Homological infinity of 4D universe for every 3-manifold

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Abstract. This article is an explanation on recent investigations on homological infinity of a 4D universe for every 3-manifold, namely a boundary-less connected oriented 4-manifold with every closed connected oriented 3-manifold embedded, and homological infinity of a 4D punctured universe, namely a boundary-less connected oriented 4-manifold with every punctured 3-manifold embedded. Types 1, 2 and full 4D universes are introduced as fine notions of a 4D universe. After introducing some topological indexes for every (possibly non-compact) oriented 4-manifold, we show the infinity on the topological indexes of every 4D universe and every 4D punctured universe. Further, it is observed that a full 4D universe is produced by collision modifications between 3-sphere fibers in the 4D spherical shell (i.e., the 3-sphere bundle over the real line) embedded properly in any 5-dimensional open manifold and the second rational homology groups of every 4D universe and every 4D punctured universe are always infinitely generated over the rationals.

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1. Introduction

Throughout this paper, by a *closed 3-manifold* we mean a closed connected oriented 3-manifold M, and by a *punctured 3-manifold* the punctured manifold M^0 of a closed 3-manifold M. Let $\mathbb{M}$ be the set of (oriented homeomorphism types of) closed 3-manifolds M, and $\mathbb{M}^0$ the set of (oriented homeomorphism types of) punctured 3-manifolds M^0. It is known that the sets $\mathbb{M}$ and $\mathbb{M}^0$ are countable sets (see for example [13, 16]).

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By a *4D universe* or simply a *universe*, we mean a boundary-less connected oriented 4-manifold with every closed 3-manifold M embedded, and by a *4D punctured universe* or simply a *punctured universe* a boundary-less connected oriented 4-manifold with every punctured 3-manifold M^0 embedded. Every universe and every punctured universe are open 4-manifolds since for every compact (orientable or non-orientable) 4-manifold, there is a punctured 3-manifold which is not embeddable in it (see [6, 23]).

For a boundary-less connected oriented 4-manifold X, we note that there are two types of embeddings $k : M \to X$. An embedding $k : M \to X$ is *of type 1* if the complement $X \setminus k(M)$ is connected, and *of type 2* if the complement $X \setminus k(M)$ is disconnected. If there is a type 1 embedding $k : M \to X$, then there is an element $x \in H_1(X;\mathbb{Z})$ with the intersection number $\mathrm{Int}_U(x, k(M)) = +1$, so that the intersection form

$$\mathrm{Int}_X : H_1(X;\mathbb{Z}) \times H_3(X;\mathbb{Z}) \to \mathbb{Z}$$

induces an epimorphism

$$I_d : H_d(X;\mathbb{Z}) \to \mathbb{Z}$$

for $d = 1, 3$ such that the composite $I_3 k_* : H_3(M;\mathbb{Z}) \to H_3(X;\mathbb{Z}) \to \mathbb{Z}$ is an isomorphism and the composite $I_1 k_* : H_1(M;\mathbb{Z}) \to H_1(X;\mathbb{Z}) \to \mathbb{Z}$ is the 0-map (see [6, 12]). By using the concepts of embeddings of types 1 and 2, special kinds of universes are considered in [12]: Namely, a universe U is a *type 1 universe* if every $M \in \mathbb{M}$ is type 1 embeddable in U, and a *type 2 universe* if every $M \in \mathbb{M}$ is type 2 embedded in U. A universe U is a *full universe* if U is a type 1 universe and a type 2 universe. In Theorem 2.1, a full universe U will be constructed in every open 5-manifold W from the spherical shell $S^3 \times \mathbb{R}$ by infinitely many collision modifications on 3-sphere fibers of $M \times \mathbb{R}$.

Actually, there exist quite many 4D universes and 4D punctured universes. The following comparison theorem between them is established in [12, Theorem 2.1]:

Comparison Theorem.

(1)
$$\begin{array}{ccccc} & & \text{Type 1 universe} & & \\ & \nearrow & & \searrow & \\ \text{Full universe} & & & & \text{Universe} \to \text{Punctured universe.} \\ & \searrow & & \nearrow & \\ & & \text{Type 2 universe} & & \end{array}$$

(2) Type 1 universe $\not\to$ Full universe.

(3) Type 2 universe $\not\to$ Full universe.

(4) Universe $\not\to$ Type 1 universe.

(5) Universe $\not\to$ Type 2 universe.

(6) Punctured universe $\not\to$ Universe.

Examples showing the assertions (2)-(6) will be given in Section 2.

Let X be a non-compact oriented 4-manifold. Let $\beta_2(X)$ be the $\mathbb{Q}$-dimension of the second rational homology group $H_2(X;\mathbb{Q})$. For the intersection form

$$\mathrm{Int} : H_d(X;\mathbb{Z}) \times H_{4-d}(X;\mathbb{Z}) \to \mathbb{Z},$$

we define the *dth null homology* of X to be the subgroup

$$O_d(X;\mathbb{Z}) = \{x \in H_d(X;\mathbb{Z}) | \, \mathrm{Int}(x, H_{4-d}(X;\mathbb{Z})) = 0\}$$

of the dth homology group $H_d(X;\mathbb{Z})$ and the *dth non-degenerate homology* of X to be the quotient group

$$\hat{H}_d(X;\mathbb{Z}) = H_d(X;\mathbb{Z})/O_d(X;\mathbb{Z}),$$

which is a free abelian group by [12, Lemma 3.1]. Let $\hat{\beta}_d(X)$ be the $\mathbb{Z}$-rank of $\hat{H}_d(X;\mathbb{Z})$.

For an abelian group G, let $G^{(2)} = \{x \in G | \, 2x = 0\}$, which is a direct sum of some copies of $\mathbb{Z}_2$. For $M^0 \in \mathbb{M}^0$, let $\delta(M^0 \subset X)$ be the minimal $\mathbb{Z}$-rank of the image of the homomorphism

$$k^0_* : H_2(M^0;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z})$$

for all embeddings $k^0 : M^0 \to X$. Let $\rho(M^0 \subset X)$ be the minimal $\mathbb{Z}_2$-rank of the homomorphism image group

$$\mathrm{Im}[k^0_* : H_2(M^0;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z})]^{(2)}$$

for all embeddings $k^0 : M^0 \to X$ with $\mathbb{Z}$-rank $\delta(M^0 \subset X)$.

Note that in [12], the $\mathbb{Z}$-rank condition in the definitions of $\rho(M^0 \subset X)$ and $\rho(M \subset X)$ was erroneously omitted.

By taking the value 0 for the non-embeddable case, we define the following topological invariants of X:

$$\begin{aligned} \delta_0(X) &= \sup\{\delta(M^0 \subset X) | \, M^0 \in \mathbb{M}^0\}, \\ \rho_0(X) &= \sup\{\rho(M^0 \subset X) | \, M^0 \in \mathbb{M}^0\}. \end{aligned}$$

For $M \in \mathbb{M}$, let $\delta(M \subset X)$ be the minimal $\mathbb{Z}$-rank of the image of the homomorphism

$$k_* : H_2(M;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z})$$

for all embeddings $k : M \to X$. Let $\rho(M \subset X)$ be the minimal $\mathbb{Z}_2$-rank of the homomorphism image group

$$\mathrm{Im}[k_* : H_2(M;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z})]^{(2)}$$

for all embeddings $k : M \to X$ with $\mathbb{Z}$-rank $\delta(M \subset X)$. By taking the value 0 for the non-embeddable case, we define the following invariants of X:

$$\begin{aligned} \delta(X) &= \sup\{\delta(M \subset X) | \, M \in \mathbb{M}\}, \\ \rho(X) &= \sup\{\rho(M \subset X) | \, M \in \mathbb{M}\}. \end{aligned}$$

Restricting all embeddings $k : M \to X$ to all embeddings $k : M \to X$ of type i for $i = 1, 2$, we obtain the topological indexes $\delta_i(X)$ and $\rho_i(X)$ $(i = 1, 2)$ of X in place of $\delta(X)$ and $\rho(X)$.

For a universe or punctured universe U, the following topological invariants

$$\hat{\beta}_d(U)(d=1,2),\ \delta(U),\ \delta_i(U)\,(i=0,1,2),\ \rho(U),\ \rho_i(U)\,(i=0,1,2),\ \beta_2(U),$$

called the *topological indexes* of U and taking values in the set $\{0,1,2,\dots,+\infty\}$ are used to investigate the topological shape of U (see [12]). The results on a universe or punctured universe U given in [10, 12, 14] are explained as follows:

• For a punctures universe U, we have $\beta_2(U) = +\infty$ and one of the topological indexes $\hat{\beta}_2(U)$, $\delta_0(U)$, $\rho_0(U)$ is $+\infty$. Further, in every case, there is a punctured spin universe U with the other topological indexes taken 0.

• For a type 1 universe U, we have $\beta_2(U) = +\infty$ and one of the topological indexes $\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ is $+\infty$. The condition $\hat{\beta}_1(U) \geq 1$ always holds, but in the case of $\rho_1(U) = +\infty$, the condition $\hat{\beta}_1(U) = +\infty$ holds. Further, in every case, there is a type 1 spin universe U with the other topological indexes on $\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ taken 0.

• For a type 2 universe U, we have $\beta_2(U) = +\infty$ and one of the topological indexes $\hat{\beta}_2(U)$, $\delta_2(U)$ is $+\infty$. Further, in every case, there is a type 2 spin universe U with the other topological index taken 0.

