

ON THE FUNDAMENTAL CLASS OF AN INFINITE CYCLIC COVERING

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(Received November 11, 1997)

(Revised June 24, 1998)

1. Introduction

We consider a compact, connected, oriented PL n -manifold M with $H^1(M; \mathbf{Z}) = \text{Hom}(H_1(M; \mathbf{Z}), \mathbf{Z}) \neq 0$. We restrict ourselves to the case that $n \geq 3$ unless otherwise stated. Every element $\gamma \in H^1(M; \mathbf{Z})$ determines a PL map $f_\gamma : M \rightarrow S^1$ which is unique up to homotopy. By transverse regularity, the preimage $V = (f_\gamma)^{-1}(q)$ for any non-vertex point $q \in S^1$ is a (possibly disconnected) oriented proper bicollared $(n-1)$ -submanifold of M . [Note on the orientation of V : Take a small interval $I_q \subset S^1$ around the point q and orient it by the orientation of S^1 . Then $(f_\gamma)^{-1}(I_q)$ is naturally identified with $V \times I_q$, which is oriented by the orientation of M . Since I_q and $V \times I_q$ are oriented, V can be oriented uniquely.] We call this V a *leaf* of γ . The homology class $[V] \in H_{n-1}(M, \partial M; \mathbf{Z})$ corresponds to γ by the Poincaré duality

$$\cap[M] : H^1(M; \mathbf{Z}) \cong H_{n-1}(M, \partial M; \mathbf{Z}).$$

The Pontrjagin-Thom construction explains how to construct the element γ from an oriented proper bicollared $(n-1)$ -submanifold V of M . From now on, we assume that the element γ is an indivisible element, that is, $\gamma : H_1(M; \mathbf{Z}) \rightarrow \mathbf{Z}$ is an epimorphism. We take the infinite cyclic covering $p : \tilde{M} \rightarrow M$ associated with γ . Since γ is indivisible, \tilde{M} is connected. Let $\exp : \mathbf{R} \rightarrow S^1$ be the infinite cyclic covering defined by $\exp(x) = e^{2\pi i x}$. Then the map $f_\gamma : M \rightarrow S^1$ lifts to a proper map $\tilde{f}_\gamma : \tilde{M} \rightarrow \mathbf{R}$ with $t\tilde{f}_\gamma = \tilde{f}_\gamma t$, where t denotes a generator of the infinite cyclic covering transformation group, such that $t(x) = x + 1$ for all $x \in \mathbf{R}$. Let M_V be the compact manifold obtained from M by splitting it along V . Let V and V' be the two copies of V in ∂M_V . Let $V^* = (\tilde{f}_\gamma)^{-1}(x_q)$ and $M^* = (\tilde{f}_\gamma)^{-1}[x_q, x_q + 1]$ for any point $x_q \in \mathbf{R}$ with $\exp(x_q) = q$. Then $t^m V^* = (\tilde{f}_\gamma)^{-1}(x_q + m)$ and the covering projection p defines a canonical identification

¹Dedicating this paper to Professor Fuichi Uchida on his 60th birthday.

of the pair $(M^*, tV^* + (-V^*))$ with the pair $(M_V, V' + (-V))$ with orientation counted. Further, the submanifold V^* of \tilde{M} is *equivariant*, that is, $t^m V^* \cap V^* = \emptyset$ for any $m \neq 0$, and the homology class $\mu = [V^*] \in H_{n-1}(\tilde{M}, \partial\tilde{M}; \mathbf{Z})$ has the properties

$$(t-1)\mu = [tV^* - V^*] = 0 \quad \text{and} \quad p_*(\mu) = [V] = \gamma \cap [M],$$

where $H_{n-1}(\tilde{M}, \partial\tilde{M}; \mathbf{Z})$ is naturally regarded as a module over the ring $\Lambda = \mathbf{Z}[t, t^{-1}]$ and $p_* : H_{n-1}(\tilde{M}, \partial\tilde{M}; \mathbf{Z}) \rightarrow H_{n-1}(M, \partial M; \mathbf{Z})$ denotes the homomorphism induced from p . Since the Λ -torsion part $TH_{n-1}(\tilde{M}, \partial\tilde{M}; \mathbf{Z})$ of $H_{n-1}(\tilde{M}, \partial\tilde{M}; \mathbf{Z})$ is Λ -isomorphic to $\Lambda/(t-1)$ by the first duality of [3], we see that the homology class μ is uniquely determined by the covering $p : \tilde{M} \rightarrow M$, which we call the *fundamental class* of the covering $p : \tilde{M} \rightarrow M$ (cf. [3; Lemma 6.1]).

