KAWAUCHI, A.
KOBE J. MATH.,
4 (1987) 31-41

ON THE INTEGRAL HOMOLOGY OF INFINITE CYCLIC COVERINGS OF LINKS

By Akio KAWAUCHI

(Received September 2, 1985)

§ 0. Introduction

We consider a polygonal oriented link ℓ in the Euclidean 3-space R^3. Let $E(\ell)=R^3-\ell$. We have a unique infinite cyclic covering space $\tilde{E}(\ell)$ over $E(\ell)$ associated with the epimorphism $\gamma: \pi_1(E(\ell))\to\langle t\rangle$ sending each meridian element*) of ℓ to t, where $\langle t\rangle$ is the infinite cyclic group generated by a letter t. For the integral group ring $\Lambda=Z\langle t\rangle$, the integral homology $H_1(\tilde{E}(\ell))$, which we denote by $H(\ell)$, forms a finitely generated Λ-module. *Throughout this paper, Λ-modules will mean finitely generated Λ-modules, unless otherwise stated.* For a Λ-module H, we use the following notations: DH=the (unique) maximal finite Λ-submodule of H (cf. [Ka]$_4$), $D_0H=\cap_{i=1}^{+\infty}(t-1)^iDH$, TH= the Λ-torsion part of H, $BH=H/TH$, $\beta H=\text{rank}_\Lambda H$, eH=the minimal number of elements generating H over Λ, and $E^qH=\text{Ext}^q_\Lambda(H, \Lambda)$. Note that $D_0H=(t-1)^N DH$ for a large positive integer N and hence $t-1$: $D_0H\cong D_0H$. By convention, $eH=0$ if $H=0$. It is known that $E^qH=0$ for $q\geq 3$ and there are natural Λ-isomorphisms $E^2H\cong E^2DH\cong\text{Hom}_Z(DH, Q/Z)$ and H has the Λ-projective dimension ≤ 1 if and only if $DH=0$ (cf. [Ka, §3]$_4$, Levine [L]). By the identity $\lambda(t)\cdot x=\lambda(t^{-1})x$ for $\lambda(t)\in\Lambda$ and $x\in H$, H has another Λ-module structure. We denote this Λ-module by $\bar{H}$. When $H=H(\ell)$, we denote DH, D_0H, TH, BH, βH and eH by $D(\ell)$, $D_0(\ell)$, $T(\ell)$, $B(\ell)$, $\beta(\ell)$ and $e(\ell)$, respectively. Let $\mu(\ell)$ be the number of components of ℓ. It is well-known that $\beta(\ell)\leq\mu(\ell)-1$ and the equality holds for, e.g., a slice link ℓ in the strong sense (cf. [Ka]$_1$). Our first purpose is to observe that there are many links ℓ with $D(\ell)\neq 0$. Actually, we characterize $D_0(\ell)$ for all links ℓ.

THEOREM I. *For all links ℓ we have $eE^2D_0(\ell)\leq\beta(\ell)\leq\mu(\ell)-1$. Conversely, given a finite Λ-module D with $t-1$: $D\cong D$, then for any integers r, s with $eE^2D\leq r\leq s$ we have a link ℓ such that $\mu(\ell)=s+1$, $\beta(\ell)=r$ and $D(\ell)=D_0(\ell)\cong D$.*

---

*) "a meridian element of ℓ" means "an element of $\pi_1(E(\ell))$ represented by a loop homotopic to a meridian of ℓ" (The orientation of a meridian of ℓ is uniquely specified by those of ℓ and R^3).

Remarks. (I.1) For all links ℓ we also have $eE^2D(\ell)\leq\beta(\ell)$ and $e(D(\ell)/(t-1)D(\ell))\leq\mu(\ell)-1-\beta(\ell)$ (cf. Lemma 1.1). In particular, if $\beta(\ell)=0$, then $D(\ell)=0$. If $\beta(\ell)=\mu(\ell)-1$, then $D(\ell)=D_0(\ell)$. So, by Theorem I, we can characterize $D(\ell)$ for all links ℓ with $\mu(\ell)\leq 2$. Hillman's link in [H, Fig. V.1], say ℓ_H, is a ribbon link in the strong sense with $\beta(\ell_H)=\mu(\ell_H)-1=1$ and $D_0(\ell_H)=D(\ell_H)\cong\Lambda/(3, t-2)$. The link ℓ with $r=s$, constructed in the proof of Theorem I is also a ribbon link in the strong sense.

(I.2) A link ℓ is *reversible* if there is an auto-homeomorphism h of R^3 with $h(\ell)=\ell$ such that h is orientation-preserving and $h\mid\ell$ is orientation-reversing or such that h is orientation-reversing and $h\mid\ell$ is orientation-preserving. For a reversible link ℓ, there is a Λ-isomorphism $H(\ell)\cong\overline{H(\ell)}$. So, we obtain the following:

(I.2.1) If ℓ is reversible, then there is a Λ-isomorphism $D_0(\ell)\cong\overline{D_0(\ell)}$.

From this and Theorem I, we obtain many non-reversible links. For example, any links ℓ with $D_0(\ell)=\Lambda/(m, t-2)$ for any odd integer $m\geq 5$ are not reversible.

(I.3) If there is a disconnected Seifert surface for ℓ, then we say that ℓ is *weakly split* (cf. [Muras, Chap. V]). This is equivalent to saying that there is an epimorphism $\pi_1(E(\ell))\to\langle t_1, t_2\rangle$ sending each meridian element of ℓ to a conjugate of t_1 or t_2 (cf. Smythe [Sm]), where $\langle t_1, t_2\rangle$ is the free group with basis t_1, t_2. We say that ℓ is *homologically split* if there is an epimorphism $\pi_1(E(\ell))\to\langle t_1, t_2\rangle$ sending each meridian element of ℓ to an element congruent to t_1 or t_2 modulo the commutator subgroup $\langle t_1, t_2\rangle'$. Clearly, a weakly split link is homologically split, but the converse is not true (cf. [Sm]). We note the following:

(I.3.1) If ℓ is homologically split, then $eE^2D_0(\ell)<\beta(\ell)$.

