# Gauss Diagrams, Unknotting Numbers and Trivializing Numbers of Spatial Graphs☆

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## Abstract

In this paper we introduce Gauss diagrams and four kinds of unknotting numbers of a spatial graph. R. Hanaki introduced the notion of pseudo diagrams and the trivializing numbers of knots, links and spatial graphs whose underlying graphs are planar. We generalize the trivializing numbers without the assumption that the underlying graphs are planar. Finally we discuss relations among the unknotting numbers and the trivializing numbers.

*Keywords:* Spatial graph, Gauss diagram, Γ-unknotting number, trivializing number, based trivializing number

## 1. Introduction

Spatial graphs are graphs embedded in 3-dimensional Euclidean space $\mathbb{R}^3$. As knots and links are encoded by Gauss diagrams with chords on circles, it is natural to extend similar ideas to encode spatial graphs using Gauss diagrams. In this paper, we first discuss a way to represent spatial graph diagrams using Gauss diagrams. In [1][2][3], T. Fleming and B. Mellor generalized the concept of Gauss codes for virtual spatial graphs. It seems important to encode spatial graphs using Gauss diagrams or Gauss codes when we treat them by computer, and it would be expected that many invariants of spatial graphs can be computed in terms of Gauss diagrams.

We introduce Gauss diagrams for spatial graphs in Section 2. The construction is similar to that of Gauss diagrams for knots. Some typical examples of the moves on Gauss diagrams corresponding to Reidemeister moves on spatial graph diagrams are shown. Some moves, RII moves and RIV moves, may change Gauss diagrams of spatial graph

diagrams to Gauss diagrams that do not correspond to spatial graph diagrams. We discuss this in Section 3. In Section 4, we discuss Gauss diagrams for based spatial graphs.

In Section 5, we introduce four kinds of unknotting numbers of a spatial graph: the unknotting number, the based unknotting number, the $\Gamma$-unknotting number and the based $\Gamma$-unknotting number. This is based on the idea discussed in [8]. R. Hanaki [5] introduced the notion of pseudo diagrams and the trivializing numbers for knots, links, and spatial graphs, where he assumed that the underlying graphs of spatial graphs are planar. A. Henrich et al. [6] gave a method of computing the trivializing number of a regular projection of a knot using Gauss diagram. In Section 6, we define four kinds of trivializing numbers: the trivializing number, the based trivializing number, the $\Gamma$-trivializing number, and the based $\Gamma$-trivializing number. In Section 7, we give inequalities among the unknotting numbers and the trivializing numbers.

## 2. Spatial graphs and Gauss diagrams

A *spatial graph* is a finite graph in $\mathbb{R}^3$. Two spatial graphs are said to be *equivalent* if they are ambiently isotopic. For simplicity, throughout this paper, we assume that a spatial graph is connected, there are no degree-0 vertices and no degree-1 vertices, and that there is at least one vertex whose degree is greater than two. Furthermore, we assume that a spatial graph is oriented, i.e., the edges are oriented.

Similar to a diagram of a knot, a regular projection of a spatial graph $G$ is obtained by projecting $G$ to a plane so that the multiple points are transverse double points away from vertices. A diagram of $G$ is a regular projection of $G$ in $\mathbb{R}^2$ with over/under information at each double point. A double point with over/under information is called a *crossing*. L. H. Kauffman [7] and D. N. Yetter [9] proved that two diagrams present equivalent spatial graphs if and only if they are related by the moves shown in Figure 1. We refer to these moves as Reidemeister moves for spatial graph diagrams.

A *Gauss diagram* of a knot diagram is an oriented circle identified with the source circle of an embedding into $\mathbb{R}^3$ whose image is the knot, and some chords attached to the circle whose endpoints correspond to over crossings and under crossings of the crossings. The chords are oriented from over crossings to under crossings. When a crossing of the diagram is denoted by $c$, then the endpoints of the corresponding chord will be denoted by $\overline{c}$ and $\underline{c}$, where $\overline{c}$ is the over crossing and $\underline{c}$ is the under crossing. Chords are assigned signs which are equal to the signs of the crossings. See Figure 2.

Reidemeister moves for Gauss diagrams are shown in Figure 3. When we apply moves of type II to Gauss diagrams of knots, we may obtain Gauss diagram that do not correspond to knot diagrams. Such Gauss diagrams correspond to virtual knot diagrams. In [4], M.Goussarov, M.Polyak, and O.Viro studied such Gauss diagrams.

We introduce the Gauss diagram of a spatial graph diagram. Let $G$ be a spatial graph and $D$ a diagram of $G$. The Gauss diagram is constructed as follows:

1. Corresponding to the edges of $D$, provide downward oriented strands in $\mathbb{R}^2$. These strands are identified with the edges of $D$ and are referred to as the same names.
2. For each crossing $c$ of $D$, let $\overline{c}$ and $\underline{c}$ denote the points on the strands corresponding to the over crossing $\overline{c}$ and the under crossing $\underline{c}$ of the crossing $c$, respectively. Connect the points $\overline{c}$ and $\underline{c}$ by a chord, which will be denoted by $\gamma_c$. We assume that the chord $\gamma_c$ is oriented from $\overline{c}$ to $\underline{c}$, and it is assigned a sign which is the sign of $c$ in $D$.
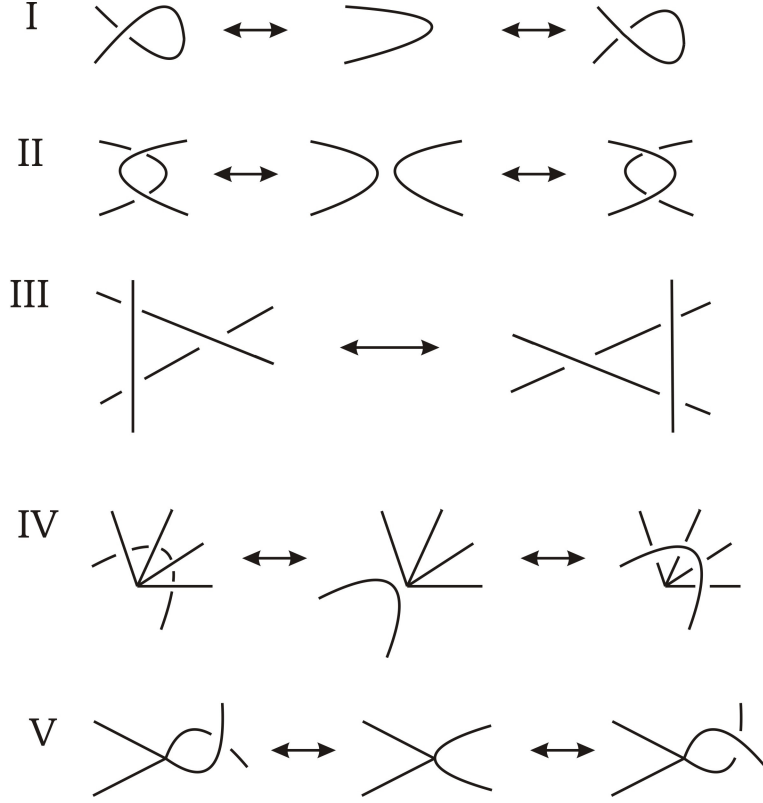
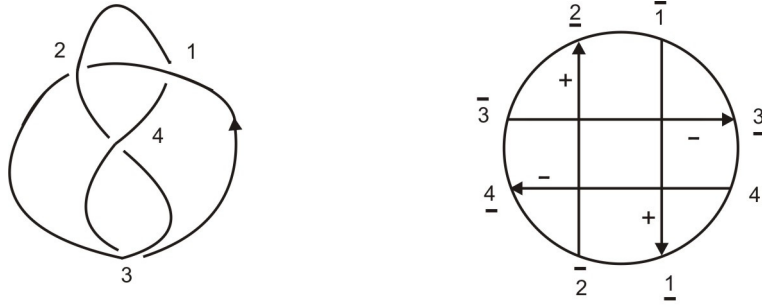Figure 1: Reidemeister moves for spatial graph diagrams



