ON LINKING SIGNATURE INVARIANTS OF SURFACE-KNOTS

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ABSTRACT

We show that the linking signature of a closed oriented 4-manifold with infinite cyclic first homology is twice the Rochlin invariant of an exact leaf with a spin support if such a leaf exists. In particular, the linking signature of a surface-knot in the 4-sphere is twice the Rochlin invariant of an exact leaf of an associated closed spin 4-manifold with infinite cyclic first homology. As an application, we characterize a difference between the spin structures on a homology quaternion space in terms of closed oriented 4-manifolds with infinite cyclic first homology, so that we can obtain examples showing that some different punctured embeddings into S^4 produce different Rochlin invariants for some homology quaternion spaces.

Keywords : quadratic function, linking signature, surface-knot, Rochlin invariant, exact leaf, spin structure, homology quaternion space

0. Introduction

A *quadratic function* on a finite abelian group G is a function

$$q : G \longrightarrow Q/Z$$

such that $q(-x) = q(x)$ for all $x \in G$ and the pairing $\ell : G \times G \to Q/Z$ defined by the identity $\ell(x,y) = q(x+y) - q(x) - q(y)$ is a non-singular symmetric bilinear form which we call the *linking* induced from q. We note that $2q(x) = \ell(x,x)$ and $q(2^m x) = 2^{2m} q(x)$ for every integer $m \geq 1$ and $x \in G$. A quadratic function $q : G \times G \to Q/Z$ is said to be *isomorphic* to a quadratic function $q' : G' \times G' \to Q/Z$ if there is an isomorphism $f : G \cong G'$ such that $q = q'f$. The *linking signature* $\sigma(q)$ of q is a rational number modulo one which is defined by the Gauss sum identity

$$GS(q) = \sum_{x \in G} \exp(2\pi\sqrt{-1} \cdot q(x)) = \sqrt{|G|}\exp(2\pi\sqrt{-1} \cdot \sigma(q))$$

(see [**8**]). The linking signature $\sigma(q) \in Q/Z$ is an invariant of a quadratic function q up to isomorphisms and has $8\sigma(q) = 0$ in Q/Z in general. A closed connected oriented 4-manifold M with $H_1(M;Z) \cong Z$ is simply called a Z^{H_1}-*manifold*. By a result of [**7**], the Z^{H_1}-manifold $M \# n(S^2 \times S^2)$ admits an exact leaf V if n is greater than a constant depending on M. In this paper, we consider the torsion quadratic function

$$q : G_{\tilde{M}} \longrightarrow Q/Z$$

of the infinite cyclic covering space $\tilde{M}$ over a Z^{H_1}-manifold M belonging to a generator $\gamma \in H^1(M;Z)$. This quadratic function q was first defined in [**8**], although the induced torsion linking

$$\ell : G_{\tilde{M}} \times G_{\tilde{M}} \longrightarrow Q/Z$$

had been defined in [**4**] (see also [**6**]). We denote the linking signature $\sigma(q)$ by $\sigma(\tilde{M}) = \sigma^\gamma(M)$. For our purpose, we consider a Z^{H_1}-manifold M containing an exact leaf V with a *spin support* M^* in M, that is, a compact spin 4-manifold neighborhood M^* of V in M such that $H_1(\partial M^*;Z)$ is a free abelian group. For a spin structure ι on V induced from any spin structure on M^*, we take the Rochlin invariant $\mu(V,\iota)$, arranged to have the value in Q/Z. Then, as the main result of this paper, we shall show that *the linking signature* $\sigma^\gamma(M)$ *is equal to* $2\mu(V,\iota)$. This theorem will be exactly stated in §2 and proved in §3. In §1, some facts of the torsion quadratic function, the torsion linking, and the linking signature are explained.

As a concrete object, we consider an oriented surface-knot F in a closed connected oriented 4-manifold M_1 with $H_1(M_1;Z)=0$ such that F admits a Seifert hypersurface in M_1 (in other words, such that $[F]=0\in H_2(M_1;Z)$). For standard examples, we take $M_1=S^4$. By [**8**] we still have the torsion quadratic function

$$q_F : G_F \longrightarrow Q/Z$$

and the torsion linking

$$\ell_F : G_F \times G_F \longrightarrow Q/Z$$

of the surface-knot F. This torsion linking ℓ_F is a natural generalization of the Farber-Levine linking of an S^2-knot in S^4 (see Farber [**2**], Levine [**13**]). We denote the linking signature $\sigma(q_F)\in Q/Z$ by $\sigma(F)$. Let E_F be the compact exterior of F in M_1, i.e., $E_F=\mathrm{cl}(M_1-N_F)$ for a trivial normal bundle N_F of F in M_1. We have the first homology $H_1(E_F;Z)\cong Z$ with a unique meridian generator. We choose a trivialization $N_F=F\times D^2$ so that the natural composite

$$H_1(F\times 1;Z)\longrightarrow H_1(\partial E_F;Z)\longrightarrow H_1(E_F;Z)\cong Z$$

is the zero map under the identification $\partial E_F=\partial N_F=F\times S^1$. Let V_0 be the handlebody such that $\partial V_0=F$. Let M_ϕ be the closed 4-manifold obtained from E_F and $V\times S^1$ by attaching the boundaries by a homeomorphism $\phi:\partial E_F=F\times S^1\to\partial V_0\times S^1$ which preserves the S^1-factor. Then M_ϕ is a Z^{H_1}-manifold. We refer the Z^{H_1}-manifold $M_{\phi,n}=M_\phi\# n(S^2\times S^2)$ as a Z^{H_1}-manifold *associated with* the surface-knot F. Choosing ϕ carefully, we can make M_ϕ and hence $M_{\phi,n}$ spin. Let V be an exact leaf of the spin Z^{H_1}-manifold $M_{\phi,n}$ for a large integer n. Let ι be a spin structure on V induced from any spin structure on $M_{\phi,n}$. In this case, our main theorem implies the identity

$$\sigma(F)=2\mu(V,\iota).$$

We shall also note in Remark 2.6 that the Rochlin invariant $\mu(V,\iota)$ itself is not an invariant for a high genus surface-knot F, although it is an invariant when F is an S^2-knot in S^4 by a result of Ruberman[**14**]. Our main result is applied in §4 to characterize a difference between the spin structures ι on a homology quaternion space V by constructing Z^{H_1}-manifolds M such that V is an exact leaf of M with a spin support M^* in M and ι is a spin structure on V induced from a spin structure on M^*. Using this characterization, we shall obtain examples showing that some different punctured embeddings into S^4 produce different Rochlin invariants for the quaternion space and a homology quaternion space.

Finally, the torsion linking and the torsion quadratic function of a surface-link are also defined in [**8**], and the surface-link version of this paper will be discussed in [**9**].