For example, Hillman's link ℓ_H has $eE^2D(\ell_H)=\beta(\ell_H)=1$ and is not homologically split, shown by Hillman [H], using different methods. For each $r\geq 1$ there are many links ℓ with $eE^2D_0(\ell)=\beta(\ell)=r$, e.g., a ribbon link ℓ in the strong sense with $\beta(\ell)=\mu(\ell)-1=r$ and $D_0(\ell)=\bigoplus_r \Lambda/(m, t-2)$ for odd $m\geq 3$ (cf. (I.1)), which are not homologically split. To prove (I.3.1), let ℓ be homologically split. Then there is a Λ-epimorphism $H(\ell)\to\Lambda$. Hence $B(\ell)\cong B_1\oplus\Lambda$ for a torsion-free Λ-module B_1. By [Ka, §3]$_4$ there is a Λ-epimorphism $E^0E^0B_1\to E^2E^1B_1$ and $E^0E^0B_1$ is Λ-free of rank $\beta B_1=\beta(\ell)-1$. $E^2E^1B_1\cong E^2E^1B(\ell)$. The inclusion $D_0E^1B(\ell)\subset E^1B(\ell)$ induces a Λ-epimorphism $E^2E^1B(\ell)\to E^2D_0E^1B(\ell)$. We see from Lemma 1.1 that $\overline{D_0(\ell)}\cong D_0E^1B(\ell)$. So, $E^2D_0E^1B(\ell)\cong E^2\overline{D_0(\ell)}$. Then $eE^2D_0(\ell)=eE^2\overline{D_0(\ell)}=eE^2D_0E^1B(\ell)\leq eE^2E^1B(\ell)=eE^2E^1B_1\leq eE^0E^0B_1=\beta(\ell)-1$, as desired.

Next, we study the invariant $e(\ell)$. If H has a square Λ-presentation matrix, then let mH be the minimal size of such a matrix. Otherwise, let $mH=+\infty$.

By convention, $mH=0$ if $H=0$. Clearly, $mH \geq eH$. A *presentation* for H is a Λ-epimorphism $\psi: P \to H$ with P, Λ-free of finite rank. We say that H is *normal* if $e(\ker\psi) \leq eP$ for any presentation $\psi: P \to H$. We have $mH=eH$ for all normal H. Let $m(\ell)=mH(\ell)$. Nakanishi $[N]_1$ introduced $m(\ell)$ when ℓ is a knot. For subsets $A \subset R^3$ and $J \subset R$, let $AJ=\{(x, t) \in R^4 \mid x \in A, t \in J\}$. Let $F \subset R^3[0, 2]$ be a compact oriented proper polyhedral locally flat surface. By [K/S/S, I], F is ambient isotopic to a surface with only elementary critical points (concerning the height function $R^3[0, 2] \to [0, 2]$ sending (x, t) to t). When F has only elementary critical points, let $c_0(F)$, $c_1(F)$ and $c_2(F)$ be the numbers of minimal, saddle and maximal points, respectively.

THEOREM II. (1) *For all links ℓ, $H(\ell)$ is normal and in particular, $m(\ell)=e(\ell)$,*
(2) *If two links $\ell_i \subset R^3[i]$, $i=0, 2$, bound a compact oriented proper surface $F \subset R^3[0, 2]$ with only elementary critical points, then we have $|c_2(F)+e(\ell_2)-c_0(F)-e(\ell_0)| \leq c_1(F)$.*

REMARKS. (II.1) If H is of Λ-projective dimension ≤ 1, then $\ker\psi$ of any presentation $\psi: P \to H$ is Λ-free by Seshadri [Se] and hence H is normal. However, this does not mean Theorem II (1), since there are many links ℓ with $E^2H(\ell) \neq 0$ by Theorem I.
(II.2) An analogue of Theorem II (1) to higher dimensional links does not hold in general. For example, a finite knot module $H(\neq 0)$, occuring as the first Alexander module of a high-dimensional knot must have $mH=+\infty$. In fact, by Crowell [C] and Levine [L], the following conditions on a knot module H are equivalent (cf. Weber [W]): (i) $mH<+\infty$, (ii) H is Z-torsion-free, (iii) H is of Λ-projective dimension ≤ 1, (iv) $mH=eH$, (v) H is normal.
(II.3) Several estimates analogous to Theorem II (2) are known (cf. [H/K/N/S], $[Ka]_3$, $[N]_1$, $[N]_2$, [N/N] and [Sa]), but the present estimate combined with Theorem II (1) generalizes or improves them.
(II.4) Assume that a link ℓ' is obtained from a link ℓ by hyperbolic transformations along m bands (cf. [K/S/S, I]). Then we can construct a surface $F \subset R^3[0, 2]$ with no critical points except m saddle points such that $F \cap R^3[0]=\ell[0]$ and $F \cap R^3[2]=\ell'[2]$ (cf. [K/S/S, I)]. By Theorem II (2), we obtain the following:
(II.4.1) If ℓ' is obtained from ℓ by hyperbolic transformations along m bands, then $|e(\ell')-e(\ell)| \leq m$.