• For any universe U, we have $\beta_2(U) = +\infty$ and one of the topological indexes $\hat{\beta}_2(U)$, $\delta(U)$, $\rho(U)$ is $+\infty$. In the case of $\rho(U) = +\infty$, the condition $\hat{\beta}_1(U) = +\infty$ is added. Further, in every case, there is a spin universe U with the other topological indexes on $\hat{\beta}_2(U)$, $\delta(U)$ and $\rho(U)$ taken 0.

• For a full universe U, we have $\beta_2(U) = +\infty$ and one of the topological indexes $\hat{\beta}_2(U)$, $\delta(U)$ is $+\infty$. The condition $\hat{\beta}_1(U) \geq 1$ always holds. Further, in every case, there is a full spin universe U with the other topological index on $\hat{\beta}_2(U)$ and $\delta(U)$ taken 0.

In this paper, the most recent result $\beta_2(U) = +\infty$ for every universe or punctured universe U in [14] is especially emphasized.

If a closed 3-manifold M is a model of our living 3-space and a smooth map $t : M \to \mathbb{R}$ for the the real line $\mathbb{R}$ is a time function, then there is a smooth embedding $M \to M \times \mathbb{R}$ sending every point $x \in M$ to the point $(x, t(x)) \in M \times \mathbb{R}$. The product $M \times \mathbb{R}$, regarded as the M-bundle over $\mathbb{R}$, is called the *spacetime* of M. Since every closed 3-manifold M embedded in U admits a trivial normal line bundle $M \times \mathbb{R}$ in U, every universe is considered as a "classifying space"for the spacetime of every 3-space model M. The smooth embedding $M \to M \times \mathbb{R}$ given by a time function $t : M \to \mathbb{R}$ is of type 2 (see [6]).

A standard physical spacetime model called the *hyersphere world-universe model* (see for example [20]) is topologically the product $S^3 \times \mathbb{R}$, called the *4D spherical shell* or simply the *spherical shell*. In Section 4, the spherical shell $S^3 \times \mathbb{R}$ is assumed to be properly and smoothly embedded in an open 5-manifold W. Then we define a collision modification on two distinct 3-sphere fibers $S^3_t, S^3_{t'}$ $(t, t' \in \mathbb{R}, t \neq t')$ of the spherical shell $S^3 \times \mathbb{R}$ and show in Theorem 2.1 that *a universe U is constructed in W from the spherical shell $S^3 \times \mathbb{R}$ by infinitely many collision modifications on 3-sphere fibers of $S^3 \times \mathbb{R}$.* It may be something interesting to mention that there are 5-dimensional physical universe models such as Kaluza-Klein model (see [2, 18]) and Randall-Sundrum model [21, 22] and an argument on the physical collision of a brane in the bulk space such as [17].

As the final note in the introduction, it would be interesting to observe that the infinity in every case of a 4D universe comes from the existence of the connected sums of copies of the trefoil knot, which occurs frequently next to the trivial knot (see [1, 24, 25]). In fact, the closed 3-manifolds contributing to the infinities in [12] are called c-efficient 3-manifolds which are the connected sums of the homology handles obtained from the 3-sphere S^3 by the 0-surgery along the connected sums of certain copies of the trefoil knot. The closed 3-manifolds contributing to the infinity $\beta_2(U) = +\infty$ are the connected sums of homology 3-tori constructed from the 3-torus T^3 by replacing the standard solid torus generators with the exteriors of the connected sums of certain copies of the trefoil knot.

2. Examples on distinctions of a 4D punctured universe and 4D universes

In the following comparison theorem, the assertion (1) is obvious by definitions. We will give examples showing the assertions (2)-(5).

Theorem 2.1 (Comparison Theorem).

$$\begin{array}{ccccc} & & \text{Type 1 universe} & & \\ & \nearrow & & \searrow & \\ (1)\ \text{Full universe} & & & & \text{Universe} \to \text{Punctured universe}. \\ & \searrow & & \nearrow & \\ & & \text{Type 2 universe} & & \end{array}$$

(2) Type 1 universe $\not\to$ Full universe.

(3) Type 2 universe $\not\to$ Full universe.

(4) Universe $\not\to$ Type 1 universe.

(5) Universe $\not\to$ Type 2 universe.

(6) Punctured universe $\not\to$ Universe.

To see (3) and (4), we note that the stable 4-space

$$S\mathbb{R}^4 = \mathbb{R}^4 \#_{i=1}^{+\infty} S^2 \times S^2_i$$

considered in [7] is a type 2 spin universe because every closed 3-manifold M bounds a simply connected spin 4-manifold whose double is the connected sum of some copies of $S^2 \times S^2$. Since $H_1(S\mathbb{R}^4;\mathbb{Z}) = 0$, we see that any closed 3-manifold cannot be type 1 embedded in $S\mathbb{R}^4$ (as observed in the introduction), showing (3) and (4). To see (2) and (5), we consider a type 1 spin universe

$$U_{SP} = \mathbb{R}^4 \#_{i=1}^{+\infty} M_i \times S^1$$

which we call the S^1-*product universe.*

An argument on a linking form, namely a non-singular symmetric bilinear form $\ell : G \times G \to \mathbb{Q}/\mathbb{Z}$ on a finite abelian group G is used. The linking form ℓ is *split* if ℓ is hyperbolic, i.e., G is a direct sum $H' \oplus H''$ with $\ell(H', H') = \ell(H'', H'') = 0$ or ℓ is the orthogonal sum of a linking form $\ell_H : H \times H \to \mathbb{Q}/\mathbb{Z}$ and its inverse $-\ell_H : H \times H \to \mathbb{Q}/\mathbb{Z}$. Then we have the following lemma:

Lemma 2.2. If a closed 3-manifold M with $H_1(M;\mathbb{Z})$ a finite abelian group is type 2 embeddable in the S^1-product universe U_{SP}, then the linking form

$$\ell : H_1(M;\mathbb{Z}) \times H_1(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

is split.

The proof of Lemma 2.2 is given by the following arguments (see [12] for the detailed proof):

(2.2.1) If $H_1(M;\mathbb{Z})$ is a finite abelian group, then M is type 2 embedded in an S^1-semi-product 4-manifold X consisting of the connected summands $M_i \times S^1$ $(i = 1, 2, \ldots, m)$ such that there is a point $p_i \in S^1$ with $(M_i \times p_i) \cap M = \emptyset$ for every i.

By (2.2.1), for $I = [0, 1]$ we may consider that M is type 2 embedded in the connected sum

$$Y = M_1 \times I \# M_2 \times I \# \ldots \# M_m \times I,$$

so that M splits Y into two compact 4-manifolds A and B whose boundaries ∂A and ∂B have the form

$$\partial A = M \cup \partial_A Y, \quad \partial B = (-M) \cup \partial_B Y,$$

where

$$\begin{aligned} \partial_A Y &= M_1 \times \partial I \cup M_2 \times \partial I \cup \cdots \cup M_s \times \partial I, \\ \partial_B Y &= M_{s+1} \times \partial I \cup M_{s+2} \times \partial I \cup \cdots \cup M_m \times \partial I. \end{aligned}$$

Then we have the following observation:

(2.2.2) The following natural sequence

$$(\#) \quad 0 \to \mathrm{tor} H_2(A, M \cup \partial_A Y;\mathbb{Z}) \xrightarrow{\partial_*} \mathrm{tor} H_1(M \cup \partial_A Y;\mathbb{Z}) \xrightarrow{i_*} \mathrm{tor} H_1(A;\mathbb{Z}) \to 0$$

on the homology torsion parts is a split exact sequence.

The lens space $L(p,q)$ with $p \neq 0, \pm 1$ is not type 2 embeddable in U_{SP} by Lemma 2.2, showing (2) and (5). To see (6), for $I = [0,1]$ we consider a punctured spin universe

$$U_{IP} = \mathbb{R}^4 \#_{i=1}^{+\infty} \mathrm{int}(M_i^0 \times I),$$

which we call the *I-product punctured universe*. Suppose that there is an embedding $k : M \to U_{IP}$ for a closed 3-manifold $M \in \mathbb{M}$. We note that every element of $H_1(U_{IP};\mathbb{Z})$ is represented by the sum of 1-cycles in $\mathrm{int}(M_i^0 \times I)$ for a finite number of i which can be moved to be disjoint from $k(M)$. This means that the intersection number $\mathrm{Int}(M, H_1(U_{IP};\mathbb{Z})) = 0$, showing that the embedding k is not of type 1 and hence k must be of type 2. The inclusion $U_{IP} \subset U_{SP}$ is obtained by taking $I \subset S^1$. Then the composite embedding $M \xrightarrow{k} U_{IP} \subset U_{SP}$ is still of type 2, because the boundary $\partial(M_i^0 \times I)$ is connected. Thus, if $H_1(M;\mathbb{Z})$ is a finite abelian group, then the linking form $\ell : H_1(M;\mathbb{Z}) \times H_1(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ splits by Lemma 2.2. Thus, the lens space $L(p,q)$ with $p \neq 0, \pm 1$ is not embeddable in U_{IP}, implying that U_{IP} is not any universe, showing (6).

3. Independence on some topological indexes of a 4D universe and a 4D punctured universe

In this section, the following lemma is shown:

Lemma 3.1.
(1) There is a punctured spin universe U such that anyone of the topological indexes $\hat{\beta}_2(U)$, $\delta_0(U)$, $\rho_0(U)$ is $+\infty$ and the other topological indexes are taken 0.

(2) There is a type 1 universe U such that anyone of the topological indexes $\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ is $+\infty$ and the other topological indexes are taken 0.

(3) There is a type 2 spin universe U such that anyone of the topological indexes $\hat{\beta}_2(U)$, $\delta_2(U)$ is $+\infty$. and the other topological index is taken 0.

