FAITHFUL EQUIVALENCE OF EQUIVALENT RIBBON SURFACE-LINKS

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Abstract. A chord graph in 3-space is constructed from a ribbon surface-link in 4-space. In earlier papers, the three moves on the diagrams of chord graphs (namely, the chord diagrams) are introduced to describe the faithful equivalence of a ribbon surface-link. In this paper, it is shown that any two equivalent ribbon surface-links are faithfully equivalent, so that any chord diagrams of any two equivalent ribbon surface-links are connected by a finite number of these three moves. By combining it with an earlier result, it is shown that any two TOP-equivalent ribbon surface-links are equivalent. In other words, there is no exotic ribbon surface-link, generalizing an earlier result on the trivial ribbon surface-knot. As another earlier result, the three moves on the chord diagrams were modified into the 16 moves on the chord diagrams without base crossing. In this paper, further modified moves of the 16 moves on the chord diagrams without base crossing are also introduced to describe how the set of ribbon torus-links is produced from the set of welded virtual links.

1. Introduction

A *surface-link* is the union of closed oriented surfaces F smoothly embedded in the 4-space $\mathbf{R}^4$. A surface-link F is *equivalent* to a surface-link F' if there is an orientation-preserving diffeomorphism $f : \mathbf{R}^4 \to \mathbf{R}^4$ sending F to F' orientation-preservingly. Then the map f is called an *equivalence*. In Section 2, it is reviewed from [**8**] that every surface-link is equivalent to the closed realizing surface of a band surgery sequence (on finitely many mutually disjoint bands) $o' \to o$ for trivial links o' and o, and a ribbon surface-link F is presented as the surface-link $F(o;\alpha)$ constructed uniquely from a *chord graph* $(o;\alpha)$ which consists of a trivial link $o = \cup_{i=0}^{n} o_i$, called a based loop system, and a chord system $\alpha = \cup_{j=1}^{s}\alpha_j$ spanning o in $\mathbf{R}^3$. In fact, the ribbon surface-link $F(o;\alpha)$ is obtained from the trivial S^2-link constructed from o by surgery along the 1-handles with cores α. A *chord diagram* is a spatial graph diagram $C = C(o;\alpha)$ of a chord graph $(o;\alpha)$ of a ribbon surface-link F in $\mathbf{R}^2$. It is noted that the based loop diagram of a chord diagram is just a diagram of a trivial link and hence may have crossings in the chord diagram. Two chord diagrams C and C' are *equivalent* if C is deformed into C' by a finite

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sequence of moves in the set of *fundamental moves* defined by the following three moves M_0, M_1 and M_2.

Move $\mathbf{M}_0$. This move is called the *Reidemeister move* which consists of the Reidemeister moves R_1, R_2, R_3, gR_4, gR_5 by regarding a chord graph as a spatial trivalent graph diagram, illustrated in Fig. 1 where the roles of the based loops and the chords are not changed although a distinction of a based loop and a chord is omitted.

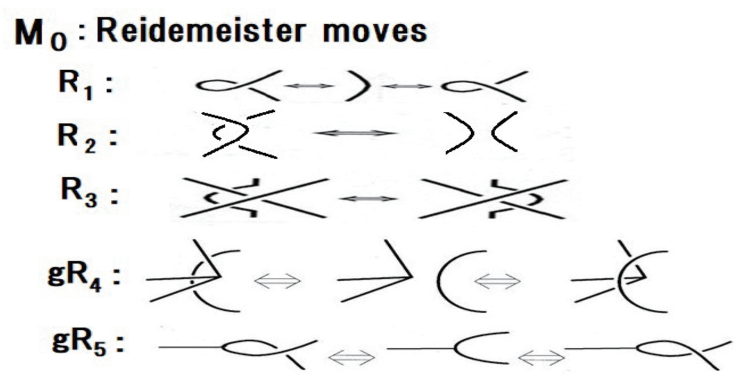

FIGURE 1. Move M_0: Reidemeister moves R_1, R_2, R_3, gR_4, gR_5 for trivalent graph diagrams

Move $\mathbf{M}_1$. This move is called the *fusion-fission move*, illustrated in Fig. 2, where the fusion operation is done only for a chord between different based loops.

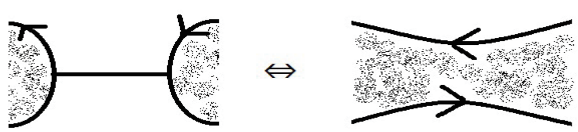

FIGURE 2. Fusion-fission M_1

Move $\mathbf{M}_2$. This move is called the *chord move* which consists of moves on chords, illustrated in Fig. 3.

By convention, a closed chord is regarded as a chord with a based loop constructed from the birth-death move shown in Fig. 4.

An equivalence f from a ribbon surface-link $F(o;\alpha)$ to a ribbon surface-link $F(o';\alpha')$ is *faithful* if f sends every meridian of $N(\alpha)$ to a null-homotopic loop in $F(o';\alpha')\cup N(\alpha')$. As the main result in [**4, 5**], it is shown that a ribbon surface-link $F(o;\alpha)$ is faithfully equivalent to a ribbon surface-link $F(o';\alpha')$ if and only if any

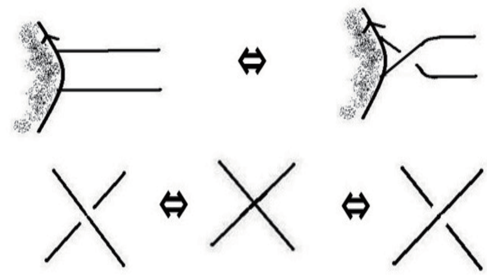

FIGURE 3. Chord moves M_2

FIGURE 4. The birth-death move

chord diagram $C(o';\alpha')$ is obtained from any chord diagram $C(o;\alpha)$ by a finite number of the moves M_0, M_1, M_2. The following theorem is our main theorem.

Theorem 2.4. A ribbon surface-link $F(o;\alpha)$ is faithfully equivalent to a ribbon surface-link $F(o';\alpha')$ if they are equivalent.

The proof of Theorem 2.4 is given in Section 3. By combining this theorem with the main result in [**4, 5**], we obtain the result that *a ribbon surface-link* $F(o;\alpha)$ *is equivalent to a ribbon surface-link* $F(o';\alpha')$ *if and only if any chord diagram* $C(o';\alpha')$ *is obtained from any chord diagram* $C(o;\alpha)$ *by by a finite number of the moves* M_0, M_1, M_2 (see Corollary 2.5). A surface-link F is *TOP-equivalent* to a surface-link F' if there is an orientation-preserving homeomorphism $f : \mathbf{R}^4 \to \mathbf{R}^4$ sending F to F' orientation-preservingly. Then the map f is called an *TOP-equivalence*. Note that a diffeomorphism (or a self-homeomorphism) of $\mathbf{R}^4$ used for an equivalence (or a TOP-equivalence) can be assumed to have a compact support.

The proof of Theorem 2.4 goes well even if we use a TOP-equivalence in place of an equivalence because by [**1**] there is no difference in the technical level used in this paper between an equivalence and a TOP-equivalence, as noted in [**4**]. Thus, we see that *any two TOP-equivalent ribbon surface-links are equivalent* meaning, in other words, that *there is no exotic ribbon surface-link* (see Corollary 2.6) generalizing the result on a trivial surface-knot in [**4, 5**].

The idea of the proof of Theorem 2.4 is to deform a ribbon surface-link into a special position to construct a double ribbon surface-link (see Lemma 3.1 later). It is also observed in Assertion 3.5 that this technique also makes possible to give a new proof of [**4**, Corollary 4.7] different from the proof given in [**5**, Theorem]. In Section 4, we discuss a chord diagram without base crossing which is a chord diagram such that the disk system bounded by the based loop system in $\mathbf{R}^2$ do not overlap and cuts the chords with mutually disjoint proper simple arcs. The moves m^i $(i = 1, 2, \ldots, 15)$ on chord diagrams without base crossing were introduced in [**6**] to describe the relations on faithfully equivalent ribbon surface-links (see Lemma 4.2). In this section, the moves m^i $(i = 1, 2, \ldots, 15)$ are modified into the moves consisting of the moves corresponding to the welded virtual links and some additional moves B^i $(j = 1, 2, 3, 4, 5, 6)$ in Fig. 23 (see Theorem 4.3).

2. Moves on chord diagrams

For a subset $A \subset \mathbf{R}^3$ and an interval $J \subset \mathbf{R}$, we use the notation

$$AJ = \{(x,t) |\, x \in A, t \in J\}. \quad \text{In particular, } \mathbf{R}^4 = \mathbf{R}^3(-\infty, +\infty).$$

Let $\ell' \to \ell$ be a band surgery from a link ℓ' to a link ℓ (see Fig. 5).

