F) = \theta(E,F) = 0$, $\theta(F,E) = 1$, $\psi = \alpha^2$ に対応する Q_α 上の加群 $\mathcal{A}$ を記述する ((3) は同様)。

前に与えた $\mathcal{S}_\lambda$ で $\lambda = \alpha$ としたもの

$$\mathcal{S}_\alpha = (Q_\alpha \times \mathbb{C}^2, \eta, \tau),$$
$$\eta_{XY} = \alpha X, \quad \tau_{XY} = 1 - \alpha XYX^{-1}$$

を考える (これは正則な加群でない)。

$\mathcal{A} := \{(X, \boldsymbol{v}) \in \mathcal{S}_\alpha \mid X\boldsymbol{v} = \alpha\boldsymbol{v}\}$ は $\mathcal{S}_\alpha$ の部分加群。
$\rightarrow$ これが (2) の表現に対応。

実は $\mathcal{S}_\alpha$ は完全可約でなく、また $\mathcal{B} := \mathcal{S}_\alpha/\mathcal{A}$ は無限小 s 多様体の表現に対応しない ($\psi = 1$ である表現は trivial だが $\mathcal{B}$ は trivial でない)

また、$\mathcal{A}$, $\mathcal{B}$ は $\mathrm{As}(Q_\alpha)$ の表現に対応しない ($\mathrm{As}(Q_\alpha)$ の表現は (1) のもの)

Polars of disconnected compact Lie groups

Makiko Sumi Tanaka

This presentation is based on the author's collaboration with Hiroyuki Tasaki.

A Riemannian symmetric space M is an Riemannian manifold equipped with the point symmetry s_x at each point x in M, that is, s_x is an involutive isometry of M and x is an isolated fixed point of s_x. A *polar* of a Riemannian symmetric space M with respect to x is a connected component of the fixed point set of s_x. A subset A of M is called an *antipodal set* if it holds that $s_x(y) = y$ for any points x, y in A. An antipodal set is finite. A compact Lie group G is a Riemannian symmetric space with respect to a biinvariant Riemannian metric. The point symmetry s_g at a point g in G is given by $s_g(h) = gh^{-1}g$ ($h \in G$). A maximal antipodal set of G containing the identity element is an abelian subgroup isomorphic to a product of some copies of $\mathbb{Z}_2$. A polar of G with respect to the identity element is simply called a polar of G.

Tasaki and the author classified maximal antipodal sets of some classical compact symmetric spaces and their quotient spaces in [4] by using their embeddings into compact connected Lie groups as polars. They also used their former results on the classification of maximal antipodal subgroups of classical compact Lie groups and their quotient groups in [3]. In order to proceed with the classification of maximal antipodal sets of other compact Riemannian symmetric spaces M, we need (i) to realize M as a polar of a disconnected compact Lie group G, and (ii) to classify maximal antipodal subgroups of G. Relating to (i), Tasaki and the author gave explicit descriptions of polars of disconnected compact Lie groups in [5]. Polars of connected compact Lie groups are well-understood by the detailed studies of Chen-Nagano [1] and Nagano [2]. In this presentation some results in [5] are explained.

References

[1] B.-Y. Chen and T. Nagano, Totally geodesic submanifolds of symmetric spaces, II, Duke Math. J. **45** (1978), 405–425.

[2] T. Nagano, The involutions of compact symmetric spaces, Tokyo J. Math. **11** (1988), 57–79.

[3] M. S. Tanaka and H. Tasaki, Maximal antipodal subgroups of some compact classical Lie groups, J. Lie Theory **27** (2017), 801–829.

[4] M. S. Tanaka and H. Tasaki, Maximal antipodal sets of compact classical symmetric spaces and their cardinalities I, Differ. Geom. Appl. **73** (2020), 101682 (32 pages).

[5] M. S. Tanaka and H. Tasaki, Polars of disconnected compact Lie groups, submitted.

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非連結コンパクトLie群の極地

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研究集会「カンドルと対称空間」

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オンライン開催（**Zoom**）

田崎博之氏（筑波大学）との共同研究

内容

1. 研究の背景

2. 極大対蹠集合の分類における基本原理

3. 非連結コンパクト**Lie**群の極地

4. 応用例

1. 研究の背景

M：**Riemann**多様体

M：**Riemann対称空間** $\underset{\text{def}}{\Leftrightarrow}$ $\forall x \in M, \exists s_x : M$の等長変換　s.t. (i) $s_x \circ s_x = \mathrm{id}$, (ii) xはs_xの孤立不動点

s_x：xにおける点対称

例：ユークリッド空間E^n、球面S^n、射影空間P^n

Mが連結 $\Rightarrow$ s_xは一意的、

$$s_x \circ s_y = s_{s_x(y)} \circ s_x \quad (x, y \in M)\text{が成立}$$

連結**Riemann**対称空間はカンドル

$F(s_x, M) = \{y \in M | s_x(y) = y\}$の連結成分を$x$に関する極地**(polar)**という

$\{x\}$：自明な極地

$F(s_x, E^n) = \{x\}, F(s_x, S^n) = \{x, -x\}$,

$F(s_x, P^n) = \{x\} \cup P^{n-1}$

Mが非コンパクト型 $\Rightarrow F(s_x, M) = \{x\}$

以下ではMがコンパクトの場合を考える

A：Mの部分集合

A：対蹠集合**(antipodal set)**

$\underset{\text{def}}{\Leftrightarrow}$ $\forall x, y \in A, \ \ s_x(y) = y$

対蹠集合は有限集合

Mの**2-number**

$\#_2 M = \max\{|A| \mid A : \text{対蹠集合}\}$

例：$\{x, -x\}$ は S^n の極大な対蹠集合で $\#_2 S^n = 2$
$\{\langle e_1\rangle, \ldots, \langle e_{n+1}\rangle\}$ は P^n の極大な対蹠集合で
$\#_2 P^n = n+1$
Chen-Nagano (1988) 2-numberの詳細な研究

$N \subset M$：全測地的部分多様体（誘導計量に関する N の測地線は M の測地線）
M の点対称 s_x は x を始点とする測地線 $\gamma(t)$ に対して
$s_x(\gamma(t)) = \gamma(-t)$
$x \in N$ のとき s_x は N を保ち N の点対称を定める
$\Rightarrow$ N は誘導計量に関して**Riemann**対称空間

正次元の極地は全測地的部分多様体
正次元の極地はコンパクト**Riemann**対称空間
A：M の対蹠集合　　$x \in A$
$A \subset F(s_x, M) = \bigcup_{j=1}^{r} M_j^+$　　極地への分解
$A \cap M_j^+$ は M_j^+ の対蹠集合
$\#_2 M \leq \sum_{j=1}^{r} \#_2 M_j^+$
M：対称 R 空間 $\Rightarrow$ 等号成立（**Takeuchi 1989)**

コンパクト型**Hermite**対称空間 M の**2**つの実形 L_1, L_2 の交叉は離散的ならば対蹠集合
L_1, L_2 が M の等長変換で合同ならば交叉は大対蹠集合
(T.-Tasaki 2012)

対蹠集合 A は $|A| = \#_2 M$ のとき大対蹠集合 (**great antipodal set**)
例 : $\mathbb{C}P^1 = S^2$ はコンパクト型**Hermite**対称空間で大円 $\mathbb{R}P^1 = S^1$ はその実形（対合的反正則等長変換の不動点集合）
2つの異なる大円 S^1 は対蹠的な2点 $\{x, -x\}$ で交わり、これは S^1 の大対蹠集合
大対蹠集合は極大対蹠集合、逆は一般には成立しない
対称 R 空間の極大対蹠集合は大対蹠集合で等長変換を除いて一意的 **(T.-Tasaki 2013)**

目的 : コンパクト**Riemann**対称空間の極大対蹠集合の構造の理解、応用
そのためにコンパクト**Riemann**対称空間の極大対蹠集合を分類（現在進行中）
古典型コンパクト**Lie**群とその商群の極大対蹠部分群の分類 **(Griess 1991, Yu 2013, T.-Tasaki 2017)**
例外型コンパクト**Lie**群　G_2, F_4 については既知
いくつかの古典型コンパクト**Riemann**対称空間の極大対蹠集合の分類 **(T.-Tasaki 2020)**

コンパクト**Lie**群Gには両側不変**Riemann**計量が存在して**Riemann**対称空間になる
G_0：単位連結成分
G_0上で点対称が$s_x(y) = xy^{-1}x$により一意的に定まる
Gの群構造だけで定まっているので自然にG全体に拡張できる
例：$U(n), SU(n), O(n), SO(n), Sp(n)$
A：Gの対蹠集合，$e \in A$
$\Rightarrow \forall x, y \in A, x^2 = y^2 = e, xy = yx$
$e \in A$が極大対蹠集合ならばAは部分群で$\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$と同型

$\#_2 G = 2^r$，rはGの2-**rank**
例：$U(n)$の極大対蹠部分群は$\{\mathrm{diag}(\pm 1, \cdots, \pm 1)\}$に共役、$\#_2 U(n) = 2^n$
$U(4)/\{\pm 1_4\}$には元の個数が2^5と2^4の極大対蹠部分群がある、$\#_2 U(4)/\{\pm 1_4\} = 2^5$

2. 極大対蹠集合の分類における基本原理

コンパクト**Lie**群の極大対蹠部分群の分類を利用してコンパクト**Riemann**対称空間の極大対蹠集合を分類する際の基本原理について述べる

G：コンパクト**Lie**群　G_0：単位連結成分　e：単位元
eに関する極地を単にGの極地とよぶ
$g \in G, I_g(x) = gxg^{-1} (x \in G)$　I_gはGの等長変換
M：Gの極地　　$x_0 \in M$
$M = \{I_g(x_0) | g \in G_0\}$

$\mathrm{Iso}_0(M) = \{I_g|_M \mid g \in G_0\}$
（$\mathrm{Iso}_0(M)$：Mの等長変換群の単位連結成分）
$A \subset M$：対蹠集合
$A \cup \{e\}$：Gの対蹠集合
$\exists \tilde{A}$：極大対蹠部分群 $A \cup \{e\} \subset \tilde{A}$
AがMで極大　$\Rightarrow$ $A = M \cap \tilde{A}$
$B_1, \ldots, B_k$：Gの極大対蹠部分群のG_0共役類の代表元
$\exists g \in G_0, 1 \leq \exists s \leq k$ s.t. $\tilde{A} = I_g(B_s)$
$A = M \cap \tilde{A} = M \cap I_g(B_s) = I_g(M \cap B_s)$
Aは$M \cap B_s$に$\mathrm{Iso}_0(M)$合同