(4) There is a spin universe U such that anyone of the topological indexes $\hat{\beta}_2(U)$, $\delta(U)$, $\rho(U)$ is $+\infty$ and the other topological indexes are taken 0.

(5) There is a full spin universe U such that anyone of the topological indexes $\hat{\beta}_2(U)$, $\delta(U)$ is $+\infty$ and the other topological index is taken 0.

The proof of Lemma 3.1 is given by the following Examples 3.2-3.4.

Example 3.2. The stable 4-space $S\mathbb{R}^4 = \mathbb{R}^4 \#_{i=1}^{+\infty} S^2 \times S_i^2$ has the following property:

(3.2.1) For every $M \in \mathbb{M}$, there is a type 2 embedding $k : M \to S\mathbb{R}^4$ inducing the trivial homomorphism $k_* = 0 : H_2(M;\mathbb{Z}) \to H_2(S\mathbb{R}^4;\mathbb{Z})$.

Thus, $U = S\mathbb{R}^4$ is a punctured and type 2 spin universe with $\hat{\beta}_2(U) = +\infty$, $\hat{\beta}_1(U) = 0$, $\delta_0(U) = \delta_2(U) = 0$ and $\rho_0(U) = \rho_2(U) = 0$. Further, $U_S = S^1 \times S^3 \# S\mathbb{R}^4$ is a punctured, type 1, type 2, full spin universe with

$$\begin{aligned} &\hat{\beta}_2(U_S) = +\infty,\ \hat{\beta}_1(U_S) = 1,\\ &\delta_0(U_S) = \delta_1(U_S) = \delta_2(U_S) = \delta(U_S) = 0,\\ &\rho_0(U_S) = \rho_1(U_S) = \rho_2(U_S) = \rho(U_S) = 0. \end{aligned}$$

Example 3.3. For any 3-manifolds $M_i \in \mathbb{M}$ $(i = 1, 2, \dots)$, let W_i be a spin 4-manifold obtained from $M_i \times I$ by attaching 2-handles on $M_i \times 1$ along a basis for $H_1(M_i \times 1;\mathbb{Z})/(\text{torsions})$ to obtain that $H_1(W_i;\mathbb{Z})$ is a torsion abelian group, where $I = [0, 1]$. Then the natural homomorphism

$$H_2(M_i \times I;\mathbb{Z}) \to H_2(W_i;\mathbb{Z})$$

is an isomorphism, so that $H_2(W_i;\mathbb{Z})$ is a free abelian group. We construct the open 4-manifolds

$$U_T = \mathbb{R}^4 \#_{i=1}^{+\infty} \mathrm{int} W_i \quad \text{and} \quad U_{ST} = S^1 \times S^3 \# U_T.$$

The open 4-manifold U_T is a punctured and type 2 spin universe with

$$\begin{aligned} &\hat{\beta}_2(U_T) = \hat{\beta}_1(U_T) = 0,\\ &\delta_0(U_T) = \delta_2(U_T) = +\infty,\\ &\rho_0(U_T) = \rho_2(U_T) = 0. \end{aligned}$$

The open 4-manifold U_{ST} is a punctured, type 1, type 2 and full spin universe with

$$\begin{aligned} &\hat{\beta}_2(U_{ST}) = 0,\ \hat{\beta}_1(U_{ST}) = 1,\\ &\delta_0(U_{ST}) = \delta_1(U_{ST}) = \delta_2(U_{ST}) = \delta(U_{ST}) = +\infty,\\ &\rho_0(U_{ST}) = \rho_1(U_{ST}) = \rho_2(U_{ST}) = \rho(U_{ST}) = 0. \end{aligned}$$

Example 3.4. Let $\mathbb{Z}_{/2} = \mathbb{Z}[\frac{1}{2}]$ be a subring of $\mathbb{Q}$. The *4-dimensional solid torus with three meridian disks* is a spin 4-manifold $D(T^3)$ with boundary the 3-dimensional torus T^3 which is obtained from the 4-disk D^4 by attaching the three 0-framed 2-handles along the Borromean rings L_B in the 3-sphere $S^3 = \partial D^4$ (see [11, 19]). For $s \geq 2$, let $D(sT^3)$ be the disk sum of s copies of $D(T^3)$. Then the boundary $\partial D(sT^3)$ is the connected sum $\# sT^3$of s copies of T^3. For $s = 0$, we understand $D(sT^3) = S^4$ and $\# sT^3 = \emptyset$. Let

$$\mathbf{\Sigma} = S^1 \times S^3 \# D(sT^3) \quad \text{and} \quad \hat{\mathbf{\Sigma}} = S^4 \# D(sT^3) = D(sT^3).$$

The 4-manifolds $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$ are called the *standard Samsara 4-manifold* and the *standard reduced Samsara 4-manifold* on S^3, respectively. A *Samsara 4-manifold* on $M \in \mathbb{M}$ is a compact oriented spin 4-manifold Σ with $\partial\Sigma = \# sT^3$

and with $\mathbb{Z}_{/2}$-homology of the standard Samsara 4-manifold $\boldsymbol{\Sigma}$ for some $s \geq 0$ such that there is a type 1 embedding $k : M \to \Sigma$ inducing the trivial homomorphism

$$k_* = 0 : H_2(M; \mathbb{Z}_{/2}) \to H_2(\Sigma; \mathbb{Z}_{/2}) = {\mathbb{Z}_{/2}}^{3s}.$$

A *reduced Samsara 4-manifold* on a punctured 3-manifold M^0 is a compact oriented spin 4-manifold $\hat{\Sigma}$ with $\partial\hat{\Sigma} = \# sT^3$ and with $\mathbb{Z}_{/2}$-homology of the standard reduced Samsara 4-manifold $\hat{\boldsymbol{\Sigma}}$ for some s such that there is a punctured embedding

$$k^0 : M^0 \to \hat{\Sigma}$$

inducing the trivial homomorphism

$$k^0_* = 0 : H_2(M^0; \mathbb{Z}_{/2}) \to H_2(\hat{\Sigma}; \mathbb{Z}_{/2}) = {\mathbb{Z}_{/2}}^{3s}.$$

The number s is called the *torus number* of a Samsara 4-manifold Σ or a reduced Samsara 4-manifold $\hat{\Sigma}$. In [10], the following result is shown:

Theorem 3.5. For every closed 3-manifold M, there is a reduced (closed or bounded) Samsara 4-manifold $\hat{\Sigma}$ on M^0 with the $\mathbb{Z}_2$-torsion relation

$$\beta_2^{(2)}(\hat{\Sigma}; \mathbb{Z}) = \beta_1(M; \mathbb{Z}_2).$$

Further, for every positive integer n, there are infinitely many closed 3-manifolds M such that every reduced (closed or bounded) Samsara 4-manifold $\hat{\Sigma}$ on M^0 has the $\mathbb{Z}_2$-torsion relation

$$\beta_2^{(2)}(\hat{\Sigma}; \mathbb{Z}) \geq \beta_1(M; \mathbb{Z}_2) = n.$$

For every closed 3-manifold M, there is a (closed or bounded) Samsara 4-manifold Σ on M with

$$\beta_2^{(2)}(\Sigma; \mathbb{Z}) = \beta_1(M; \mathbb{Z}_2).$$

Further, for every positive integer n, there are infinitely many closed 3-manifolds M such that every (closed or bounded) Samsara 4-manifold Σ on M has

$$\beta_2^{(2)}(\Sigma; \mathbb{Z}) \geq \beta_1(M; \mathbb{Z}_2) = n.$$

Note that any information on the torus number s is not given in Theorem 3.5. It can be seen from [14] that a large number is needed for the torus number s of any Samsara 4-manifold Σ on a certain closed 3-manifold M.

Let Σ_i be a Samsara 4-manifold on every $M_i \in \mathbb{M}$ $(i = 1, 2, 3, \dots)$. Let $\mathbb{R}^4_+$ be the upper-half 4-space with boundary the 3-space $\mathbb{R}^3$. Let

$$\Sigma\mathbb{R}^4_+ = \mathbb{R}^4_+ \#_{i=1}^{+\infty} \Sigma_i$$

be the 4-manifold obtained from $\mathbb{R}^4_+$ by making the connected sums of the closed Σ_i's with $\mathrm{int}\mathbb{R}^4_+$ and the disk sums with the bounded Σ_i's and $\mathbb{R}^4_+$ along a 3-disk in $\partial\Sigma_i$ and a 3-disk in $\partial\mathbb{R}^4_+ = \mathbb{R}^3$. The open 4-manifold $U_{SM} =$

$\mathrm{int}(\Sigma\mathbb{R}^4_+)$ is called a *Samsara universe*, which is a punctured and type 1 spin universe with

$$\begin{aligned}\hat{\beta}_2(U_{SM}) = 0,\ \hat{\beta}_1(U_{SM}) = +\infty,\\ \delta_0(U_{SM}) = \delta_1(U_{SM}) = 0,\\ \rho_0(U_{SM}) = \rho_1(U_{SM}) = +\infty.\end{aligned}$$

Let $\mathbf{\Sigma}\mathbb{R}^4_+$ be the 4-manifold obtained from $\mathbb{R}^4_+$ by making the connected sums with countably many copies of $S^1 \times S^3$ with $\mathrm{int}\mathbb{R}^4_+$ and the disk sums with countably many copies of $D(T^3)$ and $\mathbb{R}^4_+$ along a 3-disk in $\partial D(T^3)$ and a 3-disk in $\partial\mathbb{R}^4_+ = \mathbb{R}^3$, and

$$\mathbf{\Sigma}\mathbb{R}^4 = \mathrm{int}(\mathbf{\Sigma}\mathbb{R}^4_+).$$

Every Samsara universe U_{SM} has the same $\mathbb{Z}_{/2}$-cohomology as $\mathbf{\Sigma}\mathbb{R}^4$, so that $\hat{\beta}_2(U_{SM}) = 0$. By definition, we have $\delta(U_{SM}) = 0$. If U_{SM} is a type 2 universe, then U_{SM} would be a full universe. Then, by [14] as stated in the introduction (see Section 5), $\hat{\beta}_2(U_{SM})$ or $\delta(U_{SM})$ must be $+\infty$, which is impossible. Thus, any Samsara universe U_{SM} is not any type 2 universe.