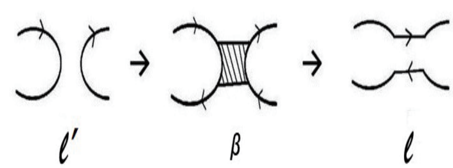

FIGURE 5. A band surgery

The *realizing surface* of a band surgery $\ell' \to \ell$ by mutually disjoint bands $\beta_1, \beta_2, \ldots, \beta_m$ is a surface F_a^b in $\mathbf{R}^3[a,b]$ defined by

$$F_a^b \cap \mathbf{R}^3[t] = \begin{cases} \ell[t] & (\frac{a+b}{2} < t \leq b), \\ (\ell' \cup_{i=1}^m \beta_i)[t] & (t = \frac{a+b}{2}), \\ \ell'[t] & (a \leq t < \frac{a+b}{2}). \end{cases}$$

For a division $a = a_0 < a_1 < \cdots < a_m = b$ of $[a,b]$, let $G = \cup_{i=1}^m F_{a_{i-1}}^{a_i}$ be the realizing surface of a band surgery sequence $\ell_0 \to \ell_1 \to \cdots \to \ell_m$. Let ℓ_0 and ℓ_m be trivial links with d' and d any mutually disjoint bounding disk systems in $\mathbf{R}^3$, respectively. Then the *closed realizing surface* in $\mathbf{R}^3[a,b]$ of the band surgery sequence $\ell_0 \to \ell_1 \to \cdots \to \ell_m$ is the closed surface

$$\widehat{G} = d'[a] \cup G \cup d[b],$$

which is called a *surface-link in* $[a,b]$ *normal form* (see Fig. 6). The surfaces $\widehat{G}_{\backslash d'} = \mathrm{cl}(\widehat{G} \backslash d'[a])$ and $\widehat{G}^{\backslash d} = \mathrm{cl}(\widehat{G} \backslash d[b])$ are the *lower-open* realizing surface and the *upper-open* realizing surface in $\mathbf{R}^3[a,b]$ of the band surgery sequence $\ell_0 \to \ell_1 \to \cdots \to \ell_m$, respectively. The following result is given in [**8**].

Lemma 2.1. Every surface-link F is equivalent to the closed realizing surface $\widehat{G}$ of a band surgery sequence $o' \to o$ on trivial links o' and o.

A surface-link F in $\mathbf{R}^4$ is *ribbon* if F is the closed realizing surface of a band surgery sequence $o \to \ell \to o$ with o a trivial link in $\mathbf{R}^3$ such that the band surgery $\ell \to o$ is the inverse of $o \to \ell$, which is illustrated in Fig. 7 (cf. [**9**]). In other words,

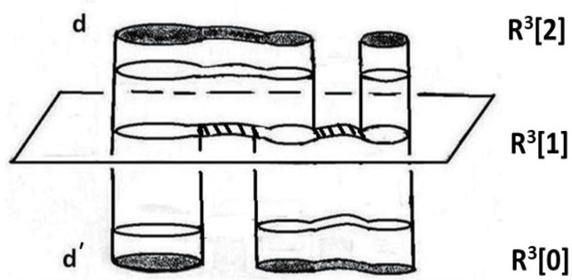

FIGURE 6. A surface-knot in $[0,2]$ normal form with all the bands in $\mathbf{R}^3[1]$

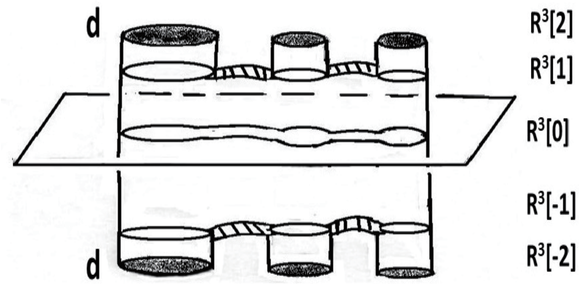

FIGURE 7. A ribbon S^2-knot in $[-2,2]$ normal form

a ribbon surface-link in $[a,b]$ normal form is a ribbon surface-link obtained from the trivial S^2-link $d[a] \cup o[a,b] \cup d[b]$ by surgery along the 1-handles $N(\alpha) = \beta[a', b']$ with $a < a' < b' < b$, where d is a disk system with $\partial d = o$ and β is a band system spanning o. Let α be a centerline system of the band system β. The following observation is given in [**4**].

Lemma 2.2. Every ribbon surface-link F in $[a,b]$ normal form is constructed uniquely from an oriented trivial link $o = \cup_{i=0}^{n} o_i$ and an arc system $\alpha = \cup_{j=1}^{s} \alpha_j$ spanning o in $\mathbf{R}^3$, where o_i and α_j denote the components of o and α, respectively.

The ribbon surface-link F in Lemma 2.2 is denoted by $F(o;\alpha)$. The pair $(o;\alpha)$ is called a *chord graph* $\mathbf{R}^3$ of the ribbon surface-link $F = F(o;\alpha)$ with o the *based loop system* and α the *chord system*. The *core graph* of a connected chord graph $(o;\alpha)$ is the graph $\Lambda(o;\alpha)$ obtained from the abstract graph of $(o;\alpha)$ by shrinking the based loops to the respective vertexes. Note that a ribbon surface-link $F = F(o;\alpha)$ is a ribbon surface-knot of genus g if and only if the chord graph $(o;\alpha)$ is a connected graph whose core graph $\Lambda(o;\alpha)$ is homotopy equivalent to the bouquet $\bigvee_g S^1$ of g copies of the circle S^1, where $\bigvee_0 S^1$ is regarded as a point. The surfaces $F(o;\alpha)_{\backslash d} = \mathrm{cl}(F(o;\alpha) \setminus d[a])$ and $F(o;\alpha)^{\backslash d} = \mathrm{cl}(F(o;\alpha) \setminus d[b])$ in $\mathbf{R}^3[a,b]$ are called the *lower-open ribbon surface-link* and the *upper-open ribbon surface-link* of the chord graph $(o;\alpha)$, respectively. A *chord diagram* is a spatial graph diagram

$C = C(o;\alpha)$ of a chord graph $(o;\alpha)$ of a ribbon surface-link F in $\mathbf{R}^2$. It is noted that the based loop diagram of a chord diagram is just a diagram of a trivial link and hence may have crossings in the chord diagram. Two chord diagrams C and C' are *equivalent* if C is deformed into C' by a finite sequence of moves in the set of *fundamental moves* defined by the following three moves M_0, M_1 and M_2.

Move $\mathbf{M}_0$. This move is called the *Reidemeister move* which consists of the Reidemeister moves R_1, R_2, R_3, gR_4, gR_5 by regarding a chord graph as a spatial trivalent graph diagram, illustrated in Fig. 1 where the roles of the based loops and the chords are not changed although a distinction of a based loop and a chord is omitted.

Move $\mathbf{M}_1$. This move is called the *fusion-fission move*, illustrated in Fig. 2, where the fusion operation is done only for a chord between different based loops.

Move $\mathbf{M}_2$. This move is called the *chord move* which consists of moves on chords, illustrated in Fig. 3.

It is also observed in [**4**] that under the use of the move M_0, the move M_1 is equivalent to a combination move of the *elementary fusion-fission move* $M_{1.0}$ illustrated in the left-hand side of the move m^4 in Fig. 13, the *chord slide move* $M_{1.1}$ and the *chord pass move* $M_{1.2}$ illustrated in Fig. 8. The birth-death move in Fig. 4 is obtained from these moves, unless a closed chord is involved.

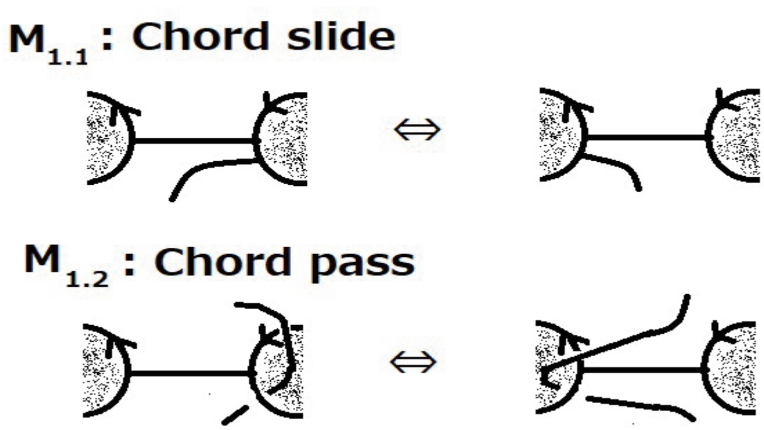

FIGURE 8. Chord slide $M_{1.1}$ and Chord pass $M_{1.2}$

An equivalence f from a ribbon surface-link $F(o;\alpha)$ to a ribbon surface-link $F(o';\alpha')$ is *faithful* if f sends every meridian of $N(\alpha)$ to a null-homotopic loop in $F(o';\alpha') \cup N(\alpha')$. The following lemma was a main result in [**3, 4**]:

Lemma 2.3. A ribbon surface-link $F(o;\alpha)$ is faithfully equivalent to a ribbon surface-link $F(o';\alpha')$ if and only if any chord diagram $C(o';\alpha')$ is obtained from any chord diagram $C(o;\alpha)$ by a finite number of the moves M_0, M_1, M_2.

The following theorem is our main theorem, which is shown in Section 3.

Theorem 2.4. A ribbon surface-link $F(o;\alpha)$ is faithfully equivalent to a ribbon surface-link $F(o';\alpha')$ if they are equivalent.

By combining this theorem with Lemma 2.3, the following corollary is obtained:

Corollary 2.5. A ribbon surface-link $F(o;\alpha)$ is equivalent to a ribbon surface-link $F(o';\alpha')$ if and only if any chord diagram $C(o';\alpha')$ is obtained from any chord diagram $C(o;\alpha)$ by a finite number of the moves M_0, M_1, M_2.