Mの極大対蹠集合の$\mathrm{Iso}_0(M)$合同類の代表元は
$M \cap B_1, \ldots, M \cap B_k$のいずれか

[T.-Tasaki 2017]で古典型コンパクト**Lie**群とその商群の極大対蹠部分群の共役類を求め、代表元の具体的表示を与えた。**[T.-Tasaki 2020]**で上記の基本原理を使って古典型コンパクト**Lie**群（の商群）の極地として実現されるコンパクト**Riemann**対称空間Mの極大対蹠集合の$\mathrm{Iso}_0(M)$合同類を求め、代表元の具体的表示を与えた。連結コンパクト**Lie**群の極地としては実現されないコンパクト**Riemann**対称空間がある。

3. 非連結コンパクトLie群の極地

極地が１点$\{x\}$のとき極という
G：コンパクト**Lie**群　G_0：単位連結成分
$x \in F(s_e, G)$　$G^+(x)$：xを含む極地
$G^+(x) = \{I_g(x) \mid g \in G_0\}$
極の全体　$Z_G(G_0) \cap F(s_e, G)$
（$Z_G(G_0)$はGにおけるG_0の中心化群）
$F(s_e, G) = (Z_G(G_0) \cap F(s_e, G)) \cup \bigcup_{i=1}^{k} G^+(x_i)$
$\mathrm{dim} G^+(x_i) > 0,\ G^+(x_i) \cap G^+(x_j) = \emptyset\ (i \neq j)$

$G = G_0 \cup \bigcup_{\lambda \in \Lambda} G_\lambda$　　連結成分への分解

$G_0 \cap F(s_e, G)$ については **Chen-Nagano** が研究

$G_\lambda \cap F(s_e, G)$ を調べる

$x_\lambda \in G_\lambda \cap F(s_e, G) \neq \emptyset \qquad G_\lambda = G_0 x_\lambda$

I_{x_λ} は G_0 の対合的自己同型写像

T_λ : $F(I_{x_\lambda}, G_0)$ の単位連結成分の極大トーラス

G_0 の G_0 への捩れた共役作用 $g.h = ghI_{x_\lambda}(g)^{-1}$ の性質（**Hermann** 作用の性質）から次を得る

<u>命題１</u>　$G_\lambda = \bigcup_{g \in G_0} g(x_\lambda T_\lambda)g^{-1}$

$G_\lambda \cap F(s_e, G) = \bigcup_{g \in G_0} g\{x \in x_\lambda T_\lambda \,|\, x^2 = e\}g^{-1}$

$\{x \in x_\lambda T_\lambda \,|\, x^2 = e\}$ を決定し、G_0 共役軌道の具体的表示を与える

<u>命題２</u>　$G_\lambda \cap F(s_e, G) \neq \emptyset$ ならば

(1) $G_0 \cup G_\lambda$ は部分群

(2) $x_\lambda \in G_\lambda \cap F(s_e, G)$ に対して、$G_0 \cup G_\lambda$ は半直積 $G_0 \rtimes \langle I_{x_\lambda} \rangle$ に同型

$G_0 \rtimes \langle I_{x_\lambda} \rangle = \{(g, \mathrm{id}) \,|\, g \in G_0\} \cup \{(g, I_{x_\lambda}) \,|\, g \in G_0\}$
　連結成分への分解

命題２の証明：$G_\lambda G_\lambda = G_0 x_\lambda G_0 x_\lambda = G_0 G_0 = G_0$
$\varphi : G_0 \rtimes \langle I_{x_\lambda} \rangle \to G_0 \cup G_\lambda$を$\varphi(g, \mathrm{id}) = g, \varphi(g, I_{x_\lambda}) = gx_\lambda$で定義すると$\varphi$は**Lie**群の同型写像

G：連結コンパクト**Lie**群　　e：Gの単位元
σ：Gの対合的自己同型写像
$M = \{g \in G \mid \sigma(g) = g^{-1}\}$
$\widehat{e} = (e, \mathrm{id})$：$G \rtimes \langle \sigma \rangle$の単位元

命題３　$F(s_{\widehat{e}}, G \rtimes \langle \sigma \rangle) = (F(s_e, G), \mathrm{id}) \cup (M, \sigma)$
特に、(M, σ)の各連結成分は$G \rtimes \langle \sigma \rangle$の極地

命題３の証明：
$F(s_{\widehat{e}}, G \rtimes \langle \sigma \rangle) = F(s_{\widehat{e}}, (G, \mathrm{id})) \cup F(s_{\widehat{e}}, (G, \sigma))$
$F(s_{\widehat{e}}, (G, \mathrm{id})) = (F(s_e, G), \mathrm{id})$
$F(s_{\widehat{e}}, (G, \sigma)) = (M, \sigma)$が次からわかる
$g \in G,\ s_{\widehat{e}}(g, \sigma) = (g, \sigma) \Leftrightarrow$
$(g, \sigma) = (g, \sigma)^{-1} = (\sigma(g^{-1}), \sigma) \Leftrightarrow$
$\sigma(g) = g^{-1}$

4. 応用例

$U(n)$：ユニタリ群　　1_n：単位元

$$F(s_{1_n}, U(n)) = \{x \in U(n) \,|\, x^2 = 1_n\} = \bigcup_{i=0}^{n} \{g\, x_i\, g^{-1} \,|\, g \in U(n)\}$$

$$x_i = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{i}, \underbrace{1, \ldots, 1}_{n-i})$$

$U(n)$の極地

$\{1_n\}$, $\{-1_n\}$,

$U(n)/(U(i) \times U(n-i))$ $(1 \leq i \leq n-1)$

複素**Grassmann**多様体

$\tau(g) = \bar{g}$　　$U(n)$の対合的自己同型写像

$G = U(n) \rtimes \langle \tau \rangle$　　$\langle \tau \rangle = \{e, \tau\}$

$G = \{(g, e) \,|\, g \in U(n)\} \cup \{(g, \tau) \,|\, g \in U(n)\}$

連結成分への分解

Gの演算

$(g, e)(h, e) = (gh, e)$　　$(g, e)(h, \tau) = (gh, \tau)$

$(g, \tau)(h, e) = (g\tau(h), \tau)$　$(g, \tau)(h, \tau) = (g\tau(h), e)$

(g, e)をg, (g, τ)を$g\tau$とも書く

$G = U(n) \cup U(n)\tau$

$\tau g = (1_n, \tau)(g, e) = (\tau(g), \tau) = (\bar{g}, \tau) = \bar{g}\tau$

$\widehat{e} : G = U(n) \rtimes \langle \tau \rangle$の単位元

$F(s_{\widehat{e}}, G) = (F(s_{\widehat{e}}, G) \cap U(n)) \cup (F(s_{\widehat{e}}, G) \cap U(n)\tau)$

$F(s_{\widehat{e}}, G) \cap U(n) = F(s_{1_n}, U(n))$

$= \bigcup_{i=0}^{n} \{g\, x_i\, g^{-1} \mid g \in U(n)\}$（極地への分解）

極と複素**Grassmann**多様体

$F(s_{\widehat{e}}, G) \cap U(n)\tau$について命題1を利用して調べる

$T : F(\tau, U(n)) = O(n)$の極大トーラス

$U(n)\tau = \bigcup_{g \in U(n)} g(\tau T)g^{-1}$

$F(s_{\widehat{e}}, G) \cap U(n)\tau = \bigcup_{g \in U(n)} g\{x \in \tau T \mid x^2 = 1_n\}g^{-1}$

$\{x \in \tau T \mid x^2 = 1_n\}$を調べる

$$T = \left\{ \begin{bmatrix} R(\theta_1) & & & \\ & \ddots & & \\ & & R(\theta_k) & \\ & & & (1) \end{bmatrix} \,\middle|\, \theta_1, \ldots, \theta_k \in \mathbb{R} \right\}$$

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad k = \lfloor \tfrac{n}{2} \rfloor$$

$t \in T \quad \tau t = \bar{t}\tau = t\tau \quad (\tau t)^2 = \tau^2 t^2 = t^2$

$\{x \in \tau T \mid x^2 = 1_n\} = \tau\{t \in T \mid t^2 = 1_n\}$

$$= \tau \left\{ \left[\begin{matrix} \epsilon_1 1_2 & & & \\ & \ddots & & \\ & & \epsilon_k 1_2 & \\ & & & (1) \end{matrix} \right] \middle| \epsilon_1, \ldots, \epsilon_k = \pm 1 \right\}$$

・$t \in T, g \in U(n)$に対して$g(\tau t)g^{-1} = g\, t\, {}^t\! g\, \tau$

・$(i1_2)(-1_2)(i1_2) = 1_2$から$t \in T, t^2 = 1_n$に対して$\exists g \in U(n)$ s.t. $g\, t\, {}^t\! g = 1_n$

これらのことから$\forall t \in T, t^2 = 1_n$に対して次が成立

$\{g(\tau t)g^{-1} \mid g \in U(n)\} = \{g\, t\, {}^t\! g \mid g \in U(n)\}\tau$

$= \{g\, 1_n\, {}^t\! g \mid g \in U(n)\}\tau$

したがって

$F(s_{\hat{e}}, G) \cap U(n)\tau = \bigcup_{g \in U(n)} g\{x \in \tau T \mid x^2 = 1_n\}g^{-1}$

$= \{g(\tau t)g^{-1} \mid t \in T, t^2 = 1_n, g \in U(n)\}$

$= \{g\, 1_n\, {}^t\! g \mid g \in U(n)\}\tau$

これが命題3の記号のもとでの(M, σ)

$U(n)$の1_nにおけるイソトロピー部分群

$g\, 1_n\, {}^t\! g = 1_n \Leftrightarrow {}^t\! g = g^{-1} = {}^t\overline{g} \Leftrightarrow g \in O(n)$

$F(s_{\hat{e}}, G) \cap U(n)\tau \cong U(n)/O(n)$、特に連結

$U(n)/O(n)$は連結コンパクト**Lie**群の極地としては実現されない

On regular polytopes quandles

Ayumu Inoue

Recall that polyhedral quandles are defined relating to rotational symmetries of regular polyhedra. In a similar way, we have quandles related to rotational symmetries of some regular polytopes as follows. Consider the following three sets consisting of points in $\mathbb{R}^4$ whose convex hulls are respectively known as 16-, 24-, and 600-cells:

$$\begin{aligned} V_{16} &= \{\pm \boldsymbol{e}_1, \pm \boldsymbol{e}_2, \pm \boldsymbol{e}_3, \pm \boldsymbol{e}_4\}, \\ V_{24} &= \{\pm \boldsymbol{e}_i \pm \boldsymbol{e}_j \mid 1 \le i < j \le 4\}, \\ V_{600} &= \{\pm \boldsymbol{e}_1, \pm \boldsymbol{e}_2, \pm \boldsymbol{e}_3, \pm \boldsymbol{e}_4\} \cup \left\{ \frac{1}{2}(\pm \boldsymbol{e}_1 \pm \boldsymbol{e}_2 \pm \boldsymbol{e}_3 \pm \boldsymbol{e}_4) \right\} \\ & \qquad \cup \left\{ \frac{1}{2}\left(\pm \phi\, \boldsymbol{e}_{\sigma(1)} \pm \boldsymbol{e}_{\sigma(2)} \pm \phi^{-1} \boldsymbol{e}_{\sigma(3)}\right) \,\middle|\, \sigma \in A_4 \right\}. \end{aligned}$$

