Complexities of a knitting pattern

Akio Kawauchi

ABSTRACT. A knitting pattern is a tangle diagram in the square and a knitting is a tessellation of a knitting pattern in the plane. In this paper, a classification of the knitting patterns is done in terms of virtual links and some complexities of a knitting pattern by some topological invariants are studied. Generalizations to a chain pattern and a graph-tangle pattern are also discussed.

1. Introduction

A knitting pattern is constructed from a link diagram in the torus with every component essential by cutting along the meridian and the longitude of the torus, which is defined in Section 2. A knitting is the lift of the link diagram to the universal covering plane of the torus, namely, a tessellation of the knitting pattern by the covering transformation group of the plane. Thus, a knitting is determined uniquely by a knitting pattern. 10 examples of knitting patterns and knittings are given in Section 2. In Section 3, equivalence of a knitting pattern is defined and stated in terms of virtual link diagrams (see Theorem 3.3) and the virtual number of a knitting pattern is defined. In Section 4, the crossing number, the (enhanced) warping degree, the (enhanced) unknotting number, the cross-index and the linking degree are introduced to investigate a complexity of a knitting pattern. The calculation results on the 10 examples of knitting patterns in Section 2 are list in the end of Section 4. In Section 5, we state how to construct a knitting pattern as Theorem 5.1. As related patterns, a chain pattern and a graph-tangle pattern are discussed in Sections 6 and 7, respectively.

2. A knitting pattern

Let $S^1 = R^1/Z$ be the circle, and $T^2 = S^1 \times S^1$ the torus. Let $Q = [0, 1]^2$ be the square obtained from the torus T^2 by cutting along the meridian $S^1 \times 0$ and the longitude $0 \times S^1$ (see Fig. 1). Let K_T denote a link diagram in the torus T^2. Let m and n be nonnegative integers with $m + n > 0$.

Definition 2.1. A (m, n)-*knitting pattern* is a tangle diagram K in the square Q obtained from a link diagram K_T in T^2 by cutting along $S^1 \times 0$ and $0 \times S^1$ such that

2010 *Mathematics Subject Classification.* 68R10.

Key words and phrases. Knitting pattern, Virtual link, Crossing number, Warping degree, Unknotting number, Cross-index, Linking degree.

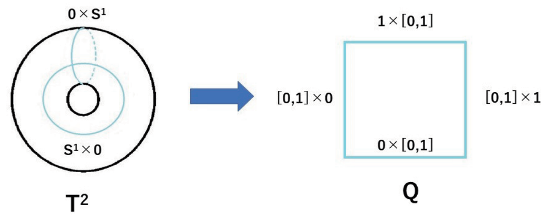


FIGURE 1. From the torus T^2 to the square Q

(1) any component of K_T is *essential* (that is, not null-homotopic in T^2), and
(2) the link diagram K_T meets $0 \times S^1$ and $S^1 \times 0$ (avoiding $(0,0)$) transversely with m and n points, respectively (see Fig 2).

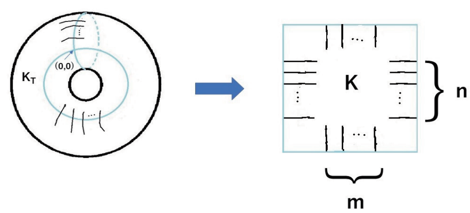


FIGURE 2. From a link diagram in T^2 to a (m,n)-knitting pattern

An example of the procedure from a link diagram in T^2 to a $(2,2)$-knitting pattern is illustrated in Fig. 3.

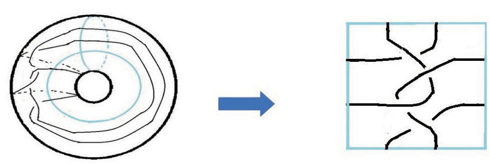


FIGURE 3. From a link diagram in T^2 to a $(2,2)$-knitting pattern

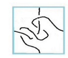
Definition 2.2. A *periodic knitting* (or simply a *knitting*) is the lift $\tilde{K}$ of a link diagram K_T to the universal covering plane R^2 of the torus T^2, namely, a tessellation of the knitting pattern (Q,K) by the fixed covering transformation group Z^2.

Here are examples on (m,n)-knitting patterns K with $0 \le m \le n \le 3$ and $m+n \ge 2$.

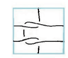
Example I. The knitting $\tilde{K}$ of a $(1,1)$-knitting pattern $K =$  is given in Fig. 4.

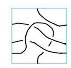
Example II. The knitting $\tilde{K}$ of a $(0,2)$-knitting pattern $K =$  is given in Fig. 5.

Example III. The knitting $\tilde{K}$ of another $(0,2)$-knitting pattern $K =$  is given in Fig. 6.

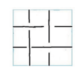
Example IV. The knitting $\tilde{K}$ of another $(1,1)$-knitting pattern $K =$  is given in Fig. 7.

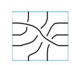
Example V. The knitting $\tilde{K}$ of a $(1,2)$-knitting pattern $K =$  is given in Fig. 8.

Example VI. The knitting $\tilde{K}$ of another $(1,2)$-knitting pattern $K =$  is given in Fig. 9.

Example VII. The knitting $\tilde{K}$ of a $(1,3)$-knitting pattern $K =$  is given in Fig. 10.