As we explain in Section 1, we have the following result generalizing the result on the trivial ribbon surface-knot in [**4, 5**]:

Corollary 2.6. Any two TOP-equivalent ribbon surface-links are equivalent. In other words, there is no exotic ribbon surface-link.

3. Proof of Theorem 2.4

In this section, the proof of Theorem 2.4 is given. Let $F(o;\alpha)$ and $F(o';\alpha')$ be equivalent ribbon surface-links in $[-1,1]$ normal forms by an equivalence f. By the moves M_1 and M_2, we can assume the following (1)-(3) for every component of $F(o;\alpha)$ and $F(o';\alpha')$, whose situations are used in an argument of [**4**].

(1) The based loops o and o' are identical: $o = o'$ and have the $n+1$ components o_i $(i = 0, 1, 2, \ldots, n)$ for some n.

(2) The chords α and α' have the same number of chords α_j and α'_j with identical boundaries $\partial\alpha_j = \partial\alpha'_j$ for all j $(j = 1, 2, \ldots, s)$ for $s \geq n$,

(3) The chords α connect the based loops o as in Fig. 9. Namely, for every j with $1 \leq j \leq n$ the chord α_j joins the based loop o_j to the based loop o_0, referred to as a *non-self-connecting chord*, and for every j with $n+1 \leq j \leq s$ the chord α_j joins the based loop o_0 itself, referred to as a *self-connecting chord*.

The following lemma is useful in our argument:

Lemma 3.1. Let $F(o;\alpha)$ and $F(o;\alpha')$ be equivalent ribbon surface-links by an equivalence f. Then there are ribbon surface-links $F(o\cup e;\alpha\cup\delta)$ and $F(o\cup e;\alpha'\cup\delta')$ in $[0,1]$ normal forms such that

(1) Every chord in δ with δ' connects from a loop of o to a loop of e and every loop of e is connected with a loop of o by just one chord of δ and just one chord of δ',

(2) Any chord diagrams $C(o\cup e;\alpha\cup\delta)$ and $C(o\cup e;\alpha'\cup\delta')$ are moved into any chord diagrams $C(o;\alpha)$ and $C(o;\alpha')$ by the moves M_0, M_1, M_2, respectively, so that the

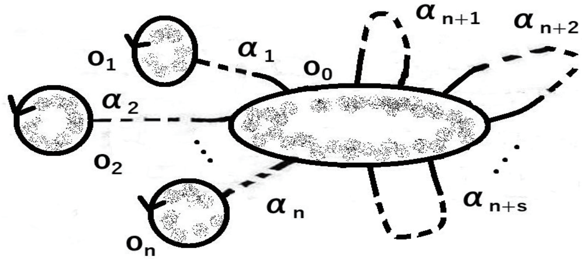

FIGURE 9. A specification of the chords α joining the based loops o

ribbon surface-links $F(o \cup e; \alpha \cup \delta)$ and $F(o \cup e; \alpha' \cup \delta')$ are faithfully equivalent to $F(o;\alpha)$ and $F(o;\alpha')$, respectively, and

(3) The equivalence f is isotopically deformed into an equivalence $f^* : \mathbf{R}^4 \to \mathbf{R}^4$ from $F(o \cup e; \alpha \cup \delta)$ to $F(o \cup e; \alpha' \cup \delta')$ with $f^*(x,t) = (x,t)$ for all $x \in \mathbf{R}^3$ and $t \in (-\infty, 0] \cup [1, +\infty)$.

Proof of Lemma 3.1. Assume that $F(o;\alpha)$ and $F(o;\alpha')$ are in $[0,1]$ normal forms and the restriction of the equivalence f to $\mathbf{R}^3[2,+\infty)$ is the identity map. Assume that $f(d[1]) = d[1]$ for a disk system d bounded by o. Let p be a point system in d with one point in every disk of d. Then $J = p[1,2]$ is an arc system. A 2-*sphere system around* J is a collection of mutually disjoint 2-spheres S in $\mathbf{R}^3(1,2)$ such that every 2-sphere component S_1 of S is the boundary of a 3-ball neighborhood of a point $p_1[t_1] \in p[t_1]$ in $(\mathbf{R}^3[t_1] \setminus p[t_1]) \cup \{p_1[t_1]\}$ for some level $t_1\,(1 < t_1 < 2)$. The point $p_1[t_1]$ is called the *center* of the 2-sphere S_1. By an argument of [**4**, Lemma 4.6], we can consider that $f(J)$ is in $\mathbf{R}^3[1-\varepsilon, 2]$ for a small $\varepsilon > 0$ and transversely intersects the interior of the 3-disk system $d[1-\varepsilon, 1]$ in finite points. In the 3-disk system $d[1-\varepsilon,1]$, we choose a 3-ball neighborhood system N of the intersection point system and an arc system J_N consisting of an arc from every component of the boundary 2-sphere system ∂N to the boundary $\partial d[1-\varepsilon, 1]$. By deforming the 2-sphere system ∂N and the arc system J_N, we have a 2-sphere system S' around J with distinct centers and an arc system δ' consisting of mutually disjoint arcs joining S' with $d[1]$ such that the union $f(J) \cup F(o;\alpha')$ is deformed into the union $F' \cup J$ for the surface-link F' obtained from $F(o;\alpha')$ and the 2-sphere system S' by tubing along δ' by an isotopic deformation g of $\mathbf{R}^4$ keeping $\mathrm{cl}(F(o;\alpha') \setminus d[1]) \cup \mathbf{R}^3(-\infty,0] \cup \mathbf{R}^3[2,+\infty)$ fixed. Note that the preimage $S = (gf)^{-1}(S')$ is a 2-sphere system around J and the preimage $\delta = (gf)^{-1}(\delta')$ is an arc system consisting of mutually disjoint arcs joining S with $d[1]$. We assume that the homeomorphism gf is identical on a neighborhood $N(J)$ of J in $\mathbf{R}^3[1,2]$ and the 2-sphere systems S and S' are in $N(J)$. Let p_S and $p_{S'}$ be distinct point systems in S and S' with one point in every 2-sphere of S and S', respectively. Let J_S and $J_{S'}$ be the vertical arc systems in $\mathbf{R}^3(1,2]$ joining p_S and $p_{S'}$ with the corresponding points in $\mathbf{R}^3[2]$, respectively.

The surface in $\mathbf{R}^3[0,2]$ obtained from $d[0] \cup o[0,1] \cup d[1] \cup J \cup S \cup J_S$ by replacing every arc in J and J_S with a tube is deformed into the surface $d[0] \cup o[0,2] \cup d^e[0] \cup e[0,2]$ in $\mathbf{R}^3[0,2]$ for a trivial loop system e and a disk system d^e bounded by e in

$\mathbf{R}^3$ by an isotopic deformation h of $\mathbf{R}^3(-\infty, 2]$. Let α and δ denote again the chord systems on the based loop system $o \cup e$ in $\mathbf{R}^3$ which are obtained from $h(\alpha)$ and $h(\delta)$ by a homotopy argument of an arc system (see [**9**, Lemma 4.11]), respectively.

Similarly, the surface in $\mathbf{R}^3[0,2]$ obtained from $d[0] \cup o[0,1] \cup d^e[1] \cup J \cup S' \cup J_{S'}$ by replacing every arc in J and $J_{S'}$ with a tube is deformed into the surface $d[0] \cup o[0,2] \cup d^e[0] \cup e[0,2]$ in $\mathbf{R}^3[0,2]$ for the same trivial loop system e and the same disk system d^e in $\mathbf{R}^3$ by an isotopic deformation h' of $\mathbf{R}^3(-\infty, 2]$ so that $h'|_{\mathbf{R}^3[2]} = h|_{\mathbf{R}^3[2]}$. Let α' and δ' denote again chord systems on the based loop system $o \cup e$ in $\mathbf{R}^3$ which are obtained from $h'(\alpha')$ and $h'(\delta')$ by a homotopy argument of an arc system (see [**9**, Lemma 4.11]), respectively.

By construction, any chord diagrams $C(o \cup e; \alpha \cup \delta)$ and $C(o \cup e; \alpha' \cup \delta')$ are moved into any chord diagrams $C(o;\alpha)$ and $C(o;\alpha')$ by the moves M_0, M_1, M_2, respectively, so that the ribbon surface-links $F(o \cup e; \alpha \cup \delta)$ and $F(o \cup e; \alpha' \cup \delta')$ are faithfully equivalent to $F(o;\alpha)$ and $F(o;\alpha')$, respectively.

