Knot Theory For Spatial Graphs Attached To A Surface

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ABSTRACT. Beside a survey on several unknotting notions of a spatial graph done earlier by the author, unknotting notions on a spatial graph with degree one vertices attached to a surface are introduced.

1. Introduction

In the ICTS program KNOT-2013 given at the Indian Institute of Science Education and Research (IISER) Mohali, the author delivered the following three lectures:

[Lecture 1] Topology for spatial graphs without degree one vertices

[Lecture 2] Unknotting notions on the spatial graphs

[Lecture 3] Spatial graphs with degree one vertices attaching to a surface

Topics on the first two lectures come from the author's earlier papers [**17, 18**] whose overview is also explained in this article. Topics on the third lecture are motivated to know knotting structures of a model tying two objects with different scales, or more concretely to understand knotting structures on a spatial graph whose degree one vertices are attached to a surface. For example, one asks the following question:

QUESTION 1.1. In what sense, the string in Figure 1 is "knotted" or "unknotted" ?

In the unknotting notions of this article, the answer will be "β-unknotted, but knotted, γ-knotted, Γ-knotted and (γ, Γ)-knotted", whose proof will be done in Section 8.

A protein attached to a cell surface such as a prion protein whose topological models are in Figure 2 (see[**19**]), and a string-shaped virus attached to a cell surface such as a virus of EBOLA haemorrhagic fever in Figure 3 [1] are scientific examples.

In a research of proteins, molecules, or polymers, it is important to understand geometrically and topologically spatial graphs possibly with degree one vertices

[1]http://www.scumdoctor.com/Japanese/disease-prevention/infectious-diseases/virus/ebola /Pictures-Of-The-Effects-Of-Ebola.html (Content Provider: CDC/ Dr. Frederick A. Murphy)
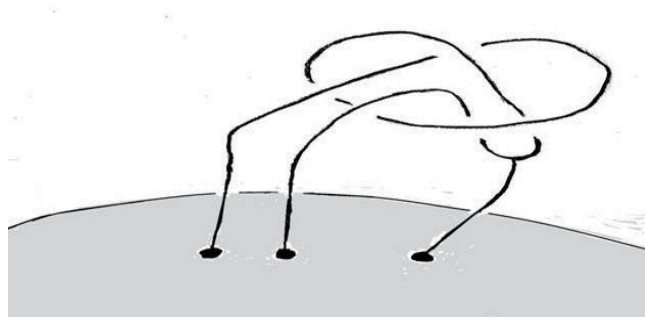
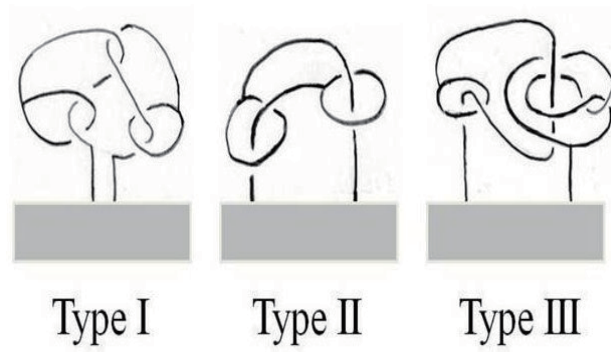FIGURE 1. A graph with degree one vertices attached to a surface
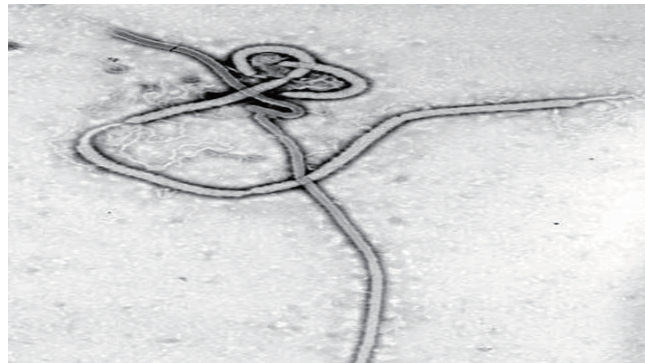

FIGURE 2. Topological models of prion proteins

FIGURE 3. A string-shaped virus of EBOLA haemorrhagic fever

including knotted arcs. From this reason, some numerical topological invariants of a spatial graph generalizing the warping degree and the unknotting number of knots and links are introduced.

In Section 2, the equivalence of a spatial graph without degree one vertices is explained. In Section 3, a monotone diagram, the warping degree, the complexity and the cross-index for a spatial graph without degree one vertices are explained. In Section 4, an unknotted graph and the induced unknotting number are explained for a spatial graph without degree one vertices and for a spatial graph with degree one vertices attaching to a surface. In Section 5, a β-unknotted graph and the induced unknotting number are explained for a spatial graph without degree one vertices and for a spatial graph with degree one vertices attaching to a surface. In Section 6, a homological invariant of an infinite cyclic covering of a spatial graph is discussed to estimate the β-unknotting number. In Section 7, a γ-unknotted graph and the induced unknotting number are explained for a spatial graph without degree one vertices and for a spatial graph with degree one vertices attaching to a surface. In Section 8, a Γ-unknotted graph and the induced unknotting number are explained for a spatial graph without degree one vertices and for a spatial graph with degree one vertices attaching to a surface. In Section 9, the values taken by these unknotting numbers are investigated. In Section 10, a notion of the knotting probability of a spatial graph with degree one free vertices is explained.
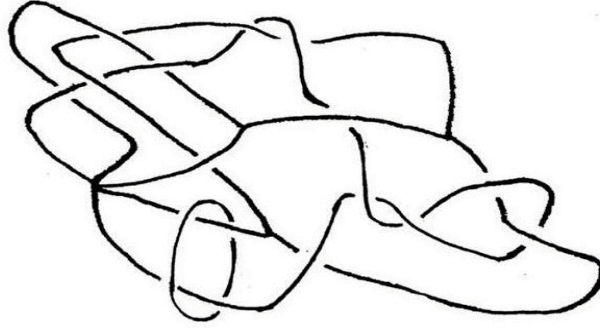

FIGURE 4. A diagram of a spatial graph

2. Equivalence of a spatial graph without degree one vertices

We begin with some basic terminologies on spatial graphs. Throughout this article, we do not consider graphs with degree zero vertices. A *spatial graph* of Γ is the image G of a topological embedding $\Gamma \to \mathbf{R}^3$ such that there is an orientation-preserving homeomorphism $h : \mathbf{R}^3 \to \mathbf{R}^3$ sending G to a polygonal graph in $\mathbf{R}^3$. We consider a spatial graph G by ignoring the degree two vertices which are useless in our topological argument. Let $v(G)$ be the set of vertices of degree ≥ 3 in G, and $v_1(G)$ the set of vertices of degree one in G. Let $\Gamma_i(i = 1, 2, \ldots, r)$ be an ordered set of the connected components of Γ, and $G_i = G(\Gamma_i)$ the corresponding spatial subgraph of $G = G(\Gamma)$. The spatial graph G is called a *link* if Γ is the disjoint union of finitely many loops, and it is *trivial* if it is the boundary of mutually disjoint disks. A *knot* is a link with one component. For a general reference of knots, links and spatial graphs, see the book [**15**] (specially, Chapter 15). A spatial graph G is *equivalent* to a spatial graph G' if there is an orientation-preserving homeomorphism $h : \mathbf{R}^3 \to \mathbf{R}^3$ such that $h(G) = G'$.

For a spatial graph G with $v_1(G) = \emptyset$, let $[G]$ be the class of spatial graphs G' which are equivalent to G. A *diagram DG* of a spatial graph G with $v_1(G) = \emptyset$ in $\mathbf{R}^3$ is an the image of G into a plane P under an orthogonal projection

$$\text{proj} : \mathbf{R}^3 \to P$$

with only double point singularities on edges of G together with the upper-lower crossing information (see Figure 4).

The fundamental result stated in L. H. Kauffman's paper [8] that the equivalence of spatial graphs can be described in terms of *generalized Reidemeister moves* (see Figure 5) on the diagrams of spatial graphs is explained here as Theorem 2.1 together with a simplified proof. We note that only the moves I, II,III are needed for knots and links in which case the moves I, II,III are simply called the *Reidemeister moves*.

THEOREM 2.1 (Equivalence Theorem). *Two spatial graphs G and G' with $v_1(G) = v_1(G') = \emptyset$ are equivalent if and only if any diagram DG of G is deformed into any diagram DG' of G' by a finite sequence of the generalized Reidemeister moves.*
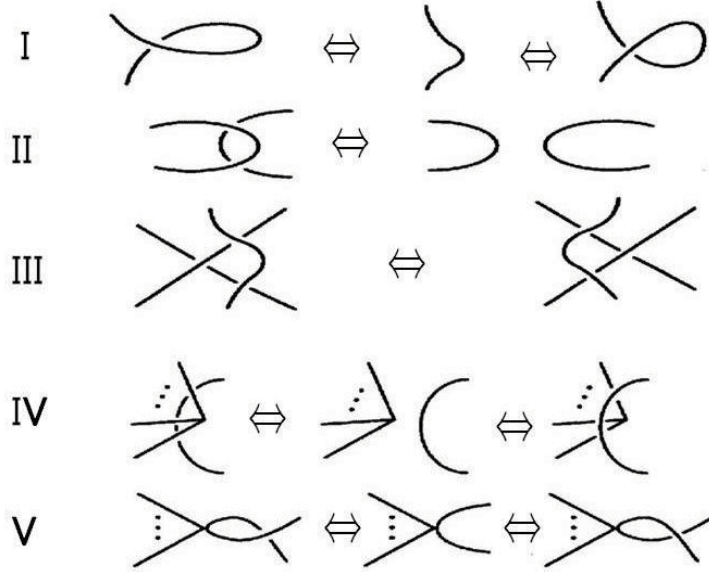
FIGURE 5. Generalized Reidemeister moves

PROOF. Let G and G' be equivalent spatial graphs, regarded as polygonal graphs. After some generalized Reidemeister moves on DG and DG', we can assume that there is an orientation-preserving homeomorphism $h : \mathbf{R}^3 \to \mathbf{R}^3$ such that $h(G) = G'$ and the restriction $h|_B$ is the identity $1_B : B \to B$ for a 3-ball neighborhood B of the set $v(G)$ of vertices of degree ≥ 3, so that in particular

we have $v(G) = v(G')$. Thus, there is a one-parameter family of piecewise-linear homeomorphisms

$$h_t : \mathbf{R}^3 \to \mathbf{R}^3 \quad (0 \le t \le 1)$$

such that h_0 is the identity $1_{\mathbf{R}^3} : \mathbf{R}^3 \to \mathbf{R}^3$, $h_1(G) = G'$ and $h_t|_{v(G)}$ is the identity on the set $v(G)$ for all t ($0 \le t \le 1$). Then we see from [6] that G' is obtained from G by a finite number of cellular moves, that is, a combination of a finite number of 2-simplex moves in Figures 6, 7. By a slight leaning of the plane P used for the orthogonal projection proj : $\mathbf{R}^3 \to P$, any diagram DG of G is deformed into any diagram DG' of G' by a finite sequence of the generalized Reidemeister moves. $\square$
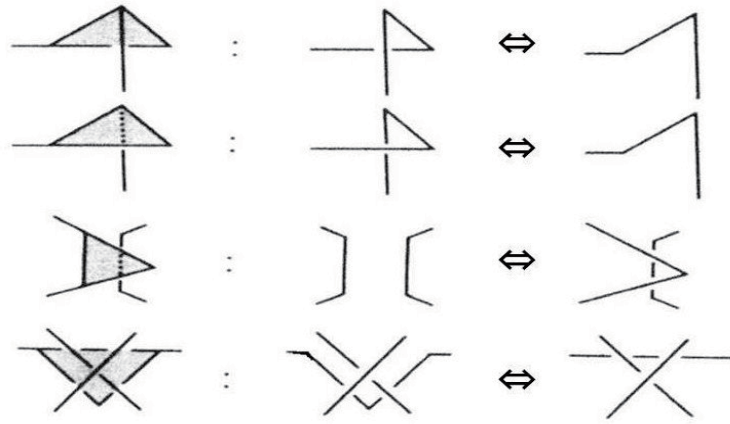