In this paper, we study how any two leaves of an indivisible element $\gamma \in H^1(M; \mathbf{Z})$ are connected via a sequence of embedded-handle surgeries in M , in other words, how any two equivariant compact proper bicollared $(n-1)$ -submanifolds of \tilde{M} representing the fundamental class μ are connected via a sequence of equivariant embedded-handle surgeries in \tilde{M} . Our main application is then done for a high-dimensional manifold-link theory. An oriented *manifold-link* F in a compact connected oriented n -manifold M is a closed, possibly disconnected, oriented $(n-2)$ -submanifold of the interior $\text{int } M$ of M such that there exists a bicollared compact oriented $(n-1)$ -manifold V in M with $\partial V \supset F$ and $\partial V - F \subset \partial M$. We call such an $(n-1)$ -submanifold V a *Seifert hypersurface* for F in M . We denote $\partial V - F$ by $\partial_M V$ which is \emptyset or a closed $(n-2)$ -submanifold of ∂M . This Seifert hypersurface V determines a unique indivisible element $\gamma_V \in H^1(M - F; \mathbf{Z}) = \text{Hom}(H_1(M - F; \mathbf{Z}), \mathbf{Z})$ sending each oriented meridian of F in M to $1 \in \mathbf{Z}$. Our argument is applied for Seifert hypersurfaces of an oriented manifold-link in an $n(\geq 3)$ -manifold with connected boundary (possibly, empty) as follows:

THEOREM 1.1. *Assume that the boundary ∂M is connected (possibly, \emptyset). Let V' be a Seifert hypersurface for an oriented manifold-link F in M . Then every Seifert hypersurface V for F in M such that $\gamma_V = \gamma_{V'}$ and $\partial_M V = \partial_M V'$ (with orientation counted) is obtained from V' by a combination of a sequence of embedded-handle surgeries and an ambient isotopy of M relative to $F \cup \partial M$.*

When ∂M is disconnected, the corresponding assertion does not hold even for some Seifert hypersurfaces of a trivial S^{n-2} -knot in $\text{int}(M)$ (see Example 3.3). The terminology "embedded-handle surgery" is precisely explained in Section 2. Some basic result (Theorem 2.1) and its consequences are also stated in Section 2. In Section 3, the connectivity of a leaf is discussed. In Section 4, the proofs of Theorems 1.1 and 2.1 are given.

Several results similar to the present results have been given by C. Kearton and W.B.R. Lickorish [5], P.M. Rice [6], and K. Yokoyama [8] in several methods. Our present method is a generalization of Rice's method in [6] which directly investigates the intersection of two Seifert surfaces of a classical knot. This method was generalized to that of a classical link in [4; Chapter 5] to show the topological invariance of the S-equivalence class of a Seifert matrix of a classical link.

Throughout this paper, spaces and maps are in the PL category, unless otherwise stated.

2. Basic Result

Let k be an integer with $1 \leq k \leq n-1$. Let D^k denote the unit k -disk. A k -embedded-handle on a compact oriented $(n-1)$ -submanifold V' of M is an embedding

$$\varphi : D^k \times D^{n-k} \rightarrow \text{int}(M)$$

whose image we denote by h^k such that

$$\varphi(D^k \times D^{n-k}) \cap \text{int}(V') = \varphi((\partial D^k) \times D^{n-k})$$

which we denoted by \dot{h}^k . Then the manifold $V'' = \text{cl}(V' - \dot{h}^k) \cup \text{cl}(\partial \dot{h}^k - \dot{h}^k)$ is said to be obtained from V' by a k -embedded-handle surgery (or simply a k -EHS) in M . We note that the manifold V' is also obtained from V'' by an $(n-k)$ -EHS in M . Further, if V' is a leaf of γ , then V'' is also a leaf of γ . A compact oriented $(n-1)$ -submanifold V of M is obtained from V' by a sequence of embedded-handle surgeries (or simply by a SEHS) if there is a sequence of $(n-1)$ -submanifolds V_j ($j = 1, 2, \dots, s$) of M such that $V' = V_1$, $V = V_s$ and for every $j = 1, 2, \dots, s-1$, V_{j+1} is obtained from V_j by a k_j -EHS where $1 \leq k_j \leq n-1$. If $k_j = k$ for all j , then the SEHS is also called a k -SEHS. By a standard argument on general position, we can arrange every SEHS so that each j -SEHS appears after every i -SEHS for all i and j with $i < j$. The following theorem is a basic result for our purpose:

THEOREM 2.1. *For a given leaf V' of γ , a leaf V of γ such that $\partial V = \partial V'$ (with orientation counted) is obtained from V' by a combination of a SEHS and an ambient isotopy of M relative to ∂M if and only if $[-V \cup V'] = 0$ in $H_{n-1}(M; \mathbf{Z})$.*

When $n = 2$, the same result holds, which is discussed in Remark 4.2.