For example, we have $e(k_B) \leq 4$ for all knots k_B obtained from the Borromean ring, say ℓ_B, by fusions, because $H(\ell_B) \cong \Lambda/(t-1)^2 \oplus \Lambda/(t-1)^2$. This gives a restriction to k_B. Note that the Robertello/Arf invariant, $R(k_B) \neq 0$ and the Alexander polynomials of k_B's are not restrictive in the sense that they can have

all of the Alexander polynomials of knots k with $R(k)\neq 0$ (cf. Murakami [Murak]).
(II.5) Let ℓ^* be a link diagram on a plane R^2 representing a link ℓ. For an integer $s\neq 0$, a $T(s)$-operation on ℓ^* is a replacement of a 2-string trivial braid in ℓ^* by a 2-string braid of s full twists (See Fig. 1). Let $u_w(\ell^*)$ be the minimal number of $T(s)$-operations, where s is not fixed, on the link diagram ℓ^* needed to transform ℓ into a trivial link (of $\mu(\ell)$ components). The *weak unlinking number* $u_w(\ell)$ of ℓ is the minimal number of $u_w(\ell^*)$ taken over all link diagrams ℓ^* representing ℓ (cf. Kinoshita [Ki]). Restricting the $T(s)$-operations to $T(\pm 1)$-operations, we have the usual unlinking number $u(\ell)$ of ℓ. Clearly, $u(\ell)\geq u_w(\ell)$. The inequality $u(\ell)+\mu(\ell)-1\geq e(\ell)$ is well-known by a surgery description method (cf. Nakanishi [N]$_1$). We observe the following:

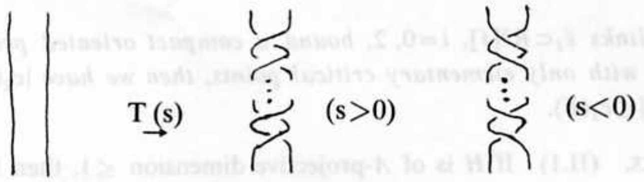

Fig. 1.

(II.5.1) For all links ℓ, $u_w(\ell)+\mu(\ell)-1\geq e(\ell)$.

To see this, let $u=u_w(\ell)$. Fig. 2 shows that there are u oriented links ℓ_i with $\mu(\ell_i)=2$, $i=1, 2,\dots, u$, such that the fusion of the split union $\ell^+=\ell+\ell_1+\cdots+\ell_u$ along some $2u$ bands produce a trivial link 0 with $\mu(0)=\mu(\ell)$. By (II.4.1), $|e(\ell^+)-e(0)|\leq 2u$. Note that $H(\ell^+)\cong H(\ell)\oplus(\oplus_{i=1}^u H(\ell_i))\oplus(\overset{u}{\oplus} \Lambda)$. Then $e(\ell^+)=e(H(\ell)\oplus(\oplus_{i=1}^u H(\ell_i)))+u\geq e(\ell)+u$ (cf. Lemma 2.5). Since $e(0)=\mu(\ell)-1$, it follows that $e(\ell)+u-\mu(\ell)+1\leq 2u$, i.e., $e(\ell)\leq u_w(\ell)+\mu(\ell)-1$.

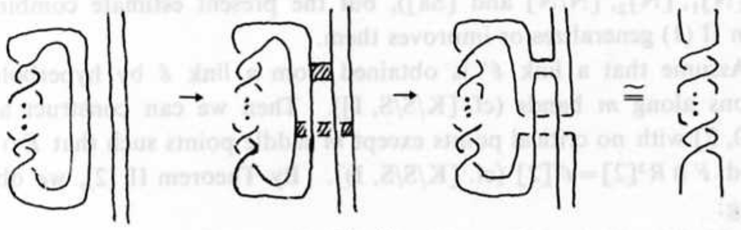

Fig. 2.

§1. Proof of Theorem I.

Lemma 1.1. *For all links ℓ we have the inequalities $eE^2D(\ell)\leq\beta(\ell)$, $e(D(\ell)/(t-1)D(\ell))\leq\mu(\ell)-1-\beta(\ell)$ and $e(E^1B(\ell)/(t-1)E^1B(\ell))\leq 2(\mu(\ell)-1-\beta(\ell))$ and a Λ-isomorphism $\overline{D_0(\ell)}\cong D_0E^1B(\ell)$.*

If $\beta(\ell)=\mu(\ell)-1$, then Lemma 1.1 *means that $D(\ell)=D_0(\ell)$, $E^1B(\ell)=D_0E^1B(\ell)$ and $\overline{D(\ell)}\cong E^1B(\ell)$, for $E^1B(\ell)$ is finite* (cf. [Ka, §3]$_4$).