Here, $\boldsymbol{e}_i \in \mathbb{R}^4$ denotes the column vector whose j-th entry is δ_{ij}, ϕ the golden ratio $\left(1+\sqrt{5}\right)/2$, and A_4 the alternating group on $\{1,2,3,4\}$. Associated with $v \in V_C$ ($C \in \{16,\ 24,\ 600\}$), define the 4×4 matrix R_v as follows:

► $C = 16$

$$R_{\pm \boldsymbol{e}_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad R_{\pm \boldsymbol{e}_2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{\pm \boldsymbol{e}_3} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad R_{\pm \boldsymbol{e}_4} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

► $C = 24$

$$R_{\pm(\boldsymbol{e}_1+\boldsymbol{e}_2)} = R_{\pm(\boldsymbol{e}_1-\boldsymbol{e}_2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$R_{\pm(\boldsymbol{e}_3+\boldsymbol{e}_4)} = R_{\pm(\boldsymbol{e}_3-\boldsymbol{e}_4)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_{\pm(\boldsymbol{e}_1+\boldsymbol{e}_3)} = R_{\pm(\boldsymbol{e}_2+\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

$$R_{\pm(\boldsymbol{e}_1-\boldsymbol{e}_3)} = R_{\pm(\boldsymbol{e}_2-\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix},$$

$$R_{\pm(\boldsymbol{e}_1+\boldsymbol{e}_4)} = R_{\pm(\boldsymbol{e}_2-\boldsymbol{e}_3)} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$R_{\pm(\boldsymbol{e}_2+\boldsymbol{e}_3)} = R_{\pm(\boldsymbol{e}_1-\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

► $C = 600$

$$\begin{aligned} R_{\pm\boldsymbol{e}_1} &= R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1+\boldsymbol{e}_2+\phi^{-1}\boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1-\boldsymbol{e}_2-\phi^{-1}\boldsymbol{e}_3)} \\ &= R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1+\phi\boldsymbol{e}_2+\boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1-\phi\boldsymbol{e}_2-\boldsymbol{e}_3)} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & \phi & -\phi^{-1} \\ 0 & \phi & -\phi^{-1} & 1 \\ 0 & \phi^{-1} & -1 & -\phi \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} R_{\pm\boldsymbol{e}_2} &= R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\phi\boldsymbol{e}_2+\phi^{-1}\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\phi\boldsymbol{e}_2+\phi^{-1}\boldsymbol{e}_4)} \\ &= R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1+\phi^{-1}\boldsymbol{e}_2+\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1-\phi^{-1}\boldsymbol{e}_2+\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & \phi^{-1} & \phi \\ 0 & 2 & 0 & 0 \\ -\phi^{-1} & 0 & -\phi & 1 \\ \phi & 0 & -1 & -\phi^{-1} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} R_{\pm\boldsymbol{e}_3} &= R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\phi^{-1}\boldsymbol{e}_3-\phi\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\phi^{-1}\boldsymbol{e}_3-\phi\boldsymbol{e}_4)} \\ &= R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1+\phi\boldsymbol{e}_3-\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1-\phi\boldsymbol{e}_3-\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & 1 & 0 & -\phi \\ -1 & -\phi & 0 & -\phi^{-1} \\ 0 & 0 & 2 & 0 \\ -\phi & \phi^{-1} & 0 & 1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} R_{\pm\boldsymbol{e}_4} &= R_{\pm\frac{1}{2}(\boldsymbol{e}_2-\phi\boldsymbol{e}_3+\phi^{-1}\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_2-\phi\boldsymbol{e}_3-\phi^{-1}\boldsymbol{e}_4)} \\ &= R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_2-\boldsymbol{e}_3+\phi\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_2-\boldsymbol{e}_3-\phi\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} -\phi & 1 & \phi^{-1} & 0 \\ -1 & -\phi^{-1} & -\phi & 0 \\ -\phi^{-1} & -\phi & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \end{aligned}$$

$$
\begin{aligned}
&R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3+\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\phi^{-1}\boldsymbol{e}_3+\phi\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_2-\phi^{-1}\boldsymbol{e}_3-\boldsymbol{e}_4)} \\
&\quad = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1+\phi\boldsymbol{e}_2-\boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1-\boldsymbol{e}_2+\phi\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & 1 & 0 & \phi \\ 0 & 1 & -\phi & -\phi^{-1} \\ -\phi & -1 & -\phi^{-1} & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\boldsymbol{e}_2-\boldsymbol{e}_3-\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\phi^{-1}\boldsymbol{e}_2+\phi\boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1-\phi^{-1}\boldsymbol{e}_2-\boldsymbol{e}_4)} \\
&\quad = R_{\pm\frac{1}{2}(\boldsymbol{e}_2+\phi\boldsymbol{e}_3+\phi^{-1}\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1+\boldsymbol{e}_3-\phi^{-1}\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & \phi^{-1} & -\phi \\ -1 & -\phi^{-1} & \phi & 0 \\ 1 & 1 & 1 & 1 \\ -1 & \phi & 0 & -\phi^{-1} \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\phi^{-1}\boldsymbol{e}_2+\phi\boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_2+\boldsymbol{e}_3+\phi\boldsymbol{e}_4)} \\
&\quad = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1+\phi\boldsymbol{e}_3+\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1-\boldsymbol{e}_2-\phi\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & 0 & \phi & -1 \\ -\phi & -\phi^{-1} & 0 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & \phi & \phi^{-1} & 1 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\boldsymbol{e}_2-\boldsymbol{e}_3+\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\phi\boldsymbol{e}_2-\phi^{-1}\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1-\boldsymbol{e}_3+\phi^{-1}\boldsymbol{e}_4)} \\
&\quad = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1+\boldsymbol{e}_2-\phi^{-1}\boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_2+\phi^{-1}\boldsymbol{e}_3-\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ \phi^{-1} & 1 & 0 & -\phi \\ -\phi & 1 & -\phi^{-1} & 0 \\ 0 & -1 & -\phi & -\phi^{-1} \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3-\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\phi^{-1}\boldsymbol{e}_2-\phi\boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_2+\boldsymbol{e}_3-\phi\boldsymbol{e}_4)} \\
&\quad = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1+\boldsymbol{e}_2-\phi\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1-\phi\boldsymbol{e}_3+\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & \phi & -1 & 0 \\ 0 & -\phi^{-1} & -1 & -\phi \\ -\phi & 0 & 1 & -\phi^{-1} \\ -1 & -1 & -1 & 1 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\phi\boldsymbol{e}_2-\phi^{-1}\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1+\boldsymbol{e}_3+\phi^{-1}\boldsymbol{e}_4)} \\
&\quad = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_2+\phi^{-1}\boldsymbol{e}_3+\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1-\boldsymbol{e}_2+\phi^{-1}\boldsymbol{e}_3)} = \frac{1}{2}\begin{pmatrix} 1 & -\phi^{-1} & \phi & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & -\phi^{-1} & \phi \\ 1 & \phi & 0 & -\phi^{-1} \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\phi^{-1}\boldsymbol{e}_3+\phi\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_2-\phi^{-1}\boldsymbol{e}_3+\boldsymbol{e}_4)} \\
&\quad = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1+\boldsymbol{e}_2+\phi\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1-\phi\boldsymbol{e}_2+\boldsymbol{e}_3)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & 0 & \phi & 1 \\ -1 & 1 & -1 & 1 \\ 0 & -\phi & -\phi^{-1} & 1 \\ \phi & \phi^{-1} & 0 & 1 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\phi^{-1}\boldsymbol{e}_2-\phi\boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1+\phi^{-1}\boldsymbol{e}_2-\boldsymbol{e}_4)} \\
&\quad = R_{\pm\frac{1}{2}(\boldsymbol{e}_2+\phi\boldsymbol{e}_3-\phi^{-1}\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1-\boldsymbol{e}_3-\phi^{-1}\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -\phi^{-1} & 1 & -\phi \\ -\phi^{-1} & \phi & 1 & 0 \\ -\phi & 0 & -1 & -\phi^{-1} \end{pmatrix}.
\end{aligned}
$$

For each $v \in V_C$, let $r_v : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation sending $\boldsymbol{x}$ to $R_v\boldsymbol{x}$. We note that r_v is respectively a $(2\pi/3)$-, $(\pi/2)$-, or $(2\pi/5)$-rotation about a plane in which v and the origin of $\mathbb{R}^4$ lie, if C is 16, 24 or 600. Then the set $\{(v, r_v) \mid v \in V_C\}$ equipped with the binary operation $*$ given by $(v, r_v) * (w, r_w) = (r_w(v), r_{r_w(v)})$ forms a quandle. We call this quandle the *16-*, *24-*, or *600-cell quandle* respectively if C is 16, 24 or 600. We have the following claims related to 16-, 24-, and 600-cells quandles (see [1] for more details):

Theorem 1. *The 16-, 24-, or 600-cell quandle is respectively isomorphic to the knot quandle of the 3-, 4-, or 5-twist-spun trefoil.*

Theorem 2. *Each of 16-, 24-, and 600-cells quandles is isomorphic to no conjugation quandle.*

References

[1] A. Inoue, *The knot quandle of the twist-spun trefoil is a central extension of a Schläfli quandle*, preprint, available at https://arxiv.org/abs/2104.13065.