The homeomorphism $f' = h'(gf)h^{-1} : \mathbf{R}^3(-\infty, 2] \to \mathbf{R}^3(-\infty, 2]$ gives an equivalence $f'' : \mathbf{R}^4 \to \mathbf{R}^4$ from the ribbon surface-link $F(o \cup e; \alpha \cup \delta)$ in $[0,2]$ normal form to the ribbon surface-link $F(o \cup e; \alpha' \cup \delta')$ in $[0,2]$ normal form such that $f''|_{\mathbf{R}^3[2,+\infty)} = 1$, $f''(d[0]) = d[0]$ and $f''(d^e[0]) = d^e[0]$. Further, by an argument of [**4**, Lemma 4.2] (as in [**4**, Lemma 4.3]), the equivalence f'' satisfies that $f''|_{u[0,2] \cup u^e[0,2]} = 1$, where u is the union of an arc $u_i \subset o_i \setminus o_i \cap (\alpha \cup \delta)$ for every i and u^e is the union of an arc $u^e_j \subset e_j \setminus e_j \cap \delta$ for every component e_j of e. Take a point $q_i \in u_i$ and a point $q^e_j \in u^e_j$ for every i and j. Let V be a 3-ball in $\mathbf{R}^3$ such that the 4-ball $V[0,2]$ contains $F(o \cup e; \alpha \cup \delta)$ and $F(o \cup e; \alpha' \cup \delta')$ in the interior. Assume that $f'''|_{\partial V[-1,2]} = 1$ for the boundary 3-sphere $\partial V[-1,2]$ of the 4-ball $V[-1,2]$. Then the arcs $f''(q_i[-1,0])$ and $f''(q^e_j[-1,0])$ are ∂-relatively isotopic to $q_i[-1,0]$ and $q^e_j[-1,0]$ for all i,j, respectively, by an ambient isotopy of $V[-1,2]$ keeping the boundary $\partial V[-1,2]$ fixed. This is because the arcs $q_i[-1,0]$ and $q^e_j[-1,0]$ are respectively deformed into arcs parallel to the arc $q_i[0,2] \cup \omega_i[2] \cup v_i[-1,2] \cup \omega_i[-1]$ in a neighborhood of $\partial V[-1,2] \cup u_i[0,2]$ in the 4-ball $V[-1,2]$ and the arc $q^e_j[0,2] \cup \omega^e_j[2] \cup v^e_j[-1,2] \cup \omega^e_j[-1]$ in a neighborhood of $\partial V[-1,2] \cup u^e_j[0,2]$ in the 4-ball $V[-1,2]$, where v^e_j is a point in ∂V and ω^e_j is an arc in V joining q^e_j and v^e_j. By replacing the arcs $q_i[-1,0]$ and $q^e_j[-1,0]$ by tubes, we have an equivalence $f''' : \mathbf{R}^4 \to \mathbf{R}^4$ from $F(o \cup e; \alpha \cup \delta)$ to $F(o \cup e; \alpha' \cup \delta')$ in $[-1,2]$ normal forms such that $f^*(x,t) = (x,t)$ for all $x \in \mathbf{R}^3$ and $t \in (-\infty, -1] \cup [2, +\infty)$. By shrinking $[-1,2]$ into $[0,1]$, the desired equivalence f^* is obtained from f'''. This completes the proof. □

The *double ribbon surface-link* $DF(o;\alpha)$ of $F(o;\alpha)$ is the closed realizing surface in $[-1,1]$ normal form on the band surgery sequence $o \to \ell \to o \to \ell \to o$ where the latter-half band surgery sequence $o \to \ell \to o$ is the inverse of the first-half band surgery sequence $o \to \ell \to o$. Let τ be the involution on $\mathbf{R}^4 = \mathbf{R}^3(-\infty, +\infty)$ defined by $\tau(x,t) = (x,-t)$ for all $x \in \mathbf{R}^3$ and $t \in (-\infty, +\infty)$. The double ribbon surface-link $DF(o;\alpha)$ is invariant under the τ-action on $\mathbf{R}^4$.

Lemma 3.2. The double ribbon surface-link $DF(o;\alpha)$ is equivalent to a ribbon surface-link $F(o;\alpha \cup \gamma)$ in $[-1,1]$ normal form by a τ-equivariant equivalence g (i.e., an equivalence g with $\tau g = g\tau$), where $\gamma = \cup_{j=1}^{s} \gamma_j$ is an additional trivial chord system on o such that the component γ_j is situated as in Fig. 10 for every j.

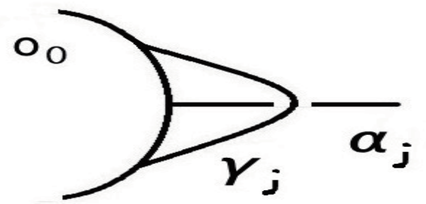

FIGURE 10. The situation of the trivial chord γ_j

Proof of Lemma 3.2. The lower-open ribbon surface-link $F(o;\alpha)_{\backslash d}$ in $\mathbf{R}^3[0,1]$ is deformed into the lower-open realizing surface $\widehat{G}_{\backslash d}$ in $\mathbf{R}^3[0,1]$ of a band surgery sequence $o \to \ell \to \tilde{o}$ where the bands on $\ell \to \tilde{o}$ are deformed to be disjoint from the bands on $o \to \ell$ by replacing the bands with narrow bands and by sliding bands. This deformation is done by a homeomorphism $h : \mathbf{R}^3[0,1] \to \mathbf{R}^3[0,1]$ with one-parameter family of homeomorphisms $h_t : \mathbf{R}^3 \to \mathbf{R}^3$ $(t \in [0,1])$ such $h(x,t) = (h_t(x),t)$ for all $x \in \mathbf{R}^3$ and $t \in [0,1]$ and h_0 is the identity. The closed realizing surface $\widehat{G}$ in $[-1,1]$ normal form on the band surgery sequence $\tilde{o} \to \ell \to o \to \ell \to \tilde{o}$ is a ribbon surface-link $F(o;\alpha\cup\gamma)$, where $\gamma = \cup_{j=1}^n \gamma_j$ is a centerline system of the bands on $\ell \to \tilde{o}$. By construction, there is a τ-equivariant equivalence g from $DF(o;\alpha)$ to $F(o;\alpha\cup\gamma)$. □

A faithful equivalence between equivalent ribbon surface-links is basically obtained from the following lemma:

Lemma 3.3. Let $F(o;\alpha)$ and $F(o;\alpha')$ be ribbon surface-links in $[-1,1]$ normal forms. Then any τ-equivariant equivalence g from $F(o;\alpha)$ to $F(o;\alpha')$ is a faithful equivalence.

Proof of Lemma 3.3. Let μ be a τ-equivariant meridian loop of $N(\alpha)$ which is the union of an arc a in $\mathbf{R}^3[0,1]$ and the arc $\tau(a)$ in $\mathbf{R}^3[-1,0]$. Then the arcs $g(a)$ and $g\tau(a)$ are homotopic to the same arc in $\mathbf{R}^3[0]$ by a homotopy in $F(o;\alpha')\cup N(\alpha')$ keeping $\partial g(a) = \partial g\tau(a)$ fixed. Thus, the image $g(\mu)$ is null-homotopic in $F(o;\alpha')\cup N(\alpha')$, showing that g is a faithful equivalence. □

Proof of Theorem 2.4 is done as follows:

Proof of Theorem 2.4. Let $F(o;\alpha)$ and $F(o';\alpha')$ be equivalent ribbon surface-links in $[-1,1]$ normal forms by an equivalence f. We show that $F(o;\alpha)$ and $F(o';\alpha')$ are faithfully equivalent. By Lemma 3.1, there are ribbon surface-links $F(o\cup e;\alpha\cup\delta)$ and $F(o\cup e;\alpha'\cup\delta')$ in $[0,1]$ normal forms which are respectively faithfully equivalent to $F(o;\alpha)$ and $F(o;\alpha')$, and equivalent by an equivalence f^* with $f^*|_{\mathbf{R}^3(-\infty,0]\cup\mathbf{R}^3[1,+\infty)} = 1$. Since the faithful equivalence is an equivalence relation, we may show that the ribbon surface-links $F(o\cup e;\alpha\cup\delta)$ and $F(o\cup e;\alpha'\cup\delta')$ are faithfully equivalent. By the τ-equivariant equivalence g uniquely constructed from this equivalence f^*, the double ribbon surface-links $DF(o\cup e;\alpha\cup\delta)$ and $DF(o\cup e;\alpha'\cup\delta')$ are equivalent. For simplicity, for the remainder of the proof, the chord graphs $(o\cup e;\alpha\cup\delta)$ and $(o\cup e;\alpha'\cup\delta')$ are written as $(o;\alpha)$ and $(o';\alpha')$, respectively. Then by Lemma 3.2 the double ribbon surface-link $DF(o;\alpha)$ is equivalent to a ribbon surface-link $F(o;\alpha\cup\gamma)$ in $[-1,1]$ normal form by a τ-equivariant equivalence for an additional trivial chord system $\gamma=\cup_{j=1}^n\gamma_j$ on o. Similarly, the double ribbon surface-link $DF(o;\alpha')$ is equivalent to a ribbon surface-link $F(o;\alpha'\cup\gamma')$ in $[-1,1]$ normal form by a τ-equivariant equivalence for an additional trivial chord system $\gamma'=\cup_{j=1}^n\gamma'_j$ on o. Then the ribbon surface-links $F(o;\alpha\cup\gamma)$ and $F(o;\alpha'\cup\gamma')$ in $[-1,1]$ normal form are equivalent by a τ-equivariant equivalence. By Lemma 3.3, the ribbon surface-links $F(o;\alpha\cup\gamma)$ and $F(o;\alpha'\cup\gamma')$ are faithfully equivalent by a τ-equivalent faithful equivalence $g^*:\mathbf{R}^4\to\mathbf{R}^4$. By the moves M_0 and M_2, the chord systems $\gamma=\cup_{j=1}^n\gamma_j$ and $\gamma'=\cup_{j=1}^n\gamma'_j$ are deformed into trivial self-connecting chord systems attaching to the based loop o_0 with $\partial\gamma_j=\partial\gamma'_j$ for all j. By an argument of [**4**, Lemma 4.2] (as in [**4**, Lemma 4.3]) for $F(o;\alpha\cup\gamma)$ and $F(o;\alpha'\cup\gamma')$ instead of $F(o;\alpha)$ and $F(o;\alpha')$, we have a faithful equivalence $h:\mathbf{R}^4\to\mathbf{R}^4$ from $F(o;\alpha\cup\gamma)$ to $F(o;\alpha'\cup\gamma')$ in $[-2,2]$ normal forms with $h|\,\mathbf{R}^3(-\infty,-3]\cup\mathbf{R}^3[3,+\infty)=1$ such that