FIGURE 6. 2-simplex moves on generalized Reidemeister moves I, II, III
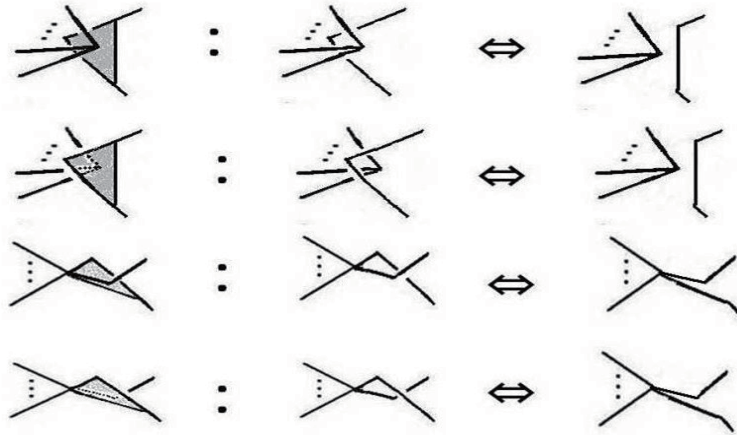

FIGURE 7. 2-simplex moves on generalized Reidemeister moves IV, V

Let $[DG]$ be the class of diagrams obtained from a diagram DG of a spatial graph G with $v_1(G) = \emptyset$ by the generalized Reidemeister moves, which is identified with the class $[G]$ by the equivalence theorem. The fundamental topological problems on spatial graphs are stated as follows, which are natural generalizations of the fundamental problems of knot theory:

(1) Study what kinds of spatial graphs there are. List them up to equivalences.

(2) Determine whether two given spatial graphs of a graph Γ are equivalent or not.

A basic question on the relationship between a spatial graph and knot theory is to ask how a spatial graph is related to knot theory. A *constituent knot* (or a *constituent link*, resp.) of a spatial graph G is a knot (or link, resp.) contained in G. The following proposition is direct from the definition of equivalence.

PROPOSITION 2.2. If two spatial graphs G^* and G are equivalent, then there is a graph-isomorphism $f : G^* \to G$ such that every constituent knot or link L^* of G^* is equivalent to the corresponding constituent knot or link $f(L^*)$ of G.

For an integer $d \geq 3$, a θ_d-*curve* is a spatial graph with 2 vertices and d edges each of which is tying the 2 vertices. A θ_3-curve is simply called a θ-curve. Any θ-curve equivalent to the θ-curve in Figure 8 is called a *trivial θ-curve*, which has the three trivial constituent knots. The θ-curve in Figure 9 has one trefoil constituent knot and two trivial constituent knots, which is a nontrivial θ-curve. Kinoshita's θ-curve in Figure 10 is known to be a nontrivial θ-curve with only trivial constituent knots. The arbitrary property of the constituent knot families of θ_d-curves is known by S. Kinoshita [20, 21].
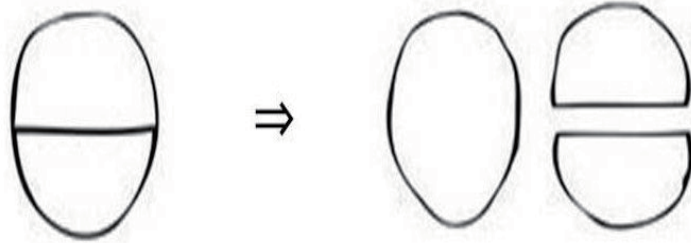
FIGURE 8. A trivial θ-curve and the three constituent knots

On the other hand, the following theorem is known by J. H. Conway and C. McA. Gordon in [2]:

PROPOSITION 2.3 (Conway-Gordon Theorem). Every spatial 6-complete graph K_6 contains a nontrivial constituent link. Every spatial 7-complete graph K_7 contains a nontrivial constituent knot.

The Conway-Gordon theorem and the following proposition suggest that the constituent knots or links are helpless to define an unknotted spatial graph for a general finite graph Γ.
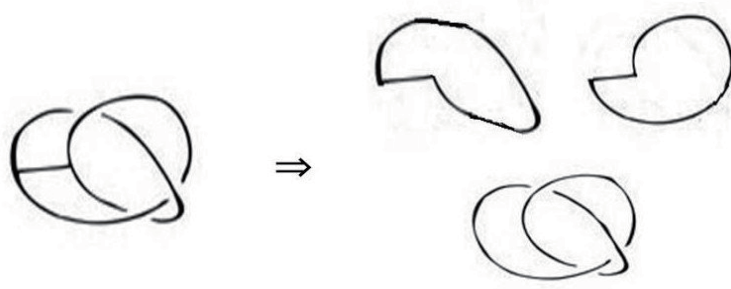
FIGURE 9. A nontrivial θ-curve and the three constituent knots
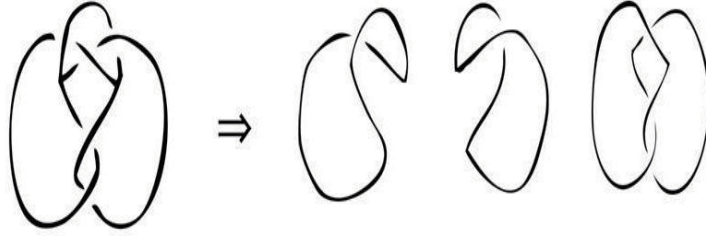
FIGURE 10. Kinoshita's θ-curve and the three constituent knots

PROPOSITION 2.4. For every spatial graph G with $v_1(G) = \emptyset$ except a knot or link, there is an infinite family of spatial graphs G^* (up to equivalences) with a graph-isomorphism $f : G^* \to G$ such that every constituent knot or link L^* of G^* is equivalent to the corresponding constituent knot or link $f(L^*)$ of G.

To show Proposition 2.4, we introduce a construction of topological imitations in [12] in a simplified setting. Let $S^3 = \mathbf{R}^3 \cup \{\infty\}$ be the 3-sphere, and $I = [-1, 1]$ the interval.

DEFINITION 2.5. A map $q : (S^3, G^*) \to (S^3, G)$ is a *normal imitation* if the composite

$$q : (S^3, G^*) \to \text{Fix}(\alpha) \subset (S^3, G) \times I \overset{\text{projection}}{\to} (S^3, G)$$

for an involution α on $(S^3, G) \times I = (S^3 \times I, G \times I)$ such that $\alpha(x, t) = (x, -t)$ for any $(x, t) \in S^3 \times \{\pm 1\} \cup N(G) \times I$, where $N(G)$ is a regular neighborhood of G in S^3.

The following properties of a normal imitation are standard (see [11]).

PROPERTIES 2.6. Let $q : (S^3, G^*) \to (S^3, G)$ be a normal imitation, and $N(G)$ a normal regular neighborhood of G in S^3. Then the statements (0)-(4) hold.

(0) The preimage $N(G^*) = q^{-1}(N(G))$ is a regular neighborhood of G^* such that the restriction $q|_{N(G^*)} : N(G^*) \to N(G)$ is a homeomorphism and $q(E(G^*)) =$

$E(G)$ for the exteriors $E(G^*) = \mathrm{cl}(S^3 \backslash N(G^*))$ and $E(G) = \mathrm{cl}(S^3 \backslash N(G))$ of the spatial graphs G^* and G, respectively.

(1) The map $q_1 : (S^3, G_1^*) \to (S^3, G_1)$ defined by q for any spatial graph G_1 in $N(G)$ and $G_1^* = q^{-1}(G_1)$ is a normal imitation.

(2) We have the same linking number $\mathrm{Link}_{S^3}(L^*) = \mathrm{Link}_{S^3}(L)$ for any oriented 2-component links L in $N(G)$ and $L^* = q^{-1}(L)$.

(3) The homomorphism $q_\# : \pi_1(S^3 \backslash G^*) \to \pi_1(S^3 \backslash G)$ on fundamental group is an epimorphism whose kernel $\mathrm{Ker}(q_\#)$ is a perfact group, i.e.,

$$\mathrm{Ker}(q_\#) = [\mathrm{Ker}(q_\#), \mathrm{Ker}(q_\#)].$$

(4) For normal imitations $q : (S^3, G^*) \to (S^3, G)$ and $q^* : (S^3, G^{**}) \to (S^3, G^*)$, there is a normal imitation $q^{**} : (S^3, G^{**}) \to (S^3, G)$.

The Kinoshita-Terasaka knot is an example of a normal imitation of a trivial knot (see [**11**]). We say that a normal imitation $q : (S^3, G^*) \to (S^3, G)$ is *homotopy-trivial* if there is a 1-parameter family $\{q_s\}_{0 \le s \le 1}$ of normal imitations $q_s : (S^3, G^*) \to (S^3, G)$ such that $q_0 = q$ and q_1 is a homeomorphism. The following notion is useful in constructing several nontrivial knots, links and spatial graphs.

DEFINITION 2.7. A normal imitation $q : (S^3, G^*) \to (S^3, G)$ is an *AID imitation* if the restriction

$$q|_{(S^3, \mathrm{cl}(G^* \backslash \alpha^*))} : (S^3, \mathrm{cl}(G^* \backslash \alpha^*)) \to (S^3, \mathrm{cl}(G \backslash \alpha))$$

is homotopy-trivial for every pair of an edge α of G and an edge α^* of G^* with $q(\alpha^*) = \alpha$.

The following proposition is a main result on the existence of AID imitations in [**12**].

PROPOSITION 2.8. For any spatial graph G with $v_1(G) = \emptyset$, there is an infinite family of AID imitations $q : (S^3, G^*) \to (S^3, G)$ such that the fundamental groups $\pi_1(E(G^*))$ of the exteriors $E(G^*)$ of the spatial graphs G^* with $v_1(G^*) = \emptyset$ are mutually non-isomorphic.

Proposition 2.4 is a direct consequence of Proposition 2.8. Further, combining Proposition 2.8 with a result in [**13**], we can add an additional property that every spatial graph G^* is obtained from G by one crossing change.