In general, any two leaves V and V' of the same element γ have $[V] = [V']$ in $H_{n-1}(M, \partial M; \mathbf{Z})$. This implies that if the boundary ∂M is \emptyset or connected, then

we have that $[-V \cup V'] = 0$ in $H_{n-1}(M; \mathbf{Z})$, because $[\partial M] = 0$ in $H_{n-1}(M; \mathbf{Z})$. Thus, by Theorem 2.1, we have the following corollary:

COROLLARY 2.2. *If the boundary ∂M is \emptyset or connected, then for a given leaf V' of γ , every leaf V of γ with $\partial V = \partial V'$ (with orientation counted) is obtained from V' by a combination of a SEHS and an ambient isotopy of M relative to ∂M .*

We consider the case that ∂M has two or more connected components B_i ($i = 0, 1, \dots, r$). We push B_i into $\text{int}(M)$ by using a boundary-collar so that the resulting B_i (say B'_i) meets a leaf V' of γ with a closed orientable $(n-2)$ -manifold transversely in $\text{int}(M)$ (cf. M.A. Armstrong and E.C. Zeeman [1]). We do an orientation-preserving cut on $V' \cup \varepsilon B'_i$ ($\varepsilon = \pm 1$) in $\text{int}(M)$ to obtain an $(n-1)$ -manifold V'' , which is still a leaf of γ with $\partial V'' = \partial V'$ and is said to be obtained from V' by making an *addition* (or more precisely, an ε -*addition*) of B_i to V' . Then we have

$$[-V \cup V''] = [-V \cup V'] + \varepsilon[B_i] \quad (\varepsilon = \pm 1)$$

in $H_{n-1}(M; \mathbf{Z})$ for any leaf V of γ with $\partial V = \partial V'$. Since $[-V \cup V']$ is generated by $[B_i]$ ($i = 0, 1, \dots, r$) and $[\partial M] = \sum_{i=0}^r [B_i] = 0$ in $H_{n-1}(M; \mathbf{Z})$, we see that $[-V \cup V']$ is generated by the homology classes $[B_i]$ ($i = 1, \dots, r$) with any one homology class, say $[B_0]$, removed. Thus, we obtain from Theorem 2.1 the following corollary:

COROLLARY 2.3. *Assume that the boundary ∂M has two or more connected components B_i ($i = 0, 1, \dots, r$). Then for a leaf V' of γ , every leaf V of γ such that $\partial V = \partial V'$ (with orientation counted) is obtained from V' by a combination of a SEHS and an ambient isotopy relative to ∂M after making some additions of B_i ($i = 1, 2, \dots, r$) to V' .*

We consider a more special case that for every $i \neq 0$, $B_i = S^1 \times F_i$ with F_i a closed connected oriented $(n-2)$ -manifold and $V' \cap B_i = \varepsilon(P_i \times F_i)$ (with orientation counted) for a finite subset $P_i \subset S^1$. Note that we do not impose any condition on B_0 . Then the leaf V'' of γ obtained from V' by an ε -addition of B_i to V' is isotopic to V' by an ambient isotopy of M which, on B_i , slides the set P_i onto itself along S^1 and keeps the $(n-2)$ -submanifold F_i fixed, and also keeps the outside of a boundary-collar of B_i in M fixed (see Fig. 1). Then we say that V'' is obtained from V' by doing a *winding operation* on B_i for V' . The following corollary is direct from Corollary 2.3:

COROLLARY 2.4. *For the connected components B_i ($i = 0, 1, \dots, r$) of the boundary ∂M , assume that for every $i \neq 0$, $B_i = S^1 \times F_i$ with F_i a closed connected oriented $(n-2)$ -manifold and $V' \cap B_i = \varepsilon(P_i \times F_i)$ (with orientation*

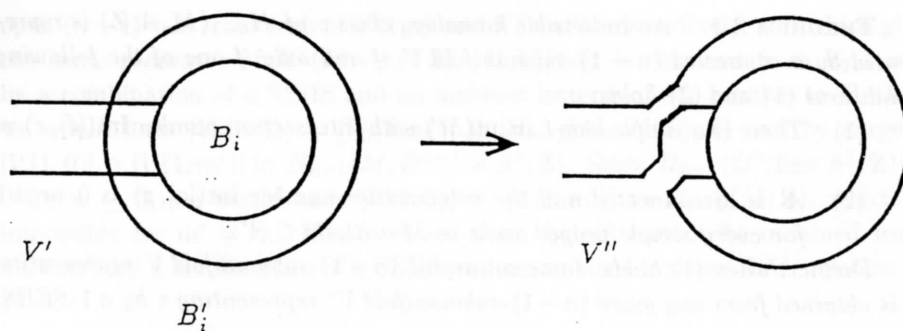


Fig. 1.

counted) for a non-empty finite subset P_i of S^1 . Then every leaf V of γ such that $\partial V = \partial V'$ (with orientation counted) is obtained from the given leaf V' of γ by a combination of a SEHS and an ambient isotopy relative to ∂M after doing some winding operations on B_i ($i = 1, 2, \dots, r$) for V' . In particular, V is obtained from V' by a combination of a SEHS and an ambient isotopy of M relative to B_0 .

In general, there exist two leaves V and V' of γ with $\partial V = \partial V'$ (with orientation counted) such that V cannot be obtained from V' by a combination of a SEHS and an ambient isotopy of M . Some simple example is given in Example 3.3.