1. The torsion linking, the torsion quadratic function, and the linking signature

Let $\Lambda = Z[Z] = Z[t, t^{-1}]$. Let W be a compact connected oriented 4-manifold which admits an infinite cyclic connected covering $p : \tilde{W} \to W$ belonging to an indivisible element $\gamma \in H^1(W; Z)$. Let A and A' be $\emptyset$ or compact 3-submanifolds of ∂W such that $A' = \mathrm{cl}(\partial W - A)$. For a subspace W' of W, let $\tilde{W}' = p^{-1}(W')$. For a finitely generated Λ-module H, let DH be the maximal finite Λ-submodule of H (see [**4**;§3]), $\mathrm{t}H$ the Z-torsion part of H, and TH the Λ-torsion part of H. Let $BH = H/TH$. Let $E^q(H) = Ext^q_\Lambda(H, \Lambda)$. By an argument of [**4**] we have a t-anti epimorphism

$$\theta_{A,A'} : DH_1(\tilde{W}, \tilde{A}; Z) \to E^1(BH_2(\tilde{W}, \tilde{A}'; Z))$$

which is an invariant of $(\tilde{W}, \tilde{A}, \tilde{A}')$ or (W, A, A', γ). We denote the kernels of $\theta_{A,A'}$ and $\theta_{A',A}$ by $DH_1(\tilde{W}, \tilde{A}; Z)^\theta$ and $DH_1(\tilde{W}, \tilde{A}'; Z)^\theta$, respectively. The second duality of [**4**] then says that there is a t-isometric non-singular bilinear form

$$\ell : DH_1(\tilde{W}, \tilde{A}; Z)^\theta \times DH_1(\tilde{W}, \tilde{A}'; Z)^\theta \longrightarrow Q/Z$$

which is an invariant of $(\tilde{W}, \tilde{A}, \tilde{A}')$ or (W, A, A', γ). By taking $A = \emptyset$ and $A' = \partial W$, let $\hat{D}H_1(\tilde{W}; Z)^\theta$ denote the following quotient finite Λ-module:

$$DH_1(\tilde{W}; Z)^\theta / \mathrm{Im}(\tilde{i}_* : H_1(\partial\tilde{W}; Z) \to H_1(\tilde{W}; Z)) \cap DH_1(\tilde{W}; Z)^\theta,$$

where $\tilde{i}_*$ denotes the natural homomorphism. Then we have the following lemma (see [**8**]):

Lemma 1.1. The bilinear form ℓ induces a t-isometric linking

$$\hat{\ell} : \hat{D}H_1(\tilde{W}; Z)^\theta \times \hat{D}H_1(\tilde{W}; Z)^\theta \longrightarrow Q/Z. \quad \blacksquare$$

The linking $\hat{\ell}$ is an invariant of $\tilde{W}$ or (W, γ) and called the *torsion linking* of $\tilde{W}$ or (W, γ). We say that H is $(t-1)$-*divisible* if $t-1$ is an automorphism of H. For a finitely generated $(t-1)$-divisible Λ-module H, it is well-known that the Z-torsion part $\mathrm{t}H$ of H is equal to DH, originally due to M. A. Kervaire [**12**] (cf. [**4**;§3]). For DH, let

$$D_0H = \cap_{n=1}^\infty (t-1)^n DH,$$
$$D_1H = \{x \in DH \mid \exists n \geq 1, (t-1)^n x = 0\}.$$

Then we have a natural splitting $DH = D_0H \oplus D_1H$, so that D_0H is a unique maximal $(t-1)$-divisible finite Λ-submodule of H (see [**8**]). We denote by $G(\tilde{W})$ the unique maximal Λ-submodule $D_0(\hat{D}H_1(\tilde{W})^\theta)$ of $\hat{D}H_1(\tilde{W})^\theta$. The restriction ℓ_G of $\hat{\ell}$ to $G(\tilde{W})$ induces a t-isometric linking

$$\ell_G : G(\tilde{W}) \times G(\tilde{W}) \longrightarrow Q/Z,$$

which we call the $(t-1)$-*divisible torsion linking* of $\tilde{W}$ or (W, γ), which leads to the following definition (see [**8**]):

Definition 1.2. The *torsion quadratic function* of $\tilde{W}$ or (W,γ) is the function

$$q : G(\tilde{W}) \longrightarrow Q/Z$$

defined by $q(x) = \ell_G(x, (1-t)^{-1}x)$. ■

It is direct to see that q is an invariant of $\tilde{W}$ or (W,γ). We have

$$q(-x) = q(x) \quad \text{and} \quad q(x+y) - q(x) - q(y) = \ell_G(x,y).$$

Thus, q is a quadratic function inducing ℓ_G. The linking signature $\sigma(q) \in Q/Z$ is given by the Gauss sum identity

$$GS(q) = \sum_{x \in G(\tilde{W})} \exp(2\pi\sqrt{-1} \cdot q(x)) = \sqrt{|G(\tilde{W})|} \exp(2\pi\sqrt{-1} \cdot \sigma(q)),$$

which is denoted by $\sigma(\tilde{W}) = \sigma^\gamma(W)$ and called by the *linking signature* of $\tilde{W}$ or (W,γ). For every prime p, let $G(\tilde{W})_p$ be the p-torsion subgroup of $G(\tilde{W})$. Then we see that the linking $\ell_G : G(\tilde{W}) \times G(\tilde{W}) \to Q/Z$ is the unique orthogonal sum of the linkings $\ell_p : G(\tilde{W})_p \times G(\tilde{W})_p \to Q/Z$ induced from ℓ_G for all primes p. The restricted function $q_p : G(\tilde{W})_p \to Q/Z$ of q is a quadratic function inducing ℓ_p. We denote $\sigma(q_p)$ by $\sigma_p(\tilde{W})$ and call it the p-*local linking signature*. Then we have the identity

$$\sigma(\tilde{W}) = \sum_p \sigma_p(\tilde{W})$$

where the summation $\sum_p$ ranges over all primes p. Further, the p-primary component $G(\tilde{W})_p$ has a homogeneous orthogonal splitting $\oplus_{i=1}^{\infty} G(\tilde{W})_p^i$ with respect to ℓ_p where $G(\tilde{W})_p^i$ is a direct sum of copies of Z_{p^i}. The restricted function $q_p^i : G(\tilde{W})_p^i \to Q/Z$ of q_p is a quadratic function inducing the linking $\ell_p^i : G(\tilde{W})_p^i \times G(\tilde{W})_p^i \to Q/Z$ induced from ℓ_p. We denote the linking signature $\sigma(q_p^i)$ by $\sigma_p^i(\tilde{W})$ and call it the *ith p-local signature* of $\tilde{W}$ or (W,γ). By definition,

$$\sigma_p(\tilde{W}) = \sum_{i=1}^{\infty} \sigma_p^i(\tilde{W}).$$

It turns out that $\sigma_p^i(\tilde{W})$ is an invariant of $\tilde{W}$ and takes a value in Q/Z as follows:

$$\sigma_p^i(\tilde{W}) = \begin{cases} 0 & \text{if } p \text{ is any prime and } i \text{ is even} \\ 0 \text{ or } \frac{1}{2} & \text{if } p = 2 \text{ and } i \text{ is odd} \\ 0, \frac{1}{2}, \text{ or } \pm\frac{1}{4} & \text{if } p \text{ and } i \text{ are odd.} \end{cases}$$