Example VIII. The knitting $\tilde{K}$ of a $(2,2)$-knitting pattern $K =$  is given in Fig. 11.

Example IX. The knitting $\tilde{K}$ of another $(2,2)$-knitting pattern $K =$  is given in Fig. 12.

Example X. The knitting $\tilde{K}$ of a $(2,3)$-knitting pattern $K =$  is given in Fig. 13.

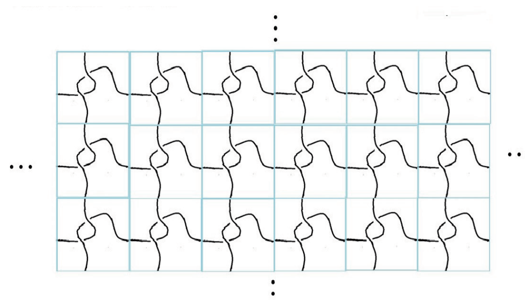


FIGURE 4. Knitting of a $(1,1)$-knitting pattern

3. Equivalence of a knitting pattern.

Two classical link diagrams are *equivalent* if one is moved into the other by a finite number of Reidemeister moves in the 2-sphere S^2 (namely, by the relations R1, R2 and R3 in Fig. 14). Two link diagrams K_T and K'_T in T^2 are *TR-equivalent* if there is an orientation-preserving diffeomorphism g of T^2 such that $g(K_T)$ is moved into K'_T by a finite number of Reidemeister moves in T^2 , namely by the relations R1,R2 and R3 in Fig. 14 in T^2. We put the following definition.

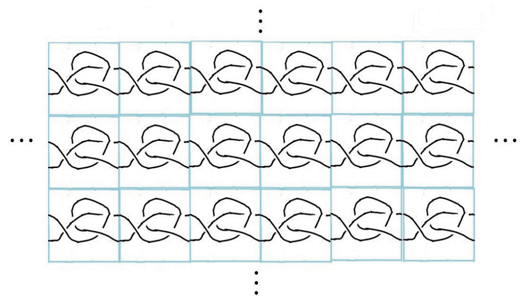


FIGURE 5. Knitting of a $(0, 2)$-knitting pattern

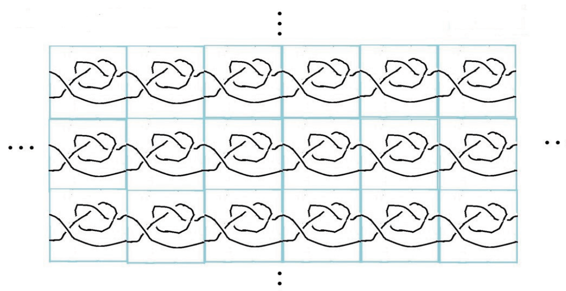


FIGURE 6. Knitting of another $(0, 2)$-knitting pattern

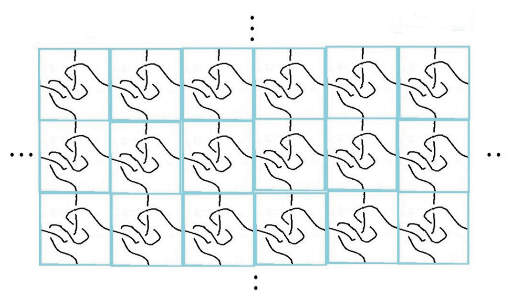


FIGURE 7. Knitting of a $(1, 1)$-knitting pattern

Definition. A (m, n)-knitting pattern K and a (m', n')-knitting pattern K' are *equivalent* if K_T is TR-equivalent to K'_T in T^2.

In the universal covering space R^2 of T^2, this definition is interpreted that the knitting $\tilde{K}$ of K is Z^2-equivalent to the knitting $\tilde{K}'$ of K', meaning that the image $\tilde{g}(\tilde{K})$ by a Z^2-equivariant diffeomorphism $\tilde{g}$ of R^2 is moved into $\tilde{K}'$ by a finite number of Z^2-equivariant Reidemeister moves. Let $T^2[-1, 1]$ be the product of T^2 and the interval $[-1, 1]$. Let $K_T[*]$ be a geometric realization of the link diagram K_T in the interior of $T^2[-1, 1]$. A *geometric realization* of a knitting $\tilde{K}$ is the lift