1.2 Proof of Lemma 1.1. Let $E^+=E^+(\ell)$ be the compact manifold obtained from $S^3=R^3\cup\{\infty\}$ by removing an open tubular neighborhood of ℓ and $\tilde{E}^+=\tilde{E}^+(\ell)$, an infinite cyclic covering space over E^+ determined by the covering space $\tilde{E}(\ell)$. Clearly, $H_1(\tilde{E}^+)\cong H(\ell)$. Let $H=H_1(\tilde{E}^+,\partial\tilde{E}^+)$. By the Second Duality Theorem of [Ka]$_4$, we have $\overline{D(\ell)}\cong\overline{DH_1(\tilde{E}^+)}\overset{\theta}{\cong}E^1BH$ and $\overline{DH}\overset{\theta}{\cong}E^1BH_1(\tilde{E}^+)\cong E^1B(\ell)$. By [Ka, §3]$_4$, there is a Λ-epimorphism $E^0E^0BH\to E^2E^1BH$ and E^0E^0BH is Λ-free of rank $\beta H=\beta(\ell)$. Since $E^2E^1BH\cong E^2\overline{D(\ell)}$, it follows that $eE^2D(\ell)=eE^2\overline{D(\ell)}=eE^2E^1BH\leq\beta(\ell)$. The Wang exact sequence (cf. [Mi]) $H\xrightarrow{t-1}H\to H_1(E^+,\partial E^+)\to 0$ induces an exact sequence (S') $0\to BH\xrightarrow{t-1}BH\to H'\to 0$, where H' is the quotient of $H_1(E^+,\partial E^+)$ by the image, T', of TH. Note that $\operatorname{rank}_Z H'=\beta(\ell)$, and $H_1(E^+,\partial E^+)$ and T' are Z-free of ranks $\mu(\ell)-1$ and $\mu(\ell)-1-\beta(\ell)$, respectively. The sequence (S') induces an exact sequence $E^1BH\xrightarrow{t-1}E^1BH\to E^2H'\to 0$. So, $D(\ell)/(t-1)D(\ell)\cong E^1BH/(t-1)E^1BH\cong E^2H'$. The exact sequence $0\to T'\to H_1(E^+,\partial E^+)\to H'\to 0$ induces an epimorphism $E^1T'\to E^2H'$. Since E^1T' is Z-free of rank $\mu(\ell)-1-\beta(\ell)$, we see that $e(D(\ell)/(t-1)D(\ell))=eE^2H'\leq\mu(\ell)-1-\beta(\ell)$. The exact sequence (S) $H_1(\partial\tilde{E}^+)\xrightarrow{i_*}H_1(\tilde{E}^+)\xrightarrow{j_*}H\xrightarrow{\partial}\tilde{H}_0(\partial\tilde{E}^+)\to 0$ induces a short exact sequence (S'') $0\to B(\ell)\to BH\to H''\to 0$, where H'' is the quotient of $\tilde{H}_0(\partial\tilde{E}^+)$ by $T''=\partial TH$. Since the boundary map $H_1(E^+,\partial E^+)\to\tilde{H}_0(\partial E^+)$ sends T' onto the image of T'' under the isomorphism $\tilde{H}_0(\partial\tilde{E}^+)\cong\tilde{H}_0(\partial E^+)$, we have that $\operatorname{rank}_Z T''\leq\operatorname{rank}_Z T'=\mu(\ell)-1-\beta(\ell)$. The sequence (S'') induces an exact sequence $E^1BH\to E^1B(\ell)\to E^2H''\to 0$ and hence an exact sequence $E^1BH/(t-1)E^1BH\to E^1B(\ell)/(t-1)E^1B(\ell)\to E^2H''\to 0$. Since $eE^2H''\leq\mu(\ell)-1-\beta(\ell)$ [In fact, there is an epimorphism $E^1T''\to E^2H''$ and E^1T'' is Z-free of rank $\leq\mu(\ell)-1-\beta(\ell)$] and $e(E^1BH/(t-1)E^1BH)=eE^2H'\leq\mu(\ell)-1-\beta(\ell)$, it follows that $e(E^1B(\ell)/(t-1)E^1B(\ell))\leq 2(\mu(\ell)-1-\beta(\ell))$. Next, we show that the map $D_0j_*\colon D_0H_1(\tilde{E}^+)\to D_0H$ induced from j_* in the sequence (S) is a Λ-isomorphism. Clearly, D_0j_* is injective, for $(t-1)H_1(\partial\tilde{E}^+)=0$ and $t-1\colon D_0H_1(\tilde{E}^+)\cong D_0H_1(\tilde{E}^+)$. Let $x\in D_0H$ and write $x=(t-1)x'$ for $x'\in D_0H$. There is a $y'\in H_1(\tilde{E}^+)$ so that $j_*(y')=x'$. For $y=(t-1)y'$, we have $j_*(y)=x$ and $y\in DH_1(\tilde{E}^+)$ [In fact, take positive integers m, m' so that $(t^m-1)x'=m'x'=0$. Then $(t^m-1)y=m'y=0$]. Choose a positive integer N so that $D_0H_1(\tilde{E}^+)=(t-1)^N\cdot DH_1(\tilde{E}^+)$ and $(t-1)^N$ acts trivially on D_0H. Then $(t-1)^Ny\in D_0H_1(\tilde{E}^+)$ and $D_0j_*((t-1)^Ny)=x$. Thus, D_0j_* is a Λ-isomorphism. Therefore, $\overline{D_0(\ell)}\cong\overline{D_0H_1(\tilde{E}^+)}\cong\overline{D_0H}\overset{\theta}{\cong}D_0E^1BH_1(\tilde{E}^+)\cong D_0E^1B(\ell)$. This completes the proof.

Let L be a polyhedral locally flat oriented 2-link (of 2-spheres) in R^4, the number of whose components is denoted by $\mu(L)$. Let $E(L)=R^4-L$ and $\tilde{E}(L)$, the infinite cyclic covering space over $E(L)$ associated with the epimorphism $\pi_1(E(L))\to$

$\langle t_1, t_2\rangle$ sending each meridian element of L to t. Let $H(L)=H_1(\tilde{E}(L))$, $TH=TH(L)$, $B(L)=BH(L)$ and $\beta(L)=\beta H(L)$.

LEMMA 1.3. *Assume that a link $\ell\subset R^3$ is the middle cross-sectional link of a 2-link $L\subset R^4$ in a normal form (cf. [K/S/S, I]). Then the inclusion $E(\ell)\subset E(L)$ induces a Λ-isomorphism $B(\ell)\cong B(L)$.*

PROOF. The natural map $H(\ell)\to H(L)$ is onto, for the natural map $\pi_1(E(\ell))\to\pi_1(E(L))$ is onto by the van Kampen theorem. Since $H_2(E(L), E(\ell))=0$, we see from the Wang exact sequence (cf. [Mi]) that $BH_2(\tilde{E}(L), \tilde{E}(\ell))=0$. Thus, the exact sequence $H_2(\tilde{E}(L), \tilde{E}(\ell))\to H(\ell)\to H(L)\to 0$ induces an isomorphism $B(\ell)\cong B(L)$, completing the proof.