Let F be a surface-knot in a closed connected oriented 4-manifold M_1 such that $H_1(M_1;Z) = 0$ and $[F] = 0 \in H_2(M_1;Z)$, and E_F the compact exterior of F in M_1. The *torsion linking*

$$\ell_F : G_F \times G_F \to Q/Z$$

of this surface-knot F is the torsion linking $\ell_G : G(\tilde{E}_F) \times G(\tilde{E}_F) \to Q/Z$ for the infinite cyclic covering space $\tilde{E}_F$ over E_F belonging to the element $\gamma \in H^1(E_F; Z)$ sending the meridian of F to 1, and the *torsion quadratic function*

$$q_F : G_F \to Q/Z$$

of F is the torsion quadratic function $q : G(\tilde{E}_F) \to Q/Z$. We define

$$\sigma(F) = \sigma(\tilde{E}_F), \quad \sigma_p(F) = \sigma_p(\tilde{E}_F), \quad \sigma_p^i(F) = \sigma_p^i(\tilde{E}_F).$$

Let $\tilde{M}_{\phi,n} \to M_{\phi,n}$ be the infinite cyclic covering belonging to γ under the identification $H^1(\tilde{E}_F; Z) = H^1(M_{\phi,n}; Z)$, which extends the infinite cyclic covering $\tilde{E}_F \to E_F$. As observed in [**8**;Proposition 4.4], the torsion quadratic function $q_F : G_F \to Q/Z$ is isomorphic to the torsion quadratic function $q : G(\tilde{M}_{\phi,n}) \to Q/Z$ for all n, so that

$$\sigma(F) = \sigma(\tilde{M}_{\phi,n}), \quad \sigma_p(F) = \sigma_p(\tilde{M}_{\phi,n}), \quad \sigma_p^i(F) = \sigma_p^i(\tilde{M}_{\phi,n})$$

for all primes p and all positive integers i.

2. Identifying the linking signature with twice the Rochlin invariant

A *leaf* of a Z^{H_1}-manifold M is a bicollared 3-submanifold V of M such that V represents a generator of $H_3(M; Z) \cong H^1(M; Z) \cong Z$. The following definition of exact leaf is found in [**7**] together with two equivalent definitions:

Definition 2.1. A leaf V of a Z^{H_1}-manifold M is *exact* if the natural semi-exact sequence

$$0 \to \mathrm{t}H_2(\tilde{M}, \tilde{V}; Z) \xrightarrow{\tilde{\partial}} \mathrm{t}H_1(\tilde{V}; Z) \xrightarrow{\tilde{i}_*} \mathrm{t}H_1(\tilde{M}; Z)$$

induced from the homology exact sequence of the pair $(\tilde{M}, \tilde{V})$ is exact. ■

Further, we say that a Z^{H_1}-manifold M is *exact* if there is an exact leaf V of M. The following lemma is proved in [**7**]:

Lemma 2.2. For every Z^{H_1}-manifold M, we have a non-negative integer n such that the connected sum $M \# n(S^2 \times S^2)$ is exact. ■

For a closed oriented 3-manifold V, we have a linking form

$$\ell_V : \mathrm{t}H_1(V; Z) \times \mathrm{t}H_1(V; Z) \longrightarrow Q/Z$$

on the Z-torsion part $\mathrm{t}H_1(V; Z)$ of $H_1(V; Z)$ defined by the Poincaré duality. Given a spin structure ι on V, we have a unique quadratic function

$$q_V^\iota : \mathrm{t}H_1(V; Z) \longrightarrow Q/Z,$$

such that

$$q_V^\iota(x + y) - q_V^\iota(x) - q_V^\iota(y) = \ell_V(x, y)$$

(see [**8**;Lemma 1.1]). By [**8**;Corollary 1.4], the linking signature $\sigma(q_V^\iota) \in Q/Z$ given by the Gauss sum identity

$$GS(q_V^\iota) = \sum_{x \in \mathrm{t}H_1(V;Z)} \exp(2\pi\sqrt{-1} \cdot q_V^\iota(x)) = \sqrt{|\mathrm{t}H_1(V; Z)|} \exp(2\pi\sqrt{-1} \cdot \sigma(q_V^\iota))$$

has $8\sigma(q_V^\iota) = 0 \in Q/Z$. We call this invariant the *spin linking signature* of (V, ι) and denote it by $s(V, \iota)$. The *Rochlin invariant* $\mu(V, \iota) \in Q/Z$ of (V, ι) is defined by the identity

$$\mu(V, \iota) = -\mathrm{sign}(U)/16 \in Q/Z$$

for any smooth spin 4-manifold (U, ι_U) bounded by (V, ι). By [**8**;Lemma 1.3], we have

$$s(V, \iota) = 2\mu(V, \iota).$$

To state our main theorem, we generalize the concept of a spin Z^{H_1}-manifold as follows: A leaf V of a Z^{H_1}-manifold M admits a *spin support* M^* in M if M^* is a compact spin 4-manifold neighborhood of V in M such that $\mathrm{t}H_1(\partial M^*; Z) = 0$. For example, let V be a leaf of a spin Z^{H_1}-manifold M', and M the connected sum of M' and any closed non-spin 4-manifold W with $H_1(W; Z) = 0$. Then V is a leaf of the non-spin Z^{H_1}-manifold M with a spin support in M. Using this concept, we state our main theorem (proved in §3) as follows:

Theorem 2.3. Let V be an exact leaf of a Z^{H_1}-manifold M with a spin support M^* in M. For any spin structure ι on V induced from any spin structure on M^*, we have

$$\sigma(\tilde{M}) = s(V, \iota) = 2\mu(V, \iota). \quad \blacksquare$$

Let F be a surface-knot in a closed spin 4-manifold M_1 such that $H_1(M_1; Z) = 0$ and F admits a Seifert hypersurface in M_1. By our choice of a trivialization $N_F = F \times D^2$, the surface $F \times 1$ bounds a bicollared 3-submanifold V_F in E_F. By Poincaré duality over Z_2, we have a Z_2-symplectic basis x_i, y_i $(i = 1, 2, \ldots, m)$ for $H_1(F \times 1; Z_2)$ whose Z_2-intersection numbers have $x_i \cdot x_j = y_i \cdot y_j = 0$ and $x_i \cdot y_j = \delta_{i,j}$ for all $i, j = 1, 2, \ldots, m$ and such that x_i bounds a Z_2-chain in V_F for all i. We represent x_i and y_i by circles S_i^x and S_i^y embedded in $F \times 1$ such that $S_i^x \cap S_j^x = S_i^y \cap S_j^y = S_i^x \cap S_j^y = \emptyset$ for all i, j with $i \neq j$ and $S_i^x \cap S_i^y =$ one point for all i. We choose a homeomorphism $\phi : \partial E_F = F \times S^1 \to \partial V_0 \times S^1$ preserving the S^1-factor such that $\phi(S_i^x)$ is a meridian disk in $V_0 \times 1$. Then we have the following lemma:

Lemma 2.4. The Z^{H_1}-manifold M_ϕ is spin. $\blacksquare$

Proof. We consider the following part

$$H_2(E_F; Z_2) \to H_2(M_\phi; Z_2) \to H_2(M_\phi, E_F; Z_2) \xrightarrow{\partial} H_1(E_F; Z_2)$$

of the exact sequence of the pair (M_ϕ, E_F). Using the excision isomorphism

$$H_2(M_\phi, E_F; Z_2) \cong H_2(V_0 \times S^1, F \times S^1; Z_2),$$

we see that $H_2(M_\phi; Z_2)$ is generated by Z_2-cycles C in $H_2(E_F; Z_2)$ and Z_2-cycles C_i' $(i = 1, 2, \ldots, m)$ in M_ϕ such that C_i' is the sum of a Z_2-chain in V_F bounded by S_i^x and a medidian disk in V_0 bounded by $\phi(S_i^x)$. Since E_F is spin, we have the Z_2-intersection number $C \cdot C = 0$ for every Z_2-cycle in E_F. By construction, we also have the Z_2-intersection number $C_i' \cdot C_i' = 0$ for all i. These mean that M_ϕ is spin. $\square$

Combining Lemma 2.4 with Theorem 2.3, we obtain the following corollary from the identity $\sigma(F) = \sigma(\tilde{M}_{\phi,n})$:

Corollary 2.5. Let $M_{\phi,n}$ be any spin Z^{H_1}-manifold associated with any surface-knot F in M_1 which admits an exact leaf V, and ι a spin structure on V induced from any spin structure on $M_{\phi,n}$. Then we have

$$\sigma(F) = s(V,\iota) = 2\mu(V,\iota). \quad \blacksquare$$

We consider an S^2-knot K in S^4. Let V be a closed oriented 3-manifold obtained from a Seifert hypersurface V_K for K in S^4 by adding a 3-ball, and ι the spin structure on V induced from S^4. Then Ruberman [**14**] showed that the Rochlin invariant $\mu(V,\iota) \in Q/Z$ is independent of a choice of V_K and hence an invariant of K. A geometric proof of this fact is also easily derived from the fact in [**5**] that any two Seifert hypersurfaces for K are connected by a surgery sequence on embedded 1-handles or 2-handles, because the surgery trace of every embedded 1-handle or 2-handle on a Seifert hypersurface V_K is in S^4 and hence has the signature zero. By definition, V is an exact leaf of the (unique) spin Z^{H_1}-manifold M_ϕ associated with K. Hence we have

$$\sigma(K) = s(V,\iota) = 2\mu(V,\iota)$$

by Corollary 2.5. By this evidence, one may expect that $\mu(V,\iota)$ itself is an invariant for a positive genus surface-knot F. However, the following remark shows that this is not true for a high genus surface-knot:

Remark 2.6. It is well-known that every homology 3-sphere V can embedded smoothly into the connected sum $\#n(S^2 \times S^2)$ for a positive integer n. For our purpose, we take any V such that $\mu(V,\iota_V) = \frac{1}{2}$, where ι_V denotes the unique spin structure on V. We note that the Z^{H_1}-manifold $M = S^1 \times S^3 \# n(S^2 \times S^2)$ is a Z^{H_1}-manifold associated with a trivial surface-knot of genus n. Since V separates $\#n(S^2 \times S^2)$ into two submanifolds, we see that V is a leaf of the Z^{H_1}-manifold M. The factor S^3 of the connected summand $S^1 \times S^3$ of M gives a leaf of M. Since $H_1(S^3;Z) = H_1(V;Z) = 0$, we see from Lemma 4.2 later that S^3 and V are exact leaves of M. However, we have $\mu(S^3,\iota_{S^3}) = 0$ and $\mu(V,\iota_V) = \frac{1}{2}$. $\blacksquare$

In spite of this example, we can re-use the Rochlin invariant as an invariant of a positive-genus surface-knot F together with a self-orthogonal Λ-submodule X of $BH_2(\tilde{E}_F;Z)$ (see [**10**]).

3. Proof of Theorem 2.3

Let V be a leaf of a Z^{H_1}-manifold M. Let $\mu \in H_3(\tilde{M};Z)$ be the fundamental class of the covering $p : \tilde{M} \to M$, that is a homology class represented by a lift of the leaf V to $\tilde{M}$ (see [**5**]). Unless a confusion might occur, this lift is also denoted by V. Let $\tau H^2(\tilde{M};Z)$ be the image of the Bockstein homomorphism $\delta_{Q/Z} : H^1(\tilde{M};Q/Z) \to H^2(\tilde{M};Z)$. Let $\mathrm{t}H_1(V;Z)^\theta$ be the subgroup of $\mathrm{t}H_1(V;Z)$ given by $(\cap[V])i^*(\tau H^2(\tilde{M};Z))$ in the following commutative diagram:

$$\begin{array}{ccc}
\tau H^2(\tilde{M};Z) & \xrightarrow{\cap\mu} & \mathrm{t}H_1(\tilde{M};Z) \\
{\scriptstyle i^*}\big\downarrow & & \big\uparrow{\scriptstyle i_*} \\
\mathrm{t}H^2(V;Z) & \xrightarrow[\cong]{\cap[V]} & \mathrm{t}H_1(V;Z).
\end{array}$$

It is shown in [**6**;Lemma 4.1 and Theorem 4.2] that if V is an exact leaf of M, then there is an orthogonal splitting $\mathrm{t}H_1(V;Z) = \mathrm{t}H_1(V;Z)^\theta \oplus \mathrm{Ker}i_*$ with respect to the linking ℓ_V. Further, we have the following (1) and (2):

(1) The map i_* induces an isomorphism from the restricted linking

$$\ell_V|_{\mathrm{t}H_1(V;Z)^\theta} : \mathrm{t}H_1(V;Z)^\theta \times \mathrm{t}H_1(V;Z)^\theta \longrightarrow Q/Z$$

to the torsion linking

$$\ell_G : G(\tilde{M}) \times G(\tilde{M}) \longrightarrow Q/Z$$

of $\tilde{M}$ with $G(\tilde{M}) = \mathrm{t}H_1(\tilde{M};Z)^\theta$.

(2) We have $\mathrm{Ker}i_* = K_+ \oplus K_-$ and $\ell_V(K_+,K_+) = \ell_V(K_-,K_-) = 0$ for

$$K_\pm = \mathrm{Im}(\partial : H_2(\tilde{M}_\pm, \tilde{V}_\pm; Z) \to \mathrm{t}H_1(\tilde{V}_\pm;Z)) \cap \mathrm{t}H_1(V;Z)$$

where $\tilde{M}_\pm$ are 4-submanifolds obtained by $\tilde{M}$ splitting along V such that $\tilde{M} = \tilde{M}_+ \cup \tilde{M}_+$, $\tilde{M}_+ \cap \tilde{M}_- = V$, $\tilde{M}_\pm \supset t^{\pm 1}V$, and $\tilde{V}_\pm = \tilde{M}_\pm \cap \tilde{V}$.

We are now in a position of the proof of Theorem 2.3.

Proof of Theorem 2.3. Let V be an exact leaf of a Z^{H_1}-manifold M with a spin support M^* in M. Let ι be the spin structure on V induced from any spin structure on M^*. Let $\tilde{M}^*_\pm = \tilde{M}_\pm \cap \tilde{M}^*$. We first show the following:

(1) *The linking signature of the restricted quadratic function* $q^\iota_V|_{\mathrm{Ker}i_*}$ *is* 0.