Let

$$\hat{\Sigma}\mathbb{R}^4_+ = \mathbb{R}^4_+ \#_{i=1}^{+\infty} \hat{\Sigma}_i$$

be the 4-manifold obtained from $\mathbb{R}^4_+$ by making the connected sums of the closed $\hat{\Sigma}_i$'s with $\mathrm{int}\mathbb{R}^4_+$ and the disk sums with the bounded $\hat{\Sigma}_i$'s and $\mathbb{R}^4_+$ along a 3-disk in $\partial\hat{\Sigma}_i$ and a 3-disk in $\partial\mathbb{R}^4_+ = \mathbb{R}^3$. The open 4-manifold

$$U_{RS} = \mathrm{int}(\hat{\Sigma}\mathbb{R}^4_+)$$

is called a *reduced Samsara universe*, which is a punctured spin universe with the following topological indexes

$$\begin{aligned}\hat{\beta}_2(U_{RS}) = \hat{\beta}_1(U_{RS}) = 0,\\ \delta_0(U_{RS}) = 0,\\ \rho_0(U_{RS}) = +\infty.\end{aligned}$$

Let $\hat{\mathbf{\Sigma}}\mathbb{R}^4_+$ be the 4-manifold obtained from $\mathbb{R}^4_+$ by making the disk sums with countably many copies of $D(T^3)$, and

$$\hat{\mathbf{\Sigma}}\mathbb{R}^4 = \mathrm{int}(\hat{\mathbf{\Sigma}}\mathbb{R}^4_+).$$

Every reduced Samsara universe U_{RS} has the same $\mathbb{Z}_{/2}$-homology as $\hat{\mathbf{\Sigma}}\mathbb{R}^4$. By [10, (3.3.1)], we can show that if a closed 3-manifold M with $H_1(M;\mathbb{Z})$ a finite abelian group is embedded in U_{RS}, then the linking form

$$\ell_p : H_1(M;\mathbb{Z})_p \times H_1(M;\mathbb{Z})_p \to \mathbb{Q}/\mathbb{Z}$$

restricted to the p-primary component $H_1(M;\mathbb{Z})_p$ of $H_1(M;\mathbb{Z})$ for every odd prime p is hyperbolic. Thus, U_{RS} is not any universe. Further, from [10, 3.3], we can see that $\mathbf{\Sigma}\mathbb{R}^4$ and $\hat{\mathbf{\Sigma}}\mathbb{R}^4$ are not any punctured universe.

4. A 4D full universe obtained by a collision modification of the spherical shell

Let W be an open connected oriented 5-manifold. Let X and X' be two disjoint compact connected oriented 4-manifolds smoothly embedded in W. By isotopic deformations $\tilde{i}: X \to W$ and $\tilde{i}': X' \to W$ of the inclusion maps $i: X \subset W$ and $i': X' \subset W$, we consider that the images $\tilde{i}X$ and $\tilde{i}'X'$ meet tangently and opposite-orientedly in W with a compact 4-submanifold V, where V is assumed to be in the interiors of the 4-manifolds X and X'. We call V a *collision field* of the 4-manifolds X and X' in the 5-manifold W. A *collision modification of X and X' in W with a collision field V* is the 4-manifold

$$X'' = \mathrm{cl}(\tilde{i}X \setminus V) \bigcup \mathrm{cl}(\tilde{i}'X' \setminus V).$$

This collision modification gives a standard procedure to construct a new 4-manifold X'' from X and X' through a regular neighborhood of V in W. In the spherical shell $S^3 \times \mathbb{R}$ embedded properly and smoothly in an open 5-manifold W, we understand that a *collision modification on distinct 3-sphere fibers S^3_t* and $S^3_{t'}$) of $S^3 \times \mathbb{R}$ in W is a collision modification of the disjoint compact spherical shells $S^3 \times I$ and $S^3 \times I'$ in W with a collision field V for any disjoint closed interval neighborhoods I and I' of the points t and t' in $\mathbb{R}$, respectively. In the following theorem, it is explained how a full universe is constructed from the spherical shell $M \times \mathbb{R}$ by infinitely many collision modifications on distinct 3-sphere fibers of $S^3 \times \mathbb{R}$.

Theorem 4.1. Assume that the spherical shell $M \times \mathbb{R}$ is embedded properly in a 5-dimensional open manifold W. Then a full universe U is produced in W by infinitely many collision modifications on distinct 3-sphere fibers of the sphere shell $S^3 \times \mathbb{R}$.

An outline of the proof given in [14] is as follows: By a collision modification of $S^3 \times I$ and $S^3 \times I'$ in W with a collision field $V = S^1 \times D^3$ the 4D solid torus, we have the connected sum $S^3 \times I \# S^3 \times I' \# S^2 \times S^2$, by which the spherical shell $S^3 \times \mathbb{R}$ changes into an open 4-manifold $S^3 \times \mathbb{R} \# S^2 \times S^2 \# S^1 \times S^3$. Continuing this modification, we have an open 4-manifold U which is the connected sum of $S^3 \times \mathbb{R}$ and infinitely many copies of $S^2 \times S^2$ and $S^1 \times S^3$. This open 4-manifold U is a full universe.

5. A non-compact version of the signature theorem for an infinite cyclic covering

In this section, we explain a non-compact 4-manifold version of the infinite cyclic covering signature theorem in [14] which is given in [12] and needed to our purpose as a mathematical tool.

Let Y be a non-compact oriented 4-manifold with boundary a closed 3-manifold B. Assume that $\hat{\beta}_2(Y) < +\infty$. We say that a homomorphism

$\gamma : H_1(Y;\mathbb{Z}) \to \mathbb{Z}$ is *end-trivial* if there is a compact submanifold Y' of Y such that the restriction $\gamma|_{\mathrm{cl}(Y\backslash Y')} : H_1(Y\backslash Y';\mathbb{Z}) \to \mathbb{Z}$ is the zero map. For any end-trivial homomorphism $\gamma : H_1(Y;\mathbb{Z}) \to \mathbb{Z}$, we take the infinite cyclic covering $(\tilde{Y},\tilde{B})$ of (Y,B) associated with γ. Then $H_2(\tilde{Y};\mathbb{Q})$ is a (possibly, infinitely generated) Γ-module for the principal ideal domain $\Gamma = \mathbb{Q}[t,t^{-1}]$ of Laurent polynomials with rational coefficients. Consider the Γ-intersection form

$$\mathrm{Int}_\Gamma : H_2(\tilde{Y};\mathbb{Q}) \times H_2(\tilde{Y};\mathbb{Q}) \to \Gamma$$

defined by $\mathrm{Int}_\Gamma(x,y) = \sum_{m=-\infty}^{+\infty} \mathrm{Int}(x,t^{-m}y)t^m$ for $x,y \in H_2(\tilde{Y};\mathbb{Q})$. Then we have the identities:

$$\begin{aligned} \mathrm{Int}_\Gamma\left(\overline{f(t)}x,y\right) &= \mathrm{Int}_\Gamma(x,f(t)y) = f(t)\mathrm{Int}_\Gamma(x,y), \\ \mathrm{Int}_\Gamma(y,x) &= \overline{\mathrm{Int}}_\Gamma(x,y), \end{aligned}$$

where $\overline{}$ denotes the involution of Γ sending t to t^{-1}. Let

$$O_2(\tilde{Y};\mathbb{Q})_\Gamma = \{x \in H_2(\tilde{Y};\mathbb{Q}) | \, \mathrm{Int}_\Gamma(x, H_2(\tilde{Y};\mathbb{Q})) = 0\}$$

and

$$\hat{H}_2(\tilde{Y};\mathbb{Q})_\Gamma = H_2(\tilde{Y};\mathbb{Q})/O_2(\tilde{Y};\mathbb{Q})_\Gamma,$$

which is a torsion-free Γ-module. We show the following lemma:

Lemma 5.1. If $\hat{\beta}_2(Y) < +\infty$, then $\hat{H}_2(\tilde{Y};\mathbb{Q})_\Gamma$ is a free Γ-module of finite rank.

Let $A(t)$ be a Γ-Hermitian matrix representing the Γ-intersection form Int_Γ on $\hat{H}_2(\tilde{Y};\mathbb{Q})_\Gamma$. For $x \in (-1,1)$ let $\omega_x = x+\sqrt{1-x^2}i$, which is a complex number of norm one. For $a \in (-1,1)$ we define the signature invariant of $\tilde{Y}$ by

$$\tau_{a\pm 0}(\tilde{Y}) = \lim_{x\to a\pm 0} \mathrm{sign} A(\omega_x).$$

The signature invariants $\sigma_a(\tilde{B})$ ($a \in [-1,1]$) of $\tilde{B}$ are defined in [4, 5, 6, 9] by the quadratic form

$$b : \mathrm{Tor}_\Gamma H_1(\tilde{B};\mathbb{Q}) \times \mathrm{Tor}_\Gamma H_1(\tilde{B};\mathbb{Q}) \to \mathbb{Q}$$

on the Γ-torsion part $\mathrm{Tor}_\Gamma H_1(\tilde{B};\mathbb{Q})$ of $H_1(\tilde{B};\mathbb{Q})$ defined in [3]. For $a \in [-1,1]$, let

$$\sigma_{[a,1]}(\tilde{B}) = \sum_{a\le x\le 1} \sigma_x(\tilde{B}),$$

$$\sigma_{(a,1]}(\tilde{B}) = \sum_{a< x\le 1} \sigma_x(\tilde{B}).$$

The following theorem is a non-compact version of the signature theorem given in [5].