3. Connectivity of a leaf

We consider an indivisible homology class z of $H_{n-1}(M, A; \mathbf{Z})$, where A is \emptyset or a compact submanifold of ∂M such that $A' = \text{cl}(\partial M - A)$ is also \emptyset or a compact $(n-1)$ -submanifold of ∂M . We note that z is always represented by a possibly disconnected, compact bicollared oriented $(n-1)$ -submanifold V with $(V, \partial V) \subset (M, A)$. In fact, by the Poincaré duality isomorphism $\cap[M] : H^1(M, A'; \mathbf{Z}) \cong H_{n-1}(M, A; \mathbf{Z})$ and a natural isomorphism $H^1(M, A'; \mathbf{Z}) \cong [M_{A'}, S^1]$, we obtain a map $f : M_{A'} \rightarrow S^1$ unique up to homotopy from z , where $M_{A'}$ is M (if $A' = \emptyset$) or the quotient space obtained from M by identifying A' with one point $p_{A'}$ (if $A' \neq \emptyset$). Then the preimage $V = f^{-1}(q)$ of any non-vertex point $q \in S^1$ (with $q \neq f(p_{A'})$ if $A' \neq \emptyset$) is a desired one. Further, if z is a non-zero, divisible homology class, then we can see from Poincaré duality that z cannot be represented by any connected one. In this section, we consider when an indivisible homology class z is represented by a connected $(n-1)$ -submanifold V . We have the following theorem (where we assume $n \geq 3$):

THEOREM 3.1. *An indivisible homology class z of $H_{n-1}(M, A; \mathbf{Z})$ is represented by a connected $(n-1)$ -submanifold V if and only if one of the following conditions (1) and (2) holds:*

- (1) *There is a simple loop ℓ in $\text{int}(M)$ with intersection number $\text{Int}([\ell], z) = \pm 1$.*
- (2) *A' is disconnected and the intersection number $\text{Int}([a], z)$ is 0 or ± 1 for every simple proper arc a in M with $\partial a \subset A'$.*

Further, when (1) holds, some connected $(n-1)$ -submanifold V representing z is obtained from any given $(n-1)$ -submanifold V' representing z by a 1-SEHS.

We refer an explanation of the $n = 2$ version of this theorem to Remark 3.4. In the case (2), we must have $\text{Int}([\ell], z) = 0$ for every simple loop ℓ in $\text{int}(M)$, since otherwise we can easily construct a proper arc a' in M with $\partial a' \subset A'$ such that $\text{Int}([a'], z)$ is an integer except 0 and ± 1 , which is a contradiction. This also implies that there is a simple proper arc a' in M with $\partial a' \subset A'$ and $\text{Int}([a'], z) = \pm 1$ by Poincaré duality. The following corollary is direct from Theorem 3.1 (1):

COROLLARY 3.2. *For $n \geq 3$, every leaf V' of an indivisible element $\gamma \in H^1(M; \mathbf{Z})$ is modified into a connected leaf V of γ by a 1-SEHS.*

The following example contains the reason why some codimension one immersed sphere cannot be homologous to an embedded sphere. For $n = 3$, compare with the sphere theorem (see for example J. Hempel [2]).

EXAMPLE 3.3. Let M be the connected sum of $S^{n-1} \times S^1$ and a compact connected oriented n -manifold M_1 with ∂M_1 disconnected ($n \geq 3$). Let B be a connected component of $\partial M = \partial M_1$. Let $z(m, m')$ be the homology class $m[S^{n-1} \times p] + m'[B]$ in $H_{n-1}(M; \mathbf{Z})$ for a point $p \in S^1$ and integers m and m' . The homology class $z(m, m')$ is always representable by a closed bicollared $(n-1)$ -submanifold of M and, further when B is an $(n-1)$ -sphere, by an immersed $(n-1)$ -sphere in M . It is indivisible if and only if m and m' are coprime. From Theorem 3.1, we can see that the indivisible homology class $z(m, m')$ is representable by a closed connected orientable bicollared $(n-1)$ -submanifold $V(m, m')$ of M if and only if $m = 0, \pm 1$. We note that $V(1, 0)$ and $V(1, m')$ ($m' \neq 0$) are leaves of the same indivisible element of $H^1(M; \mathbf{Z})$. However, $V(1, 0)$ cannot be obtained from $V(1, m')$ by a combination of a SEHS and an ambient isotopy of M , because $V(1, 0)$ and $V(1, m')$ represent distinct homology classes of $H_{n-1}(M; \mathbf{Z})$. Let F be a trivial S^{n-2} -knot in M_1 . Since a surgery of M_1 along F replacing a regular neighborhood $N(F) = S^{n-2} \times D^2$ of F with $D^{n-1} \times S^1$ produces M , we may consider that F admits Seifert hypersurfaces $V_0(1, 0)$ and $V_0(1, m')$ in M_1 which are obtained by removing an