Figure 2: A diagram of the figure eight knot and its Gauss diagram

3. For each vertex, $X$, of $D$, consider a sequence of edges incident to $X$ appearing in anticlockwise direction, say $(\alpha_1\alpha_2\ldots\alpha_n)$. We denote this sequence by $ES(X)$ and call it the *edge sequence* at $X$. When $ES(X) = (\alpha_1\alpha_2\ldots\alpha_n)$, we often abbreviate it as $X(\alpha_1\alpha_2\ldots\alpha_n)$. The edge sequence $ES(X)$ is considered up to cyclic permutation.

The diagram depicted in Figure 4(a) is a diagram of $Kinoshita's$ $\theta$-curve, and its Gauss diagram is depicted in Figure 4(b).

When the edges of $G$ are ordered, we usually draw the Gauss diagram so that the strands appear in this order from the left. When the edges of $G$ are not ordered, the Gauss diagram should be considered up to changing the order of the strands in figure.

Reidemeister moves for spatial graph diagrams are translated into moves on Gauss diagrams, which we call Reidemeister moves for Gauss diagrams of spatial graphs. See
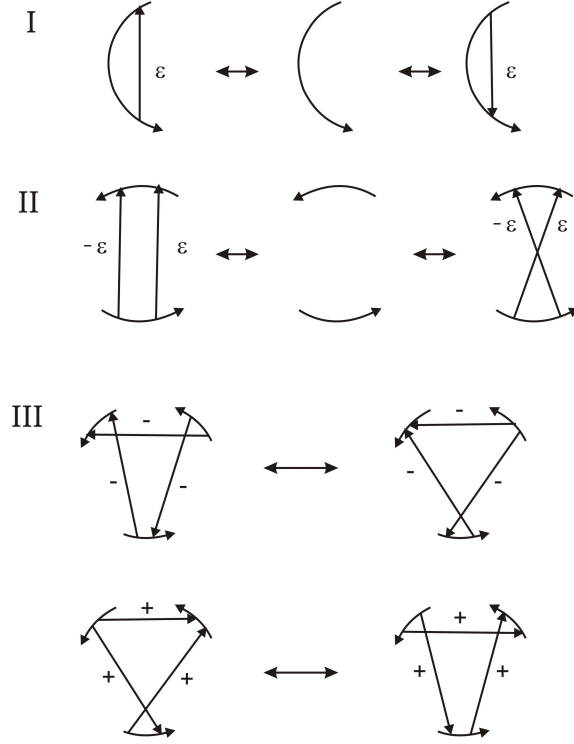
Figure 3: Reidemeister moves for Gauss diagrams of knots



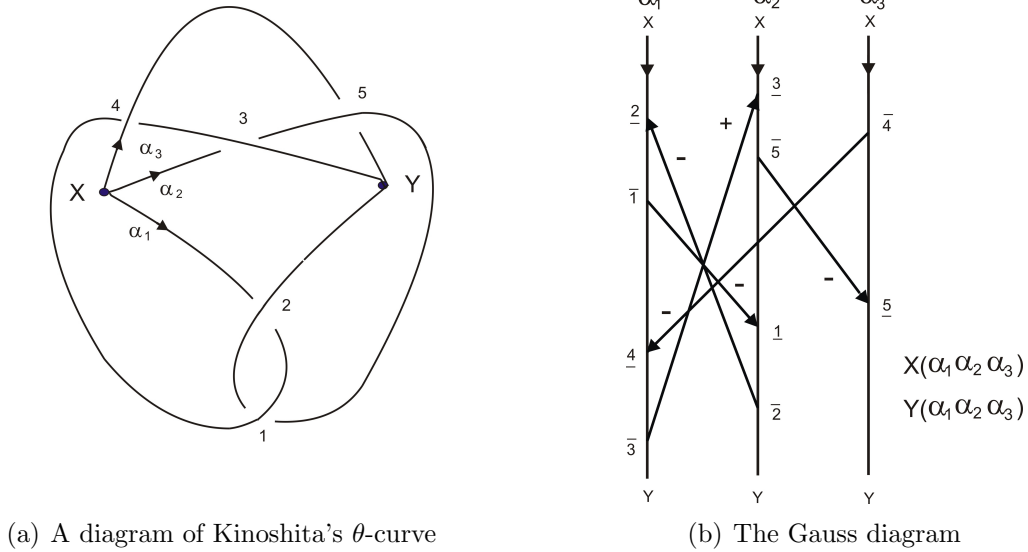(a) A diagram of Kinoshita's $\theta$-curve        (b) The Gauss diagram