Theorem 5.2 (A non-compact version of the signature theorem).

$$\tau_{a-0}(\tilde{Y}) - \mathrm{sign}Y = \sigma_{[a,1]}(\tilde{B}),$$
$$\tau_{a+0}(\tilde{Y}) - \mathrm{sign}Y = \sigma_{(a,1]}(\tilde{B}).$$

The proof is in [12]. Let $\kappa_1(\tilde{B})$ denote the $\mathbb{Q}$-dimension of the kernel of the homomorphism $t-1 : H_1(\tilde{B};\mathbb{Q}) \to H_1(\tilde{B};\mathbb{Q})$. By Theorem 5.2, we have

$$\sigma_{(a,1]}(\tilde{B}) + \mathrm{sing}Y = \tau_{a+0}(\tilde{Y}) = \tau_{a+0}(\tilde{Y}').$$

On the other hand, in [6, Theorem 1.6], it is shown that

$$|\tau_{a+0}(\tilde{Y}'))| - \kappa_1(\partial\tilde{Y}') \leq \hat{\beta}_2(Y').$$

Since $\hat{\beta}_2(Y') = \hat{\beta}_2(Y)$ and $\partial\tilde{Y}' = \tilde{B} \cup \tilde{B}_0$ with $\sigma_{(a,1]}(\tilde{B}_0) = \kappa_1(\tilde{B}_0) = 0$, we have the following corollary:

Corollary 5.3. For every $a \in (-1,1)$,

$$|\sigma_{(a,1]}(\tilde{B})| - \kappa_1(\tilde{B}) \leq |\mathrm{sign}Y| + \hat{\beta}_2(Y) \leq 2\hat{\beta}_2(Y).$$

Let M' be a compact connected oriented 3-manifold M', and U a possibly non-compact connected oriented 4-manifold. An embedding $k' : M' \to U$ is said to be *loose* if the kernel

$$K(M') = \ker(k'_* : H_2(M';\mathbb{Z}) \to H_2(U;\mathbb{Q})) \neq 0.$$

It is known that if the boundary $\partial M'$ of M' is $\emptyset$ or connected, then every indivisible $x \in K(M')$ is represented by a closed connected oriented surface F in M' which we call a *null-surface* of the loose embedding k' (see [8]). Then we have $sk'_*[F] = 0$ in $H_2(U;\mathbb{Z})$ for a positive integer s, which is assumed to be taken to be the smallest positive integer. We consider a loose embedding $k^0 : M^0 \to U$ for $M^0 \in \mathbb{M}^0$ which is regarded as the inclusion map $k^0 : M^0 \subset U$, and F as a null-surface of k^0. We use the following lemma:

Lemma 5.4. For a tubular neighborhood N_F of F in U, there is a compact connected oriented 3-manifold V in $\mathrm{cl}(U \backslash N_F)$ such that $[\partial V] = s[F]$ in $H_2(N_F;\mathbb{Z})$.

Let $E_M = \mathrm{cl}(U \backslash M^0 \times [-1,1]) \subset E = \mathrm{cl}(U \backslash N_F)$. For a null-surface F of a loose embedding $k^0 : M^0 \subset U$, we define a homomorphism

$$\gamma : H_1(E_M;\mathbb{Z}) \xrightarrow{i_*} H_1(E;\mathbb{Z}) \xrightarrow{\mathrm{Int}_V} \mathbb{Z}$$

by using V in Lemma 5.4, where i_* is a natural homomorphism and Int_V is defined by the identity $\mathrm{Int}_V(x) = \mathrm{Int}(x,V)$ for $x \in H_1(E;\mathbb{Z})$. We have the following lemma:

Lemma 5.5. i_* and Int_V are onto, so that γ is onto.

The homomorphism γ is called a *null-epimorphism* (associated with an null-surface F) of a loose embedding k^0. We also need the following lemma:

Lemma 5.6. Every null-epimorphism $\gamma : H_1(E_M;\mathbb{Z}) \to \mathbb{Z}$ of a loose embedding $k^0 : M^0 \to U$ is end-trivial.

Let α be the reflection on the double $DM^0(=\partial E_M)$ of M^0 exchanging the two copies of M^0 orientation-reversely. A meridian m of F in $M^0\times[-1,1]$ is deformed in $M^0\times[-1,1]$ into a loop m' in $DM^0 = \partial E_M$ with $\alpha(m') = -m'$. Since $\mathrm{Int}_V([m]) = s$, the following lemma is directly obtained:

Lemma 5.7. We have $\dot\gamma(x_F) = s$ and $\alpha_*(x_F) = -x_F$ for the element $x_F = [m'] \in H_1(\partial E_M;\mathbb{Z})$ and the restriction $\dot\gamma : H_1(DM^0;\mathbb{Z}) \to \mathbb{Z}$ of γ.

Corollary 5.8. If s is odd, then the $\mathbb{Z}_2$-reduction $\dot\gamma_2 : H_1(DM^0;\mathbb{Z}) \to \mathbb{Z}_2$ of $\dot\gamma$ is not α-invariant.

A homomorphism $\dot\gamma : H_1(DM^0;\mathbb{Z}) \to \mathbb{Z}$ satisfying the conclusion of Corollary 5.5 is called a $\mathbb{Z}_2$-*asymmetric* homomorphism in [4, 5].

Let $M = M(k)$ be the homology handle obtained from the 3-sphere S^3 by the 0-surgery along an oriented knot k (see [11] for a general reference of knots), and $\tilde M$ the infinite cyclic connected covering of M associated with a generator $\dot\gamma_M \in H^1(M;\mathbb{Z})$. Let $\sigma_{[a,1]}(k) = \sigma_{[a,1]}(\tilde M)$ and $\sigma_{(a,1]}(k) = \sigma_{(a,1]}(\tilde M)$ for every $a \in (-1,1)$ (see [9]). The signature invariant $\sigma_{[a,1]}(k)$ of a knot k is *critical* if $\sigma_{[a,1]}(k) \neq 0$ and $\sigma_{[x,1]}(k) = 0$ for every $x \in (a,1)$.

To confirm that $\beta_2(U) = +\infty$ for any universe or punctured universe U, a property of the signature invariants of a homology 3-torus generalizing a property of the 3-torus T^3 is needed, which we introduce from now.

For the 3-torus $T^3 = S^1\times S^1\times S^1$, let C_i $(i = 1,2,3)$ be disjointly embedded circles in T^3 representing a $\mathbb{Z}$-basis for $H_1(T^3;\mathbb{Z})$ such that C_1, C_2, C_3 are isotopic to $S^1\times 1\times 1$, $1\times S^1\times 1$, $1\times 1\times S^1$ in T^3, respectively. Let $N(C_i)$ be a tubular neighborhood of C_i in T^3 with a fixed meridian-longitude system for $i = 1,2,3$.

A *homological 3-torus* is a closed 3-manifold $M = M(k_1,k_2,k_3) \in \mathbb{M}$ obtained from T^3 and 3 knots k_1,k_2,k_3 in S^3 by replacing $N(C_1), N(C_2), N(C_3)$ with the compact knot exteriors $E(k_1), E(k_2), E(k_3)$ so that the meridian-longitude system of $\partial N(C_i)$ is identified with the longitude-meridian system of k_i in $E(k_i)$ for $i = 1,2,3$. The cup product $a\cup b\cup c \in H^3(M;\mathbb{Z})$ of a $\mathbb{Z}$-basis a,b,c of $H^1(M;\mathbb{Z})$ representing the dual elements of the meridians of k_i $(i = 1,2,3)$ is a generator of $H^3(M;\mathbb{Z}) \cong Z$, which is a property inherited from a well-known property of the 3-torus T^3.

It is convenient to note that the cup product $a'\cup b'\cup c' \in H^3(M;\mathbb{Q})$ of any $\mathbb{Q}$-base change a',b',c' of a,b,c in $H^1(M;\mathbb{Q})$ is a generator of $H^3(M;\mathbb{Q}) \cong$

$\mathbb{Q}$ and hence the elements $a' \cup b', b' \cup c', c' \cup a' \in H^2(M;\mathbb{Q})$ form a $\mathbb{Q}$-basis of $H^2(M;\mathbb{Q})$ orthogonally dual to the $\mathbb{Q}$-basis c', a', b' of $H^1(M;\mathbb{Q})$, respectively [To see this, note that $u \cup v = -v \cup u$ and, in particular, $u \cup u = 0$ for all $u, v \in H^1(M;\mathbb{Q})$].

For an integer $m > 0$, let $\mathbb{T}_m$ be the collection of 3-manifolds consisting of the connected sums of m homological 3-tori.