$$h|\,d[-2]\cup u[-2,2]\cup o[1,2]\cup d[2]=1$$

for the union u of an arc $u_i\subset o_i\setminus o_i\cap(\alpha\cup\gamma)$ for every i. Note that the band systems β and β' with the centerline chord systems α and α', respectively are situated in the levels $t=\pm1$, so that $N(\alpha)=\beta[-1,1]$ and $N(\alpha')=\beta'[-1,1]$. We put copies of the chord systems $\alpha\cup\gamma$ and $\alpha'\cup\gamma'$ in the upper bands of $F(o;\alpha\cup\gamma)$ and $F(o;\alpha'\cup\gamma')$ in $\mathbf{R}^3[1]$, respectively, which are denoted by $\bar\alpha\cup\bar\gamma$ and $\bar\alpha'\cup\bar\gamma'$. Let $\Gamma(\alpha\cup\gamma)$ be the graph obtained from the chord graph $(o;\alpha\cup\gamma)$ in $\mathbf{R}^3$ by deleting the interior of u, and $\Gamma(\alpha'\cup\gamma')$ the graph obtained similarly from the chord graph $(o;\alpha'\cup\gamma')$ in $\mathbf{R}^3$ (see Fig. 11). Let $\bar\Gamma(\alpha\cup\gamma)$ and $\bar\Gamma(\alpha'\cup\gamma')$ be the graphs obtained by putting $\Gamma(\alpha\cup\gamma)$ and $\Gamma(\alpha'\cup\gamma')$ in $F(o;\alpha\cup\gamma)\cap\mathbf{R}^3[1]$ and $F(o;\alpha'\cup\gamma')\cap\mathbf{R}^3[1]$, respectively. By [**4**, Lemma 4.4], the faithful equivalence $h:\mathbf{R}^4\to\mathbf{R}^4$ induces a map

$$h_*:\bar\Gamma(\alpha\cup\gamma)\to\bar\Gamma(\alpha'\cup\gamma')$$

preserving the degree one vertexes and graph-components and also inducing an isomorphism on the fundamental groups of the corresponding components. On the self-connecting chords, we use the fact in [**10**] that every automorphism of a free group of a finite rank is generated by the *elementary Nielsen transformations*, meaning

(1) Exchange of two basis elements,
(2) Replacement of a basis element by its inverse, and
(3) Replacement of a basis element by the product of it and another basis element.

Let $\epsilon(o)$ be a boundary-collar of the based loop system o in the disk system d. This means that after a deformation of h given by handle slides on the 1-handles $N(\alpha)=\beta[-1,1]$ attaching to the 3-manifolds $\epsilon(o)[-3,3]$, the self-connecting chords $h(\bar\alpha_j)$ and $h(\bar\gamma_j)$ are homotopic to ones of the self-chords $\pm\bar\alpha'_{j'}$, $\pm\bar\gamma_{j''}$ for all j',j'' up to homotopies relative to the cylinders $o[-3,3]$. By ignoring the orientations of the self-chords and by changing the orientations of $\bar\alpha'_{j'}$ and $\bar\gamma'_{j''}$ if necessary, we can consider that $h(\bar\alpha_j)$ and $h(\bar\gamma_j)$ are homotopic to ones of the self-chords $\bar\alpha'_{j'}$ or $\bar\gamma'_{j''}$ for all j',j'' up to homotopies relative to the cylinders $o[-3,3]$. If $h(\bar\gamma_j)$

is homotopic to $\bar{\alpha}'_{j'}$ up to a homotopy relative to the cylinders $o[-3,3]$, then the chord $\bar{\alpha}'_{j'}$ must be a trivial chord. Hence if $h(\bar{\gamma}_j)$ is homotopic to $\bar{\alpha}'_{j'}$ (or $\bar{\gamma}'_{j''}$ with $j'' \neq j$, respectively), then by a deformation of h interchanging $\bar{\alpha}'_{j'}$ (or $\bar{\gamma}'_{j''}$, respectively) and $\bar{\gamma}'_j$, the image $h(\bar{\gamma}_j)$ is homotopic to $\bar{\gamma}'_j$ up to a homotopy relative to the cylinders $o[-3,3]$ for all j. This also means that $h(\bar{\alpha}_j)$ must be homotopic to one of $\bar{\alpha}'_{j_k}$ and hence by changing the indexes of $\bar{\alpha}'_{j_k}$, the image $h(\bar{\alpha}_j)$ is homotopic to $\bar{\alpha}'_j$ up to a homotopy relative to the cylinders $o[-3,3]$, for all j. This condition is just equal to the assumption of [**5**, Theorem]. Hence by [**4, 5**], any chord graph $C(o;\alpha)$ is deformed into a chord graph $C(o;\alpha')$ by a finite number of the moves M_0, M_1, M_2. This means that $F(o;\alpha)$ is faithfully equivalent to $F(o';\alpha')$. This completes the proof of Theorem 2.4. □

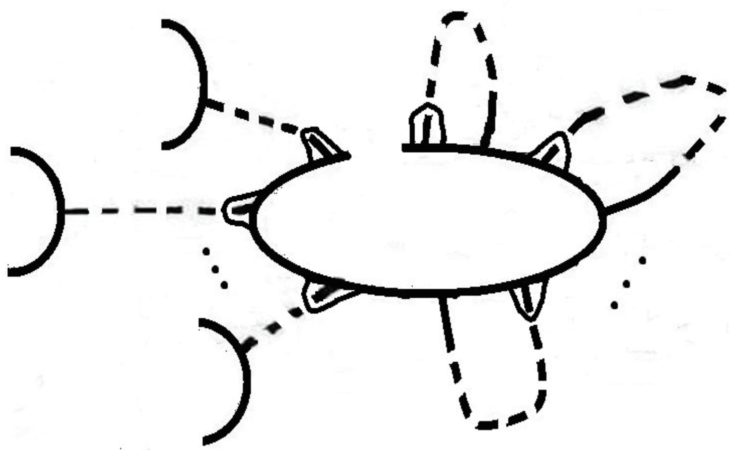

FIGURE 11. The graph Γ

By the proof of Theorem 2.4, we have a stronger faithful equivalence on isotopically deformed ribbon surface-links $F(o\cup e;\alpha\cup\delta)$ and $F(o\cup e;\alpha'\cup\delta')$, as stated in the following corollary:

Corollary 3.4. Let $F(o;\alpha)$ and $F(o;\alpha')$ be equivalent ribbon surface-links. Then there is a faithful equivalence $f^*:\mathbf{R}^4\to\mathbf{R}^4$ with $f^*(x,t)=(x,t)$ for all $x\in\mathbf{R}^3$ and $t\in(-\infty,0]\cup[1,+\infty)$ from a ribbon surface-link $F(o\cup e;\alpha\cup\delta)$ to a ribbon surface-link $F(o\cup e;\alpha'\cup\delta')$ in $[0,1]$ normal forms such that

(1) Every chord in δ and δ' connect from a loop of o to a loop of e and every loop of e is connected with a loop of o by just one chord of δ and just one chord of δ',

(2) Any chord diagrams $C(o\cup e;\alpha\cup\delta)$ and $C(o\cup e;\alpha'\cup\delta')$ are moved into any chord diagrams $C(o;\alpha)$ and $C(o;\alpha')$ by the moves M_0, M_1, M_2, respectively, and

(3) The faithful equivalence f^* sends every meridian of the 1-handle system $N=N(\alpha\cup\delta)$ of the chord system $\alpha\cup\delta$ on $F(o\cup e;\alpha\cup\delta)$ to a meridian of the 1-handle system $N'=N(\alpha'\cup\delta')$ of the chord system $\alpha'\cup\delta'$ on $F(o\cup e;\alpha'\cup\delta')$. In particular,

the image $f^*(\alpha \cup \delta)$ of the chord system $\alpha \cup \delta$ is homotopic to the chord system $\alpha' \cup \delta'$ on the 1-handle system N' by a homotopy in N' relative to the attaching disk systems.

Proof of Corollary 3.4. For proof, it suffices to show that every meridian of the 1-handle system N is sent to a meridian of the 1-handle system N' since a technique of removing the extra trivial chord systems occurring in the proof of Theorem 2.4 is shown there. Since f^* defines a diffeomorphism $\mathbf{R}^3(0,1) \to \mathbf{R}^3(0,1)$, every meridian disk of N is sent to a singular disk in $(\alpha' \cup \delta')[0,1] \cup N'$ which is homotopy equivalent to a graph. Hence every meridian disk of N is sent to a meridian of N'. □

The following Assertion 3.5 is the same assertion as [**4**, Corollary 4.7], which is shown in [**5**, Theorem]. Using Corollary 3.4, we can have a different proof of this assertion.