The following characterizes $B(L)$ for all 2-links L:

LEMMA 1.4. *For a 2-link $L\subset R^4$, we have $\beta(L)=\mu(L)-1$ and $t-1$: $E^1B(L)\cong E^1B(L)$. Conversely, given a finitely generated, torsion-free Λ-module B with $\beta B=r$ and $t-1$: $E^1B\cong E^1B$, then there is a 2-link $L\subset R^4$ with $\mu(L)=r+1$ and $H(L)=B(L)\cong B$.*

PROOF. Let ℓ be the middle cross-sectional link of L in a normal form. Then $\mu(\ell)=\mu(L)$. By Lemma 1.3, $B(\ell)\cong B(L)$ and $\beta(\ell)=\beta(L)$. Since ℓ is a ribbon link in the strong sense, we have $\beta(\ell)=\mu(\ell)-1$. By Lemma 1.1, $t-1$: $E^1B(\ell)\cong E^1B(\ell)$. Hence $\beta(L)=\mu(L)-1$ and $t-1$: $E^1B(L)\cong E^1B(L)$. Next, note that B is of Λ-projective dimension ≤ 1 (cf. [Ka, §3]$_4$). Let $0\to P_1\to P_0\to B\to 0$ be a Λ-projective (i.e., Λ-free by [Se]) resolution for B with $eP_1=d$, $eP_0=d+r$ for some d. By [Ka, §3]$_4$, there is a Λ-exact sequence $0\to B\to E^0E^0B\to E^2E^1B\to 0$. By assumption, $t-1$: $E^2E^1B\cong E^2E^1B$, so that $B/(t-1)B\cong E^0E^0B/(t-1)E^0E^0B$. Since E^0E^0B is Λ-free of rank r, $B/(t-1)B$ is Z-free of rank r and the induced exact sequence $0\to P_1/(t-1)P_1\to P_0/(t-1)P_0\to B/(t-1)B\to 0$ splits. Let $P(t)=(p_{ij}(t))$ be a $d\times(d+r)$-matrix representing the Λ-monomorphism $P_1\to P_0$. By elementary row and column operations on $P(t)$, we can assume that $p_{ij}(1)=\delta_{ij}$ for all i, j. Let $p_{ij}(t)=\sum_{n=-\infty}^{+\infty} a(i, j, n)t^n$, $a(i, j, n)\in Z$. The construction of our 2-link is similar to the Kervaire/Levine construction of a 2-knot (cf. [Ke], [L]). Construct a group presentation $G=(y_1,\dots, y_{d+r}, z\,|\,R_1,\dots, R_d)$ where

$$R_1 = \textstyle\prod_{n=-\infty}^{+\infty}(z^n y_1 z^{-n})^{a(1,1,n)}\cdots\prod_{n=-\infty}^{+\infty}(z^n y_{d+r} z^{-n})^{a(1,d+r,n)}$$
$$\cdots$$
$$R_d = \textstyle\prod_{n=-\infty}^{+\infty}(z^n y_1 z^{-n})^{a(d,1,n)}\cdots\prod_{n=-\infty}^{+\infty}(z^n y_{d+r} z^{-n})^{a(d,d+r,n)}.$$

Let γ: $G\to\langle t\rangle$ be the epimorphism defined by $\gamma(y_j)=1$ and $\gamma(z)=t$, and $\tilde{G}$, the kernel of γ. By the Fox free calculus, $\gamma(\partial R_i/\partial y_j)=p_{ij}(t)$ and $\gamma(\partial R_i/\partial z)=0$, so that

$H_1(\tilde{G})$ has $P(t)$ as a Λ-presentation matrix (cf. [Ka]$_2$). Hence $H_1(\tilde{G}) \cong B$. Let $W = \#_{j=1}^{d} S^1 \times S_j^3 \# R^4$. Take a trivial 2-link L_0 with components S_i^2, $i=0, 1, \ldots, r$, in the factor R^4 of W. Identify $\pi_1(W - L_0)$ with the free group $\langle y_1, \ldots, y_{d+r}, z\rangle$ so that y_j $(1 \leq j \leq d)$, $y_{d+i}z$ $(1 \leq i \leq r)$ and z are represented by loops homotopic to $S^1 \times *_j (\subset S^1 \times S_j^3)$, meridians of S_i^2 and S_0^2, respectively. Let $C_1, \ldots, C_d$ be mutually disjoint simple closed curves in $W - L_0$ such that C_i is homotopic to a loop representing the element $R_i \in \pi_1(W - L_0)$. Let $T_1, \ldots, T_d$ be mutually disjoint tubular neighborhoods of $C_1, \ldots, C_d$. Since $R_i(y_1, \ldots, y_{d+r}, 1) = y_i$, $i = 1, \ldots, d$, the 4-manifold obtained from W by surgery along $T_1, \ldots, T_d$ (with any framings) is R^4. Let $L \subset R^4$ be a 2-link resulting from $L_0 \subset R^4$ by the surgery. Then $\pi_1(E(L)) \cong G$ and $H(L) \cong H_1(\tilde{G}) \cong B$. This completes the proof.