For an application of the signature invariants $\sigma_{[a,1]}(\tilde{B})$, consider the disjoint union $B = M \times 1 \cup M \times (-1)$ for a closed 3-manifold $M \in \mathbb{T}_m$, where $M \times 1$ and $M \times (-1)$ are respectively identified with M and the same 3-manifold as M but with orientation reversed. A homomorphism $\dot{\gamma} : H_1(B;\mathbb{Z}) \to \mathbb{Z}$ is *asymmetric* if there is no system of elements $x_1, x_2, \dots, x_n \in H_1(M;\mathbb{Z})$ $(n = 3m)$ representing a $\mathbb{Q}$-basis for $H_1(M;\mathbb{Q})$ such that $\dot{\gamma}(x_i) = \pm\alpha_*(x_i)$ for all i, where α denotes the standard orientation-reversing involution on B switching $M \times 1$ to $M \times (-1)$.

The following calculation is used in our argument.

Lemma 5.9. For positive integers d and m, let $(k_{i,1}, k_{i,2}, k_{i,3})$ $(i = 1, 2, \dots, m)$ be a sequence of triplets of knots used for the closed 3-manifold $M \in \mathbb{T}_m$ such that

(1) the signature invariants $\sigma_{[a,1]}(k_{i,1}), \sigma_{[a,1]}(k_{i,2}), \sigma_{[a,1]}(k_{i,3})$ are critical for all i $(i = 1, 2, \dots, m)$, and

(2) $|\sigma_{[a,1]}(k_{1,1})| > 2d + 4m$, and for all i, i', j, j' $(i, i' = 2, 3, \dots, 3m; j, j' = 1, 2, 3)$,

$$|\sigma_{[a,1]}(k_{i,j})| > \sum_{(i,j)>(i',j')} |\sigma_{[a,1]}(k_{i',j'})| + 2d + 4m,$$

where $(i,j) > (i',j')$ denotes the dictionary order.

Then for any asymmetric homomorphism $\gamma : H_1(B;\mathbb{Z}) \to \mathbb{Z}$, there is a number $b \in (-1, 1)$ such that

$$\kappa_1(\tilde{B}) \leq 4m \quad \text{and} \quad |\sigma_{[b,1]}(\tilde{B})| > 2d + 4m.$$

Example 5.10. Let k be a trefoil knot. Then the connected sum $k_{1,1}$ of $d + 2m + 1$ copies of k has the critical signature invariant

$$|\sigma_{[\frac{1}{2},1]}(k_{1,1})| = 2d + 4m + 2.$$

Further continuing connected sums of copies of k, we obtain a sequence of triplets of knots $(k_{i,1}, k_{i,2}, k_{i,3})$ $(i = 1, 2, \dots, m)$ used for the closed 3-manifold $M \in \mathbb{T}_m$ satisfying the assumptions (1) and (2) of Lemma 3.1 with $a = \frac{1}{2}$.

The following estimate on a $\mathbb{Q}$-subspace of the first cohomology $H^1(M;\mathbb{Q})$ of a closed 3-manifold M in $\mathbb{T}_m$ is technically useful:

Lemma 5.11. Let Δ be a $\mathbb{Q}$-subspace of $H^1(M;\mathbb{Q})$ of codimension $c(= 3m - \dim_{\mathbb{Q}} \Delta)$, and $\Delta^{(2)}$ the $\mathbb{Q}$-subspace of $H^2(M;\mathbb{Q})$ consisting of the cup product $u \cup v \in H^2(M;\mathbb{Q})$ for all $u, v \in \Delta$. Then $\dim_{\mathbb{Q}} \Delta^{(2)} \geq 2m - c$.

We call the $\mathbb{Q}$-space $\Delta^{(2)}$ the *cup product space* of the $\mathbb{Q}$-space Δ.

The following corollary is used to confirm the non-vanishing of the second rational homology of a bounded Samsara 4-manifold.

Corollary 5.12. For a (possibly non-compact) oriented 4-manifold X and $m > 0$, assume that a closed 3-manifold $M \in \mathbb{T}_m$ is a boundary component of X. Let d be the $\mathbb{Q}$-dimension of the kernel of the natural homomorphism $i_* : H_1(M;\mathbb{Q}) \to H_1(X;\mathbb{Q})$. Then we have $\beta_2(X) \geq \max\{2m - d, d\} \geq m$.

6. Infinities on the topological indexes of a 4D universe and a 4D punctured universe

In this section, the following result in [12] is explained.

Lemma 6.1.
(1) For a punctured spin universe U, anyone of the topological indexes $\hat{\beta}_2(U)$, $\delta_0(U)$, $\rho_0(U)$ must be $+\infty$.

(2) For a type 1 universe U, anyone of the topological indexes $\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ must be $+\infty$.

(3) For a type 2 spin universe U, anyone of the topological indexes $\hat{\beta}_2(U)$, $\delta_2(U)$ must be $+\infty$.

(4) For a universe U, anyone of the topological indexes $\hat{\beta}_2(U)$, $\delta(U)$, $\rho(U)$ must be $+\infty$.

(5) For a full spin universe U, anyone of the topological indexes $\hat{\beta}_2(U)$, $\delta(U)$ must be $+\infty$.

An outline of Lemma 6.1 given in [12] is as follows:

Confirmation of (1). For any positive integers n, c, let k_i $(i = 1, 2, \ldots, n)$ be knots whose signatures $\sigma(k_i)$ $(i = 1, 2, \ldots, n)$ have the condition that

$$|\sigma(k_1)| > 2c \quad \text{and} \quad |\sigma(k_i)| > \sum_{j=1}^{i-1} |\sigma(k_j)| + 2c \quad (i = 2, 3, \ldots, n).$$

Let $M_i = \chi(k_i, 0)$ and $M = M_1 \# M_2 \# \ldots \# M_n$. We call M a *c-efficient 3-manifold of rank* n. The following calculation is done in [6, Lemma 1.3]:

(6.1.2) Every c-efficient 3-manifold M of any rank n has

$$|\sigma_{(-1,1]}(\widetilde{DM^0}))| > 2c$$

for every $\mathbb{Z}_2$-asymmetric homomorphism $\dot{\gamma} : H_1(DM^0;\mathbb{Z}) \to \mathbb{Z}$.

Suppose that a punctured universe U has

$$\hat{\beta}_2(U) = c < +\infty,\ \delta_0(U) = b < +\infty,\ \rho_0(U) = b' < +\infty.$$

Let M be a c-efficient 3-manifold of any rank $n > b + b'$. Suppose that M^0 is embedded in U with $\mathbb{Z}$-rank b of the image. For the inclusion $k^0 : M^0 \subset U$, the kernel

$$K(M^0) = \ker[k^0_* : H_2(M^0;\mathbb{Z}) \to H_2(U;\mathbb{Q})]$$

is a free abelian group of some rank $d = n - b > b'$. Then there is a basis x_i $(i = 1, 2, \ldots, n)$ of $H_2(M^0;\mathbb{Z})$ such that x_i $(i = 1, 2, \ldots, d)$ is a basis of $K(M^0)$. Since $\rho_0(U) = b' < d$, we can find an indivisible element x in the basis x_i $(i = 1, 2, \ldots, d)$ after a base change such that the multiplied element rx for an odd integer r is represented by the boundary cycle of a 3-chain in U. Taking a closed connected oriented surface F in M^0 representing x, we have a null-epimorphism $\gamma : H_1(E_M;\mathbb{Z}) \to \mathbb{Z}$ (associated with an null-surface F) of the loose embedding k^0 whose restriction $\dot{\gamma} : H_1(DM^0;\mathbb{Z}) \to \mathbb{Z}$ is a $\mathbb{Z}_2$-asymmetric homomorphism. Then we obtain from (6.1.1) a contradiction that

$$2c < |\sigma_{(-1,1]}(\widetilde{DM^0})| \leq 2c$$

because $\hat{\beta}_2(E_M) \leq \hat{\beta}_2(U) = c$ and $\kappa_1(\widetilde{DM^0}) = 0$. Thus, at least one of $\hat{\beta}_2(U)$, $\delta_0(U)$, $\rho_0(U)$ must be $+\infty$.

Confirmation of (2). Let U be a type 1 universe. We always have $\hat{\beta}_1(U) \geq 1$. Since U is also a punctured universe, at least one of $\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ must be $+\infty$ by (1). Suppose that a type 1 universe U has

$$b = \hat{\beta}_2(U) < +\infty,\ c = \delta_1(U) < +\infty,\ s = \hat{\beta}_1(U) < +\infty.$$

Then we show that there is a 3-manifold M which is not type 1 embeddable in U. Let $\hat{H}_1(U;\mathbb{Z}) = \mathbb{Z}^s$. Let U_u $(u = 1, 2, \ldots, 2^s - 1)$ be the connected double coverings of U induced from the epimorphisms $\mathbb{Z}^s \to \mathbb{Z}_2$. Let $\mathbb{M}_u$ be the subset of $\mathbb{M}$ consisting of M such that a type 1 embedding $k : M \to U$ is trivially lifted to $k_u : M \to U_u$. Since every type 1 embedding $M \to U$ lifts to U_u trivially for some u, we see that

$$\bigcup_{u=1}^{2^s-1} \mathbb{M}_u = \mathbb{M}.$$

Let U' be a compact 4-submanifold of U such that $U'' = \mathrm{cl}(U \backslash U')$ is trivially lifted to U_u for all u. Let U'_u and U''_u be the lifts of U' and U'' to U_u. Let

$$b' = \max\{\beta_2(U'_u)|\, u = 1, 2, \ldots, 2^{s-1}\}.$$

(6.1.3) $\mathrm{rank}(\mathrm{im}(k_u)_*) \leq b + b'$ for any u.

