A chord graph constructed from a ribbon surface-link

Akio Kawauchi

This paper is dedicated to Professors Taizo Kanenobu, Makoto Sakuma and Yasutaka Nakanishi on their 60-year-old birthdays.

Abstract. A chord graph in 3-space is constructed from a ribbon surface-link in 4-space. In an earlier paper, the three moves on the diagrams of chord graphs (namely, the chord diagrams) are introduced to correspond them to the faithful equivalences of ribbon surface-links by a canonical bijection, suggesting a relaxed spatial graph theory in 3-space. Links, virtual links and welded virtual links are also considered as chord graphs of ribbon torus-links in 4-space by Satoh's correspondence. At the level of diagrams, the image of this transformation of every (virtual) link diagram is a chord diagram without base crossing. In this paper, the three moves on the chord diagrams are modified into the 16 moves on the chord diagrams without base crossing to relate welded virtual links to the faithful equivalences of ribbon surface-links more efficiently.

1. Introduction

A *chord graph* is a trivalent graph $(o;\alpha)$ in $\mathbf{R}^3$ consisting of a trivial link o, called the *based loops*, and finitely many, mutually disjoint arcs α spanning o, called the *chords*. A *chord diagram* is a spatial graph diagram $D = D(o;\alpha)$ of a chord graph $(o;\alpha)$ in $\mathbf{R}^2$. Two chord diagrams D and D' are *equivalent* if D is deformed into D' by a finite sequence of moves in the set of *fundamental moves* defined by the following three moves M_0, M_1 and M_2.

Move $\mathbf{M}_0$. This move is called the *Reidemeister move* which consists of the Reidemeister moves R_1, R_2, R_3, gR_4, gR_5 by regarding a chord graph as a spatial trivalent graph diagram, illustrated in Fig. 1 where the roles of the based loops and the chords are not changed although a distinction of a based loop and a chord is omitted.

Move $\mathbf{M}_1$. This move is called the *fusion-fission move*, illustrated in Fig. 2, where the fusion operation is done only for a chord between different based loops.

2010 *Mathematics Subject Classification.* Primary 57Q45; Secondary 57M25.

Key words and phrases. Chord graph, Ribbon surface-link, Welded virtual link, Chord diagram, Fundamental moves, Base crossing, Chord diagram pass, Base pass.

This work was supported by JSPS KAKENHI Grant Number 24244005.

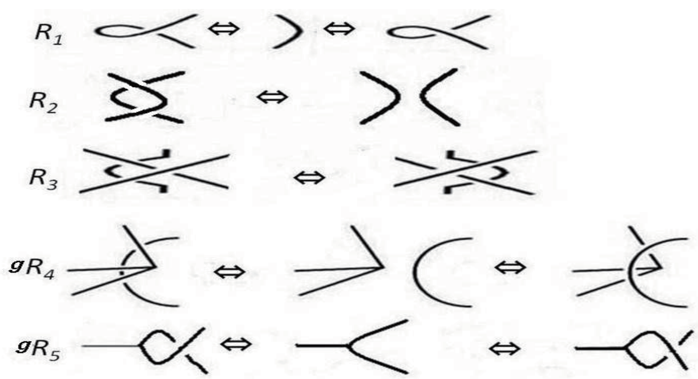

FIGURE 1. Move M_0: Reidemeister moves R_1, R_2, R_3, gR_4, gR_5 for trivalent graph diagrams

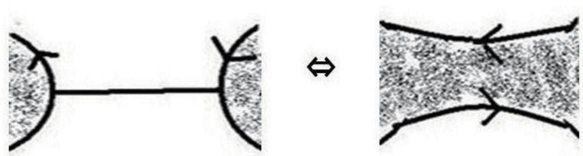

FIGURE 2. Fusion-fission M_1

Move $\mathbf{M}_2$. This move is called the *chord move* which consists of moves on chords, illustrated in Fig. 3.

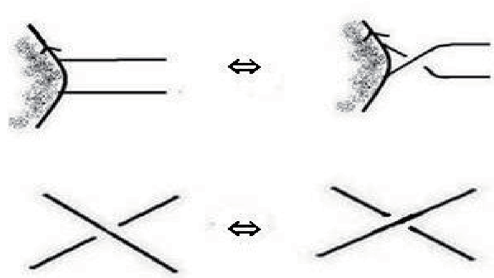

FIGURE 3. Chord moves M_2

It is proved in [**Kaw**][1] that the equivalence class of a chord diagram corresponds to the faithful equivalence of a ribbon surface-link in 4-space $\mathbf{R}^4$ by a canonical correspondence, which is explained in § 2. This suggests a relaxed spatial graph theory in 3-space. It is also observed in [**Kaw**] that under the use of the move M_0, the move M_1 is equivalent to the combination move of the *elementary fusion-fission*

[1]A full proof of Corollary 4.7 of [**Kaw**] is given in [**Kaw'**].

move $M_{1.0}$, the *chord slide move* $M_{1.1}$ and the *chord pass move* $M_{1.2}$ illustrated in Fig. 5. The *birth-death move* illustrated in Fig. 4 is obtained from these moves, unless a closed chord is involved.