Let k be a 1-knot in V with $[k] \in K_+$. Then there is a 2-chain C in $\tilde{M}_+$ with $\partial C = k$ and $[C] \in \mathrm{t}H_2(\tilde{M}_+, \tilde{V}_+; Z)$. Then C meets $\tilde{M}^*_+$ in a 2-chain C^* such that $c = \partial C^* - k$ is a torsion cycle in ∂M^*. Since $\partial \tilde{M}^*$ is a trivial lift of ∂M^* and $\mathrm{t}H_1(\partial M^*_+;Z) = 0$, we have $\mathrm{t}H_1(\partial \tilde{M}^*_+;Z) = 0$. Hence c is null-homologous in ∂M^*_+. Let $\hat{C}$ be a 2-chain in $\tilde{M}^*_+$ with $\partial \hat{C} = k$ obtained from the 2-chain C^* by adding a 2-chain in $\partial \tilde{M}^*_+$ with boundary $-c$. Let k' be a longitude of k in V given by the spin structure ι. For a 2-chain $\hat{C}'$ in $\tilde{M}_+$ with $\partial \hat{C}' = k'$ obtained from $\hat{C}$ by moving k to k' locally, the Z-intersection number $s(\hat{C}, \hat{C}')$ in $\tilde{M}_+$ is even by using a property of the spin structure on $\tilde{M}^*_+$. Let $\mathrm{cl}_Q(\hat{C})$ and $\mathrm{cl}_Q(\hat{C}')$ be rational 2-cycles in $\tilde{M}_+$ obtained from $\hat{C}$ and $\hat{C}'$ by adding rational 2-chains c_Q and c'_Q in V with $\partial c_Q = -k$ and $\partial c'_Q = -k'$, respectively. Since $\mathrm{cl}_Q(\hat{C})$ and $\mathrm{cl}_Q(\hat{C}')$ are rationally homologous (in $\tilde{M}^*$) to rational 2-cycles in $\tilde{V}_+ \cup \partial \tilde{M}^*_+$, we see that the Q-intersection number $s_Q(\mathrm{cl}_Q(\hat{C}), \mathrm{cl}_Q(\hat{C}')) = s_Q(\mathrm{cl}_Q(\hat{C}), \hat{C}') = 0$. This means that the Z-intersection number $s(\hat{C}, \hat{C}') = -\mathrm{Link}_Q(k,k')$. Therefore, we have

$$\begin{aligned} q^\iota_V([k]) &= \frac{\mathrm{Link}_Q(k,k')}{2} \pmod 1 \\ &= -\frac{s(\hat{C}, \hat{C}')}{2} \pmod 1 = 0. \end{aligned}$$

Thus, $q_V^\iota(K_+) = 0$. Similarly, $q_V^\iota(K_-) = 0$. By a result of hyperbolic quadratic function in [**8**;Corollary 2.5], the linking signature of the quadratic function $q_V^\iota|_{\mathrm{Ker} i_*}$ is 0, as desired. Next, we show the following :

(2) *For any elements $x \in G(\tilde{M})$ and $y \in tH_1(V;Z)^\theta$ with $i_*(y) = x$, we have*

$$q_V^\iota(y) = \ell_G(x, (1-t)^{-1}x) = q(x).$$

As a result, we see that the linking signature of the restricted quadratic function $q|_{\mathrm{t}H_1(V;Z)^\theta}$ is equal to $\sigma(\tilde{M})$.

Let $z = (1-t)^{-1}x \in G(\tilde{M})$. Let k and k_t be 1-knots in V with $[k], [k_t] \in \mathrm{t}H_1(V;Z)^\theta$ such that $i_*([k]) = z$ and $i_*([k_t]) = tz$. Since $i_*[k - k_t] = (1-t)z = x$, we have $y = [k - k_t]$. Let U be the compact manifold obtained from M by splitting along V which we identify with a canonical lift to $\tilde{M}$ such that $\partial U = V - tV$. Using that V is an exact leaf and $\tilde{i}_*[k_t - tk] = 0$, we have a 2-chain $\tilde{C}$ in $\mathrm{t}H_2(\tilde{M}, \tilde{V}; Z)$ such that $\partial\tilde{C} = k_t - tk$. Considering the intersection of $\tilde{C}$ with U, we have a 2-chain C in $\mathrm{t}H_2(U, \partial U; Z)$ such that $\partial C = (k_t + k_-) - (tk + tk_+)$ for some 1-knots $k_\pm$ in V with $[k_\pm] \in K_\pm$ and $k_t \cap k_- = k \cap k_+ = \emptyset$. Let U^* be the compact manifold obtained from M^* by splitting along V. Then C meets U^* in a 2-chain C^* such that $c = \partial C^* - \partial C$ is a torsion and hence null-homologous 1-cycle in ∂M^*. Let $\hat{C}$ be a 2-chain in U^* with $\partial\hat{C} = \partial C$ obtained from the 2-chain C^* by adding a 2-chain in ∂M^* with boundary $-c$. Let k'_t, k'_-, k', k'_+ be longitudes of k_t, k_-, k, k_+ given by the spin structure ι, respectively. Then for a 2-chain $\hat{C}'$ in U^* with $\partial\hat{C}' = (k'_t + k'_-) - (tk' + tk'_+)$ obtained from $\hat{C}$ by moving k_t, k_-, k, k_+ to k'_t, k'_-, k', k'_+ locally, respectively, we have that $s(\hat{C}, \hat{C}')$ is an even integer. Let $\mathrm{cl}_Q(\hat{C})$ and $\mathrm{cl}_Q(\hat{C}')$ be rational 2-cycles in U^* obtained from $\hat{C}$ and $\hat{C}'$ by adding rational 2-chains in $V \cup tV$ with boundaries $-(k_t + k_-) + (tk + tk_+)$ and $-(k'_t + k'_-) - (tk' + tk'_+)$, respectively. Since $\mathrm{cl}_Q(\hat{C})$ and $\mathrm{cl}_Q(\hat{C}')$ are rationally homologous (in U^*) to rational 2-cycles in $V \cup tV \cup \partial M^*$, we see that the Q-intersection number $s_Q(\mathrm{cl}_Q(\hat{C}), \mathrm{cl}_Q(\hat{C}')) = s_Q(\mathrm{cl}_Q(\hat{C}), \hat{C}') = 0$. This means that

$$\begin{aligned}
-s(\hat{C}, \hat{C}') &= \mathrm{Link}_Q(k_t + k_-, k'_t + k'_-) - \mathrm{Link}_Q(k + k_+, k' + k'_+) \\
&= \mathrm{Link}_Q(k_t, k'_t) + 2\mathrm{Link}_Q(k_t, k_-) + \mathrm{Link}_Q(k_-, k'_-) \\
&\qquad - \mathrm{Link}_Q(k, k') - 2\mathrm{Link}_Q(k, k_+) - \mathrm{Link}_Q(k_+, k'_+).
\end{aligned}$$

Since $\frac{s(\hat{C},\hat{C}')}{2}$, $\mathrm{Link}_Q(k_t, k_-)$, $\frac{\mathrm{Link}_Q(k_-,k'_-)}{2}$, $\mathrm{Link}_Q(k, k_+)$, and $\frac{\mathrm{Link}_Q(k_+,k'_+)}{2}$ are all 0 (mod 1), it follows that

$$\frac{\mathrm{Link}_Q(k_t, k'_t)}{2} = \frac{\mathrm{Link}_Q(k, k')}{2} \quad (\mathrm{mod}\ 1).$$