Assertion 3.5. If ribbon surface-links $F(o;\alpha)$ and $F(o;\alpha')$ are faithfully equivalent, then after a finite number of the moves M_0, M_1, M_2 on the chord diagrams $C(o;\alpha)$ and $C(o;\alpha')$, the chords α are homotopic to the chords α' in $\mathbf{R}^3$ by a homotopy relative to the based loops o.

Proof of Assertion 3.5. By Corollary 3.4, there is a diffeomorphism $f^* : \mathbf{R}^3[-3,3] \to \mathbf{R}^3[-3,3]$ with $f^*((o \cup e)[-3,3]) = (o \cup e)[-3,3]$ such that the chords $f^*(\bar{\alpha} \cup \bar{\delta})$ are homotopic to the chords $\bar{\alpha}' \cup \bar{\delta}$ by a homotopy relative to the cylinders $o[-3,3]$ and the chord diagrams $C(o \cup e; \alpha \cup \delta)$ and $C(o \cup e; \alpha' \cup \delta')$ are moved into the chord diagrams $C(o;\alpha)$ and $C(o;\alpha')$ by the moves M_0, M_1, M_2, respectively. Since the chords $\alpha \cup \delta$ are homotopic to the chords $\alpha' \cup \delta'$ in $\mathbf{R}^3$ by a homotopy relative to the based loops o, showing Assertion 3.5. □

4. Moves on chord diagrams without base crossing

Let $\mathbb{F}$ be the set of the equivalence classes of ribbon surface-links. For the set $\mathbb{C}$ of chord diagrams, we have a canonical map $\kappa : \mathbb{C} \to \mathbb{F}$. The following lemma is a paraphrase of Corollary 2.5.

Lemma 4.1. The map $\kappa : \mathbb{C} \to \mathbb{F}$ defines a bijection $\kappa_* : \mathbb{C}/(M_0, M_1, M_2) \to \mathbb{F}$.

A *chord diagram without base crossing* is a chord diagram such that the disks bounded by the based loops in $\mathbf{R}^2$ do not overlap and cut the chords with only mutually disjoint proper simple arcs. Let $\mathbb{B}$ be the subset of the set $\mathbb{C}$ consisting a chord diagram without base crossing. We impose the following conventions on the set $\mathbb{B}$, where the *elementary move* m^0 is a fundamental move in $\mathbb{C}$ illustrated in Fig. 12.

Conventions.

(1) Mutually disjoint disks bounding a based loop system in the plane are oriented counterclockwise by observing the left-hand move of the elementary move m_0.

(2) A chord attaches to the disk system from outside.

(3) The right move of the elementary move m_0 is always assumed.

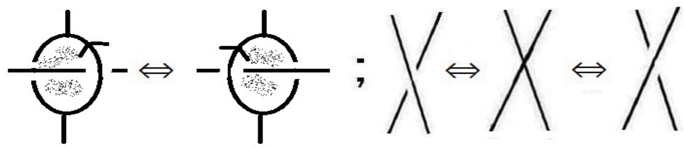

FIGURE 12. The elementary move m^0

The inclusion $\iota : \mathbb{B} \subset \mathbb{C}$ induces an onto map

$$\mathbb{B} \to \mathbb{C}/(M_0, M_1, M_2)$$

since every based loop system is a trivial link and hence every chord diagram is deformed into a chord diagram without based loop crossing by the move M_0. We provide the moves m^i $(i = 1, 2, \ldots, 15)$ on $\mathbb{B}$ illustrated in Fig. 13. Note that the shadow on the move m^9 in [**6**] is changed because of the convention (2) and the mirror image move of the move m^{13} in [**6**] is added in m^{13} because of the convention (1). In these figures, it is also noted that every based loop without indication of an orientation can have any orientation which is preserved by the moves. The following lemma was the main result of [**6**] because a chord attached to the disk system from inside can be always avoided.

Lemma 4.2. The inclusion $\iota : \mathbb{B} \to \mathbb{C}$ induces a bijection

$$\iota_* : \mathbb{B}/\left(m^i\,(i = 1, 2, 3, \ldots, 15)\right) \to \mathbb{C}/(M_0, M_1, M_2).$$

Note that the move m^* in Fig. 14 is a consequence of m^4, the left move of m^8 and the last two moves of m^9 in $\mathbb{B}$, which is shown in Fig. 15.

Let $\widetilde{\mathbb{B}}$ be the subset of $\mathbb{B}$ consisting of a chord diagram without base crossing such that every simple proper arc of a chord cut by the disk system intersects the interior of a disk system at one point. In this set $\widetilde{\mathbb{B}}$, we can assume that the situation around a based loop is illustrated in Fig. 16 where we grant a based disk without any chord in the interior or a based disk without attaching any chord from the left-hand or right-hand side in the picture. To make this assumption, we need the *repair move* matching the up-down relations on the arcs in every based disk such as in Fig. 17 which is obtained from the move M_1 and the *regular change move* in Fig. 18 obtained by the move M_0. For convenience, we use the notation in Fig. 19.

Let $\mathbb{D}$ be the set of virtual link diagrams (see [**2**]). A chord graph (o, α) is *of genus one* if the core graph $\Lambda(o; \alpha)_1$ of every connected component $(o, \alpha)_1$ of (o, α) is homotopy equivalent to a circle. Any chord diagram of a chord graph of genus one is also said to be *of genus one*. Let $\mathbb{A}$ be the subset of $\widetilde{\mathbb{B}}$ consisting of a chord diagram of genus one without base crossing. The map

$$\tau : \mathbb{D} \to \mathbb{A}$$

is defined by sending every virtual link diagram $D \in \mathbb{D}$ to a regular chord diagram C_D replacing a neighborhood of every real crossing with the diagram in the right-hand side of Fig. 20 (where the arrow is omitted when the chord orientation is assumed) and then by replacing a neighborhood of every virtual crossing with the

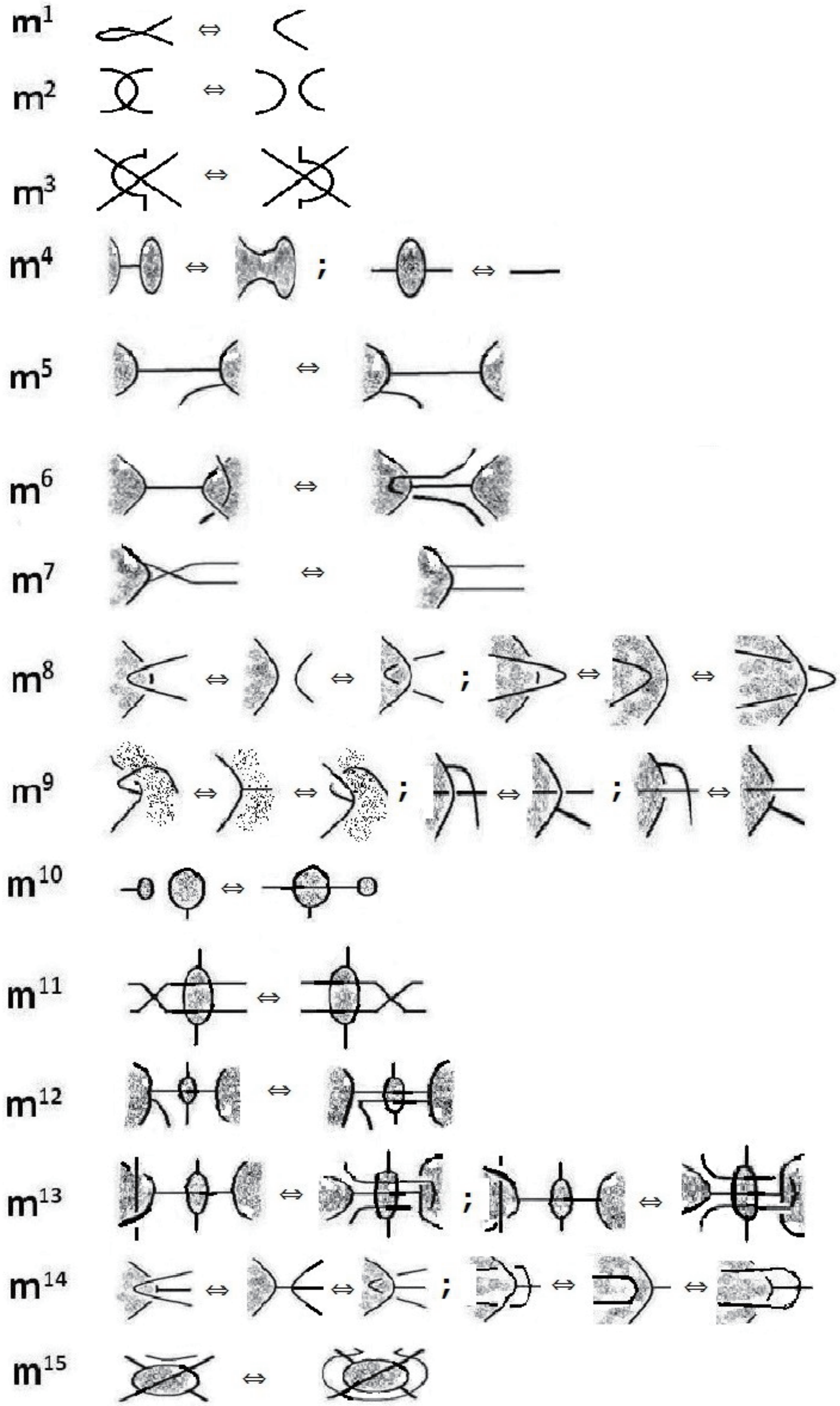

FIGURE 13. The moves $m^1, m^2, \dots, m^{15}$ on chord diagrams without base crossing