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正多胞体が定めるカンドルについて

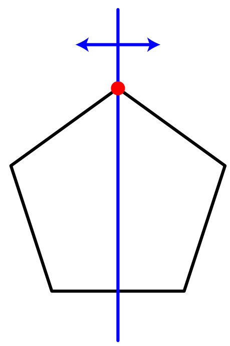
井上 歩
(ayminoue@tsuda.ac.jp)

津田塾大学

2020 年 12 月 18 日

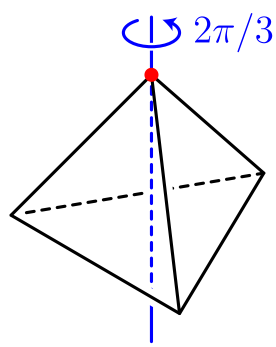
カンドルは対称性と相性が良い：

カンドルは対称性と相性が良い：



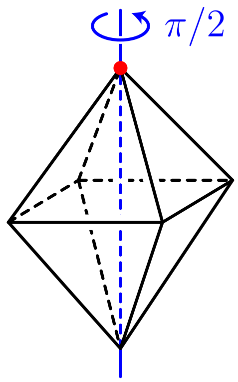
2 面体カンドル

カンドルは対称性と相性が良い：



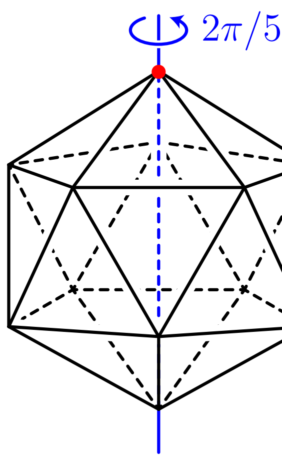
4 面体カンドル

カンドルは対称性と相性が良い：



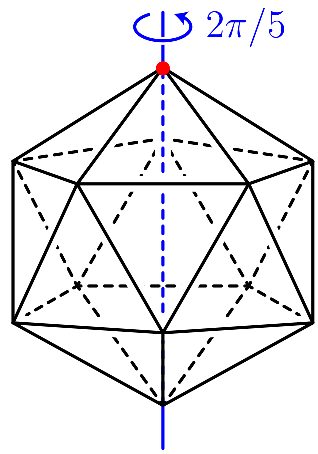
8 面体カンドル

カンドルは対称性と相性が良い：



20 面体カンドル

カンドルは対称性と相性が良い：



20 面体カンドル

次元を上げて考えてみる．

Schläfli の記号

- $\{p\}$: 正 p 角形
- $\{p, q\}$: 各頂点に $\{p\}$ が q 個ずつ集まった正多面体

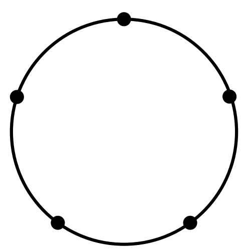
  $\{3, 3\}$: 正 4 面体（3 単体） $\{4, 3\}$: 正 6 面体
  $\{3, 4\}$: 正 8 面体 $\{5, 3\}$: 正 12 面体 $\{3, 5\}$: 正 20 面体
- $\{p, q, r\}$: 各辺（稜）に $\{p, q\}$ が r 個ずつ集まった正多胞体

  $\{3, 3, 3\}$: 正 5 胞体（4 単体）
  $\{4, 3, 3\}$: 正 8 胞体（超立方体）
  $\{3, 3, 4\}$: 正 16 胞体 $\{3, 4, 3\}$: 正 24 胞体
  $\{5, 3, 3\}$: 正 120 胞体 $\{3, 3, 5\}$: 正 600 胞体

$\vdots$

正多角形，正多面体，正多胞体は，
それぞれ，$\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{S}^3$ のタイル貼りと見なせる：

正多角形，正多面体，正多胞体は，
それぞれ，$\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{S}^3$ のタイル貼りと見なせる：



正 5 角形

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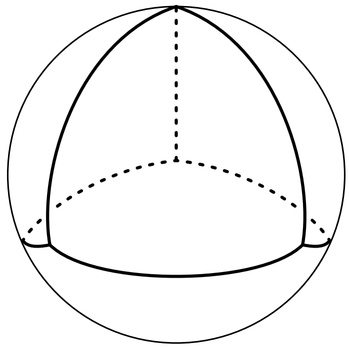
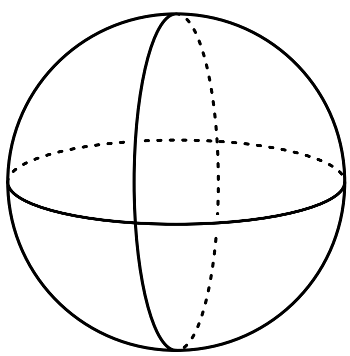
正多角形，正多面体，正多胞体は，
それぞれ，$\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{S}^3$ のタイル貼りと見なせる：



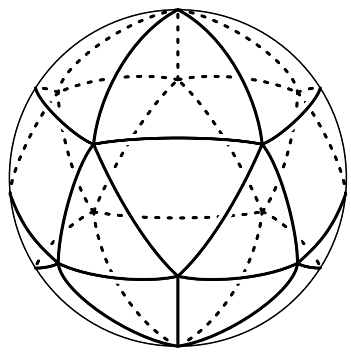
正 4 面体

正多角形，正多面体，正多胞体は，
それぞれ，$\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{S}^3$ のタイル貼りと見なせる：



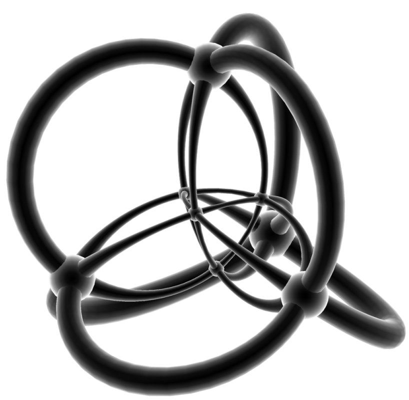
正 8 面体

正多角形，正多面体，正多胞体は，
それぞれ，$\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{S}^3$ のタイル貼りと見なせる：



正 20 面体

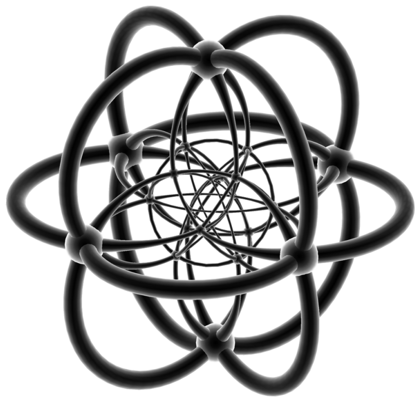
正多角形，正多面体，正多胞体は，
それぞれ，$\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{S}^3$ のタイル貼りと見なせる：



正 16 胞体

(出典：`http://www.math.cmu.edu/~fho/jenn/polytopes/16-cell.png`)

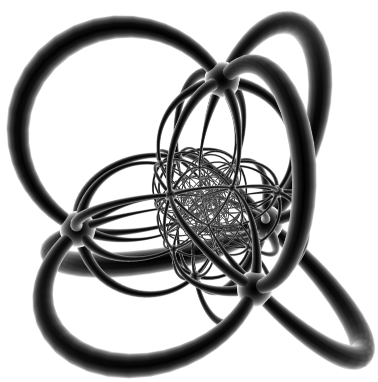
正多角形，正多面体，正多胞体は，
それぞれ，$\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{S}^3$ のタイル貼りと見なせる：



正 24 胞体

(出典：http://www.math.cmu.edu/~fho/jenn/polytopes/24-cell.png)

正多角形，正多面体，正多胞体は，
それぞれ，$\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{S}^3$ のタイル貼りと見なせる：



正 600 胞体

(出典：https://www.math.cmu.edu/~fho/jenn/polytopes/600-cell.png)

多面体カンドルとは，

$\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ のタイル貼りに対する
頂点を中心とする θ 回転変換全体（θ は固定）

がなすカンドルである．

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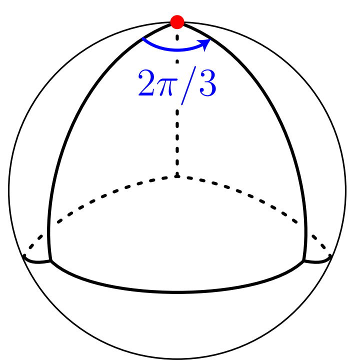
多面体カンドルとは，

$\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ のタイル貼りに対する
頂点を中心とする θ 回転変換全体（θ は固定）

がなすカンドルである．



4 面体カンドル

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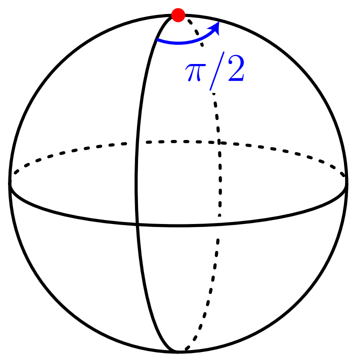
多面体カンドルとは，

$\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ のタイル貼りに対する
頂点を中心とする θ 回転変換全体（θ は固定）

がなすカンドルである．

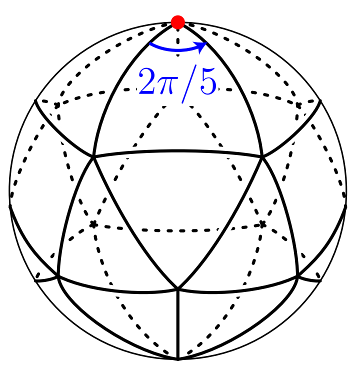


8 面体カンドル

多面体カンドルとは，

$\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ のタイル貼りに対する
頂点を中心とする θ 回転変換全体（θ は固定）

がなすカンドルである．

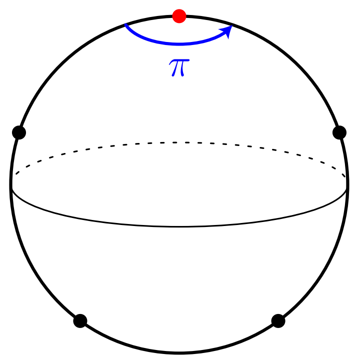


20 面体カンドル

多面体カンドルとは，

$\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ のタイル貼りに対する
頂点を中心とする θ 回転変換全体（θ は固定）

がなすカンドルである．

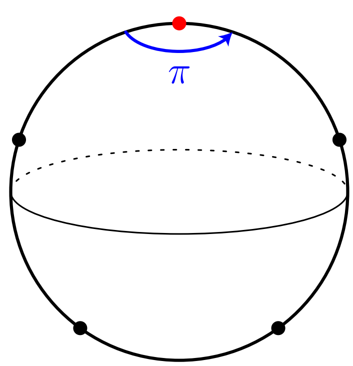


2 面体カンドル

多面体カンドルとは，

$\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ のタイル貼りに対する
頂点を中心とする θ 回転変換全体（θ は固定）