Figure 4

Figures 5 and 6, where some typical examples of moves are depicted. Especially, for simplicity, we only show RIV moves there in the case that the edge and the vertex involved in the corresponding Reidemeister moves of type IV are $\alpha_k$ and a vertex $X$ with $ES(X) = (\alpha_1\alpha_2\ldots\alpha_n)$. The RV moves in Figure 6 correspond to Reidemeister moves of type V. The left two moves are Reidemeister moves of type V that involve one edge $\alpha_i$ of the graph. For the other four moves, $\alpha_i$ and $\alpha_j$ are a pair of consecutive adjacent edges at the vertex $X$, i.e, $ES(X) = (\ldots\alpha_i\alpha_j\ldots)$ or $(\ldots\alpha_j\alpha_i\ldots)$.
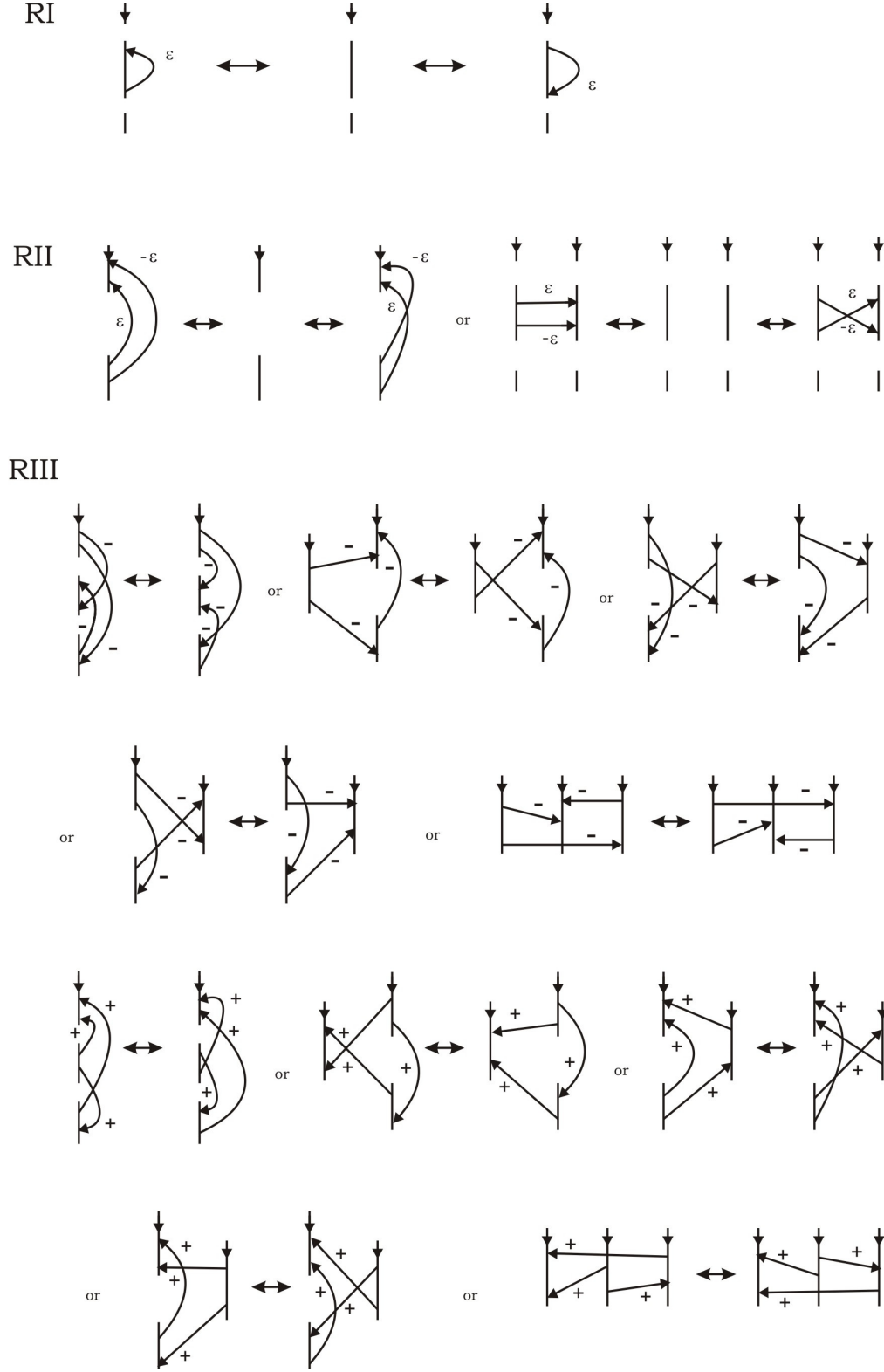
Figure 5: Moves on Gauss diagrams

## 3. On RIV moves and S-regions

When we apply an RII move or an RIV move to the Gauss diagram of a spatial graph diagram, the result might be a Gauss diagram that does not correspond to a spatial graph
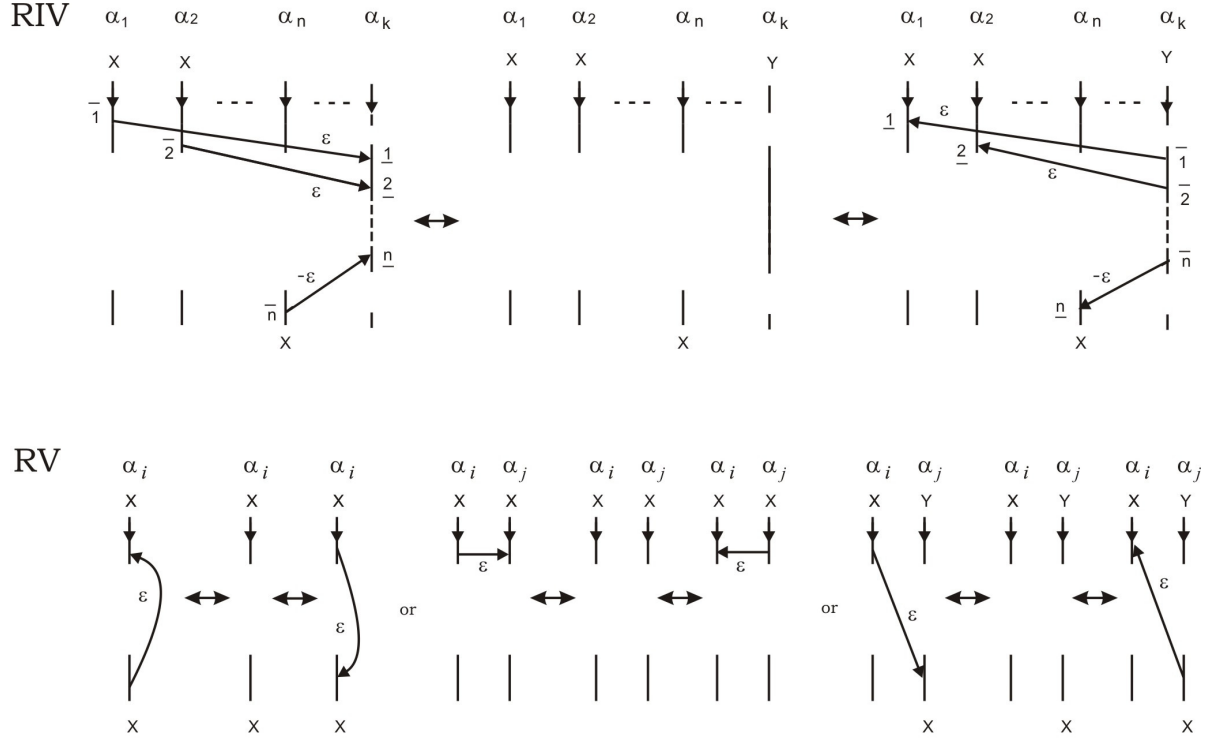
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Figure 6: Moves on Gauss diagrams