3. A monotone diagram, the warping degree, the complexity and the cross-index for a spatial graph without degree one vertices

Let G_i $(i = 1, 2, \ldots, r)$ be the connected components of a spatial graph G with $v_1(G) = \emptyset$. Let T_i be a maximal tree of G_i. By definition, $T_i = \emptyset$ if G_i is a knot, and T_i is one vertex if G_i has just one vertex of degree ≥ 3. The union $T = \cup_{i=1}^r T_i$ is called a *basis* of G, and the pair (G, T) a *based* spatial graph. The spatial graph G is obtained from a basis T by adding *edges* (consisting of arcs or loops) α_k $(k = 1, 2, \ldots, m)$. Let D be a diagram of G. Let DT and $D\alpha_k$ be the subdiagrams of D corresponding to the basis T and the edge α_k, respectively. The diagram D is a *based diagram* on a basis T and denoted by $(D; T)$ if there are no
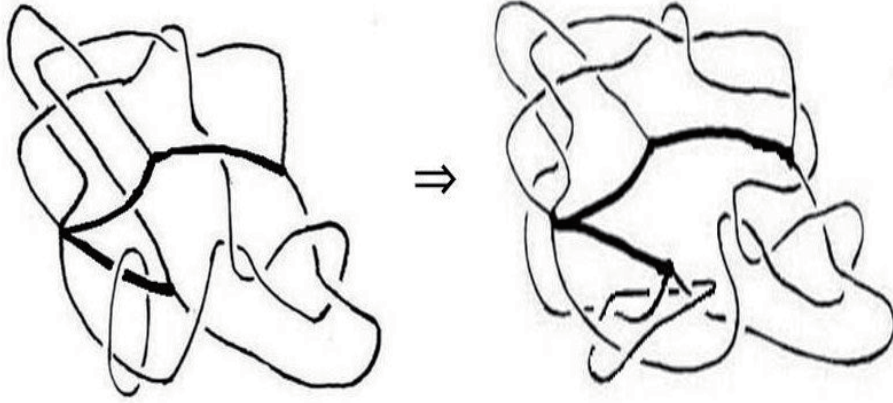
FIGURE 11. Deforming the diagram of Figure 4 into a based diagram

crossing points of D belonging to DT. Every diagram can be deformed into a based diagram by a finite sequence of the generalized Reidemeister moves (see Figure 11).

An edge diagram $D\alpha_k$ is *monotone* if there is an orientation on the edge α_k such that a point going along the oriented diagram $D\alpha_k$ from the origin vertex meets first the upper crossing point at every crossing point (see Figure 12), where a suitable non-crossing point is taken as a starting point if α_k is a loop.
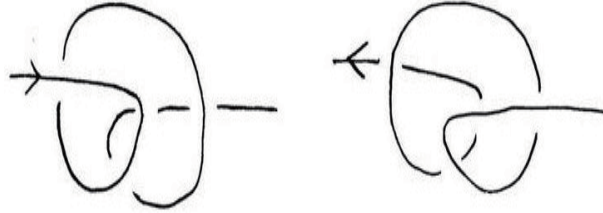

FIGURE 12. Monotone edge diagrams

A sequence of the edges α_k $(k = 1, 2, \ldots, m)$ for a based spatial graph (G, T) is *regularly ordered* if any edge belonging to a connected based graph component (G_i, T_i) is ordered to be smaller than any edge belonging to a connected based graph component $(G_{i'}, T_{i'})$ for every $i < i'$. A based diagram $(D; T)$ of a based spatial graph (G, T) is *monotone* if there is a regularly ordered edge sequence α_k $(k = 1, 2, \ldots, m)$ of (G, T) such that the edge diagram $D\alpha_k$ is monotone for all k and the edge diagram $D\alpha_k$ is upper than the edge diagram $D\alpha_{k'}$ for every $k < k'$. The *warping degree* $d(D; T)$ of a based diagram $(D; T)$ is the least number of crossing changes on the edge diagrams $D\alpha_k (k = 1, 2, \ldots, m)$ needed to obtain a monotone diagram from $(D; T)$ (see Figure 13). For $T = \emptyset$, we denote $d(D; T)$ by $d(D)$. When the edges α_k $(k = 1, 2, \ldots, m)$ are previously oriented, we can also

define the *oriented warping degree* $\vec{d}(D; T)$ (or $\vec{d}(D)$ for $T = \emptyset$) of a based diagram $(D; T)$ by considering only the crossing changes on the edge or loop diagrams $D\alpha_k$ ($k = 1, 2, \ldots, m$) along the specified orientations. Similar notions on links have been discussed by W. B. R. Lickorish and K. C. Millett [22], S. Fujimura [4], T. S. Fung [5], M. Okuda [26] and M. Ozawa [27] considering the ascending number of an oriented link. A. Shimizu [29, 30] also established a relationship between the warping degrees and the crossing number of a knot or link diagram. In particular, A. Shimizu characterized the alternating knot diagrams by establishing the inequality

$$\vec{d}(D) + \vec{d}(-D) \leq c(D) - 1$$

for every knot diagram D with crossing number $c(D) > 0$, where the equality holds if and only if D is an alternating diagram. For the present applications, we note the following relationships

$$\vec{d}(D\alpha) + \vec{d}(-D\alpha) = c(D\alpha), \quad d(D\alpha) = \min\{\vec{d}(D\alpha), \vec{d}(-D\alpha)\}$$

for an oriented edge diagram $D\alpha$ and the oppositely oriented edge diagram $-D\alpha$, where $c(D\alpha)$ denotes the crossing number of $D(\alpha)$. For example,

$$d\left(\; \raisebox{-0.5em}{\includegraphics[height=2em]{fig}} \; \right) = 1,$$

for

$$\vec{d}\left(\; \raisebox{-0.5em}{\includegraphics[height=2em]{fig}} \; \right) = 1 \quad \text{and} \quad \vec{d}\left(\; \raisebox{-0.5em}{\includegraphics[height=2em]{fig}} \; \right) = 3.$$

The *warping degree* $d(G)$ of a spatial graph G with $v_1(G) = \emptyset$ is the minimum of the warping degrees $d(D; T)$ for all based diagrams $(D; T) \in [DG]$. The *complexity* of a based diagram (D, T) is the pair $cd(D; T) = (c(D; T), d(D; T))$ together with the dictionary order. This notion was introduced in [16] for an oriented ordered link diagram. A. Shimizu observed that the dictionary order on $cd(D; T)$ is equivalent to the numerical order on $c(D; T)^2 + d(D; T)$ by using the inequality $d(D; T) \leq c(D; T)$.

The *complexity* of a spatial graph G with $v_1(G) = \emptyset$ is the minimum $\gamma(G) = (c_\gamma(G), d_\gamma(G))$ (in the dictionary order) of the complexities $cd(D; T)$ for all based diagrams $(D; T) \in [DG]$, where the topological invariants $c_\gamma(G)$ and $d_\gamma(G)$ are called the γ-*crossing number* and the γ-*warping degree* of G, respectively.

The *crossing number* of a spatial graph G with $v_1(G) = \emptyset$ is a non-negative integer given by $c(G) = \min_{D \in [DG]} c(D)$. By definition, we have the inequality

$$c(G) \leq c_\gamma(G).$$

The following properties (1) and (2) motivate a reason why we call $\gamma(G)$ the complexity of a spatial graph G with $v_1(G) = \emptyset$:

(1) If $d_\gamma(G) > 0$, then there is a crossing change on any based diagram $(D; T)$ of G with $cd(D; T) = \gamma(G)$ to obtain a spatial graph G' with $\gamma(G') < \gamma(G)$ (see Figure 13). If $d_\gamma(G) = 0$, then G is equivalent to G' with a monotone diagram $(D'; T')$ with $c(D'; T') = c_\gamma(G)$.

(2) If $c_\gamma(G) > 0$, then there is a spatial graph G' with $c_\gamma(G') < c_\gamma(G)$, so that $\gamma(G') < \gamma(G)$, by any splice on any based diagram (D, T) of G with $cd(D; T) = \gamma(G)$ (see Figure 13). If the crossing number $c_\gamma(G) = 0$, then $c(G) = 0$, i.e., G is equivalent to a graph in a plane $\subset \mathbf{R}^3$.
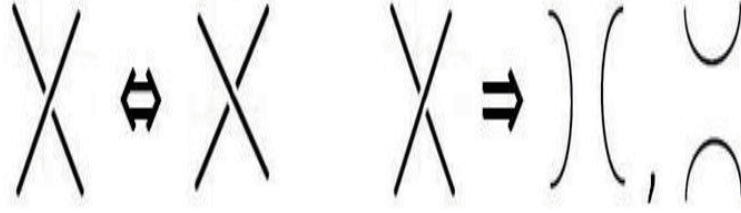
FIGURE 13. A crossing change in the left hand side and a splice in the right hand side

Let $D\alpha_k$ $(k = 1, 2, \ldots, m)$ be the edge diagrams of a based diagram $(D; T)$ of a spatial graph G with $v_1(G) = \emptyset$. For $k \neq k'$, let $\varepsilon(k, k')$ be 0 or 1 according to whether the crossing number between $D\alpha_k$ and $D\alpha_{k'}$ is even or odd (see Figure 14).
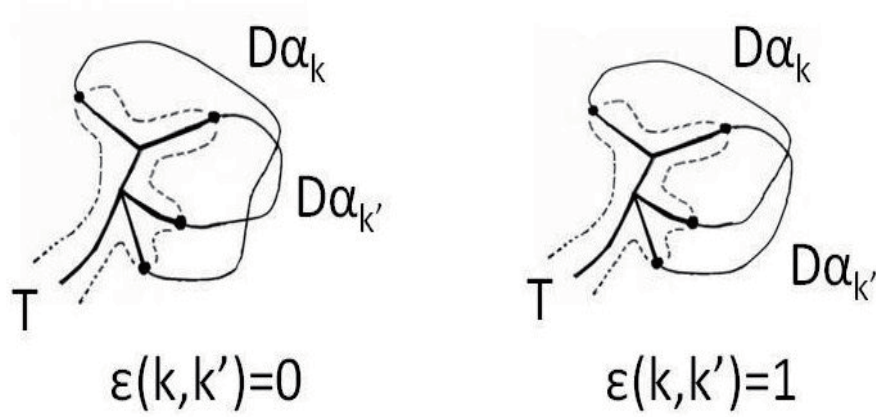
FIGURE 14. Cross indices of two kinds of edges

The *cross index* of a based diagram $(D; T)$ is the number

$$\varepsilon(D; T) = \sum_{1 \leq k < k' \leq m} \varepsilon(k, k').$$

We show the following lemma:

LEMMA 3.1. *Let G be a spatial graph of a finite graph Γ without degree one vertices. Then the number $\varepsilon(D; T)$ is independent of any choices, any crossing changes and any Reidemeister moves I, II, III of a based diagram $(D, T) \in [DG]$ fixing the basis diagram DT. Further, the inequality*

$$c(D; T) \geq \varepsilon(D; T)$$

holds and there is a spatial graph G_* of the finite graph Γ with a based diagram $(D_*; T) \in [DG_*]$ with the same basis T such that

$$c(D_*; T) = \varepsilon(D_*; T) = \varepsilon(D; T).$$

PROOF. By definition, we have $c(D; T) \geq \varepsilon(D; T)$. By crossing changes and Reidemeister moves, we can reduce the number $c(D; T)$ to attain the number $\varepsilon(D; T)$.   □