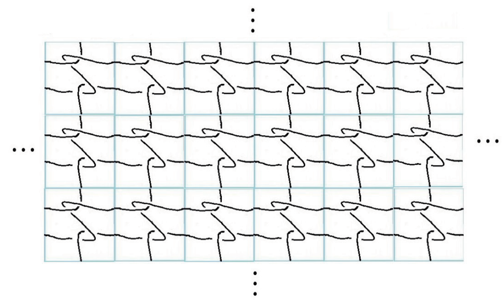


FIGURE 8. Knitting of a $(1,2)$-knitting pattern

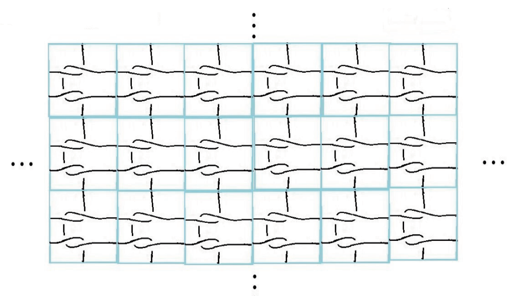


FIGURE 9. Knitting of another $(1,2)$-knitting pattern

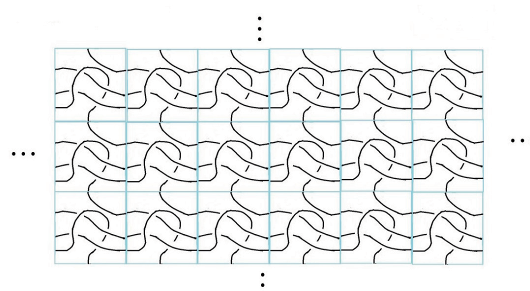


FIGURE 10. Knitting of a $(1,3)$-knitting pattern

$\tilde{K}[*]$ of $K_T[*]$ to the universal covering $R^2[-1,1]$ of $T^2[-1,1]$. The following lemma clarifies the definition of the equivalence of a knitting pattern.

Lemma 3.1. A (m,n)-knitting pattern K is equivalent to a (m',n')-knitting pattern K' if and only if there is an orientation-preserving, boundary-preserving Z^2-equivariant diffeomorphism $\tilde{f}: R^2[-1,1] \to R^2[-1,1]$ sending $\tilde{K}[*]$ to $\tilde{K}'[*]$.

Proof. If a (m,n)-knitting pattern K is equivalent to a (m',n')-knitting pattern K', then there is an orientation-preserving, boundary-preserving diffeomorphism $f: T^2[-1,1] \to T^2[-1,1]$ sending $K_T[*]$ to $K'_T[*]$, which is equivalent to that there is

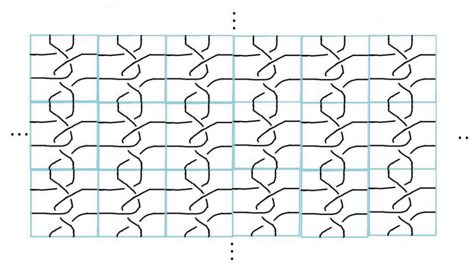


Figure 11. Knitting of a $(2,2)$-knitting pattern

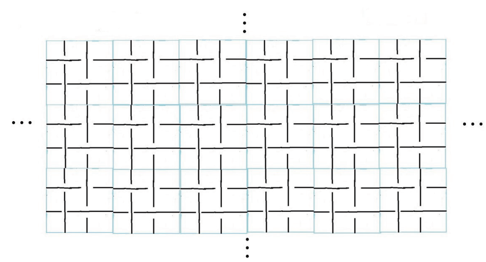


Figure 12. Knitting of another $(2,2)$-knitting pattern

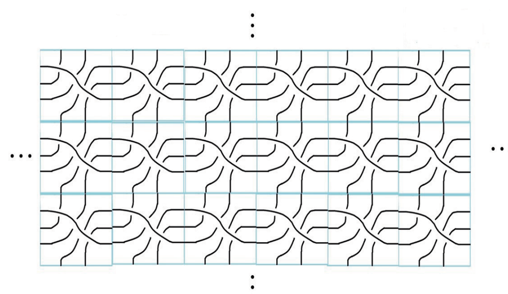
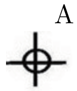


Figure 13. Knitting of a $(2,3)$-knitting pattern

an orientation-preserving, boundary-preserving Z^2-equivariant diffeomorphism $\tilde{f}$: $R^2[-1,1] \to R^2[-1,1]$ sending $\tilde{K}[*]$ to $\tilde{K}'[*]$ by lifting $T^2[-1,1]$ to $R^2[-1,1]$. Since an orientation-preserving, boundary-preserving diffeomorphism $f : T^2[-1,1] \to T^2[-1,1]$ sending $K_T[*]$ to $K'_T[*]$ means that K_T is TR-equivalent to K'_T in T^2 (see [**6, 8, 13**]), the desired result is obtained. $\square$

A *virtual link diagram* is a link diagram with additional virtual crossings like  which was introduced by Kauffman [**11**]. The *virtual link* is the equivalence class of a virtual link diagram divided by the equivalence relation generated by the *virtual Reidemeister moves* on virtual link diagrams illustrated in Fig. 14. In other words, two virtual link diagrams are *VR-equivalent* if one is obtained from the other by a finite number of the virtual Reidemeister moves. A virtual link diagram is *classical* if it is VR-equivalent to a classical link diagram. Otherwise, *non-classical*.

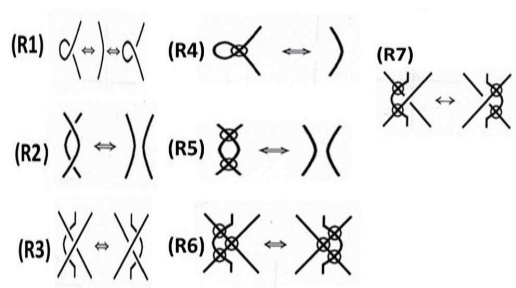


FIGURE 14. Virtual Reidemeister moves

We construct a virtual link diagram K_V from a (m, n)-knitting pattern K as shown in Fig. 15, where the product mn is the number of virtual crossing points in the virtual link diagram K_V and called the *virtual number* $v(K)$ of K. See Fig. 16 for a concrete example.