diagram. To avoid this situation, we need to identify which part of strands we can apply an RII move or an RIV move so that the result is the Gauss diagram of a spatial graph diagram.

For this purpose, we introduce a concept here.

**Definition 3.1.** A *region* of a spatial graph diagram $D$ is a connected component of the complement of the regular projection in $\mathbb{R}^2$ from which $D$ is obtained. For a vertex $X$ of $D$, an *S-region* at $X$ is a region of $D$ such that the boudary of the region contains $X$.

Note that for an S-region of a vertex $X$, an RIV move is applicable by using any edge of the boudary of the region so that the result is the Gauss diagram of a spatial graph diagram.

In the rest of this section, we discuss how to find the boundary of an S-region in terms of a Gauss diagram.

Let $D$ be a spatial graph diagram of a spatial graph $G$ and let $X$ be a vertex. Let $G_D$ be the Gauss diagram of $D$ and $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the edges of $G$ or the corresponding strands of $G_D$. Our aim is to identify S-regions at $X$ of $D$ in $G_D$. Let $ES(X) = (\alpha_{x1}\alpha_{x2}\ldots\alpha_{xn})$ be the edge sequence at $X$.

Step 1: Choose any two consecutive strands $\alpha_{xi}$ and $\alpha_{x(i+1)}$.
Step 2: Move along the strand $\alpha_{xi}$ (from $X$), till we reach an endpoint of a chord or an endpoint of the strand.
Step 3: (a) Suppose that we moved downward in step 2.

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1. If we reach an over crossing $\bar{c}$ of a chord $\gamma_c$, then move along the chord $\gamma_c$ to reach the under crossing $\underline{c}$, move along the strand downward (or upward, resp.) if the sign of chord is positive (or negative, resp.), till we reach another endpoint of a chord or an endpoint of the strand.

2. If we reach an under crossing $\underline{c}$ of a chord $\gamma_c$, then move along the chord $\gamma_c$ to reach the over crossing $\bar{c}$, move along the strand upward (or downward, resp.) if the sign of chord is positive (or negative, resp.), till we reach another endpoint of a chord or an endpoint of the strand.

(b) Suppose that we moved upward in step 2.
Do the same argument with the case (a) above, with 'upward' and 'downward' switched.

(c) If we reach a vertex $Y$ different from vertex $X$ along strand $\alpha$, then move along the adjacent strand from vertex $Y$, which is consecutive to $\alpha$ and is located in clockwise direction with respect to $\alpha$, till we reach an endpoint of the chord or the endpoint of the strand.

Step 4: Repeat step 3, till we reach vertex $X$ along the strand $\alpha_{x(i+1)}$. Then we obtain a sequence $X_{\alpha_{xi}}b_1 b_2 \ldots b_k X_{\alpha_{x(i+1)}}$, where

$$
b_i = \begin{cases} \bar{c} \text{ or } \underline{c}, & \text{for some crossing } c, \text{ or} \\ Y_\alpha & \text{for some vertex } Y \text{ which is an endpoint of a strand } \alpha. \end{cases}
$$

This sequence describes the loop presenting the S-region at $X$ between the edges $\alpha_{xi}$ and $\alpha_{x(i+1)}$ in $D$.

Step 5: Find all loops by applying step 1 to step 4 for $i = 1, 2, \ldots, n$. When $i = n$, assume $\alpha_{n+1} = \alpha_1$. We obtain all loops presenting boundaries of S-regions at vertex $X$ in $D$.

Figure 7 depicts the Gauss diagram of the spatial graph diagram shown in Figure 9. Applying the procedure described above, we have three loops as in Figure 8. They present the S-regions $S_1$, $S_2$ and $S_3$ at vertex $X$ in Figure 9.
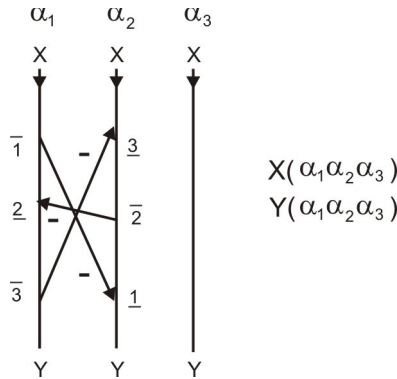