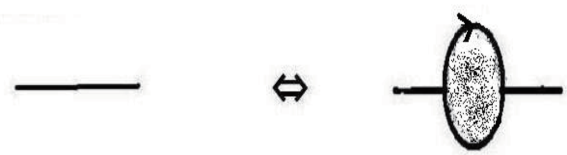

FIGURE 4. The birth-death move

By convention, a closed chord is regarded as a chord with a based loop constructed from the birth-death move.

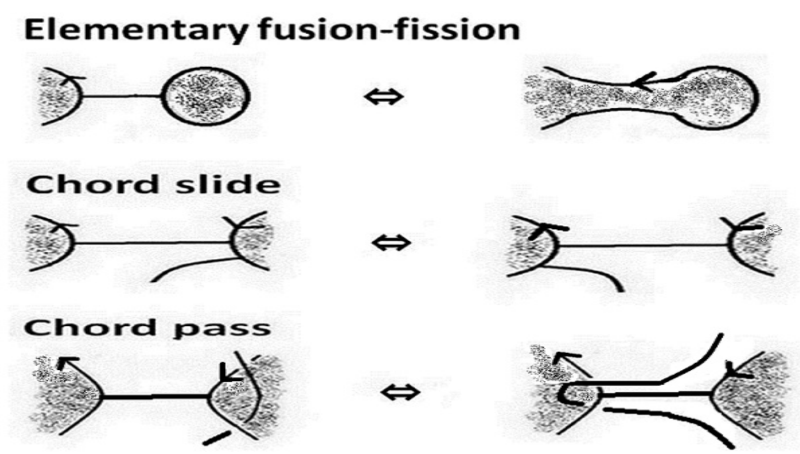

FIGURE 5. Elementary fusion-fission $M_{1.0}$, Chord slide $M_{1.1}$ and Chord pass $M_{1.2}$

A *chord diagram without base crossing* is a chord diagram such that the disks bounded by the based loop system in $\mathbf{R}^2$ do not overlap and meet the chords with mutually disjoint proper simple arcs. Every chord diagram is deformed into a chord diagram without based loop crossing by the Reidemeister move M_0, because the based loop diagram is a diagram of a trivial link.

By an argument in [**Kaw**] using the result of Satoh [**S**], every (virtual) link diagram is canonically transformed into a chord diagram without base crossing of a ribbon torus-link which induces a mapping from the set of (virtual) link onto the set of faithful equivalence classes of ribbon torus-links.

The purpose of this paper is to relpace the moves M_0, M_1, M_2 on the chord diagrams the 16 moves $m^i\,(i = 0, 1, 2, \ldots, 15)$ on the chord diagrams without base crossing, which serves to relate welded virtual links to the faithful equivalences of ribbon torus-links more efficiently.

In § 2, it is explained how to transform a (welded virtual) link diagram into a chord diagram without base crossing. In § 3, it is explained how to transform a chord diagram into a ribbon surface-link in 4-space. In § 4, it is explained how to modify the moves M_0, M_1, M_2 into the 16 moves m^i $(i = 0, 1, 2, \ldots, 15)$ on the chord diagrams without base crossing.

2. 2. How to transform a (welded virtual) link diagram into a chord diagram without base crossing

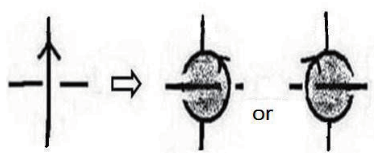

FIGURE 6. Transforming a neighborhood of a real crossing point into a part of a chord diagram

A transformation τ from a virtual link diagram D (see [**Kau**]) into a chord diagram C_D without base crossing is defined by replacing a neighborhood of every real crossing with either one of the two diagrams in the right hand of Fig. 6 and then by replacing a neighborhood of every virtual crossing by either one of the two diagrams in the right hand of Fig. 7.

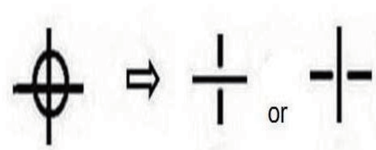

FIGURE 7. Transforming a neighborhood of a virtual crossing point into a part of a chord diagram

It is noted that every based loop of C_D is passed through by one chord. In general, such a chord diagram is called a *regular chord diagram*. Further, if every chord of C_D is oriented as it is given in Fig. 8, then every chord of C_D is compatibly oriented. In general, such a regular chord diagram is called an *oriented regular chord diagram*. For an oriented regular chord diagram, an orientation to one chord is sufficient to specify the orientations on all the chords. The chord system of a chord diagram is *simple* if there is no crossings between the chords [2]. If D is a link diagram, then the chord diagram C_D is an oriented regular chord diagram with simple chord system, and if D is a virtual link diagram, then the chord diagram C_D is an oriented regular chord diagram.