1.5 Proof of Theorem I. The inclusion $D_0(\ell) \subset D(\ell)$ induces a Λ-epimorphism $E^2D(\ell) \to E^2D_0(\ell)$. So, by Lemma 1.1, $eE^2D_0(\ell) \leq eE^2D(\ell) \leq \beta(\ell) \leq \mu(\ell) - 1$. Let D be a finite Λ-module with $eE^2D \leq r \leq s$ and $t-1\colon D \cong D$. First, let $r = s$. Since $eE^2D \leq r$, there is a presentation $\psi\colon P \to E^2D$ with $eP = r$. Let $B = \ker \psi$, which is Λ-torsion-free of rank r. Note that $E^1B \cong E^2E^2D \cong D$. By Lemma 1.4, there is a 2-link $L \subset R^4$ with $\mu(L) = r+1$ and $H(L) \cong B$. Let $\ell \subset R^3$ be the middle cross-sectional link of L in a normal form. Then $\mu(\ell) = \mu(L) = r+1$ and ℓ is a ribbon link in the strong sense. By Lemma 1.3, $B(\ell) \cong B$. By Lemma 1.1, $\overline{D_0(\ell)} = \overline{D(\ell)} \cong E^1B \cong D$. Let $\bar{\ell}$ be the link obtained from ℓ by reversing the orientation of ℓ. Then $D(\bar{\ell}) \cong \overline{D(\ell)} \cong D$. For $s = r+m$, $m \geq 1$, let ℓ_m be a link with $\mu(\ell_m) = m+1$, illustrated in Fig. 3.

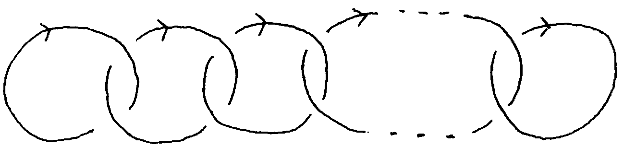

Fig. 3.

Then $H(\ell_m) \cong \underset{m}{\oplus} \Lambda/(t-1)$. Let $\bar{\ell}'$ be a link obtained from a split union $\bar{\ell} + \ell_m$ by a simple fusion so that $H(\bar{\ell}') \cong H(\bar{\ell}) \oplus H(\ell_m)$. Since $\mu(\bar{\ell}') = \mu(\bar{\ell}) + m = s+1$, $\beta(\bar{\ell}') = \beta(\bar{\ell}) = r$ and $D(\bar{\ell}') \cong D(\bar{\ell}) \cong D$, the link $\bar{\ell}'$ gives a desired one. This completes the proof.

Remark 1.6. There is an argument on 2-links analogous to Remark (I.3). A 2-link L in R^4 is *weakly split* if there is a disconnected Seifert manifold for L in R^4, which is equivalent that there is an epimorphism $\pi_1(E(L)) \to \langle t_1, t_2\rangle$ sending each meridian element of L to a conjugate of t_1 or t_2. A 2-link L in R^4 is *homologically split* if there is an epimorphism $\pi_1(E(L)) \to \langle t_1, t_2\rangle$ sending each meridian element of L to an element congruent to t_1 or t_2 modulo $\langle t_1, t_2\rangle'$.

(1.6.1) If L is homologically split, then $eE^2E^1B(L)<\beta(L)$.

The proof is basically contained in that of (I.3.1) and omitted. Let D be a finite Λ-module with $t-1: D\cong D$ and $eE^2D=r$. By an argument in 1.5, we have a 2-link L with $\beta(L)=\mu(L)-1=r$ and $E^1B(L)\cong D$. For example, taking $D=\bigoplus_r \Lambda/(m, t-2)$ (m, odd ≥ 3), we see that for each $r\geq 1$ there are many 2-links L with $eE^2E^1B(L)=\beta(L)=\mu(L)-1=r$, that are not homologically split.

§2. Proof of Theorem II

Let W be a compact connected oriented 4-manifold with $H_1(W)=Z$. Let (A, A') be a splitting of ∂W, i.e., A, A' are compact 3-submanifolds of ∂W or empty such that $A'=\mathrm{cl}_{\partial W}(\partial W-A)$. Let $\tilde{W}$ be the infinite cyclic connected covering space over W and $(\tilde{A}, \tilde{A}')$, the lift of (A, A').

LEMMA 2.1. *If $H_1(\tilde{W})=0$ and $\tilde{A}$ is connected, then $H_2(\tilde{W}, \tilde{A}')$ is Λ-free.*

PROOF. $H_1(\tilde{W}, \tilde{A})=0$ and $H_0(\tilde{W}, \tilde{A})\cong\Lambda/(t-1)$ (if $A=\emptyset$) or 0 (if $A\neq\emptyset$). By Reidemeister duality (cf. [Ka, §2]$_4$), there is a Λ-isomorphism $H^q_\Lambda(\tilde{W}, \tilde{A}')\cong\overline{H_{4-q}(\tilde{W}, \tilde{A})}$ for all q. By the universal coefficient exact sequence in [Ka, §1]$_4$, the sequence $0\to BH^q_\Lambda(\tilde{W}, \tilde{A}')\to E^0H_q(\tilde{W}, \tilde{A}')\to E^2H_{q-1}(\tilde{W}, \tilde{A}')\to TH^{q+1}_\Lambda(\tilde{W}, \tilde{A}')\to E^1H_q(\tilde{W}, \tilde{A}')\to 0$ is Λ-exact for all q. We have $E^1H_2(\tilde{W}, \tilde{A}')=E^2H_2(\tilde{W}, \tilde{A}')=0$. By [Ka, §3]$_4$, $H_2(\tilde{W}, \tilde{A}')$ is Λ-free, completing the proof.

Let G be a finitely generated group with an epimorphism $\gamma: G\to\langle t\rangle$ and $\tilde{G}$, the kernel of γ.