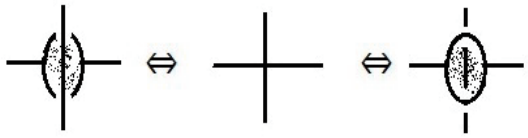

FIGURE 14. The move m^*

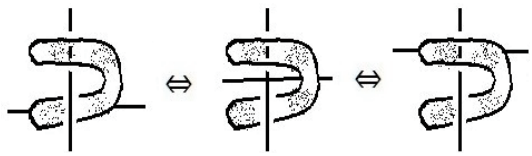

FIGURE 15. Showing the move m^*

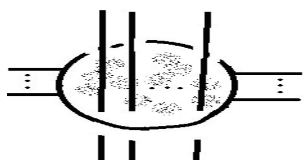

FIGURE 16. A disk with chords

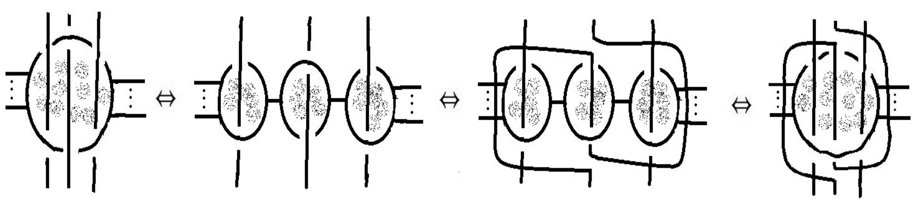

FIGURE 17. Matching the up-down relations on the arcs

diagram in the right-hand side of Fig. 21. As illustrated in Fig. 22, the Reidemeister moves $R_i\,(i = 1, 2, \ldots, 8)$ on the set $\mathbb{D}$ of virtual link diagrams change into the moves $cR_i\,(i = 1, 2, \ldots, 8)$ on the set $\mathbb{A}$. The set $\mathbb{D}/\,(R_i\,(i = 1, 2, \ldots, 8))$ is called the *set of welded virtual links* with an element a *welded virtual link*. Consider the moves $cR_i\,(i = 1, 2, \ldots, 8)$ as the moves on the set $\widetilde{\mathbb{B}}$. Further, we introduce the moves $B^j\,(j = 1, 2, \ldots, 6)$ on the set $\widetilde{\mathbb{B}}$, as illustrated in Fig. 23. Then we show the following theorem.

Theorem 4.3. The map $\tilde{\kappa} : \widetilde{\mathbb{B}} \to \mathbb{F}$ induces a bijection

$$\tilde{\kappa}_* : \widetilde{\mathbb{B}}/\left(cR_i\,(i = 1, 2, \ldots, 8), B^j\,(j = 1, 2, \ldots, 6)\right) \to \mathbb{F}.$$

Proof of Theorem 4.3. It is directly confirmed that the moves $cR_i\,(i = 1, 2, \ldots, 8)$ are consequences of the moves M_0, M_1, M_2. By Lemma 4.1, this fact is equivalent

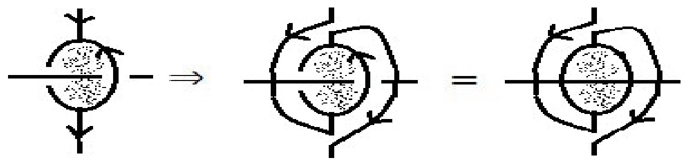

FIGURE 18. A regular change move

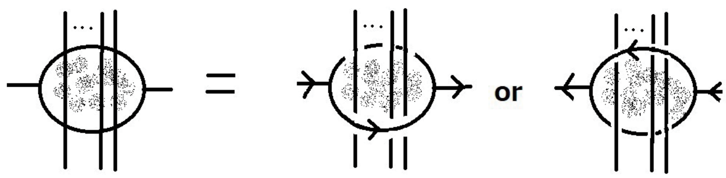

FIGURE 19. A notation on a based loop

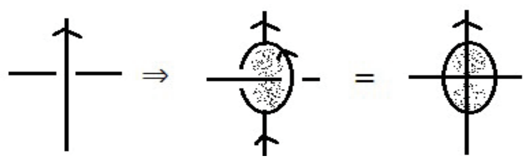

FIGURE 20. Transforming a neighborhood of a real crossing point into a part of a chord diagram

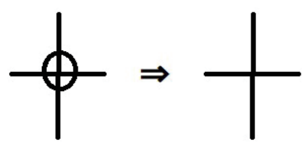

FIGURE 21. Transforming a neighborhood of a virtual crossing point into a part of a chord diagram

to the fact that the moves $R_i\,(i = 1, 2, \ldots, 8)$ on the set $\mathbb{D}$ of virtual link diagrams do not change the equivalence class of the corresponding ribbon torus-links shown by [**11**] using T. Yajima's ribbon torus-knot description in [**13**]. The moves $B^i\,(i = 1, 2, \ldots, 6)$ are also consequences of the moves M_0, M_1, M_2. We also note that the birth-death move is obtained from the moves $cR_i\,(i = 1, 2, \ldots, 8), B^j\,(j = 1, 2, \ldots, 6)$ since any closed chord is not allowed and hence the chord diagram component on the left figure of Fig. 4 contains at least two based loops.

The surjectivity of the map $\tilde{\kappa}_*$ is obtained from Lemma 4.2 and the repair move as in Fig. 17 which is obtained from the moves B^1 and B^2.

Assume that a chord diagram $C \in \widetilde{\mathbb{B}}$ is moved into a chord diagram $C' \in \widetilde{\mathbb{B}}$ by a finite number of the moves $m^i\,(i = 1, 2, \ldots, 15)$. If an upper or lower simple

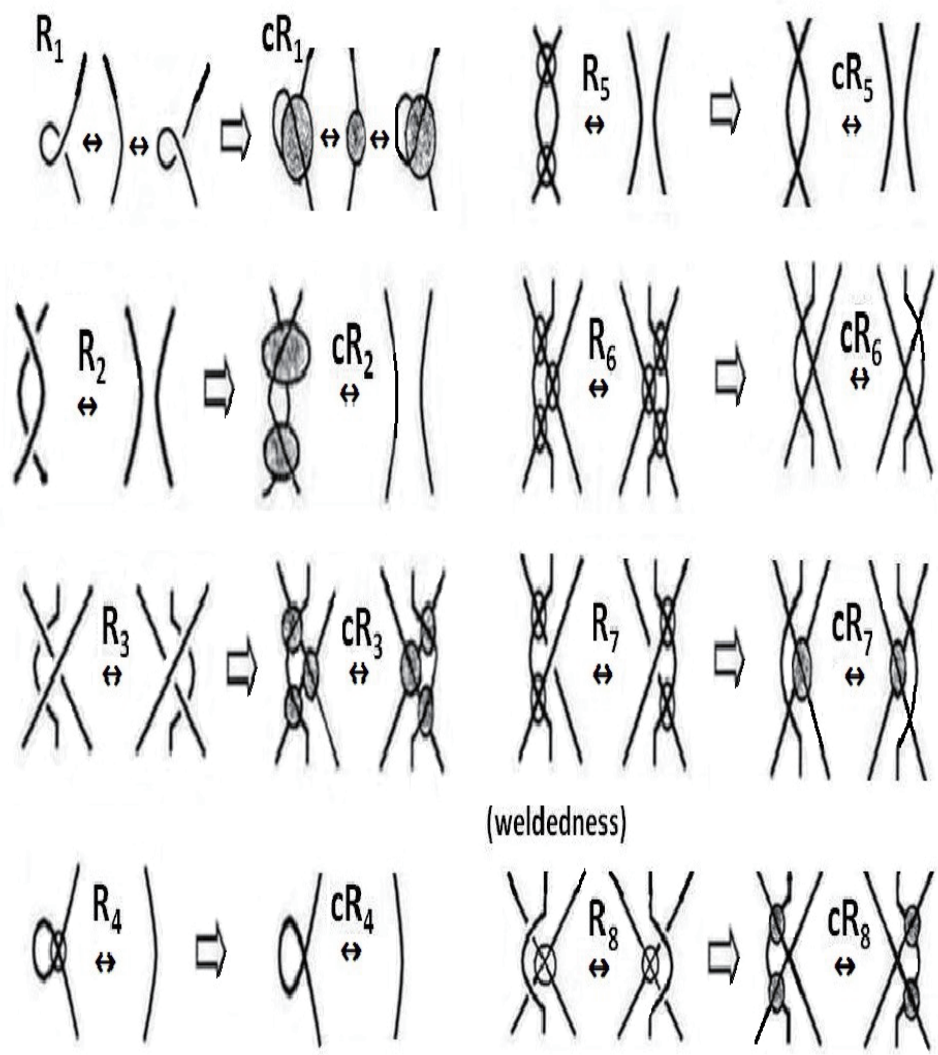

FIGURE 22. Replacements of the Reidemeister moves

proper arc of a chord cut by the disk system arises by the left move of the move m^8 or m^{14} in the process of moves from C to C', then construct a chord crossing given in the middle diagram of Fig. 14 by the move B^1, namely M_1 (which comes from the moves $m^i\,(i=4,5,6)$ by an argument in Section 2) and m^*. Then there is a process of moves from C to C' using the move B^2 but avoiding the use of the left moves of m^8 and m^{14}. Then we use the following observations:

(1) By the assumption of the based loop as in Fig.16, the right move of m^8 is restricted to the move in Fig. 24 which is attained by the moves B^1 and cR_2.