To avoid ambiguities on an oriented regular chord diagram C_D constructed from a virtual link diagram D, we also use the replacements given in Fig. 9. Then the diagram resulting from C_D is called the *flat chord diagram* of C_D and denoted

[2]By general position any crossings among three or more chords can be resolved into double crossings

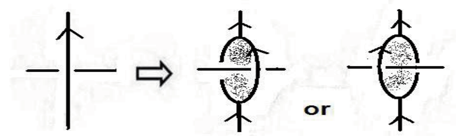

FIGURE 8. An oriented chord diagram

by $\bar{C}_D$. It is noted that the flat chord diagram $\bar{C}_D$ recovers the virtual link diagram D uniquely by taking the upper arcs on the based loops of any chord diagram C_D inducing $\bar{C}_D$ and then by replacing the crossing points on the chords of $\bar{C}_D$ with virtual crossing points.

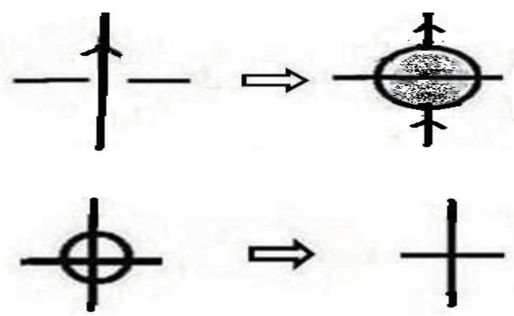

FIGURE 9. Transforming a neighborhood of a crossing point and a virtual crossing point into a flat chord diagram

Let $\mathbb{D}$ be the set of link diagrams, and $\mathbb{D}_{\mathbb{V}}$ the set of virtual link diagrams. Then we have $\mathbb{D} \subset \mathbb{D}_{\mathbb{V}}$. Let $\bar{\mathbb{C}}_{\mathbb{D}}$ be the set of flat chord diagrams obtained from the set $\mathbb{D}$ of link diagrams, and $\bar{\mathbb{C}}_{\mathbb{V}}$ the set of flat chord diagrams obtained from the set $\mathbb{D}_{\mathbb{V}}$ of virtual link diagrams. Then we have $\bar{\mathbb{C}}_{\mathbb{D}} \subset \bar{\mathbb{C}}_{\mathbb{V}}$. Let

$$\bar{\tau} : (\mathbb{D}_{\mathbb{V}}, \mathbb{D}) \to (\bar{\mathbb{C}}_{\mathbb{V}}, \bar{\mathbb{C}}_{\mathbb{D}})$$

be the bijection defined by sending every D to $\bar{C}_D$. The Reidemeister moves of classical, virtual and welded virtual link diagrams are transformed into moves on the flat chord diagrams as they are given in Fig. 10, where the Reidemeister moves $R_i\,(i = 1, 2, \ldots, 8)$ change into the moves $cR_i\,(i = 1, 2, \ldots, 8)$ on the flat chord diagrams, where orientations of the chords, naturally preserved are omitted.

The sets of links, virtual links and welded virtual links are by definition given by the quotient sets $\mathbb{D}/(R_1, R_2, R_3)$, $\mathbb{D}_{\mathbb{V}}/(R_i\,(i = 1, 2, \ldots, 7))$ and $\mathbb{D}_{\mathbb{V}}/(R_i\,(i = 1, 2, \ldots, 8))$, respectively. The following lemma is direct from construction.

Lemma 2.1. The bijection $\bar{\tau}$ induces bijections:

$$\begin{aligned} \bar{\tau}_* : \mathbb{D}/(R_1, R_2, R_3) &\to \bar{\mathbb{C}}_{\mathbb{D}}/(cR_1, cR_2, cR_3), \\ \bar{\tau}_* : \mathbb{D}_{\mathbb{V}}/(R_i\,(i = 1, 2, \ldots, 7)) &\to \bar{\mathbb{C}}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 7)), \\ \bar{\tau}_* : \mathbb{D}_{\mathbb{V}}/(R_i\,(i = 1, 2, \ldots, 8)) &\to \bar{\mathbb{C}}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 8)). \end{aligned}$$