The minimum of cross indexes $\varepsilon(D; T)$ for all bases T of G is an invariant of the finite graph Γ which is called the *cross index* of Γ and denoted by $\varepsilon(\Gamma)$.
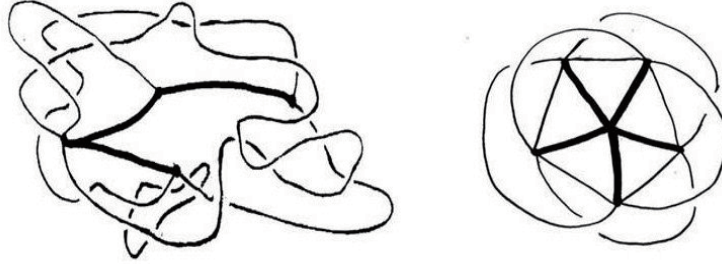

FIGURE 15. Unknotted spatial graphs

4. An unknotted graph and the induced unknotting number

We define that a spatial graph G with $v_1(G) = \emptyset$ is *unknotted* if the warping degree $d(G) = 0$. For example, see Figure 15 for some unknotted spatial graphs, where the figure in the left hand side is an unknotted spatial graph obtained from the based diagram of Figure 11 by crossing changes. This notion is related to some notions by T. Endo-T. Otsuki [3], R. Shinjo [31] and M. Ozawa and Y. Tsutsumi [27]. By definition, a link G is unknotted in this sense if and only if G is a trivial link. A θ_d-curve for every $d \geq 3$ is unknotted if and only if it is equivalent to a θ_d-curve embedded in a plane $\subset \mathbf{R}^3$.

The following properties on spatial graphs without degree one vertices are shown in [18].

PROPERTY 4.1. For every spatial graph G with $v_1(G) = \emptyset$ of a finite graph Γ, there are finitely many crossing changes on DG to make G with $d(G) = 0$.

PROPERTY 4.2. For every given finite graph Γ without degree one vertices, there are only finitely many spatial graphs G with $d(G) = 0$ of Γ up to equivalences.

PROPERTY 4.3. For a spatial graph G of every finite connected graph Γ without degree one vertices and with a vertex of degree ≥ 3 such that $d(G) = 0$, there is a tree basis T of G such that the spatial graph G/T obtained from G by shrinking T into a point is equivalent to a bouquet of circles embeddable in a plane P.

PROPERTY 4.4. A spatial graph G of every finite connected graph Γ without degree one vertices and with a vertex of degree ≥ 3 such that $d(G) = 0$ is deformed into a tree basis T of G by a sequence of edge reductions in Figure 16.
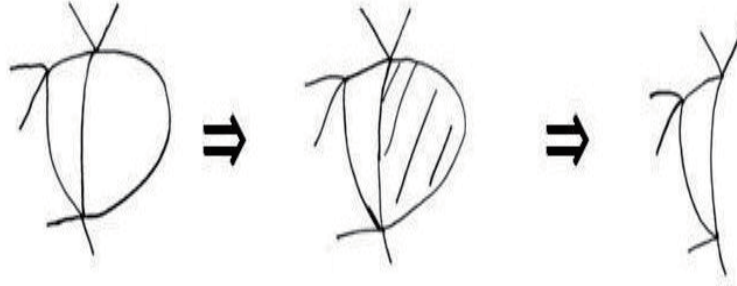
FIGURE 16. An edge reduction

PROPERTY 4.5. For a spatial graph G of every finite connected graph Γ without degree one vertices and with a vertex of degree ≥ 3 such that $d(G) = 0$, there is a tree basis T of G such that every edge (arc or loop) attaching to T is in a trivial constituent knot.

For example, an unknotted spatial 6-complete graph K_6 with a constituent Hopf link and an unknotted spatial 7-complete graph K_7 with a constituent trefoil knot are illustrated in Figure 17.

An unknotted K_6 An unknotted K_7
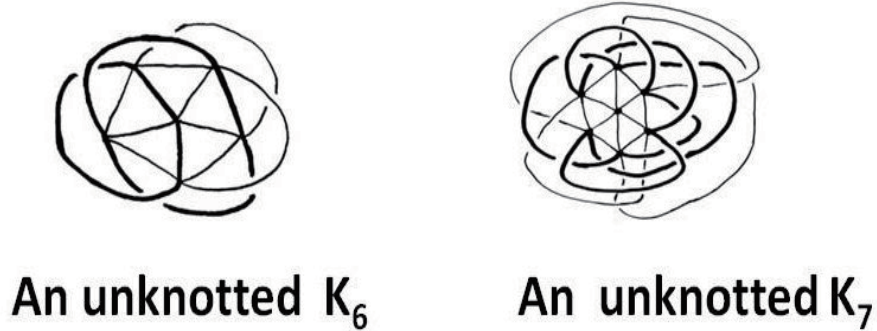
FIGURE 17. An unknotted K_6 and an unknotted K_7

Let O be the set of unknotted spatial graphs of a finite graph Γ without degree one vertices. The *unknotting number* of a spatial graph G of Γ is the distance $u(G)$ from G to the set O by crossing changes on the edges attaching to a basis T of G:

$$u(G) = \rho(G, O).$$

Next, this unknotting notion is generalized to a spatial graph with degree one vertices attached to a surface. Let F be a compact surface in $\mathbf{R}^3$ with the connected components F_j $(j = 1, 2, \ldots, s)$. A *spatial graph on F* of a finite graph Γ is a spatial graph G of Γ such that

(1) G meets F with $G \cap F = v_1(G)$,

(2) $G \setminus v_1(G)$ is contained in a connected component of $\mathbf{R}^3 \setminus F$, and

(3) there is a homeomorphism $h : \mathbf{R}^3 \to \mathbf{R}^3$ such that $h(G \cup F)$ is a compact polyhedron in $\mathbf{R}^3$.

Further, we impose the following mild conditions (4)-(5) on the spatial graph G and the surface F:

(4) F does not need $\partial F = \emptyset$.

(5) Although we grant that Γ, G or F are disconnected, assume that $|F_j \cap v_1(G)| \geq 2$ for every j.

A spatial graph G on a surface F is *equivalent* to a spatial graph G' on a surface F' if there is an orientation-preserving homeomorphism $h : \mathbf{R}^3 \to \mathbf{R}^3$ such that $h(F \cup G) = F' \cup G'$. A *shrinked spatial graph* of a spatial graph G on a surface F is a spatial graph $\hat{G}$ with $v_1(\hat{G}) = \emptyset$ in $\mathbf{R}^3$ obtained from G by shrinking a 2-cell Δ_j with

$$F_j \supset \Delta_j \supset F_j \cap V_1(G)$$

into a point for every j. We put the following definition.

DEFINITION 4.6. A spatial graph G on a surface F is *unknotted* if there is an unknotted shrinked spatial graph $\hat{G}$ of G.

We note that if F_j is a 2-sphere or 2-cell for every j, then the equivalence class $[\hat{G}]$ of all shrinked spatial graphs $\hat{G}$ of a spatial graph G on a surface F does not depend on any choices of 2-cells Δ_j $(j = 1, 2, \ldots, s)$. However, in a general surface F, the equivalence class $[\hat{G}]$ depends on a choice of 2-cells Δ_j $(j = 1, 2, \ldots, s)$ (see Figure 18).
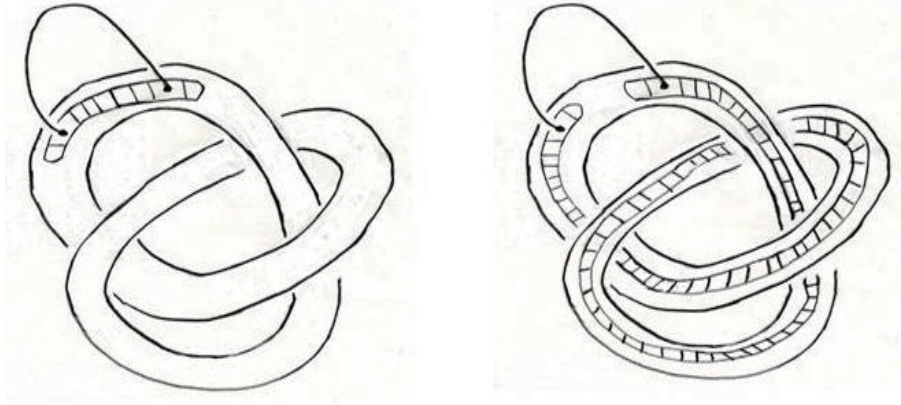
FIGURE 18. A trivial shrinked knot and a trefoil shrinked knot obtained by choices of a 2-cell

Every shrinked spatial graph $\hat{G}$ is a spatial graph of the same graph $\hat{\Gamma}$ obtained from Γ by shrinking $F_j \cap V_1(G)$ into a point for every j. The resulting finite graph $\hat{\Gamma}$ without degree one vertices is called the *finite shrinked graph of* Γ *associated with* F. From this observation and Property 4.2, we see the following lemma:

LEMMA 4.7. *For any given finite graph Γ with degree one vertices and any given surface F in $\mathbf{R}^3$, there are only finitely many unknotted spatial graphs G of Γ on the surface F up to equivalences.*

Let O_F be the set of unknotted spatial graphs of a finite graph Γ on a surface F. The *unknotting number* of a spatial graph G of a finite graph Γ on a surface F is the distance $u(G)$ from the set $\{\hat{G}\}$ of all shrinked spatial graphs $\hat{G}$ to the set O_F by crossing changes on the edges attaching to a basis:

$$u(G) = \rho(\{\hat{G}\}, O_F).$$

5. A β-unknotted graph and the induced unknotting number

Let G be a spatial graph G with $v_1(G) = \emptyset$, and T a basis of G with T_i $(i = 1, 2, \ldots, r)$ the connected components. Let B be the disjoint union of mutually disjoint 3-ball regular neighborhoods B_i of T_i in S^3 $(i = 1, 2, \ldots, r)$. Let $B^c = \text{cl}(S^3 \backslash B)$ be the complement domain of B, and $L = B^c \cap G$ be an m-string tangle in B^c consisting of mutually disjoint m arcs which is called the *complementary tangle* of the based graph (G, T). We put the following definition.

DEFINITION 5.1. A spatial graph G with $v_1(G) = \emptyset$ is *β-unknotted* if there is a basis T of G whose complementary tangle (B^c, L) is trivial, meaning that L is in a compact punctured 2-sphere properly embedded in B^c.

Here are some observations on β-unknotted spatial graphs.