Figure 7

**Remark 3.2.** To identify which part of strands we can apply an RII move so that the result is the Gauss diagram of a spatial graph diagram, we need to find loops corresponding to boundaries of regions, not necessary $S$-regions. Such loops can be found by a similar procedure for $S$-regions by starting any point of a stand and moving upward or downward.
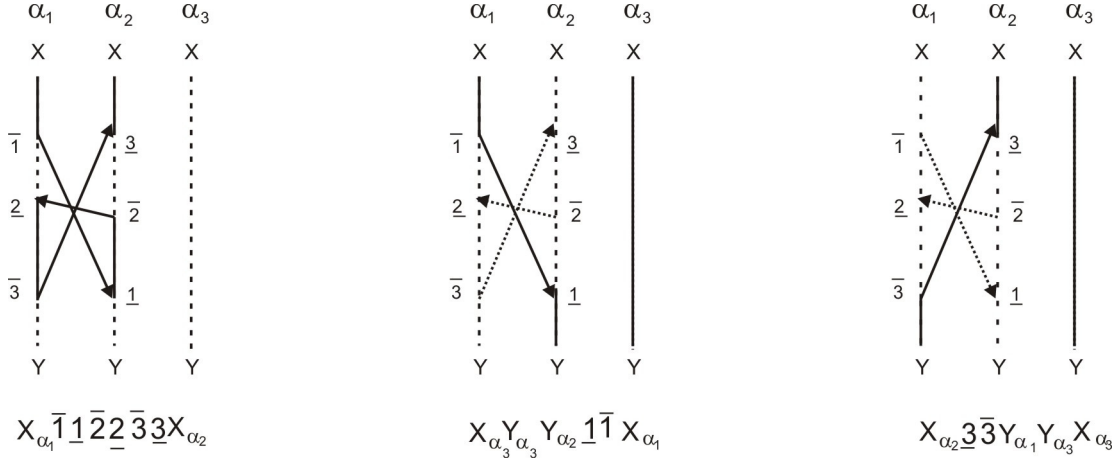
Figure 8: Loops in a Gauss diagram presenting S-regions

$$X_{\alpha_1}\overline{1}\underline{1}\,\overline{2}\underline{2}\,\overline{3}\underline{3}X_{\alpha_2} \qquad X_{\alpha_3}Y_{\alpha_3}Y_{\alpha_2}\underline{1}\overline{1}X_{\alpha_1} \qquad X_{\alpha_2}\underline{3}\overline{3}Y_{\alpha_1}Y_{\alpha_3}X_{\alpha_3}$$



Figure 9: A spatial graph diagram and S-regions at $X$

## 4. Based spatial graphs and Gauss diagrams

In this section, we recall the definitions of a based spatial graph, a based diagram, a monotone based diagram and the warping degree of a based diagram. We also discuss Gauss diagrams of based diagrams of spatial graphs.

A *basis* of a spatial graph $G$ is a maximal tree $T$ of $G$. The pair $(G,T)$ is called a *based spatial graph*. Let $\alpha_i(i = 1, 2, \ldots, m)$ be the edges of $G - T$. For a diagram $D$ of $G$, we denote by $DT$ and $D\alpha_i$ the sub-diagrams of $D$ corresponding to $T$ and $\alpha_i$, respectively. We call $DT$ the *tree diagram* and $D\alpha_i$ an *edge diagram*. Since we assume that a spatial graph is oriented, edge diagrams are oriented.

A diagram $D$ is called a *based diagram* (with basis $T$) if there are no crossings of $D$ on $DT$. A based diagram $D$ with basis $T$ is denoted by $(D;T)$. It is easily seen that for any based spatial graph $(G,T)$ and for any diagram $D$ of $G$, one can apply Reidemeister moves so that the result is a based diagram of $(G,T)$.

We say that an edge diagram $D\alpha_i$ is *monotone* if we walk along the oriented diagram $D\alpha_i$, for each crossing of $D\alpha_i$, we meet it at the first time as over crossing and then as under crossing. A based diagram $(D;T)$ of a based spatial graph $(G,T)$ is *monotone* if every edge diagram $D\alpha_i$ is monotone and after changing the numbering of $\alpha_i$ $(i = 1, 2, \ldots, m)$ suitably, for any $i$ and $j$ with $1 \leq i < j \leq m$, the edge diagram $D\alpha_i$ is over the edge diagram $D\alpha_j$.

**Definition 4.1 ([8]).** Let $(D;T)$ be a based diagram of a based spatial graph $(G,T)$. The *warping degree* $d(D;T)$ of $(D;T)$ is the least number of crossing changes on the edge

diagrams needed to obtain a monotone diagram from $(D; T)$.

By definition, the warping degree of $(D; T)$ is zero if and only if $(D; T)$ is monotone. When an order of edges of $G - T$ is given and fixed, the warping degree of $(D; T)$ with respect to the order is also defined. Obviously, this restricted version of the warping degree is greater than or equal to the warping degree without the restriction of the order of edges.

Let $(G, T)$ be a based spatial graph. Let $D$ be a diagram of $G$ and let $G_D$ be the Gauss diagram of $D$.

**Definition 4.2.** The $T$-*strands* are strands of $G_D$ corresponding to the tree diagrams of $D$, and hence corresponding to edges of $T$. The $R$-*strands* are strands of $G_D$ corresponding to edge diagrams of $D$, and hence corresponding to edges of $G - T$.

If $D$ is a based diagram with basis $T$, i.e., there are no crossings of $D$ over $DT$, then there are no chords of $G_D$ meeting the $T$-strands. Conversely, for a diagram $D$ of $G$, if there are no chords of $G_D$ meeting the $T$-strands, then the diagram $D$ is a based diagram with basis $T$.

Let $(G, T)$ be a based spatial graph. Given a diagram of $G$, one can transform it into a based diagram by Reidemeister moves. Such a transformation may be interpreted in terms of Gauss diagrams.

Figure 10(a) is the Gauss diagram of the spatial graph diagram shown in Figure 11(a). There are three vertices, $X$, $Y$ and $Z$. Strands $\alpha_1$ and $\alpha_2$ form a basis, and we regard $\alpha_1$ and $\alpha_2$ as $T$-strands. The RV move from (a) to (b) in Figure 10 corresponds to the move from (a) to (b) in Figure 11. The move from (b) to (c) in Figure 11 is a combination of a move of type IV and a move of type II. This corresponds to the move from (b) to (c) in Figure 10 which is a combination of an RIV move and an RII move.
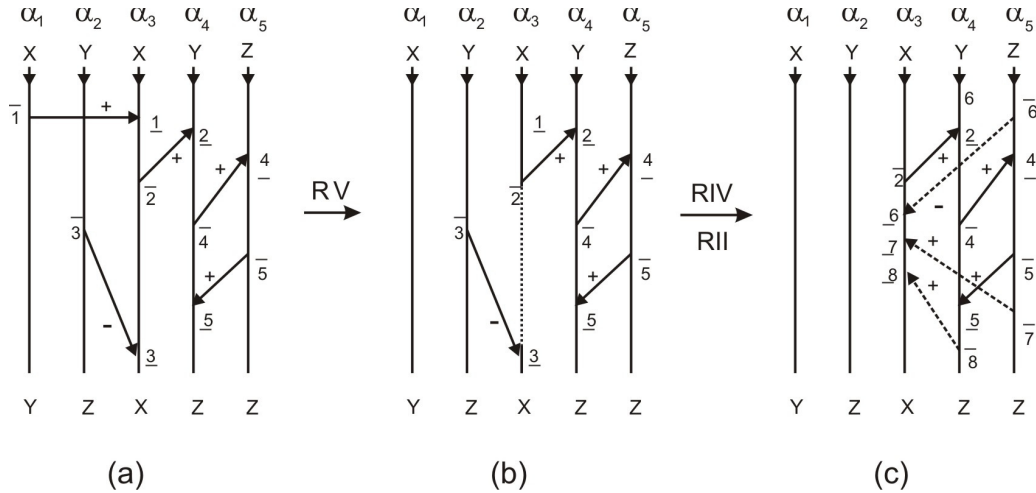