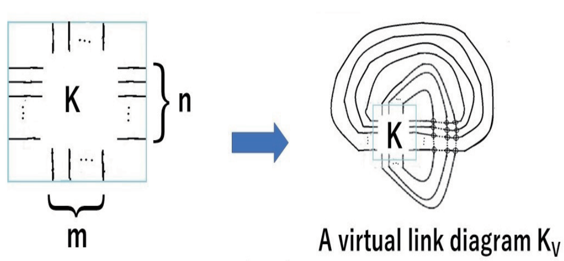


FIGURE 15. Constructing a virtual link diagram K_V with virtual number $v(K) = mn$ from a (m, n)-knitting pattern K

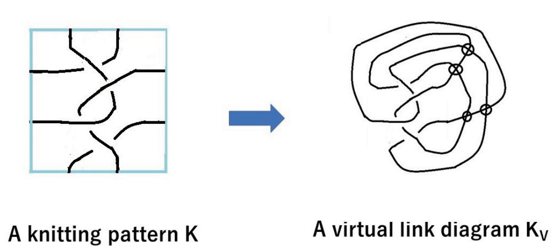


FIGURE 16. Constructing a virtual link diagram K_V with virtual number $v(K) = 4$ from a $(2, 2)$-knitting pattern K

From the following lemma (whose proof is omitted since it is easily obtained), we see that every string of a knitting is a proper line in the plane.

Lemma 3.2. Let K be a tangle in Q with the same antipodal points in ∂Q, and K_T and K_V the link diagram in T^2 and the virtual link diagram, obtained from K by connecting the antipodal points in ∂Q, respectively. Then the following (1)-(3) are mutually equivalent.

(1) Any component of a link diagram K_T is not null-homotopic in T^2.
(2) Any oriented loop obtained from strings in K by connecting the antipodal points in ∂Q has a non-zero intersection number with one of the 4 edges of ∂Q.
(3) Any oriented component of the virtual link diagram K_V has a nonzero intersection number with one of the 4 edges of ∂Q.

Two link diagrams K_T and K'_T in T^2 are *AR-equivalent* if K_T and K'_T are moved into link diagrams K_A and K'_A in annuli A and A' in T^2 by a finite number of Reidemeister moves, respectively such that K_A is sent to K'_A by an orientation-preserving diffeomorphism $g_A : A \to A'$.

Theorem 3.3. For a (m, n)-knitting pattern K and a (m', n')-knitting pattern K', we have the following (1) and (2).

(1) If K_T is AR-equivalent to K'_T, then the virtual link diagrams K_V and K'_V are VR-equivalent and classical.

(2) K is equivalent to K' if and only if either K_T is AR-equivalent to K'_T or K_V and K'_V are VR-equivalent and non-classical.

Proof. (1) is obtained from [**2, 7**] since K_T and K'_T belong to a stably equivalence class of a classical link. (2) is obtained by [**19**] stating a relationship between a virtual link and a minimal stable equivalence of a link under an interpretation of a virtual link in [**2, 7**]. □

Note that there are many computable topological invariants for the VR-equivalence of virtual link diagrams, e.g., Jones polynomial ([**10**]), Miyazawa polynomial ([**21**], Writhe polynomial ([**3, 4, 5, 27**]), etc. Also, we mention that a classification of knots with at most 5 crossings in T^2 were made in [**1**].

4. Topological complexities of a tangle diagram

In this section, the square $Q = [0, 1]^2$ is regarded as a disk. An *r-string tangle diagram* is the union K of r arcs α_i $(i = 1, 2, \ldots, r)$ properly immersed and transversely intersected in the disk Q whose double point singularities are in the interior of Q and have over and under information. Two tangle diagrams are *equivalent* if one is moved into the other by a finite number of Reidemeister moves in the interior of the disk Q , namely by the relations R1, R2 and R3 in Fig. 14. Let $[K]$ be the equivalence class of a tangle diagram K. Tangle diagram invariants used for a complexity of a tangle diagram are introduced in the following (1)-(5):

(1) Crossing number. The *crossing number* of a tangle diagram K is the number $c(K)$ of the crossing points of K. Let

$$c[K] = \min\{c(K')|K' \in [K]\}.$$

A tangle diagram K in the disk Q is *reduced alternating* if an over-crossing and an under-crossing appear alternately for every string of K and any 1-string tangle diagram with a crossing is not contained in the tangle diagram (Q, K). Note that even if K is a reduced alternating knitting pattern, the knitting $\tilde{K}$ need not be an alternating diagram. For alternating knitting patterns in the exmaples of Section 2, the knittings of Examples II, III, VIII and IX are alternating, but the knittings of Examples I, IV and V are not alternating. We have the following observation.

Lemma 4.1. If a tangle diagram K is a reduced alternating tangle diagram, then $c[K] = c(K)$.