open $(n-1)$ -ball from each of $V(1,0)$ and $V(1,m')$, respectively, and which give the same generator of $H^1(M_1 - F; \mathbf{Z})$. If $V_0(1,0)$ is obtained from $V_0(1,m')$ by a combination of a SEHS and an ambient isotopy of M_1 relative to F , then we would have $[V_0(1,0)] = [V_0(1,m')]$ in $H_{n-1}(M_1, N(F); \mathbf{Z})$ and by surgery $[V(1,0)] = [V(1,m')]$ in $H_{n-1}(M, D^{n-1} \times S^1; \mathbf{Z})$. Since $H_{n-1}(D^{n-1} \times S^1; \mathbf{Z}) = 0$ for $n \geq 3$, we would have $[V(1,0)] = [V(1,m')]$ in $H_{n-1}(M; \mathbf{Z})$, which is impossible for $m' \neq 0$. Hence we see that $V_0(1,0)$ cannot be obtained from $V_0(1,m')$ by a combination of a SEHS and an ambient isotopy of M_1 relative to F .

PROOF OF THEOREM 3.1. Assume that there is a connected $(n-1)$ -submanifold V representing z . Then the manifold M_V obtained from M by splitting it along V has at most two connected components. When M_V is connected, there is a simple loop ℓ in $\text{int}(M)$ which meets with V in one point transversely, so that $\text{Int}(\ell, V) = \pm 1$ and the case (1) occurs. Assume that M_V is disconnected. Then M_V has two connected components. Let A'_i ($i = 0, 1, \dots, r$) be the connected components of A' . We note that every loop ℓ in $\text{int}(M)$ is homologous to a sum of loops not meeting with V since V is connected, so that $\text{Int}(\ell, V) = 0$. This means that every simple proper arc a in M joining some A'_i and some A'_j has $\text{Int}(a, V) = 0$ or ± 1 according to whether A'_i and A'_j belong to the same connected component of M_V or not. Then the case (2) occurs. Conversely, assume that z has the condition (1) or (2). Let V' be an $(n-1)$ -submanifold representing z . We take mutually disjoint simple proper arcs a_i ($i = 1, 2, \dots, r$) in M with $\partial a_i \subset A'$ and simple loops ℓ_j ($j = 1, 2, \dots, s$) in $\text{int}(M)$ such that

- (i) these arcs and loops meet V' transversely,
- (ii) these arcs and loops represent a basis of the quotient group $\text{b}H_1(M, A'; \mathbf{Z})$ of $H_1(M, A'; \mathbf{Z})$ by the torsion subgroup, and
- (iii) $\text{Int}([a_i], z) = 0$ ($i = 1, 2, \dots, r$) and $\text{Int}([\ell_j], z) = \delta_{j1}$ ($j = 1, 2, \dots, s$) in the case (1), or $\text{Int}([a_i], z) = \delta_{i1}$ ($i = 1, 2, \dots, r$) and $\text{Int}([\ell_j], z) = 0$ ($j = 1, 2, \dots, s$) in the case (2).

Then by a 1-SEHS (along these arcs and loops), we obtain from V' an $(n-1)$ -submanifold V'' representing z meeting in a single point only with ℓ_1 in the case (1) or only with a_1 in the case (2). Let V''_0 be the connected component of V'' meeting with ℓ_1 in the case (1) or a_1 in the case (2). Since the intersection pairing

$$\text{Int} : \text{b}H_1(M, A'; \mathbf{Z}) \times H_{n-1}(M, A; \mathbf{Z}) \rightarrow \mathbf{Z}$$

is non-singular by Poincaré duality, we see from (ii) that each connected component of $V'' - V''_0$ represents the zero element of $H_{n-1}(M, A; \mathbf{Z})$. Hence $[V''_0] = [V''] = z$. For the first half assertion, we can take V''_0 as V .

In the case (1), we can take a simple arc α joining the loop ℓ_1 with a connected

component V_1'' of $V'' - V_0''$ in M . Let b_1 and b_2 be the two arcs obtained from ℓ_1 by cutting it by the two points $\ell_1 \cap (\alpha \cup V_0'')$. Then the arc $b_1 \cup \alpha$ or $b_2 \cup \alpha$ connects V_0'' and V_1'' with coherent orientation. By the 1-EHS along a 1-handle, we can connect V_0'' and V_1'' . By induction, we can obtain a connected $(n-1)$ -submanifold V representing z from V'' by a 1-SEHS. This completes the proof.

REMARK 3.4. We observe here the version of Theorem 3.1 in the case $n = 2$ is false in general. Let $A_i (i = 1, 2, \dots, u)$ be the connected components of A with orientations induced from M . When $n = 2$, the intersection number $\text{Int}(A_i, z)$ is well-defined for an indivisible homology class $z \in H_1(M, A; \mathbf{Z})$. We define

$$I_A(z) = \sum_{i=1}^u |\text{Int}(A_i, z)|,$$

which is a non-negative even integer. By convention, $I_A(z) = 0$ for $A = \emptyset$. Then we have the following proposition (whose proof is obvious):

An indivisible homology class $z \in H_1(M, A; \mathbf{Z})$ is represented by a connected 1-manifold if and only if we have $I_A(z) \leq 2$ in addition to the condition (1) or (2) of Theorem 3.1.