For any positive integers n, c, we take n knots k_i $(1 \leq i \leq n)$ whose local signatures $\sigma_{(a,1)}(k_i)$ $(1 \leq i \leq n)$ have the condition that there are numbers $a_i \in (-1,1)$ $(i = 1,2,\dots,n)$ such that

$$|\sigma_{(a_1,1]}(k_1)| > 2c, \ \ |\sigma_{(a_i,1]}(k_i)| > \sum_{j=1}^{i-1} \left|\sigma_{(a,1]}(k_j)\right| + 2c \quad (i = 2,3,\dots,n)$$

for every $a \in (-1,1)$ (see [9]). Let $M_i = \chi(k_i, 0)$ be the 0-surgery manifold along k_i, and $M = M_1 \# M_2 \# \dots \# M_n$. We call M a strongly *c-efficient 3-manifold of rank n*. For this 3-manifold M, we say that a homomorphism $\dot\gamma : H_1(DM^0;\mathbb{Z}) \to \mathbb{Z}$ is *symmetric* if $\dot\gamma|_{\alpha(M_i^0)} = \pm\dot\gamma|_{M_i^0}$ for all i, where α is the reflection on the double DM^0. Otherwise, $\dot\gamma$ is said to be an *asymmetric homomorphism*. The following calculation is also seen from [6, Lemma 1.3]:

(6.1.4) For every strongly c-efficient 3-manifold M of any rank n and every asymmetric homomorphism $\dot\gamma : H_1(DM^0;\mathbb{Z}) \to \mathbb{Z}$, we have a number $a \in (-1,1)$ such that

$$|\sigma_{(a,1]}(\widetilde{DM^0}))| > 2c.$$

For example, if M is constructed from the knots k_i $(i = 1,2,\dots,n)$ with k_i the ic^+-fold connected sum of the trefoil knot for any fixed integer $c^+ > c$, then M is a strongly c-efficient 3-manifold of rank n. We show that every strongly c-efficient 3-manifold M of rank $> b+b'$ is not type 1 embedded in U. Suppose that M is type 1 embedded in U and lifts trivially in U_u. Let $U(M)$ and $U_u(M) = U(M) \cup tU(M)$ be the 4-manifolds obtained respectively from U and U_u by splitting along M, where t denotes the double covering involution. Let $\partial U(M) = M_0 \cup -M_1$ and $\partial U_u(M) = M_0 \cup -M_2$, where M_0, M_1, M_2 are the copies of M. Since the natural homomorphism $H_2(M;\mathbb{Z}) \to H_2(U;\mathbb{Q})$ is not injective, there is a non-zero element $[C] \in H_2(M;\mathbb{Z})$ such that $C = \partial C$ for a 3-chain D in U_u and $C = \partial D_*$ for a 3-chain D_* in U which is the image of D under the covering projection $U_u \to U$. The 3-chains $D$and D_* define 3-chains D', D'' and D''' in $U(M)$ such that

$$\begin{aligned}
\partial D' &= C_1'' - (C_0 + C_0'),\\
\partial D'' &= C_1' - C_0'',\\
\partial D''' &= (C_1' + C_1'') - (C_0 + C_0' + C_0'')
\end{aligned}$$

for some 2-cycles C_u, C_u', C_u'' in M_u $(u = 0,1)$. Since $\hat\beta_2(U(M)) \leq c$, the non-zero end-trivial homomorphism $\dot\gamma : H_1(DM^0;\mathbb{Z}) \to \mathbb{Z}$ defined by any 3-chain in $U(M)$ must be symmetric by Corollary 4.3 and (6.2.2) because every strongly c-efficient 3-manifold M has $\kappa_1(\widetilde{DM^0}) = 0$. Let

$$[C] = \sum_{i=1}^{m} a_i x_i, \ \ [C'] = \sum_{i=1}^{m} a_i' x_i, \ \ [C''] = \sum_{i=1}^{m} a_i'' x_i$$

in $H_1(M;\mathbb{Z})$ with x_i a generator of $H_1(M_i;\mathbb{Z}) \cong \mathbb{Z}$. By the symmetry conditions on D', D'' and D''', we have the following relations:

$$a_i'' = \varepsilon_i(a_i + a_i'),\ a_i' = \varepsilon_i' a_i'',\ a_i' + a_i'' = \varepsilon_i''(a_i + a_i' + a_i''),$$

where $\varepsilon_i, \varepsilon_i', \varepsilon_i'' = \pm 1$ for all i. Then we have

$$(1+\varepsilon_i')a_i'' = \varepsilon_i''(\varepsilon_i + 1)a_i''.$$

If $\varepsilon_i\varepsilon_i' = -1$, then we have $a_i'' = a_i' = a_i = 0$ for all i. If $\varepsilon_i\varepsilon_i' = 1$, then we have $a_i = 0$ for all i. Hence we have $[C] = 0$, contradicting that $[C] \neq 0$. Hence M is not type 1 embeddable in U.

Confirmation of (3). Let U be a type 2 universe. Suppose that

$$\hat{\beta}_2(U) = c < +\infty, \quad \delta_2(U) = b < +\infty.$$

Let $M \in \mathbb{M}$ be a c-efficient 3-manifold of any rank $n > b$. Let $k : M \subset U$ be a type 2 embedding which is a loose embedding. Let U' and U'' be the 4-manifolds obtained from U by splitting along M. For U' or U'', say U', we have a null-surface F in M and a positive (not necessarily odd) integer r such that the natural homomorphism $H_2(M;\mathbb{Z}) \to H_2(U';\mathbb{Z})$ sends $r[F]$ to 0. Taking the minimal positive integer r, we have a compact connected oriented 3-manifold V in U' with $\partial V = rF$. This 3-manifold V defines an end-trivial epimorphism $\gamma : H_1(U';\mathbb{Z}) \to \mathbb{Z}$ whose restriction $\dot{\gamma} : H_1(M;\mathbb{Z}) \to \mathbb{Z}$ is equal to $r\dot{\gamma}_F$ for the epimorphism $\dot{\gamma}_F : H_1(M;\mathbb{Z}) \to \mathbb{Z}$ defined by F. Let $\tilde{M}$ and $\tilde{M}_F$ denote the infinite cyclic coverings of M induced from $\dot{\gamma}$ and $\dot{\gamma}_F$, respectively. Let $(1 \leq) i_1 < i_2 < \cdots < i_s (\leq n)$ be the enumeration of i such that the $\mathbb{Z}_2$-reduction of $\dot{\gamma}_F$ restricted to the connected summand M_i of M is non-trivial. By a calculation made in [6, Lemma 1.3], we have

$$\sigma_{(-1,1]}(\tilde{M}_F) = \sum_{j=1}^{s} \sigma(K_{i_j}),$$

so that $|\sigma_{(-1,1]}(\tilde{M}_F)| > 2c$. By [6, Lemma 1.3], we also have

$$\sigma_{(-1,1]}(\tilde{M}_F) = \sigma_{(a,1]}(\tilde{M})$$

for some $a \in (-1,1)$. Then, since $\hat{\beta}_2(U') \leq \hat{\beta}_2(U) = c$ and $\kappa_1(\tilde{M}) = 0$, we obtain from Corollary 4.3 a contradiction that

$$2c < |\sigma_{(a,1]}(\tilde{M})| \leq 2c.$$

Hence $\hat{\beta}_2(U)$ or $\delta_2(U)$ must be $+\infty$.

Confirmation of (4). Let U be a universe. Assume that

$$\hat{\beta}_2(U) = c < +\infty \quad \text{and} \quad \delta(U) < +\infty.$$

By the proof of (3), for every infinite family of strongly c-efficient 3-manifolds of infinitely many ranks n any member must be type 1 embeddable to U. By the proof of (2), we have $\rho(U) = +\infty$ and $\hat{\beta}_1(U) = +\infty$.

Confirmation of (5). Since a full universe U is a type 1 and type 2 universe, the desired result follows from (2) and (3).

7. Infinities of the second rational homology groups of every 4D universe and every 4D punctured universe

In this section, it is shown that $\beta_2(U) = +\infty$ for any universe or punctured universe U. More precisely, the following theorem is shown.

Theorem 7.1. Let X be a non-compact oriented 4-manifold with the second Betti number $\beta_2(X) < +\infty$. Then there is a punctured 3-manifold $M^0 \in \mathbb{M}^0$ which is not embeddable in X.

The following corollary is direct from Theorem 7.1.

Corollary 7.2. For any universe or punctured universe U, we have $\beta_2(U) = +\infty$.

An outline of the proof of Theorem 7.1 given in [14] is as follows:

Let $\beta_2(X) = d < +\infty$. We show that there is $M \in \mathbb{M}$ such that M^0 is not embeddable in X. Suppose M^0 is in X for an $M \in \mathbb{M}$ with $\beta_1(M) = n$. The 2-sphere $S^2 = \partial M^0$ is a null-homologous 2-knot in X. Let X_M be the 4-manifold obtained from X by replacing a tubular neighborhood $N(K) = S^2 \times D^2$ by the product $D^3 \times S^1$. Then we have

$$\beta_2(X_M) = \beta_2(X) = d$$

and the closed 3-manifold M is embedded in X_M by a type 1 embedding.

We show that there is an $M \in \mathbb{T}_m$ with $m > d$ non-embeddable in X_M by a type 1 embedding.

Let X' be the 4-manifold obtained from X_M by splitting along M, and $B = \partial X' = M \times 1 \cup M \times (-1)$.

For the homomorphisms $i'_*, i_* : H_2(M;\mathbb{Q}) \to H_2(X';\mathbb{Q})$ induced from the natural maps $i' : M \to M \times (-1) \to X'$, $i : M \to M \times 1 \to X'$, let

$$C = \mathrm{im} i'_* \cap \mathrm{im} i* \subset H_2(X';\mathbb{Q}),\ C'_* = (i'_*)^{-1}(C),\ C_* = (i_*)^{-1}(C).$$

The following lemma is needed:

Lemma 7.3. Every closed 3-manifold $M \in \mathbb{T}_m$ with $m > d$ satisfies one of the following (1)-(3).