Proof. Put the tangle diagram K in the upper-half plane R^2_+. Let K^* be the tangle diagram in the lower-half plane R^2_- obtained from K by reflecting K in the line $\partial R^2_+ = \partial R^2_-$ and then changing upper and lower information on every crossing. Then the tangle sum $K + K^*$ is a reduced alternating link diagram in the plane R^2. By the Murasugi-Kauffman theorem [**10, 22**], the crossing number $c(K \cup K^*)$ of the reduced alternating link diagram $K \cup K^*$ in S^2 is a Reidemeister move invariant and equal to $2c(K)$, showing that $c[K] = c(K)$. □

FIGURE 17. Monotone arc diagrams

(2) Warping Degree. This notion is discussed in many papers [**14, 15, 16, 20, 23, 24, 25, 26**], where we use the string version in [**14, 15**]. Let $K = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_r$ be an r-string tangle diagram in Q. An arc component diagram α_i is *monotone* if a point going along α_i from one of the end points meets first the upper crossing point at every crossing (see Fig. 17). For any strings α_i and α_j with $i \neq j$, it is denoted by $\alpha_i > \alpha_j$ that α_i is upper than α_j at every crossing. The tangle diagram K is *monotone* if every 1-string tangle diagram is monotone and $\alpha_{i_1} > \alpha_{i_2} > \cdots > \alpha_{i_r}$ for a permutation $(i_1, i_2, \dots, i_r)$ of $(1, 2, \dots, r)$. The *warping degree* of an r-string tangle diagram K is the minimal number $d(K)$ of crossing changes on K needed to transform K into a monotone tangle diagram. Let $d[K] = \min\{d(K')|\, K' \in [K]\}$.

(3) Unknotting number. This notion is introduced in [**15, 16**] as a generalization of the usual unknotting number of a link. An r-string tangle diagram K is *unknotted* if $d[K] = 0$. The *unknotting number* of an r-string tangle diagram K is the minimal number $u(K)$ of crossing changes on K needed to obtain an unknotted tangle diagram. Let $u[K] = \min\{u(K')|\, K' \in [K]\}$. Let $u^s(K) = \sum_{i=1}^r u(\alpha_i)$, and $u^s[K] = \min\{u^s(K')|\, K' \in [K]\}$. We have $u[K] \geq u^s[K]$.

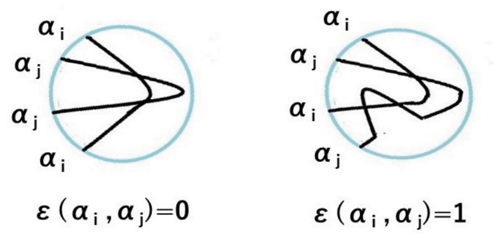


FIGURE 18. The elementary cross-index

(4) Cross-index. This notion is introduced in [**16, 18**]. The *elementary cross-index* of any strings α_i and α_j with $i \neq j$ of an r-string tangle diagram K is the number $\varepsilon(\alpha_i, \alpha_j)$ being 0 or 1 according to whether or not the set $\partial\alpha_j$ is contained in one component of $\partial Q \setminus \partial\alpha_i$ (see Fig. 18). The *cross-index* of K is the number

$$c^*(K) = \sum_{i<j} \varepsilon(\alpha_i, \alpha_j),$$

which is independent of a finite number of Reidemeister moves on K and thus, let $c^*[K] = c^*(K)$.

A *warping crossing point set* of K is a set $\mathbf{p} = \{p_1, p_2, \dots, p_d\}$ of crossing points of K such that the cross changes on the set $\mathbf{p}$ make K a monotone tangle diagram. For any strings α_i and α_j with $i \neq j$, let $d(\alpha_i, \alpha_j; \mathbf{p})$ be the number of crossing points between α_i and α_j belonging to the set $\mathbf{p}$, and $d^s(K; \mathbf{p})$ the number of self-crossing points belonging to the set $\mathbf{p}$. Let

$$d^+(K; \mathbf{p}) = d^s(K; \mathbf{p}) + \sum_{i<j} \max\{d(\alpha_i, \alpha_j; \mathbf{p}), \varepsilon(\alpha_i, \alpha_j)\}.$$

The *enhanced warping degree* of K is the minimal number $d^+(K)$ of $d^+(K; \mathbf{p})$ for all warping crossing point sets $\mathbf{p}$.

An *unknotting crossing point set* of K is a set $\mathbf{q} = \{q_1, q_2, \dots, q_u\}$ of crossing points of K such that the crossing changes on the set $\mathbf{q}$ make K an unknotted tangle diagram. For any strings α_i and α_j with $i \neq j$, let $u(\alpha_i, \alpha_j; \mathbf{q})$ be the number of crossing points between α_i and α_j belonging to the set $\mathbf{q}$, and $u^s(K; \mathbf{q})$ the number of self-crossing points belonging to the set $\mathbf{q}$. Let

$$u^+(K; \mathbf{q}) = u^s(K; \mathbf{q}) + \sum_{i<j} \max\{u(\alpha_i, \alpha_j; \mathbf{q}), \varepsilon(\alpha_i, \alpha_j)\}.$$

The *enhanced unknotting number* of K is the minimal number $u^+(K)$ of $u^+(K; \mathbf{q})$ for all unknotting crossing point sets $\mathbf{q}$ of K.

Let $d^+[K] = \min\{d^+(K') | K' \in [K]\}$, $u^+[K] = \min\{u^+(K')|K' \in [K]\}$. Then we have $d^+[K] \geq d[K]$ and $u^+[K] \geq u[K]$.