4. Proofs of Theorems 1.1 and 2.1

4.1 PROOF OF THEOREM 2.1. The "only if" part is obvious. We show the "if" part. By Corollary 3.2, we assume that V is connected and V' has no closed manifold component. We take mutually disjoint, oriented, simple proper arcs $a_i (i = 1, 2, \dots, r)$ in M and simple loops ℓ and $\ell_j (j = 1, 2, \dots, s)$ in $\text{int}(M)$ with the following properties (i), (ii) and (iii):

- (i) these arcs and loops meet V and V' transversely and $\partial a_i \subset \partial M - \partial V (i = 1, 2, \dots, r)$,
- (ii) these arcs and loops represent a system of generators for the free abelian group $bH_1(M, \partial M; \mathbf{Z})$, and
- (iii) $\text{Int}(a_i, V) = 0 (i = 1, 2, \dots, r)$, $\text{Int}(\ell, V) = 1$, and $\text{Int}(\ell_j, V) = 0 (j = 1, 2, \dots, s)$.

Since $[-V \cup V'] = 0$ in $H_{n-1}(M; \mathbf{Z})$, we see from (iii) that $\text{Int}(a_i, V') = 0 (i = 1, 2, \dots, r)$, $\text{Int}(\ell, V') = 1$, and $\text{Int}(\ell_j, V') = 0 (j = 1, 2, \dots, s)$. By a 1-SEHS along these arcs and loops, we can assume that the loop ℓ meets each of V and V' at a single point, and the other loops $\ell_j (j = 1, 2, \dots, s)$ and the arcs $a_i (i = 1, 2, \dots, r)$ do not meet $V \cup V'$. Let N be a tubular neighborhood of ℓ in M meeting V and V' with $(n-1)$ -disks. We may deform V and V' along ℓ so that these $(n-1)$ -disks coincide, i.e., $V \cap N = V' \cap N$ by an ambient isotopy of M relative to the outside of a regular neighborhood of N . Let $E = \text{cl}(M - N)$, $V_E = V \cap E$ and $V'_E = V' \cap E$. When $\partial M \neq \emptyset$, we choose a proper oriented arc

a_{r+1} in E connecting ∂N and some component of ∂M with $\partial a_{r+1} \cap \partial V_E = \emptyset$ and meeting V_E and V'_E transversely. We note that $\partial N = S^1 \times S^{n-2}$ and $\partial N \cap \partial V_E = \partial N \cap \partial V'_E = p \times S^{n-2}$ for some $p \in S^1$. We do winding operations on ∂N for V_E and V'_E to obtain $\text{Int}(a_{r+1}, V_E) = \text{Int}(a_{r+1}, V'_E) = 0$. We note that these operations are ambient isotopic deformations of V and V' in M relative to the outside of a regular neighborhood of N . By a 1-SEHS along a_{r+1} , we can deform V_E and V'_E so that $a_{r+1} \cap V = a_{r+1} \cap V' = \emptyset$. Since the arcs a_i ($i = 1, 2, \dots, r+1$) and the loops ℓ_j ($j = 1, 2, \dots, s$) represent a system of generators for the free abelian group $\text{bH}_1(E, \partial E; \mathbf{Z})$ and they do not meet with $-V_E \cup V'_E$, we see from the non-singular intersection form

$$\text{Int} : \text{bH}_1(E, \partial E; \mathbf{Z}) \times H_{n-1}(E; \mathbf{Z}) \rightarrow \mathbf{Z}$$

that the homology class $[-V_E \cup V'_E] \in H_{n-1}(E; \mathbf{Z})$ must be zero. Thus, it is sufficient to prove the theorem under the following assumption:

ASSUMPTION 4.1.1. $\partial M \supset \partial V' = \partial V \neq \emptyset$, and there are mutually disjoint, oriented, simple proper arcs a_i ($i = 1, 2, \dots, r$) in M and simple loops ℓ_j ($j = 1, 2, \dots, s$) in $\text{int}(M)$ such that these arcs and loops do not meet V and V' , and represent a system of generators for the free abelian group $\text{bH}_1(M, \partial M; \mathbf{Z})$.

Let b_k ($k = 1, 2, \dots, u$) be simple proper oriented arcs or simple oriented loops in V which represent a basis for $\text{bH}_1(V, \partial V; \mathbf{Z})$. Let b_k^+ be a slight translation of b_k into a positive normal direction obtained by using a collar of V in M . Since $[-V \cup V'] = 0$ in $H_{n-1}(M; \mathbf{Z})$, we have $\text{Int}(b_k^+, -V \cup V') = 0$. Using that $b_k^+ \cap V = \emptyset$, we see that $\text{Int}(\text{int}(b_k), V') = 0$. By an argument of transversality in [1], we can assume that the intersection $L = \text{int}(V) \cap \text{int}(V')$ is a closed (possibly disconnected) orientable bicollared $(n-2)$ -submanifold of V and V' . Then by a 1-SEHS on V' along b_k , we may consider that $\text{int}(b_k) \cap V' = \emptyset$ ($k = 1, 2, \dots, u$). Let L_1 be any connected component of L . Since $b_k \cap L_1 = \emptyset$ for all k , the non-singular intersection form