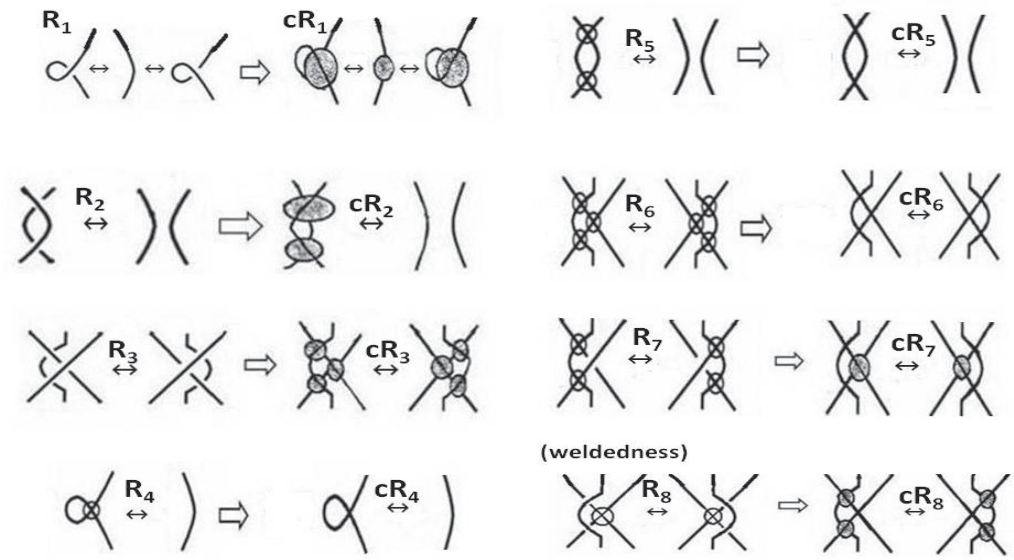

Figure 10. Replacements of the Reidemeister moves

The following lemma is known by [**GPV**] for (1) and [**R**] for (2).

Lemma 2.2. (1) The quotient map

$$\mathbb{D}/(R_1, R_2, R_3) \to \mathbb{D}_{\mathbb{V}}/(R_i\,(i = 1, 2, \ldots, 7))$$

is injective.
(2) The composite quotient map

$$\mathbb{D}/(R_1, R_2, R_3) \to \mathbb{D}_{\mathbb{V}}/(R_i\,(i = 1, 2, \ldots, 7)) \to \mathbb{D}_{\mathbb{V}}/(R_i\,(i = 1, 2, \ldots, 8))$$

is injective.

It is known that the quotient map

$$\mathbb{D}_{\mathbb{V}}/(R_i\,(i = 1, 2, \ldots, 7)) \to \mathbb{D}_{\mathbb{V}}/(R_i\,(i = 1, 2, \ldots, 8))$$

is not injective, so that the quotient map

$$\bar{\mathbb{C}}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 7)) \to \bar{\mathbb{C}}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 8))$$

is not injective. For example, see [**Kaw**, Fig.15] for how a non-trivial virtual knot diagram in $\mathbb{D}_{\mathbb{V}}/(R_i\,(i = 1, 2, \ldots, 7))$ is equivalent to a trivial knot in $\mathbb{D}_{\mathbb{V}}/(R_i\,(i = 1, 2, \ldots, 8))$ in the flat chord diagram level. On the other hand, it is noted by Lemma 2.2 that the composite map

$$\bar{\mathbb{C}}_{\mathbb{D}}/(cR_1 - cR_3) \to \bar{\mathbb{C}}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 7)) \overset{non-injective}{\to} \bar{\mathbb{C}}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 8))$$

is injective. Let $\mathbb{C}_{\mathbb{D}}$ be the set of chord diagrams obtained from the set $\mathbb{D}$ of link diagrams, and $\mathbb{C}_{\mathbb{V}}$ the set of chord diagrams obtained from the set $\mathbb{D}_{\mathbb{V}}$ of virtual link diagrams. Then we have $\mathbb{C}_{\mathbb{D}} \subset \mathbb{C}_{\mathbb{V}}$. The *elementary move* m^0 in Fig. 11 identifying the two diagrams of the right hand sides in Fig. 6 and in Fig. 7 is introduced to obtain the following natural bijective transformation

$$\tau : (\bar{\mathbb{C}}_{\mathbb{V}}, \bar{\mathbb{C}}_{\mathbb{D}}) \to (\mathbb{C}_{\mathbb{V}}/(m^0), \mathbb{C}_{\mathbb{D}}/(m^0))$$

defined by Fig. 6 and Fig. 7, by which the pair $(\bar{\mathbb{C}}_{\mathbb{V}}, \bar{\mathbb{C}}_{\mathbb{D}})$ is identified with the quotient pair $(\mathbb{C}_{\mathbb{V}}/(m^0), \mathbb{C}_{\mathbb{D}}/(m^0))$. Let $\mathbb{C}$ be the set of full oriented chord diagrams, and $\mathbb{C}[0]$ the set of full oriented chord diagrams without base crossing. Then we have

$$\mathbb{C}_{\mathbb{D}} \subset \mathbb{C}_{\mathbb{V}} \subset \mathbb{C}[0] \subset \mathbb{C}.$$

Since the elementary move m^0 is in the moves M_0 and M_2, the following lemma is checked without difficulty by comparing the moves $cR_i\,(i = 1, 2, \ldots, 8)$ with the moves M_0, M_1, M_2.

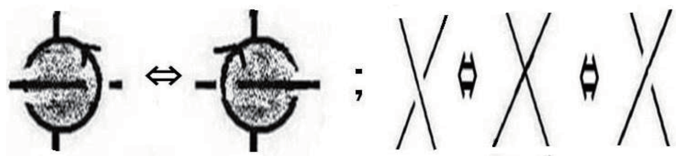

FIGURE 11. Move m^0

Lemms 2.3. The moves $cR_i\,(i = 1, 2, \ldots, 8)$ on the set $\mathbb{C}_{\mathbb{V}}$ of oriented chord diagrams are the consequences of the moves M_0, M_1, M_2 on $\mathbb{C}$.