Figure 10

The following lemma is obvious from the definition, and we omit the proof. (In the lemma we assume that an order of edges of $G$ is given and the strands are drawn in the order from the left.)
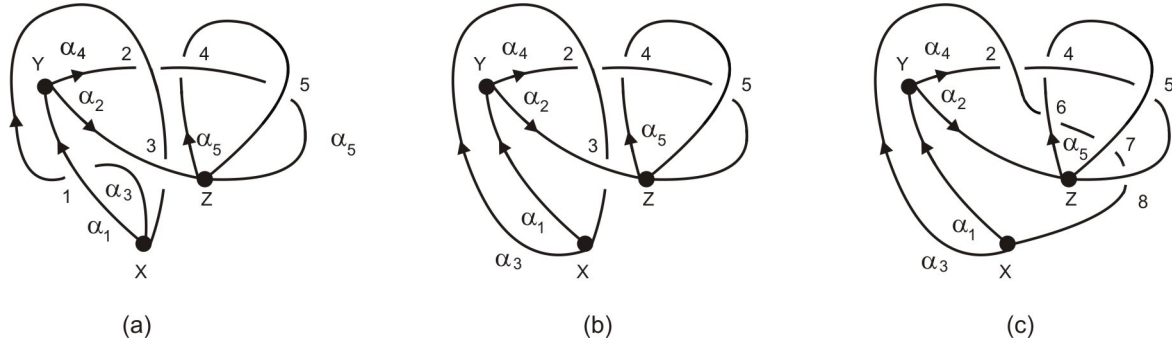
Figure 11

**Lemma 4.3.** *Let $D$ be a spatial graph diagram and $G_D$ the Gauss diagram of $D$. In $G_D$, if every chord is oriented from left to right when the endpoints are on distinct strands and if every chord is oriented downward when the endpoints are on the same strand, then the diagram $D$ is monotone. The converse is also true.*

Let $(G, T)$ be a based spatial graph and $(D; T)$ a based diagram of $(G, T)$. Let $G_D$ be the Gauss diagram of $D$. By Lemma 4.3, we see that $d(D; T)$ is the minimum number of changes of the orientations of chords in $G_D$ so that every chord is oriented as in the lemma.
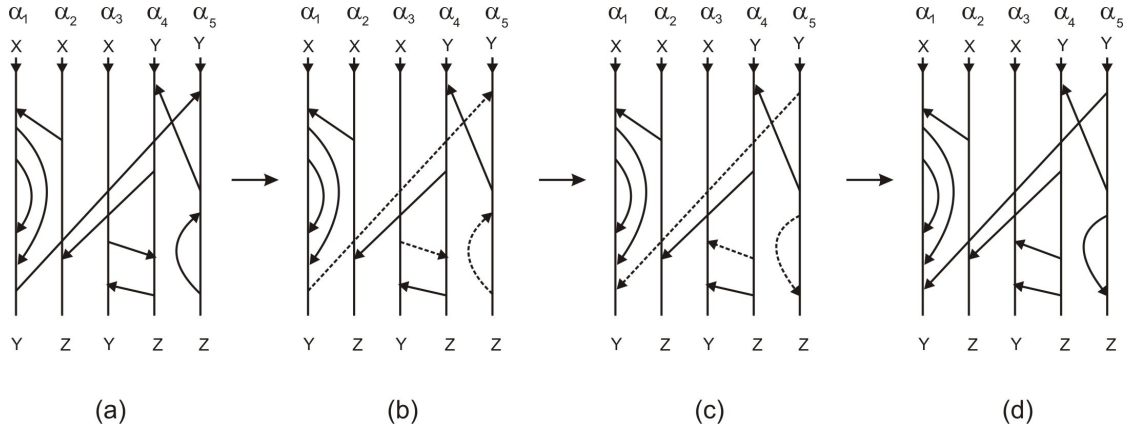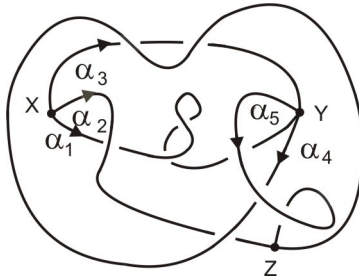


Figure 12



Figure 13: A spatial graph diagram corresponding to the Gauss diagram in Figure 12(d)

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The spatial graph diagram in Figure 13 corresponds to the Gauss diagram in Figure 12(d), which is monotone with respect to an order $(\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1)$ of the edges. When we redraw the Gauss diagram so that the strands appear in the order $(\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1)$ from the left, then the condition of Lemma 4.3 is satisfied.

The warping degree (with a fixed order of edges) of the diagram in Figure 13 is three when we fix the order $(\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1)$. This is seen from that the least number of changes of the orientations of chords in Figure 12(a) so that the condition of Lemma 4.3 holds is three after we redraw the Gauss diagram using the order $(\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1)$.

## 5. Unknotting numbers

In this section we introduce four kinds of unknotting numbers of a spatial graph: the unknotting number, the based unknotting number, the $\Gamma$-unknotting number, and the based $\Gamma$-unknotting number.

A based spatial graph $(G, T)$ is *unknotted* if there is a based diagram $(D; T)$ which is monotone.