がなすカンドルである．



2 面体カンドル

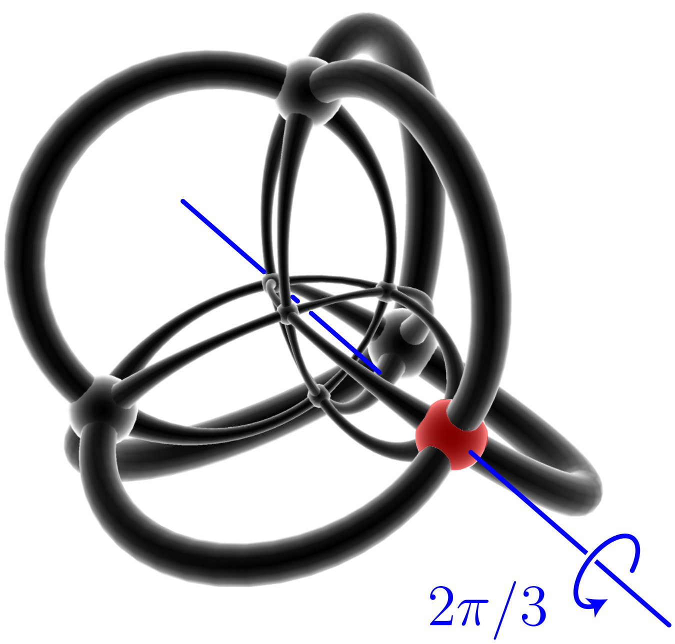
多胞体カンドルも同様に定義できる！

約束

$V_n \subset \mathbb{S}^3 \subset \mathbb{R}^4$: 正 n 胞体の頂点全体の集合
$R_v \in SO(4)$: 頂点 $v \in V_n$ を中心とする正 n 胞体の θ 回転変換
（θ は固定）

$v * w := R_w v$ $(v, w \in V_n)$
$\rightsquigarrow (V_n, *)$: n 胞体カンドル

16 胞体カンドル



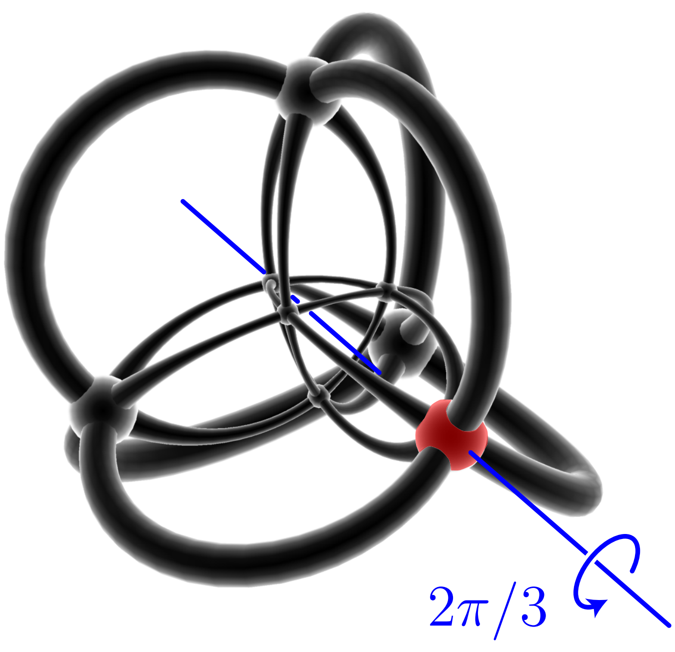
16 胞体カンドル

$V_{16} = \{\pm \boldsymbol{e}_1, \pm \boldsymbol{e}_2, \pm \boldsymbol{e}_3, \pm \boldsymbol{e}_4\}$

$$R_{\pm \boldsymbol{e}_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad R_{\pm \boldsymbol{e}_2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{\pm \boldsymbol{e}_3} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad R_{\pm \boldsymbol{e}_4} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

16 胞体カンドル



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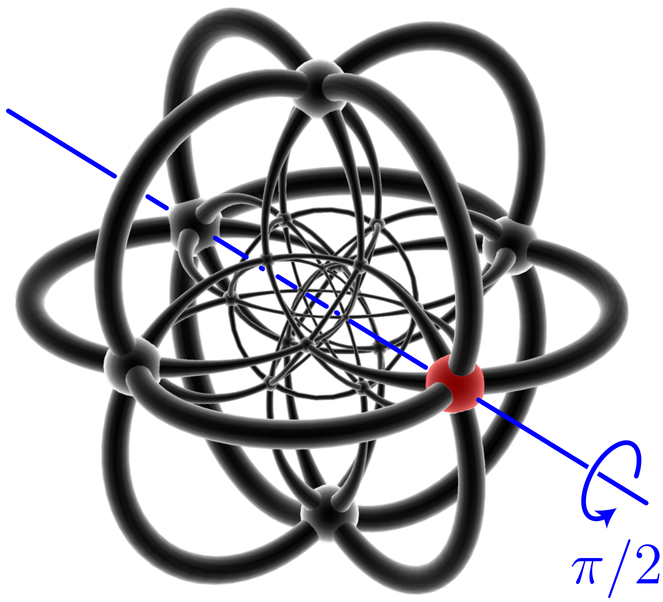
24 胞体カンドル



24 胞体カンドル

$V_{24} = \{\pm \boldsymbol{e}_i \pm \boldsymbol{e}_j \mid 1 \leq i < j \leq 4\}$

$$R_{\pm(\boldsymbol{e}_1+\boldsymbol{e}_2)} = R_{\pm(\boldsymbol{e}_1-\boldsymbol{e}_2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$R_{\pm(\boldsymbol{e}_3+\boldsymbol{e}_4)} = R_{\pm(\boldsymbol{e}_3-\boldsymbol{e}_4)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_{\pm(\boldsymbol{e}_1+\boldsymbol{e}_3)} = R_{\pm(\boldsymbol{e}_2+\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

24 胞体カンドル

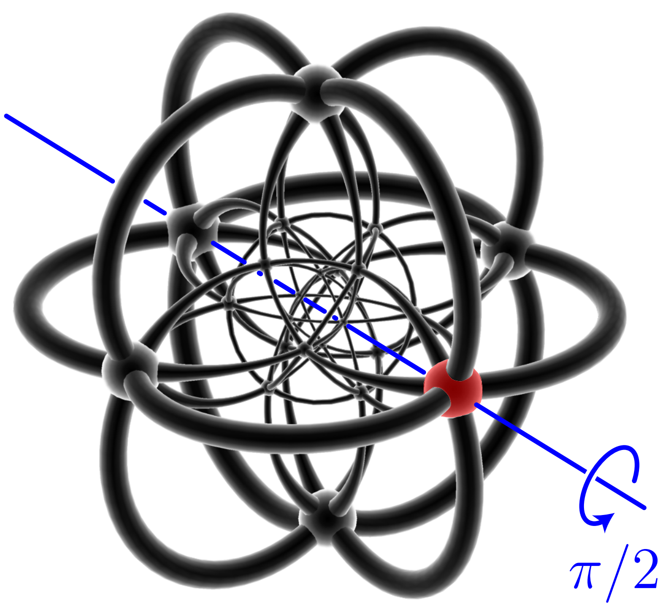
$V_{24} = \{\pm \boldsymbol{e}_i \pm \boldsymbol{e}_j \mid 1 \le i < j \le 4\}$

$$R_{\pm(\boldsymbol{e}_1-\boldsymbol{e}_3)} = R_{\pm(\boldsymbol{e}_2-\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix},$$

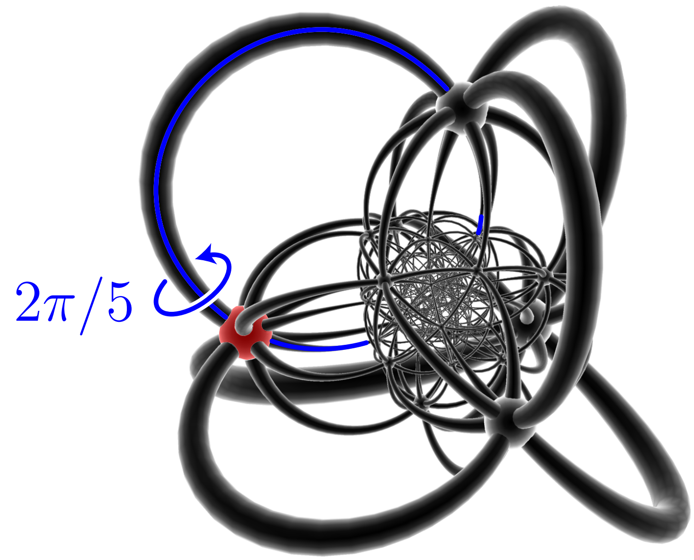
$$R_{\pm(\boldsymbol{e}_1+\boldsymbol{e}_4)} = R_{\pm(\boldsymbol{e}_2-\boldsymbol{e}_3)} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$R_{\pm(\boldsymbol{e}_2+\boldsymbol{e}_3)} = R_{\pm(\boldsymbol{e}_1-\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

24 胞体カンドル



600 胞体カンドル



600 胞体カンドル

$$V_{600} = \{\pm \boldsymbol{e}_1, \pm \boldsymbol{e}_2, \pm \boldsymbol{e}_3, \pm \boldsymbol{e}_4\} \cup \left\{ \tfrac{1}{2}(\pm \boldsymbol{e}_1 \pm \boldsymbol{e}_2 \pm \boldsymbol{e}_3 \pm \boldsymbol{e}_4) \right\}$$

$$\cup \left\{ \tfrac{1}{2} \left(\pm \phi \boldsymbol{e}_{\sigma(1)} \pm \boldsymbol{e}_{\sigma(2)} \pm \phi^{-1} \boldsymbol{e}_{\sigma(3)} \right) \;\middle|\; \sigma \in A_4 \right\} \qquad \left(\phi = \tfrac{1+\sqrt{5}}{2} : \text{黄金比} \right)$$

$$R_{\pm \boldsymbol{e}_1} = R_{\pm\frac{1}{2}(\phi \boldsymbol{e}_1 + \boldsymbol{e}_2 + \phi^{-1} \boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi \boldsymbol{e}_1 - \boldsymbol{e}_2 - \phi^{-1} \boldsymbol{e}_3)}$$

$$= R_{\pm\frac{1}{2}(\phi^{-1} \boldsymbol{e}_1 + \phi \boldsymbol{e}_2 + \boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi^{-1} \boldsymbol{e}_1 - \phi \boldsymbol{e}_2 - \boldsymbol{e}_3)} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & \phi & -\phi^{-1} \\ 0 & \phi & -\phi^{-1} & 1 \\ 0 & \phi^{-1} & -1 & -\phi \end{pmatrix},$$

$$R_{\pm \boldsymbol{e}_4} = R_{\pm\frac{1}{2}(\boldsymbol{e}_2 - \phi \boldsymbol{e}_3 + \phi^{-1} \boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_2 - \phi \boldsymbol{e}_3 - \phi^{-1} \boldsymbol{e}_4)}$$

$$= R_{\pm\frac{1}{2}(\phi^{-1} \boldsymbol{e}_2 - \boldsymbol{e}_3 + \phi \boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi^{-1} \boldsymbol{e}_2 - \boldsymbol{e}_3 - \phi \boldsymbol{e}_4)} = \frac{1}{2} \begin{pmatrix} -\phi & 1 & \phi^{-1} & 0 \\ -1 & -\phi^{-1} & -\phi & 0 \\ -\phi^{-1} & -\phi & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