LEMMA 2.2. *If G is isomorphic to the fundamental group of an orientable 3-manifold, then the Λ-module $H_1(\tilde{G})$ is normal.*

PROOF. It is known that $H_1(\tilde{G})=H_1(\tilde{M})$ for an infinite cyclic connected covering space $\tilde{M}$ over a compact orientable 3-manifold M such that ∂M is not empty and contains no 2-spheres (cf. [Ka]$_2$). For a presentation $\psi: P\to H_1(\tilde{M})$, let $g_1,\dots, g_r\in H_1(\tilde{M})$ be the image of a Λ-basis of P under ψ. Let $T_1,\dots, T_r$ be solid tori in $\mathrm{Int}\,\tilde{M}$ whose cores represent $g_1,\dots, g_r$ and such that $t^nT_i\cap T_j=\emptyset$ except when $i=j$ and $n=0$. The images $T'_1,\dots, T'_r$ of $T_1,\dots, T_r$ under the covering projection $\tilde{M}\to M$ are mutually disjoint solid tori in $\mathrm{Int}\,M$. Construct a 4-manifold $W=M\times[0, 1]\cup D^2\times D^2_1\cup\cdots\cup D^2\times D^2_r$ identifying $T'_i\times 1$ (with any framing) with $(\partial D^2)\times D^2_i$, $i=1,\dots, r$. There is an infinite cyclic connected covering $\tilde{W}\to W$ extending the covering $\tilde{M}\to M$ (identified with the covering $\tilde{M}\times 0\to M\times 0$). By construction, $H_1(\tilde{W})=0$, so that $H_1(W)=Z$. Note that $\partial\tilde{W}$ is connected. By Lemma 2.1, $H_2(\tilde{W})$ is Λ-free. By assumption on M, the Euler characteristic, $\chi(M)\leq 0$. Since $H_q(W)\cong Z$ (if $q=0, 1$), 0 (if $q=3, 4$) and $\chi(W)=$

$\chi(M)+r$, it follows that $H_2(W)$ is Z-free of rank $\leq r$. Then we see from the Wang exact sequence that $eH_2(\tilde{W})\leq r$. On the other hand, by excision, $H_2(\tilde{W}, \tilde{M})\cong \oplus_{i=1}^r H_2(D^2\times D_i^2, (\partial D^2)\times D_i^2)\otimes_Z \Lambda\cong \overset{r}{\oplus} \Lambda$. Hence it follows from the exact sequence $H_2(\tilde{W})\to H_2(\tilde{W}, \tilde{M})\xrightarrow{\partial} H_1(\tilde{M})\to 0$ that $H_1(\tilde{G})=H_1(\tilde{M})$ is normal, since ∂: $H_2(\tilde{W}, \tilde{M})\to H_1(\tilde{M})$ is a geometric realization of ψ: $P\to H_1(\tilde{M})$. This completes the proof.

2.3 PROOF OF THEOREM II (1). It is direct from Lemma 2.2.

REMARK 2.4. Let G be a group with an element of infinite order. Let $\tilde{G}_1=\ker\gamma$ for a finitely generated subgroup G_1 of G with an epimorphism γ: $G_1\to\langle t\rangle$. The Λ-module $H_1(\tilde{G}_1)$ is *produced in* G. By Lemma 2.2 and an argument of [Ka]$_2$, we have a necessary condition for G to be isomorphic to the fundamental group of a 3-manifold, which is different from that of [Ka]$_2$ and stated as follows:

(2.4.1) If G is isomorphic to the fundamental group of an orientable 3-manifold, then any Λ-module produced in G is normal.
(2.4.2) If G is isomorphic to the fundamental group of a non-orientable 3-manifold, then there exists an index 2 subgroup G' of G such that any Λ-module produced in G' is normal.

LEMMA 2.5. *For a Λ-submodule H' of a finitely generated Λ-module H, let H/H' be of Λ-projective dimension ≤ 1. Then*

$$eH' + \beta(H/H') \leq eH \leq eH' + e(H/H').$$

PROOF. The right hand side inequality is obvious. Let ψ: $P\to H$ be a presentation with $eP=eH$. Let $P'=\psi^{-1}H'$. Then $P/P'\cong H/H'$. Since it is of Λ-projective dimension ≤ 1, P' is Λ-projective, i.e., Λ-free by [Se] of rank $eH-\beta(H/H')$. Using that $\psi\,|\,P'$: $P'\to H'$ is onto, we see that $eH'\leq eP'=eH-\beta(H/H')$. This completes the proof.

For example, if H/H' is Λ-free, then $\beta(H/H')=e(H/H')$ and hence $eH=eH'+e(H/H')$. The following special case of Theorem II (2) is obtained from a combination of Theorem II (1) and Nakanishi's inequality in [N]$_2$, but we shall give it a different proof.