Then

$$
\begin{aligned}
q_V^\iota(y) &= \frac{\mathrm{Link}_Q(k-k_t, k'-k'_t)}{2} \pmod 1 \\
&= \frac{\mathrm{Link}_Q(k,k')}{2} - \mathrm{Link}_Q(k_t,k) + \frac{\mathrm{Link}_Q(k_t,k'_t)}{2} \pmod 1 \\
&= \ell_V([k],[k]) - \ell_V([k_t],[k]) \\
&= \ell_V([k]-[k_t],[k]) \\
&= \ell_G(x,z) \\
&= \ell_G(x,(1-t)^{-1}x) \\
&= q(x).
\end{aligned}
$$

By (1) and (2) and the identity

$$s(V,\iota) = \sigma(q_V^\iota|_{\mathrm{Ker}\, i_*}) + \sigma(q_V^\iota|_{\mathrm{t}H_1(V;Z)^\theta}),$$

the identity $s(V,\iota)=\sigma(\tilde{M})$ is obtained. □

4. An application to spin structures on a homology quaternion space

A *homology quaternion space* is a closed connected oriented 3-manifold V such that $H_1(V;Z)\cong Z_2\oplus Z_2$ and the linking $\ell_V : H_1(V;Z)\times H_1(V;Z)\to Q/Z$ is hyperbolic. Then we have $\ell_V(x,x)=\ell_V(y,y)=0$ and $\ell_V(x,y)=\frac{1}{2}$ for every Z_2-basis x,y for $H_1(V;Z)$. By [**8**;Corollary 2.5], the spin linking signature $s(V,\iota)=0$ or $\frac{1}{2}$ for every spin structure ι on V. We represent x and y by disjoint knots k_x and k_y in V, respectively. By [**3**;Lemma 1.1], we have unique longitudes L_x and L_y on the tubular neighbourhoods $T(k_x)$ and $T(k_y)$ such that two paralells (with the same orientation) of L_x and L_y bound compact oriented surfaces F_{2x} and F_{2y} in $\mathrm{cl}(V-T(k_x))$ and $\mathrm{cl}(V-T(k_y))$, respectively. Using $\ell_V(x,y)=\frac{1}{2}$, we can assume that k_x meets F_{2y} transversely in one point and k_y meets F_{2x} transversely in one point. There are four spin structures on a homology quaternion space V which we can specify for a Z_2-basis x,y of $H_1(V;Z)$ as follows:

Let ι_{00} be the spin structure on V such that L_x and L_y are the Z_2-longitudes on k_x and k_y on this spin structure, respectively.
Let ι_{01} be the spin structure on V such that L_x is the Z_2-longitude on k_x and L_y is not the Z_2-longitude on k_y on this spin structure.
Let ι_{10} be the spin structure on V such that L_x is not the Z_2-longitude on k_x and L_y is the Z_2-longitude on k_y on this spin structure.
Let ι_{11} be the spin structure on V such that L_x and L_y are the non-Z_2-longitudes on k_x and k_y on this spin structure, respectively.

For the spin linking signature $s(V,\iota)$ of a spin homology quaternion space (V,ι), we have the following characterization result:

Theorem 4.1. For a homology quaternion space V and a spin structure ι on V, the following statements are mutually equivalent:
(1) $s(V,\iota)=0$.
(2) $\iota=\iota_{00}$, ι_{01}, or ι_{10} for any Z_2-basis x,y for $H_1(V;Z)$.
(3) There is a Z_2-basis x,y for $H_1(V;Z)$ on which $\iota=\iota_{00}$.

(4) There is a Z^{H_1}-manifold M with $H_1(\tilde{M};Z) = 0$ which contains V as an exact leaf with a spin support M^* in M, and ι is induced from a spin structure on M^*.
(5) $\mu(V,\iota) = 0, \frac{1}{2}$.

The following statements are also mutually equivalent:
(1′) $s(V,\iota) = \frac{1}{2}$.
(2′) $\iota = \iota_{11}$ for any Z_2-basis x, y for $H_1(V;Z)$.
(3′) There is a Z^{H_1}-manifold M with $H_1(\tilde{M};Z) \neq 0$ which contains V as a leaf with a spin support M^* in M, and ι is induced from a spin structure on M^*.
(4′) There is a Z^{H_1}-manifold M with $H_1(\tilde{M};Z) \neq 0$ which contains V as an exact leaf with a spin support M^* in M, and ι is induced from a spin structure on M^*.
(5′) $\mu(V,\iota) = \pm\frac{1}{4}$. ■

Proof. Because $s(V,\iota) = 0$ if and only if there is a Z_2-basis x, y of $H_1(V;Z)$ such that $q_V^\iota(x) = q_V^\iota(y) = 0$ (see [**8**;Corollary 2.5]), we have (1) ⇔ (2) ⇔ (3) and (1′) ⇔ (2′). Using $s(V;\iota) = 2\mu(V,\iota)$, we see that (5) ⇔ (1) and (5′) ⇔ (1′). Thus, it suffices to show that

$$\begin{gathered}(3) \Rightarrow (4) \Rightarrow (5),\\ (2') \Rightarrow (3') \Rightarrow (4') \Rightarrow (5'),\end{gathered}$$

To see that (3) ⇒ (4), we use the fact that there is a Z_2-basis x, y of $H_1(V;Z)$ such that $q_V^\iota(x) = q_V^\iota(y) = 0$. This means that L_x and L_y are Z_2-longitudes of k_x and k_y on ι. By taking homeomorphisms $f_{\pm 1} : D^2 \times D^2 \to h_{\pm 1}$, we construct a 4-manifold

$$W = V \times [-1,1] \cup h_{-1} \cup h_1$$

where we identify $T(k_x) \times (-1)$ with $f_{-1}((\partial D^2) \times D^2)$ and $T(k_y) \times 1$ with $f_1((\partial D^2) \times D^2)$ so that $L_x \times (-1)$ and $L_y \times 1$ correspond to $f_{-1}(\partial D^2 \times p)$ and $f_1(\partial D^2 \times p)$ for a point $p \in \partial D^2$. Then W is a spin 4-manifold with $H_1(W;Z) = 0$ and ∂W is the disjoint union of two closed 3-manifolds V_{-1} and V_1 such that $H_1(V_{\pm 1};Z) \cong Z$ where $k_y \times (-1)$ and $k_x \times 1$ represent generators of $H_1(V_{-1};Z)$ and $H_1(V_1;Z)$, respectively. Let M be the double of W. Then M is a spin Z^{H_1}-manifold with $H_1(\tilde{M};Z) = 0$. We show that $V = V \times 0$ of a copy of W in M is an exact leaf of M. Let M_V be the 4-manifold obtained from M by splitting along V. By construction, we have $H_1(M_V;Z) \cong Z_2 \oplus Z_2$. Then the boundary operator $\tilde{\partial}' : H_2(\tilde{M}, \tilde{M}_V;Z) \to H_1(\tilde{M}_V;Z)$ is onto, for $H_1(\tilde{M};Z) = 0$. Since

$$H_2(M, M_V;Z) \cong H_2(V \times I, V \times \partial I;Z) \cong Z_2 \oplus Z_2$$

by excision, we see that $H_2(\tilde{M}, \tilde{M}_V;Z)$ and $H_1(\tilde{M}_V;Z)$ are Λ-isomorphic to the same Λ-module $\Lambda_2 \oplus \Lambda_2$ with $\Lambda_2 = Z_2 \otimes \Lambda$, so that the Λ-epimorphism $\tilde{\partial}' : H_2(\tilde{M}, \tilde{M}_V;Z) \to H_1(\tilde{M}_V;Z)$ is an isomorphism by a Noetherian ring property. By [**7**;Theorem 2.1], this means that V is an exact leaf of M. Thus, we have (3) ⇒ (4). The assertion (4) ⇒ (5) is direct from Theorem 2.3, because $\sigma(\tilde{M}) = 0$.