NOTE 5.2. There are infinitely many β-unknotted spatial graphs G of the θ-curve Γ up to equivalences (see Figure 19).
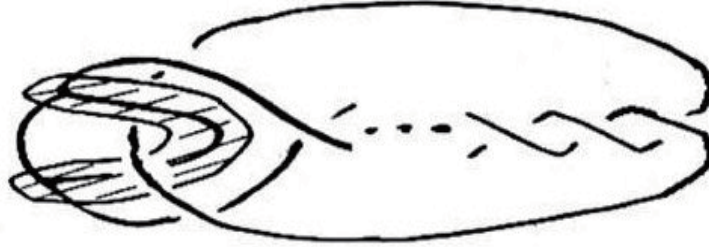
FIGURE 19. An infinite family of β-unknotted θ-curves

NOTE 5.3. Triviality of the complementary tangle (B^c, L) of a based spatial graph (G, T) with $v_1(G) = \emptyset$ depends on a choice of a basis T in general (see Figure 20).

NOTE 5.4. If a spatial graph G with $v_1(G) = \emptyset$ is β-unknotted, then G is a *free* spatial graph, namely a spatial graph with the fundamental group $\pi_1(\mathbf{R}^3 \backslash G)$ a free group. However, the converse is not true (see Figure 21).

FIGURE 20. Dependence of a choice of a basis

FIGURE 21. A free β-knotted spatial graph

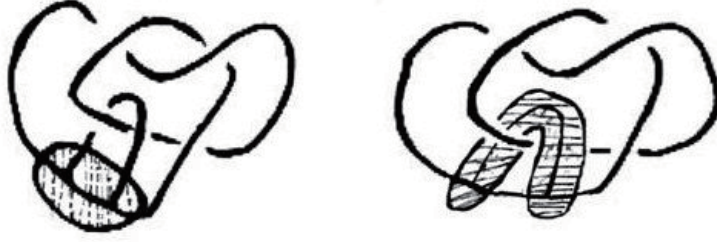Let O_β be the set of β-unknotted spatial graphs of a finite graph Γ without degree one vertices. The β-*unknotting number* of a spatial graph G of Γ is the distance $u_\beta(G)$ from G to O_β by crossing changes on edges attaching to a basis T of G:

$$u_\beta(G) = \rho(G, O_\beta).$$

Next, this β-unknotting notion is generalized to a spatial graph with degree one vertices attached to a surface.

DEFINITION 5.5. A spatial graph G on a surface F is β-*unknotted* if there is a β-unknotted shrinked spatial graph $\hat{G}$ in $\mathbf{R}^3$.

By definition, we have:

$$\text{unknotted} \quad \Rightarrow \quad \beta\text{-unknotted.}$$

Let $O_{F,\beta}$ be the set of β-unknotted spatial graphs of Γ on a surface F. The β-*unknotting number* of a spatial graph G of Γ on a surface F is the distance $u_\beta(G)$ from the set $\{\hat{G}\}$ of all shrinked spatial graphs $\hat{G}$ of G to the set $O_{F,\beta}$ by crossing changes on the edges attaching to a basis:

$$u_\beta(G) = \rho(\{\hat{G}\}, O_{F,\beta}).$$

6. A homological invariant of an infinite cyclic covering of a spatial graph without degree one vertices

Let G be a spatial graph with $v_1(G) = \emptyset$ in $S^3 = \mathbf{R}^3 \cup \{\infty\}$. Let T be a basis of G, and α_k $(k = 1, 2, \ldots, m)$ the edges attaching to T which are suitably

oriented. Let $E(G) = \text{cl}(S^3 - N(G))$ be the compact 3-manifold for a regular neighborhood $N(G)$ of G in S^3, which is called the *exterior* of G. Let

$$\chi : H_1(E(G)) \to \mathbf{Z}$$

be the epimorphism sending the meridians of α_k $(k = 1, 2, \ldots, m)$ to $1 \in \mathbf{Z}$. Let $\tilde{E}(G) \to E(G)$ be the infinite cyclic covering of $E(G)$ associated with χ. Let $\Lambda = \mathbf{Z}[t, t^{-1}]$. The homology $H_1(\tilde{E}(G))$ is a finitely generated Λ-module which we denote by $M(G, T; \chi)$. Let

$$\Lambda^a \to \Lambda^b \to M(G, T; \chi) \to 0$$

be an exact sequence (over Λ) for nonnegative integers b', b with $b' \geq b$. A matrix $A(G, T; \chi)$ over Λ representing the homomorphism $\Lambda^{b'} \to \Lambda^b$ is called a *presentation matrix* of the module $M(G, T; \chi)$. For a nonnegative integer $d \leq b$, the dth ideal $E_d(G, T; \chi)$ of the Λ-module $M(G, T; \chi)$ is defined to be the ideal generated by all the $(b - d)$-minors of $A(G, T; \chi)$, and for $d > b$, we define $E_d(G, T; \chi) = \Lambda$. The ideals $E_d(G, T; \chi)$ $(d = 0, 1, 2, 3, \ldots)$ are invariants of the Λ-module $M(G, T; \chi)$. Let Δ_d be a generator of the smallest principal ideal containing the ideal $E_d(G, T; \chi)$. Then the Laurent polynomial $\Delta_d \in \Lambda$ is called the *dth Alexamder polynomial* of the Λ-module $M(G, T; \chi)$. If G is a knot K (with $T = \emptyset$), then the 0th Alexander polynomial $\Delta_0 \in \Lambda$ is denoted by $\Delta_K(t)$ and called the *Alexander polynomial of* the *knot K*.

Assume that a spatial graph G^* is obtained from G by n crossing changes on α_k $(k = 1, 2, \ldots, m)$. Then χ induces the epimorphism $\chi^* : H_1(E(G^*)) \to \mathbf{Z}$. Let $m(G, T; \chi)$ and $m(G^*, T; \chi^*)$ be the minimal numbers of Λ-generators of the Λ-modules $M(G, T; \chi)$ and $M(G^*, T; \chi^*)$, respectively. The following lemma is a generalization of a result of [14] in the case of a knot or link and announced in [18, Lemma 3.3]:

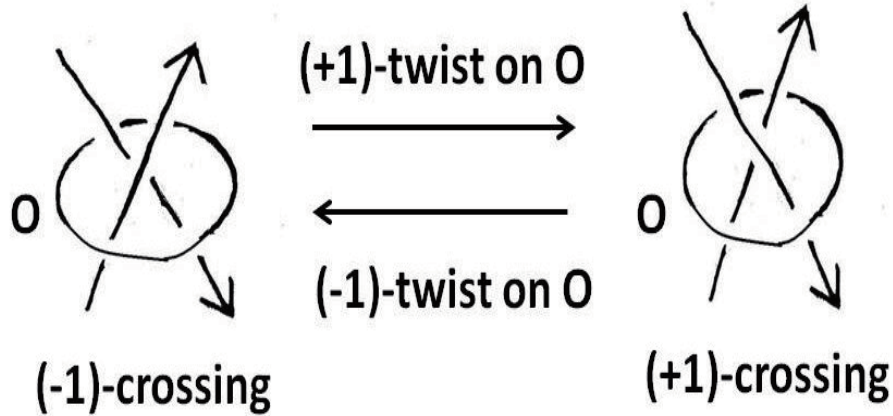LEMMA 6.1. $|m(G, T; \chi) - m(G^*, T; \chi^*)| \leq n.$

FIGURE 22. A zero-linking twist for a crossing change is the result of a (± 1)-framed 2-handle surgery along the loop O

PROOF. We note that the exterior $E(G^*)$ is obtained from the exterior $E(G)$ by surgeries of (± 1)-framed 2-handles $D^2 \times D_k^2$ $(k = 1, 2, \ldots, n)$ along zero-linking loops like a loop O in Figure 22. Let

$$W = E(G) \times [0, 1] \bigcup_{k=1}^{n} D^2 \times D_k^2$$

be the compact 4-manifold which is the surgery trace from $E(G)$ to $E(G^*)$ on the 2-handles $D^2 \times D_k^2$ $(k = 1, 2, \ldots, n)$, which is also the surgery trace from $E(G^*)$ to $E(G)$ on the "dual 2-handles" $D^2 \times D_k^2$ $(k = 1, 2, \ldots, n)$ (see Figure 23). By construction, χ and χ^* extend to an epimorphism $\chi^+ : H_1(W) \to \mathbf{Z}$. Let $(\tilde{W}; \tilde{E}(G), \tilde{E}(G^*))$ be the infinite cyclic covering triad of the triad $(W; E(G), E(G^*))$ associated with χ^+. Let $m(W; \chi^+)$ be the minimal number of Λ-generators of the Λ-module $H_1(\tilde{W})$. Because the natural homomorphisms $\pi_1(E(G)) \to \pi_1(W)$ and $\pi_1(E(G^*)) \to \pi_1(W)$ are onto, so that the natural homomorphisms $H_1(\tilde{E}(G)) \to H_1(\tilde{W})$ and $H_1(\tilde{E}(G^*)) \to H_1(\tilde{W})$ are onto. Thus, we have

$$m(W; \chi^+) \leq m(G, T; \chi) \quad \text{and} \quad m(W; \chi^+) \leq m(G^*, T; \chi^*).$$

By the exact sequence

$$H_2(\tilde{W}, \tilde{E}(G)) \to H_1(\tilde{E}(G)) \to H_1(\tilde{W}) \to 0$$

of the pair $(\tilde{W}, \tilde{E}(G))$ and the computation $H_2(\tilde{W}, \tilde{E}(G)) \cong \Lambda^n$ with a Λ-basis represented by the 2-handle cores $D^2 \times 0_k$ $(k = 1, 2, \ldots, n)$, we obtain

$$m(G, T; \chi) \leq n + m(W; \chi^+) \leq n + m(G^*, T; \chi^*).$$

Similarly, we have:

$$m(G^*, T; \chi^*) \leq n + m(W; \chi^+) \leq n + m(G, T; \chi).$$

Thus, we have
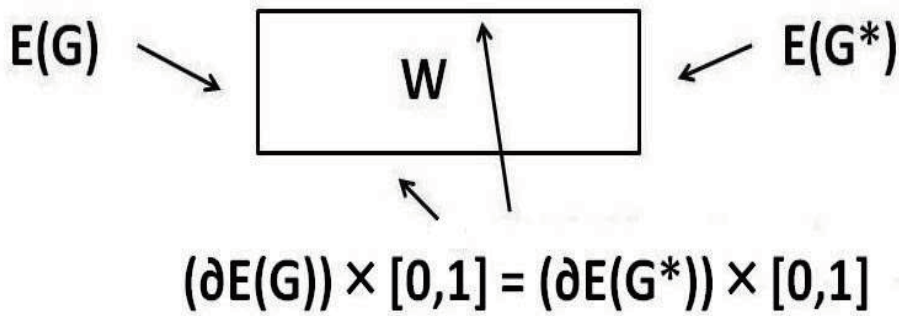
$$|m(G, T; \chi) - m(G^*, T; \chi^*)| \leq n.$$

$\square$

FIGURE 23. The surgery trace

7. A γ-unknotted spatial graph and the induced unknotting number

First, let G be a spatial graph with $v_1(G) = \emptyset$. Let $\gamma(G) = (c_\gamma(G), d_\gamma(G))$ be the complexity of G. A spatial graph G is γ-*unknotted* if $d_\gamma(G) = 0$.