By Lemms 2.3, the induced map

$$\tau_* : \mathbb{C}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 8)) \to \mathbb{C}/(M_0, M_1, M_2)$$

is obtained. It is noted that this map τ_* is not injective. For example, if a link ℓ is not equivalent to the reflected inverse $-\ell^*$ of ℓ, namely ℓ is a negative amphicheiral link, then the links ℓ and $-\ell^*$ are not equivalent in $\bar{\mathbb{C}}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 8))$, but equivalent in $\mathbb{C}/(M_0, M_1, M_2)$. See [**Kaw**, Proposition 3.4] for this fact.

Problem. Introduce new relations $\{cR_*\}$ to the set $\bar{\mathbb{C}}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 8))$ so that the map

$$\tau_* : \bar{\mathbb{C}}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 8)) \to \mathbb{C}/(M_0, M_1, M_2)$$

induces an injection

$$\tau_{**} : \bar{\mathbb{C}}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 8), \{cR_*\}) \to \mathbb{C}/(M_0, M_1, M_2).$$

As it will be noted in § 3, the induced map

$$\tau_* : \bar{\mathbb{C}}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 8)) \to \mathbb{C}/(M_0, M_1, M_2)$$

is not surjective. On this problem, the main result (Theorem 4.1) of this paper will show that the set $\mathbb{C}$ with the relations M_0, M_1, M_2 can be replaced by the subset $\mathbb{C}[0]$ with the relations $m^i\,(i = 0, 1, 2, \ldots, 15)$.

3. 3. How to transform a chord graph into a ribbon surface-link in 4-space

A *surface-link* in $\mathbf{R}^4$ is a closed oriented (possibly disconnected) surface F embedded in $\mathbf{R}^4$ by a smooth embedding or a piecewise-linear locally flat embedding. When F is connected, it is called a *surface-knot* in $\mathbf{R}^4$. Two surface-links F and F' in $\mathbf{R}^4$ are *equivalent* if there is an orientation-preserving smooth or piecewise-linear homeomorphism $f : \mathbf{R}^4 \to \mathbf{R}^4$ sending F to F' orientation-preservingly. A surface-link F in $\mathbf{R}^4$ is *trivial* If F is equivalent to a surface standard embedded in the 3-subspace $\mathbf{R}^3 \subset \mathbf{R}^4$. Let $\mathbb{F}$ be the set of equivalence classes of surface-links. An *embedded 1-handle* on a surface-link F in $\mathbf{R}^4$ is the image $h = \mathrm{im}(\psi)$ of an embedding

$$\psi : D^2 \times I \to \mathbf{R}^4$$

such that

$$F \cap h = \psi(D^2 \times \{0, 1\}),$$

where D^2 is the unit disk and $I = [0, 1]$. The *surface-link obtained from F by surgery along the embedded 1-handle* h is the surface-link

$$F' = \mathrm{cl}(F \setminus F \cap h) \cup \mathrm{cl}(\partial h \setminus F \cap h)$$

in $\mathbf{R}^4$. A surface-link F in $\mathbf{R}^4$ is *ribbon* if F is obtained from a trivial S^2-link

$$O = S^2_0 \cup S^2_1 \cup \cdots \cup S^2_n$$

in $\mathbf{R}^4$ by surgery along embedded 1-handles $h_j \, (j = 1, 2, \ldots, s)$ on O. Let $\mathbb{F}^r$ be the set of equivalence classes of ribbon surface-links which is a subset of the set $\mathbb{F}$ of all surface-links.

For the real line $\mathbf{R}$, 4-space $\mathbf{R}^4$ is considered as

$$\mathbf{R}^4 = \{(x, t) |\, x \in \mathbf{R}^3, t \in \mathbf{R}\}.$$

For a subset $A \subset \mathbf{R}^3$ and an interval J, we use the notation

$$AJ = \{(x, t) |\, x \in A, t \in J\}.$$

The transformation from a chord graph $(o; \alpha)$ in $\mathbf{R}^3$ to a ribbon surface-link $F(o; \alpha)$ in $\mathbf{R}^4$ is constructed as follows: Let d be a disk system for the trivial link o in $\mathbf{R}^3$. Let $O = d[-2] \cup o[-2, 2] \cup d[2]$ be a trivial S^2-link in $\mathbf{R}^4$. Let $\beta(\alpha)$ be the disjoint union of bands in $\mathbf{R}^3$ spanning o and thickening the chords α, where the bands $\beta(\alpha)$ should have their orientations coherent to the orientations of o, but any full twisting ambiguities are granted. Let $h(\alpha) = \beta(\alpha)[-1, 1]$ be the embedded 1-handles on O. Then the desired ribbon surface-link $F(o; \alpha)$ in $\mathbf{R}^4$ is obtained from the trivial S^2-link O by surgery along the embedded 1-handles $h(\alpha)$ on O. It is noted that the ribbon surface-link $F(o; \alpha)$ is uniquely obtained from the chord graph $(o; \alpha)$ in $\mathbf{R}^3$ up to equivalences. In fact, the independence on choices of a disk system d is given by Horibe-Yanagawa's lemma in [**KSSI**] and the independence on choices of full twists of the bands $\beta(\alpha)$ is given [**?**, Theorem 1.2]. Thus, the map