$$\text{Int} : \text{bH}_1(V, \partial V; \mathbf{Z}) \times H_{n-2}(V; \mathbf{Z}) \rightarrow \mathbf{Z}$$

implies that L_1 bounds a compact $(n-1)$ -submanifold V_1 in V . We take L_1 to be innermost in V , i.e., $(L - L_1) \cap V_1 = \emptyset$. Let $c : V_1 \times [-1, 1] \rightarrow M$ be a PL embedding such that $c(x, 0) = x$ for all $x \in V_1$ and $c(L_1 \times [-1, 1]) \subset V'$. We consider the $(n-1)$ -manifold $V'' = V' - c(L \times (-1, 1)) \cup c(V_1 \times \{-1, 1\})$. The $(n-1)$ -manifold V'' has at most one closed manifold component.

We first assume that V'' has no closed manifold component. Then we modify $c(L_1 \times [-1, 1])$ into $c(V_1 \times \{-1, 1\})$ by a SEHS along the n -manifold $c(V_1 \times [-1, 1])$, so that the leaf V'' of γ without a closed manifold component is obtained from V' by a SEHS. By construction, we have $\text{int}(V) \cap \text{int}(V'') = L - L_1$.

Next, we assume that V'' contains a closed manifold component V_1'' . Let $V^* = V'' - V_1''$. Without loss of generality, we consider that $c(V_1 \times 1) \subset V_1''$. Since $V_1'' \cap a_i = V_1'' \cap \ell_j = \emptyset$ for all i and j , the non-singular intersection form

$$\text{Int} : bH_1(M, \partial M; \mathbf{Z}) \times H_{n-1}(M; \mathbf{Z}) \rightarrow \mathbf{Z}$$

implies that V_1'' is the boundary of a compact connected n -submanifold M_1 in M . Since $\partial M \supset \partial V = \partial V^* \neq \emptyset$ and V^* has no closed manifold component, we have $M_1 \cap V^* = \emptyset$. We modify the $(n-1)$ -submanifold $\partial M_1^+ - c(\text{int}(V_1) \times (-1))$ of V' into $c(V_1 \times (-1))$ by a SEHS on V' along the n -manifold $M_1^+ = M_1 \cup c(V_1 \times [-1, 1])$. Thus, we obtain from V' a leaf V^* of γ by a SEHS such that V^* has no closed manifold component and $\text{int}(V) \cap \text{int}(V^*) = L - L_1$.

Using that the arcs a_i ($i = 1, 2, \dots, r$) and loops ℓ_j ($j = 1, 2, \dots, s$) do not meet V'' and V^* , we can continue this process to obtain that $L = \emptyset$, that is, $(-V) \cup V'$ is a closed connected $(n-1)$ -manifold. Then, since it is null-homologous in M , it bounds a compact connected n -manifold in M . Using it, we see that V is obtained from V' by a SEHS. This completes the proof of Theorem 2.1.

REMARK 4.2. The $n = 2$ version of Theorem 2.1 is true, although the proof is somewhat different from 4.1. We show the "if" part since the "only if" part is obvious. Let V and V' be leaves of an indivisible element $\gamma \in H^1(M; \mathbf{Z})$ such that $\partial V = \partial V'$ (with orientation counted) and $[-V \cup V'] = 0$ in $H_1(M; \mathbf{Z})$. Then we show that V is obtained from V' by a combination of a SEHS and an ambient isotopy relative to ∂M . We may assume that M is bounded. We take mutually disjoint oriented simple proper arcs a_i ($i = 1, 2, \dots, r$) in M meeting V and V' transversely and being disjoint from ∂V such that the surface M^* obtained from M by cutting along the arcs is a disk. We modify V and V' by 1-handle (= band) surgeries along the arcs so that $|a_i \cap V| = |\text{Int}(a_i, V)|$ and $|a_i \cap V'| = |\text{Int}(a_i, V')|$ for every $i = 1, 2, \dots, r$. Let V^* and $(V')^*$ be the proper oriented 1-manifolds in the disk M^* obtained from V and V' by cutting along the arcs, respectively. Since $\text{Int}(a_i, V) = \text{Int}(a_i, V')$ for all i , we can overlap the parts of V^* and $(V')^*$ contained in a boundary collar of M^* by an ambient isotopy of M^* relative to ∂V . Then by band surgeries, V^* and $(V')^*$ are deformed so that they coincide up to trivial loops in M^* . Thus, to complete the proof, it suffices to show that $V + O$ is deformed into V by band surgeries in M for the boundary O of a disk D in $\text{int}(M) - V$. Let ℓ be a simple oriented loop in $\text{int}(M)$ such that $\text{Int}(\ell, V) = m > 0$ and $\ell \cap D$ is an arc. By band surgeries, V is deformed into V'' with $|\ell \cap V''| = \text{Int}(\ell, V'') = m$. Then we can do an oriented band surgery along an arc in ℓ joining O and V'' . The result of this surgery on $V'' + O$ is ambient isotopic to V'' relative to ∂M . We can recover V from V'' by the inverse band surgeries. Hence $V + O$ is deformed into V by band surgeries in M . Thus, we see

that V is obtained from V' by a combination of a SEHS and an ambient isotopy relative to ∂M .