Given a spatial graph G with $v_1(G) = \emptyset$, let $[D(G, \gamma)]$ be the set of based diagrams $(D; T) \in [DG]$ such that $c(D; T) = c_\gamma(G)$. Let $O(G, \gamma)$ be the set of γ-unknotted spatial graphs represented by a based diagram $(D; T)$ with $cd(D; T) = \gamma(G) = (c_\gamma(G), 0)$. Let $O(\gamma)$ be the union of the set $O(G, \gamma)$ for all spatial graphs G of Γ.

The γ-*unknotting number* of a spatial graph G with $v_1(G) = \emptyset$ is the distance $u_\gamma(G)$ from G to the set O of unknotted spatial graphs by crossing changes on the based diagrams $(D; T) \in [D(G, \gamma)]$:

$$u_\gamma(G) = \rho([D(G, \gamma)], O).$$

By definition, $u_\gamma(G) = 0$ if and only if G is γ-unknotted.

Next, this γ-unknotting notion is generalized to a spatial graph with degree one vertices attached to a surface.

DEFINITION 7.1. A spatial graph G on a surface F is γ-*unknotted* if there is a γ-unknotted shrinked spatial graph $\hat{G}$ in $\mathbf{R}^3$.

By definition, we have

$$\gamma\text{-unknotted} \quad \Rightarrow \quad \text{unknotted} \quad \Rightarrow \quad \beta\text{-unknotted}.$$

The γ-*unknotting number* of a spatial graph G on a surface F is the minimum $u_\gamma(G)$ of the γ-unknotting numbers $u_\gamma(\hat{G})$ for the set $\{\hat{G}\}$ of all shrinked spatial graphs $\hat{G}$ of G:

$$u_\gamma(G) = \min_{\hat{G} \in \{\hat{G}\}} u_\gamma(\hat{G}).$$

8. A Γ-unknotted spatial graph and the induced unknotting number

For a finite graph Γ without degree one vertices, let $\gamma(\Gamma)$ be the minimum of the complexities $\gamma(G)$ of all spatial graphs G of Γ. A spatial graph G of Γ is Γ-*unknotted* if $\gamma(G) = \gamma(\Gamma)$. Writing $\gamma(\Gamma) = (c_\gamma(\Gamma), d_\gamma(\Gamma))$, we have

$$d_\gamma(\Gamma) = 0.$$

Thus,

$$\Gamma\text{-unknotted} \Rightarrow \gamma\text{-unknotted} \Rightarrow \text{unknotted} \Rightarrow \beta\text{-unknotted}.$$

By definition, it is seen that $c_\gamma(\Gamma) = 0$ if and only if Γ is a plane graph and a spatial plane graph G is Γ-unknotted if and only if G is equivalent to a graph in a plane. Let O_Γ be the set of Γ-unknotted spatial graphs of Γ. Then we have

$$O_\beta \supset O \supset O_\Gamma.$$

The Γ-*unknotting number* of a spatial graph G of a finite graph Γ without degree one vertices is the distance $u_\Gamma(G)$ from G to the set O_Γ of Γ-unknotted spatial graphs of Γ by crossing changes on the edges attaching to a basis of G:

$$u_\Gamma(G) = \rho(G, O_\Gamma).$$

By definition, $u_\Gamma(G) = 0$ if and only if G is Γ-unknotted.

The (γ, Γ)-*unknotting number* $u_{\gamma,\Gamma}(G)$ of a spatial graph G of a finite graph Γ without degree one vertices is the distance from the set $[D(G, \gamma)]$ to O_Γ by crossing changes on the edges attaching to a basis:

$$u_{\gamma,\Gamma}(G) = \rho([D(G, \gamma)], O_\Gamma).$$

By definition, $u_{\gamma,\Gamma}(G) = 0$ if and only if G is (γ, Γ)-unknotted, and

(γ, Γ)-unknotted $\Rightarrow$ Γ-unknotted

$\Rightarrow$ γ-unknotted $\Rightarrow$ unknotted $\Rightarrow$ β-unknotted.

Next, the Γ-unknotting and (γ, Γ)-unknotting notions are generalized to a spatial graph with degree one vertices attached to a surface. Let Γ be a finite graph with degree one vertices.

DEFINITION 8.1. A spatial graph G of Γ on a surface F is Γ-*unknotted* if there is a $\hat{\Gamma}$-unknotted shrinked spatial graph $\hat{G}$ in $\mathbf{R}^3$ for the finite shrinked graph $\hat{\Gamma}$ of Γ associated with F.

The Γ-*unknotting number* of a spatial graph G on a surface F is the minimum $u_\Gamma(G)$ among the $\hat{\Gamma}$-unknotting numbers $u_{\hat{\Gamma}}(\hat{G})$ for the set $\{\hat{G}\}$ of all shrinked spatial graphs $\hat{G}$ of the finite shrinked graph $\hat{\Gamma}$ of Γ associated with F:

$$u_\Gamma(G) = \min_{\hat{G} \in \{\hat{G}\}} u_{\hat{\Gamma}}(\hat{G}).$$

The (γ, Γ)-*unknotting number* of a spatial graph G on a surface F is the minimum $u_{\gamma,\Gamma}(G)$ among the $(\gamma, \hat{\Gamma})$-unknotting numbers $u_{\gamma,\hat{\Gamma}}(\hat{G})$ for the set $\{\hat{G}\}$ of all shrinked spatial graphs $\hat{G}$ of the finite shrinked graph $\hat{\Gamma}$ of Γ associated with F:

$$u_\Gamma(G) = \min_{\hat{G} \in \{\hat{G}\}} u_{(\gamma,\hat{\Gamma})}(\hat{G}).$$

Since the introduction of all the unknotting notions is finished, we answer here Question 1.1 in the introduction.

ANSWER TO QUESTION 1.1. For the spatial graph G in Figure 1 on a surface F where only a disk part D of F is illustrated in Figure 1. The shrinked spatial graph $\hat{G} = G/D$ illustrated in Figure 24 is β-unknotted (see Figure 20) and hence G is β-unknotted.
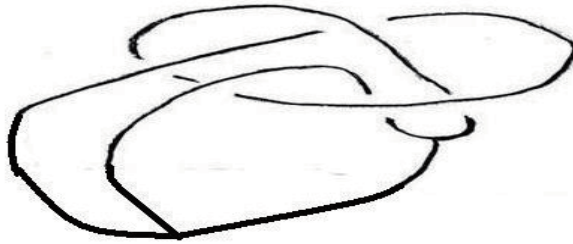

FIGURE 24. The shrinked spatial graph $\hat{G} = G/D$

On the other hand, the shrinked spatial graph $\hat{G}$ has a trefoil knot as a constituent knot. Any shrinked spatial graph of the spatial graph G on the surface F

is a degree 3 vertex connected sum $\hat{G}(\theta)$ of $\hat{G}$ and a θ-curve (see Moriuchi [**23**]), which has the trefoil knot as a connected direct summand. Hence $\hat{G}(\theta)$ is knotted, so that the spatial graph G on the surface F is (γ, Γ)-knotted, Γ-knotted, γ-knotted and knotted. $\qquad\square$

9. The values taken by these unknotting numbers

We show the following two theorems on the values taken by the unknotting numbers defined in Sections 4-8:

THEOREM 9.1. *The unknotting numbers*

$$u_\beta(G), u(G), u_\gamma(G), u_\Gamma(G), u_{\gamma,\Gamma}(G)$$

of any spatial graph G on any surface F satisfy the following inequalities:

$$u_\beta(G) \leq u(G) \leq \{u_\gamma(G), u_\Gamma(G)\} \leq u_{\gamma,\Gamma}(G).$$

Further, these unknotting numbers are distinct for some spatial graphs G on the 2-sphere $F = S^2$. In particular, the large-small relation on $u_\gamma(G)$ and $u_\Gamma(G)$ depends on a choice of spatial graphs G on $F = S^2$.

THEOREM 9.2. *For any given finite graph Γ, any surface F in $\mathbf{R}^3$ and any integer $n \geq 1$, there are infinitely many spatial graphs G of Γ on F such that*

$$u_\beta(G) = u(G) = u_\gamma(G) = u_\Gamma(G) = u_{\gamma,\Gamma}(G) = n.$$

We show Theorem 9.1.

PROOF OF THEOREM 9.1. The inequalities are direct from definitions. We show that these invariants are mutually distinct. Let G be a spatial graph on $F = S^2$ which is illustrated in Figure 25. The shrinked spatial graph $\hat{G}$ has $c_\gamma(\hat{G}) = 2$ and hence
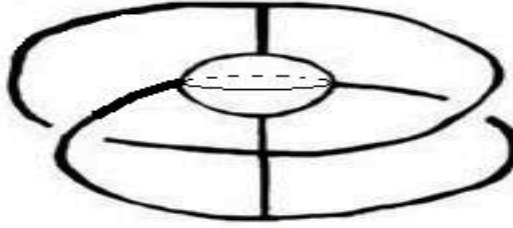
$$u_\beta(G) = u(G) = u_\gamma(G) = 0.$$

FIGURE 25. A γ-unknotted spatial graph on S^2

On the other hand, we have

$$u_\Gamma(G) = u_{\gamma,\Gamma}(G) = 1,$$

because the shrinked spatial graph $\hat{G}$ is a spatial graph of a plane graph $\hat{\Gamma}$ which has a Hopf link as a constituent link and hence not Γ-unknotted. Let G be a spatial graph on $F = S^2$, illustrated in Figure 26. Then the shrinked spatial graph $\hat{G}$ is the knot 10^8 which is known by Y. Nakanishi [**25**] and S. A. Bleiler [**1**] to be $u(10^8) = 2$

and $u_\gamma(10^8) = 3$ by the crossing changes at the dotted crossings in Figure 26. Since every knot or link K has $u_\beta(K) = u_\Gamma(K)$ and $u_\gamma(K) = u_{\gamma,\Gamma}(K)$ by definition, we have

$$u_\beta(G) = u(G) = u_\Gamma(G) = 2 < u_\gamma(G) = u_{\gamma,\Gamma}(G) = 3.$$
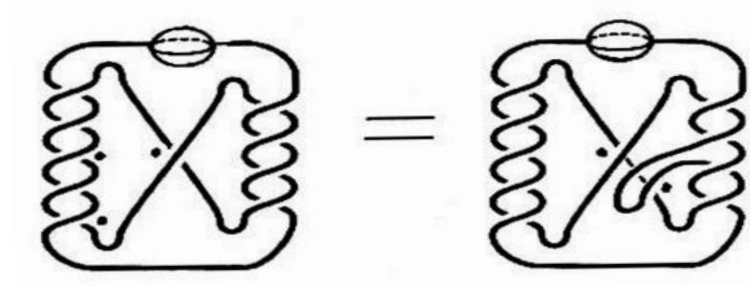

FIGURE 26. A spatial arc of the knot 10^8 on S^2

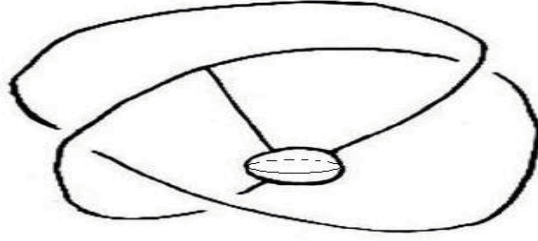FIGURE 27. A θ-like spatial graph on S^2

Next, let G be a spatial graph on $F = S^2$, illustrated in Figure 27. The shrinked spatial graph $\hat{G}$ is a β-unknotted spatial θ-curve (see Figure 28). Thus, $u_\beta(G) = 0$.
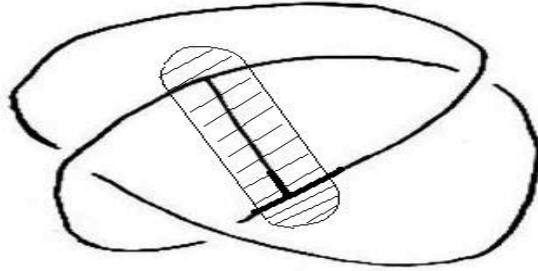

FIGURE 28. A β-unknotted θ-curve

On the other hand, if the θ-curve $\hat{G}$ is unknotted, then $\hat{G}$ would be isotopic to a graph in a plane $\subset \mathbf{R}^3$, which is impossible since $\hat{G}$ has a trefoil knot as a constituent knot. Thus, we have

$$u(G) = u\gamma(G) = u_\Gamma(G) = u_{\gamma,\Gamma}(G) = 1.$$

This completes the proof of Theorem 9.1. $\hfill\square$

Next, we show Theorem 9.2.