$$\kappa : \mathbb{C} \to \mathbb{F}^r$$

from a chord diagram $C(o; \alpha)$ to the equivalence class of the ribbon surface-link $F(o; \alpha)$ is obtained. This map κ is surjective, since it is shown in [**KSSII**, Lemma 4.11] that every ribbon surface link in $\mathbf{R}^4$ is equivalent to $F(o; \alpha)$ for a chord graph $(o; \alpha)$. Let $F(o; h(\alpha)) = F(o; \alpha) \cup h(\alpha)$. A ribbon surface-link $F(o; \alpha)$ is *faithfully equivalent* to a ribbon surface-link $F(o'; \alpha')$ if there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from

$F(o;\alpha)$ to $F(o';\alpha')$ sending every meridian (i.e., belt 1-sphere) of the 1-handles $h(\alpha)$ to a null-homotopic curve in $F(o';h(\alpha'))$. The faithful equivalence is an equivalence relation on ribbon surface-links (see [**Kaw**]). Let $\mathbb{F}^{fr}$ be the set of faithful equivalence classes of ribbon surface-links. There is a natural surjection

$$\mathbb{F}^{fr} \to \mathbb{F}^r,$$

but it is unknown whether or not this map is a bijection for all ribbon surface-links. The following lemma is the main theorem of [**Kaw**].

Lemma 3.1. The map $\kappa : \mathbb{C} \to \mathbb{F}^r$ defines a bijection

$$\kappa_* : \mathbb{C}/(M_0, M_1, M_2) \to \mathbb{F}^{fr}.$$

The image of the composite map

$$\kappa_*\tau_* : \mathbb{C}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 8)) \to \mathbb{C}/(M_0, M_1, M_2) \to \mathbb{F}^{fr}$$

is directly seen to consist of the faithful equivalence classes of ribbon torus-links, which is earlier given by Satoh [**S**] by a different method replacing the faithful equivalence by the equivalence. This means that the map

$$\tau_* : \mathbb{C}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 8)) \to \mathbb{C}/(M_0, M_1, M_2)$$

is not surjective, as was announced in § 2. Let $\mathbb{F}^{fr}_1$ be the subset of $\mathbb{F}^{fr}$ consisting of the faithful equivalence classes of ribbon torus-links, and $\mathbb{C}_1$ the subset of $\mathbb{C}$ such that $\kappa(\mathbb{C}_1)$ consists of ribbon torus-links. Then the map $\tau_* : \mathbb{C}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 8)) \to \mathbb{C}/(M_0, M_1, M_2)$ defines a surjection

$$\tau^1_* : \mathbb{C}_{\mathbb{V}}/(cR_i\,(i = 1, 2, \ldots, 8)) \to \mathbb{C}_1/(M_0, M_1, M_2)$$

and the map $\kappa : \mathbb{C} \to \mathbb{F}^r$ defines a bijection

$$\kappa^1_* : \mathbb{C}_1/(M_0, M_1, M_2) \to \mathbb{F}^{fr}_1.$$

4. 4. How to modify the moves on a chord diagram into the moves on a chord diagram without base crossing

For the set $\mathbb{C}$ of chord diagrams, let $\mathbb{C}[0]$ be the subset of $\mathbb{C}$ consisting chord diagrams without base crossing. Then the inclusion $\mathbb{C}[0] \subset \mathbb{C}$ induces a surjection

$$\iota : \mathbb{C}[0] \to \mathbb{C}/(M_0, M_1, M_2)$$

which is seen from the move M_0. We provide the moves $m^i\,(i = 1, 2, \ldots, 7)$ on $\mathbb{C}[0]$ illustrated in Fig. 12 and the moves $m^i\,(i = 7, 8, \ldots, 15)$ on $\mathbb{C}[0]$ illustrated in Fig. 13. It is noted that in these figures, the based loop without indication of an orientation can have any orientation which is preserved by the move. Then we show the following theorem.

Theorem 4.1. For the 16 moves $m^i\,(i = 0, 1, 2, \ldots, 15)$ on $\mathbb{C}[0]$ illustrated in Fig. 11, Fig. 12 and Fig. 13, the surjection $\iota : \mathbb{C}[0] \to \mathbb{C}/(M_0, M_1, M_2)$ induces a bijection

$$\iota_* : \mathbb{C}[0]/(m^i\,(i = 0, 1, 2, \ldots, 15)) \to \mathbb{C}/(M_0, M_1, M_2).$$

Proof of Theorem 4.1. Let $C = C(o;\alpha)$ and $C' = C(o';\alpha')$ be chord diagrams without base crossing such that C' is obtained from C by a finite number of the