4.3 PROOF OF THEOREM 1.1. We may consider that a regular neighborhood N of F in M is a trivial disk-bundle over F with a trivialization $f : F \times (D^2, \mathbf{0}) \cong (N, F)$, where D^2 denotes the 2-disk and $\mathbf{0}$, the origin. By the uniqueness of PL normal disk bundles (cf. C.T.C. Wall [7]), we can assume that $V_N = V \cap N$ and $V'_N = V' \cap N$ give cross-sections of this bundle, that is, there are embeddings $g, g' : F \times [0, 1] \rightarrow F \times D^2$ preserving the F -factors identically such that

- (1) $g(x \times ([0, 1], \mathbf{0})) = x \times ([\mathbf{0}, \mathbf{1}], \mathbf{0})$ for all $x \in F$, where $[\mathbf{0}, \mathbf{1}]$ denotes the line-segment in D^2 with endpoints $\mathbf{0}$ and a point $\mathbf{1} \in S^1$ (independent of x),
- (2) $g'(F \times [0, 1]) \cap F \times \mathbf{0} = g'(F \times \mathbf{0})$ and $g'(F \times [0, 1]) \cap F \times S^1 = g'(F \times \mathbf{1})$,
- (3) $fg(F \times [0, 1]) = f(F \times [\mathbf{0}, \mathbf{1}]) = V_N$ and $fg'(F \times [0, 1]) = V'_N$.

Further, replacing D^2 with a smaller 2-disk, we can assume that $g'(x \times [0, 1]) = x \times [\mathbf{0}, \mathbf{1}_x]$ for all $x \in S^1$ where $[\mathbf{0}, \mathbf{1}_x]$ is a line-segment in D^2 with an endpoint $\mathbf{1}_x \in S^1$ depending on x . Let $F_1 = g(F \times \mathbf{1})$, $F'_1 = g'(F \times \mathbf{1})$, $E = \text{cl}(M - N)$, $V_E = V \cap E$, and $V'_E = V' \cap E$. Since V_E and V'_E are leaves of the same element $\gamma = \gamma_V|_E = \gamma_{V'}|_E \in H^1(E; \mathbf{Z})$, we have $[fF_1] = [fF'_1]$ in $H_{n-2}(\partial E; \mathbf{Z})$, so that $[F_1] = [F'_1]$ in $H_{n-2}(F \times S^1; \mathbf{Z})$. By the assumption on g , this homology class is Poincaré dual to the element $\gamma_0 \in H^1(F \times S^1; \mathbf{Z})$ represented by the product bundle projection $p_0 : F \times S^1 \rightarrow S^1$. Since F_1 and F'_1 are leaves of the same element γ_0 , we have $\gamma_0|_{F'_1} = \gamma_0|_{F_1} = 0$, which implies that the composite maps $s = p_0(g|_{F \times \mathbf{1}})$, $s' = p_0(g'|_{F \times \mathbf{1}}) : F \times \mathbf{1} \rightarrow S^1$ induce the same (trivial) homomorphism in homology and hence are homotopic. This homotopy induces an isotopy preserving the F -factors identically between the embeddings $g|_{F \times \mathbf{1}}, g'|_{F \times \mathbf{1}} : F \times \mathbf{1} \rightarrow F \times S^1$, which extends, by a cone extension, to an isotopy between the embeddings $g, g' : F \times [0, 1] \rightarrow F \times D^2$ which keeps $g|_{F \times \mathbf{0}} = g'|_{F \times \mathbf{0}} : F \times \mathbf{0} \rightarrow F \times \mathbf{0}$ fixed. By the isotopy extension theorem, this isotopy extends to an ambient isotopy of $F \times D^2$. From this, we can construct an ambient isotopy of M relative to $\partial M \cup F$ sending V'_N onto V_N . Now we consider that $(N, V_N) = (N, V'_N) = F \times (D^2, [\mathbf{0}, \mathbf{1}])$ and $(\partial N, V_N \cap \partial N) = (\partial N, V'_N \cap \partial N) = F \times (S^1, \mathbf{1})$. Since V_E and V'_E are leaves of the same element $\gamma \in H^1(E; \mathbf{Z})$ with $\partial V_E = \partial V'_E$ and the winding operations on the components of ∂N for V'_E extend to an ambient isotopy of N relative to F , we can apply Corollary 2.4 to obtain the desired result, taking E , V_E , and V'_E as M , V , and V' , respectively. This completes the proof of Theorem 1.1.

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