PROOF OF THEOREM 9.2. Assume that Γ and F are connected for simplicity. Let F be in the interior of a 3-ball $B \subset S^3$, and $S^2 = \partial B$. Let G_0 be a Γ-unknotted graph on S^2 in $B^c = \mathrm{cl}(S^3 - B)$. For a disk $\Delta_0 \subset S^2$ with $v_1(G_0) \subset S^2$, let $\hat{G}_0 = G_0/\Delta_0$ be the shrinked spatial graph which is a $\hat{\Gamma}$-unknotted spatial graph in S^3 with a monotone based diagram $(D_0; T_0)$ such that the crossing number $c(D_0; T_0)$ is equal to the cross index $\varepsilon(\hat{\Gamma})$ by Lemma 3.1, where $\hat{\Gamma}$ is the finite shrinked graph of Γ associated with the disk Δ_0. Let $K(n)$ be the n-fold connected sum of a trefoil knot K, and $DK(n)$ a diagram of the knot $K(n)$ with minimal crossing number. Since $c(DK(n)) \leq 3n$ is obvious, we obtain the crossing number $c(DK(n)) = 3n$ from the following lemma (which is shown later).

LEMMA 9.3. *Let K'' be the connected sum of the knot $K(n)$ and a (possibly trivial) knot K'. Then any diagram D'' of the knot K'' has the crossing number $c(D'') \geq 3n$.*
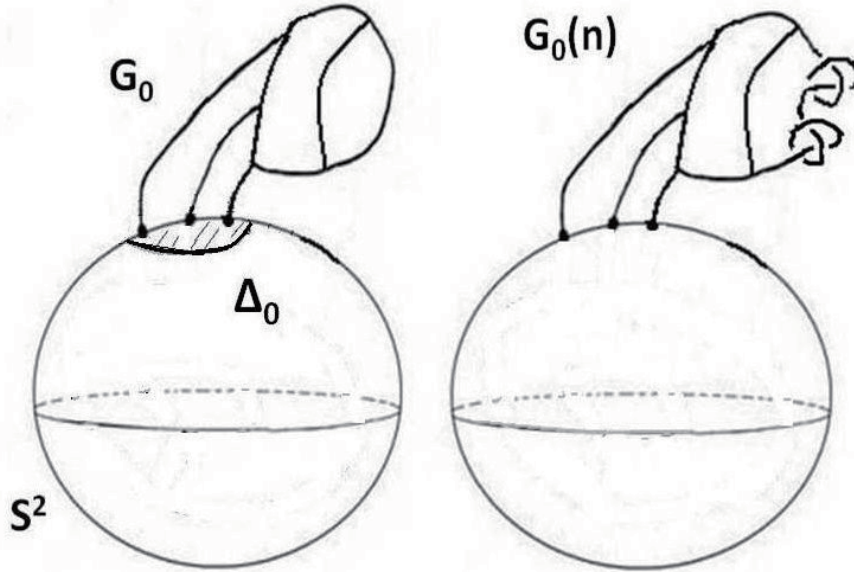

FIGURE 29. Spatial graphs G_0 and $G_0(n)$ on S^2

Let $(D_0(n); T_0)$ be a based diagram of a spatial graph $\hat{G}_0(n)$ obtained from the based diagram $(D_0; T_0)$ by taking a connected sum $D\alpha_0 \# DK(n)$ of an edge

diagram $D\alpha_0$ of $(D_0; T_0)$ and the knot diagram $DK(n)$ so that we have the crossing number

$$c(D_0(n); T_0) = c(D_0; T_0) + c(DK(n) = \varepsilon(\hat{\Gamma}) + 3n$$

(see Figure 29). Then we show that $(D_0(n); T_0) \in [D(\hat{G}_0(n), \gamma)]$. In fact, every based diagram $(D'; T') \in [D(\hat{G}_0(n), \gamma)]$ has the cross index $\varepsilon(D'; T') \geq \varepsilon(\hat{\Gamma})$ and an edge α' of the based diagram $(D'; T')$ has the knot $K(n)$ as a connected summand (see Figure 30 for the case that the connected sum edge $D\alpha_0 \# DK(n)$ belongs to a tree T'' which will be deformed into the basis T').
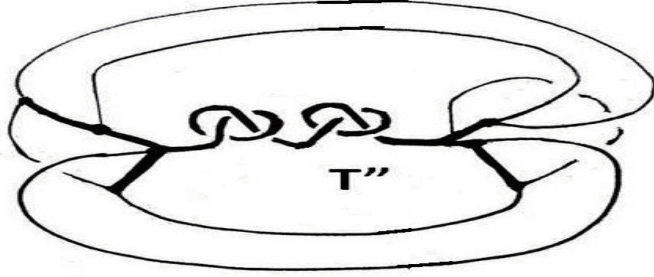

FIGURE 30. A basis T'' containing the connected sum edge $D\alpha \# DK(n)$

By the definition of the cross index and Lemma 9.3, we have

$$c(D', T') \geq \varepsilon(\hat{\Gamma}) + 3n,$$

showing that the based diagram $(D_0(n); T_0)$ belongs to the set $[D(\hat{G}_0(n), \gamma)]$. By the unknotting number $u(K(n)) \leq n$, we have

$$u_{\gamma,\Gamma}(G_0(n)) = u_{\gamma,\Gamma}(\hat{G}_0(n)) \leq n.$$

We modify the spatial graph $G_0(n)$ on S^2 to construct a spatial graph $G_1(n)$ on F by taking in B a 1-handle H connecting the 2-cell $\Delta_0 \subset S^2$ and a 2-cell $\Delta_1 \subset F$ and then adding d parallel arcs in H to $G_0(n)$ for $d = |v_1(G_0(n))|$. See Figure 31 for this situation.

The shrinked spatial graph $G_1(n)/\Delta_1$ is identical to the shrinked spatial graph $\hat{G}_0(n)$, so that by definition we have

$$u_{\gamma,\Gamma}(G_1(n)) \leq u_{\gamma,\Gamma}(\hat{G}_0(n)) \leq n.$$

Let $G = G_1(n)$. We show that $u_\beta(G) \geq n$. Let $u_\beta(G) = u_\beta(\hat{G})$ for a shrinked spatial graph $\hat{G} = G/\Delta$ for a 2-cell Δ in F. Assume that $u_\beta(G) = n^*$ and a β-unknotted spatial graph $\hat{G}^*$ is obtained from the spatial graph $\hat{G}$ by n^* crossing changes on the edges α_k $(k = 1, 2, \ldots, m)$ attaching to a basis T in $\hat{G}$. To orient the edges α_k $(k = 1, 2, \ldots, m)$, the following two cases are considered.

Case (1). The connected sum edge $D\alpha_0 \# DK(n)$ belongs to the edges α_k $(k = 1, 2, \ldots, m)$.

Case (2). The connected sum edge $D\alpha_0 \# DK(n)$ belongs to the basis T.

In Case (1), we orient all the edges α_k $(k = 1, 2, \ldots, m)$ in any orientation. In Case (2), the connected sum edge $D\alpha_0 \# DK(n)$ splits T into two subtrees $T^{(1)}$ and
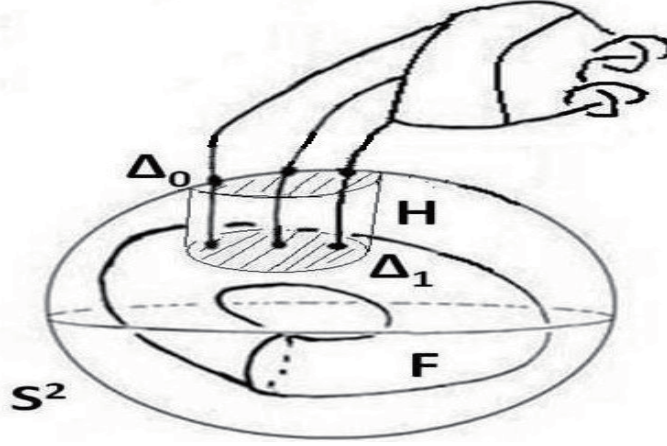
FIGURE 31. A spatial graph $G_1(n)$ on F

$T^{(2)}$. For the edges α_k $(k = 1, 2, \ldots, u)$ connecting $T^{(1)}$ and $T^{(2)}$, we orient by the orientations starting from the vertices in $T^{(1)}$ to the vertices in $T^{(2)}$ and for the remaining edges α_k $(k = u + 1, u + 2, \ldots, m)$ we orient in any orientation. Then the epimorphism $\chi : H_1(E(\hat{G})) \to \mathbf{Z}$ is defined by sending every oriented meridian to $1 \in \mathbf{Z}$. By Lemma 6.1, we have

$$|m(\hat{G}, T; \chi) - m(\hat{G}^*, T; \chi^*)| \leq n^*,$$

where χ^* denotes the induced epimorphism $\chi^* : H_1(E(\hat{G}^*)) \to \mathbf{Z}$. We note that $m(\hat{G}^*, T; \chi^*) = m - 1$ since $\pi_1(E(\hat{G}^*))$ is a free group of rank m and hence

$$M(\hat{G}^*, T; \chi^*) = H_1(\tilde{E}(\hat{G}^*)) \cong \Lambda^{m-1}.$$

We calculate the number $m(\hat{G}, T; \chi)$. The spatial graph $G_1(n)/\Delta_1 = \hat{G}_0(n)$ has the basis inherited from T and the oriented edges inherited from the oriented edges α_k $(k = 1, 2, \ldots, m)$ attaching to T. We have

$$M(\hat{G}_0(n), T; \chi) = H_1(\tilde{E}(G_0(n))) \cong \left\{ \begin{array}{ll} \Lambda^{m-1} \oplus (\Lambda/(\Delta_K(t)))^n, & \text{in Case (1)} \\ \Lambda^{m-1} \oplus (\Lambda/(\Delta_K(t^u)))^n, & \text{in Case (2),} \end{array} \right.$$

where $\Delta_K(t) = t^2 - t + 1$. Hence, in either case, we have

$$m(\hat{G}_0(n), T; \chi) = m - 1 + n.$$

Then we have the following lemma (proved later).