(5) Linking degree. For any strings α_i and α_j with $i \neq j$ of an r-string tangle diagram K, let $\lambda(\alpha_i, \alpha_j)$ be the absolute value of the sum of the signs ± 1 of the crossing points between α_i and α_j for any orientation of K. The *linking degree* of

K is the integer

$$\lambda(K) = \lceil \frac{\lambda(\alpha_i, \alpha_j)}{2} \rceil \quad (\lceil\,\rceil \text{ denoted the ceiling function}),$$

which is independent of a finite number of Reidemeister moves on K, and thus, let $\lambda[K] = \lambda(K)$. By definition, we have $\lambda[K] \geq c^*[K]$. On these tangle diagram invariants, we have the following inequalities.

Lemma 4.2. $c[K] \geq d^+[K] \geq u^+[K] \geq u^s[K] + \lambda[K] \geq u^s[K] + c^*[K]$.

Proof. The inequalities $c[K] \geq d^+[K] \geq u^+[K]$ are obtained directly from the definitions. To see that $u^+[K] \geq u^s[K] + \lambda[K]$, let $u^s(K; \mathbf{q})$ be the number of self-crossing points in an unknotting crossing point set $\mathbf{q}$ with $u^+(K; \mathbf{q}) = u^+(K)$. Then since $u^s(K; \mathbf{q}) \geq u^s(K)$, we have $u^+(K) - u^s(K) \geq \lambda(K) = \lambda[K]$ and hence $u^+(K) \geq u^s[K] + \lambda[K]$, so that $u^+[K] \geq u^s[K] + \lambda[K]$. Since $\lambda[K] \geq c^*[K]$, the desired inequalities are obtained. □

The calculation results on the knitting patterns K of Examples I-X are given in Calculation Table, where c, v, d^+, u^+, λ, u^s and c^* denote the values the crossing number $c[K]$, the virtual number $v(K)$ considered as $v[K]$, the enhanced warping degree $d^+[K]$, the enhanced unknotting number $u^+[K]$, the linking degree $\lambda[K]$, the self-unknotting number $u^s[K]$ and the cross-index $c^*[K]$.

Calculation Table on Examples I-X

	c	v	d^+	u^+	λ	c^*	u^s	total
I	2	1	1	1	1	0	0	6
II	4	0	1	1	1	1	0	8
III	4	0	2	2	1	1	1	11
IV	4	1	2	2	2	0	0	11
V	4	2	2	2	2	0	0	12
VI	4	2	2	2	2	0	0	12
VII	4	3	2	2	2	0	0	13
$VIII$	4	4	3	3	3	2	0	19
IX	4	4	4	4	4	4	0	24
X	4	6	4	4	4	4	0	26

5. Constructing a knitting pattern

As a standard terminology on a tangle (see [**13**]), we say that a tangle diagram K in Q is *not prime* if one of (1)-(3) is satisfied.

(1) After Reidemeister moves R1, R2, R3 on K, there is a proper arc in Q which is disjoint from K and separates K into two tangle diagrams.
(2) K is the connected sum of a tangle K' and a non-trivial knot diagram.
(3) After Reidemeister moves on K, there is a proper arc in Q which meets K in a single point and separates K into two tangle diagrams K' and K'' with $c[K'] > 0$ and $c[K''] > 0$.

Otherwise, K is *prime*. In the following theorem, we explain how to construct a knitting pattern.

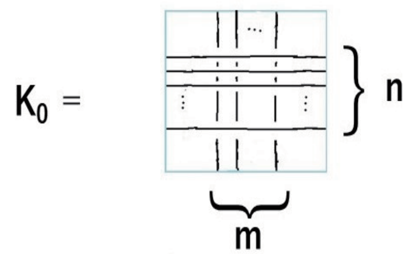


FIGURE 19. The standard (m,n)-knitting pattern

Theorem 5.1. Every (m,n)-knitting pattern K in the square $Q=[0,1]^2$ is constructed from the *standard* (m,n)*-knitting pattern* K_0 illustrated in Fig. 19 by a finite number of the following operations:

(OP1) A permutation on the ends of K_0 in ∂Q.
(OP2) A Reidemister move R1, R2 or R3 in int$Q$.
(OP3) A crossing change in int$Q$.

Further, let n,m,e,f be any non-negative integers such that $m+n\geq 2$, $\frac{(m+n)(m+n-1)}{2}\geq e$, $g\geq e$ and $f+g>0$. Then we have (1) and (2).

(1) There is a prime (m,n)-knitting pattern K^* in Q such that

$$c^*[K^*]=e,\ u^s(K^*)=f,\ \lambda[K^*]=g,\ c[K^*]\geq d^+[K^*]\geq u^+[K^*]=f+g.$$

(2) There is a prime reduced alternating (m,n)-knitting pattern K^{**} in Q such that

$$c^*[K^{**}]=e, u^s(K^{**})=f, \lambda[K^{**}]=g,\ c[K^{**}]\geq d^+[K^{**}]\geq u^+[K]\geq f+g.$$