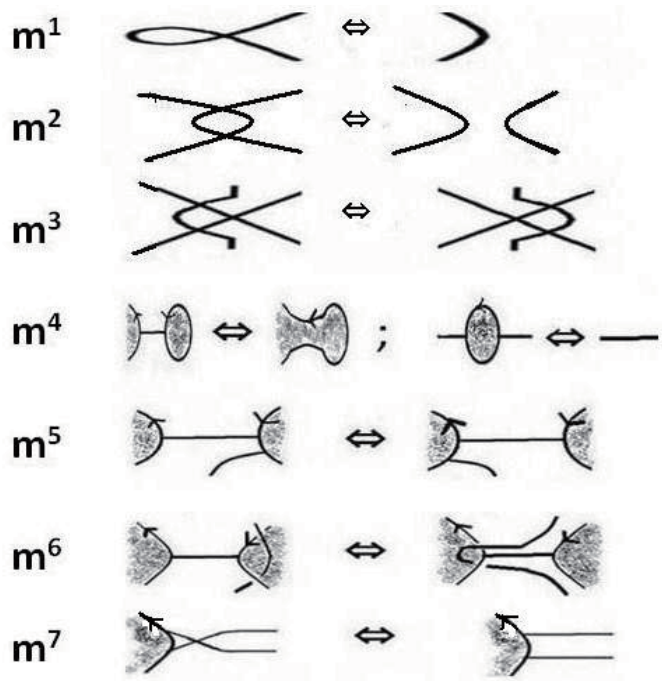

FIGURE 12. The moves $m^1, m^2, \ldots, m^7$ on chord diagrams without base crossing

M_0, M_1, M_2 moves. Assume that in every step of the moves, any self-attaching chord (i.e., any chord attaching to only one based loop) is not contained if necessary, by increasing a based loop and that the chord graph is non-splittable by using M_2-moves. Then, by using the moves on M_0, it is assumed that every move on M_1 or M_2 is performed on the set $\mathbb{C}[0]$ of chord diagrams without base crossing. Assume that C and C' are chord diagrams without base crossing such that C' is obtained from C by a finite number of the M_0-moves. Then there is an orientation-preserving piecewise-linear homeomorphism $h : \mathbf{R}^3 \to \mathbf{R}^3$ sending the trivalent graph $(o; \alpha)$ to the trivalent graph $(o'; \alpha')$ preserving o to o'. Let d and d' be disk systems bounded by o and o' in $\mathbf{R}^3$ and embedded in the plane $\mathbf{R}^2$. Assume that a neighborhood annulus system $A(o)$ of o in d is sent to a neighborhood annulus system $A(o')$ of o' in d' by h. Further, assume that the chord systems α and α' transversely meet the disk systems d and d' in $A(o)$ and $A(o')$, respectively. Consider a transverse intersection of $h(d)$ and d' consisting simple loops in $h(d \setminus A(o))$ and $d' \setminus A(o')$. Then the disk system $h(d)$ is deformed so that $h(d) \cap d' = h(o) = o'$ by a finite sequence of a 3-cell collapsing not meeting $h(\alpha) = \alpha'$ in the 3-sphere compactification $S^3 = \mathbf{R}^3 \cup \{\infty\}$. This means that in a diagram level, only the chord move M_2, a disk system replacement not meeting $h(\alpha) = \alpha'$ and a move called the *chord diagram pass* and illustrated in Fig. 14 where the box denotes a subdiagram including some based loops are needed to make $h(d) = d'$.

Now, assume that $h(d) = d'$ and $h(\alpha) = \alpha'$. Let $G(\alpha)$ and $G(\alpha')$ be the spatial graphs in $\mathbf{R}^3$ obtained from the complexes $\alpha \cup d$ and $\alpha' \cup d'$, respectively by shrinking

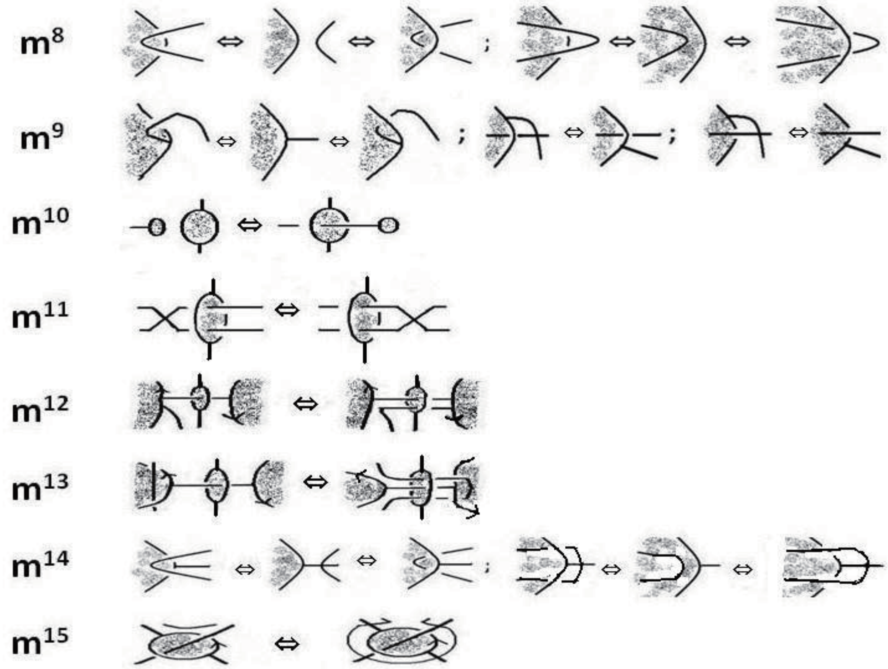

FIGURE 13. The moves $m^8, m^9, \ldots, m^{15}$ on chord diagrams without base crossing