井上 歩 (津田塾大学) 正多胞体が定めるカンドルについて 2020 年 12 月 18 日 9 / 16

600 胞体カンドル

$$V_{600} = \{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \pm\boldsymbol{e}_3, \pm\boldsymbol{e}_4\} \cup \left\{\tfrac{1}{2}(\pm\boldsymbol{e}_1 \pm \boldsymbol{e}_2 \pm \boldsymbol{e}_3 \pm \boldsymbol{e}_4)\right\}$$

$$\cup \left\{ \tfrac{1}{2}\left(\pm\phi\,\boldsymbol{e}_{\sigma(1)} \pm \boldsymbol{e}_{\sigma(2)} \pm \phi^{-1}\boldsymbol{e}_{\sigma(3)}\right) \,\middle|\, \sigma \in A_4 \right\} \qquad \left(\phi = \tfrac{1+\sqrt{5}}{2} \text{ : 黄金比}\right)$$

$$R_{\pm\boldsymbol{e}_3} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\phi^{-1}\boldsymbol{e}_3-\phi\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\phi^{-1}\boldsymbol{e}_3-\phi\boldsymbol{e}_4)}$$

$$= R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1+\phi\boldsymbol{e}_3-\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1-\phi\boldsymbol{e}_3-\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & 1 & 0 & -\phi \\ -1 & -\phi & 0 & -\phi^{-1} \\ 0 & 0 & 2 & 0 \\ -\phi & \phi^{-1} & 0 & 1 \end{pmatrix},$$

$$R_{\pm\boldsymbol{e}_2} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\phi\boldsymbol{e}_2+\phi^{-1}\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\phi\boldsymbol{e}_2+\phi^{-1}\boldsymbol{e}_4)}$$

$$= R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1+\phi^{-1}\boldsymbol{e}_2+\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1-\phi^{-1}\boldsymbol{e}_2+\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & \phi^{-1} & \phi \\ 0 & 2 & 0 & 0 \\ -\phi^{-1} & 0 & -\phi & 1 \\ \phi & 0 & -1 & -\phi^{-1} \end{pmatrix},$$

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$$\cup \left\{ \tfrac{1}{2}\left(\pm\phi\,\boldsymbol{e}_{\sigma(1)} \pm \boldsymbol{e}_{\sigma(2)} \pm \phi^{-1}\boldsymbol{e}_{\sigma(3)}\right) \,\middle|\, \sigma \in A_4 \right\} \qquad \left(\phi = \tfrac{1+\sqrt{5}}{2} \text{ : 黄金比}\right)$$

$$R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3+\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\phi^{-1}\boldsymbol{e}_3+\phi\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_2-\phi^{-1}\boldsymbol{e}_3-\boldsymbol{e}_4)}$$

$$= R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1+\phi\boldsymbol{e}_2-\boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1-\boldsymbol{e}_2+\phi\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & 1 & 0 & \phi \\ 0 & 1 & -\phi & -\phi^{-1} \\ -\phi & -1 & -\phi^{-1} & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\boldsymbol{e}_2-\boldsymbol{e}_3-\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\phi^{-1}\boldsymbol{e}_2+\phi\boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1-\phi^{-1}\boldsymbol{e}_2-\boldsymbol{e}_4)}$$

$$= R_{\pm\frac{1}{2}(\boldsymbol{e}_2+\phi\boldsymbol{e}_3+\phi^{-1}\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1+\boldsymbol{e}_3-\phi^{-1}\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & \phi^{-1} & -\phi \\ -1 & -\phi^{-1} & \phi & 0 \\ 1 & 1 & 1 & 1 \\ -1 & \phi & 0 & -\phi^{-1} \end{pmatrix},$$

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$$V_{600} = \{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \pm\boldsymbol{e}_3, \pm\boldsymbol{e}_4\} \cup \left\{\tfrac{1}{2}(\pm\boldsymbol{e}_1 \pm \boldsymbol{e}_2 \pm \boldsymbol{e}_3 \pm \boldsymbol{e}_4)\right\}$$

$$\cup \left\{ \tfrac{1}{2}\left(\pm\phi\,\boldsymbol{e}_{\sigma(1)} \pm \boldsymbol{e}_{\sigma(2)} \pm \phi^{-1}\boldsymbol{e}_{\sigma(3)}\right) \,\middle|\, \sigma \in A_4 \right\} \qquad \left(\phi = \tfrac{1+\sqrt{5}}{2} \text{ : 黄金比}\right)$$

$$R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\phi^{-1}\boldsymbol{e}_2+\phi\boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_2+\boldsymbol{e}_3+\phi\boldsymbol{e}_4)}$$
$$= R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1+\phi\boldsymbol{e}_3+\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1-\boldsymbol{e}_2-\phi\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & 0 & \phi & -1 \\ -\phi & -\phi^{-1} & 0 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & \phi & \phi^{-1} & 1 \end{pmatrix},$$

$$R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\boldsymbol{e}_2-\boldsymbol{e}_3+\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\phi\boldsymbol{e}_2-\phi^{-1}\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1-\boldsymbol{e}_3+\phi^{-1}\boldsymbol{e}_4)}$$
$$= R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1+\boldsymbol{e}_2-\phi^{-1}\boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_2+\phi^{-1}\boldsymbol{e}_3-\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ \phi^{-1} & 1 & 0 & -\phi \\ -\phi & 1 & -\phi^{-1} & 0 \\ 0 & -1 & -\phi & -\phi^{-1} \end{pmatrix},$$

600 胞体カンドル

$$V_{600} = \{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \pm\boldsymbol{e}_3, \pm\boldsymbol{e}_4\} \cup \left\{\tfrac{1}{2}(\pm\boldsymbol{e}_1 \pm \boldsymbol{e}_2 \pm \boldsymbol{e}_3 \pm \boldsymbol{e}_4)\right\}$$

$$\cup \left\{ \tfrac{1}{2}\left(\pm\phi\,\boldsymbol{e}_{\sigma(1)} \pm \boldsymbol{e}_{\sigma(2)} \pm \phi^{-1}\boldsymbol{e}_{\sigma(3)}\right) \,\middle|\, \sigma \in A_4 \right\} \qquad \left(\phi = \tfrac{1+\sqrt{5}}{2} \text{ : 黄金比}\right)$$

$$R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3-\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\phi^{-1}\boldsymbol{e}_2-\phi\boldsymbol{e}_3)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_2+\boldsymbol{e}_3-\phi\boldsymbol{e}_4)}$$
$$= R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1+\boldsymbol{e}_2-\phi\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1-\phi\boldsymbol{e}_3+\boldsymbol{e}_4)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & \phi & -1 & 0 \\ 0 & -\phi^{-1} & -1 & -\phi \\ -\phi & 0 & 1 & -\phi^{-1} \\ -1 & -1 & -1 & 1 \end{pmatrix},$$

$$R_{\pm\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\boldsymbol{e}_1-\phi\boldsymbol{e}_2-\phi^{-1}\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1+\boldsymbol{e}_3+\phi^{-1}\boldsymbol{e}_4)}$$
$$= R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_2+\phi^{-1}\boldsymbol{e}_3+\boldsymbol{e}_4)} = R_{\pm\frac{1}{2}(\phi\boldsymbol{e}_1-\boldsymbol{e}_2+\phi^{-1}\boldsymbol{e}_3)} = \frac{1}{2}\begin{pmatrix} 1 & -\phi^{-1} & \phi & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & -\phi^{-1} & \phi \\ 1 & \phi & 0 & -\phi^{-1} \end{pmatrix},$$

600 胞体カンドル

$$V_{600} = \{\pm e_1, \pm e_2, \pm e_3, \pm e_4\} \cup \left\{ \tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\}$$

$$\cup \left\{ \tfrac{1}{2}\left(\pm \phi\, e_{\sigma(1)} \pm e_{\sigma(2)} \pm \phi^{-1} e_{\sigma(3)}\right) \,\middle|\, \sigma \in A_4 \right\} \qquad \left(\phi = \tfrac{1+\sqrt{5}}{2}：\text{黄金比}\right)$$

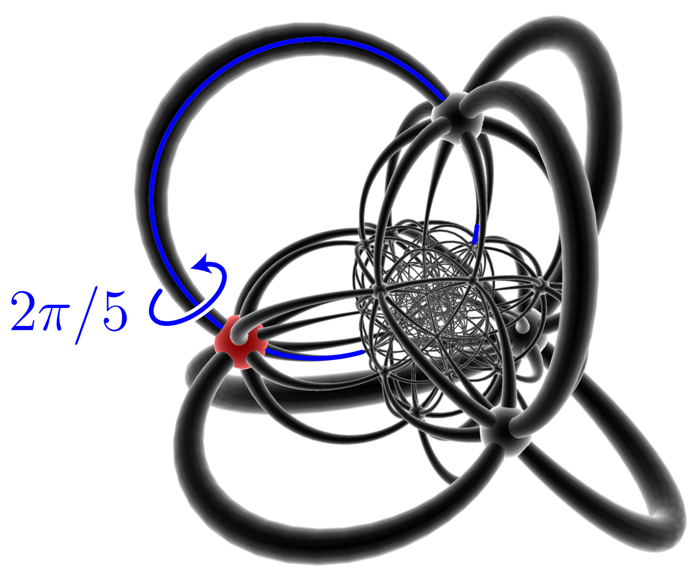
$$R_{\pm\frac{1}{2}(e_1 - e_2 + e_3 + e_4)} = R_{\pm\frac{1}{2}(e_1 + \phi^{-1} e_3 + \phi e_4)} = R_{\pm\frac{1}{2}(\phi e_2 - \phi^{-1} e_3 + e_4)}$$

$$= R_{\pm\frac{1}{2}(\phi^{-1} e_1 + e_2 + \phi e_4)} = R_{\pm\frac{1}{2}(\phi^{-1} e_1 - \phi e_2 + e_3)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & 0 & \phi & 1 \\ -1 & 1 & -1 & 1 \\ 0 & -\phi & -\phi^{-1} & 1 \\ \phi & \phi^{-1} & 0 & 1 \end{pmatrix},$$

$$R_{\pm\frac{1}{2}(e_1 + e_2 + e_3 - e_4)} = R_{\pm\frac{1}{2}(e_1 - \phi^{-1} e_2 - \phi e_3)} = R_{\pm\frac{1}{2}(\phi e_1 + \phi^{-1} e_2 - e_4)}$$

$$= R_{\pm\frac{1}{2}(e_2 + \phi e_3 - \phi^{-1} e_4)} = R_{\pm\frac{1}{2}(\phi e_1 - e_3 - \phi^{-1} e_4)} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -\phi^{-1} & 1 & -\phi \\ -\phi^{-1} & \phi & 1 & 0 \\ -\phi & 0 & -1 & -\phi^{-1} \end{pmatrix}$$

600 胞体カンドル



考察 1

16, 24, 600 胞体カンドルは共役カンドルではない.