Proof. Let K be any given (m,n)-knitting pattern. Transform the end of every string of the standard (m,n)-knitting pattern K_0 to coincide with the end of every string of K by (OP1). Then by (OP2) and (OP3) the resulting (m,n)-knitting pattern K_0' is deformed into K, showing the first half of the theorem. To see (1) and (2), first notice that a (m,n)-knitting pattern $K(O)$ without crossing is constructed from K_0 by (OP1) and (OP2). Then construct a (m,n)-knitting pattern K' with $c^*(K')=e$ and $d(K')=0$ from $K(O)$ by (OP1). In fact, if $1\leq e\leq m+n-1$, then $c^*(K')=e$ and $d(K')=0$ is easily obtained by moving an end of one string of $K(O)$. In general, by iterating this operation, we obtain $c^*(K')=e$ and $d(K')=0$ because for every integer e such that $m+n-1<e\leq\frac{(m+n)(m+n-1)}{2}$, there are non-negative integers s,e' such that

$$e=(m+n-1)+\cdots+(m+n-s-1)+e' \quad\text{and}\quad 0\leq e'\leq m+n-s-2.$$

Next, let K'' be a local connected sum of a string of K' and the f-fold connected sum of a 3-crossing trefoil knot diagram. Then the (m,n)-knitting pattern K'' has $c^*(K'')=e$ and $u^s(K'')=f$. Further, construct a (m,n)-knitting pattern K''' from K'' by making a g-full twists locally on any two strings of K'' so that K''' is a (m,n)-knitting pattern with $c^*[K''']=e$, $u^s[K''']=f$, $\lambda[K''']=g$ and $u^+[K''']=f+g$.

Note that K''' is non prime for $f > 0$. To obtain (1), for $u^+[K'''] = f + g > 0$ there is a technique in [**12**] deforming K''' with a prime (m, n)-knitting pattern K^* with $c^*[K^*] = e$, $u^s[K^*] = f$, $\lambda[K^*] = g$ and $u^+[K^*] = f + g$. To obtain (2), we can construct a prime reduced alternating K^{**} with $\lambda[K^{**}] = g$ from K''' by using distinct strings of K'''. Then $c^*[K^{**}] = e, u^s(K^{**}) = f$ and $u^+[K^{**}] \geq f + g$. $\square$

6. Other related patterns (1): A chain pattern

A (m, n)-*chain pattern* is a (m, n)-tangle diagram K in Q obtained from a link diagram K_T in the torus T^2 by cutting along $S^1 \times 0$ and $0 \times S^1$ such that

(1) every knot component of K_T meets $S^1 \times 0$ and/or $0 \times S^1$, and there is a knot component of K_T which is null-homotopic in T^2, and
(2) K_T meets $S^1 \times 0$ and $0 \times S^1$ (avoiding $(0, 0)$) transversely with m and n points, respectively.

A *chain* is the lift $\tilde{K}$ of a link diagram K_T in T^2 to the universal covering plane R^2 of T^2, determined uniquely from the knitting pattern K. For a complexity of a (m, n)-chain pattern K, the equivalence, the crossing number $c[K]$, the warping degree $d[K]$, the unknotting number $u[K]$ and the cross-index $c^*[K]$ are similarly defined. For example, the chain $\tilde{K}$ of the $(2, 4)$-chain pattern $K =$  is given in Fig. 20.

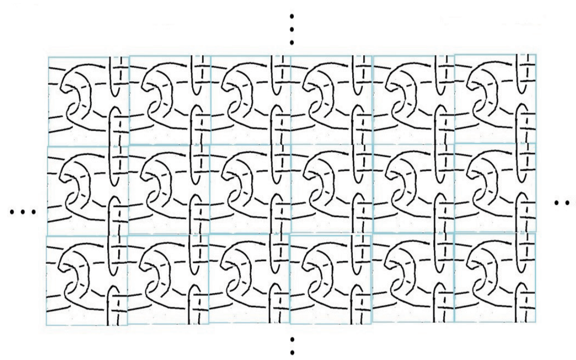


FIGURE 20. Chain of a $(2, 4)$-chain pattern

7. Other related patterns (2): A graph-tangle pattern

A (m, n)-*graph-tangle pattern* is a graph-tangle diagram K in Q obtained from a spatial graph diagram K_T in T^2 by cutting along $S^1 \times 0$ and $0 \times S^1$ such that

(1) every spatial graph component of K_T meets $S^1 \times 0$ and/or $0 \times S^1$, and
(2) K_T meets $S^1 \times 0$ and $0 \times S^1$ (avoiding $(0, 0)$) transversely with m and n points, respectively.

A *graph-tangle* is the lift $\tilde{K}$ of a spatial graph diagram K_T in T^2 to the universal covering plane R^2 of T^2, determined uniquely from the graph-tangle pattern K. For example, the graph-tangle $\tilde{K}$ of the $(2,2)$-graph-tangle pattern $K =$  is given in Fig. 21. For a complexity of a (m,n)-graph-tangle pattern K, the equivalence, the crossing number $c[K]$, the warping degree $d[K]$, the unknotting number $u[K]$ and the cross index $c^*[K]$ are defined by using an idea coming from [**16**]. For this explanation, let (S^2, K_S) be the connected spatial graph diagram obtained from the graph-tangle pattern (Q,K) by shrinking the boundary circle ∂Q to a vertex (which may be of degree 2). Then the crossing number $c[K]$, the warping degree $d[K]$, the unknotting number $u[K]$ and the cross-index $c^*[K]$ of the $(2,2)$-graph-tangle pattern K in Fig. 21 is defined to be the minima of $c[K'], d[K'], u[K']$ and $c^*[K']$ for the equivalence classes $[K']$ of all the tangle patterns K' obtained from (S^2, K_S) by deleting the interior of a regular neighborhood of a maximal tree of K_S in S^2 (see Fig. 22).