(2) The first move of m^9 is attained by B^1 and cR_1.

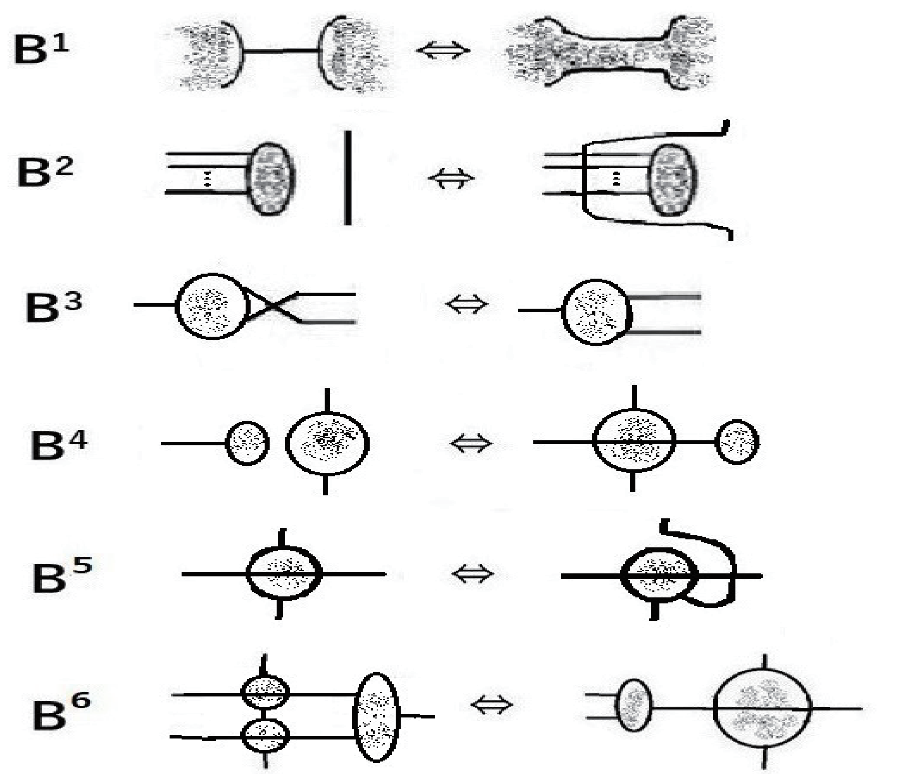

FIGURE 23. The moves B^j ($j = 1, 2, \dots, 6$)

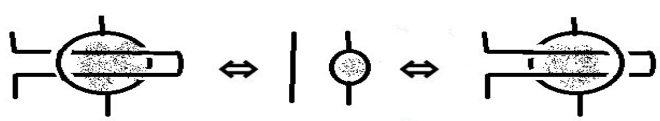

FIGURE 24. A restriction of the move m^8

FIGURE 25. The move B'^5

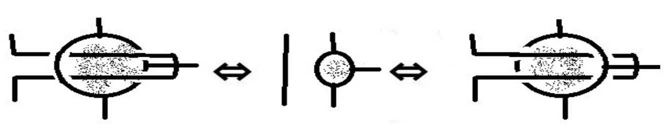

FIGURE 26. A restriction of the move m^{14}

(3) The second and third moves of m^9 are restricted to the move B^5 and the move B'^5 in Fig. 25 which is obtained from the move B^5 by using the moves B^2, B^3.

(4) The move m^{14} is restricted to the move in Fig. 26 which is attained by the moves B^1, B^5 and cR_2.

Then we see that the inclusion $\tilde{\iota} : \widetilde{\mathbb{B}} \to \mathbb{C}$ induces a bijection

$$\tilde{\iota}_* : \widetilde{\mathbb{B}}/\left(m^i\,(i=1,2,\dots,15, i \neq 8,9,14), cR_1, cR_2, B^1, B^2, B^3, B^5\right) \to \mathbb{C}/(M_0, M_1, M_2).$$

By combining this bijection with Lemma 4.1, we have the composite bijection

$$\kappa_* \tilde{\iota}_* : \widetilde{\mathbb{B}}/\left(m^i\,(i=1,2,\dots,15, i \neq 8,9,14), cR_1, cR_2, B^1, B^2, B^3, B^5\right) \to \mathbb{F}.$$

Note that the moves $m^i\,(i = 4,5,6)$ are obtained from the move B^1 (cf. [**4**, Lemma 3.1]). The moves $m^1, m^2, m^3, m^7, m^{10}$ correspond to $cR_4, cR_5, cR_6, B^3, B^4$, respectively. The move m^{11} is obtained from cR_8 and B^1. By using B^1, the move m^{12} is regarded as the move B'^6 in Fig. 27 which is obtained from the move B^1 and B^6. The move m^{15} is obtained from cR_5 and cR_7. For the remaining move m^{13}, use the move cR'_3 illustrated in Fig. 28 obtained from cR_3 by using B^1. Then the move m^{13} is obtained from B^1, cR_2, cR'_3 as in Fig. 29 where the mirror case is similarly obtained. Thus, we have shown the desired result. □

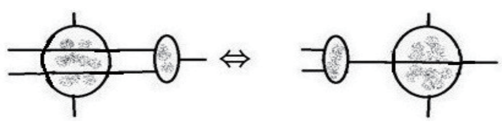

FIGURE 27. The move B'^6

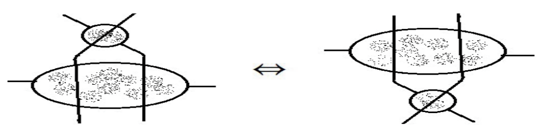

FIGURE 28. The move cR'_3

Let $\mathbb{F}_1$ be the subset of $\mathbb{F}$ consisting of the equivalent classes of ribbon torus-links. For the set $\mathbb{A}$ of chord diagrams of genus one without base crossing, we see from [**4, 11**] that the composite map

$$\kappa_A \tau : \mathbb{D} \to \mathbb{F}_1$$

for the map $\tau : \mathbb{D} \to \mathbb{A}$ and the restriction $\kappa_A : \mathbb{A} \to \mathbb{F}_1$ of the map $\tilde{\kappa}$ to $\mathbb{A}$ is an onto map from the set $\mathbb{D}/\left(R_i\,(i=1,2,\dots,8)\right)$ of welded virtual links onto the set $\mathbb{F}_1$. We obtain the following result as a corollary of Theorem 4.3.

Corollary 4.4. The map $\tau : \mathbb{D} \to \mathbb{A}$ induces an onto map

$$\tau_* : \mathbb{D}/\left(R_i\,(i=1,2,\dots,8)\right) \to \mathbb{A}/\left(cR_i\,(i=1,2,\dots,8), B^j\,(j=1,2,\dots,6)\right)$$

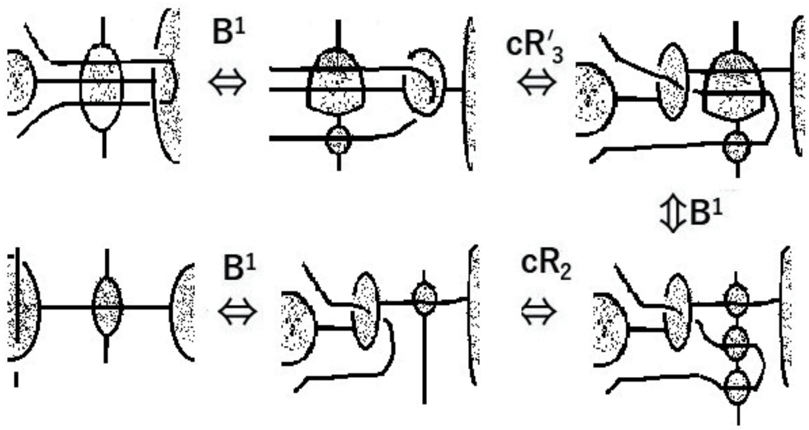

FIGURE 29. The moves for m^{13}

and the map $\kappa_A : \mathbb{A} \to \mathbb{F}_1$ induces a bijection

$$\kappa_{A*} : \mathbb{A}/\left(cR_i\,(i = 1, 2, \ldots, 8), B^j\,(j = 1, 2, \ldots, 6)\right) \to \mathbb{F}_1.$$

In other words, two welded virtual links D and D' represent equivalent ribbon torus-links if and only if the chord diagrams $\tau(D)$ and $\tau(D')$ are changed into each other in $\mathbb{A}$ by a finite number of the moves $cR_i\,(i = 1, 2, \ldots, 8), B^j\,(j = 1, 2, \ldots, 6)$.

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