考察 1

16, 24, 600 胞体カンドルは共役カンドルではない.

Theorem (Clark–Saito–Vendramin 2016)

$(X, *)$: 連結なカンドル

$$G := \langle g_x \ (x \in X) \mid g_y^{-1} g_x g_y g_{x*y}^{-1} \ (x, y \in X),\ g_x^{n_x} \ (x \in X) \rangle$$
$$(n_x := \min\{n \in \mathbb{N} \mid (* \, x)^n = \mathrm{id}\})$$

$(X, *)$ は共役カンドル $\Leftrightarrow$ $\rho : X \to G$ $(\rho(x) = g_x)$ は単射

考察 1

16, 24, 600 胞体カンドルは共役カンドルではない．

∵ 16 胞体カンドルについて，

$$\begin{aligned} e_1 * e_3 &= e_2, \quad \cdots ① \\ e_2 * e_1 &= e_3, \quad \cdots ② \\ e_3 * e_2 &= -e_1 \quad \cdots ③ \end{aligned}$$

より，

$$g_{e_2} g_{e_1} \overset{②}{=} g_{e_1} g_{e_3} \overset{①}{=} g_{e_3} g_{e_2} \overset{③}{=} g_{e_2} g_{-e_1}$$

であるから，

$$g_{e_1} = g_{-e_1}$$

を得る． □

考察 1

16, 24, 600 胞体カンドルは共役カンドルではない．

∵ 24 胞体カンドルについて，

$$\begin{aligned} (e_1 + e_2) * (e_2 - e_3) &= e_2 + e_4, \\ (e_2 + e_4) * (e_1 + e_2) &= e_2 - e_3, \\ (e_2 - e_3) * (e_2 + e_4) &= -e_1 + e_2 \end{aligned}$$

より，

$$g_{e_1+e_2} = g_{-e_1+e_2}$$

を得る． □

考察 1

16, 24, 600 胞体カンドルは共役カンドルではない．

$\because$ 600 胞体カンドルについて，

$$\begin{aligned} &\boldsymbol{e}_1 * \left(-\tfrac{1}{2}(\phi^{-1}\boldsymbol{e}_1 + \phi\boldsymbol{e}_2 - \boldsymbol{e}_3)\right) = -\tfrac{1}{2}(\phi^{-1}\boldsymbol{e}_1 + \phi\boldsymbol{e}_3 - \boldsymbol{e}_4), \\ &\left(-\tfrac{1}{2}(\phi^{-1}\boldsymbol{e}_1 + \phi\boldsymbol{e}_3 - \boldsymbol{e}_4)\right) * \boldsymbol{e}_1 = -\tfrac{1}{2}(\phi^{-1}\boldsymbol{e}_1 + \phi\boldsymbol{e}_2 - \boldsymbol{e}_3), \\ &\left(-\tfrac{1}{2}(\phi^{-1}\boldsymbol{e}_1 + \phi\boldsymbol{e}_2 - \boldsymbol{e}_3)\right) * \left(-\tfrac{1}{2}(\phi^{-1}\boldsymbol{e}_1 + \phi\boldsymbol{e}_3 - \boldsymbol{e}_4)\right) \\ &\qquad = -\tfrac{1}{2}(\phi^{-1}\boldsymbol{e}_1 - \phi\boldsymbol{e}_2 - \boldsymbol{e}_3) \end{aligned}$$

より，

$$g_{\boldsymbol{e}_1} = g_{-\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1 - \phi\boldsymbol{e}_2 - \boldsymbol{e}_3)}$$

を得る． $\square$

考察 2

16, 24, 600 胞体カンドルに現れる回転変換全体は
自身の共役に関して閉じている．

X_n : n 胞体カンドルに現れる回転変換全体の集合

$X_{16} = \{R_{\boldsymbol{e}_1}, R_{\boldsymbol{e}_2}, R_{\boldsymbol{e}_3}, R_{\boldsymbol{e}_4}\}$

$X_{24} = \{R_{\boldsymbol{e}_1+\boldsymbol{e}_2}, R_{\boldsymbol{e}_3+\boldsymbol{e}_4}, R_{\boldsymbol{e}_1+\boldsymbol{e}_3}, R_{\boldsymbol{e}_1-\boldsymbol{e}_3}, R_{\boldsymbol{e}_1+\boldsymbol{e}_4}, R_{\boldsymbol{e}_2+\boldsymbol{e}_3}\}$

$$X_{600} = \left\{ \begin{array}{l} R_{\boldsymbol{e}_1}, R_{\boldsymbol{e}_4}, R_{\boldsymbol{e}_3}, R_{\boldsymbol{e}_2}, \\ R_{\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3+\boldsymbol{e}_4)}, R_{\frac{1}{2}(\boldsymbol{e}_1-\boldsymbol{e}_2-\boldsymbol{e}_3-\boldsymbol{e}_4)}, R_{\frac{1}{2}(\boldsymbol{e}_1-\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4)}, R_{\frac{1}{2}(\boldsymbol{e}_1-\boldsymbol{e}_2-\boldsymbol{e}_3+\boldsymbol{e}_4)}, \\ R_{\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3-\boldsymbol{e}_4)}, R_{\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_4)}, R_{\frac{1}{2}(\boldsymbol{e}_1-\boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_4)}, R_{\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4)} \end{array} \right\}$$

考察2

16, 24, 600 胞体カンドルに現れる回転変換全体は
自身の共役に関して閉じている．

X_n : n 胞体カンドルに現れる回転変換全体の集合

$X_{16} = \{R_{e_1}, R_{e_2}, R_{e_3}, R_{e_4}\}$ (4 面体カンドル)

$X_{24} = \{R_{e_1+e_2}, R_{e_3+e_4}, R_{e_1+e_3}, R_{e_1-e_3}, R_{e_1+e_4}, R_{e_2+e_3}\}$ (8 面体カンドル)

$$X_{600} = \left\{ \begin{array}{l} R_{e_1}, R_{e_4}, R_{e_3}, R_{e_2}, \\ R_{\frac{1}{2}(e_1+e_2-e_3+e_4)}, R_{\frac{1}{2}(e_1-e_2-e_3-e_4)}, R_{\frac{1}{2}(e_1-e_2+e_3-e_4)}, R_{\frac{1}{2}(e_1-e_2-e_3+e_4)}, \\ R_{\frac{1}{2}(e_1+e_2-e_3-e_4)}, R_{\frac{1}{2}(e_1+e_2+e_3+e_4)}, R_{\frac{1}{2}(e_1-e_2+e_3+e_4)}, R_{\frac{1}{2}(e_1+e_2+e_3-e_4)} \end{array} \right\}$$

(20 面体カンドル)

Theorem (Clark–Saito 2016)

16, 24 (, 600) 胞体カンドルは，それぞれ，
4, 8 (, 20) 面体カンドルのアーベル拡大である：

$$V_{16} \xrightarrow{2:1} X_{16}, \quad V_{24} \xrightarrow{4:1} X_{24}, \quad (V_{600} \xrightarrow{10:1} X_{600})$$

※ $V_{16} \cong \mathsf{SmallQuandle}(8,1)$, $V_{24} \cong \mathsf{SmallQuandle}(24,2)$

Theorem (Clark–Saito 2016)

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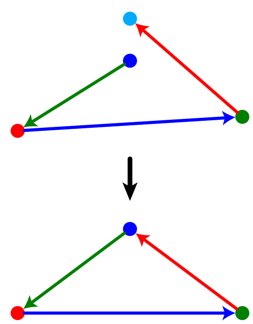
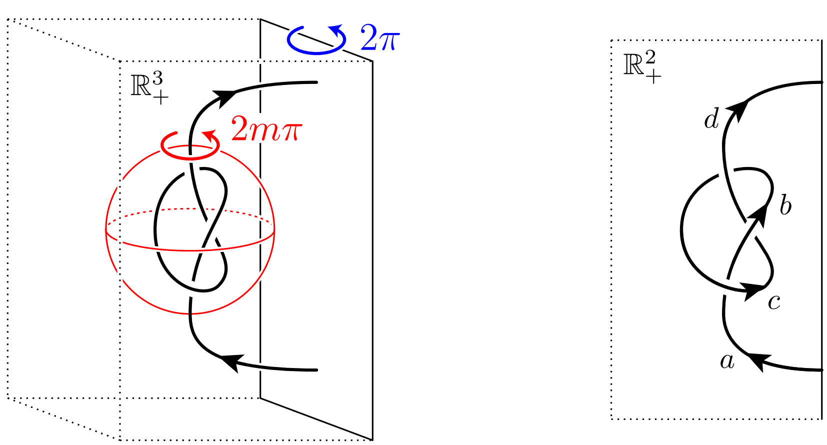


$$\boldsymbol{e}_1 * \boldsymbol{e}_3 = \boldsymbol{e}_2,$$
$$\boldsymbol{e}_2 * \boldsymbol{e}_1 = \boldsymbol{e}_3,$$
$$\boldsymbol{e}_3 * \boldsymbol{e}_2 = -\boldsymbol{e}_1$$

$$R_{\boldsymbol{e}_1} * R_{\boldsymbol{e}_3} = R_{\boldsymbol{e}_2},$$
$$R_{\boldsymbol{e}_2} * R_{\boldsymbol{e}_1} = R_{\boldsymbol{e}_3},$$
$$R_{\boldsymbol{e}_3} * R_{\boldsymbol{e}_2} = R_{\boldsymbol{e}_1} (= R_{-\boldsymbol{e}_1})$$