To see that (2′) ⇒ (3′), we take knots k_x, k_y, and k_{x+y} in V representing x, y, $x + y$. By the assumption of (2′), the longitudes L_x on $T(k_x)$, L_y on $T(k_y)$, and L_{x+y} on $T(k_{x+y})$ are non-Z_2 longitudes on ι, respectively. We construct an orientable 4-manifold W' from $V \times [0,1]$ by identifying $T(k_x) \times 0$ with $T(k_y) \times 1$ and $T(k_y) \times 0$ with $T(k_{x+y}) \times 1$ so that $L_x \times 0$ and $L_y \times 0$ coincide with $L_y \times 1$ and $L_{x+y} \times 1$, respectively. Then W' is a spin 4-manifold. To calculate $H_1(\partial W'; Z)$, let $V_{(x,y)} = \mathrm{cl}(V - T(k_x) \cup T(k_y))$ and $V_{(y,x+y)} = \mathrm{cl}(V - T(k_y) \cup T(k_{x+y}))$. Examining the relations between k_x and F_{2y} and k_y and F_{2x}, we see that the meridians of $T(k_x)$ and $T(k_y)$ are homologous to $2L_y$ and $2L_x$ in $V_{(x,y)}$, respectively. Then we have $H_1(V_{(x,y)}; Z) \cong Z \oplus Z$ with a basis represented by L_x and L_y. Similarly, we have $H_1(V_{(y,x+y)}; Z) \cong Z \oplus Z$ with a basis represented by L_y and L_{x+y}. Thus, $H_1(\partial W'; Z) \cong Z \oplus Z \oplus Z$ with a basis represented by $L_x \times 0$, $L_y \times 0$, and a simple loop L such that $L \cap V_{(x,y)} \times 0$ is an arc and hence $L \cap V_{(y,x+y)} \times 1$ is also an arc. Attaching a 2-handle $D^2 \times D^2$ to W' along a tubular neighborhood $T(L)$ of L in $\partial W'$ with a Z_2-longitude given by a spin structure on W', we obtain a spin 4-manifold W^* such that $H_1(W^*; Z) \cong Z$ and $H_1(\partial W^*; Z) \cong Z \oplus Z$ with a basis represented by $L_x \times 0$ and $L_y \times 0$. To examine the homology $H_1(\tilde{W}^*; Z)$, let W_V^* be a 4-manifold obtained from W^* by splitting along $V = V \times \frac{1}{2}$, and $V_\pm$ the two copies of V in W_V^*. From construction, the natural homomorphisms $H_1(V_\pm; Z) \to H_1(W_V^*; Z)$ are isomorphisms. Thus, the natural homomorphism $H_1(V; Z) \to H_1(\tilde{M}^*; Z)$ is an isomorphism. Since the 3-dimensional bordism group

$$\Omega_3(S^1 \times S^1) \cong \oplus_{p+q=3} H_p(S^1 \times S^1; Z) \otimes \Omega_q = 0$$

by Conner-Floyd [**1**], there is a compact orientable (not necessarily spin) 4-manifold X such that $\partial X = \partial W^*$ and the natural homomorphism $H_1(\partial X; Z) \to H_1(X; Z)$ is an isomorphism. Then $M = M^* \cup X$ is a Z^{H_1}-manifold such that V is a leaf of M with a spin support M^* in M. From construction, the natural homomorphism $H_1(\tilde{M}^*; Z) \to H_1(\tilde{M}; Z)$ is an isomorphism, and hence $H_1(\tilde{M}; Z) \neq 0$, showing that $(2') \Rightarrow (3')$. To see that $(3') \Rightarrow (4')$, we use the following two lemmas proved later:

Lemma 4.2. For every leaf V of a Z^{H_1}-manifold M and every field $\mathbb{F}$, the natural homomorphism $i_* : H_1(V; \mathbb{F}) \to H_1(\tilde{M}; \mathbb{F})$ is an epimorphism. In particular, if $H_1(V; Z)$ is finite, then the natural homomorphism $i_* : H_1(V; Z) \to H_1(\tilde{M}; Z)$ is an epimorphism. ■

Lemma 4.3. A leaf V of a Z^{H_1}-manifold M is exact if the natural homomorphism $i_* : H_1(V; Z) \to H_1(\tilde{M}; Z)$ is a monomorphism. ■

By lemma 4.2, $i_* : H_1(V; Z) \to H_1(\tilde{M}; Z)$ is an epimorphism for a homology quaternion space V. If i_* is not an isomorphism, then we must have $H_1(\tilde{M}; Z) \cong Z_2$, because $H_1(V; Z) \cong Z_2 \oplus Z_2$ and $H_1(\tilde{M}; Z) \neq 0$. However, this is impossible since $H_1(\tilde{M}; Z)$ is $(t-1)$-divisible. Thus, i_* is an isomorphism and by Lemma 4.3 V is exact and the assertion that $(3') \Rightarrow (4')$ is proved. To show that $(4') \Rightarrow (5')$, we may assume by the preceding argument that $i_* : H_1(V; Z) \to H_1(\tilde{M}; Z)$ is an isomorphism. By Theorem 2.3, we have $2\mu(V, \iota) = \sigma(V, \iota) = \sigma(\tilde{M})$. To calculate $\sigma(\tilde{M})$ directly, we note that the elements x, tx for any non-zero element $x \in H_1(\tilde{M}; Z)$ form a Z_2-basis for $G(\tilde{M}) = H_1(\tilde{M}; Z)$, for $G(\tilde{M})$ is $(t-1)$-divisible. Further, we see that $\ell_G(x, x) = \ell_G(tx, tx) = 0$, $\ell_G(x, tx) = 1/2$, and $(1-t)^{-1}e = te$, which imply that $q(x) = q(tx) = q(x + tx) = \frac{1}{2}$. Hence we have

$$GS(q) = -2 = 2\exp(2\pi\sqrt{-1} \cdot \frac{1}{2})$$

and $\sigma(\tilde{M}) = \sigma(q) = \frac{1}{2}$, showing $(4') \Rightarrow (5')$. $\square$