**Definition 5.1 (cf. [8]).**    1. A spatial graph $G$ is *unknotted* if it is unknotted as a based spatial graph for a basis.
   2. The *unknotting number* $u(G)$ of a spatial graph $G$ is the minimal number of crossing changes needed to obtain a diagram of an unknotted spatial graph from a diagram of $G$.
   3. The *based unknotting number* $u_b(G)$ of a spatial graph $G$ is the minimal number of crossing changes needed to obtain a diagram of an unknotted spatial graph from a based diagram of $G$.

For a based diagram $(D; T)$ of $G$, let $c(D; T)$ denote the number of crossings of $(D; T)$. The *complexity* $cd(D; T)$ of $(D; T)$ is the pair $(c(D; T), d(D; T))$. The *complexity* $\gamma(G)$ of $G$ is the minimum (in the dictionary order) of the complexities $cd(D; T)$ for all based diagrams $(D; T)$ of $G$. (Here we consider all possible basis $T$ for $G$.) For a graph $\Gamma$, let $\gamma(\Gamma)$ be the minimum of $\gamma(G)$ for all spatial graphs $G$ whose underlying graph is $\Gamma$.

**Definition 5.2 (cf. [8]).**    1. A spatial graph $G$ is $\Gamma$-*unknotted* if $\gamma(G) = \gamma(\Gamma)$, where $\Gamma$ is the underlying graph of $G$.
   2. The $\Gamma$-*unknotting number* $u^{\Gamma}(G)$ of a spatial graph $G$ is the minimal number of crossing changes needed to obtain a diagram of a $\Gamma$-unknotted spatial graph from a diagram of $G$.
   3. The *based* $\Gamma$-*unknotting number* $u_b^{\Gamma}(G)$ of a spatial graph $G$ is the minimal number of crossing changes needed to obtain a diagram of a $\Gamma$-unknotted spatial graph from a based diagram of $G$.

In the definitions above, the based unknotting number $u_b(G)$ and the based $\Gamma$-unknotting number $u_b^{\Gamma}(G)$ are new in this paper, and the other notions are found in [8].

**Remark 5.3.** (cf. [8])    (1) For any spatial graph $G$,

$$u(G) \leq u_b(G), \quad u^{\Gamma}(G) \leq u_b^{\Gamma}(G), \quad u(G) \leq u^{\Gamma}(G), \quad \text{and} \quad u_b(G) \leq u_b^{\Gamma}(G).$$

The first two inequalities are trivial. The last two inequalities follow from the fact that a Γ-unknotted spatial graph is unknotted.

(2) When $G$ is a spatial graph whose underlying graph is planar, the following three conditions are equivalent: (i) $G$ is Γ-unknotted, (ii) $G$ is unknotted, and (iii) $G$ is equivalent to a spatial graph contained in a plane.

## 6. Trivializing numbers

In this section we introduce four kinds of trivializing numbers of a spatial graph: the trivializing number, the based trivializing number, the Γ-trivializing number, and the based Γ-trivializing number.

First we recall the notion of a pseudo diagram and the trivializing number for a knot introduced by R. Hanaki in [5]. A *pseudo diagram $Q$* of a knot means a regular projection of a knot such that some (or no) double points are equipped with over/under information and the other double points are not. Double points with over/under information are called *crossings* and double points without over/under information are called *pre-crossings*. A regular projection itself and a knot diagram are special cases of pseudo diagrams. By *resolving* a pre-crossing, we mean giving over/under information to the pre-crossing. A diagram $D$ is said to be *obtained from $Q$* if $D$ is obtained from $Q$ by resolving pre-crossings. A pseudo diagram $Q$ is called *trivial* (or *unknotted*) if every diagram obtained from $Q$ is a diagram of a trivial knot. The *trivializing number $tr(S)$* of a regular projection $S$ of a knot is the minimum number of pre-crossings of $S$ needed to obtain a trivial pseudo diagram by resolving the pre-crossings. A. Henrich et al. [6] showed that the trivializing number of a regular projection for a knot can be computed using Gauss diagrams. The Gauss diagram of a regular projection has chords without orientations and signs. Such Gauss diagrams are called *chord diagrams* in [6]. The *trivializing number $tr(K)$* of a knot $K$ is the minimum among $tr(S)$ for all regular projections $S$ of $K$.

R. Hanaki [5] also defined a pseudo diagram and the trivializing number for a spatial graph when the underlying graph is planar. A spatial graph $G$ whose underlying graph is planar is called *trivial* (or *unknotted*) if $G$ is equivalent to a spatial graph contained in a plane.

A *pseudo diagram $Q$* of a spatial graph is a regular projection of a spatial graph such that some (or no) double points are equipped with over/under information and the other double points are not. A pseudo diagram $Q$ of a spatial graph whose underlying graph is planar is said to be *trivial* (or *unknotted*) if every diagram obtained from $Q$ is a diagram of a trivial spatial graph. The *trivializing number $tr(S)$* of a regular projection $S$ of a spatial graph whose underlying graph is planar is the minimum number of pre-crossings of $S$ needed to obtain a trivial pseudo diagram by resolving the pre-crossings. The *trivializing number $tr(G)$* of a spatial graph $G$ whose underlying graph is planar is the minimum among $tr(S)$ for all regular projections $S$ of $G$.

In this section, we generalize these notions to spatial graphs without the assumption that underlying graphs are planar.

**Definition 6.1.** A pseudo diagram $Q$ of a spatial graph is *unknotted* (or Γ-*unknotted*, resp.) if every diagram obtained from $Q$ is a diagram of an unknotted spatial graph (or a Γ-unknotted spatial graph, resp.).

Note that when the underlying graph is planar, $Q$ is unknotted if and only if it is $\Gamma$-unknotted, and this is equivalent to that $Q$ is unknotted in Hanaki's sense.

**Definition 6.2.** For a regular projection $S$ of a spatial graph, the the *trivializing number* (or the $\Gamma$-*trivializing number*, resp.) of $S$ is the minimum number of pre-crossings of $S$ needed to obtain an unknotted pseudo diagram (or a $\Gamma$-unknotted pseudo diagram, resp.) from $S$ by resolving the pre-crossings. We denote it by $tr(S)$ (or $tr^{\Gamma}(S)$, resp.).

**Definition 6.3.** For a spatial graph $G$, the *trivializing number* (or the $\Gamma$-*trivializing number*, resp.) of $G$ is the minimum among $tr(S)$ (or $tr^{\Gamma}(S)$, resp.) for all regular projections $S$ of $G$. We denote it by $tr(G)$ (or $tr^{\Gamma}(G)$, resp.).