考察３

16, 24, 600 胞体カンドルは，それぞれ，3-, 4-, 5-twist-spun trefoil ($\tau^m 3_1$, $m = 3, 4, 5$) の結び目カンドルと同型である．



$$Q(\tau^m 3_1) = \langle a, c \mid (a * c) * a = c,\ c *^m a = c \rangle_Q$$

考察３

16, 24, 600 胞体カンドルは，それぞれ，3-, 4-, 5-twist-spun trefoil ($\tau^m 3_1$, $m = 3, 4, 5$) の結び目カンドルと同型である．

$$Q(\tau^m 3_1) = \langle a, c \mid (a * c) * a = c,\ c *^m a = c \rangle_Q$$

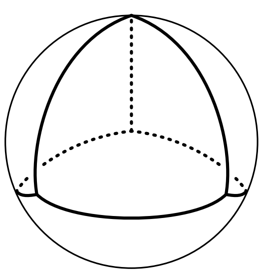
∵ 写像 $\varphi : Q(\tau^m 3_1) \to V_*$ を

$$\varphi(a) = \begin{cases} \boldsymbol{e}_1 & (m = 3), \\ \boldsymbol{e}_1 + \boldsymbol{e}_2 & (m = 4), \\ \boldsymbol{e}_1 & (m = 5); \end{cases} \quad \varphi(c) = \begin{cases} \boldsymbol{e}_2 & (m = 3), \\ \boldsymbol{e}_2 + \boldsymbol{e}_4 & (m = 4), \\ -\frac{1}{2}(\phi^{-1}\boldsymbol{e}_1 + \phi\boldsymbol{e}_3 - \boldsymbol{e}_4) & (m = 5) \end{cases}$$

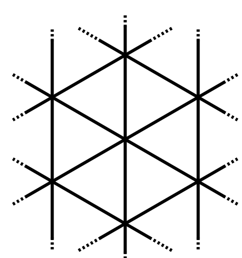
と定めると，φ は同型写像になる． $\square$

Schläfli カンドル

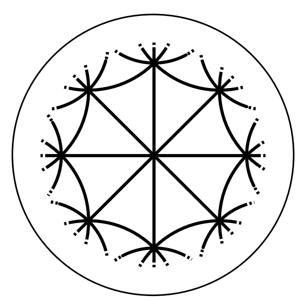
Schläfli の記号は，$\mathbb{E}^n$, $\mathbb{H}^n$ のタイル貼りも記述できる：



$\{3, 3\}$



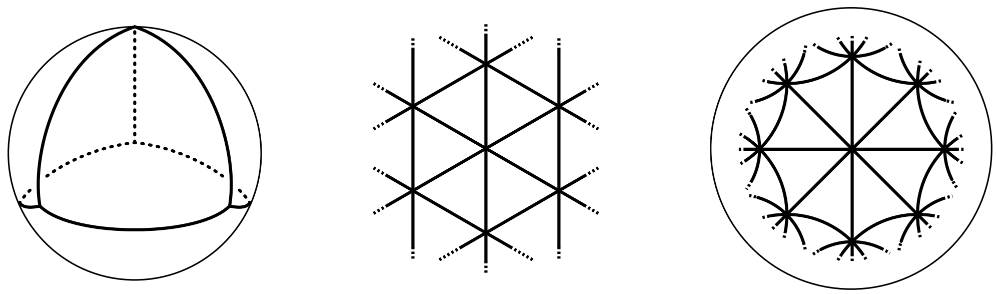
$\{3, 6\}$



$\{3, 8\}$

Schläfli カンドル

Schläfli の記号は，$\mathbb{E}^n$, $\mathbb{H}^n$ のタイル貼りも記述できる：



$\{3,3\}$ $\{3,6\}$ $\{3,8\}$

V : タイル貼りの頂点全体の集合

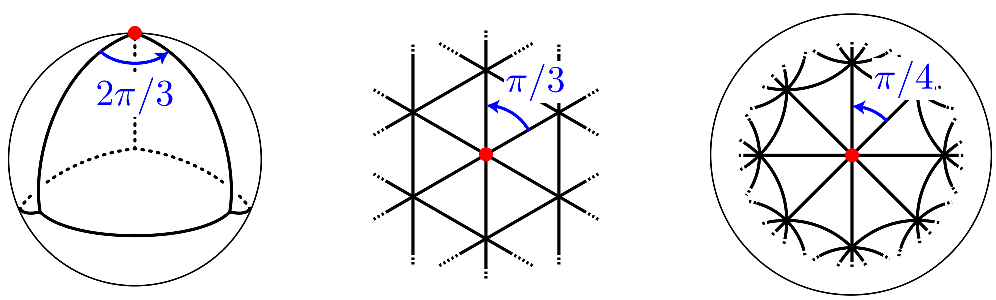
R_v : 頂点 $v \in V$ を中心とするタイル貼りの θ 回転変換（θ は固定）

$v * w := R_w v \ (v, w \in V)$

$\rightsquigarrow (V, *)$: Schläfli カンドル

W_m : タイル貼り $\{3, m\}$ の頂点全体の集合

$\theta = \frac{2\pi}{m}$ とすると，Schläfli カンドル $(W_m, *)$ が定まる：



W_3 W_6 W_8

注意

W_2, W_3, W_4, W_5 は，それぞれ，位数 3 の 2 面体カンドル，
4 面体カンドル，8 面体カンドル，20 面体カンドルに他ならない．

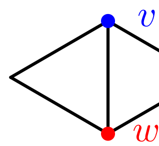
考察 4

$Q(\tau^m 3_1) = \langle a, c \mid (a * c) * a = c,\, c *^m a = c \rangle_Q$

写像 $\varphi : Q(\tau^m 3_1) \to W_m$ $(m \geq 2)$ を

$$\varphi(a) = v, \;\; \varphi(c) = w$$

と定めると，φ は全射準同形である．



予想

$Q(\tau^m 3_1)$ は W_m のアーベル拡大である $(m \geq 2)$．

On a quandle derivative

ATSUSHI ISHII

ABSTRACT. We introduce the notion of a derivative for quandles. The derivative is defined with an Alexander pair, which corresponds to an extension of a quandle. By using the derivative, we obtain a knot invariant such as the twisted Alexander polynomial. This is a joint work with Kanako Oshiro.

1 Alexander pairs and derivatives

Let $(Q, \triangleleft)$ be a quandle. Let R be a ring. The pair (f_1, f_2) of $f_1, f_2 : Q \times Q \to R$ is an *Alexander pair* if f_1 and f_2 satisfy the following conditions:

- For any $a \in Q$, $f_1(a, a) + f_2(a, a) = 1$.
- For any $a, b \in Q$, $f_1(a, b)$ is invertible.
- For any $a, b, c \in Q$,

$$\begin{aligned}
&f_1(a \triangleleft b, c) f_1(a, b) = f_1(a \triangleleft c, b \triangleleft c) f_1(a, c),\\
&f_1(a \triangleleft b, c) f_2(a, b) = f_2(a \triangleleft c, b \triangleleft c) f_1(b, c), \text{ and}\\
&f_2(a \triangleleft b, c) = f_1(a \triangleleft c, b \triangleleft c) f_2(a, c) + f_2(a \triangleleft c, b \triangleleft c) f_2(b, c).
\end{aligned}$$

We note that f_1 and f_2 correspond to η and τ in [1], respectively.

Let $S = \{x_1, \dots, x_n\}$ be a finite set, and let $X = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_m \rangle$ be a finitely presented quandle. Let $F_{\mathsf{Qnd}}(S)$ be the free quandle on S, and $\mathrm{pr} : F_{\mathsf{Qnd}}(S) \to X$ the canonical projection. We often omit "pr" to represent $\mathrm{pr}(a)$ as a. Let (f_1, f_2) be an Alexander pair of maps $f_1, f_2 : X \times X \to R$. Put $f = (f_1, f_2)$.

Definition 1. The *f-derivative with respect to x_j* for $j \in \{1, \dots, n\}$ is the map $\frac{\partial_f}{\partial x_j} : F_{\mathsf{Qnd}}(S) \to R$ satisfying

$$\frac{\partial_f}{\partial x_j}(a \triangleleft b) = f_1(a, b) \frac{\partial_f}{\partial x_j}(a) + f_2(a, b) \frac{\partial_f}{\partial x_j}(b), \qquad \frac{\partial_f}{\partial x_j}(x_i) = \delta_{ij}$$

for any $a, b \in F_{\mathsf{Qnd}}(S)$ and $i \in \{1, \dots, n\}$, where δ_{ij} is the Kronecker delta.

2 Alexander matrices and invariants

Let Q be a quandle. Let R be a ring. Let (f_1, f_2) be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$. Let $X = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_m \rangle$ be a finitely presented quandle, and $\rho : X \to Q$ a quandle representation. We define

$$A(X, \rho; f_1, f_2) = \begin{pmatrix} \frac{\partial_{f \circ \rho^2}}{\partial x_1}(r_1) & \cdots & \frac{\partial_{f \circ \rho^2}}{\partial x_n}(r_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial_{f \circ \rho^2}}{\partial x_1}(r_m) & \cdots & \frac{\partial_{f \circ \rho^2}}{\partial x_n}(r_m) \end{pmatrix},$$

where

$$f \circ \rho^2 = (f_1 \circ (\rho \times \rho), f_2 \circ (\rho \times \rho)), \qquad \frac{\partial_f}{\partial x_j}(r_a = r_b) = \frac{\partial_f}{\partial x_j}(r_a) - \frac{\partial_f}{\partial x_j}(r_b).$$

Suppose that R is a GCD domain. The *dth Alexander invariant* $\Delta_d(A)$ of A is the greatest common divisor of all $(n-d)$-minors of $m \times n$ matrix A if $n-m \leq d < n$, and

$$\Delta_d(A) = \begin{cases} 0 & \text{if } d < n-m, \\ 1 & \text{if } n \leq d. \end{cases}$$

Theorem 1. *Let $X = \langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle$ and $X' = \langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle$ be finitely presented quandles, and let $\rho : X \to Q$ and $\rho' : X' \to Q$ be quandle representations. If $(X, \rho) \cong (X', \rho')$, then we have*

$$\Delta_d(A(X, \rho; f_1, f_2)) \doteq \Delta_d(A(X', \rho'; f_1, f_2)),$$

where the symbol $\doteq$ indicates equality up to a unit factor.

References

[1] N. Andruskiewitsch and M. Graña, *From racks to pointed Hopf algebras*, Adv. Math. **178** (2003), no. 2, 177–243.

[2] A. Ishii and K. Oshiro, *Derivatives with Alexander pairs for quandles*, preprint.

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