Proof of Lemma 4.2. By [**11**], the natural homomorphism

$$i_*^{\mathbb{F}} : H_1(V; \mathbb{F}) \to H_1(\tilde{M}; \mathbb{F})$$

is onto for every field $\mathbb{F}$. Taking $\mathbb{F} = Q$, we see from $H_1(V; Q) = 0$ that $H_1(\tilde{M}; Z)$ is a Z-torsion Λ-module. Hence $H_1(\tilde{M}; Z)$ is finite because it is $(t-1)$-divisible. If $i_* : H_1(V; Z) \to H_1(\tilde{M}; Z)$ is not onto, then the cokernel $\mathrm{coker}(i_*)$ is a non-trivial finite abelian group and we have a prime p such that $\mathrm{coker}(i_*) \otimes Z_p \neq 0$. Then the homomorphism

$$i_* \otimes 1 : H_1(V; Z) \otimes Z_p \longrightarrow H_1(\tilde{M}; Z) \otimes Z_p$$

which is identical to $i_*^{Z_p} : H_1(V; Z_p) \to H_1(\tilde{M}; Z_p)$ is not onto. Thus, we have a contradiction. $\square$

Proof of Lemma 4.3. Let M_V be the manifold obtained from M by splitting along V. As a part of the exact sequence of the pair $(\tilde{M}, \tilde{M}_V)$, we have the following exact sequence:

$$H_2(\tilde{M}, \tilde{M}_V; Z) \xrightarrow{\tilde{\partial}'} H_1(\tilde{M}_V; Z) \xrightarrow{\tilde{i}'_*} H_1(\tilde{M}; Z).$$

Further, by excision we have an isomorphism

$$H_2(\tilde{M}, \tilde{M}_V; Z) \cong H_2(\tilde{V} \times I, \tilde{V} \times \partial I; Z) (\cong H_1(\tilde{V}; Z)).$$

Using that $i_* : H_1(V; Z) \to H_1(\tilde{M}; Z)$ is injective, we see that the boundary operator $\tilde{\partial}' : H_2(\tilde{M}, \tilde{M}_V; Z) \to H_1(\tilde{M}_V; Z)$ is injective, and thus the exact sequence above implies that the following semi-exact sequence

$$0 \to \mathrm{t}H_2(\tilde{M}, \tilde{M}_V; Z) \xrightarrow{\tilde{\partial}'} \mathrm{t}H_1(\tilde{M}_V; Z) \xrightarrow{\tilde{i}'_*} \mathrm{t}H_1(\tilde{M}; Z)$$

is exact. By [**7**;Theorem 2.1], this means that V is an exact leaf of M. $\square$

Here are two examples showing that some different punctured embeddings into S^4 produce different Rochlin invariants for the quaternion space and a homology quaternion space.

Example 4.4. Let V be the quaternion space, which is the boundary ∂N of a tubular neighborhood N of the real projective plane P^2 embedded smoothly in S^4. Then a punctured 3-manifold V^o of V is the interior of a Seifert hypersurface of a trivial 2-knot K_0 in S^4. Let ι_0 be the spin structure on V induced from the inclusion $V \subset S^4$. By a 2-handle surgery along K_0, we see that V is a leaf of the spin Z^{H_1}-manifold $M_0 = S^1 \times S^3$. Since $BH_2(\tilde{M}_0; Z) = 0$, we see that V is necessarily an exact leaf of M_0 by [**7**]. By construction, the spin structure ι_0 on V coincides with the one induced from any spin structure on M_0. Since the spin 3-manifold (V, ι_0) is the boundary of a compact spin 4-submanifold of S^4 (which has zero signature), we have $\mu(V; \iota_0) = 0$. Using $H_1(\tilde{M}_0; Z) = 0$, we see from Theorem 4.1 that there is a Z_2-basis x, y for $H_1(V; Z)$ with $\iota_0 = \iota_{00}$. On the other hand, a punctured 3-manifold V^o of the quaternion space V is a fiber of a fibered S^2-knot K in S^4 (for example, consider the 3-twist spun trefoil by E. C. Zeeman [**15**]). Let ι_1 be the spin structure on V determined by the inclusion $V^0 \subset S^4 - K \subset S^4$. The quaternion space V is a fiber of a spin fiber bundle M_K over S^1 with $H_*(M_K; Z) = H_*(S^1 \times S^3; Z)$, obtained from S^4 by a 2-handle surgery along K, and V is an exact leaf of M_K by [**7**] since $BH_2(\tilde{M}_K; Z) = 0$. By construction, the spin structure ι_1 on V coincides with the one induced from any spin structure on M_K. Since $H_1(\tilde{M}_K; Z) \cong H_1(V; Z) \neq 0$, we see from Theorem 4.1 that $\mu(V, \iota_1) = \pm\frac{1}{4}$ and $\iota_1 = \iota_{11}$ on any Z_2-basis x, y for $H_1(V; Z)$. ■

Example 4.5. Let V^P be the homology quaternion space $V \# P$ where V is the quaternion space and P is the Poincaré homology 3-sphere with $\mu(P, \iota_P) = \frac{1}{2}$ for the unique spin structure ι_P on P. A punctured 3-manifold P^o of P is a fiber of a fibered 2-knot K_P in S^4 (for example, consider the 5-twist spun trefoil by [**15**]). A punctured 3-manifold $(V^P)^0$ of V^P is the interior of a Seifert hypersurface for the S^2-knot $K_0 \# K_P = K_P$ in S^4. Let ι_0^P be the spin structure on V^P determined by the inclusion $(V^P)^0 \subset S^4 - K_P \subset S^4$, which is equal to the spin structure determined from ι_0 in Example 4.4 by construction. The homology quaternion space V^P is a leaf of a spin Z^{H_1}-manifold M_0^P with $H_*(M_0^P; Z) = H_*(S^1 \times S^3; Z)$, obtained from S^4 by a 2-handle surgery along K_P, and V^P is an exact leaf of M_0^P by [**7**] since $BH_2(\tilde{M}_0^P; Z) = 0$. We note that the spin structure ι_0^P is the one induced from any spin structure on M_0^P and $H_1(\tilde{M}_0^P; Z) = 0$. In this case, we see that $\mu(V^P; \iota_0^P) = \frac{1}{2}$, and there is a Z_2-basis x, y for $H_1(V^P; Z)$ with $\iota_0^P = \iota_{00}$ as it is shown in Theorem 4.1. On the other hand, a punctured manifold $(V^P)^0$ of V^P is a fiber of a fibered S^2-knot $K \# K_P$ in S^4, where K denotes the S^2-knot in Example 4.4. Let ι_1^P be the spin structure on V^P determined by the inclusion $(V^P)^0 \subset S^4 - K \# K_P \subset S^4$, which is equal to the spin structure determined from ι_1 in Example 4.4 by construction. The homology quaternion space V^P is a fiber of a spin fiber bundle M_K^P over S^1 with $H_*(M_K^P; Z) = H_*(S^1 \times S^3; Z)$, obtained from S^4 by a 2-handle surgery along $K \# K_P$, and V^P is an exact leaf of M_K^P by [**7**] since $BH_2(\tilde{M}_K^P; Z) = 0$. We note that the spin structure ι_1^P is the one induced from any spin structure on M_K^P and $H_1(\tilde{M}_K^P; Z) \cong H_1(V^P; Z) \neq 0$. In this case, we see that

$$\mu(V^P, \iota_1^P) = \mu(V, \iota_1) + \frac{1}{2} = \mp\frac{1}{4},$$

and $\iota_1^P = \iota_{11}$ on any Z_2-basis x, y for $H_1(V^P; Z)$ as it is shown in Theorem 4.1. ■

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