LEMMA 9.4. *There is a short exact sequence*

$$0 \to M(\hat{G}_0(n), T; \chi) \to M(\hat{G}, T; \chi) \to M \to 0$$

for a $\mathbf{Z}$-torsion-free, Λ-torsion module M such that $t - 1 : M \to M$ is an automorphism.

Let DM be the maximal finite Λ-submodule of a finitely generated Λ-module M (see [9]). The following lemma is also proved later although it is implicitly shown in [10].

LEMMA 9.5. *Let M' be a Λ-submodule of a finitely generated Λ-module M. Let b' and b be the minimal numbers of Λ-generators of M' and M, respectively. If the maximal finite Λ-submodule $D(M/M')$ of M/M' is 0, then we have $b' \leq b$.*

By Lemmas 9.4 and 9.5, we have

$$m(\hat{G}_0(n), T; \chi) \leq m(\hat{G}, T; \chi),$$

because $DM = 0$, so that

$$
\begin{aligned}
n &= (m - 1 + n) - (m - 1) \\
&= m(\hat{G}_0(n), T; \chi) - m(\hat{G}^*, T; \chi^*) \\
&\leq m(\hat{G}, T; \chi) - m(\hat{G}^*, T; \chi^*) \\
&\leq n^*.
\end{aligned}
$$

Hence $u_\beta(G) \geq n$ and

$$u_\beta(G) = u(G) = u_\gamma(G) = u_\Gamma(G) = u_{\gamma,\Gamma}(G) = n.$$

This completes the proof of Theorem 9.2 except the proofs of Lemmas 9.3, 9.4 and 9.5. □

The proofs of Lemmas 9.3, 9.4 and 9.5 are given as follows.

PROOF OF LEMMA 9.3. It is well-known that the span of the Jones polynomial $V_{K''}(t)$ of the knot K'' is smaller than or equal to $c(D'')$ (see Murasugi [24], Kauffman [7]). Since

$$V_{K''}(t) = V_K(t)^n \cdot V_{K'}(t), \quad V_K(t) = t + t^3 - t^4$$

by taking a positive trefoil knot as K, we see that $c(D'') \geq 3n$. □

PROOF OF LEMMA 9.4. The spatial graph $\hat{G}$ is a degree d vertex connected sum of the spatial graph $\hat{G}_0(n)$ and a θ_d-curve Θ relative to the vertex v_1 obtained from $v_1(G_0(n))$ and a vertex v_2 of Θ (see [23]). In precise, $\hat{G}$ is the union of $G_0(n)' = \mathrm{cl}(\hat{G}_0(n) \backslash B_1 \cap \hat{G}_0(n)) \subset B_1^c$ and $\Theta' = \mathrm{cl}(\Theta \backslash B_2 \cap \Theta) \subset B_2^c$ where B_i is a 3-ball regular neighborhood of v_i in S^3 for $i = 1, 2$. Then the exterior $E(\hat{G})$ is the union of the exteriors $E(\hat{G}_0(n))$ and $E(\Theta)$ with as the intersection part a compact dth punctured 2-sphere $S(d)$ in the boundaries $\partial E(\hat{G}_0(n))$ and $\partial E(\Theta)$. Let $S(d)^c = \mathrm{cl}(\partial E(\Theta) \backslash S(d))$. Let $\tilde{E}(\hat{G}_0(n))$, $\tilde{E}(\Theta)$, $\tilde{S}(d)$ and $\tilde{S}(d)^c$ be the connected lifts of $E(\hat{G}_0(n))$, $E(\Theta)$, $S(d)$ and $S(d)^c$ to the infinite cyclic covering $\tilde{E}(\hat{G})$ of $E(\hat{G})$, respectively. By excision, there is a natural isomorphism

$$H_d(\tilde{E}(\hat{G}), \tilde{E}(\hat{G}_0(n))) \cong H_d(\tilde{E}(\Theta), \tilde{S}(d)).$$

Since $H_1(E(\Theta), S(d)) = H_1(E(\Theta), S(d)^c) = 0$, we see from the Wang exact sequence that $M = H_1(\tilde{E}(\Theta), \tilde{S}(d))$ and $M^c = H_1(\tilde{E}(\Theta), \tilde{S}(d)^c)$ are finitely generated Λ-modules such that $t - 1 : M \to M$ and $t - 1 : M^c \to M^c$ are automorphisms, implying that M and M^c are Λ-torsion modules whose $\mathbf{Z}$-torision parts $\tau(M)$ and $\tau(M^c)$ are equal to the maximal finite Λ-modules DM and DM^c, respectively (see [9]). By the second duality theorem in [9],

$$\tau(M) = DM \cong Ext_\Lambda^1(M^c/\mathrm{Tor}_\Lambda(M^c); \Lambda) = 0.$$

(*Note*: Though we have also $\tau(M^c) = DM^c = 0$, we do not use this fact.) Then the homology exact sequence of the pair $(\tilde{E}(\hat{G}), \tilde{E}(\hat{G}_0(n)))$ induces a desired exact sequence. $\qquad\square$

PROOF OF LEMMA 9.5. For a Λ-epimorphism $f : \Lambda^b \to M$, let

$$B' = f^{-1}(M') \subset \Lambda^b,$$

which is a finitely generated Λ-module mapped onto M' by f. Since the quotient Λ-module Λ^b/B' is isomorphic to M/M', which has a Λ-projective dimension ≤ 1 since the maximal finite Λ-submodule $D(M/M')$ of M/M' is 0 (see [**9**]). Hence $B' \cong \Lambda^{c'}$ for some nonnegative integer c', implying that $b' \leq c' \leq b$. $\qquad\square$

10. Knotting dynamics of a spatial graph with degree one free vertices

In this section, we consider a spatial graph G with degree one vertices $v_1, v_2, \ldots,$ $v_d(d \geq 1)$ neither of which is not attached to any surface. These degree one vertices are referred to as *free vertices*. We explain here knotting dynamics of a spatial graph G with degree one free vertices by applying the knotting notions on the spatial graphs without degree one vertices associated with G. This notion is introduced in [**17, 18**]. We need to impose a mild restriction on a spatial graph with degree one free vertices. A spatial graph G with degree one free vertices is *normal* if G has the following properties (1) and (2) where $V = \{v_1, v_2, \ldots, v_d\}$:

(1) There is a set $X = \{x_1, x_2, \ldots, v_d\}$ of mutually distinct d points in $G\backslash V$ such that the line segments $|v_i x_i|\, (i = 1, 2, \ldots, d)$ are mutually disjoint and intersect G only in the set $V \cup X$. (We call the set X a *coupling with* V.)

(2) There are only finitely many equivalence classes of the spatial graphs (without degree one vertices)

$$G_X = G \bigcup_{i=1}^{d} |v_i x_i|$$

for all couplings X with V.

Every polygonal spatial graph G with degree one free vertices which is not in a plane is normal and if G is normal in a plane $\subset \mathbf{R}^3$, then the spatial graph G_X without degree one vertices is always a Γ-unknotted spatial graph for every coupling X with V. For every normal spatial graph G with degree one free vertices and every coupling X with V, the *unknotting number* $u(G_X)$ of the spatial graph G_X without degree one vertices is defined in Section 4. An analysis on the dynamics of the invariant $u(G_X)$ for every coupling X with V will be useful in studying a knotted structure of the normal spatial graph G with degree one free vertices. The *unknotting number* $u(G)$ of a normal spatial graph G with degree one free vertices is defined to be

$$u(G) = \max\{u(G_X)|\, X \text{ is a coupling with } V\}.$$

Let $\bar{n}_G$ be the number of distinct equivalence classes on the spatial graphs G_X for all couplings X with V, and n_G the number of distinct equivalence classes of spatial graphs G_X with $u(G_X) > 0$ for all couplings X with V. The *knotting probability* of a normal spatial graph G with degree one free vertices is defined by the fraction

$$p(G) = \frac{n_G}{\bar{n}_G},$$

and we say that G is a $(p(G) \times 100)\%$-*knotted graph.* For example, we consider a spatial polygonal (normal) arc G with ordered vertices

$$v_1 = (0,0,0), \quad p_1 = (3,1,0), \quad p_2 = (3,2,1),$$
$$p_3 = (2,3,1), \quad p_4 = (1,2,1), \quad v_2 = (1,0,0),$$
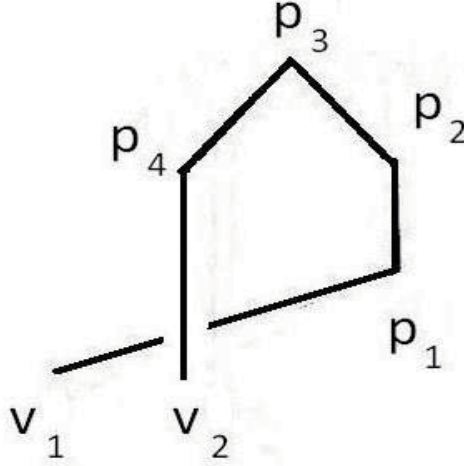
which is illustrated in Figure 32.

FIGURE 32. A normal spatial arc

It turns out that the spatial graphs G_X for all couplings $X = \{x_1, x_2\}$ with $V = \{v_1, v_2\}$ are classified into three equivalence classes consisting of an unknotted handcuff graph, an unknotted θ-curve, and a knotted handcuff graph of unknotting number one caused from the observation that the line segment $|v_1 x_1|$ taking x_1 in an open line segment (p_2, p_3) or (p_2, q) with the midpoint q of the line segment $|p_1 p_2|$ meets at an interior point of the triangle $|v_2 x_2 p_4|$ taking $x_2 = p_3$. This check is relatively easily done because the spatial graph G_X is a θ-curve or a handcuff graph for every normal spatial arc G and every coupling X with V, so that G_X is unknotted if and only if G_X is equivalent to a graph in a plane $\subset \mathbf{R}^3$. Thus, we have the unknotting number $u(G) = 1$ and the knotting probability $p(G) = \frac{1}{3}$. In other words, the normal spatial arc G is a $\frac{100}{3}\%$-knotted arc with $u(G) = 1$.

In similar ways, the γ-*unknotting number* $u_\gamma(G_X)$, the Γ-*unknotting number* $u^\Gamma(G_X)$, and the (γ, Γ)-*unknotting number* $u_{\gamma, \Gamma}(G_X)$ and their related notions are defined for every normal spatial graph G with free degree one vertices. Detailed studies on the knotting probability of a normal spatial arc will be done elsewhere.

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