(1) The homomorphism i'_* or i_* is not injective,

(2) The homomorphisms i'_* and i_* are injective and $C'_* = C_* = 0$ or $C'_* \neq C_*$.
(3) The homomorphisms i'_* and i_* are injective and $C'_* = C^* \neq 0$ which has no $\mathbb{Q}$-basis $x_1, x_2, \ldots, x_s$ with $i'_*(x_i) = \pm i_*(x_i)$ for all i.

By assuming Lemma 7.3, an outline of the proof of Theorem 7.1 is as follows.

If i'_* and i_* are injective and $C_* = C'_* = 0$, then the natural homomorphism $H_2(M;\mathbb{Q}) \to H_2(X_M;\mathbb{Q})$ is injective. Since

$$\beta_1(M) = n = 3m > \beta_2(X_M) = \beta_2(X) = d,$$

we have a contradiction. Hence (2) implies $C'_* \neq C_*$. Then in either case, there is an end-trivial homomorphism $\gamma : H_1(X';\mathbb{Z}) \to \mathbb{Z}$ such that the restriction $\dot\gamma : H_1(B;\mathbb{Z}) \to \mathbb{Z}$ of γ is asymmetric. To see this, we use an analogous argument of [12, Section 5]. The inclusion $k : B \to X'$ is called a *loose embedding* if the homomorphism $k_* : H_2(B;\mathbb{Z}) \to H_2(X';\mathbb{Q})$ is not injective. By Lemma 7.3, the inclusion k is a loose embedding and there is a closed oriented 2-manifold F in B, called a *null-surface*, such that F bounds a compact connected oriented 3-manifold V in X' and the Poincaré dual element $\dot\gamma \in H^1(B;\mathbb{Z})$ of the homology class $[F] \in H_2(B;\mathbb{Z})$ is asymmetric. Then the 3-manifold V defines an end-trivial homomorphism

$$\gamma : H_1(X';\mathbb{Z}) \to \mathbb{Z}$$

by the intersection number $\mathrm{Int}_{X'}(x,[V]) \in \mathbb{Z}$ for every $x \in H_1(X';\mathbb{Z})$. Then the element $\dot\gamma \in H^1(B;\mathbb{Z})$ is a restriction of γ. Since

$$\hat\beta_2(X') \le \hat\beta_2(X) \le \beta_2(X) = d,$$

the inequality (3.1) of the signature theorem implies

$$|\sigma_{[a,1]}(\tilde B)| - \kappa_1(\tilde B) \le 2d$$

for all $a \in (-1,1)$. By a choice of a closed 3-manifold $M \in \mathbb{T}_m$ in Lemma 5.9, there is a number $b \in (-1,1)$ such that $|\sigma_{[b,1]}(\tilde B)| - \kappa_1(\tilde B) > 2d$, which is a contradiction. This completes the outline of the proof of Theorem 7.1 except for the proof of Lemma 7.3.

An outline of the proof of Lemma 7.3 is as follows:

Let $\tilde X$ be the infinite cyclic cover of X associated with the fundamental region $(X'; M \times (-1), M \times 1)$. Let $n = 3m$. Suppose that the following assertion is true:

(*) The homomorphisms i_* and i'_* are injective and $C'_* = C^* \neq 0$, which has a $\mathbb{Q}$-basis $x_1, x_2, \dots, x_s$ with $i'_*(x_i) = \pm i_*(x_i)$ for all i.

Then by the Mayer-Vietoris exact sequence, we have

$$H_2(\tilde X;\mathbb{Q}) \cong \Gamma^{d'} \oplus (\Gamma/(t+1))^{c(+)} \oplus (\Gamma/(t-1))^{c(-)},$$

for some non-negative integers d' and $c(\pm)$ such that

$$\dim_{\mathbb{Q}} C = c(+) + c(-) \le n,\ n - (c(+) + c(-)) \le d',\ d' + c(-) \le d,$$

so that $n - c(+) \le d$. Let Y be a compact 4-manifold such that $M \subset Y \subset X$ and the Γ-torsion part $\mathrm{Tor}_\Gamma H_2(\tilde Y;\mathbb{Q})$ of the homology Γ-module $H_2(\tilde Y;\mathbb{Q})$

has

$$\mathrm{Tor}_\Gamma H_2(\tilde{Y};\mathbb{Q}) = \mathrm{Tor}_\Gamma H_2(\tilde{X};\mathbb{Q}) \cong (\Gamma/(t+1))^{c(+)} \oplus (\Gamma/(t-1))^{c(-)}.$$

By the duality in [3], we have

$$\mathrm{Tor}_\Gamma H_1(\tilde{Y},\partial\tilde{Y};\mathbb{Q}) \cong (\Gamma/(t+1))^{c(+)} \oplus (\Gamma/(t-1))^{c(-)}.$$

Let

$$H_*(\tilde{Y},\partial\tilde{Y};\mathbb{Q}) = \mathrm{Tor}_\Gamma H_*(\tilde{Y},\partial\tilde{Y};\mathbb{Q}) \oplus FH_*(\tilde{Y},\partial\tilde{Y};\mathbb{Q})$$

be any splitting of a finitely generated Γ-module into the Γ-torsion part and Γ-free part, and

$$H^*(\tilde{Y},\partial\tilde{Y};\mathbb{Q}) = T^*(\tilde{Y},\partial\tilde{Y};\mathbb{Q}) \oplus F^*(\tilde{Y},\partial\tilde{Y};\mathbb{Q})$$

the $\mathbb{Q}$-dual splitting. Let $T^1(\tilde{Y},\partial\tilde{Y};\mathbb{Q})_{t+1}$ be the $(t+1)$-component of $T^1(\tilde{Y},\partial\tilde{Y};\mathbb{Q})$. For the natural homomorphism $k^* : T*(\tilde{Y},\partial\tilde{Y};\mathbb{Q}) \to H^*(M;\mathbb{Q})$, consider the following commutative square on cup products:

$$\begin{array}{ccc} T^1(\tilde{Y},\partial\tilde{Y};\mathbb{Q})_{t+1} \times T^1(\tilde{Y},\partial\tilde{Y};\mathbb{Q})_{t+1} & \overset{\cup}{\longrightarrow} & H^2(\tilde{Y},\partial\tilde{Y};\mathbb{Q}) \\ k^*\otimes k^* \downarrow & & k^* \downarrow \\ H^1(M;\mathbb{Q}) \times H^1(M;\mathbb{Q}) & \overset{\cup}{\longrightarrow} & H^2(M;\mathbb{Q}). \end{array}$$

Let Ω be the $\mathbb{Q}$-subspace of $H^2(M;\mathbb{Q})$ generated by the elements $k*(u\cup v) \in H^2(M;\mathbb{Q})$ for all $u,v \in T^1(\tilde{Y},\partial\tilde{Y};\mathbb{Q})_{t+1}$. Let

$$\begin{array}{rcl} j^* : H^2(\tilde{Y},\partial\tilde{Y};\mathbb{Q}) & \to & H^2(\tilde{Y};\mathbb{Q}), \\ (k')^* : T^2(\tilde{Y};\mathbb{Q}) & \to & H^2(M;\mathbb{Q}) \end{array}$$

be the natural homomorphisms. By a transfer argument of [4, Lemma 1.4], the homomorphism $(k')^* : T^2(\tilde{Y};\mathbb{Q}) \to H^2(M;\mathbb{Q})$ is injective. Since

$$k*(u\cup v) = (k')^* j^*(u\cup v) \in (k')^* T^2(\tilde{Y};\mathbb{Q})_{t-1},$$

we have

$$\Omega \cap (k')^* T^2(\tilde{Y};\mathbb{Q})_{t+1} = 0.$$

Hence the quotient map

$$\Omega \to H^2(M;\mathbb{Q})/(k')^* T^2(\tilde{Y};\mathbb{Q})_{t+1}$$

is injective. Since

$$T^2(\tilde{Y};\mathbb{Q})_{t+1} \cong (\Gamma/(t+1))^{c(+)},$$

we have

$$\dim_\mathbb{Q} \Omega \le \dim_\mathbb{Q} H^2(M;\mathbb{Q})/(k')^* T^2(\tilde{Y};\mathbb{Q})_{t+1} = 3m - c(+) \le d.$$

On the other hand, by a transfer argument of [4, Lemma 1.4], the homomorphism

$$k^* : T^1(\tilde{Y},\partial\tilde{Y};\mathbb{Q})_{t+1} \to H^1(M;\mathbb{Q})$$

is injective. Since

$$\dim_\mathbb{Q} T^1(\tilde{Y},\partial\tilde{Y};\mathbb{Q})_{t+1} = c(+),$$

the image $\Delta = k^*T^1(\tilde{Y}, \partial\tilde{Y}; \mathbb{Q})_{t+1}$ of the homomorphism k^* is a $\mathbb{Q}$-subspace of $H^1(M; \mathbb{Q})$ of codimension $d' = 3m - c(+) \leq d$. Since the cup product space $\Delta^{(2)}$ of Δ is equal to Ω, we have

$$\dim_{\mathbb{Q}} \Omega \geq 2m - d' \geq 2m - d.$$

Hence $2m - d \leq d$, that is $m \leq d$. This contradicts the inequality $m > d$. Thus, the assertion (*) is false. This completes the outline of the proof of Lemma 7.3.

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