LEMMA 2.6. *When $c_2(F)=c_0(F)=0$, we have $|e(\ell_2)-e(\ell_0)|\leq c_1(F)$.*

PROOF. We may assume that $e(\ell_2)\geq e(\ell_0)$. Let $E^+(F)$ be the compact 4-manifold obtained from $S^3\times[0, 2]$, $S^3=R^3\cup\{\infty\}$, by removing an open tubular neighborhood of F. For $i=0, 2$ let $E^+(\ell_i)=E^+(F)\cap S^3\times i$ (cf. 1.2). Let W be a 4-manifold constructed in the proof of Lemma 2.2 where we take

$M=E^+(\ell_0)$, $\tilde{M}=\tilde{E}^+(\ell_0)$ and for ψ: $P\to H_1(\tilde{M})$, $eP=eH_1(\tilde{M})=e(\ell_0)$. Construct a 4-manifold $W'=W\cup E^+(F)$ identifying two copies of $E^+(\ell_0)$. Since the natural map $\pi_1(E^+(\ell_0))\to\pi_1(E^+(F))$ is onto by the van Kampen theorem, it follows that $H_1(W')=Z$ and $H_1(\tilde{W}')=0$ for the connected infinite cyclic covering sapce $\tilde{W}'$. Note that the covering $\tilde{W}'\to W'$ is an extension of the coverings $\tilde{E}^+(\ell_i)\to E^+(\ell_i)$, $i=0, 2$. By Lemma 2.1, $H_2(\tilde{W}', \tilde{E}^+(\ell_2))$ is Λ-free. Since $H_2(E^+(F), E^+(\ell_2))\to H_2(W', E^+(\ell_2))\to H_2(W', E^+(F))$ is exact and $H_2(E^+(F), E^+(\ell_2))\cong \bigoplus_{c_1(F)} Z$ and $H_2(W', E^+(F))\cong H_2(W, E^+(\ell_0))\cong \bigoplus_{e(\ell_0)} Z$, we see from the Wang exact sequence that $eH_2(\tilde{W}', \tilde{E}^+(\ell_2))=\text{rank}_Z H_2(W', E^+(\ell_2))\le c_1(F)+e(\ell_0)$. But, the boundary map ∂: $H_2(\tilde{W}', \tilde{E}^+(\ell_2))\to H_1(\tilde{E}^+(\ell_2))$ is onto. Hence $e(\ell_2)\le c_1(F)+e(\ell_0)$. This completes the proof.

2.7 PROOF OF THEOREM II (2). F can be deformed into a surface F' in $R^3[0, 2]$ with only elementary critical points such that for each i, $i=0, 2$, $F'\cap R^3[i]=\ell_i'$ is a split union of ℓ_i and a trivial link of $c_i(F)$ components and $c_1(F')=c_1(F)$ and $c_i(F')=0$ (cf. [K/S/S, I, §3]). By Lemma 2.6, $|e(\ell_2')-e(\ell_0')|\le c_1(F)$. $H(\ell_i')\cong H(\ell_i)\oplus(\bigoplus_{c_i(F)} \Lambda)$ and by Lemma 2.5, $e(\ell_i')=e(\ell_i)+c_i(F)$, $i=0, 2$. Hence $|e(\ell_2)+c_2(F)-e(\ell_0)-c_0(F)|\le c_1(F)$. This completes the proof.

References

[C] R. H. Crowell: The group G'/G'' of a knot group G, Duke Math. J. **30** (1963), 349–354.

[H] J. A. Hillman: Alexander Ideals of Links, Lecture Notes in Math. **895**, Springer-Verlag, 1981.

[H/K/N/S] F. Hosokawa/A. Kawauchi/Y. Nakanishi/M. Sakuma: Note on critical points of surfaces in 4-space, Kobe J. Math. **1** (1984), 151–152.

[Ka]₁ A. Kawauchi: On the Alexander polynomials of cobordant links, Osaka J. Math. **15** (1978), 151–159.

[Ka]₂ A. Kawauchi: A test for the fundamental group of a 3-manifold, J. Pure Applied Algebra **28** (1983), 189–196.

[Ka]₃ A. Kawauchi: On critical points of proper surfaces in the upper half 4-space (unpublished version), Osaka City Univ. (1984).

[Ka]₄ A. Kawauchi: Three dualities on the integral homology of infinite cyclic coverings of manifolds, Osaka J. Math. **23** (1986), 633–651.

[Ke] M. A. Kervaire: Les noeuds de dimensions supérieures, Bull. Soc. Math. France **93** (1965), 225–271.

[Ki] S. Kinoshita: On Wendt's theorem of knots, Osaka Math. J. **9** (1957), 61–66.

[K/S/S] A. Kawauchi/T. Shibuya/S. Suzuki: Descriptions on surfaces in four-space, I, Math. Sem. Notes, Kobe Univ. **10** (1982), 75–125; II, ibid. **11** (1983), 31–69.

[L] J. Levine: Knot modules. I, Trans. Amer. Math. Soc. **229** (1977), 1–50.

[Mi] J. W. Milnor: Infinite cyclic coverings, Conference on the Topology of Manifolds, Prindle, Weber & Schmidt, 1968, 115–133.

[Murak] H. Murakami: On the Conway polynomial of a knot with T-genus one, Kobe J.

Math. **2** (1985), 117–121.

[Muras] K. Murasugi: Lecture notes, University of Toronto, 1970.

[N]$_1$ Y. Nakanishi: A note on unknotting number, Math. Sem. Notes, Kobe Univ. **9** (1981), 99–108.

[N]$_2$ Y. Nakanishi: A remark on critical points of link cobordism, Kobe J. Math. **3** (1986), 209–212.

[N/N] Y. Nakanishi/Y. Nakagawa: On ribbon knots, Math. Sem. Notes, Kobe Univ. **10** (1982), 423–430.

[Sa] M. Saito: Minimal number of saddle points of properly embedded surfaces in the 4-ball, Math. Sem. Notes, Kobe Univ. **11** (1983), 345–348.

[Se] C. S. Seshadri: Triviality of vector bundles over the affine space K^2, Proc. Nat. Acad. U. S. A. **44** (1958), 456–458.

[Sm] N. F. Smythe: Boundary links, Topology Seminar, Wisconsin, 1965, 69–72.

[W] C. Weber: Torsion dans les modules d'Alexander, Knot Theory, Lecture Notes in Math. **685**, Springer-Verlag, 1977, 300–308.

Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka, 558, Japan