A regular projection $S$ of a spatial graph $G$ is called a *based projection* of $G$ if there is a basis $T$ of $G$ such that there are no double points of $S$ on the image of $T$ in $S$. A pseudo diagram $Q$ of a spatial graph $G$ is called a *based pseudo diagram* if there is a basis $T$ of $G$ such that there are no crossings and no pre-crossings of $Q$ on the image of $T$ in $Q$.

**Definition 6.4.** For a spatial graph $G$, the *based trivializing number* (or the *based* $\Gamma$-*trivializing number*, resp.) of $G$ is the minimum among $tr(S)$ (or $tr^{\Gamma}(S)$, resp.) for all based projections $S$ of $G$. We denote it by $tr_b(G)$ (or $tr_b^{\Gamma}(G)$, resp.).



(a)  A   $\Gamma$-unknotted pseudo diagram $Q$
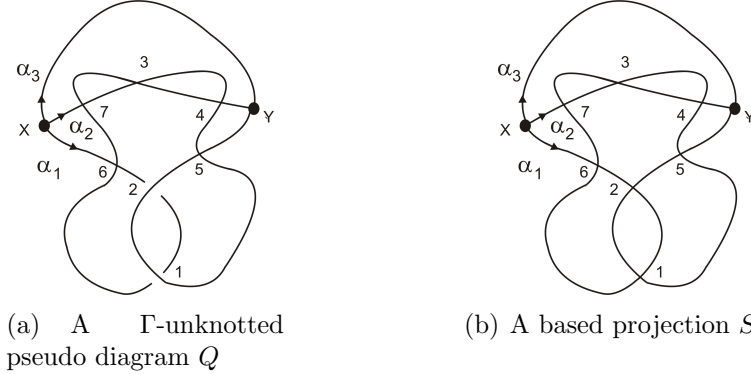
(b) A based projection $S$

Figure 14

Let $Q$ be the based pseudo diagram illustrated in Figure 14(a) and let $S$ be the based projection illustrated in Figure 14(b). The diagram $Q$ is $\Gamma$-unknotted. Since $Q$ is obtained from $S$ by resolving two pre-crossings, $tr^{\Gamma}(S) \leq 2$. It is directly verified that $tr^{\Gamma}(S)$ is neither 0 nor 1. Thus $tr^{\Gamma}(S) = 2$.

## 7. Inequalities among $\Gamma$-trivializing numbers and $\Gamma$-unknotting numbers

For any spatial graph $G$, there is an inequality between the $\Gamma$-unknotting number $u^{\Gamma}(G)$ and the $\Gamma$-trivializing number $tr^{\Gamma}(G)$ and a similar inequality for their based version.

**Theorem 7.1.** *For any spatial graph $G$,* $u^{\Gamma}(G) \leq \dfrac{tr^{\Gamma}(G)}{2}$ *and* $u_b^{\Gamma}(G) \leq \dfrac{tr_b^{\Gamma}(G)}{2}$.

PROOF. Let $n = tr^\Gamma(G)$ (or $n = tr_b^\Gamma(G)$, resp.) and let $S$ be a regular projection (or a based projection, resp.) of $G$ with $n = tr^\Gamma(G) = tr^\Gamma(S)$ (or $n = tr_b^\Gamma(G) = tr^\Gamma(S)$, resp.). Let $c_1, c_2, \ldots, c_n$ be pre-crossings of $S$ such that we obtain a $\Gamma$-unknotted pseudo diagram, say $Q$, from $S$ by resolving them.

Let $D$ be a diagram (or a based diagram, resp.) of $G$ that is obtained from $S$. Let $C$ be the subset of $\{c_1, c_2, \ldots, c_n\}$ such that $D$ and $Q$ have distinct over/under information at each pre-crossing in $C$ and they have the same over/under information at each pre-crossings in $\{c_1, c_2, \ldots, c_n\} - C$. Put $k = \#C$. Let $D'$ be the diagram obtained from $D$ by crossing changes over $C$. Then $D'$ is a diagram of a $\Gamma$-unknotted spatial graph. Thus, $u^\Gamma(G) \leq k$ (or $u_b^\Gamma(G) \leq k$, resp.).

Let $\overline{Q}$ be the pseudo diagram obtained from $S$ by resolving the same pre-crossings $c_1, c_2, \ldots, c_n$ in the opposite way to $Q$. Note that $\overline{Q}$ is also a $\Gamma$-unknotted pseudo diagram. Changing $Q$ with $\overline{Q}$ if necessary, we may assume that $0 \leq k \leq n/2$. Therefore we have $u^\Gamma(G) \leq tr^\Gamma(G)/2$ (or $u_b^\Gamma(G) \leq tr_b^\Gamma(G)/2$, resp.). $\qquad\square$

**Corollary 7.2.** Let $G$ be a spatial handcuff graph. Let $K_1$ and $K_2$ be the constituent knots of $G$ and put $L = K_1 \cup K_2$. Then

$$u(L) + u(K_1) + u(K_2) \leq u^\Gamma(G) \leq \frac{tr^\Gamma(G)}{2},$$

where $u(K_1)$ and $u(K_2)$ are unknotting numbers of $K_1$ and $K_2$, and $u(L)$ is the unlinking number of $L$, which is the minimum number of crossing changes between $K_1$ and $K_2$ needed to split $K_1$ and $K_2$.

PROOF. Let $n = u^\Gamma(G)$. Let $D$ be a diagram of $G$ such that there is a subset $C$ of crossings of $D$ with $\#C = n$ and the diagram obtained from $D$ by crossing changes over $C$ is a diagram $D'$ of a $\Gamma$-unknotted spatial graph $G'$. Let $K_1'$, $K_2'$ and $L'$ be the constituent knots and the constituent link of $G'$ corresponding to $K_1$, $K_2$ and $L$. Then $K_1'$ and $K_2'$ are trivial knots and $L'$ is a trivial link.

Divide $C$ into $C_1$, $C_2$ and $C_3$ such that crossings belonging to $C_1$ are crossings of $K_1$, crossings belonging to $C_2$ are crossings of $K_2$, and crossings belonging to $C_3$ are crossings between $K_1$ and $K_2$. Then $u(K_1) \leq \#C_1$, $u(K_2) \leq \#C_2$ and $u(L) \leq \#C_3$. Thus we have $u(L) + u(K_1) + u(K_2) \leq u^\Gamma(G)$. The other inequality follows from Theorem 7.1. $\qquad\square$

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