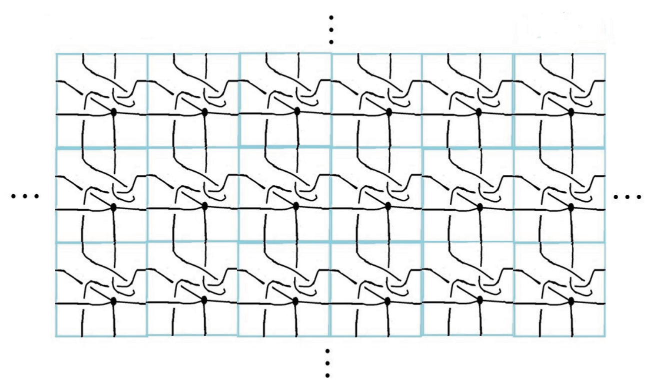


FIGURE 21. Graph-tangle of a $(2,2)$-graph-tangle pattern

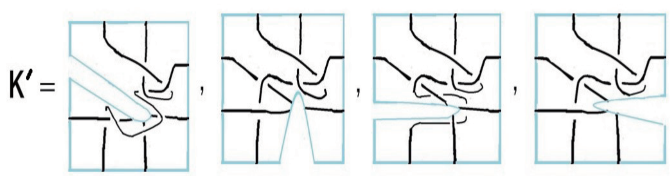


FIGURE 22. Graph-tangles K'

Acknowledgements. The author would like to thank the referee for pointing out lots of careless mistakes.

References

[1] A. Akimova and S. V. Matveev, Classification of genus 1 virtual knots having at most five classical crossings, J. Knot Theory Ramifications, 23 (2014) 1450031[19 pages].

[2] J. S. Carter, S. Kamada, and M. Saito, Stable equivalence of knots on surfaces and virtual knot cobordisms, J. Knot Theory Ramifications 11 (2002), 311-322.
[3] Z. Cheng and H. Gao, A polynomial invariant of virtual links, J. Knot Theory Ramifications, 22 (2013) 1341002 (33 pages).
[4] H. Dye: Vassiliev invariants from parity mappings, J. Knot Theory Ramifications, 22 (2013), 1340008.
[5] Y. H. Im, S. Kim, and D. S. Lee: The parity writhe polynomials for virtual knots and flat virtual knots, J. Knot Theory Ramifications, 22 (2013) 1250133 [20 pages].
[6] N. Kamada, The crossing number of alternating link diagrams on a surface, KNOTS '96 (Tokyo), 377382, World Sci. Publ., River Edge, NJ, 1997.
[7] N. Kamada, S. Kamada, Abstract link diagrams and virtual knots. J. Knot Theory Ramifications 9 (2000), no. 1, 93106.
[8] S. Kamada, A. Kawauchi and T. Matumoto, Combinatorial moves on ambient isotopic submanifolds in a manifold, J. Math. Soc. Japan 53 (2001),321-331.
[9] L. H. Kauffman, Virtual knot theory, European J. Combin. 20 (1999), 663-690.
[10] L. H. Kauffman, State models and the Jones polynomial, Topology 26 (1987), 395-407.
[11] L. H. Kauffman, Virtual knot theory, European J. Combin., 20(1999), 663-690.
[12] A. Kawauchi, Almost identical link imitations and the skein polynomial, in:Knots 90, pp.465-476, Walter de Gruyter, Berlin-New York,1992.
[13] A. Kawauchi, A survey of knot theory, Birkhäuser (1996).
[14] A. Kawauchi, Lectures on knot theory (in Japanese), Kyoritu Shuppan (2007).
[15] A. Kawauchi, On a complexity of a spatial graph, Bussei Kenkyu 92-1 (2009-4), 16-19.
[16] A. Kawauchi, On transforming a spatial graph into a plane graph, Progress of Theoretical Physics Supplement, No. 191 (2011), 225-234.
[17] A. Kawauchi, Knot theory for spatial graphs attached to a surface, Proceedings of the ICTS Program: Knot Theory and its Applications, Contemporary Mathematics, 670 (2016), 141-169, Amer. Math. Soc., Providence, RI, USA.
[18] A. Kawauchi, A. Shimizu, and Y. Yaguchi, Cross-index of a graph, preprint.
[19] G. Kuperberg, What is a virtual link? Algebr. Geom. Topol. 3 (2003), 587591.
[20] W. B. R. Lickorish and K. C. Millett, A polynomial invariant of oriented links, Topology 26 (1987), 107-141.
[21] Y. Miyazawa, Magnetic graphs and an invariant for virtual links, J. Knot Theory Ramifications, 15 (2006), 13191334.
[22] K. Murasugi, Jones polynomials and classical conjectures in knot theory, Topology 26(1987), 187-194.
[23] M. Okuda, A determination of the ascending number of some knots, thesis, Hiroshima University, 1998.
[24] M. Ozawa, Ascending number of knots and links. J. Knot Theory Ramifications 19 (2010), 15-25.
[25] A. Shimizu, The warping degree of a knot diagram, J. Knot Theory Ramifications 19(2010), 849-857.
[26] A. Shimizu, The warping degree of a link diagram, Osaka J. Math. 48 (2011), 209-231.
[27] S. Satoh and K. Taniguchi, The writhes of a virtual knot, Fund. Math., 225 (2014), 327341.

Osaka City University Advanced Mathematical Institute, Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

E-mail address: `kawauchi@sci.osaka-cu.ac.jp`