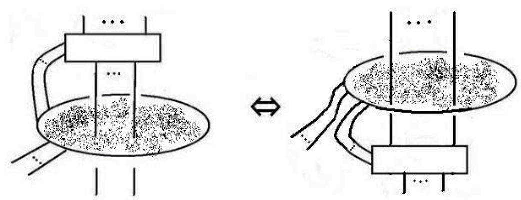

FIGURE 14. A chord diagram pass

every disk in d and d' into a point. The homeomorphism h is modified to be an orientation-preserving piecewise-linear homeomorphism $h' : \mathbf{R}^3 \to \mathbf{R}^3$ sending $G(\alpha)$ to $G(\alpha')$. Since h' is ambient isotopic to the identity, it is seen from [**Kauf**] and [**KKM**, Corollary 1.4] that any diagram of the spatial graph $G(\alpha)$ is deformed into any diagram of the spatial graph $G(\alpha')$ by the generalized Reidemeister moves on spatial graphs. However, since every edge of $G(\alpha)$ and $G(\alpha')$ comes from a chord and hence homotopically changeable by the chord move M_2, so that in a diagram level only a move illustrated in Fig. 15 and called the *base pass* is needed up to the chord move M_2.

It is noted that the move m^i $(i = 0, 1, 2, \ldots, 7)$ correspond to the M_1 and M_2 moves. Since the disk bounded by a based loop is splitted by disks by the moves m^4, m^5, m^6 such that every disk in the member meets the chords either in one point

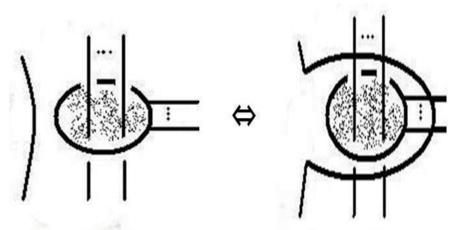

FIGURE 15. A base pass

in the interior and at most 2 points in the boundary or only in at most 3 points in the boundary, we see that the chord diagram pass and the base pass are generated by m^i $(i = 8, 9, \ldots, 15)$. Thus, the conclusion is obtained. $\square$

References

[GPV] M. Goussarov, M. Polyak and O. Viro, Finite-type invariants of classical and virtual knots, Topology 39 (2000), 1045-1068.

[HK] F. Hosokawa and A. Kawauchi, Proposals for unknotted surfaces in four-space, Osaka J. Math. 16 (1979), 233-248.

[KKM] S. Kamada, A. Kawauchi and T. Matumoto, Combinatorial moves on ambient isotopic submanifolds in a manifold, J. Math. Soc. Japan 53 (2001),321-331.

[Kauf] L. H. Kauffman, Invariants of graphs in three space, Trans. Amer. Math. Soc. 311 (1989), 697-710.

[Kau] L. H. Kauffman, Virtual knot theory, European J. Combin. 20 (1999), 663-690.

[K] A. Kawauchi, A survey of knot theory, Birkhäuser (1996).

[Kaw] A. Kawauchi, A chord diagram of a ribbon surface-link, Journal of Knot Theory and Its Ramifications 24 (2015), 1540002 (24 pages).

[Kaw'] Akio Kawauchi, *Supplement to a chord diagram of a ribbon surface-link*, J. Knot Theory Ramifications (to appear).

[KSSI] A. Kawauchi, T. Shibuya and S. Suzuki, Descriptions on surfaces in four-space, I : Normal forms, Math. Sem. Notes, Kobe Univ. 10 (1982), 75-125.

[KSSII] A. Kawauchi, T. Shibuya and S. Suzuki, Descriptions on surfaces in four-space, II: Singularities and cross-sectional links, Math. Sem. Notes, Kobe Univ. 11 (1983), 31-69.

[R] C. Rourke, What is a welded link ?, Intelligence of Low Dimensional Topology 2006, (2007), 263-270, World Sci. Publ. Co..

[S] S. Satoh, Virtual knot presentation of ribbon torus-knots, J. Knot Theory Ramifications, 9 (2000), 531- 542.

OSAKA CITY UNIVERSITY ADVANCED MATHEMATICAL INSTITUTE, SUGIMOTO, SUMIYOSHI-KU, OSAKA 558-8585, JAPAN

E-mail address: kawauchi@sci.osaka-cu.ac.jp